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Claudemir Alcantara dos Santos Junior

Regularity theory for bi-Laplacian and degenerate/singular normalized p-Laplacian equations with applications

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática, do Departamento de Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor : Prof. Boyan Slavchev Sirakov
Co-advisor: Dr. Makson Sales Santos

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To my family, for their support
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Abstract

Santos Junior, Claudemir Alcantara dos; Sirakov, Boyan (Advisor); Santos, Makson (Co-Advisor). **Regularity theory for bi-Laplacian and degenerate/singular normalized p-Laplacian equations with applications**. Rio de Janeiro, 2025. 81p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

This thesis establishes new regularity results for three classes of partial differential equations. First, we prove $C^{2,\alpha}$-regularity for a semi-linear variant of the bi-Laplacian equation in the superlinear, subquadratic setting. Second, we investigate a dead core problem for a degenerate/singular normalized p-Laplacian with power-type degeneracy law. We derive geometric results, and along the set $\partial\{u > 0\} \cap B_1$, we establish regularity estimates, including non-degeneracy, free boundary porosity, and a Strong Maximum Principle for the critical case. Finally, we establish sharp Hölder regularity for the gradient of viscosity solutions to degenerate/singular normalized p-Laplacian equations, where the degeneracy covers a general class, and prove Sobolev estimates for the homogeneous equation.

Keywords

Regularity theory; Degenerate equations; Singular equations; Dead core problems; Hölder regularity.

Resumo

Santos Junior, Claudemir Alcantara dos; Sirakov, Boyan; Santos, Makson. **Teoria de regularidade para o bi-Laplaciano e equações degeneradas/singulares do p-Laplaciano normalizado com aplicações**. Rio de Janeiro, 2025. 81p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Esta tese estabelece novos resultados de regularidade para três classes de equações diferenciais parciais. Primeiramente, provamos a regularidade $C^{2,\alpha}$ para uma variante semilinear da equação do bi-Laplaciano no regime superlinear e subquadrático. Em segundo lugar, investigamos um problema de núcleo morto para um p-Laplaciano normalizado degenerado/singular com degenerescência do tipo potência. Derivamos resultados geométricos e, ao longo do conjunto $\partial\{u > 0\} \cap B_1$, estabelecemos estimativas de regularidade, incluindo não-degenerescência, porosidade da fronteira livre e um Princípio do Máximo Forte para o caso crítico. Finalmente, estabelecemos regularidade de Hölder ótima para o gradiente de soluções no sentido da viscosidade do operador p-Laplaciano normalizado degenerado/singular, na qual a degenerescência é dada por uma lei geral, e provamos estimativas de Sobolev para a equação homogênea.

Palavras-chave

Teoria da regularidade; Equações degeneradas; Equações singulares; Problemas de núcleo morto; Regularidade de Hölder.

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The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living

Henri Poincaré, *Science and Method.*

1 Introduction

This thesis is dedicated to the development of regularity results for a triad of partial differential equations over three distinct regimes. It includes regularity estimates for a bi-Laplacian equation in the superlinear and subquadratic regime [2]; geometric regularity estimates for the normalized p-Laplacian operator, which exhibit either degenerate or singular behaviour when the gradient vanishes, where a polynomial behaviour gives the degeneracy law [1]; as well a interior regularity result for a class of degenerate/singular normalized p-Laplacian with a general law of degeneracy [4].

In the first part of this thesis, we examine the regularity of weak solutions to the semi-linear bi-Laplacian equation

$$\Delta^2 u = f(x, u, Du) \quad \text{in } \Omega, \tag{1-1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain, $\Delta^2 u := \Delta(\Delta u)$ denotes the bi-Laplacian operator, and the nonlinearity $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies a polynomial growth condition. Our findings report on $C^{2,\sigma}$-regularity estimates for the solutions to (1-1), depending on the growth regime of the nonlinearity.

Our reasoning in the Chapter 3 relies on a reduction argument. Namely, we write the fourth-order PDE as a system of two second-order equations. This strategy is inspired by ideas introduced in the works [41, 42] of Evans. The gist of those papers is to design new PDE methods to address the Mather minimization principle in dynamics, as well as the weak KAM theory.

Our approach stems from this class of ideas, as we write the single equation (1-1) as a coupling. Indeed, the integrability available for D^2u in the class of weak solutions allows us to define a function $v \in L^q(\Omega)$ as $v := \Delta u$. The definition of weak solution to (1-1) then implies that v is a *very* weak solution to the Poisson equation $\Delta v = f$. As a consequence, we render the bi-Laplacian equation as the system

$$\begin{cases} \Delta u = v & \text{in } \Omega, \\ \Delta v = f(x, u, Du) & \text{in } \Omega. \end{cases}$$

The unknown for the system is a pair (u, v) in a suitable functional space, where u is a strong solution for the first equation in the system, while v satisfies the second one in the very weak sense.

In this context, the regularity of the solutions to (1-1) benefits from the interplay between the regularity estimates available at the level of the equations taken on their own. An L^q-regularity theory for very weak solutions builds upon standard Sobolev embeddings to produce improved integrability for v in Lebesgue spaces. In turn, the integrability of v affects the regularity of u. Indeed, we prove that very weak solutions to

$$\Delta v = f(x,u,Du) \quad \text{in} \quad \Omega$$

are in $W^{2,s}_{\text{loc}}(\Omega)$, for some $s \in (d/2, d]$. Hence, u solves a Poisson equation with right-hand side in $C^{0,2-d/s}_{\text{loc}}(\Omega)$. This fact unlocks a Schauder regularity theory for the solutions to (1-1).

In the Chapter 4, our goal is to investigate geometric regularity estimates for problems governed by a quasi-linear elliptic model in non-divergence form, given by

$$|Du(x)|^{\gamma}\Delta_p^{\mathrm{N}} u(x) = f(x,u) \quad \text{in} \quad B_1, \tag{1-2}$$

which may exhibit either degenerate(or singular) behaviour when the gradient vanishes, with $\gamma \in (-1,\infty)$, $p \in (1,\infty)$. The right-hand side of the equation (1-2) is the mapping $u \mapsto f(x,u) \lesssim \mathfrak{a}(x)u_+^m$, with $m \in [0,\gamma+1)$, does not decay sufficiently fast at the origin.

The results presented in Chapter 4 are motivated in part by the work of Teixeira [70]. We begin by showing that non-negative bounded viscosity solutions of a variation of (1-2), obtained by multiplying the source term by a small constant, can be flattened. As a consequence, we are in a position to use an inductive process to obtain an improved geometric C^{κ}_{loc} regularity along the set $\partial\{u>0\} \cap B_1$, which is the free boundary of the model. That is, for a positive universal constant $C > 0$, we establish

$$\sup_{B_\rho(z_0)} u \leq C\rho^{\kappa}.$$

The ideas in [33] enable us to prove a Comparison Principle, which we use to compare the solution of (1-2) with a suitable barrier function. This ensures the existence of a small universal constant $c > 0$ such that

$$c\rho^{\kappa} \leq \sup_{B_\rho(z_0)} u,$$

allowing us to control the decay rate of viscosity solutions.

Our strategy to obtain the measure properties in Chapter 4 is to combine the improved regularity with the non-degeneracy estimate of solutions to (1-2), in order to show that the region $\{u>0\}$ maintains a uniform positive density along the free boundary. In particular, the formation of cusps at free boundary points is prevented. Additionally, we can show that the set $\partial\{u > 0\} \cap B_1$ is a $(\tau/2)$-porous set. Furthermore, we obtain a Liouville-type result for entire solutions, given that the growth of these solutions at infinity is controlled. Inspired by the work of Araújo, Leitão, and Teixeira [6], we explore the borderline equation given by

$$|Du(x)|^\gamma \Delta_p^{\mathrm{N}} u(x) = \mathfrak{a}(x) u_+^{\gamma+1}(x), \tag{1-3}$$

where the non-degeneracy estimate fails to hold, and consequently, this creates a more challenging scenario. However, in this context, we are still able to prove a Strong Maximum Principle; that is, we show that a nonnegative bounded viscosity solution to the borderline equation is either strictly positive or identically zero in the domain.

In the last part of this thesis, we investigate the regularity theory for viscosity solutions to a class of degenerate/singular normalized p-Laplace equations of the form

$$-\Phi(x, |Du|)\Delta_p^{\mathrm{N}} u = f(x) \quad \text{in} \quad B_1, \tag{1-4}$$

where the degeneracy law $\Phi(\cdot, \cdot)$ satisfies a set of assumptions described in the Chapter 2, $p \in (1, +\infty)$ and $f \in L^\infty(B_1)$. We emphasize that the degeneracy law $\Phi(\cdot, \cdot)$ covers many important cases in the literature, such as power type and variable exponent degeneracies, as discussed in Section 5.1. In particular, we establish sharp Hölder regularity for the gradient of viscosity solutions. In the case where p is close to 2, we obtain an improved regularity and prove Sobolev estimates for the homogeneous equation.

Chapter 5 follows a classical strategy, where we prove a compactness estimate for both the degenerate and singular cases using a technical lemma for viscosity solutions; see [33]. This is followed by a stability result, which leads to an approximation lemma. Consequently, we can find an affine function that approximates the solution u of (1-4) in a suitable sense at small scales, thereby establishing $C^{1,\alpha}-$regularity.

We emphasize that the ideas presented are similar when p is sufficiently close to 2. However, in this case, the exponent $\tilde{\alpha}$ is no longer bounded from below, since the limiting equation corresponds to the Laplacian operator.

Our third main result in Chapter 5 concerns Sobolev regularity in the case where $f \equiv 0$. In this setting, we show that solutions belong to the space $W^{2,q}(B_{1/2})$ for every $q \in [1, +\infty)$. See Theorem 5.13. We emphasize that our findings are novel even in the classical p-Laplace case, where $\Phi(x, t) = t^{p-2}$. In particular, Theorem 5.13 implies that solutions to

$$-\Delta_p u = 0 \quad \text{in } B_1,$$

are of class $C^{1,\alpha}$ for every $\alpha \in (0,1)$, provided p is sufficiently close to 2.

2
Main assumptions and preliminary material

In this chapter we enunciate the thesis's main assumptions, and gather some definitions and auxiliary results. Since the right-hand side term f satisfies different conditions within Chapters 3, 4 and 5, we start by providing these assumptions, which concern with the growth regime and the regularity of the source term.

2.1
Assumptions

2.1.1
Assumptions for the source term

We start this subsection with the assumptions on the nonlinearity f which are essential in the chapter 3.

A 2.1 (Growth condition) *We assume that the function $f : \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ satisfies the following polynomial growth condition*

$$|f(x, r, \xi)| \leq h(x) + C\left(|r|^\sigma + |\xi|^\theta\right), \tag{2-1}$$

for $h \in L^d(B_1)$ and fixed constants $C > 0$ and

$$\sigma, \theta \in [1, 2). \tag{2-2}$$

We emphasize that the main result in Chapter 3 depends on the growth regime of f. However, we do not require that the source term belong to $C(\Omega)\cup L^\infty(\Omega)$.

Next, we give an assumption which plays a crucial role in Chapter 4, and pertains to the bounded *Thiele modulus* for a modulating function $\mathfrak{a}$.

A 2.2 (Thiele modulus) *We assume that $\mathfrak{a} : \Omega \to \mathbb{R}_+$ (referred to as the Thiele modulus or modulating function) is a continuous and bounded function (strictly positive and finite). Specifically, there exist constants $\Lambda_\mathfrak{a} \geq \lambda_\mathfrak{a} > 0$ such that*

$$\Lambda_\mathfrak{a} \geq \sup_\Omega \mathfrak{a}(x) \geq \mathfrak{a}(x) \geq \inf_\Omega \mathfrak{a} \geq \lambda_\mathfrak{a}.$$

The next assumption is related to the right-hand side of the equations analysed in Chapter 5.

A 2.3 (Source term) *We assume that $f \in C(B_1) \cap L^\infty(B_1)$.*

Although we assume f to be continuous, it is a mere technicality; none of the estimates presented will depend on the modulus of continuity of f.

2.1.2
Assumptions for the degeneracy laws

We proceed with the definition of almost increasing and almost decreasing.

Definition 2.4 *Let $v : [0, \infty) \to [0, \infty)$. We say that v is almost increasing if there is a constant $M \geq 1$ such that*

$$v(t_1) \leq M v(t_2) \quad \textit{for} \quad 0 \leq t_1 < t_2. \tag{2-3}$$

Similarly, we say that v is almost decreasing if there is a constant $M \geq 1$ such that

$$v(t_2) \leq M v(t_1) \quad \textit{for} \quad 0 \leq t_1 < t_2. \tag{2-4}$$

Next, we enunciate the assumptions related to the law $\Phi(\cdot, \cdot)$ that are satisfied throughout this thesis.

A 2.5 (Growth property) *We suppose there exist constants $-1 < \gamma \leq \mu$ such that for every $x \in B_1$ the map*

$$t \mapsto \frac{\Phi(x,t)}{t^\gamma}$$

is almost increasing, and the map

$$t \mapsto \frac{\Phi(x,t)}{t^\mu}$$

is almost decreasing.

A 2.6 (Degeneracy/Singularity law) *We assume that $\Phi : [0, \infty) \to [0, \infty)$ is a continuous function and there exist a constant $B \geq 1$ such that*

$$\frac{1}{B} \leq \Phi(x, 1) \leq B \quad \textit{for all} \quad x \in B_1. \tag{2-5}$$

2.2
Definitions of Solutions

Due to the nature of the problems investigated in this thesis and the diversity of approaches between chapters, it became necessary to employ different notions of solution. We begin with the definitions that will be used in the chapter 3. First, we introduce the weak solution for the bi-Laplacian equation in a distributional sense.

Definition 2.7 (Local weak solution of the bi-Laplacian equation) *We say that a function $u \in W^{2,2}_{loc}(\Omega)$ is a local weak solution to*

$$\Delta^2 u = f(x,u,Du) \quad in\ \Omega, \tag{2-6}$$

if

$$\int_\Omega \Delta u\, \Delta\varphi\, dx = \int_\Omega f(x,u,Du)\,\varphi\, dx,$$

for every $\varphi \in C_c^\infty(\Omega)$.

Next, we will be concerned with Poisson's equation

$$\Delta w = g \quad \text{in } \Omega, \tag{2-7}$$

and explore two notions of solution, those of very weak and L^q-strong solution. For completeness, we recall these notions in what follows.

Definition 2.8 (Very weak solution of the Poisson equation) *Let* $g \in L^1_{loc}(\Omega)$. *A function* $w \in L^1_{loc}(\Omega)$ *is a very weak solution to* (2-7) *if*

$$\int_\Omega w\, \Delta\varphi\, dx = \int_\Omega g\,\varphi\, dx,$$

for every $\varphi \in C_c^\infty(\Omega)$.

Definition 2.9 (L^q-strong solution of the Poisson equation) *Let* $g \in L^q(\Omega)$ *for* $q > 1$. *We say that* $w \in W^{2,q}(\Omega)$ *is an* L^q-*strong solution to* (2-7) *if*

$$\Delta w(x) = g(x), \quad a.e.\ x \in \Omega.$$

Since our approach in the chapter 3 relies on rewriting the equation (2-6) as the system

$$\begin{cases} \Delta u = v & \text{in } \Omega, \\ \Delta v = f(x,u,Du) & \text{in } \Omega. \end{cases} \tag{2-8}$$

We proceed by defining a notion of solution to the system in (2-8), relating (2-6) with the latter.

Definition 2.10 *The pair* $(u,v) \in W^{2,q}(\Omega) \times L^q(\Omega)$ *is a solution to* (2-8) *if* u *is an* L^q-*strong solution to the first equation in* (2-8) *whereas* v *is a very weak solution to the second equation in* (2-8).

Next, we present the definitions to be employed in Chapter 4. We begin with the definition of a viscosity solution for the degenerate case.

Definition 2.11 (Viscosity Solution - case $\gamma > 0$) *Let* f *be a continuous function in* $C(\Omega \times \mathbb{R}_+)$. *An upper semicontinuous function* $u \in C(\Omega)$ *is called a viscosity subsolution (resp. supersolution) of*

$$|Du|^\gamma \Delta_p^{\mathrm{N}} u = f(x, u(x)), \tag{2-9}$$

if whenever $\phi \in C^2(\Omega)$ *and* $u - \phi$ *has a local maximum (resp. minimum) at* $x_0 \in \Omega$*, the following conditions are satisfied:*

1) *If* $D\phi(x_0) \neq 0$*, then*

$$|D\phi(x_0)|^\gamma \Delta_p^{\mathrm{N}} \phi(x_0) \geq f\left(x_0, \phi(x_0)\right) \quad \Big(\textit{resp.} \quad |D\phi(x_0)|^\gamma \Delta_p^{\mathrm{N}} \phi(x_0) \leq f\left(x_0, \phi(x_0)\right)\Big)\,;$$

2) *If* $D\phi(x_0) = 0$*, then*

   a) *For* $p \geq 2$

$$\begin{aligned} &\Delta\phi(x_0) + (p-2)\lambda_{\max}\left(D^2\phi(x_0)\right) \geq f\left(x_0, \phi(x_0)\right) \\ &\Big(\textit{resp.} \quad \Delta\phi(x_0) + (p-2)\lambda_{\min}\left(D^2\phi(x_0)\right) \leq f\left(x_0, \phi(x_0)\right)\Big) \end{aligned}$$

   where $\lambda_{\max}\left(D^2\phi(x_0)\right)$ *e* $\lambda_{\min}\left(D^2\phi(x_0)\right)$ *are the largest and smallest eigenvalues of the matrix* $D^2\phi(x_0)$ *respectively.*

   b) *For* $1 < p < 2$

$$\begin{aligned} &\Delta\phi(x_0) + (p-2)\lambda_{\min}\left(D^2\phi(x_0)\right) \geq f\left(x_0, \phi(x_0)\right) \\ &(\textit{resp.} \quad \Delta\phi(x_0) + (p-2)\lambda_{\max}\left(D^2\phi(x_0)\right) \leq f\left(x_0, \phi(x_0)\right)). \end{aligned}$$

Finally, we say that u *is a solution to* (2-9) *in the viscosity sense if it satisfies the conditions for both subsolutions and supersolutions.*

The following definition for the case $\gamma < 0$ is adapted from the one introduced by Julin and Juutinen in [50] for the singular p-Laplacian.

Definition 2.12 ([11, Definition 2.2]) *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain,* $1 < p < \infty$*, and* $-1 < \gamma < 0$*. A function* u *is a viscosity subsolution of* (2-9) *if* $u \not\equiv \infty$ *and if for all* $\varphi \in C^2(\Omega)$ *such that* $u - \varphi$ *attains a local maximum at* x_0 *and* $D\varphi(x) \neq 0$ *for* $x \neq x_0$*, one has*

$$\lim_{r\to 0} \inf_{\substack{x \in B_r(x_0),\\ x \neq x_0}} \left(-|D\varphi(x)|^\gamma \Delta_p^{\mathrm{N}} \varphi(x)\right) \leq -f(x_0, \varphi(x_0)).$$

A function u *is a viscosity supersolution of* (2-9) *if* $u \not\equiv \infty$ *and for all* $\varphi \in C^2(\Omega)$ *such that* $u - \varphi$ *attains a local minimum at* x_0 *and* $D\varphi(x) \neq 0$ *for* $x \neq x_0$*, one has*

$$\lim_{r\to 0} \sup_{\substack{x \in B_r(x_0),\\ x \neq x_0}} \left(-|D\varphi(x)|^\gamma \Delta_p^{\mathrm{N}} \varphi(x)\right) \geq -f(x_0, \varphi(x_0)).$$

We say that u is a viscosity solution of (2-9) in Ω if it is both a viscosity subsolution and a viscosity supersolution.

We emphasize that Birindelli and Demengel introduced an alternative definition of solutions in [18, 19, 20]. This definition represents a variation of the standard notion of viscosity solutions for (2-9), which avoids the use of test functions with vanishing gradients at the testing point.

Definition 2.13 ([11, Definition 2.3]) *Let $-1 < \gamma < 0$ and $p > 1$. A lower semicontinuous function u is a viscosity supersolution of (2-9) in Ω if for every $x_0 \in \Omega$ one of the following conditions holds:*

i) Either for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at x_0 and $D\varphi(x_0) \neq 0$, we have

$$-|D\varphi(x_0)|^{\gamma} \Delta_p^{\mathrm{N}} \varphi(x_0) \geq -f(x_0, \varphi(x_0)).$$

ii) Or there exists an open ball $B_\eta(x_0) \subset \Omega$, $\eta > 0$, such that u is constant in $B_\eta(x_0)$ and $-f(x, u) \leq 0$ for all $x \in B_\eta(x_0)$.

An upper semicontinuous function u is a viscosity subsolution of (2-9) in Ω if for all $x_0 \in \Omega$ one of the following conditions holds:

i) Either for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ attains a local maximum at x_0 and $D\varphi(x_0) \neq 0$, we have

$$-|D\varphi(x_0)|^{\gamma} \Delta_p^{\mathrm{N}} \varphi(x_0) \leq -f(x_0, \varphi(x_0)).$$

ii) Or there exists an open ball $B_\eta(x_0) \subset \Omega$, $\eta > 0$, such that u is constant in $B_\eta(x_0)$ and $-f(x, u) \geq 0$ for all $x \in B_\eta(x_0)$.

Now, we introduce the concept of viscosity solution to be used in the chapter 5.

Definition 2.14 (Viscosity solution) *Let $1 < p < \infty$ and $f \in C(B_1) \cap L^\infty(B_1)$. We say that $u \in C(B_1)$ is a viscosity subsolution to (5-1) if for every $x_0 \in B_1$, either there exists η such that u is constant in $B_\eta(x_0)$ and $0 \leq f(x)$ for all $x \in B_\eta(x_0)$, or for any $\varphi \in C^2(B_1)$ such that $u - \varphi$ has a local maximum at x_0 and $D\varphi(x_0) \neq 0$, we have*

$$-\Phi(x_0, |D\varphi(x_0)|)\Delta_p^N \varphi(x_0) \leq f(x_0).$$

Similarly, we say that $u \in C(B_1)$ is a viscosity supersolution to (5-1) *if for every $x_0 \in B_1$, either there exists η such that u is constant in $B_\eta(x_0)$ and $0 \geq f(x)$ for all $x \in B_\eta(x_0)$, or for any $\varphi \in C^2(B_1)$ such that $u-\varphi$ has a local minimum at x_0 and $D\varphi(x_0) \neq 0$, we have*

$$-\Phi(x_0, |D\varphi(x_0)|)\Delta_p^N \varphi(x_0) \geq f(x_0).$$

We say that a function $u \in C(B_1)$ is a viscosity solution if it is both a viscosity sub and supersolution. Finally, $u \in C(B_1)$ is a normalized viscosity solution if $\sup_{B_1} |u| \leq 1$.

2.3
Auxiliary results

The results in this section are organised following the order of the chapters. First, we ensure that the definitions related to chapter 4 for the singular scenario are equivalent.

Proposition 2.15 ([11, Proposition 2.4]) *Definitions 2.12 and 2.13 are equivalent.*

Next, we observe that the singular case can be reformulated as the analysis of the normalized p-Laplacian problem incorporating a lower-order term. The proof follows the same ideas as the one in [11, Lemma 2.5], and for this reason, we will omit the proof here.

Lemma 2.16 *Let $\gamma \in (-1, 0)$ and $p > 1$. Assume that u is a viscosity solution to*

$$|Du(x)|^\gamma \Delta_p^{\mathrm{N}} u(x) = f(x, u) \quad in \quad \Omega.$$

Then, u is a viscosity solution to

$$\Delta_p^{\mathrm{N}} u = f(x, u)|Du|^{-\gamma} \quad in \quad \Omega.$$

Theorem 2.17 ([11, Theorem 1.1] and [72, Theorem 1.1]) *Assume that $\gamma > -1$, $p > 1$, and $f \in L^\infty(\Omega) \cap C(\Omega)$ be Hölder continuous when $\gamma < 0$. Then, there exists $\alpha = \alpha(p, d, \gamma) > 0$ such that any viscosity solution u of*

$$|Du|^\gamma \Delta_p^{\mathrm{N}} u = f(x) \quad in \quad \Omega$$

belongs to $C^{1,\alpha}_{loc}(\Omega)$, and for any $\Omega' \subset\subset \Omega$,

$$[u]_{C^{1,\alpha}(\Omega')} \leq \mathrm{C} = \mathrm{C}(p, d, n_\Omega, \gamma, n', \|u\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}),$$

where $n_\Omega = \mathrm{diam}(\Omega)$ *and* $n' = \mathrm{dist}(\Omega', \partial\Omega)$. *In particular, we have the following gradient estimate:*

$$\|Du\|_{L^\infty(\Omega')} \leq \mathrm{C}(p, d, n_\Omega, \gamma, n') \left(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}^{\frac{1}{\gamma+1}} \right).$$

Remark 2.18 *In* **[11, Theorem 1.1]**, *after the proof of the theorem, there exists a note in which the authors explain that in the case* $-1 < \gamma < 0$, *the Hölder regularity assumption of* f *can be removed.*

The following lemma will be instrumental in chapter 4 regarding the proof of the improvement of flatness. It is a sort of cutting lemma.

Lemma 2.19 (Cutting Lemma, [11, Lemma 2.6]) *Let* $\gamma > -1$ *and* $p > 1$. *Assume that* w *is a viscosity solution of*

$$-|Dw + \xi|^\gamma \left(\Delta w - (p-2) \left\langle D^2 w \frac{(Dw+\xi)}{|Dw+\xi|}, \frac{(Dw+\xi)}{|Dw+\xi|} \right\rangle \right) = 0.$$

Then, w *is a viscosity solution of*

$$-\Delta w - (p-2) \left\langle D^2 w \frac{(Dw+\xi)}{|Dw+\xi|}, \frac{(Dw+\xi)}{|Dw+\xi|} \right\rangle = 0.$$

Theorem 2.20 ([68, Theorem 5.9]) *A function* u *is a viscosity solution to*

$$-\Delta_p^{\mathrm{N}} u = 0 \quad in \quad \Omega$$

if and only if it is a weak solution to

$$-\Delta_p u = 0 \quad in \quad \Omega.$$

Next, we state a lemma that will establish the Comparison principle in chapter 4, and subsequently, this result leads to the compactness of solutions in chapter 5.

Lemma 2.21 ([33, Theorem 3.2]) *Let* B_1 *be a unit ball in* $\mathbb{R}^d$. *Let* $\Omega \subset B_1$, $u, v \in C(B_1)$ *and* φ *be twice continuously differentiable in* $\Omega \times \Omega$. *Define*

$$w(x, y) \coloneqq u(x) + v(y) \quad \forall (x, y) \in \Omega \times \Omega,$$

and assume that $w - \varphi$ attains the maximum at $(x_0, y_0) \in \Omega \times \Omega$. Then for each $\varepsilon > 0$, there exists $X, Y \in \mathcal{S}(d)$ such that

$$(D_x\varphi(x_0, y_0), X) \in \overline{\mathcal{J}}^{2,+}u(x_0), \quad (D_y\varphi(x_0, y_0), Y) \in \overline{\mathcal{J}}^{2,+}v(y_0).$$

Moreover, the block diagonal matrix with entries X and Y satisfies

$$-\left(\frac{1}{\varepsilon} + \|B\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq B + \varepsilon B^2,$$

where $B = D^2\varphi(x_0, y_0) \in \mathcal{S}(d)$.

Now, we present a characterization of solutions for the singular case $\gamma \in (-1, 0)$ in chapter 5. More precisely, we observe that solutions to

$$-\Phi(x, |Du|)\Delta_p^{\mathrm{N}} u = f(x) \tag{2-10}$$

are also solutions to a related equivalent equation. This characterization will help us to obtain better compactness properties in this scenario.

Proposition 2.22 *Assume that $-1 < \gamma < 0$, and that $A2.3 - A2.6$ hold true. Let u be a viscosity solution to* (2-10)*, then u solves*

$$|Du|^{-\gamma}\Phi(x, |Du|)\Delta_p^{\mathrm{N}} u = |Du|^{-\gamma} f(x) \tag{2-11}$$

in the viscosity sense.

Proof. Let us take a test function $\varphi \in C^2(B_1)$ such that $u - \varphi$ has a local maximum at $x_0 \in B_1$. If the gradient of our test function does not vanish at the point x_0, we have $|D\varphi(x_0)|^{-\gamma} > 0$, and

$$-\Phi(x_0, |D\varphi(x_0)|)\Delta_p^{\mathrm{N}}\varphi(x_0) \leq f(x_0).$$

Hence,

$$-|D\varphi(x_0)|^{-\gamma}\Phi(x_0, |D\varphi(x_0)|)\Delta_p^{N}\varphi(x_0) \leq |D\varphi(x_0)|^{-\gamma} f(x_0),$$

and, as a consequence, u is a subsolution to (2-11).

Then, let us focus our attention on the case where x_0 is a critical point of φ, *i.e.*, $D\varphi(x_0) = 0$. In this context, two scenarios are possible. For the first one, we assume that the $D^2\varphi(x_0)$ is invertible, and consequently, the point x_0 is a locally isolated critical point of φ. Thus, for every $\varepsilon > 0$ there exist a small $\rho_0 \geq \rho$ and a sequence $x_\rho \to x_0$ such that $D\varphi(x_\rho) \neq 0$ and

$$-\Phi(x_\rho, |D\varphi(x_\rho)|)\Delta_p^{\mathrm{N}}\varphi(x_\rho) \leq f(x_\rho) + \varepsilon.$$

As a consequence, $|D\varphi(x_\rho)|^{-\gamma} > 0$ and

$$-|D\varphi(x_\rho)|^{-\gamma}\Phi(x_\rho, |D\varphi(x_\rho)|)\Delta_p^{\mathrm{N}}\varphi(x_\rho) \leq |D\varphi(x_\rho)|^{-\gamma}\left(f(x_\rho)+\varepsilon\right).$$

Letting $\varepsilon, \rho \to 0$, we obtain

$$0 = -|D\varphi(x_0)|^{-\gamma}\Phi(x_0, |D\varphi(x_0)|)\Delta_p^{\mathrm{N}}\varphi(x_0) \leq |D\varphi(x_0)|^{-\gamma}f(x_0) = 0.$$

If the $D^2\varphi(x_0)$ is not invertible, we can take a semi-positive definite matrix $M \in \mathcal{S}(d)$ such that, for every $\delta > 0$, we have that $D^2\varphi(x_0) - \delta M$ is invertible. Let us define the auxiliary function

$$\varphi_\delta(x) \coloneqq \varphi(x) - \delta\langle M(x-x_0), x-x_0\rangle,$$

and apply the same argument as in the first scenario with φ replaced by φ_δ to obtain

$$0 = -|D\varphi_\delta(x_0)|^{-\gamma}\Phi(x_0, |D\varphi_\delta(x_0)|)\Delta_p^{\mathrm{N}}\varphi_\delta(x_0) \leq |D\varphi_\delta(x_0)|^{-\gamma}f(x_0) = 0.$$

Thereby, letting $\delta \to 0$, we obtain the desired result. Thus, in both scenarios we have that u is a subsolution to (2-11).

The procedure to show that u is a supersolution is analogous, and we will omit it here. Therefore, u is a viscosity solution to (2-11). ■

3
Regularity for a bi-Laplacian equation

This chapter is dedicated to examining the regularity of weak solutions to the semi-linear bi-Laplacian equation

$$\Delta^2 u = f(x, u, Du) \quad \text{in } \Omega, \tag{3-1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain, $\Delta^2 u := \Delta(\Delta u)$ denotes the bi-Laplacian operator, and the nonlinearity f satisfies a polynomial growth condition, more precisely the assumption 2.1. In the following, we give an overview of the bi-Laplacian operator and some recent developments.

3.1
Some context on bi-Laplacian equation

Elliptic equations driven by operators of higher order play an important role across disciplines in pure mathematics and find relevant applications in various realms of life and social sciences. We mention differential geometry, calculus of variations, free boundary problems, the mechanics of deformable media (mainly in the mathematical theory of elasticity) and the dynamics of slow viscous fluids; see the monograph [66] and the references therein.

From the perspective of partial differential equations (PDE), the study of bi-Laplacian equations has covered a plethora of topics. These include the existence of solutions, fundamental properties (such as the validity of the maximum principle and the positivity of the Green's function), and regularity estimates. We notice the existence of relevant literature examining those properties in connection with the geometry of the domain. Of particular interest is the fact that merely Lipschitz-regular domains entail further difficulties for the analysis. In this connection, we refer the reader to [58] and the extensive list of references therein.

Regarding regularity estimates, the study of the bi-Laplacian operator has been pursued in several contexts. In the realm of obstacle problems, it appears as the operator governing a variational inequality. More precisely, given a function $\varphi : \Omega \to \mathbb{R}$, let $u \in W^{2,2}(\Omega)$ be such that

$$\Delta^2 u \geq 0 \quad \text{in } \Omega, \tag{3-2}$$

with $u \geq \varphi$, and

$$\Delta^2 u \cdot (u - \varphi) = 0, \quad \text{in } \Omega. \tag{3-3}$$

In [46], the author proves that weak solutions to this problem are in $W^{2,\infty}(\Omega)$.

The analysis of the free boundary associated with (3-2)-(3-3) is the subject of [27]. In that paper, the authors establish the local boundedness of the Hessian and verify that $\Delta^2 u$ is a non-negative measure with finite mass. In addition, they examine the planar case, showing that solutions are C^2-regular and that the free boundary is contained in a continuously differentiable curve.

The bi-Laplacian operator has also been studied in [37] in the context of the two-phase free boundary obstacle problem

$$\begin{cases} \Delta^2 u = 0 & \text{in} \quad B_1^+ \\ u = g & \text{on} \quad (\partial B_1)^+ \\ \partial_{x_{d+1}} u = 0 & \text{on} \quad B_1' \\ \partial_{x_{d+1}} \Delta u = \lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1} & \text{on} \quad B_1', \end{cases} \tag{3-4}$$

where $p > 1$, λ_- and λ_+ are positive constants, $g \in W^{2,q}(B_1^+)$ for $q := \max\{2, p\}$, and the sets B_1^+ and B_1' are defined as

$$B_1^+ := \left\{(x, x_{d+1}) \in B_1 \subset \mathbb{R}^d \times \mathbb{R} \,|\, x_{d+1} > 0\right\}$$

and

$$B_1' := B_1 \cap \{x_{d+1} = 0\}\,.$$

The authors prove that both u and Δu are locally bounded. They also establish regularity estimates in Hölder spaces of the type $C^{p+1,\gamma}$, for every $\gamma \in (0,1)$, and show their findings are optimal in the case of $p \in \mathbb{N}$. They also consider an Almgren's type frequency formula and a Monneau monotonicity formula and perform a thorough analysis of the singular set associated with (3-4). If B_1 is replaced with $\mathbb{R}^d$, it is worth noticing the formulation in (3-4) can be regarded as a Dirichlet-to-Neumann extension, in the spirit of Caffarelli and Silvestre [28], for the operator

$$(-\Delta)^{3/2} u = \lambda_-(u^-)^{p-1} - \lambda_+(u^+)^{p-1} \quad \text{in} \quad \mathbb{R}^d,$$

with $u \to 0$ as $|x| \to \infty$.

Interior regularity estimates for the pure equation $\Delta^2 u = f$ prescribed in a domain Ω have also been pursued in the literature. Of particular interest is the analysis of polyharmonic equations of the form

$$(-\Delta)^m u = f \quad \text{in} \quad \Omega,$$

where $2 \leq d \leq 2m+1$ and $f \in C_c^\infty(\Omega)$. In [58], the authors prove that solutions to this problem satisfy

$$D^{m-\frac{d}{2}+\frac{1}{2}}u \in L^\infty(\Omega) \quad \text{if} \quad d = 2n+1,$$

and

$$D^{m-\frac{d}{2}}u \in L^\infty(\Omega) \quad \text{if} \quad d = 2n,$$

with $n \in \mathbb{N}$. In the concrete case of (3-1), were $f = f(x)$ a smooth function, the analysis would lead to $u \in L^\infty(\Omega)$ for dimensions $d = 4, 5$, and $Du \in L^\infty(\Omega)$ for dimensions $d = 2, 3$.

Concerning the semi-linear formulation of the bi-Laplacian equation, we mention the developments reported in [38]. In that paper, the authors produce regularity estimates for the weak solutions to

$$\Delta^2 u + a(x)u = g(x,u) \quad \text{in} \quad \mathbb{R}^d. \tag{3-5}$$

They work under natural assumptions on the functions $a = a(x)$ and $g = g(x,u)$, including polynomial growth conditions on g, and prove that solutions to (3-5) are in $W^{4,2}(\mathbb{R}^d) \cap W^{2,s}(\mathbb{R}^d)$ for every $1 \le s \le \infty$. Their arguments rely on asymptotic properties of the fundamental solution associated with the operator $\Delta^2 + n^2$, for $n \in \mathbb{N}$. Apparently, these methods fall short in addressing the dependence on the gradient Du, not covering the case of (3-1). To the best of our knowledge, the analysis of semi-linear bi-Laplacian equations with explicit dependence on Du has hitherto not been addressed in the literature.

3.2 Improved regularity in Hölder spaces

To begin this section, we recall a result on the regularity of very weak solutions to the Poisson equation.

Proposition 3.1 (Sobolev regularity for very weak solutions) *Fix $1 < s < \infty$. Let $w \in L^1_{loc}(\Omega)$ be a very weak solution to*

$$\Delta w = g \quad \textit{in } \Omega,$$

with $g \in L^s_{loc}(\Omega)$. Then $Dw \in W^{1,s}_{loc}(\Omega)$. If $w \in L^s_{loc}(\Omega)$, then $w \in W^{2,s}_{loc}(\Omega)$. Moreover, for $\Omega'' \Subset \Omega' \Subset \Omega$, there exists $C := C(d, s, \Omega'', \Omega') > 0$ such that

$$\|w\|_{W^{2,s}(\Omega'')} \le C \left(\|w\|_{L^s(\Omega)} + \|g\|_{L^s(\Omega)} \right). \tag{3-6}$$

Proof. To verify that $w \in W^{2,s}_{loc}(\Omega)$, we resort to [45, Theorem 3]. Once we have $w \in W^{2,s}_{\text{loc}}(\Omega)$, w becomes a strong solution to $\Delta w = g$ in Ω'. Standard results in elliptic regularity theory (see, for instance, [47, Theorem 9.11]) yield (3-6). Indeed, for $\Omega'' \Subset \Omega'$ the [47, Theorem 9.11] implies the existence of $C(d, s, \Omega'', \Omega') > 0$ such that

$$\|w\|_{W^{2,s}(\Omega'')} \leq C\left(\|w\|_{L^s(\Omega')} + \|g\|_{L^s(\Omega')}\right).$$

Therefore, the proof is complete. ■

The next lemma is concerned with the notion of solution to the system

$$\begin{cases} \Delta u = v & \text{in } \Omega, \\ \Delta v = f(x, u, Du) & \text{in } \Omega, \end{cases} \tag{3-7}$$

and will help us to verify that a weak solution to (3-1) yields a pair (u, v) in a suitable functional space, solving (3-7).

Lemma 3.2 *Let $u \in W^{2,q}(\Omega)$ be a local weak solution to*

$$\Delta^2 u = f(x, u, Du) \quad \textit{in } \Omega,$$

with $q \geq 2$. Then, there exists $v \in L^q(\Omega)$ such that (u, v) is a solution to

$$\begin{cases} \Delta u = v & \textit{in } \Omega, \\ \Delta v = f(x, u, Du) & \textit{in } \Omega, \end{cases}$$

according to Definition 2.10.

Proof. It is clear that if $u \in W^{2,q}(\Omega)$, $\Delta u \in L^q(\Omega)$. Set $v := \Delta u$ and notice that $v \in L^q(\Omega)$ is defined almost everywhere in Ω. In addition, the weak formulation of

$$\Delta^2 u = f(x, u, Du) \quad \text{in } \Omega,$$

implies

$$\int_\Omega v\, \Delta\varphi\, dx = \int_\Omega \Delta u\, \Delta\varphi\, dx = \int_\Omega f(x, u, Du)\, \varphi\, dx,$$

for every $\varphi \in C_c^\infty(\Omega)$. Hence, v is a very weak solution to

$$\Delta v = f(x, u, Du) \quad \text{in } \Omega,$$

and the proof is complete. ■

In the remainder of this section, we state and prove the main result in this chapter.

Theorem 3.3 *Let $2 \leq q \leq d$ and $u \in W^{2,q}(\Omega)$ be a local weak solution to*

$$\Delta^2 u = f(x, u, Du) \quad \textit{in } B_1.$$

Assume the nonlinearity satisfies the growth condition 2.1. Suppose further that

$$\max\{\sigma,\theta\}\frac{d}{2} < q \leq d. \tag{3-8}$$

Then $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ *for*

$$\alpha := 2 - \frac{d\max\{\sigma,\theta\}}{q} \in (0,1).$$

Moreover, for $\Omega'' \Subset \Omega' \Subset \Omega$ *there exists* $C := C(d,\sigma,\theta,q,s,\Omega'',\Omega',\Omega) > 0$ *such that*

$$\|u\|_{C^{2,\alpha}(\Omega'')} \leq C\left(\|h\|_{L^d(\Omega)} + \|u\|^{\max\{\sigma,\theta\}}_{W^{2,q}(\Omega)}\right).$$

For ease of presentation, we set $\Omega \equiv B_1$, where B_1 stands for the unit ball in $\mathbb{R}^d$; standard covering arguments ensure this reduction entails no further restrictions on the problem.

Proof. We start by choosing

$$s := \frac{q}{\max\{\sigma,\theta\}},$$

noting that, due to $\sigma,\theta \in [1,2)$ in the assumption 2.1 and (3-8), we have $s \in (d/2, q]$. Moreover, for $B_{9/10} \Subset B_1$ it follows from the Growth condition (2-1) that

$$\begin{aligned}
\|f(x,u,Du)\|_{L^s(B_{9/10})} &\leq C\left(\|h\|_{L^s(B_{9/10})} + \||u|^\sigma\|_{L^s(B_{9/10})} + \||Du|^\theta\|_{L^s(B_{9/10})}\right)\\
&\leq C\left(\|h\|_{L^s(B_{9/10})} + \|u\|^\sigma_{L^{\sigma s}(B_{9/10})} + \|Du\|^\theta_{L^{\theta s}(B_{9/10})}\right)\\
&\leq C\left(\|h\|_{L^d(B_1)} + \|u\|^{\max\{\sigma,\theta\}}_{W^{2,q}(B_1)}\right),
\end{aligned}$$

which is finite since $u \in W^{2,q}(B_1)$ and $h \in L^d(B_1)$. The constant C here depends on $d, q, B_{9/10}$ and B_1. By Lemma 3.2, there exists $v \in L^q(B_1)$ such that v is a very weak solution to

$$\Delta v = f(x,u,Du) \quad \text{in } B_1.$$

Due to Proposition 3.1, we conclude $Dv \in W^{1,s}_{\mathrm{loc}}(B_{99/100})$. But, since $d/2 < s \leq q$, we also have $v \in L^s(B_1)$ and therefore $v \in W^{2,s}_{\mathrm{loc}}(B_{99/100})$. In addition, there exists $C(d,s,B_{8/9},B_{9/10}) > 0$ such that

$$\begin{aligned}
\|v\|_{W^{2,s}(B_{8/9})} &\leq C\left(\|v\|_{L^s(B_{9/10})} + \|f(x,u,Du)\|_{L^s(B_{9/10})}\right)\\
&\leq C\left(\|h\|_{L^d(B_1)} + \|v\|_{L^s(B_1)} + \|u\|^{\max\{\sigma,\theta\}}_{W^{2,q}(B_1)}\right)\\
&\leq C\left(\|h\|_{L^d(B_1)} + \|u\|_{W^{2,q}(B_1)} + \|u\|^{\max\{\sigma,\theta\}}_{W^{2,q}(B_1)}\right),
\end{aligned}$$

where the third inequality follows from the fact that

$$\|v\|_{L^s(B_{9/10})} = \|\Delta u\|_{L^s(B_{9/10})} \leq \|u\|_{W^{2,q}(B_1)}\,.$$

Because of Gagliardo-Nirenberg-Sobolev's embedding theorem [47, Theorem 7.10], we obtain $v \in W^{1,s^*}(B_{7/8})$, and the Morrey's inequality [47, Theorem 7.17] give to us that $v \in C^{0,\alpha}(\overline{B_{7/8}})$, with

$$\alpha := 2 - \frac{d\max\{\sigma,\theta\}}{q}.$$

Since u is an L^q-strong solution to $\Delta u = v$, we have $u \in C^{2,\alpha}(\overline{B_{7/8}})$ (see [47, theorem. 9.19]). Also, by Schauder's theory, there exists a positive constant $C = C(d,\sigma,\theta,q)$, such that

$$\|u\|_{C^{2,\alpha}(B_{6/7})} \leq C\left(\|u\|_{L^\infty(B_{7/8})} + \|v\|_{C^{0,\alpha}(B_{7/8})}\right).$$

To complete the proof, we notice that

$$\begin{aligned}
\|v\|_{C^{0,\alpha}(B_{7/8})} &\leq C\,\|v\|_{W^{2,s}(B_{7/8})} \\
&\leq C\,\|f(\cdot,u,Du)\|_{L^s(B_{8/9})} \\
&\leq C\left(\|h\|_{L^d(B_1)} + \|u\|_{W^{2,q}(B_1)}^{\max\{\sigma,\theta\}}\right),
\end{aligned}$$

where the final constant $C > 0$ depends on $d, \sigma, \theta, q, s, B_{8/9}, B_{9/10}$ and B_1. Therefore, the estimate in the theorem follows. ■

Remark 3.4 A fundamental question arises in the context of assumption (3-8): one must ensure that

$$\max\{\sigma,\theta\}\frac{d}{2} < d,$$

so the range for q is non-empty. The above inequality is indeed satisfied due precisely to (2-2).

The explicit description of the modulus of continuity is appealing, as it provides asymptotic information. As $q \to d$ and the growth conditions for f approach the linear regime, the exponent $\alpha \to 1$, yielding asymptotic estimates for (3-1) in $C^{2,1}$. We also notice the explicit gains of regularity stemming from (3-1). Indeed, we start with a function $u \in W^{2,q}(\Omega)$ and the equation yields $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$, with estimates.

3.3
C^∞-regularity estimates

We conclude this chapter with a corollary to Theorem 3.3 yielding smoothness of the solutions to (3-1) in the case $\sigma = \theta = 1$, under the assumption that $h \in C^\infty(\Omega)$.

Corollary 3.5 *Let $u \in W^{2,q}(\Omega)$ be a weak solution to* (3-1)*, with $q \geq 2$ satisfying* (3-8)*. Suppose* (2-1) *is in force, with*

$$f(x, r, \xi) := h(x) + a(x)r + c(x) \cdot \xi,$$

where $h, a \in C^\infty(\Omega)$ and $c \in C^\infty(\Omega, \mathbb{R}^d)$. Suppose further there exists $C(d, \sigma, \theta, q, s, \Omega'', \Omega', \Omega) > 0$ such that

$$\|h\|_{C^\infty(\Omega)} + \|a\|_{C^\infty(\Omega)} + \|c\|_{C^\infty(\Omega,\mathbb{R}^d)} \leq C.$$

Then $u \in C^\infty_{\text{loc}}(\Omega)$. Moreover, for every $n \in \mathbb{N}$ and every multi-index ϑ with $|\vartheta| = n$, we have

$$\sup_{\Omega''} |D^\vartheta u| \leq C \left(1 + \|u\|_{W^{2,q}(\Omega)}\right).$$

Proof. Fix a direction $i \in \{1, \ldots, d\}$ and define

$$v := \frac{\partial u}{\partial x_i}.$$

Clearly, v solves

$$\Delta^2 v = g(x) + a(x)v(x) + c(x) \cdot Dv,$$

with g given by

$$g(x) := \frac{\partial}{\partial x_i} h(x) + \frac{\partial}{\partial x_i} a(x)u(x) + \frac{\partial}{\partial x_i} c(x) \cdot Du(x).$$

One easily notices that

$$|g(x) + a(x)v(x) + c(x) \cdot Dv| \leq C \left(1 + \|u\|_{W^{2,q}(\Omega)} + |v(x)| + |Dv(x)|\right).$$

Hence, Theorem 3.3 implies $v \in C^{2,\alpha}_{\text{loc}}(\Omega)$. Because the direction i is arbitrary, we conclude $u \in C^{3,\sigma}_{\text{loc}}(\Omega)$. An induction argument on the order of differentiation completes the proof. ∎

4
Geometric regularity estimates for quasi-linear elliptic models in non-divergence form with strong absorption

The model under consideration in this chapter is given by

$$|Du(x)|^{\gamma}\Delta_p^{\mathrm{N}}u(x) = f(x,u) \lesssim \mathfrak{a}(x)u_+^m(x) \quad \text{in} \quad \Omega, \tag{4-1}$$

where $p \in (1,\infty)$ is a fixed parameter (related to the elliptic nature of the operator), $\gamma \in (-1,\infty)$ governs the singularity/degeneracy of the model, and $m \in [0,\gamma+1)$ represents the strong absorption exponent of the model. Additionally, we impose the boundary condition

$$u(x) = g(x) \quad \text{on } \partial\Omega, \tag{4-2}$$

where the boundary datum is assumed to be continuous and non-negative, i.e., $0 \leq g \in C(\partial\Omega)$. In what follows, we will provide some details about the normalized p-Laplacian operator and some context for the dead core problems.

4.1
The normalized p-Laplacian operator

The normalized p-Laplace operator is defined as

$$\begin{aligned}\Delta_p^{\mathrm{N}}u &= |Du|^{2-p}\Delta_p u = \Delta u + (p-2)\Delta_\infty u\\ &= \Delta u + (p-2)\left\langle D^2u\frac{Du}{|Du|}, \frac{Du}{|Du|}\right\rangle,\end{aligned}$$

where $\Delta_p u := \mathrm{div}(|Du|^{p-2}Du)$ denotes the standard p-Laplace operator. The normalized p-Laplacian operator arises in a wide range of mathematical contexts, including stochastic processes and differential geometry. For example, it admits a geometric interpretation in connection with the mean curvature flow, as discussed in [43], and a probabilistic formulation via noisy tug-of-war games with running pay-off, as explored in [60]. Due to these diverse applications, the normalized p-Laplace operator has been the focus of considerable research over the years. Using tools from game theory, the authors in [55] obtained new regularity results for the equation

$$\Delta_p^{\mathrm{N}}u = 0,$$

which were later extended in [65] to include right-hand sides f that are positive and bounded. A treatment based on PDE techniques can be found in [29]. We also point out that parabolic counterparts of these regularity results have been developed in [15, 40, 49, 56].

We recall that elliptic PDE models governed by normalized operators with an elliptic nature have gained increasing interest in recent years due to their broad connections with: Stochastic processes (random walks); Probability theory (dynamic programming principles); Game theory (Tug-of-War games with noise); Analysis (asymptotic mean value characterizations of solutions to certain nonlinear PDEs), among others (cf. [9], [7], [8], [22], [23], [54], [57], [59], and [64] for insightful contributions).

We assert that the Normalized p-Laplacian operator is "uniformly elliptic" in the sense that

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) \le \Delta_p^{\mathrm{N}} u \le \mathcal{M}^{+}_{\lambda,\Lambda}(D^2u),$$

where

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) \coloneqq \inf_{\mathfrak{A}\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(\mathfrak{A}D^2u) \quad \text{and} \quad \mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) \coloneqq \sup_{\mathfrak{A}\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(\mathfrak{A}D^2u) \tag{4-3}$$

are the *Pucci extremal operators*, and $\mathcal{A}_{\lambda,\Lambda}$ denotes the set of symmetric $d \times d$ matrices whose eigenvalues lie within the interval $[\lambda_p^{\mathrm{N}}, \Lambda_p^{\mathrm{N}}]$. Indeed, the Normalized p-Laplacian operator can be expressed in the form

$$\Delta_p^{\mathrm{N}} u = \mathrm{Tr}\left[\left(\mathrm{Id}_n + (p-2)\frac{Du}{|Du|} \otimes \frac{Du}{|Du|}\right) D^2u\right],$$

from which it is straightforward to verify that

$$\lambda_p^{\mathrm{N}} \coloneqq \min\{1, p-1\} \quad \text{and} \quad \Lambda_p^{\mathrm{N}} \coloneqq \max\{1, p-1\}.$$

This formulation shows that the normalized p-Laplace operator can be interpreted as a uniformly elliptic operator in nondivergence form, with ellipticity constants λ_p^{N} and Λ_p^{N}. However, due to the gradient dependence and the singular behaviour at points where $Du = 0$, classical $C^{1,\alpha}$ regularity results (such as those in [25, 26]) are not directly applicable. The regularity of the gradient for the viscosity solutions to

$$-\Delta_p^{\mathrm{N}} u = f \quad \text{in } B_1 \tag{4-4}$$

was studied a subject of [10]. In that paper, the authors showed that viscosity solutions to (4-4) belong to $C^{1,\alpha}(B_{1/2})$ for some $\alpha \in (0,1)$, with appropriate estimates. The authors also derived estimates in terms of the $L^q(B_1)$ norm of f, provided $p > 2$ and $q > \max\{2, d, p/2\}$. In the borderline case, it was showed in [16] that if $f \in L(n,1)$, then any $W^{2,q}$-viscosity solution to (4-4) is differentiable. Moreover, when $f \in L^q$ with $q > d$, the solutions enjoy $C^{1,\alpha}_{\mathrm{loc}}(B_1)$ regularity for some $\alpha \in (0,1)$.

In recent years, considerable attention has also been devoted to degenerate or singular models of the form

$$-|Du|^{\gamma}\Delta_p^{\mathrm{N}} u = f(x) \quad \text{in } B_1, \tag{4-5}$$

where $\gamma > -1$. This formulation includes both the standard p-Laplace operator (when $\gamma = p-2$) and the normalized p-Laplace operator (when $\gamma = 0$) as special cases. Concerning regularity theory for viscosity solutions to (4-5), it was first shown in [21] that radial solutions belong to $C^{1,\alpha}$. This result was later extended in [11], where the authors proved local $C^{1,\alpha}$ regularity for all viscosity solutions, assuming the standard condition $f \in C(B_1) \cap L^{\infty}(B_1)$. Furthermore, for $\gamma \in [-1,0)$ and $p \in \left(1, 3 + \left(\frac{2}{n} - 2\right)\right)$, solutions were shown to be locally in $W^{2,2}(\Omega)$. For the degenerate regime $\gamma > 0$, local $W^{2,2}$ estimates were also established under the assumption that $|p-2+\gamma|$ is sufficiently small.

In addition, equations featuring Hamiltonian terms and strong absorption effects have also been investigated. In [17], the authors considered

$$|\nabla u|^{\theta}\left(\Delta_p^{\mathrm{N}} u + \langle \mathcal{B}(x), \nabla u\rangle\, \varrho(x)|\nabla u|^{\theta}\right) = f(x) \quad \text{in } B_1,$$

proving the existence of viscosity solutions as well as gradient bounds, non-degeneracy of solutions, the Strong Maximum Principle, among other results.

In a very recent work [3], the authors worked with equations of the form

$$-\sigma(|Du|)\Delta_p^{\mathrm{N}} u = f(x) \quad \text{in} \quad B_1,$$

where $\sigma(\cdot)$ is a general modulus of continuity, such that $\sigma^{-}1(\cdot)$ is Dini continuous and $f \in L^{\infty}(B_1)$. Under such conditions, it was shown that viscosity solutions are locally of class C^1.

A variant involving nonhomogeneous degeneracies was analyzed in [72], where the authors considered equations of the form

$$-\left[|Du|^{\alpha(x,u)} + a(x)|Du|^{\beta(x,u)}\right]\Delta_p^{\mathrm{N}} u = f(x) \quad \text{in } B_1,$$

under appropriate assumptions on the variable exponents $0 \le \alpha(\cdot), \beta(\cdot)$ and the continuous coefficient $0 \le a(\cdot) \in C(B_1)$. This work established local $C^{1,\alpha}$ regularity of viscosity solutions.

4.2 PDEs models with the presence of dead cores

In the following, we will highlight some pivotal contributions concerning non-uniformly elliptic dead core models (in both divergence and non-divergence forms) over the past decades.

Several elliptic models with free boundaries arise in various phenomena associated with reaction-diffusion and absorption processes, both in pure and applied sciences. Notable examples include models in chemical and biological processes, combustion phenomena, and population dynamics, among other applications. A particularly significant problem within this framework pertains to diffusion processes under a sign constraint—the well-known one-phase problems—which, in chemical and physical contexts, represent the only relevant case to be considered (cf. [32, 31, 14], and [39] and references therein for further motivation). As motivation, a class of such problems is given by

$$\begin{cases} -\mathcal{Q}u(x) + f(u)\chi_{\{u>0\}}(x) & = & 0 & \text{in} & \Omega \\ u(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases}$$

where $\mathcal{Q}$ denotes a quasi-linear elliptic operator in divergence form with p-structure for $2 \leq p < \infty$ (cf. [30] and [67] for further details), and $\Omega \subset \mathbb{R}^d$ is a regular and bounded domain. In this setting, f is a continuous and monotonically increasing reaction term satisfying $f(0) = 0$, and $0 \leq g \in C(\partial\Omega)$. In models from applied sciences, $f(u)$ represents the ratio of the reaction rate at concentration u to that at concentration one.

It is worth noting that when the nonlinearity $f \in C^1(\Omega)$ is locally $(p-1)$-Lipschitz near zero (i.e., f satisfies a Lipschitz condition of order $(p-1)$ at 0 if there exist constants $\mathrm{M}_0, \zeta > 0$ such that $f(u) \leq \mathrm{M}_0 u^{p-1}$ for $0 < u < \zeta$), it follows from the Maximum Principle that non-negative solutions must, in fact, be strictly positive (cf. [71]).

However, the function f may fail to be differentiable or to decay sufficiently fast at the origin. For instance, if $f(t)$ behaves as t^q with $0 < q < p-1$, then f does not satisfy the Lipschitz condition of order $(p-1)$ at the origin. In this scenario, problem (1.1) lacks the Strong Minimum Principle, meaning that non-negative solutions may completely vanish within an a priori unknown region of positive measure $\Omega_0 \subset \Omega$, referred to as the *Dead Core* set (cf. Díaz's monograph [39], Chapter 1, for a comprehensive survey on this topic).

As an illustration of the aforementioned discussion (cf. [32], [31], and [69]), for a domain $\Omega \subset \mathbb{R}^d$, certain (stationary) isothermal and irreversible catalytic reaction processes can be mathematically described by boundary value problems of the reaction-diffusion type, given by

$$-\Delta u(x) + \lambda_0(x) u_+^q(x) = 0 \quad \text{in } \Omega, \quad \text{and} \quad u(x) = 1 \quad \text{on } \partial\Omega,$$

where, in this context, u represents the concentration of a chemical reagent (or gas), and the non-Lipschitz kinetics corresponds to the q-th-order Freundlich isotherm. Furthermore, $\lambda_0 > 0$, known as the *Thiele Modulus*, governs the ratio between the reaction rate and the diffusion-convection rate.

For $q \in (0, 1)$, the strong absorption due to chemical reactions may outpace the supply driven by diffusion across the boundary, potentially causing the complete depletion of the chemical reagent in certain sub-regions, known as dead core zones, mathematically represented as $\Omega_0 := \{x \in \Omega : u(x) = 0\} \subset \Omega$. In these regions, no chemical reaction occurs. Consequently, understanding the qualitative and quantitative behaviour of dead core solutions is of paramount importance in chemical engineering and other applied sciences.

More specifically, for an arbitrary domain $\Omega \subset \mathbb{R}^d$, the manuscript [63] is dedicated to the study of various phenomena related to dead cores and bursts in quasi-linear partial differential equations of the form:

$$\operatorname{div}(A(|Du|)Du) = f(u) \quad \text{in} \quad \Omega, \tag{4-6}$$

where $A \in C((0,\infty))$ satisfies the conditions that the mapping $\rho \mapsto \rho A(\rho)$ is increasing and $\lim\limits_{\rho\to 0} \rho A(\rho) = 0$. Moreover, the nonlinearity f is assumed to be continuous, satisfying $f(0) = 0$, non-decreasing on $\mathbb{R}$, and strictly positive on $(0,\infty)$.

This work extends the results obtained in [62], where Pucci and Serrin investigated similar problems. The main contributions of the present paper establish the existence of solutions exhibiting dead cores. Additionally, the authors demonstrate that, under certain conditions, solutions may exhibit both a dead core and bursts within the core.

Recently, in the paper [36], Da Silva and Salort investigated diffusion problems governed by quasi-linear elliptic models of the p-Laplace type, for which a minimum principle is unavailable. Specifically, they considered the following boundary value problem:

$$\begin{cases} -\operatorname{div}(\Phi(x,u,Du)) + \lambda_0(x) f(u)\chi_{\{u>0\}} & = & 0 & \text{in} & \Omega, \\ u(x) & = & g(x) & \text{on} & \partial\Omega, \end{cases} \tag{4-7}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $\Phi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous, monotone vector field satisfying p-ellipticity and p-growth conditions, $0 \le g \in C(\partial\Omega)$, $\lambda_0 \in C(\overline{\Omega})$ is a non-negative bounded function, and f is a continuous, increasing function with $f(0) = 0$. This model allows for the existence of solutions with dead core zones, i.e., a priori unknown regions where non-negative solutions vanish identically.

In this setting, Da Silva and Salort established sharp and improved $C^{\beta(p,q)}_{\text{loc}}$ regularity estimates along free boundary points, which are points belonging to $\partial\{u > 0\} \cap \Omega$, where the regularity exponent is explicitly given by

$$\beta(p,q) := \frac{p}{p-1-q}.$$

The precise statement of these regularity results is given by the following estimate:

$$\sup_{B_r(x_0)} u(x) \le C_0 \max\left\{ \inf_{B_r(x_0)} u(x),\ r^{\beta(p,q)} \right\},$$

for a positive universal constant C_0 depending on n, p, q, λ_0. In particular, if $x_0 \in$

$\partial\{u>0\}\cap\Omega$ (i.e., x_0 is a free boundary point), then

$$\sup_{B_r(x_0)} u(x) \le C_0 r^{\beta(p,q)},$$

for all $0<r<\min\{1, \mathrm{dist}(x_0,\partial\Omega)/2\}$.

Additionally, several geometric and measure-theoretic properties, such as non-degeneracy, uniform positive density, and porosity of the free boundary, are also addressed in [36]. As an application, a Liouville-type theorem is established for entire solutions whose growth at infinity is appropriately controlled.

Subsequently, in the manuscript [35], Da Silva *et al.* investigated the regularity properties of dead core problems and their limiting behavior as $p\to\infty$ for elliptic equations of the p-Laplacian type, where $2<p<\infty$, under a strong absorption condition. Specifically, the authors consider the problem:

$$\begin{cases} -\Delta_p u(x) + \lambda_0(x)u_+^q(x) &= 0 \quad \text{in} \quad \Omega,\\ u(x) &= g(x) \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4-8}$$

where $0\le q<p-1$, $0<\inf_\Omega \lambda_0 \le \lambda_0$ is a bounded function, $\Omega\subset\mathbb{R}^d$ is a bounded domain, and $g\in C(\partial\Omega)$. The PDE in (4-8) is characterized by a strong absorption condition, and the set $\partial\{u>0\}\cap\Omega$ constitutes the free boundary of the problem.

In this context, the authors established the strong non-degeneracy of a solution u at points in $\{u>0\}\cap\Omega$ (see [35, Theorem 1.1]). Furthermore, they provided improved regularity along the free boundary $\partial\{u>0\}\cap\Omega$ for a fixed pair $p\in(2,\infty)$ and $q\in[0,p-1)$ (see [35, Theorem 1.2]).

In the manuscript [34], Da Silva *et al.* investigated reaction-diffusion problems governed by the second-order nonlinear elliptic equation:

$$F(x,Du,D^2u) + |Du|^\gamma\langle b(x),Du\rangle = \lambda_0(x)u^\mu\chi_{\{u>0\}}(x), \quad \text{in } \Omega, \tag{4-9}$$

where $\Omega\subset\mathbb{R}^d$ is a smooth, open, and bounded domain, $g\ge 0$, $g\in C(\partial\Omega)$, $b\in C(\overline{\Omega},\mathbb{R}^d)$, $0\le\mu<\gamma+1$ is the absorption factor, $\lambda_0\in C(\overline{\Omega})$, and $F:\Omega\times(\mathbb{R}^d\setminus\{0\})\times\mathcal{S}(d)\to\mathbb{R}$ is a second-order fully nonlinear elliptic operator with measurable coefficients that satisfies certain ellipticity and homogeneity conditions. Specifically,

$$|\xi|^\gamma\mathcal{M}^-_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}) \le F(x,\xi,\mathrm{X}) - F(x,\xi,\mathrm{Y}) \le |\xi|^\gamma\mathcal{M}^+_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}) \quad (\text{with } \gamma>-1)$$

for all $\mathrm{X},\mathrm{Y}\in\mathcal{S}(d)$ (the space of $d\times d$ symmetric matrices) with $\mathrm{X}\ge\mathrm{Y}$, where $\mathcal{M}^\pm_{\lambda,\Lambda}(\cdot)$ denote the *Pucci extremal operators*, defined previously in (4-3).

The authors establish the sharp asymptotic behaviour governing the dead core sets of non-negative viscosity solutions to (4-9). Furthermore, they derive a sharp regularity estimate at free boundary points, i.e., $\mathrm{C}_{\mathrm{loc}}^{\frac{\gamma+2}{\gamma+1-\mu}}$. Additionally, they obtain the exact rate at which the gradient decays at interior free boundary points, along with a sharp Liouville-type result (cf. [70] for the corresponding results in the case

$\gamma = 0$ and $b = 0$).

Finally, the last relevant free boundary model to highlight concerns the ∞-dead core problem. Specifically, in [6], Araújo *et al.* focused on investigating reaction-diffusion models governed by the ∞-Laplacian operator. For $\lambda > 0$, $0 \leq \gamma < 3$, and $0 < \phi \in C(\partial\Omega)$, let $\Omega \subset \mathbb{R}^d$ be a bounded open domain, and define

$$\mathcal{L}_{\infty}^{\gamma} v := \Delta_{\infty} v - \lambda \cdot (v_+)^{\gamma} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad v = \phi \quad \text{on} \quad \partial\Omega. \tag{4-10}$$

Here, the operator $\mathcal{L}_{\infty}^{\gamma}$ represents the ∞-diffusion operator with γ-strong absorption, and the constant $\lambda > 0$ is referred to as the *Thiele modulus*, which controls the ratio between the reaction rate and the diffusion-convection rate (cf. [39] for related results).

We must also emphasize that a significant feature in the mathematical representation of equation (4-10) is the potential occurrence of plateaus, namely sub-regions, *a priori* unknown, where the function becomes identically zero (see [35], [36], and [39] for corresponding quasi-linear dead core problems).

After proving the existence and uniqueness of viscosity solutions via Perron's method and a Comparison Principle (see [6, Theorem 3.1]), the main result in the manuscript [6] guarantees that a viscosity solution is pointwise of class $C^{\frac{4}{3-\gamma}}$ along the free boundary of the non-coincidence set, i.e., $\partial\{u > 0\}$ (see [6, Theorem 4.2]). This implies that the solutions grow precisely as $\text{dist}^{\frac{4}{3-\gamma}}$ away from the free boundary. Additionally, by utilizing barrier functions, the authors demonstrate that such an estimate is sharp in the sense that u separates from its coincidence region precisely as $\text{dist}^{\frac{4}{3-\gamma}}$ (see, e.g., [6, Theorem 6.1]).

Furthermore, the authors established some Liouville-type results. Specifically, if u is an entire viscosity solution to

$$\Delta_{\infty} u(x) = \lambda u^{\gamma}(x) \quad \text{in} \quad \mathbb{R}^d,$$

with $u(0) = 0$, and $u(x) = \mathrm{o}\left(|x|^{\frac{4}{3-\gamma}}\right)$, then $u \equiv 0$ (see [6, Theorem 4.4]).

Finally, they also addressed a refined quantitative version of the previous result. Specifically, if u is an entire viscosity solution to

$$\Delta_{\infty} u(x) = \lambda u^{\gamma}(x) \quad \text{in} \quad \mathbb{R}^d,$$

such that

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|^{\frac{4}{3-\gamma}}} < \left(\frac{\lambda(3-\gamma)^4}{4^3(1+\gamma)}\right)^{\frac{1}{3-\gamma}},$$

then $u \equiv 0$ (see [6, Theorem 5.1]).

A fundamental aspect of free boundary problems like

$$|Du(x)|^{\gamma} \Delta_p^{\mathrm{N}} u(x) = f(x, u) \lesssim \mathfrak{a}(x) u_+^m(x) \quad \text{in} \quad B_R(x_0), \tag{4-11}$$

is determining the optimal regularity of viscosity solutions. For instance, if we fix

$0 < m < \gamma + 1$ and $\mathrm{R} > r > 0$, the radial profile $\omega : B_{\mathrm{R}}(x_0) \to \mathbb{R}^+$ given by

$$\omega(x) \coloneqq \mathrm{c}\left(|x - x_0| - r\right)_+^{\sigma}, \quad \text{for} \quad \sigma = \frac{\gamma+2}{\gamma+1-m}, \tag{4-12}$$

where

$$\mathrm{c} \coloneqq \mathrm{c}(d, \gamma, m, p) = \frac{1}{\sigma^{\frac{\gamma+1}{\gamma+1-m}}} \frac{1}{[d - 1 + (p-1)(\sigma - 1)]^{\frac{1}{\gamma+1-m}}} > 0$$

is a suitable constant, is a viscosity solution to the equation

$$-|D\omega(x)|^{\gamma} \Delta_p^{\mathrm{N}} \omega(x) + \omega^q(x) = 0 \quad \text{in} \quad B_{\mathrm{R}}(x_0).$$

It is well-known that, in general, solutions to (4-11) belong to $C^{1,\kappa}_{\mathrm{loc}}$ for some $\kappa \in (0,1)$ (cf. Attouchi–Ruosteenoja [11] for further details).

4.3 Regularity across the free boundary

Let us begin this section by making a few observations regarding the scaling properties of the model, which will prove useful throughout the chapter. Suppose that u is a viscosity solution to (4-11). Let κ and r be positive constants, and define the function

$$v(x) \coloneqq \frac{u(rx)}{\kappa}.$$

Consequently, we can compute

$$Dv(x) = \frac{r}{\kappa} Du(rx) \quad \text{and} \quad D^2 v(x) = \frac{r^2}{\kappa} D^2 u(rx),$$

which implies

$$\begin{aligned} \Delta_p^{\mathrm{N}} v(x) &= \Delta v(x) + (p-2)\left\langle D^2 v(x) \frac{Dv(x)}{|Dv(x)|}, \frac{Dv(x)}{|Dv(x)|} \right\rangle \\ &= \frac{r^2}{\kappa} \Delta u(rx) + (p-2)\left\langle \frac{r^2}{\kappa} D^2 u(rx) \frac{Du(rx)}{|Du(rx)|}, \frac{Du(rx)}{|Du(rx)|} \right\rangle \\ &= \frac{r^2}{\kappa} \Delta_p^{\mathrm{N}} u(rx). \end{aligned}$$

The above calculation reveals that v is a viscosity solution to

$$|Dv(x)|^{\gamma} \Delta_p^{\mathrm{N}} v(x) = \mathfrak{a}_{\kappa,r}(x) v_+^m(x),$$

where $\mathfrak{a}_{\kappa,r}(x) \coloneqq \left(r^{2+\gamma}/\kappa^{\gamma+1-m}\right) \mathfrak{a}(rx)$. By choosing $\kappa = \|u\|_{L^\infty(B_1)}$, we may assume, without loss of generality, that solutions are normalized, i.e., $\|u\|_{L^\infty(B_1)} \leq 1$. Moreover, throughout the remainder of this chapter, we will consider the constant

$$\sigma \coloneqq \sigma(\gamma, m) = \frac{\gamma+2}{\gamma+1-m}.$$

Lemma 4.1 (Flatness Estimate) *Let $\delta > 0$ and suppose that A2.2 holds. Then, there exists $r := r(d, \delta) \in (0, 1)$ such that if $u \in C(B_1)$ is a normalized viscosity solution to*

$$\begin{cases} |Du(x)|^\gamma \Delta_p^{\mathrm{N}} u(x) &= \zeta^2 \mathfrak{a}(x) u_+^m(x) \quad \textit{in} \quad B_1, \\ u(0) &= 0, \end{cases}$$

where $\zeta \in (0, r]$, we obtain

$$\sup_{B_{1/2}} u \leq \delta. \tag{4-13}$$

Proof. Assume, for the sake of contradiction, that the conclusion of the lemma does not hold. Then, there exist $\delta_0 > 0$ and a sequence of normalized non-negative functions $(u_n)_{n\in\mathbb{N}}$ satisfying

$$\begin{cases} |Du_n(x)|^\gamma \Delta_p^{\mathrm{N}} u_n(x) &= \left(\frac{1}{n}\right)^2 \mathfrak{a}(x)(u_n)_+^m(x) \quad \text{in} \quad B_1, \\ u_n(0) &= 0. \end{cases} \tag{4-14}$$

However,

$$\sup_{B_{1/2}} u_n > \delta_0, \tag{4-15}$$

for every $n \geq 1$. Since the right-hand side of the equation (4-14) belongs to $L^\infty(B_1) \cap C(B_1)$, it follows from Theorem 2.17 that for every $n \geq 1$,

$$\|u_n\|_{C^{1,\alpha}(B_{9/10})} \leq C,$$

where $C > 0$ is a constant independent of n. Consequently, there exists a function $u_\infty \in C^{0,\alpha}(B_{8/9})$, for some $\alpha \in (0, 1)$, satisfying $0 \leq u_\infty \leq 1$ and $u_\infty(0) = 0$, such that $u_n \to u_\infty$ locally uniformly in $B_{8/9}$. By a slight adaptation of [11, Appendix], the structural stability of the problem (4-14) implies that u_∞ solves the limit problem in the viscosity sense. That is, u_∞ is a viscosity solution of

$$\begin{cases} |Du_\infty(x)|^\gamma \Delta_p^{\mathrm{N}} u_\infty(x) &= 0 \quad \text{in} \quad B_{8/9}, \\ u_\infty(0) &= 0. \end{cases}$$

By Lemma 2.19, the function u_∞ is also a viscosity solution to

$$\begin{cases} \Delta_p^{\mathrm{N}} u_\infty(x) &= 0 \quad \text{in} \quad B_{8/9}, \\ u_\infty(0) &= 0. \end{cases}$$

Finally, by Theorem 2.20 and the Strong Maximum Principle from [71], we conclude that $u_\infty \equiv 0$ in $B_{8/9}$. This contradicts (4-15) for sufficiently large n. ■

We are now in a position to present the improved regularity estimate.

Theorem 4.2 (Improved Regularity) *Suppose that A2.2 (upper bound) holds. Let $u \in C(B_1)$ be a non-negative bounded viscosity solution to*

$$|Du(x)|^\gamma \Delta_p^{\mathrm{N}} u(x) = \mathfrak{a}(x) u_+^m(x) \quad \textit{in} \quad B_1$$

and let $x_0 \in \partial\{u > 0\} \cap B_{1/2}$. Then, for any point $z \in \{u > 0\} \cap B_{1/2}$, there exists a universal constant $C_{\mathcal{G}} := C(d,\, p,\, \gamma,\, m, \Lambda_{\mathfrak{a}}) > 0$ such that

$$u(z) \leq C_{\mathcal{G}} \|u\|_{L^\infty(B_1)} |z - x_0|^{\frac{\gamma+2}{\gamma+1-m}}.$$

Proof. Assume without loss of generality that $x_0 = 0$. Define an auxiliary function

$$v_1(x) := \kappa u(\tau x),$$

where $\kappa := 1/\|u\|_{L^\infty(B_1)}$ and $\tau \in (0,1)$ is a constant to be determined later. Observe that

$$\begin{aligned} |Dv_1(x)|^\gamma \Delta_p^{\mathrm{N}} v_1(x) &= \kappa^\gamma \tau^\gamma |Du(x)|^\gamma \left[\kappa\tau^2 \Delta u(\tau x) + (p-2)\kappa\tau^2 \Delta_\infty^N u(\tau x)\right] \\ &= \kappa^{1+\gamma}\tau^{2+\gamma} |Du(\tau x)|^\gamma \Delta_p^{\mathrm{N}} u(\tau x), \end{aligned}$$

which implies that $v_1(x)$ is a viscosity solution to the problem

$$|Dv_1(x)|^\gamma \Delta_p^{\mathrm{N}} v_1(x) = \kappa^{1+\gamma-m}\tau^{2+\gamma} \mathfrak{a}_1(x) v_1^m(x) \quad \text{in} \quad B_1,$$

where $\mathfrak{a}_1(x) := \mathfrak{a}(\tau x)$. The previous lemma ensures the existence of $r := r(d,\delta) \in (0,1)$ for a given

$$\delta := 2^{-\sigma} > 0, \tag{4-16}$$

such that for any function ψ satisfying $0 \leq \psi \leq 1$, $\psi(0) = 0$, and solving

$$|D\psi(x)|^\gamma \Delta_p^{\mathrm{N}} \psi(x) = \zeta^2 \mathfrak{a}(x) \psi_+^m(x) \quad \text{in} \quad B_1,$$

in the viscosity sense, where $0 < \zeta \leq r$, it holds that

$$\sup_{B_{1/2}} \psi \leq \delta.$$

Fix this value of $r(d,\delta)$ for the given $\delta > 0$ in (4-16), and with this fixed value, select the constant

$$\tau := r\kappa^{-1/\sigma}.$$

Thus, the auxiliary function v_1 satisfies

$$\begin{cases} |Dv_1(x)|^\gamma \Delta_p^{\mathrm{N}} v_1(x) = \zeta^2 \mathfrak{a}_1(x) v_1^m(x) \quad \text{in} \quad B_1, \\ v_1(0) = 0, \\ 0 \leq v_1 \leq 1, \end{cases}$$

where $\zeta \leq r$. Applying Lemma 4-13 to v_1, we obtain

$$\sup_{B_{1/2}} v_1 \leq 2^{-\sigma}.$$

Next, define a second auxiliary function by

$$v_2(x) := 2^{\sigma} v_1\left(\frac{x}{2}\right).$$

From the definition of v_2 and the constant σ, we can observe that

$$\begin{aligned}
|Dv_2(x)|^{\gamma}\Delta_p^{\mathrm{N}} v_2(x) &= 2^{\frac{(2+\gamma)(1+\gamma)}{1+\gamma-m}} \cdot 2^{-(2+\gamma)}\left|Dv_1\left(\frac{x}{2}\right)\right|^{\gamma}\Delta_p^{\mathrm{N}} v_1\left(\frac{x}{2}\right)\\
&= 2^{\frac{(2+\gamma)(1+\gamma)-(2+\gamma)(1+\gamma-m)}{1+\gamma-m}}\zeta^2\mathfrak{a}_1\left(\frac{x}{2}\right)v_1^m\left(\frac{x}{2}\right)\\
&= 2^{\frac{(2+\gamma)m}{1+\gamma-m}}\zeta^2\cdot 2^{-\frac{(2+\gamma)m}{1+\gamma-m}}\mathfrak{a}_2(x)v_2^m(x)\\
&= \zeta^2\mathfrak{a}_2(x)v_2^m(x),
\end{aligned}$$

where $\mathfrak{a}_2(x) := \mathfrak{a}_1(x/2) = \mathfrak{a}\left((\tau/2)x\right)$. It follows that v_2 satisfies the hypotheses of Lemma 4-13, and consequently,

$$\sup_{B_{1/2}} v_2 \le 2^{-\sigma}.$$

From the definition of v_2, we obtain

$$\sup_{B_{1/4}} v_1 \le 2^{-2\cdot\sigma}.$$

Repeating this process for the function

$$v_n(x) := 2^{\sigma} v_1\left(\frac{x}{2^n}\right),$$

we derive the estimate

$$\sup_{B_{1/2^n}} v_n \le 2^{-n\cdot\sigma}.$$

Finally, for a given $0<\rho\le r/2$, let $n\ge 1$ be an integer such that $2^{-(n+1)}\le \rho/r\le 2^{-n}$. Then,

$$\begin{aligned}
\kappa\cdot\sup_{B_\rho} u(x) &\le \sup_{B_{\rho/r}} v_1(x)\\
&\le \sup_{B_{2^{-n}}} v_1(x)\\
&\le 2^{-n\cdot\sigma}\\
&= 2^{\sigma}\cdot 2^{-(n+1)\sigma}\\
&\le 2^{\sigma}\left(\frac{\rho}{r}\right)^{\sigma}\\
&\le \left(\frac{2}{r}\right)^{\sigma}\rho^{\sigma}.
\end{aligned}$$

Therefore,

$$\sup_{B_\rho} u(x) \le \left(\frac{2}{r}\right)^{\sigma}\|u\|_{L^\infty(B_1)}\rho^{\sigma}. \tag{4-17}$$

■

Remark 4.3 *An examination of the proof reveals that we only utilize the fact that $\mathfrak{a}$ is bounded from above. This observation allows us to derive the same regularity estimate even when the bounded Thiele modulus depends on u. More generally, the result holds when the right-hand side is a function $f(x,u) \in L^\infty(B_1 \times [0,\mathfrak{L}])$ for which there exists a constant $C_0 > 0$ satisfying:*

$$f(x,\mu t) \leq C_0 \mu^m f(x,t) \quad \text{for all } (x,t) \in \Omega \times [0,\mathfrak{L}_0],$$

for some sufficiently small $\mu > 0$. Consequently, the constant appearing in (4-17) also depends on $\|f\|_{L^\infty(B_1\times[0,\mathfrak{L}])}$.

4.4 Non-degeneracy and measure-theoretic estimates

The following result is fundamental for establishing the existence of viscosity solutions to our problem and for deriving certain weak geometric properties. It represents a more generalized formulation, which will prove instrumental for our objectives in future research.

Lemma 4.4 (Comparison Principle) *Let $c, h_1, h_2 \in C(\overline{\Omega})$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and increasing function satisfying $f(0) = 0$. Suppose that*

$$\begin{cases} |Du_1|^\gamma \Delta_p^{\mathrm{N}} u_1 + H(x, Du_1) + c(x) f(u_1) & \geq & h_1(x) & \text{in} & \Omega, \\ |Du_2|^\gamma \Delta_p^{\mathrm{N}} u_2 + H(x, Du_2) + c(x) f(u_2) & \leq & h_2(x) & \text{in} & \Omega, \end{cases} \tag{4-18}$$

in the viscosity sense. Furthermore, assume that $u_2 \geq u_1$ on $\partial\Omega$ and one of the following holds:

- ***(a)*** *If $c < 0$ in $\overline{\Omega}$ and $h_1 \geq h_2$ in $\overline{\Omega}$,*
- ***(b)*** *If $c \leq 0$ in $\overline{\Omega}$ and $h_1 > h_2$ in $\overline{\Omega}$.*

Then, $u_2 \geq u_1$ in Ω.

Proof. Let us argue by contradiction, for that assume there exists a constant $\mathrm{M}_0 > 0$ such that

$$\mathrm{M}_0 \coloneqq \sup_{\overline{\Omega}} (u_1 - u_2) > 0.$$

For each $\varepsilon > 0$, define

$$\mathrm{M}_\varepsilon = \sup_{\overline{\Omega}\times\overline{\Omega}} \left(u_1(x) - u_2(y) - \frac{1}{2\varepsilon}|x-y|^2 \right) < \infty. \tag{4-19}$$

Let $(x_\varepsilon, y_\varepsilon) \in \overline{\Omega} \times \overline{\Omega}$ denote the points where the supremum is achieved. Following the argument in [33, Lemma 3.1], we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \mathrm{M}_\varepsilon = \mathrm{M}_0. \tag{4-20}$$

In particular,

$$z_0 := \lim_{\varepsilon\to 0} x_\varepsilon = \lim_{\varepsilon\to 0} y_\varepsilon, \tag{4-21}$$

with $u_1(z_0) - u_2(z_0) = \mathrm{M}_0$. Since

$$\mathrm{M}_0 > 0 \geq \sup_{\partial\Omega}(u_1 - u_2),$$

it follows that $x_\varepsilon, y_\varepsilon \in \Omega'$ for some interior domain $\Omega' \Subset \Omega$ and for all sufficiently small $\varepsilon > 0$. By Theorem 2.21, there exist matrices $X, Y \in \mathcal{S}(d)$ such that

$$\left(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}, X\right) \in \overline{\mathcal{J}}^{2,+} u_1(x_\varepsilon) \quad \text{and} \quad \left(\frac{y_\varepsilon - x_\varepsilon}{\varepsilon}, Y\right) \in \overline{\mathcal{J}}^{2,-} u_2(y_\varepsilon). \tag{4-22}$$

Moreover,

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{4-23}$$

in particular, $X \leq Y$. Letting $\eta_\varepsilon = (x_\varepsilon - y_\varepsilon)/\varepsilon$ and using (4-18) and (4-22), we derive

$$\begin{aligned} h_1(x_\varepsilon) &\leq |\eta_\varepsilon|^\gamma \left(\mathrm{Tr}(X) + (p-2)\langle X\eta_\varepsilon, \eta_\varepsilon\rangle\right) + c(x_\varepsilon) f(u_1(x_\varepsilon)) + H(x_\varepsilon, \eta_\varepsilon) \\ &\leq |\eta_\varepsilon|^\gamma \left(\mathrm{Tr}(Y) + (p-2)\langle Y\eta_\varepsilon, \eta_\varepsilon\rangle\right) + c(x_\varepsilon) f(u_1(x_\varepsilon)) + H(x_\varepsilon, \eta_\varepsilon) \\ &\leq h_2(y_\varepsilon) - c(y_\varepsilon) f(u_2(y_\varepsilon)) - H(x_\varepsilon, \eta_\varepsilon) + c(x_\varepsilon) f(u_1(x_\varepsilon)) + H(x_\varepsilon, \eta_\varepsilon) \\ &\leq h_2(y_\varepsilon) + f(u_2(y_\varepsilon))(c(x_\varepsilon) - c(y_\varepsilon)) + \left[\min_{\overline{\Omega}} c\right] \left(f(u_1(x_\varepsilon)) - f(u_2(y_\varepsilon))\right) \\ &\quad + \omega(|x_\varepsilon - y_\varepsilon|)(1 + |\eta_\varepsilon|^\kappa). \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ in the above inequality and applying (4-20) and (4-21), we obtain

$$h_1(z_0) - h_2(z_0) \leq \left[\min_{\overline{\Omega}} c\right] \left(f(u_1(z_0)) - f(u_2(z_0))\right).$$

This contradicts assumptions **(a)** and **(b)**. Therefore, we conclude that $u_2 \geq u_1$ in Ω. ■

Next, we present the non-degeneracy of solutions by constructing a suitable barrier function.

Theorem 4.5 (Non-degeneracy) *Let us assume that A2.2 (lower bound) holds. Let $u \in C(B_1)$ be a non-negative bounded viscosity solution to*

$$|Du(x)|^\gamma \Delta_p^{\mathrm{N}} u(x) = \mathfrak{a}(x) u_+^m(x) \quad in \quad B_1,$$

and let $y \in \overline{\{u > 0\}} \cap B_{1/2}$. Then, there exists a universal positive constant C_{ND} such that for $r \in (0, 1/2)$ hold

$$\sup_{\overline{B}_r(y)} u \geq C_{\mathrm{ND}}(d, \gamma, m, p, \lambda_{\mathfrak{a}}) \cdot r^\sigma,$$

where

$$C_{\mathrm{ND}}(\gamma, d, m, p, \lambda_{\mathfrak{a}}) = \left[\frac{(1+\gamma-m)^{2+\gamma}\lambda_{\mathfrak{a}}}{(2+\gamma)^{1+\gamma}[(d-1)(1+\gamma-m)+(p-1)(1+m)]}\right]^{\frac{1}{1+\gamma-m}} > 0. \tag{4-24}$$

Proof. Assume without loss of generality that $y = 0$. Since $u \in C(\Omega)$, it suffices to prove the theorem for points $y \in \{u > 0\} \cap B_{1/2}$. Thus, we may assume that u is strictly positive. Define the auxiliary function

$$\Psi(x) := c|x|^{\sigma},$$

where $\sigma := (2+\gamma)/(1+\gamma-m)$ and $c = C_{\mathrm{ND}}(d, \gamma, p, m, \lambda_{\mathfrak{a}})$ is a small positive constant to be determined later. Observe that

$$D\Psi(x) = c\sigma|x|^{\sigma-2}x \quad \text{and} \quad D^2\Psi(x) = c\sigma|x|^{\sigma-2}\left[I + (\sigma-2)\frac{x}{|x|}\otimes\frac{x}{|x|}\right].$$

Consequently, we have

$$\begin{aligned}
|D\Psi(x)|^{\gamma}\Delta_p^{\mathrm{N}}\Psi(x) &= |D\Psi(x)|^{\gamma}\left[\Delta\Psi(x) + (p-2)\Delta_{\infty}^{N}\Psi(x)\right] \\
&= c^{\gamma}\sigma^{\gamma}\left[c\sigma(\sigma-2+d)|x|^{\sigma-2} + (p-2)c\sigma(\sigma-1)|x|^{\sigma-2}\right]|x|^{(\sigma-1)\gamma} \\
&= c^{1+\gamma}\sigma^{1+\gamma}[d-1+(p-1)(\sigma-1)]|x|^{(\sigma-1)\gamma+\sigma-2} \\
&= c^{1+\gamma}\sigma^{1+\gamma}[d-1+(p-1)(\sigma-1)]|x|^{m\sigma}.
\end{aligned}$$

To select the appropriate value of c, it is sufficient to choose

$$0 < c \leq \left[\frac{(1+\gamma-m)^{2+\gamma}\lambda_{\mathfrak{a}}}{(2+\gamma)^{1+\gamma}[(d-1)(1+\gamma-m)+(p-1)(1+m)]}\right]^{\frac{1}{1+\gamma-m}},$$

where $\lambda_{\mathfrak{a}}$ is a positive constant given by assumption $A2.2$. Thus, we can estimate

$$|D\Psi(x)|^{\gamma}\Delta_p^{\mathrm{N}}\Psi(x) - \mathfrak{a}(x)\Psi_+^m(x) \leq 0 = |Du(x)|^{\gamma}\Delta_p^{\mathrm{N}}u(x) - \mathfrak{a}(x)u_+^m(x) \quad \text{in} \quad B_1.$$

Now, suppose that for some $x_0 \in \partial B_r$, with $0 < r < 1/2$, we have

$$u(x_0) > \Psi(x_0). \tag{4-25}$$

It follows that

$$\sup_{\overline{B}_r} u \geq u(x_0) \geq \Psi(x_0) = c \cdot r^{\sigma}.$$

Therefore, it suffices to verify whether (4-25) holds. Suppose, for contradiction, that it does not hold, i.e., $u(x) < \Psi(x)$ for all $x \in \partial B_r$. Since u and Ψ satisfy

$$\begin{cases} |D\Psi(x)|^{\gamma}\Delta_p^{\mathrm{N}}\Psi(x) - \Psi_+^m(x) \leq |Du(x)|^{\gamma}\Delta_p^{\mathrm{N}}u(x) - u_+^m(x) & \text{in} \quad B_r, \\ \Psi(x) > u(x) & \text{on} \quad \partial B_r, \end{cases}$$

the Comparison Principle (Lemma 4.4) implies

$$\Psi(x) \geq u(x) \quad \text{in} \quad B_r.$$

In particular, at the point $y = 0$, we have $\Psi(0) = 0$, which implies

$$0 = \Psi(0) \geq u(0) > 0,$$

a contradiction. Thus, (4-25) must hold, and the proof is complete. ■

4.5
Measure properties

Next, we will deliver some measure-theoretic properties, and we establish certain properties of the phase transition. The following result demonstrates that the region $\{u > 0\}$ maintains a uniform positive density along the free boundary. In particular, the formation of cusps at free boundary points is precluded.

Corollary 4.6 (Uniform positive density) *Assume that* A2.2 *hold true, let $u \in C(B_1)$ be a non-negative bounded viscosity solution to*

$$|Du(x)|^{\gamma}\Delta_p^{\mathrm{N}} u(x) = \mathfrak{a}(x) u_+^m(x) \quad \textit{in} \quad B_1,$$

and $x_0 \in \partial\{u > 0\} \cap B_{1/2}$. Then, for all $r \in (0, 1/2)$ holds

$$\mathcal{L}^d(B_r(x_0) \cap \{u > 0\}) \geq \Theta r^d,$$

where Θ is a positive universal constant, and $\mathcal{L}^d(\mathrm{E})$ states the d-dimensional Lebesgue measure of set E.

Proof. Let $r \in (0, 1/2)$ be fixed. From Theorem 4.5, there exists a point $y_0 \in \overline{B}_r(x_0)$ such that

$$u(y_0) = \sup_{\overline{B}_r(x_0)} u \geq C_{\mathrm{ND}} \cdot r^{\sigma}.$$

On the other hand, the following inclusion holds:

$$B_{\tau r}(y_0) \subset \{u > 0\}, \tag{4-26}$$

for some sufficiently small $\tau > 0$. To verify this, suppose, for contradiction, that

$$\partial\{u > 0\} \cap B_{\tau r}(y_0) \neq \emptyset.$$

Let $z_0 \in \partial\{u > 0\} \cap B_{\tau r}(y_0)$. By Theorem 4.2, we have

$$u(y_0) \leq C_{\mathcal{G}} |y_0 - z_0|^{\sigma}.$$

Consequently,

$$C_{\mathrm{ND}} r^{\sigma} \leq u(y_0) \leq C_{\mathcal{G}} |y_0 - z_0|^{\sigma} \leq C_{\mathcal{G}} (\tau r)^{\sigma}.$$

This leads to a contradiction if we choose

$$0 < \tau < \left(\frac{C_{\mathrm{ND}}}{C_{\mathcal{G}}}\right)^{\frac{1}{\sigma}}. \tag{4-27}$$

Thus, for such τ, the inclusion (4-26) holds. Finally, we conclude that

$$\mathcal{L}^d(B_r(x_0) \cap \{u > 0\}) \geq \mathcal{L}^d(B_r(x_0) \cap B_{\tau r}(y_0)) \geq \Theta r^d,$$

where $\Theta > 0$ is a constant depending on τ and the dimension d. ■

As a result of Theorems 4.2 and 4.5, we also establish the porosity of the zero-level set.

Definition 4.7 (Porous set) *A set $\mathcal{A} \subset \mathbb{R}^d$ is said porous with porosity $\delta_{\mathcal{A}} > 0$ if there exists $\mathrm{R} > 0$ such that*

$$\forall x \in \mathcal{A},\ \forall r \in (0, \mathrm{R}),\ \exists y \in \mathbb{R}^d \text{ such that } B_{\delta_{\mathcal{A}} r}(y) \subset B_r(x) \setminus \mathcal{A}.$$

We note that a porous set with porosity $\delta_{\mathcal{A}} > 0$ satisfies

$$H_{\dim}(\mathcal{A}) \leq d - \mathrm{c}\delta_{\mathcal{A}}^d,$$

where $H_{\dim}$ states the Hausdorff dimension and $\mathrm{c} = \mathrm{c}(d) > 0$ is a dimensional constant. Particularly, a porous set has Lebesgue measure zero (see, for example, [53] and [73]). Next, we establish the porosity of the free boundary.

Corollary 4.8 (Porosity of the free boundary) *There exists a universal constant $\tau \in (0, 1]$ such that*

$$H^{d-\tau}\left(\partial\{u > 0\} \cap B_{1/2}\right) < +\infty.$$

Proof. Let x_0 be an arbitrary point in $\partial\{u > 0\}$, and let $y_0 \in \partial B_r(x_0)$. For a fixed $0 < \tau \ll 1$, to be determined later, define

$$\overline{y} = (1 - \tau)x_0 + \tau y_0,$$

where $|\overline{y} - y_0| = (\tau/2)r$. Note that for each $z \in B_{(\tau/2)r}(\overline{y})$, we have

$$|z - y_0| \leq |z - \overline{y}| + |\overline{y} - y_0| \leq \tau r.$$

Since $\overline{y}$ lies on the segment connecting x_0 and y_0, it follows that

$$\begin{aligned}|z - x_0| &\leq |z - \overline{y}| + |x_0 - \overline{y}| = |z - \overline{y}| + (|x_0 - y_0| - |\overline{y} - y_0|)\\ &\leq \frac{\tau}{2}r + \left(r - \frac{\tau}{2}r\right)\\ &= r.\end{aligned}$$

By choosing τ as in (4-27), we obtain

$$B_{(\tau/2)r}(\overline{y}) \subset B_{\tau r}(y_0) \cap B_r(x_0) \subset B_r(x_0) \setminus \partial\{u > 0\}.$$

Thus, we conclude that $\partial\{u > 0\} \cap B_{1/2}$ is a $(\tau/2)$-porous set. By [53, Theorem 2.1], the desired result follows. ■

4.6 Liouville-type Theorems

We now address the existence of a viscosity solution to the Dirichlet problem (4-1) (respectively, (4-2)). This result is derived by employing Perron's method, provided that a suitable version of the Comparison Principle holds. Specifically, consider the functions $u^\sharp$ and $u_\flat$, which satisfy the following boundary value problems:

$$\begin{cases} |Du^\sharp|^\gamma \Delta_p^{\mathrm{N}} u^\sharp &= 0 \quad \text{in} \quad \Omega,\\ u^\sharp(x) &= g(x) \quad \text{on} \quad \partial\Omega,\end{cases} \tag{4-28}$$

and

$$\begin{cases} |Du_\flat|^\gamma \Delta_p^{\mathrm{N}} u_\flat &= \|g\|^m_{L^\infty(\Omega)} \quad \text{in} \quad \Omega,\\ u_\flat(x) &= g(x) \quad \text{on} \quad \partial\Omega.\end{cases} \tag{4-29}$$

The existence of such solutions can be established using standard techniques. Moreover, it is evident that $u^\sharp$ and $u_\flat$ act as a supersolution and a subsolution, respectively, to (4-1). Therefore, by invoking the Comparison Principle (Lemma 4.4), Perron's method guarantees the existence of a viscosity solution in $C(\Omega)$ to (4-1). More precisely, we obtain the following theorem.

Theorem 4.9 (Existence and Uniqueness) *Let $f \in C([0,\infty))$ be a bounded, increasing real-valued function satisfying $f(0) = 0$. Suppose that there exist a viscosity subsolution $u_\flat \in C(\Omega) \cap C^{0,1}(\Omega)$ and a viscosity supersolution $u^\sharp \in C(\Omega) \cap C^{0,1}(\Omega)$ to the equation*

$$|Du|^\gamma \Delta_p^{\mathrm{N}} u = \mathfrak{a}(x) f(u) \quad \textit{in} \quad \Omega, \tag{4-30}$$

such that $u_\flat = u^\sharp = g \in C(\partial\Omega)$. Define the class of functions

$$\begin{aligned}\mathcal{S}_g(\Omega) := \Big\{ v \in C(\Omega) \mid v \textit{ is a viscosity supersolution to } |Du|^\gamma \Delta_p^{\mathrm{N}} u = \mathfrak{a}(x) f(u) \textit{ in } \Omega,\\ \textit{such that } u_\flat \leq v \leq u^\sharp \textit{ and } v = g \textit{ on } \partial\Omega \Big\}.\end{aligned} \tag{4-31}$$

Then, the function

$$u(x) := \inf_{\mathcal{S}_g(\Omega)} v(x), \quad \textit{for } x \in \Omega, \tag{4-32}$$

is the unique continuous (up to the boundary) viscosity solution to the problem

$$\begin{cases} |Du|^{\gamma}\Delta_p^{\mathrm{N}} u &= \mathfrak{a}(x)f(u) \quad \text{in} \quad \Omega, \\ u(x) &= g(x) \qquad\quad \text{on} \quad \partial\Omega. \end{cases} \tag{4-33}$$

Next, as an application, a qualitative Liouville-type result for entire solutions is addressed, provided that their growth at infinity can be controlled appropriately (see [18, Theorem 3.1] for related results).

Theorem 4.10 (Liouville-type Theorem I) *Suppose that* A2.2 *holds, and let* u *be a non-negative viscosity solution of*

$$|Du(x)|^{\gamma}\Delta_p^{\mathrm{N}} u(x) = \mathfrak{a}(x)u_+^m(x) \quad \text{in} \quad \mathbb{R}^d,$$

and $u(x_0) = 0$. *If*

$$u(x) = o\left(|x|^{\sigma}\right), \quad \text{as} \quad |x| \to \infty, \tag{4-34}$$

then $u \equiv 0$.

Proof. Without loss of generality, assume that $x_0 = 0$. Define the sequence

$$\varphi_n(x) := \frac{u(nx)}{n^{\sigma}},$$

where $n \in \mathbb{N}$ and $\sigma = (2+\gamma)/(1+\gamma-m)$. Since

$$D\varphi_n(x) = n^{1-\sigma}Du(nx) \quad \text{and} \quad D^2\varphi_n(x) = n^{2-\sigma}D^2u(nx),$$

it follows that

$$\begin{aligned} |D\varphi_n(x)|^{\gamma}\Delta_p^{\mathrm{N}}\varphi_n(x) &= |D\varphi_n(x)|^{\gamma}\left(\Delta\varphi_n(x) + (p-2)\Delta_\infty^N\varphi_n(x)\right) \\ &= n^{(1-\sigma)\gamma}|Du(nx)|^{\gamma}\left(n^{2-\sigma}\Delta u(nx) + n^{2-\sigma}(p-2)\Delta_\infty^N u(nx)\right) \\ &= n^{(1-\sigma)\gamma+2-\sigma}|Du(nx)|^{\gamma}\Delta_p^{\mathrm{N}}u(nx) \\ &= n^{-m\sigma}\mathfrak{a}(nx)u_+^m(nx) \\ &= \mathfrak{a}_n(x)(\varphi_n^m)_+(x), \end{aligned}$$

where $\mathfrak{a}_n(x) := \mathfrak{a}(nx)$. Thus, φ_n is a non-negative viscosity solution to

$$|D\varphi_n(x)|^{\gamma}\Delta_p^{\mathrm{N}}\varphi_n(x) = \mathfrak{a}_n(x)(\varphi_n^m)_+(x),$$

and $\varphi_n(0) = 0$ for every $n \in \mathbb{N}$. Let $r > 0$ be small, and let $x_n \in \overline{B}_r$ be such that

$$\varphi_n(x_n) = \sup_{\overline{B}_r}\varphi_n(x).$$

Observe that

$$\|\varphi_n\|_{L^\infty(B_r)} \to 0 \quad \text{as} \quad n \to \infty. \tag{4-35}$$

Indeed, if $|nx_n|$ is bounded as $n \to \infty$, then $|\varphi_n(x_n)|$ is also bounded, say by a constant B, since u is continuous. Thus,

$$\varphi_n(x_n) = \frac{u(nx_n)}{n^\sigma} \le \frac{B}{n^\sigma} \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, if $|nx_n| \to \infty$, then by hypothesis (4-34), we have

$$\varphi_n(x_n) = \frac{|x_n|^\sigma u(nx_n)}{|nx_n|^\sigma} \le r \cdot \frac{u(nx_n)}{|nx_n|^\sigma} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, by Theorem 4.2, we obtain

$$\varphi_n(x) \le \mathrm{o}(|x|^\sigma) \cdot |x|^\sigma.$$

Now, suppose there exists a point $x_0 \in \mathbb{R}^d$ such that $u(x_0) > 0$. Since (4-35) holds, choose n sufficiently large such that $x_0 \in B_{nr}$ and

$$\sup_{B_r} \varphi_n(x) \le \frac{u(x_0)}{10n^\sigma}.$$

On the other hand, we can estimate

$$\begin{aligned} \frac{u(x_0)}{n^\sigma} &\le \sup_{B_{nr}} \frac{u(x)}{n^\sigma} \\ &= \sup_{B_r} \frac{u(nx)}{n^\sigma} \\ &= \sup_{B_r} \varphi_n(x) \\ &\le \frac{u(x_0)}{10n^\sigma}, \end{aligned}$$

which leads to a contradiction. This completes the proof. ∎

Finally, we establish a sharp Liouville-type result, which asserts that any entire viscosity solution satisfying a controlled growth condition at infinity must be identically zero.

Theorem 4.11 (Liouville-type Theorem II) *Let u be a non-negative viscosity solution to*

$$|Du(x)|^\gamma \Delta_p^{\mathrm{N}} u(x) = \mathfrak{a}(x) u_+^m(x) \quad in \quad \mathbb{R}^d.$$

Then, $u \equiv 0$ provided that

$$\limsup_{|x| \to \infty} \frac{u(x)}{|x|^\sigma} < C_{\mathrm{ND}}(\gamma, d, m, p, \lambda_{\mathfrak{a}}),$$

where $C_{\mathrm{ND}}(\gamma, d, m, p, \lambda_{\mathfrak{a}}) > 0$ comes from (4-24).

Proof. Fix $R > 0$ and consider $v : \overline{B}_R \to \mathbb{R}$, the viscosity solution to the boundary value problem

$$\begin{cases} |Dv(x)|^{\gamma}\Delta_p^{\mathrm{N}} v(x) &= \mathfrak{a}(x) v_+^m(x) \quad \text{in} \quad B_R, \\ v &= \sup_{\partial B_R} u(x) \quad \text{on} \quad \partial B_R. \end{cases}$$

Theorem 4.9 ensures that the solution v is unique. Moreover, by Lemma 4.4, we have

$$v \geq u \quad \text{in} \quad B_R. \tag{4-36}$$

From the hypothesis, for $R \gg 1$ sufficiently large, it follows that

$$\sup_{\partial B_R} \frac{u(x)}{R^{\sigma}} \leq \Theta \cdot C_{\mathrm{ND}}, \tag{4-37}$$

for some $0 < \Theta < 1$. For $R \gg 1$, the radial profile (4-12) implies that the solution $v = v_R$ is given by

$$v_R(x) := C_{\mathrm{ND}} \cdot \left[|x| - R \left(\frac{\sup_{\partial B_R} u(x)}{C_{\mathrm{ND}}} \right)^{\frac{1}{\sigma}} \right]_+^{\sigma}.$$

Combining this with (4-36) and (4-37), we obtain

$$\begin{aligned} u(x) &\leq C_{\mathrm{ND}} \cdot \left[|x| - R \left(\frac{\sup_{\partial B_R} u(x)}{C_{\mathrm{ND}}} \right)^{\frac{1}{\sigma}} \right]_+^{\sigma} \\ &\leq C_{\mathrm{ND}} \cdot \left[|x| - R \left(1 - \Theta^{\frac{1}{\sigma}} \right) \right]_+^{\sigma} \to 0 \quad \text{as} \quad R \to \infty. \end{aligned}$$

This completes the proof of the theorem. ■

4.7 The critical equation

In this section, we address the borderline scenario, i.e., when $m = \gamma + 1$. In this context, we prove that a Strong Maximum Principle holds.

We also establish a precise and rigorous characterization of solutions to (4-1) when $m = \gamma+1$. It is worth noting that, in this case, the previously derived regularity estimates deteriorate. Consequently, investigating this critical regime constitutes a subtle and challenging endeavour.

Theorem 4.12 (Strong Maximum Principle) *Let $u \in C(\Omega)$ be a nonnegative, bounded viscosity solution to* (4-1) *with $m = \gamma + 1$. Then, the following dichotomy holds: either $u > 0$ or $u \equiv 0$ in Ω.*

Proof. We proceed by contradiction. Suppose there exists a point $y \in B_1$ such that $u(y) > 0$. Without loss of generality, assume $y = 0$, and define the distance from this point to the zero set of u:

$$0 < \Theta := \mathrm{dist}(0, \{u = 0\}) < \frac{1}{10} \mathrm{dist}(0, \partial B_1).$$

Define the auxiliary operator

$$\mathcal{F}^{1+\gamma}[u] := |Du(x)|^{\gamma}\Delta_p^{\mathrm{N}} u(x) - \mathfrak{a}(x)u_+^{1+\gamma}(x) = 0, \tag{4-38}$$

and observe that the function $v(x) := u(0)\chi_{B_1}$ is locally bounded. Indeed, since v is a viscosity solution to

$$\mathcal{F}^{1+\gamma}[v] = -\mathfrak{a}(x)u^{1+\gamma}(0) < 0,$$

and $u = v$ on ∂B_1, the Comparison Principle 4.4 implies $v(x) \geq u(x)$ in B_1. Consequently, $u(0) \geq u(x)$ for all $x \in B_1$.

Next, define an auxiliary function to serve as a barrier:

$$\Phi_a(x) := \begin{cases} e^{-a(\Theta/2)} - \kappa_0, & \text{in } B_{\Theta/2}, \\ e^{-a|x|^2} - \kappa_0, & \text{in } B_\Theta \setminus B_{\Theta/2}, \\ 0, & \text{in } \mathbb{R}^d \setminus B_\Theta, \end{cases} \tag{4-39}$$

where $\kappa_0 := e^{-a\Theta^2}$ for $a \geq (2/\Theta^2)$. Note that

$$|D\Phi_a(x)| = 2a|x|e^{-a|x|^2} \geq a\Theta e^{-a\Theta^2} := J_0 > 0 \quad \text{in} \quad B_\Theta \setminus B_{\Theta/2}. \tag{4-40}$$

Moreover, we compute

$$\begin{aligned}\Delta_p^{\mathrm{N}}\Phi_a(x) &= 2^\gamma a^\gamma |x|^\gamma e^{-a\gamma|x|^2}\left[\left(4a^2|x|^2e^{-a|x|^2} - 2ae^{-a|x|^2}\right) + (p-2)\left(4a^2|x|^2e^{-a|x|^2} - 2ae^{-a|x|^2}\right)\right] \\ &= (2a)^{1+\gamma}|x|^\gamma e^{-a(1+\gamma)|x|^2}(p-1)(2a|x|^2-1).\end{aligned}$$

As a result, for $x \in B_\Theta \setminus B_{\Theta/2}$, we have

$$\begin{aligned}\mathcal{F}^{1+\gamma}[\Phi_a(x)] &= (2a)^{1+\gamma}|x|^\gamma e^{-a(1+\gamma)|x|^2}(p-1)(2a|x|^2-1) - \mathfrak{a}(x)\left(e^{-a|x|^2} - \kappa_0\right)^{1+\gamma} \\ &\geq (2a)^{1+\gamma}\left(\frac{\Theta}{2}\right)^\gamma e^{-a(1+\gamma)\Theta^2}(p-1)\left(2a\left(\frac{\Theta}{2}\right)^2 - 1\right) - \|\mathfrak{a}\|_{L^\infty(B_1)}\left(e^{-a\Theta^2} - \kappa_0\right)^{1+\gamma} \\ &\geq 0.\end{aligned}$$

Now, observe that for any constant $\tau > 0$,

$$\mathcal{F}^{1+\gamma}[\tau\cdot\Phi_a(x)] = \tau^{1+\gamma}\mathcal{F}^{1+\gamma}[\Phi_a(x)] \geq 0 > \mathcal{F}^{1+\gamma}[u] \quad \text{in} \quad B_\Theta \setminus B_{\Theta/2}.$$

Thus, choosing $1 \gg \tau > 0$ such that

$$u \geq \tau \cdot \Phi_a \quad \text{on} \quad \partial B_\Theta \cup \partial B_{\Theta/2},$$

we can apply the Comparison Principle 4.4 again to obtain

$$u \geq \tau\cdot\Phi_a \quad \text{in} \quad B_\Theta \setminus B_{\Theta/2}. \tag{4-41}$$

On the other hand, we can rewrite the equation (4-38) as

$$|Du|^\gamma \Delta_p^{\mathrm{N}} u = \mathfrak{a}(x)[u^\delta(x)]\cdot u^{1+\gamma-\delta} = \overline{\mathfrak{a}}(x)u^m(x),$$

where $m := 1 + \gamma - \mu$ and the bounded Thiele modulus $\overline{\mathfrak{a}}(x) := \mathfrak{a}(x)[u^{\mu}(x)]$. Then, by Theorem 4.2 and Remark 4.3, we have

$$\sup_{B_r(y_0)} u \leq \mathrm{C} r^{2+\gamma},$$

for any $y_0 \in \partial B_\Theta \cap \partial\{u > 0\}$. Now, choose a radius $r_0 > 0$ sufficiently small such that

$$\mathrm{C} \cdot r_0^{2+\gamma} \leq \frac{1}{2}\tau J_0 \cdot r_0. \tag{4-42}$$

To conclude, combine (4-39), (4-41), and (4-42) as follows:

$$\begin{aligned}
0 < \tau r_0 J_0 &\leq \tau r_0 \inf_{B_\Theta \setminus B_{\Theta/2}} |D\Phi_a| \\
&\leq \tau r_0 \inf_{B_\Theta \cap \partial B_{r_0}(y_0)} |D\Phi_a| \\
&\leq \tau r_0 \frac{|\Phi_a(x) - \Phi_a(y_0)|}{|x - y_0|} \\
&\leq \sup_{B_\Theta \cap \partial B_{r_0}(y_0)} \tau \cdot |\Phi_a(x) - \Phi_a(y_0)| \\
&\leq \sup_{B_{r_0}(y_0)} \tau \cdot |\Phi_a(x) - \Phi_a(y_0)| \\
&\leq \sup_{B_{r_0}(y_0)} \tau \cdot \Phi_a \\
&\leq \sup_{B_{r_0}(y_0)} u \\
&\leq \mathrm{C} \cdot r_0^{2+\gamma} \\
&\leq \frac{1}{2}\tau r_0 J_0,
\end{aligned}$$

which is a contradiction. This completes the proof. ■

5
Regularity for a class of degenerate/singular normalized $p-$Laplacian equations

We investigate the regularity theory for viscosity solutions to a class of degenerate/singular normalized p-Laplace equations of the form

$$-\Phi(x,|Du|)\Delta_p^{\mathrm{N}} u = f(x) \quad \text{in} \quad B_1, \tag{5-1}$$

where the degeneracy law $\Phi(\cdot,\cdot)$ satisfies a set of assumptions described below, $p \in (1,+\infty)$ and $f \in L^{\infty}(B_1)$. We emphasize that the degeneracy law $\Phi(\cdot,\cdot)$ covers many important cases in the literature, such as power type and variable exponent degeneracies. In particular, we establish sharp Hölder regularity for the gradient of viscosity solutions. In the case where p is close to 2, we obtain an improved regularity and prove Sobolev estimates for the homogeneous equation.

In this Chapter, the constant $\alpha_0 \in (0,1)$ accounts for the Hölder regularity for the gradient of viscosity solutions to the homogeneous p-Laplace equations.

$$-\Delta_p u = 0 \quad \text{in} \quad B_1.$$

Next, we emphazise the scaling properties of the problem (5-1).

Remark 5.1 *(Scaling properties) Across this Chapter, we will suppose that certain quantities satisfy a smallness regime. This assumption is not restrictive, i.e, we can assume without loss of generality that u is a normalized viscosity solution, the L^{∞}-norm of the source term is arbitrarily small and that $B=1$. In fact, for any $\varepsilon > 0$, the function*

$$v(x) := \frac{f(x)}{K}$$

is such that v is a viscosity solution to

$$-\tilde{\Phi}(x,|Du|)\Delta_p^{\mathrm{N}} u = \tilde{f}(x) \quad \textit{in} \quad B_1,$$

where

$$\tilde{\Phi}(x,t) := \frac{\Phi(x,Kt)}{\Phi(x,K)} \quad \textit{and} \quad \tilde{f}(x) := \frac{f(x)}{\Phi(x,K)K}.$$

Observe that $\tilde{\Phi}(\cdot,\cdot)$ satisfies A2.6 with $B=1$, and A2.5 with the same constant $M \geq 1$. Moreover, A2.5 also gives

$$\|\tilde{f}\|_{L^{\infty}(B_1)} \leq \frac{1}{\Phi(x,K)K}\|f\|_{L^{\infty}(B_1)} \leq \frac{BM}{K^{1+\gamma}}\|f\|_{L^{\infty}(B_1)}.$$

Hence, recalling that $\gamma > -1$ and taking

$$K := 2\left[\|u\|_{L^{\infty}(B_1)} + \left(\varepsilon^{-1}BM\|f\|_{L^{\infty}(B_1)}\right)^{\frac{1}{1+\gamma}}\right],$$

we obtain that

$$\|v\|_{L^\infty(B_1)} \leq 1 \quad \textit{and} \quad \|\tilde{f}\|_{L^\infty(B_1)} \leq \varepsilon.$$

5.1 General degenerate/singular law

For very general classes of degeneracies, we refer to [13] in the context of fully nonlinear elliptic equations, and to [44] for the fully nonlinear nonlocal elliptic case. In [13], the authors study fully nonlinear elliptic equations of the form

$$\Phi(x, |Du|)\, F(D^2u) = f(x) \quad \text{in } B_1,$$

where $f \in L^\infty(B_1)$. They prove that viscosity solutions of this equation are locally of class $C^{1,\alpha}$, where the exponent α is

$$\alpha = \begin{cases} \min\left\{\alpha_0^-, \frac{1}{1+\mu}\right\} & \text{if } \gamma \geq 0 \\ \min\left\{\alpha_0^-, \frac{1}{1+\mu-\gamma}\right\} & \text{if } -1 < \gamma < 0. \end{cases}$$

In their setting, the constant α_0 corresponds to the Hölder regularity of the gradient of viscosity solutions to the homogeneous equation

$$F(D^2u) = 0 \quad \text{in } B_1.$$

Analogous results were obtained in [44], where the authors investigate viscosity solutions to the nonlocal equation

$$\Phi(x, |Du|)\, \mathcal{I}_\sigma(u, x) = f(x) \quad \text{in } B_1,$$

where $\mathcal{I}_\sigma$ denotes a suitable fully nonlinear nonlocal operator and $f \in L^\infty(B_1)$. Their results mirror those in the local case, but are established specifically in the degenerate setting.

We emphasize that the degeneracy function $\Phi(\cdot,\cdot)$ considered in these works is quite general. Both [13] and [44] operate under similar assumptions (see also conditions $A2.5$ and $A2.6$ described in the Chapter 2), in which Φ encompasses a broad class of growth conditions. Examples of such degeneracies include:

(1) $\gamma-$**growth**: $\Phi(x,t) = t^\gamma$, with $-1 < \gamma$, as studied in [11];

(2) **Variable** $\gamma(x)-$**growth**: $\Phi(x,t) = t^{\gamma(x)}$, with $-1 < \gamma(x)$;

(3) **Double phase growth**: $\Phi(x,t) = t^\gamma + a(x)\, t^\sigma$, with $-1 < \gamma \leq \sigma$ and $a(\cdot) \in C(B_1)$;

(4) **Variable exponent double phase**: $\Phi(x,t) = t^{\gamma(x)} + a(x)\, t^{\sigma(x)}$, with $-1 < \gamma(x) \leq \sigma(x)$, $a(\cdot) \in C(B_1)$, in the spirit of [72], where only the degenerate case was analyzed;

(5) **Borderline double phase case**: $\Phi(x,t) = t^{\gamma}+t^{\gamma}\log(1+t)$, with $-1 < \gamma \le \sigma$, $a(\cdot) \in C(B_1)$;

(6) **Multi-phase growth**: $\Phi(x,t) = t^{\gamma} + a(x)\, t^{\sigma} + b(x)\, t^{\eta}$, with $-1 < \gamma \le \sigma, \eta$, and $a(\cdot), b(\cdot) \in C(B_1)$.

5.2 Compactness of the solutions

In this section, we prove the compactness of the solution to (5-1). Let us consider the following auxiliary problem

$$-\Phi(x,|Du(x)+\xi|)\left[\Delta u(x) + (p-2)\left\langle D^2u\frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\right\rangle\right] \quad \text{in} \quad B_1, \quad (5\text{-}2)$$

where ξ is a arbitrary vector in $\mathbb{R}^d$. In order to prove the compactness result, we will analyze separately the degenerate and singular cases. For that, we begin by providing a triad of auxiliary lemmas that will help us achieve the goal of this section.

Lemma 5.2 *Let $u \in C(B_1)$ be a normalized viscosity solution to* (5-2)*, assume that $\gamma \ge 0$ and that $A2.3 - A2.6$ are in forced. Then, there exist a constant $\Gamma > 1$ such that the if $|\xi| > \Gamma$ the function $u \in C^{0,\alpha}_{loc}(B_1)$ for some universal $\alpha \in (0,1)$. Moreover, there exists a constant $C > 0$ such that*

$$\|u\|_{C^{0,\alpha}_{loc}(B_1)} \le C\left(d, p, B, M, \alpha, \|f\|_{L^\infty(B_1)}\right).$$

Proof. Let us begin by considering a modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ defined by $\omega(t) := t^{\alpha}$, for a fixed value of $\alpha \in (0,1)$. For $r \in (0, 9/10)$ and $z \in B_{r/2}$ arbitrary, define the constant

$$\mathbf{S} := \sup_{x,y\in B_r}\left(u(x) - u(y) - J_1\omega(|x-y|) - J_2\left(|x-z|^2 + |y-z|^2\right)\right).$$

with J_1 and J_2 positive constants. If $\mathbf{S} \le 0$, we obtain the desired result, *i.e.*, we find a positive universal constant such that

$$|u(x) - u(x)| \le C|x-y|^{\alpha}.$$

We will argue by contradiction. Let us suppose that for all positive constants J_1 and J_2, we can find a point $z_0 \in B_{r/2}$ such that $\mathbf{S} > 0$.

Step 1: *The choose of the appropriate value of the constant J_2.*

We proceed by constructing two auxiliary functions

$$\psi(x,y) := J_1\omega(|x-y|) + J_2\left(|x-z_0|^2 + |y-z_0|^2\right),$$

and

$$\varphi(x,y) := u(x) - u(y) - \psi(x,y).$$

Let $(x_0, y_0) \in \overline{B}_r \times \overline{B}_r$ be a point where the maximum of the function φ is achieved, and note that if $x_0 = y_0$ we get that $\mathbf{S} \leq 0$, independently of the choice of $J_2 > 0$. Therefore, we will assume without loss of generality that $|x_0 - y_0| > 0$. Since

$$\varphi(x_0, y_0) = \mathbf{S} > 0$$

and $\|u\|_{L^\infty(B_1)} \leq 1$, we have

$$\psi(x_0, y_0) < u(x_0) - u(y_0) \leq 2.$$

In particular, this implies

$$J_2 \left(|x_0 - z_0|^2 + |y_0 - z_0|^2\right) \leq 2.$$

Now we are able to choose the constant J_2 to ensure that (x_0, y_0) are interior points of $B_r \times B_r$, and to attain this goal, we can take $J_2 := (32/r)^2$. In fact, note that

$$\left(\frac{32}{r}\right)^2 |x_0 - z_0|^2 \leq 2 \quad \text{and} \quad \left(\frac{32}{r}\right)^2 |y_0 - z_0|^2 \leq 2,$$

as a consequence

$$|x_0 - z_0| \leq \frac{r}{16} \quad \text{and} \quad |y_0 - z_0| \leq \frac{r}{16}. \tag{5-3}$$

Step 2: *Applying the Lemma 2.21.*

Let us begin this step by computing

$$D_x\psi(x_0, y_0) = J_1\omega'(|x_0 - y_0|)\frac{x_0 - y_0}{|x_0 - y_0|} + 2J_2(x_0 - z_0)$$

and

$$-D_x\psi(x_0, y_0) = J_1\omega'(|x_0 - y_0|)\frac{x_0 - y_0}{|x_0 - y_0|} - 2J_2(y_0 - z_0).$$

The Lemma 2.21 ensure the existence of matrix $X, Y \in \mathcal{S}(d)$ such that for each $\varepsilon > 0$, we have

$$\left(-\frac{1}{\varepsilon} + \|B\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} B + \varepsilon B^2 & -B - \varepsilon B^2 \\ -B - \varepsilon B^2 & B + \varepsilon B^2 \end{pmatrix} \tag{5-4}$$

where B is given by

$$\begin{aligned} B &:= D^2\psi(x_0, y_0) \\ &= J_1\alpha|x_0 - y_0|^{\alpha-2}\frac{x_0 - y_0}{|x_0 - y_0|} \otimes \frac{x_0 - y_0}{|x_0 - y_0|} + J_1\alpha\frac{|x_0 - y_0|^{\alpha-1}}{|x_0 - y_0|}\left(I - \frac{x_0 - y_0}{|x_0 - y_0|} \otimes \frac{x_0 - y_0}{|x_0 - y_0|}\right) \\ &= J_1\alpha|x_0 - y_0|^{\alpha-2}\left(I + (\alpha - 2)\frac{x_0 - y_0}{|x_0 - y_0|} \otimes \frac{x_0 - y_0}{|x_0 - y_0|}\right), \end{aligned}$$

where we have used that fact that $\omega(t) = t^\alpha$. Hence, knowing that $x \otimes x = xx^T$ we can calculate

$$\begin{aligned}
B^2 &= \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2\left(I+(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right)\left(I+(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right)\\
&= \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2\left[I+2(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right]\\
&+ \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2\left[(\alpha-2)^2\left(\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right)\left(\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right)\right]\\
&= \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2\left[I+2(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right]\\
&+ \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2\left[(\alpha-2)^2\left(\frac{x_0-y_0}{|x_0-y_0|}\right)\left(\frac{x_0-y_0}{|x_0-y_0|}\right)^T\left(\frac{x_0-y_0}{|x_0-y_0|}\right)\left(\frac{x_0-y_0}{|x_0-y_0|}\right)\right]\\
&= \left(J_1\alpha|x_0-y_0|^{\alpha-2}\right)^2\left(I+\alpha(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right).
\end{aligned}$$

Thus, choosing $\varepsilon := 1/(J_1\alpha|x_0-y_0|^{\alpha-2})$, we get

$$B+\varepsilon B^2 = J_1\alpha|x_0-y_0|^{\alpha-2}\left(2I+(1+\alpha)(\alpha-2)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right).$$

Step 3: *The eigenvalues of $X-Y$.*

For a unitary vector ξ, consider vectors of the form $(\xi,\xi)\in\mathbb{R}^{2d}$ and observe that from (5-4) we get

$$\begin{pmatrix}\xi\\ \xi\end{pmatrix}^T\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix} \le \begin{pmatrix}\xi\\ \xi\end{pmatrix}^T\begin{pmatrix}B+\varepsilon B^2 & -B-\varepsilon B^2\\ -B-\varepsilon B^2 & B+\varepsilon B^2\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}. \tag{5-5}$$

As a consequence of (5-5), for any pair of unit vectors $(\xi,\xi)\in\mathbb{R}^{2d}$, we obtain $X-Y\le 0$. Moreover, we can estimate

$$\|X\|,\|Y\|\le 2J_1\alpha|x_0-y_0|^{\alpha-2}\left(4+\alpha(1-\alpha)\right). \tag{5-6}$$

Furthermore, applying inequality (5-5) with the vector $(\xi,-\xi)$, where ξ is given by $\xi := (x_0-y_0)/|x_0-y_0|$, we have

$$\langle (X-Y)\xi,\xi\rangle \le 4\langle (B+\varepsilon B^2)\xi,\xi\rangle = 4J_1\alpha^2|x_0-y_0|^{\alpha-2}(\alpha-1)<0,$$

which means that at least one eigenvalue of $X-Y$ is negative and smaller than the constant

$$4J_1\alpha^2|x_0-y_0|^{\alpha-2}(\alpha-1). \tag{5-7}$$

Throughout the remainder of the paper, we will denote this eigenvalue by λ_0.

Step 4: *The choice of Γ.*

For simplicity, let us denote

$$\xi_{x_0} := D_x\psi(x_0,y_0) \qquad \text{and} \qquad \xi_{y_0} := D_y\psi(x_0,y_0),$$

and fix an arbitrary vector $\xi \in \mathbb{R}^d$ satisfying $|\xi| > \Gamma$. Observe that since $0 < |x_0 - y_0| \leq 2$, it is sufficient (at this point) to consider

$$J_1 > \frac{2^{9-\alpha}}{r\alpha} \geq \frac{2^8}{r\alpha|x_0 - y_0|^{\alpha-1}}.$$

Now, from the definition of J_2 and the information in (5-3), we have

$$J_1\alpha|x_0 - y_0|^{\alpha-1} > 4J_2 \max\{|x_0 - z_0|, |y_0 - z_0|\}.$$

Using this estimate, we get

$$2J_1\alpha|x_0 - y_0|^{\alpha-1} \geq |\xi_{x_0}| \geq \frac{J_2}{2}|x_0 - y_0|^{\alpha-1}$$

and

$$2J_1\alpha|x_0 - y_0|^{\alpha-1} \geq |\xi_{y_0}| \geq \frac{J_2}{2}|x_0 - y_0|^{\alpha-1}.$$

Define $\nu_{x_0} := \xi_{x_0} + \xi$ and $\nu_{y_0} := \xi_{y_0} + \xi$. Thus, by choosing

$$\Gamma := \frac{9}{4}J_1\alpha|x_0 - y_0|^{\alpha-1} > 1, \tag{5-8}$$

we obtain

$$|\nu_{x_0}| \geq \frac{J_1}{4}\alpha|x_0 - y_0|^{\alpha-1} > 1, \tag{5-9}$$

and

$$|\nu_{y_0}| \geq \frac{J_1}{4}\alpha|x_0 - y_0|^{\alpha-1} > 1. \tag{5-10}$$

Then, the inequalities in the viscosity sense are

$$0 \leq \Phi(x_0, |\nu_{x_0}|)\left[\mathrm{Tr}(X + J_2 I) + (p-2)\left\langle (X + J_2 I)\frac{\nu_{x_0}}{|\nu_{x_0}|}, \frac{\nu_{x_0}}{|\nu_{x_0}|}\right\rangle\right] + f(x_0),$$

and

$$0 \geq \Phi(y_0, |\nu_{y_0}|)\left[\mathrm{Tr}(X - J_2 I) + (p-2)\left\langle (X - J_2 I)\frac{\nu_{y_0}}{|\nu_{y_0}|}, \frac{\nu_{y_0}}{|\nu_{y_0}|}\right\rangle\right] + f(y_0).$$

For $\nu \neq 0$, with $\overline{\nu} = \nu/|\nu|$ and

$$A(\nu) := I + (p-2)\overline{\nu} \otimes \overline{\nu},$$

we can rewrite the inequalities above as

$$\mathrm{Tr}\left(A(\nu_{x_0})(X + J_2 I)\right) \geq -\frac{f(x_0)}{\Phi(x_0, |\nu_{x_0}|)}$$

and

$$\mathrm{Tr}(A(\nu_{y_0})(Y - J_2 I)) \leq -\frac{f(y_0)}{\Phi(y_0, |\nu_{y_0}|)}.$$

Step 5: *Estimates for the viscosity inequality.*

Notice that we can use the linearity of the trace operator to conclude that

$$\begin{aligned}\mathrm{Tr}(A(\nu_{x_0})(X+J_2I)) - \mathrm{Tr}(A(\nu_{y_0})(Y-J_2I)) &= \underbrace{\mathrm{Tr}(A(\nu_{x_0})(X-Y))}_{(I)} \\ &\quad + \underbrace{J_2\left[\mathrm{Tr}(A(\nu_{x_0})) - \mathrm{Tr}(A(\nu_{y_0}))\right]}_{(II)} \\ &\quad + \underbrace{\mathrm{Tr}((A(\nu_{x_0}) - A(\nu_{y_0}))Y)}_{(III)}.\end{aligned}$$

At this point, we estimate each term individually. To estimate (I), recall that the eigenvalues of $A(\nu_{x_0})$ belong to the interval $[\min\{1,p-1\}, \max\{1,p-1\}]$ and (5-7) hold true. Consequently

$$\begin{aligned}\mathrm{Tr}(A(\nu_{x_0})(X-Y)) &\le \sum_{i=1}^{d} \lambda_i(A(\nu_{x_0}))\lambda_i(X-Y) \le \min\{1,p-1\}\lambda_0(X-Y) \\ &\le 4J_1\alpha^2\min\{1,p-1\}|x_0-y_0|^{\alpha-2}(\alpha-1).\end{aligned}$$

For the estimate of (II), observe that

$$\begin{aligned}J_2\left[\mathrm{Tr}(A(\nu_{x_0})) + \mathrm{Tr}(A(\nu_{y_0}))\right] &\le J_2\left(\sum_{i=1}^{d}\lambda_i(A(\nu_{x_0})) + \sum_{i=1}^{d}\lambda_i(A(\nu_{y_0}))\right) \\ &\le 2dJ_2\max\{1,p-1\}.\end{aligned}$$

At last, in order to estimate (III), note that we can write

$$\begin{aligned}A(\nu_{x_0}) - A(\nu_{y_0}) &= (p-2)(\overline{\nu}_{x_0}\otimes\overline{\nu}_{x_0} - \overline{\nu}_{y_0}\otimes\overline{\nu}_{y_0} + \overline{\nu}_{y_0}\otimes\overline{\nu}_{x_0} - \overline{\nu}_{y_0}\otimes\overline{\nu}_{x_0}) \\ &= (p-2)[(\overline{\nu}_{x_0} - \overline{\nu}_{y_0})\otimes\overline{\nu}_{x_0} - \overline{\nu}_{y_0}\otimes(\overline{\nu}_{y_0} - \overline{\nu}_{x_0})]. \qquad (5\text{-}11)\end{aligned}$$

Moreover, from the definition ξ_{x_0} and ξ_{y_0} we have

$$|\nu_{x_0} - \nu_{y_0}| = 2J_2|(x_0 - z_0) + (y_0 - z_0)| \le \frac{J_2}{4}.$$

Hence, combining this estimate with the information in (5-9) and (5-10), we get

$$\begin{aligned}|\overline{\nu}_{x_0} - \overline{\nu}_{y_0}| = \left(\frac{\nu_{x_0}}{|\nu_{x_0}|} - \frac{\nu_{y_0}}{|\nu_{y_0}|}\right) &\le 2\max\left\{\frac{|\nu_{x_0} - \nu_{y_0}|}{|\nu_{x_0}|}, \frac{|\nu_{x_0} - \nu_{y_0}|}{|\nu_{y_0}|}\right\} \\ &\le 2\left(\frac{|\nu_{x_0} - \nu_{y_0}|}{|\nu_{x_0}|} + \frac{|\nu_{x_0} - \nu_{y_0}|}{|\nu_{y_0}|}\right) \\ &\le \frac{8J_2}{J_1\alpha|x_0 - y_0|^{\alpha-1}}.\end{aligned}$$

Thereby, from (5-6), (5-11), the fact that $|\overline{\nu}_{x_0}| = |\overline{\nu}_{y_0}| = 1$, and the above estimate, we obtain

$$
\begin{aligned}
\operatorname{Tr}\left((A(\nu_{x_0}) - A(\nu_{y_0}))Y\right) &\leq d\|Y\|\|A(\nu_{x_0}) - A(\nu_{y_0})\| \\
&\leq 2d|p-2|\|Y\||\overline{\nu}_{x_0} - \overline{\nu}_{y_0}| \\
&\leq 4dJ_1\alpha|p-2|\left(4+\alpha(1-\alpha)\right)|x_0-y_0|^{\alpha-2}|\overline{\nu}_{x_0} - \overline{\nu}_{y_0}| \\
&\leq 32dJ_2|p-2|\left(4+\alpha(1-\alpha)\right)|x_0-y_0|^{-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{Tr}(A(\nu_{x_0})(X+J_2I)) - \operatorname{Tr}(A(\nu_{y_0})(Y-J_2I)) &\leq (I)+(II)+(III) \\
&\leq 4J_1\alpha^2\min\{1,p-1\}|x_0-y_0|^{\alpha-2}(\alpha-1) \\
&+ 2dJ_2\max\{1,p-1\} \\
&+ 32dJ_2|p-2|\left(4+\alpha(1-\alpha)\right)|x_0-y_0|^{-1}.
\end{aligned}
$$

On the other hand, using the information in $A2.5$ that the map $t \to \Phi(x,t)/t^{\gamma}$ is almost increasing together with the assumption $A2.3$ and $A2.6$, and equations (5-9) and (5-10), we obtain

$$
\begin{aligned}
\operatorname{Tr}\left(A(\nu_{x_0})(X+J_2I)\right) - \operatorname{Tr}(A(\nu_{y_0})(Y-J_2I)) &\geq -\frac{f(x_0)}{\Phi(x_0,|\nu_{x_0}|)} + \frac{f(y_0)}{\Phi(y_0,|\nu_{y_0}|)} \\
&\geq -\frac{M\|f\|_{L^\infty(B_1)}}{\Phi(x_0,1)|\nu_{x_0}|^{\gamma}} - \frac{M\|f\|_{L^\infty(B_1)}}{\Phi(y_0,1)|\nu_{y_0}|^{\gamma}} \\
&\geq -\frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{x_0}|^{\gamma}} - \frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{y_0}|^{\gamma}} \\
&\geq -2MB\|f\|_{L^\infty(B_1)},
\end{aligned}
$$

where, in the last inequality, we have used that $\gamma \geq 0$.

Step 6: *Choosing the value of J_1.*

At this point, we use the estimate in **Step 5** to obtain

$$
\begin{aligned}
&4J_1\alpha^2\min\{1,p-1\}|x_0-y_0|^{\alpha-2}(\alpha-1) + 2dJ_2\max\{1,p-1\} \\
&+ 2MB\|f\|_{L^\infty(B_1)} + 32dJ_2|p-2|\left(4+\alpha(1-\alpha)\right)|x_0-y_0|^{-1} \geq 0. \qquad (5\text{-}12)
\end{aligned}
$$

Observe that in this inequality only the first term is negative since $\alpha \in (0,1)$, and thus we can choose the value of J_1 larger than

$$
\frac{dJ_2\left[\max\{1,p-1\} + 16dJ_2|p-2|\left(4+\alpha(1-\alpha)\right)|x_0-y_0|^{-1}\right] + MB\|f\|_{L^\infty(B_1)}}{2\alpha^2\min\{1,p-1\}|x_0-y_0|^{\alpha-2}(\alpha-1)}.
$$

So this choice leads the sum in (5-12) to be strictly negative, which is a contradiction. Therefore, there exist constants $J_1 > 0$ and $J_2 > 0$ such that $\mathbf{S} \leq 0$, and the result follows. ■

Lemma 5.3 *Let $u \in C(B_1)$ be a normalized viscosity solution to* (5-2)*, assume that $\gamma \geq 0$ and that $A2.3 - A2.6$ are in forced. Assume that $|\xi| \leq \Gamma$. Then, there exist a constant $\alpha \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{0,\alpha}_{loc}(B_1)} \leq C(d, \alpha, B, M, \lambda_p^N, \Lambda_p^N, \|f\|_{L^\infty(B_1)}).$$

Proof. To begin note that if $|Du| \geq 2\Gamma$ we have

$$|Du + \xi| \geq ||\Gamma| - |\xi|| \geq \Gamma > 1,$$

which together with the information in the assumptions $A2.5$ and $A2.6$ implies that

$$\Phi(x, |Du + \xi|) \geq (MB)^{-1}|Du + \xi|^\gamma > (MB)^{-1}.$$

Consequently, the operator in (5-2) is non-degenerate uniformly elliptic, and moreover, $u \in C(B_1)$ is a viscosity solution to

$$\begin{cases} \mathcal{M}^+(D^2u) + MB|f| \geq 0 & \text{in} \quad B_1 \\ \mathcal{M}^-(D^2u) - MB|f| \leq 0 & \text{in} \quad B_1. \end{cases}$$

Therefore, from [48], we ensure the existence of a universal constant $\alpha \in (0,1)$ such that $u \in C^{0,\alpha}_{\text{loc}}(B_1)$ with the estimate

$$\|u\|_{C^{0,\alpha}_{loc}}(B_1) \leq C,$$

for a universal constant $C > 0$. ■

The following lemma treats the singular case, where we only need the case where $\xi = 0$.

Lemma 5.4 *Let $u \in C(B_1)$ be a viscosity solution to* (5-2)*, with $|\xi| = 0$. Assume that $-1 < \gamma < 0$ and that $A2.3 - A2.6$ are in forced. Then, u is a locally Lipschitz continuous function in B_1. Moreover, there exists a constant $C > 0$ such that*

$$\|u\|_{C^{0,1}_{loc}(B_1)} \leq C\left(d, p, B, M, \|f\|_{L^\infty(B_1)}\right).$$

Proof. The proof of this lemma follows closely the proof of Lemma 5.2 and thus we will only emphasise the main changes. First, the modulus of continuity considered in this case is given by

$$\omega(t) := \begin{cases} t - \omega_0 t^\alpha & \text{if} \quad t \leq t_0, \\ \omega(t_0) & \text{if} \quad t \geq t_0, \end{cases} \tag{5-13}$$

where $\alpha \in (1,2)$ and ω_0 is a positive constant such that

$$t_0 := \left(\frac{1}{\alpha\omega_0}\right)^{1/(\alpha-1)} > 2 \qquad \text{and} \qquad \omega_0 2^{\alpha-1} \leq 1/4. \tag{5-14}$$

We can compute

$$\omega'(t) := \begin{cases} 1-\alpha\omega_0 t^{\alpha-1} & \text{if} \quad t \le t_0, \\ 0 & \text{if} \quad t \ge t_0, \end{cases}$$

and

$$\omega''(t) := \begin{cases} -\alpha(\alpha-1)\omega_0 t^{\alpha-2} & \text{if} \quad t \le t_0, \\ 0 & \text{if} \quad t \ge t_0. \end{cases}$$

As a consequence, we get

$$\omega''(t) \le 0 \qquad \text{and} \qquad \frac{3}{4} \le \omega'(t) \le 1, \quad \forall t \in (0,2). \tag{5-15}$$

Next, in the **Step 2**, we apply Lemma 2.21 in order to ensure the existence of matrices X and Y such that

$$-\left(\frac{1}{\varepsilon}+\|B\|\right)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \begin{pmatrix} B+\varepsilon B^2 & -B-\varepsilon B^2 \\ -B-\varepsilon B^2 & B+\varepsilon B^2 \end{pmatrix}, \tag{5-16}$$

where B is given by

$$\begin{aligned} B := J_1\Bigg[&\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}I \\ &+\left(\omega''(|x_0-y_0|)-\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\Bigg], \end{aligned}$$

and

$$\begin{aligned} B^2 = J_1^2\Bigg[&\left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2 I \\ &+\left(\left(\omega''(|x_0-y_0|)\right)^2-\left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2\right)\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\Bigg]. \end{aligned}$$

Similarly, as was done in **Step 3** of the proof of Lemma 5.2, we have $X-Y \le 0$. Moreover, we can use (5-15) to obtain

$$\omega''(t)-\frac{\omega'(t)}{t} \le 0 \qquad \text{and} \qquad \omega''(t) \le 0,$$

for any $t \in (0,2)$. Therefore, we have

$$\begin{aligned} \|B\| = J_1\Bigg\|&\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}I+\left(\omega''(|x_0-y_0|)-\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right) \\ &\cdot\left(\frac{x_0-y_0}{|x_0-y_0|}\otimes\frac{x_0-y_0}{|x_0-y_0|}\right)\Bigg\| \\ \le J_1&\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}, \end{aligned}$$

and

$$\begin{aligned}
\|B^2\| = J_1^2 \Bigg\| & \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2 I + \left((\omega''(|x_0-y_0|))^2 - \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2\right) \\
& \cdot \left(\frac{x_0-y_0}{|x_0-y_0|} \otimes \frac{x_0-y_0}{|x_0-y_0|}\right) \Bigg\| \\
\le J_1^2 & \left[|\omega''(|x_0-y_0|)| + \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)\right]^2 .
\end{aligned}$$

By choosing

$$\frac{1}{\varepsilon} := 2J_1 \left[|\omega''(|x_0-y_0|)| + \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)\right],$$

using the fact that $\omega'(t) \le 1$, we obtain

$$\begin{aligned}
\|X\|, \|Y\| &\le 4\|B + \varepsilon B^2\| \le 4\left(\|B\| + \varepsilon\|B^2\|\right) \\
&\le 4\left[J_1 \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|} + \frac{J_1}{2}\left(|\omega''(|x_0-y_0|)| + \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)\right)\right] \\
&\le 4J_1 \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|} + 2\left(|\omega''(|x_0-y_0|)| + \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)\right) \\
&\le 6J_1 \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|} + 2J_1|\omega''(|x_0-y_0|)| \\
&= 6J_1|x_0-y_0|^{-1} + +2J_1|\omega''(|x_0-y_0|)|. && (5\text{-}17) \\
& && (5\text{-}18)
\end{aligned}$$

Moreover, for $\xi = (x_0-y_0)/|x_0-y_0|$, we obtain

$$\begin{aligned}
\langle (X-Y)\xi, \xi\rangle &\le 4\langle (B+\varepsilon B^2)\xi, \xi\rangle \\
&= 4J_1\left[\left\langle \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|} I\right)\xi, \xi\right\rangle + \left\langle \left(\omega''(|x_0-y_0|) - \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)(\xi\otimes\xi)\xi, \xi\right\rangle\right] \\
&+ 4J_1^2\varepsilon\left[\left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2 \langle (I)\xi,\xi\rangle + \left((\omega''(|x_0-y_0|))^2 - \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2\right)\langle(\xi\otimes\xi)\xi,\xi\rangle\right] \\
&= 4J_1\left[\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\langle I\xi,\xi\rangle + \left(\omega''(|x_0-y_0|) - \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)\langle(\xi\otimes\xi)\xi,\xi\rangle\right] \\
&+ 4J_1^2\varepsilon\left[\left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2 \langle I\xi,\xi\rangle + \left((\omega''(|x_0-y_0|))^2 - \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2\right)\langle(\xi\otimes\xi)\xi,\xi\rangle\right] \\
&= 4J_1\left[\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}|\xi|^2 + \left(\omega''(|x_0-y_0|) - \frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)|\xi|^4\right] \\
&+ 4J_1^2\varepsilon\left[\left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2 |\xi|^2 + \left((\omega''(|x_0-y_0|))^2 - \left(\frac{\omega'(|x_0-y_0|)}{|x_0-y_0|}\right)^2\right)|\xi|^4\right] \\
&= 4J_1\omega''(|x_0-y_0|) + 4J_1^2\varepsilon\left(\omega''(|x_0-y_0|)\right)^2 \\
&\le 4J_1\left(\omega''(|x_0-y_0|) + \frac{1}{2}|\omega''(|x_0-y_0|)|\right) \\
&= 2J_1\omega''(|x_0-y_0|).
\end{aligned}$$

Since $|\xi| = 0$ we know that $|\xi_{x_0}| = |\nu_{x_0}|$ and $|\xi_{y_0}| = |\nu_{y_0}|$, as a consequence, from **Step 4**, we can take at this point $J_1 \gg 1$ satisfying

$$\frac{J_1}{2} \geq \frac{2J_2 \max\{|x_0 - z_0|, |y_0 - z_0|\}}{\omega'(|x_0 - y_0|)}.$$

Then, using the previous estimate together with the information about $\omega'(t)$ in (5-15), we have

$$\frac{3J_1}{2} \geq |\nu_{x_0}| \geq \frac{3J_1}{8} \tag{5-19}$$

and

$$\frac{3J_1}{2} \geq |\nu_{y_0}| \geq \frac{3J_1}{8}. \tag{5-20}$$

Thus, the change in **Step 5** occurs first when we estimate (I). More precisely,

$$\mathrm{Tr}(A(\nu_{x_0})(X - Y)) \leq 2J_1\omega''(|x_0 - y_0|)\min\{1, p-1\}. \tag{5-21}$$

We can use the value of J_1, and the equations (5-19) and (5-20), to obtain

$$\begin{aligned}
|\overline{\nu}_{x_0} - \overline{\nu}_{y_0}| = \left(\frac{\nu_{x_0}}{|\nu_{x_0}|} - \frac{\nu_{y_0}}{|\nu_{y_0}|}\right) &\leq 2\max\left\{\frac{|\nu_{x_0} - \nu_{y_0}|}{|\nu_{x_0}|}, \frac{|\nu_{x_0} - \nu_{y_0}|}{|\nu_{y_0}|}\right\} \\
&\leq \frac{J_2}{2}\left(\frac{1}{|\nu_{x_0}|} + \frac{1}{|\nu_{y_0}|}\right) \\
&\leq \frac{8J_2}{3J_1}.
\end{aligned}$$

Therefore, from (5-17) and the previous estimate, we estimate (III) as

$$\begin{aligned}
\mathrm{Tr}((A(\nu_{x_0}) - A(\nu_{y_0}))\,Y) &\leq 2d|p-2|\|Y\||\overline{\nu}_{x_0} - \overline{\nu}_{y_0}| \\
&\leq \frac{32J_2d}{3}|p-2|\left(3|x_0 - y_0|^{-1} + |\omega''(|x_0 - y_0|)|\right).
\end{aligned}$$

The last change needed in **Step 5** is done by using the information in $A2.5$, *i.e.*, that the map $t \to \Phi(x,t)/t^{\gamma}$ is almost increasing together with the assumption $A2.3$, $A2.6$, and equations (5-19) and (5-20). We have

$$\begin{aligned}
\mathrm{Tr}\left(A(\nu_{x_0})(X + J_2I)\right) - \mathrm{Tr}(A(\nu_{y_0})(Y - J_2I)) &\geq -\frac{f(x_0)}{\Phi(x_0, |\nu_{x_0}|)} + \frac{f(y_0)}{\Phi(y_0, |\nu_{y_0}|)} \\
&\geq -\frac{M\|f\|_{L^\infty(B_1)}}{\Phi(x_0, 1)|\nu_{x_0}|^{\gamma}} - \frac{M\|f\|_{L^\infty(B_1)}}{\Phi(y_0, 1)|\nu_{y_0}|^{\gamma}} \\
&\geq -\frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{x_0}|^{\gamma}} - \frac{MB\|f\|_{L^\infty(B_1)}}{|\nu_{y_0}|^{\gamma}} \\
&\geq -\frac{2^{\gamma}MB\|f\|_{L^\infty(B_1)}}{(3J_1)^{\gamma}} - \frac{2^{\gamma}MB\|f\|_{L^\infty(B_1)}}{(3J_1)^{\gamma}} \\
&\geq -\frac{2^{\gamma+1}MB\|f\|_{L^\infty(B_1)}}{(3J_1)^{\gamma}}.
\end{aligned}$$

Thereby, in **Step 6**, we have

$$2J_1\omega''(|x_0-y_0|)\min\{1,p-1\}+2dJ_2\max\{1,p-1\}+32J_2d|p-2||x_0-y_0|^{-1} + \frac{32J_2d}{3}|p-2||\omega''(|x_0-y_0|)|+3^{-\gamma}2^{\gamma+1}MB\|f\|_{L^\infty(B_1)}J_1^{-\gamma}\geq 0,$$

recalling that $-1<\gamma<0$ and $\omega''(|x_0-y_0|)\leq 0$, and knowing that J_1 grows faster than $J_1^{-\gamma}$. Hence, by taking $J_1\gg 1$ large enough, depending only on γ,α,M,B,p,J_2 and d, we get a contradiction and conclude the proof. ■

Proposition 5.5 (Compactness estimate) *Let $u\in C(B_1)$ be a viscosity solution to* (5-2), *assume that A2.3 – A2.6 hold true. Then, we have the following results:*

1) *If $\gamma\geq 0$ and ξ is a arbitrary vector in $\mathbb{R}^d$, then $u\in C^{0,\alpha}_{loc}(B_1)$ for some universal $\alpha\in(0,1)$. Moreover, there exists a constant $C>0$ such that the estimate holds*

$$\|u\|_{C^{0,\alpha}_{loc}(B_1)}\leq C(d,\alpha,B,M,\lambda_p^N,\Lambda_p^N,\|f\|_{L^\infty(B_1)}).$$

2) *If $-1<\gamma<0$, then u is a Lipschitz continues function and hold the following estimate*

$$\|u\|_{C^{0,1}_{loc}(B_1)}\leq C\left(d,p,B,M,\|f\|_{L^\infty(B_1)}\right).$$

Proof. The proof of this proposition is a direct consequence of the previous lemmas. More precisely, the first item is a consequence of Lemma 5.2 and Lemma 5.3, and the second item is a consequence of Lemma 5.4. ■

The next result provides a type of cancellation law that might be of independent interest.

Proposition 5.6 (Cancellation law) *Let $u\in C(B_1)$ be a viscosity solution to*

$$-\Phi(x,|Du+\xi|)\left[\Delta u+(p-2)\left\langle D^2u\frac{Du+\xi}{|Du+\xi|},\frac{Du+\xi}{|Du+\xi|}\right\rangle\right]=0\quad in\quad B_1,$$

where ξ is a arbitrary vector in $\mathbb{R}^d$. Then, u solves

$$-\Delta u-(p-2)\left\langle D^2u\frac{Du+\xi}{|Du+\xi|},\frac{Du+\xi}{|Du+\xi|}\right\rangle=0\quad in\quad B_1. \tag{5-22}$$

Proof. Let us define $v:=u+\xi\cdot x$, and note that v is a viscosity solution to

$$-\Phi(x,|Dv|)\left[\Delta v+(p-2)\left\langle D^2v\frac{Dv}{|Dv|},\frac{Dv}{|Dv|}\right\rangle\right]=0\quad\text{in}\quad B_1.$$

Considering a test function $\varphi\in C^2(B_1)$ that touches v from above at x_0 and satisfies $|D\varphi(x_0)|\neq 0$, we have

$$-\Phi(x,|D\varphi(x_0)|)\left[\Delta\varphi(x_0)+(p-2)\left\langle D^2\varphi(x_0)\frac{D\varphi(x_0)}{|D\varphi(x_0)|},\frac{D\varphi(x_0)}{|D\varphi(x_0)|}\right\rangle\right]\geq 0.$$

Since $\Phi(x, |D\varphi(x_0)|) > 0$, we obtain

$$-\Delta\varphi(x_0) - (p-2)\left\langle D^2\varphi(x_0)\frac{D\varphi(x_0)}{|D\varphi(x_0)|}, \frac{D\varphi(x_0)}{|D\varphi(x_0)|}\right\rangle \geq 0 \quad \text{in} \quad B_1.$$

But, in the homogeneous case, it is sufficient to use test functions that satisfy $D\varphi(x_0) \neq 0$ (see [51, 52]), and we thus conclude that v is a subsolution to

$$-\Delta v - (p-2)\left\langle D^2 v\frac{Dv}{|Dv|}, \frac{Dv}{|Dv|}\right\rangle = 0 \quad \text{in} \quad B_1.$$

Going back to the function u, we get

$$-\Delta u - (p-2)\left\langle D^2 u\frac{Du+\xi}{|Du+\xi|}, \frac{Du+\xi}{|Du+\xi|}\right\rangle \geq 0 \quad \text{in} \quad B_1,$$

which shows that u is a subsolution to (5-22). A similar argument shows that u is also a supersolution. Therefore, u is a viscosity solution to (5-22). ■

The following result is technically pivotal for our analysis.

Proposition 5.7 (Stability) *Let (u_n) be a sequence of normalized viscosity solutions to*

$$\begin{aligned} &-\Phi_n\left(x, |Du_n+\xi_n|\right)\Bigg[\Delta u_n \\ &\qquad + (p-2)\left\langle D^2 u_n\frac{Du_n+\xi_n}{|Du_n+\xi_n|}, \frac{Du_n+\xi_n}{|Du_n+\xi_n|}\right\rangle\Bigg] = f_n(x) \quad \text{in } B_1, \end{aligned} \tag{5-23}$$

where each Φ_n satisfies the assumptions A2.6 and A2.5. Assume that there exists a function $u_\infty \in C(B_1)$ such that u_n converges to u_∞ uniformly locally in B_1. Suppose further that $\|f_n\|_{L^\infty(B_1)} \to 0$.

1. *If the sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded, then u_∞ is a viscosity solution to*
$$-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty\frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}, \frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}\right\rangle = 0 \quad \text{in} \quad B_{8/9}. \tag{5-24}$$

2. *If the sequence $(\xi_n)_{n\in\mathbb{N}}$ is unbounded, then u_∞ is a viscosity solution to*
$$-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty e_\infty, e_\infty\right\rangle = 0 \quad \text{in} \quad B_{8/9}, \tag{5-25}$$
with $|e_\infty| = 1$.

Proof. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a quadratic polynomial that touches u_∞ from above at $x_0 \in B_{8/9}$. Let us assume, without loss of generality, that $|x_0| = u_\infty(x_0) = 0$, and that the polynomial function φ is given by

$$\varphi(x) := b\cdot x + \frac{1}{2}\left\langle Mx, x\right\rangle.$$

By hypotheses, we know that $u_n \to u_\infty$, and as a consequence, we can find a sequence of functions $(\varphi_n)_{n\in\mathbb{N}}$ given by

$$\varphi_n(x) \coloneqq u_n(x_n) + b\cdot(x-x_n) + \frac{1}{2}\left\langle M(x-x_n), x-x_n\right\rangle,$$

and a sequence of points $(x_n)_{n\in\mathbb{N}}$ such that φ_n converges to φ and x_n converges to 0. Moreover, for each $n\in\mathbb{N}$, the function φ_n touches u_n from above at the point x_n.

At this point, let us assume that the sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded. Consequently, up to a subsequence, we have that ξ_n converges to some ξ_∞ in $B_{8/9}$. Since the function u_n is a viscosity solution to (5-23), we have that φ_n satisfies

$$-\Phi_n(x_n,|b+\xi_n|)\left[\operatorname{Tr}(M)+(p-2)\left\langle M\frac{b+\xi_n}{|b+\xi_n|},\frac{b+\xi_n}{|b+\xi_n|}\right\rangle\right]\leq f_n(x_n). \tag{5-26}$$

First, recall that in the homogeneous case, it is sufficient to use test functions with non-zero gradient, which implies we can assume $|b+\xi_\infty|>0$. Hence, for n large enough, we can assume $|b+\xi_n|\geq 2^{-1}|b+\xi_\infty|>0$. Thus, using the hypotheses that each Φ_n satisfies $A2.5$ and $A2.6$, we obtain

$$\Phi_n(x_n,|b+\xi_n|)\geq (MB)^{-1}|b+\xi_n|^\gamma\geq (MB)^{-1}2^{-\gamma}|b+\xi_\infty|^\gamma,$$

when $|b+\xi_\infty|\geq 1$, and

$$\Phi_n(x_n,|b+\xi_n|)\geq (MB)^{-1}|b+\xi_n|^\mu\geq (MB)^{-1}2^{-\mu}|b+\xi_\infty|^\mu,$$

when $|b+\xi_\infty|<1$. Therefore,

$$-\left[\operatorname{Tr}(M)+(p-2)\left\langle M\frac{b+\xi_n}{|b+\xi_n|},\frac{b+\xi_n}{|b+\xi_n|}\right\rangle\right]\leq\frac{f_n(x_n)}{\Phi_n(x_n,|b+\xi_n|)},$$

which implies,

$$-\left[\operatorname{Tr}(M)+(p-2)\left\langle M\frac{b+\xi_n}{|b+\xi_n|},\frac{b+\xi_n}{|b+\xi_n|}\right\rangle\right]\leq\frac{2^\mu MB\|f_n\|_{L^\infty(B_1)}}{\max\{|b+\xi_\infty|^\gamma,|b+\xi_\infty|^\mu\}},$$

By passing the limit $n\to\infty$, that u_∞ is a viscosity subsolution to

$$-\Delta u_\infty-(p-2)\left\langle D^2u_\infty\frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|},\frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}\right\rangle=0\quad\text{in}\quad B_{8/9}.$$

A similar argument shows that u_∞ is also a viscosity supersolution to (5-24), and hence a viscosity solution to (5-24).

On the other hand, assume that the sequence $(\xi_n)_{n\in\mathbb{N}}$ is unbounded. Let us define a normalized sequence $(e_n)_{n\in\mathbb{N}}$ given by $e_n\coloneqq\xi_n/|\xi_n|$, and observe that e_n, up to a subsequence, converges to some e_∞ in $B_{8/9}$. Moreover, since ξ_n is unbounded we have

$$\frac{b}{|\xi_n|}\neq -e_n\qquad\text{and}\qquad |b+\xi_n|>1,$$

for n large enough. Since u_n is a viscosity solution to (5-23), we have that φ_n satisfies (5-26), and as a consequence we get

$$-\operatorname{Tr}(M)-(p-2)\left\langle M\frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|},\frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|}\right\rangle\leq\frac{f_n(x_n)}{\Phi_n(x_n,|b+\xi_n|)}$$

in B_1. As before, by using $A2.6$ and $A2.5$, we obtain

$$\begin{aligned}&-\operatorname{Tr}(M)-(p-2)\left\langle M\frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|},\frac{b|\xi_n|^{-1}+e_n}{|b|\xi_n|^{-1}+e_n|}\right\rangle\\&\leq\frac{MB\|f_n\|_{L^\infty(B_1)}}{|b+\xi_n|^\gamma}\leq MB\|f_n\|_{L^\infty(B_1)}.\end{aligned}$$

Once more, by passing to the limit $n\to\infty$, we obtain that u_∞ is a subsolution to (5-25). With an analogous argument, we can prove that u_∞ is a viscosity supersolution to (5-25). This concludes the proof. ■

5.3 Gradient estimates

Throughout this section, we define

$$\alpha=\begin{cases}\min\left\{\alpha_0^-,\frac{1}{1+\mu}\right\} & \text{if } \gamma\geq 0\\ \min\left\{\alpha_0^-,\frac{1}{1+\mu-\gamma}\right\} & \text{if } -1<\gamma<0.\end{cases}\tag{5-27}$$

We start with an approximation lemma, connecting solutions of our equation to solutions of the $p-$Laplace equation. Recall that α_0 accounts for the $C^{1,\alpha_0}-$regularity for

$$-\Delta_p u=0.$$

Lemma 5.8 (Approximation lemma I) *Let $u\in C(B_1)$ be a normalized viscosity solution to (5-2), and assume $A2.3-A2.6$ are in force. Suppose further that $\gamma\geq 0$. Given $\delta>0$, there exists $\varepsilon>0$ such that, if*

$$\|f\|_{L^\infty(B_1)}\leq\varepsilon,$$

then we can find a function $h\in C^{1,\alpha_0}_{loc}(B_1)$ satisfying

$$\|u-h\|_{L^\infty(B_{9/10})}\leq\delta.$$

Proof. We argue by contradiction. Suppose that, for $\delta_0>0$ and sequences $(\Phi_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$, and $(f_n)_{n\in\mathbb{N}}$, we have

i) each f_n satisfies $A2.3$, and $\|f_n\|_{L^\infty}\to 0$;

ii) each $\Phi_n\in C(B_1)\times C([0,\infty])$ satisfies $A2.6$ and $A2.5$;

iii) each $u_n \in C(B_1)$ is a viscosity solution to

$$
\begin{aligned}
&-\Phi_n\left(x, |Du_n + \xi_n|\right)\Bigg[\Delta u_n\\
&\quad + (p-2)\left\langle D^2 u_n \frac{Du_n+\xi_n}{|Du_n+\xi_n|}, \frac{Du_n+\xi_n}{|Du_n+\xi_n|}\right\rangle\Bigg] = f_n(x) \quad \text{in } B_1;
\end{aligned}
$$

but, nevertheless,

$$
\|u_n - h\|_{L^\infty(B_{9/10})} > \delta_0, \tag{5-28}
$$

for all $h \in C^{1,\alpha_0}(B_{9/10})$. Proposition 5.5 ensures the existence of a function $u_\infty \in C^\alpha(B_{9/10})$ such that, up to a subsequence, u_n converge to u_∞ locally uniformly. At this point, we split the proof into two cases.

Case 1: *The sequence* $(\xi_n)_{n\in\mathbb{N}}$ *is bounded.*

In this case, we can assume that $\xi_n \to \xi_\infty$, for some $\xi_\infty \in \mathbb{R}^d$. From the stability result, Proposition 5.7, we have that u_∞ is a viscosity solution to

$$
-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty \frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}, \frac{Du_\infty+\xi_\infty}{|Du_\infty+\xi_\infty|}\right\rangle = 0 \quad \text{in} \quad B_{8/9}.
$$

Hence, $u \in C^{1,\alpha_1}_{\text{loc}}(B_{8/10})$, for some $\alpha_1 \in (0,1)$ (see, for instance, [10, Lemma 3.2]). Therefore, choosing $h \equiv u_\infty$, we get a contradiction with (5-28) for n sufficiently large.

Case 2: *The sequence* $(\xi_n)_{n\in\mathbb{N}}$ *is unbounded.*

Once more, Proposition 5.7 shows that u_∞ solves

$$
-\Delta u_\infty - (p-2)\left\langle D^2 u_\infty e_\infty, e_\infty\right\rangle = 0 \quad \text{in} \quad B_{8/9},
$$

in the viscosity sense, where e_∞ is a unitary vector. Since the operator above is linear and uniformly elliptic, and has constant coefficients, it follows from [24, Corollary 5.7] that there exists a universal $\alpha_2 \in (0,1)$ such that $u_\infty \in C^{1,\alpha_2}(B_{8/10})$. Again, by choosing $h \equiv u_\infty$, we obtain a contradiction with (5-28) for n large enough. ∎

By using Lemma 5.8, we can find an affine function that is close to u, in a suitable sense, on a fixed small scale.

Proposition 5.9 *Let $u \in C(B_1)$ be a normalized viscosity solution to* (5-2)*. Suppose that A2.3 – A2.6 are in force. Suppose further that $\gamma \geq 0$. There exists $\varepsilon > 0$ such that, if*

$$
\|f\|_{L^\infty(B_1)} \leq \varepsilon,
$$

then we can find an affine function $\ell(x) := a + b\cdot x$, with $|a|, |b| \leq C$, for a universal constant $C > 0$, such that

$$
\|u - \ell\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha},
$$

for every $\alpha \in (0, \alpha_0)$.

Proof. Let us fix $\delta > 0$ and $\rho > 0$ sufficiently small to be chosen later. From Lemma 5.8, there exists $\varepsilon > 0$ and a function $h \in C^{1,\alpha_0}_{loc}(B_1)$ satisfying

$$\|u - h\|_{L^\infty(B_{9/10})} \leq \delta.$$

Since $h \in C^{1,\alpha_0}$, there exists a positive constant C such that

$$\|h(x) - h(0) - Dh(0) \cdot x\|_{L^\infty(B_\rho)} \leq C\rho^{1+\alpha_0}.$$

Thereby, we can define the affine function $\ell := h(0) + Dh(0) \cdot x$ and use the triangle inequality to obtain

$$\begin{aligned} \|u - \ell\|_{L^\infty(B_\rho)} &\leq \|u - h\|_{L^\infty(B_\rho)} + \|h - \ell\|_{L^\infty(B_\rho)} \\ &\leq \delta + C\rho^{1+\alpha_0}. \end{aligned}$$

Now, for $0 < \alpha < \alpha_0$, we make the universal choices

$$\rho := \left(\frac{1}{2C}\right)^{\frac{1}{\alpha_0 - \alpha}} \qquad \text{and} \qquad \delta := \frac{\rho^{1+\alpha}}{2},$$

fixing also the value of $\varepsilon > 0$ through Lemma 5.8. Thus, we get

$$\|u - \ell\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha}.$$

■

We now iterate the previous result to find a sequence of affine functions approximating u on every small scale.

Proposition 5.10 *Let $u \in C(B_1)$ be a normalized viscosity solution to* (5-1) *and assume $A2.3 - A2.6$ are in force. There exists $\varepsilon > 0$ such that, if*

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon,$$

then there exists a sequence of affine function $(\ell_k)_{k\in\mathbb{N}}$, such that, for each $k \in \mathbb{N}$, ℓ_k is of the form

$$\ell_k(x) := a_k + b_k \cdot x,$$

with

$$|a_{k+1} - a_k| + \rho^k|b_{k+1} - b_k| \leq C\rho^{k(1+\alpha)},$$

for a universal constant $C > 0$, and

$$\|u - \ell_k\|_{L^\infty(B_{\rho^k})} \leq \rho^{k(1+\alpha)}.$$

Proof. We will split the proof into two cases, depending on the sign of the constant γ.

Case 1: Assume first that $\gamma \geq 0$. As usual, we will argue inductively. For $k = 1$, we set $\ell_2(x) = \ell_1(x) = \ell(x)$, where $\ell(\cdot)$ is the affine function from Proposition 5.9, and the result follows from it. Now, suppose that the conclusion holds for $k = 1, \ldots, n$, and let us prove the case $k = n+1$. We introduce the auxiliary function $v_n : B_1 \to \mathbb{R}$, defined by

$$v_n(x) := \frac{u(\rho^n x) - \ell_n(\rho^n x)}{\rho^{n(1+\alpha)}}.$$

Notice that, from the induction hypothesis, we have $\|v_n\|_{L^\infty(B_1)} \leq 1$; moreover, v_n solves

$$\begin{aligned} &- \Phi_n\left(x, |Dv_n + \rho^{-n\alpha} b_n|\right) \Bigg[\Delta v_n \\ &\quad + (p-2)\left\langle D^2 v_n \frac{Dv_n + \rho^{-n\alpha} b_n}{|Dv_n + \rho^{-n\alpha} b_n|}, \frac{Dv_n + \rho^{-n\alpha} b_n}{|Dv_n + \rho^{-n\alpha} b_n|}\right\rangle\Bigg] = f_n(x) \quad \text{in } B_1, \end{aligned}$$

in the viscosity sense, where

$$\Phi_n(x,t) := \frac{\Phi(\rho^n x, \rho^{n\alpha} t)}{\Phi(\rho^n x, \rho^{n\alpha})} \qquad \text{and} \qquad f_n(x) := \frac{\rho^{n(1-\alpha)} f(\rho^n x)}{\Phi(\rho^n x, \rho^{n\alpha})}. \tag{5-29}$$

Notice that $\Phi_n(\cdot,\cdot)$ and $f_n(\cdot)$ satisfy the hypotheses of Proposition 5.9. Indeed, we have

$$\begin{aligned} \|f_n\|_{L^\infty(B_1)} &\leq \frac{\rho^{n(1-\alpha)} M \|f\|_{L^\infty(B_1)}}{\Phi(x,1)\rho^{n\alpha\mu}} \\ &\leq M\|f\|_{L^\infty(B_1)} \rho^{n[1-\alpha(1+\mu)]} \\ &\leq M\varepsilon, \end{aligned}$$

where we have used the value of α chosen in (5-27). Moreover, a simple computation shows that the map $t \mapsto \Phi_n(x,t)/t^\gamma$ is almost increasing, the map $t \mapsto \Phi_n(x,t)/t^\mu$ is almost decreasing, and $\Phi_n(x,1) = 1$ for every x in B_1. Therefore, we can apply Lemma 5.9 to the function v_n, and obtain

$$\|v_n - \ell\|_{L^\infty(B_\rho)} \leq \rho^{1+\alpha}.$$

Rescaling back to the function u, we get

$$\|u - \ell_{n+1}\|_{L^\infty(B_{\rho^{n+1}})} \leq \rho^{(n+1)(1+\alpha)},$$

where

$$\ell_{n+1}(x) := \ell_n(x) + \rho^{n(1+\alpha)} \ell(\rho^{-n} x).$$

Moreover, we can verify that

$$|a_{n+1} - a_n| + \rho^n |b_{n+1} - b_n| \leq C\rho^{n(1+\alpha)},$$

which concludes the inductive process.

Case 2: Now, assume $-1<\gamma<0$. Recall that from Proposition 2.22 we have that u is a viscosity solution to

$$\overline{\Phi}(x,|Du|)\Delta_p^{\mathrm{N}}u=\overline{f}(x) \quad \text{in} \quad B_1, \tag{5-30}$$

where

$$\overline{\Phi}(x,t):=t^{-\gamma}\Phi(x,t) \quad \text{and} \quad \overline{f}(x):=|Du|^{-\gamma}f(x),$$

for $x\in B_1$ and $t>0$. Now, $A2.6$ and $A2.5$ imply that the map $t\mapsto\overline{\Phi}(x,t)$ is almost increasing and that the map $t\mapsto\overline{\Phi}(x,t)/t^{\mu-\gamma}$ is almost decreasing and since $\overline{\Phi}(x,1)=\Phi(x,1)$, we have

$$\overline{\Phi}(x,1)=1 \quad \text{for all} \quad x\in B_1.$$

Moreover, we obtain from Lemma 5.4 that u is a Lipschitz continuous function, and as a consequence, the gradient Du is bounded almost everywhere. Thus, we can estimate

$$\begin{aligned}\|\overline{f}(x)\|_{L^\infty(B_{9/10})}&=|Du|^{-\gamma}\cdot\|f\|_{L^\infty(B_{9/10})}\\&\leq\|f\|_{L^\infty(B_1)}\cdot\|u\|^{-\gamma}_{C^{0,1}(B_{9/10})}.\end{aligned}$$

Therefore, $\bar{\Phi}$ and $\bar{f}$ satisfy the assumptions $A2.3-A2.6$, and we can apply the same inductive argument as in the **Case 1** to the equation (5-30). Once again, we define the function

$$v_n(x):=\frac{u(\rho^n x)-\ell_n(\rho^n x)}{\rho^{n(1+\alpha)}},$$

that solves, in the viscosity sense

$$\begin{aligned}\overline{\Phi}_n\left(x,|Dv_n+\rho^{-n\alpha}b_n|\right)\Bigg[\Delta v_n&\\+(p-2)\left\langle D^2v_n\frac{Dv_n+\rho^{-n\alpha}b_n}{|Dv_n+\rho^{-n\alpha}b_n|},\frac{Dv_n+\rho^{-n\alpha}b_n}{|Dv_n+\rho^{-n\alpha}b_n|}\right\rangle\Bigg]&=\overline{f}_n(x) \quad \text{in } B_1,\end{aligned}$$

where

$$\overline{\Phi}_n(x,t):=\frac{\overline{\Phi}(\rho^n x,\rho^{n\alpha}t)}{\overline{\Phi}(\rho^n x,\rho^{n\alpha})} \quad \text{and} \quad \overline{f}_n(x):=\frac{\rho^{n(1-\alpha)}\overline{f}(\rho^n x)}{\overline{\Phi}(\rho^n x,\rho^{n\alpha})}.$$

Observe that since the map $t \mapsto \overline{\Phi}(x,t)/t^{\mu-\gamma}$ is almost decreasing, we have

$$
\begin{aligned}
\|\overline{f}_n\|_{L^\infty(B_9/10)} &= \frac{\|\overline{f_n}\|_{L^\infty(B_{9/10})}}{\overline{\Phi}(\rho^n x, \rho^{n\alpha})} \\
&\leq \frac{\rho^{n(1-\alpha)} M\|\overline{f}\|_{L^\infty(B_{9/10})}}{\overline{\Phi}(\rho^n x, 1)\rho^{n\alpha(\mu-\gamma)}} \\
&\leq M\|\overline{f}\|_{L^\infty(B_{9/10})}\rho^{n[1-\alpha(1+\mu-\gamma)]} \\
&\leq M\varepsilon,
\end{aligned}
$$

where we have again used the value of α chosen in (5-27). Now, note that

$$
\overline{\Phi}_n(x,1) = \frac{\overline{\Phi}(\rho^n x, \rho^{n\alpha})}{\overline{\Phi}(\rho^n x, \rho^{n\alpha})} = 1, \quad \text{for all} \quad x \in B_1,
$$

the map $t \mapsto \overline{\Phi}_n(x,t)/t^{\gamma}$ is almost increasing, and the map $t \mapsto \overline{\Phi}_n(x,t)/t^{\mu-\gamma}$ is almost decreasing. Therefore, we can apply the Lemma 5.8 and repeat the remainder of **Case 1** to obtain the desired result. ■

Now, we are ready to enunciate and give the proof of our first main results of this Chapter.

Theorem 5.11 *Let* $u \in C(B_1)$ *be a viscosity solution to* (5-1). *Suppose that* $A2.3 - A2.6$ *are in force. Then* $u \in C^{1,\alpha}_{loc}(B_1)$, *and there exists a constant* $C > 0$ *such that*

$$
\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C = C\left(d, p, B, \alpha, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}\right),
$$

where

$$
\alpha = \begin{cases} \min\left\{\alpha_0^-, \frac{1}{1+\mu}\right\} & \text{if } \gamma \geq 0 \\ \min\left\{\alpha_0^-, \frac{1}{1+\mu-\gamma}\right\} & \text{if } -1 < \gamma < 0. \end{cases} \tag{5-31}
$$

Here, α_0^- *denotes any number* $\beta \in (0, \alpha_0)$.

Notice that, in particular, Theorem 5.11 shows that viscosity solutions to (5-1) in the singular $p-$Laplacian case, *i.e.*, with $\Phi(x,t) = t^{p-2}$, for $p \in (1,2)$, are almost as smooth as viscosity solutions to the homogeneous equation

$$
-\Delta_p u = 0 \quad B_1,
$$

since we can take $\gamma = \mu = p - 2$.

Proof. To begin with, Proposition 5.10 ensures the existence of a sequence of affine functions $(\ell_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$,

$$
|a_{n+1} - a_n| + \rho^n|b_{n+1} - b_n| \leq C\rho^{n(1+\alpha)}
$$

and

$$
\|u - \ell_n\|_{L^\infty(B_{\rho^n})} \leq \rho^{n(1+\alpha)}.
$$

Therefore, the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences, and as a consequence, there exist a_∞ and b_∞ such that $a_n \to a_\infty$ and $b_n \to b_\infty$, as $n \to \infty$. Let us define the affine function $\ell_\infty(x) := a_\infty + b_\infty \cdot x$, and observe that

$$|a_n - a_\infty| \le C\rho^{n(1+\alpha)} \qquad \text{and} \qquad |b_n - b_\infty| \le C\rho^{n\alpha}.$$

At this point, for any $0 < r \ll 1$ sufficiently small, we can take $n \in \mathbb{N}$ such that $\rho^{n+1} < r \le \rho^n$. Consequently,

$$\begin{aligned}
\|u - \ell_\infty\|_{L^\infty(B_r)} &\le \|u - \ell_n\|_{L^\infty(B_{\rho^n})} + \|\ell_n - \ell_\infty\|_{L^\infty(B_{\rho^n})} \\
&\le \rho^{n(1+\alpha)} + |a_n - a_\infty| + |b_n - b_\infty| \cdot |x| \\
&\le \rho^{n(1+\alpha)} + C\rho^{n(1+\alpha)} + C\rho^{n\alpha} \cdot \rho^n \\
&\le C\rho^{n(1+\alpha)} \\
&\le Cr^{1+\alpha},
\end{aligned}$$

which implies the $C^{1,\alpha}-$regularity. ■

Remark 5.12 *It is important to note that α_0 can be arbitrarily small in general. As a result, even when γ and μ are close to zero, or even equal to zero, the regularity of the solutions remains bounded above by α_0. In the next result, we study the homogeneous p-Laplace equation and show that α_0 can be taken arbitrarily close to 1, as long as p is close enough to 2.*

5.4 Sobolev estimates

In this section, we derive Sobolev estimates for viscosity solutions to the $p-$Laplace equation

$$-\Delta_p u = 0 \quad \text{in} \quad B_1. \tag{5-32}$$

Recall that, from [51], viscosity solutions to the equation above are also viscosity solutions to

$$-\Delta_p^{\mathrm{N}} u = 0 \quad \text{in} \quad B_1. \tag{5-33}$$

From now on, we will work with this equation.

Let $u \in C(B_1)$ be a viscosity solution to (5-33). We argue similarly to [5, 11] and consider the regularized operator

$$\begin{cases} -\Delta v_\varepsilon - (p-2)\dfrac{\langle D^2 v_\varepsilon D v_\varepsilon, Dv_\varepsilon\rangle}{|Dv_\varepsilon|^2 + \varepsilon^2} + \lambda v_\varepsilon = \lambda u & \text{in} \quad B_{3/4} \\ v_\varepsilon = u & \text{on} \quad \partial B_{3/4}. \end{cases} \tag{5-34}$$

for some $\lambda > 0$. We added the zero-order term in the equation above to avoid issues with the uniqueness of solutions. Now, since we have a uniformly elliptic operator with no singularities, classical results ensure that solutions v_ε are C^2 classical solutions (see, for instance [47]). It is immediate to check that v_ε is bounded, independently of ε, by using the maximum principle.

We are now ready to prove Theorem, our second main result of this Chapter.

Theorem 5.13 *Let $u \in C(B_1)$ be a viscosity solution to*

$$-\Delta_p u = 0 \quad B_1.$$

Given $q \in [1,\infty)$, there exists a constant $\tilde{\varepsilon} := \tilde{\varepsilon}(q) > 0$, such that if

$$|p-2| \leq \tilde{\varepsilon},$$

then $u \in W^{2,q}_{loc}(B_1)$, for all $q \in [1,+\infty)$, and there exists a universal constant C such that

$$\|u\|_{W^{2,q}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

In particular, $u \in C^{1,\alpha}(B_{1/2})$ for any $\alpha \in (0,1)$, with the same estimate.

Proof. Let us consider the operator

$$F(D^2u, x) := \Delta u + (p-2)\frac{\langle D^2 u Dv_\varepsilon, Dv_\varepsilon\rangle}{|Dv_\varepsilon|^2 + \varepsilon^2},$$

and notice that it is uniformly elliptic, with ellipticity constants

$$\lambda = \min\{1, p-1\} \qquad \text{and} \qquad \Lambda = \max\{1, p-1\}.$$

Moreover, the oscillation of F,

$$\theta_F(x) := \sup_{M \in \mathcal{S}(d)} \frac{F(M,x) - F(M,0)}{\|M\|},$$

is such that

$$\theta_F(x) \leq 2|p-2|.$$

Since, from (5-34), v_ε solves

$$F(D^2 v_\varepsilon, x) = \lambda(v_\varepsilon - u) \in L^\infty(B_{3/4})$$

(recall that both u and v_ε are bounded), we can apply classic $W^{2,q}-$estimates (see [25], [24]) to guarantee that for a given $q \in [1,\infty)$, there exist $\tilde{\varepsilon} := \tilde{\varepsilon}(q) > 0$, such that if

$$|p-2| \leq \tilde{\varepsilon},$$

then $v_\varepsilon \in W^{2,q}$. Moreover, there exists a universal constant $C > 0$ such that

$$\|v_\varepsilon\|_{W^{2,q}(B_{1/2})} \leq C\left(\|v_\varepsilon\|_{L^\infty(B_{3/4})} + \lambda\|v_\varepsilon - u\|_{L^\infty(B_{3/4})}\right). \tag{5-35}$$

Finally, from the fact that $v_\varepsilon \to u$ locally uniformly, we also have that $u \in W^{2,q}$ and satisfies the estimate

$$\|u\|_{W^{2,q}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

In particular, $u \in C^{1,\alpha}(B_{1/2})$ for any $\alpha \in (0,1)$, with the same estimate. ■

Remark 5.14 *The proof can be adapted (cf. [12]) to show that viscosity solutions to*

$$-\Delta_p u = f(x) \quad \textit{in } B_1,$$

where $f \in L^q(B_1)$, are of class $W^{2,q}$, as long as $|p-2| \le \tilde{\varepsilon}(q)$.

A direct application of Theorem 5.13 is an improvement of the regularity estimates provided by Theorem 5.11. In particular, this result generalizes [61, Theorem 1].

Corollary 5.15 *Let $u \in C(B_1)$ be a viscosity solution to* (5-1). *Suppose that A2.3-A2.6 are in force and let*

$$\alpha = \begin{cases} \frac{1}{1+\mu} & \textit{if } \gamma \ge 0 \\ \frac{1}{1+\mu-\gamma} & \textit{if } -1 < \gamma < 0. \end{cases}$$

Given $\beta \in (0, \alpha]$, there exists a constant $\tilde{\varepsilon} := \tilde{\varepsilon}(\beta) > 0$ such that if

$$|p-2| \le \tilde{\varepsilon},$$

then $u \in C^{1,\beta}_{loc}(B_1)$, and there exists a constant $C > 0$ such that

$$\|u\|_{C^{1,\beta}(B_{1/2})} \le C = C\left(d, p, B, \alpha, \|u\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}\right).$$

Proof. The proof follows directly from Theorem 5.11 and Theorem 5.13, since for given

$$-1 < \gamma \le \mu \qquad \text{and} \qquad |p-2| \le \tilde{\varepsilon},$$

we can take

$$\alpha_0 > \frac{1}{1+\mu}$$

if $\gamma \ge 0$ and

$$\alpha_0 > \frac{1}{1+\mu-\gamma}$$

if $\gamma < 0$. ■

Remark 5.16 *Although Corollary 5.15 improves the regularity of solutions when p is close to 2, it still imposes the constraint that $\tilde{\varepsilon}$ depends on the exponent β. In particular, we cannot allow β to depend on the parameter p, as this would cause $\tilde{\varepsilon}$ to also depend on p, creating a circular dependence.*

6
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