# 3
# Model Presentation

The model presented here is a tree-structured mixture of models (Tree-MM). The core idea is to model the weight functions $g$ in (2-1) as a smooth transition regression-tree model, as in (18).

To represent mathematically a complex regression-tree model, we introduce the following notation. The root node is at position $0$ and a parent node at position $j$ generates left- and right-child nodes at positions $2j+1$ and $2j+2$, respectively. Every parent node has an associated split variable $x_{s_j t} \in \mathbf{x}_t$, where $s_j \in \mathbb{S} = \{1, 2, \ldots, q\}$. Furthermore, let $\mathbb{J}$ and $\mathbb{T}$ be the sets of indexes of the parent and terminal nodes, respectively. Then, any binary tree $\mathbb{JT}$ can be fully determined by $\mathbb{J}$ and $\mathbb{T}$.

**Definição 3.1** *The random variable $y_t \in \mathbb{R}$ follows a tree-structured mixture of models if its conditional probability density function (p.d.f.) $f(y_t|\mathbf{x}_t;\boldsymbol{\theta})$ is written as*

$$f(y_t|\mathbf{x}_t;\boldsymbol{\theta}) = \sum_{i\in\mathbb{T}} B_i(\mathbf{x}_t;\boldsymbol{\theta}_i)\pi(y_t|\mathbf{x}_t;\boldsymbol{\beta}_i'\widetilde{\mathbf{x}}_t,\sigma_i^2), \tag{3-1}$$

*where $\mathbf{x}_t \in \mathbb{R}^q$ is a vector of explanatory variables, $\boldsymbol{\theta}$ is the conditional p.d.f. parameter vector, $\pi(\cdot)$ is the Gaussian p.d.f. with parameter vector $\boldsymbol{\psi}_i = (\boldsymbol{\beta}', \sigma_i)'$, $\widetilde{\mathbf{x}}_t = (1, \mathbf{x}_t')'$,*

$$B_i\left(\mathbf{x}_t;\boldsymbol{\theta}_i\right) = \prod_{j\in\mathbb{J}} g(x_{s_{j,t}};\gamma_j,c_j)^{\frac{n_{i,j}(1+n_{i,j})}{2}} \left[1 - g(x_{s_{j,t}};\gamma_j,c_j)\right]^{(1-n_{i,j})(1+n_{i,j})}, \tag{3-2}$$

*and*

$$n_{i,j} = \begin{cases} -1 & \text{if the path to leaf } i \text{ does not include the parent node } j; \\ 0 & \text{if the path to leaf } i \text{ includes the right-child node of the parent node } j; \\ 1 & \text{if the path to leaf } i \text{ includes the left-child node of the parent node } j. \end{cases} \tag{3-3}$$

*Let $\mathbb{J}_i$ be the subset of $\mathbb{J}$ containing the indexes of the parent nodes that form the path to leaf $i$. Then, $\boldsymbol{\theta}_i$ is the vector containing all the parameters $\boldsymbol{\nu}_k = (\gamma_k, c_k)$ such that $k \in \mathbb{J}_i$, $i \in \mathbb{T}$. Furthermore,*

$$g(x_{s_k,t};\gamma_k,c_k) = \frac{1}{1+e^{-\gamma_k(x_{s_k,t}-c_k)}}. \tag{3-4}$$

It is easy to show that $\sum_{i \in \mathbb{T}} B_i(\mathbf{x}_t; \theta_i) = 1$.