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**Filipe Bellio da Nóbrega**

# On Osculating Conics in the Real Projective Plane

**Tese de Doutorado**

Thesis presented to the Programa de Pós–graduação em Matemática, do Departamento de Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Marcos Craizer
Co-advisor: Prof. Étienne Ghys

Rio de Janeiro
April 2025

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do Rio de Janeiro

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To my friends and family members who helped me get back on my feet,
to those who did not let my passion for mathematics be extinguished,
and to the ones that reminded me that life is ultimately worth living.

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The path I took during my PhD was anything but linear. I began my research in France in 2018, endured deep isolation during the COVID-19 pandemic, faced a profound existential crisis, returned to Brazil, and finally resumed my PhD at PUC in 2023, where this journey now comes to an end.

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Unfortunately, the COVID-19 pandemic hit me particularly hard. The solitude I faced during the period of worldwide uncertainty compelled me to re-evaluate my sense of purpose. It reached a point that I had to come back to Brazil in order to reconnect with myself. Back in Rio, The support from my family and friends during the critical year of 2022 was fundamental to my recovery. The unfailing support and embrace of my parents, Maria and Alberto, my aunt Lucyna, and my uncle Armínio were fundamental in restoring my self-confidence, grounded in the love they have always shown me. I love you all!

In February 2022, I co-founded Baobá, a nonprofit organization focused on popular education, alongside two of my closest friends, Maitê and Rebeca. Since then, we have maintained a strong bond, and I owe much of my emotional and professional recovery to them. Moreover, it was through my work at Baobá that I have established my new lifelong purpose: to transform as many lives as possible for the better through education. I aspire to build organizations, methodologies, and systems that expand access to quality education, particularly for those most in need, with the goal of fostering a more critical and equitable society.

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# Abstract

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We investigate how the osculating conics of a regular curve in the real projective plane evolve as one traverses the curve. The *Tait-Kneser Theorem* states that if the curve has no inflection or vertex, then the osculating circles do not intersect and are *nested*, that is, the smaller osculating circle is contained in the bounded region defined by the larger circle. We generalize this result by proving that if the curve has no inflection or sextactic point, then its osculating conics are *convexly nested*.

In addition, we compute the first and second terms of the power series of the *J-invariant* of the binary quartic related to a pair of osculating conics of an arbitrary curve. Finally, we show that given a pair of *harmonically nested* conics $u, v$, there exists a zero projective curvature logarithmic spiral that has $u$ and another conic of the pencil generated by $u$ and $v$ as its osculating conics.

## Keywords

Real pencil of conics; Convex binary quartics; Blenders; Projective spiral.

## Resumo

Nóbrega, Filipe Bellio da; Craizer, Marcos; Ghys, Étienne. **Cônicas Osculatrizes no Plano Projetivo Real**. Rio de Janeiro, 2025. 193p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nós investigamos como as cônicas osculatrizes de uma curva regular do plano projetivo real evoluem à medida que percorremos a curva. O *Teorema de Tait-Kneser* afirma que se uma curva não tem inflexão ou vértice, então seus círculos osculadores são disjuntos e *aninhados*, ou seja, o círculo menor é contido na região limitada definida pelo círculo maior. Nós generalizamos esse resultado ao provar que se uma curva não tem inflexão ou ponto sextático, então as cônicas osculatrizes são *convexamente aninhadas*.

Além disso, nós calculamos os dois primeiros termos da série de potências do *invariante-$J$* da quártica binária associada a um par de cônicas osculatrizes de uma curva arbitrária. Finalmente, nós mostramos que dado um par de cônicas *harmonicamente aninhadas*, $u, v$, existe uma espiral logarítmica de curvatura projetiva zero que tem $u$ e outra cônica do feixe gerado por $u$ e $v$ como suas cônicas osculatrizes.

## Palavras-chave

Matemática - Teses; Feixe real de cônicas; Quártica binárias convexas; Cones de polinômios homogêneos; Espirais projetivas.

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*Viel haben wir gelernt, Siddhartha, viel bleibt noch zu lernen. Wir gehen nicht im Kreise, wir gehen nach oben, der Kreis ist eine Spirale, manche Stufe sind wir schon gestiegen.*

*We have learned much, Siddhartha. There still remains much to learn. We are not going in circles, we are going upwards. The path is a spiral; we have already climbed many steps.*

*Olha, Sidarta, aprendemos muito, mas muita coisa ainda resta-nos aprender. Não andamos em círculos, estamos sempre subindo. O círculo é uma espiral. Já galgamos muitos degraus.*

**Hermann Hesse**, *Siddhartha.*

# Introduction

The main goal of this thesis is to study the behaviour of the *osculating conics* of a regular curve in the real projective plane. More precisely, we investigate how they evolve as one traverses the curve. The analogous problem for osculating circles is well known and is the subject of the so called *Tait-Kneser Theorem*, which states that if the curve has no inflection or vertex, then the osculating circles do not intersect and are nested, that is, the smaller osculating circle is contained in the bounded region defined by the larger circle. In Chapter 1 we explain in more details the case of osculating circles and also provide a new proof of Tait-Kneser Theorem.

Then, we introduce the tools we shall use to achieve the generalization for the case of osculating conics. In Chapter 2 we first present another interesting setting where a given curve is approximated by a certain family of curves, namely how the *Taylor polynomials* approach the graph of a smooth function. This revealing scenario shows us that the condition on the relative position of the osculating curves at two distinct points can be more subtle than one might expect. In the case of the Taylor polynomials, for instance, if a real function $f$ has a derivative of an odd order $n \geq 3$ that is positive in an interval $I$, then for any two points $a < b \in I$, the Taylor polynomials of degree $n-1$ that osculate the graph of $f$ at these points have a difference $T_b(x) - T_a(x)$ that must be positive and *convex* in the entire real line. The fact that this difference is positive indicates that the two approximations do not intersect, just like the osculating circles, but the less evident property is the convexity of $T_b(x) - T_a(x)$. Motivated by this example, we move to the study of the space of *homogeneous polynomials*, as we intend to use them in order to analyse the relative position of the osculating conics. More specifically, we use *binary quartics* to describe such relation. It is of particular importance a certain type of structure in the space of homogeneous polynomials called *blenders*, due to its invariance under the action of $\mathrm{PGL}(2;\mathbb{R})$ by linear change of variables. Finally, we present another property that remains invariant under projective transformations, the *cross-ratio* of the roots of the binary quartic. We explain this phenomenon in detail and also characterize the convex binary quartics via the cross-ratio of its roots.

In Chapter 3 we prove that every nondegenerate conic in the real projective plane admits a quadratic parametrization and also show that all such parametrizations belong to the same $\mathrm{PGL}(2;\mathbb{R})$ orbit. Then, by combining the quadratic parametrization of the first conic with the implicit equation of the second one, we obtain the binary quartic that describes the projective relative position of the ordered pair. This is the main tool we use to prove the novel results in the following chapters.

We define what it means for a conic to be *convexly nested* with respect to another conic in Chapter 4. This is a stronger condition than the simple nesting that we observe for the osculating circles. Then we prove one of the main theorems of this thesis: If a smooth curve in the real projective plane has no inflection nor *sextactic point*, then its osculating conics are disjoint and convexly nested.

In Chapter 5, we seek to give both algebraic and geometric descriptions of what it means for a conic to be convexly nested with respect to another conic. By employing a powerful technique of *simultaneous diagonalization*, we reduce the dimension of the space of pairs of conics, rendering our analysis much more manageable. In the end, we reach a *normal form* that depends on a single parameter $\lambda$. Next, we introduce the *algebraic invariants* of binary quartics and use them to characterize the convexity of the corresponding form. Finally, we present another original result, we give the first two terms of the power series around zero of the $J$-invariant and the $K$-invariant of the quartics arising from the osculating conics of *any* regular curve away from an inflection or sextactic point. Surprisingly, the limiting value for the $J$-invariant is $32/27$, which is greater then the extreme value that implies convexity, which is 1. This unexpected result motivated us to coin the term *harmonic nesting* for when the quartic coming from a pair of conics has its $J$-invariant greater than $32/27$.

To conclude the thesis, we explore in Chapter 6 the family of curves with constant *projective curvature*. There we face the following reciprocal problem: Given two conics in the real projective plane, under which conditions can we find a smooth curve with no inflections or sextactic points that joins these conics, that is, so that both of them are osculating conics of the curve at two distinct points? We manage to prove the following partial solution: If the two conics $u$ and $v$ are harmonically nested, then there exists a zero projective curvature logarithmic spiral that has $u$ and another conic of the pencil $\overline{uv}$ as osculating conics. We also show that for such a curve, there cannot be three osculating conics in the same pencil.

We provide in Appendix A a broad overview of the space of *pencils of conics*, both in the real and complex settings. We present and prove a complete classification of the orbits under the actions of the respective projective groups $\mathrm{PGL}(3;\mathbb{R})$ and $\mathrm{PGL}(3;\mathbb{C})$. In addition, we analyse the *stabilizer subgroup* of a generic element of the orbit and also discuss the orbits of *marked* pencils of conics, where either one or two conics of the pencil are highlighted.

As a whole, the relevance of this thesis stems from the employment of different concepts and techniques in order to deepen our understanding about the osculating conics of a regular curve in the projective plane. Besides the proofs of new theorems, the establishment of new connections between concepts provides original perspectives that we hope may be fruitful for even further developments of the theory in the future.

# 1

# Osculating Circles

## 1.1

## Local approximation of a curve using circles

Our investigation begins with a regular curve on the plane $\gamma : \mathbb{R} \to \mathbb{R}^2$ and with some set of curves $\mathcal{C}$. First, we wish to find, for each point $\gamma(s)$, the curve $c \in \mathcal{C}$ that best approximates the original one locally, that is, in a small neighbourhood of $\gamma(s)$. The nature of the problem at hand depends on which family of curves $\mathcal{C}$ we choose to begin with. We may, for example, appoint the family of lines on $\mathbb{R}^2$ as $\mathcal{C}$, in which case we get the *tangent line* at $\gamma(s)$ as the solution for the problem. Our main interest is in understanding and describing how these *curves of best contact* evolve as we traverse the curve $\gamma$. When $\mathcal{C}$ is the family of circles, the *Tait-Kneser Theorem* states how the *osculating circles* of an arc evolve in general. This classic result indicates that there might be some interesting structure to be found for each family of curves we choose to analyse. The aim of this thesis is to investigate and describe the behaviour of the *osculating conics* of a regular curve $\gamma$.

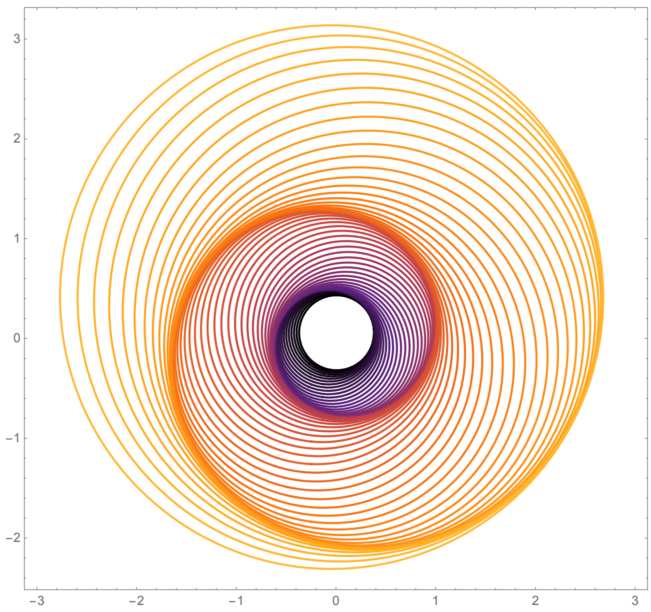


Figure 1.1: Osculating circles of a logarithmic spiral.

**Theorem 1.1.1** (Tait-Kneser)**.** *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve with no inflection point and with monotonic curvature. Then the* osculating circles *of $\gamma$ are pairwise* disjoint *and* nested*, that is, they do not intersect and the smaller circle is contained in the bounded region defined by the bigger circle.*

*Proof.* The original proof presented by Peter Tait in 1896 [Tait] is concrete and straightforward. Consider the evolute of $\gamma$ and observe the osculating circles at two points. Their centers belong to the evolute and the difference between the radii is the length of the evolute between the two centers. Since the length of the line segment joining these two points is shorter than the evolute, the sum of the length of this segment with the smaller radius is less than the bigger radius, thus the circles are nested. □

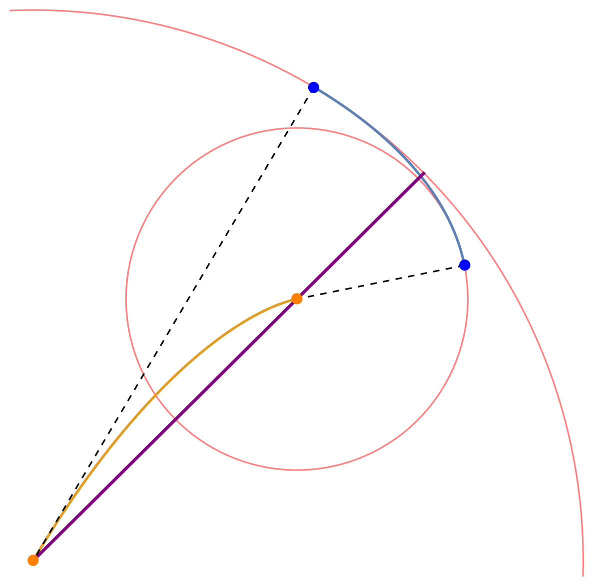

Figure 1.2: Evolute (orange) of an arc of ellipse (blue).

However, this approach does not suggest a path for generalization, so we will present a new proof in a more modern language by considering a *moduli space* of circles. During the development of this work, G. Bor, C. Jackman and S. Tabachnikov published an article [Bor] also with a new proof of Tait-Kneser Theorem employing a similar technique as the one we present below. We are grateful for the interesting conversations we had on the subject.

The idea is to define a differentiable manifold where each point represents an oriented circle in $\mathbb{R}^2$, and such that certain geometric properties in this manifold are in direct relation to the phenomena we wish to understand about the original circles. Once we understand how the proof works in the space of circles, we can proceed to the moduli spaces of conics.

## 1.2

## The Space of Circles

The moduli space of circles in $\mathbb{R}^2$, and more generally, the space of spheres $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, have been studied and described by Sophus Lie, and thus their geometrical theory was named *Lie Sphere Geometry*. The following description is found in the work of Thomas Cecil [Cecil] on the subject.

To define a circle in the plane, we need 3 parameters, as we must determine its center $(x_0, y_0) \in \mathbb{R}^2$ and its radius $r > 0$, so we anticipate a 3-dimensional manifold for the desired moduli space. Let us take $\mathbb{R}^3$ with coordinates $(x, y, z)$ and associate the plane $z = 0$ with the original plane $\mathbb{R}^2$. Now, given a circle in this plane, we can relate it to a unique point $u = (x_0, y_0, r)$ in the upper half-space $H_+ = \{(x, y, z) \in \mathbb{R}^3 \,|\, z > 0\}$ simply by joining the coordinates of its center and its radius. This relationship also works in the other direction, and we can see it geometrically: by taking any point $u$ in this half-space, if we consider the right cone whose vertex is at $u$ and having an angle of $\pi/4$ with the base plane, its intersection with the plane $z = 0$ gives us the circle associated with $u$. Thus, we already have a bijection between circles in $\mathbb{R}^2$ and the upper half-space.

The construction so far does not take into account the orientation of the circles. Fortunately, we have at our disposal the other half-space $H_- = \{(x, y, z) \in \mathbb{R}^3 \,|\, z < 0\}$, which will allow us to work with oriented circles. We can make the same association with an additional condition: if the circle is oriented in the positive direction, we take the cone with vertex $u_+ = (x_0, y_0, r)$ in $H_+$, whereas if it is oriented in the negative direction, we take the cone in the other half-space, so with vertex $u_- = (x_0, y_0, -r)$. This establishes the bijection between oriented circles in the plane and points in $H_+ \cup H_-$.

Another way to interpret this representation is to assume that we have circles of "positive radius" and circles of "negative radius", which correspond to the two possible orientations. We may further extend our model and define "circles of radius 0". Seen as limiting elements, they correspond naturally to points of $\mathbb{R}^2$, more specifically, the circle with center $p = (x_0, y_0)$ and radius 0 is just the point $(x_0, y_0) \in \mathbb{R}^2$ and it may even be described by the limiting algebraic equation: $(x - x_0)^2 + (y - y_0)^2 = 0$. Notice that the circles of radius 0 fittingly do not admit an orientation. Finally, we have a bijection between $\mathbb{R}^3$ and the circles in $\mathbb{R}^2$ with signed radius.

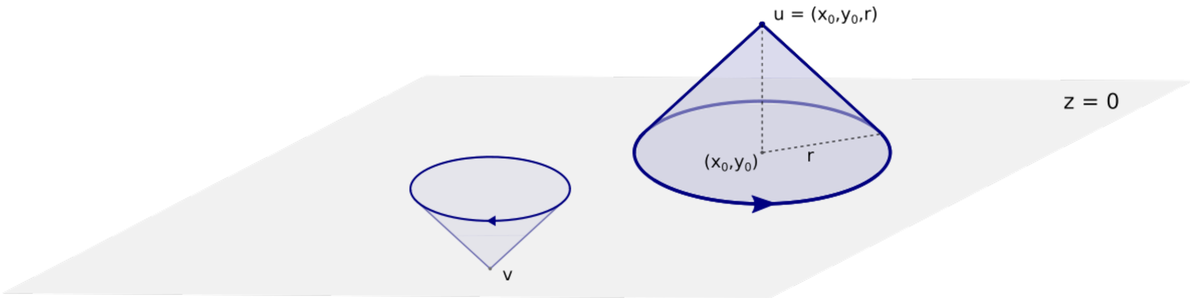


Figure 1.3: Two oriented circles represented by their points in the moduli space.

In reality, this construction is still incomplete compared to the moduli space described by Lie. He also considers circles of "infinite radius", which correspond to oriented lines. In order to include them in the representation, it is necessary to move to a projective space, as explained by Cecil in his first chapter [Cecil]. Nevertheless, the construction described above suffices for the new proof of the Tait-Kneser Theorem.

Let us then study more deeply this space of circles represented by $\mathbb{R}^3$. As we have seen, we naturally have a cone emanating from each point, which suggests the introduction of a bilinear form of signature $(2, 1)$, providing the structure of a *Lorentzian space*. The bilinear form is defined in the tangent space at each point, but we make the natural identification between elements of the tangent space and vectors of the base space, since it is also a vector space. We define the quadratic form $(\cdot\,;\,\cdot)$ as follows: let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be elements of $\mathbb{R}^3$, then:

$$(u\,;v) = u_1v_1 + u_2v_2 - u_3v_3 = u^\intercal \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} v$$

The cones described above can be expressed by an equation using the quadratic form. Take $u \in \mathbb{R}^3$; the points of the cone emanating from $u$ are the $v \in \mathbb{R}^3$ such that $(v - u\,; v - u) = 0$. Moreover, the points inside one of the two sheets of the cone satisfy $(v - u\,; v - u) < 0$, and the points outside give a positive value $(v - u\,; v - u) > 0$.

Employing the terminology of Physics, we classify the vectors in the following way:
$w \in \mathbb{R}^3$ is *timelike* if $(w\,;\,w) < 0$;
$w \in \mathbb{R}^3$ is *lightlike* if $(w\,;\,w) = 0$ ($w$ is an *isotropic* vector);
$w \in \mathbb{R}^3$ is *spacelike* is $(w\,;\,w) > 0$.

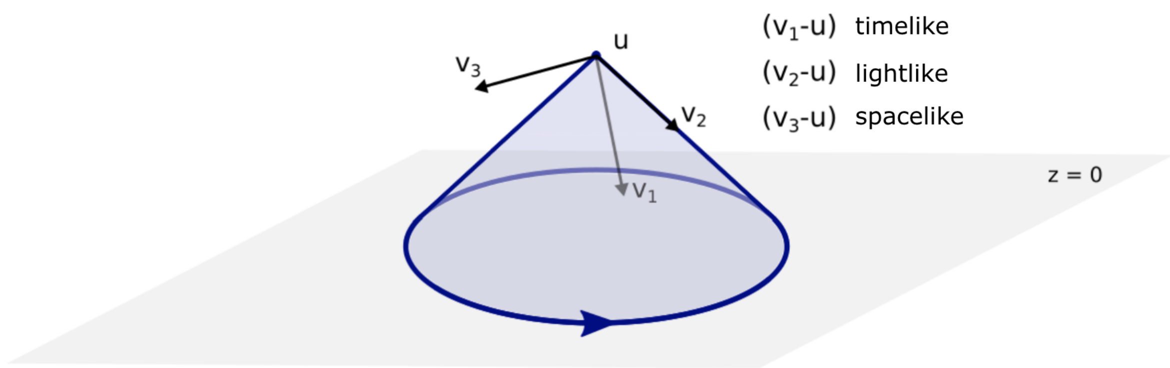


Figure 1.4: The three types of vectors defined by the quadratic form.

Now we can investigate the correspondence between the geometry of our moduli space and the properties of the circles in the original plane. This is precisely the benefit and the interest in working with moduli spaces. Firstly, the quadratic form detects the *oriented contact* between the circles. Consider two oriented circles and their representatives $u, v \in \mathbb{R}^3$. The two circles are tangent and their tangent vectors at the point of contact have the same direction if and only if $v - u$ is isotropic, that is, $(v - u\,; v - u) = 0$.

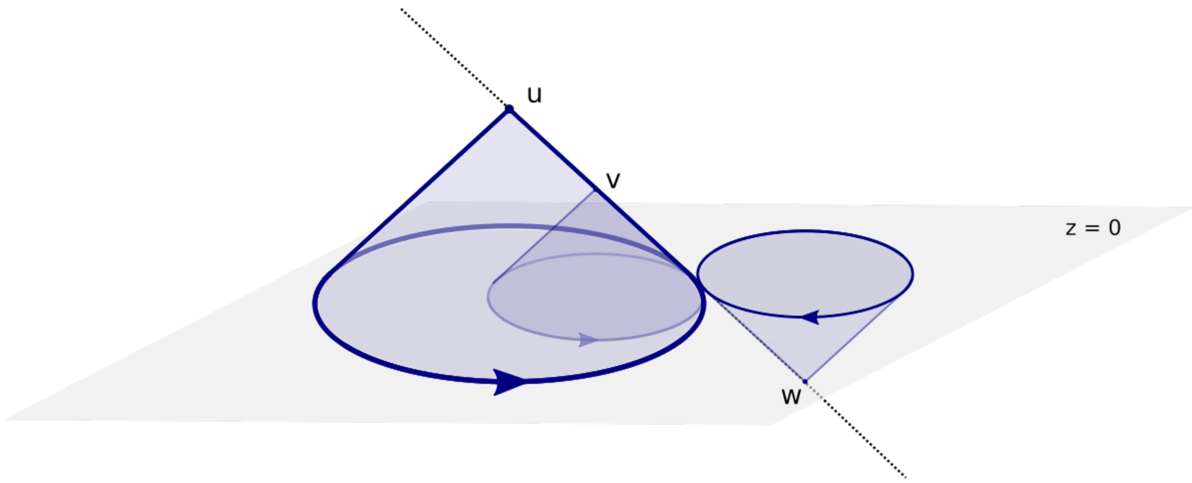


Figure 1.5: Circles with the same oriented tangent belong to a lightlike direction.

If $v - u$ is timelike, then the circles are disjoint and nested as long as $u_3$ and $v_3$ have the same sign (and therefore the circles have the same orientation), which is locally true for circles with radius $r \neq 0$.

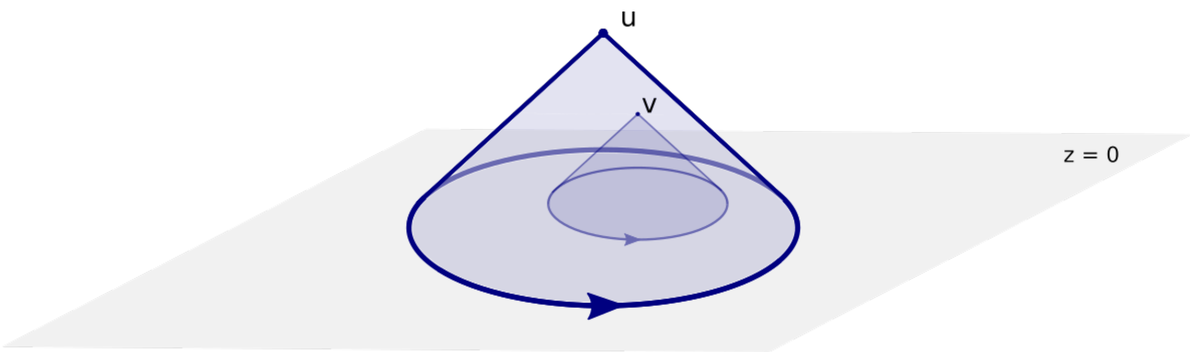


Figure 1.6: Nested circles belong to a timelike direction.

Finally, $v - u$ is spacelike if and only if the two circles have a pair of common oriented tangents. Moreover, these tangents intersect exactly at the point where the line connecting $u$ and $v$ intersects the plane $z = 0$.

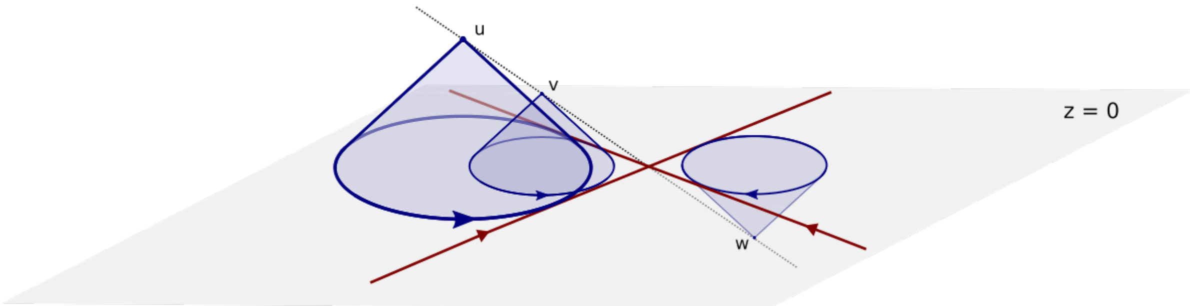


Figure 1.7: Circles with a pair of common real tangents belong to a spacelike direction.

## 1.3

## New proof of Tait-Kneser Theorem

Once familiar with the space of circles, let us state and then prove the Tait-Kneser Theorem.

**Theorem 1.3.1 (Tait-Kneser).** *Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ be a smooth curve in $\mathbb{R}^2$. If its curvature $k$ satisfies that for every $s \in (-\varepsilon, \varepsilon)$, $k(s) \neq 0$ and $k'(s) \neq 0$, then the osculating circles of $\gamma$ are pairwise disjoint and nested.*

*Proof.* We may assume that $\gamma$ is parametrized by arc length. In this case, the equation of the evolute gives us the expression for the centers of the osculating circles. Let $n(s)$ be the normal vector to $\gamma(s)$ at $s$ and $r(s) = 1/k(s)$ its radius of curvature (hence the radius of the osculating circle at $\gamma(s)$). The formula for the evolute is:

$$p(s) = \gamma(s) + r(s)n(s).$$

Therefore, we have a path in the space of circles $\Gamma(s) = \big(p(s), r(s)\big)$, where $p(s)$ provides the first two coordinates. To prove the theorem, it suffices to show that for all $a, b \in (-\varepsilon, \varepsilon)$, the vector $\Gamma(b) - \Gamma(a)$ is timelike, as this implies that the corresponding circles are disjoint and nested. In order to prove this, let us compute the derivative of $\Gamma$ using the Frenet-Serret equation $n' = -k\gamma'$:

$$p' = \gamma' + r'n + rn' = r'n.$$

Thus, we have the tangent vector $\Gamma'(s) = \big(p'(s), r'(s)\big) = r'(s)\big(n(s), 1\big)$. Since $r' = -k'/k^2$, the hypothesis $k' \neq 0$ ensures that the curve $\Gamma$ is not singular. The fact that $n(s)$ is unitary implies that $\Gamma'$ is always isotropic, since $(\Gamma' \,;\, \Gamma') = (r')^2\,(||n||^2 - 1) = 0$. This property yields the desired result, because if $\Gamma'$ is always isotropic, then $\Gamma$ cannot exit the isotropic cones. Formally, this is shown with the following inequality of integrals:

$$\begin{aligned}\Gamma(b) - \Gamma(a) = \int_a^b \Gamma'(s)ds &= \left(\int_a^b r'(s)n(s)\,ds\,,\, \int_a^b r'(s)\,ds\right)\\ &= \left(\int_a^b r'(s)n(s)\,ds\,,\, r(b) - r(a)\right).\end{aligned}$$

Therefore,

$$(\Gamma(b) - \Gamma(a)\,;\,\Gamma(b) - \Gamma(a)) < 0 \iff \left\|\int_a^b r'(s)n(s)\,ds\right\| < |r(b) - r(a)|.$$

But $||\int_a^b r'(s)n(s)\,ds|| \leq \int_a^b |r'(s)|\,||n(s)||\,ds = \int_a^b |r'(s)|\,ds$. In addition, the equality holds if and only if $n(s)$ is constant. However, this would imply $n' = -k\gamma' = 0$, which contradicts the hypothesis $k \neq 0$. Since $r'$ never changes sign, we have: $\int_a^b |r'(s)|\,ds = |\int_a^b r'(s)\,ds| = |r(b) - r(a)|$. Thus indeed $||\int_a^b r'(s)n(s)\,ds|| < |r(b) - r(a)|$, which concludes the proof.

$\square$

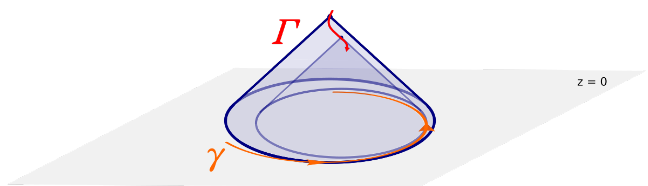


Figure 1.8: A curve $\gamma$ and its corresponding curve of osculating circles $\Gamma$.

# 2

# Convex Binary Quartics

## 2.1

## Osculating Polynomials

In their 2012 paper [Ghys], E. Ghys, S. Tabachnikov, and V. Timorin present some results on osculating curves, particularly related to the Tait-Kneser Theorem. In this article, they prove an interesting version of this theorem for the *Taylor polynomials* of a real function.

**Theorem 2.1.1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and $I$ an interval of the real line. Let $n = 2r$ be an even integer. Suppose that $f$ is $(n+1)$-times differentiable and that for all $x \in I$, $f^{(n+1)}(x) > 0$.*

*Then, for each $a, b \in I$, the graphs of the Taylor polynomials of degree $n$ at $a$ and $b$, denoted $T_a$ and $T_b$, are disjoint over the entire real line.*

*Proof.* The key idea of the proof is to consider the expression of the Taylor polynomial at $t \in I$ as a function of the base point $t$ and to compute its derivatives.

$$T_t(x) = \sum_{i=0}^{n} \frac{f^{(i)}(t)}{i!}(x-t)^i,$$

$$\frac{\partial T_t(x)}{\partial t} = \sum_{i=0}^{n} \frac{f^{(i+1)}(t)}{i!}(x-t)^i - \sum_{i=1}^{n} \frac{f^{(i)}(t)}{(i-1)!}(x-t)^{i-1} = \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Since $f^{(n+1)}(t) > 0$ for all $t \in I$, we have that $\frac{\partial T_t(x)}{\partial t} > 0$ for all $t \in I$, except for the point of contact, where $\frac{\partial T_t(t)}{\partial t} = 0$. Therefore, for any fixed $x$, $T_t(x)$ increases as a function of $t$. So by assuming $a < b$, we have that $T_a(x) < T_b(x)$ for all $x \in \mathbb{R}$. $\square$

As an example, let us examine the function $f(x) = x^3$. Its third derivative, $f'''(x) = 6$, is positive for every value of $x \in \mathbb{R}$, so all of its Taylor polynomials of degree 2 do not intersect and produce an interesting foliation of $\mathbb{R}^2$.

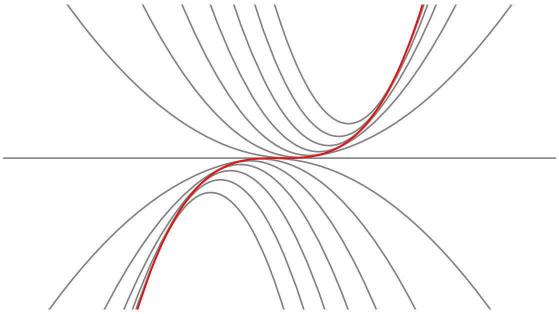

Figure 2.1: $f(x) = x^3$ and some of its Taylor polynomials of degree 2.

The theorem above shows that, under its hypothesis, the image of the polynomial $T_b(x) - T_a(x)$ is always positive. One may then ask whether for an even $n$, any positive polynomial $p$ of degree $n$ can be obtained as $p(x) = T_b(x) - T_a(x)$, for a smooth function $f : \mathbb{R} \to \mathbb{R}$ whose $(n+1)$-th derivative is positive on an interval containing $a$ and $b$. The answer to this question is that the condition that $p(x)$ is always positive is not enough to guarantee the existence of such an $f$. Indeed, the fundamental theorem of calculus shows us that

$$T_b(x) - T_a(x) = \int_a^b \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt \, .$$

The polynomial $T_b(x) - T_a(x)$ is expressed as a barycenter of polynomials of the form $(x - t)^n$, so it is not only positive but also convex for even $n \geq 2$. The search for better understanding this problem led us to the study of the spaces of homogeneous polynomials.

## 2.2

## Homogeneous Polynomials and Blenders

### 2.2.1

### Definition and first examples

We shall denote by $F_{n,d}$ the set of homogeneous polynomial of degree $d$ in $n$ variables, which are also called *algebraic forms*. For the purposes of our investigation, we will later focus on the binary forms, that is, when $n = 2$, and use $x$ and $y$ as the variables. Notice that $F_{2,d}$ is an $n+1$-dimensional vector space spanned by the basis $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$. However, many results can be established for an arbitrary number of variables as presented by B. Reznick in his article [Reznick] from 2011 where he defines some particular subsets of $F_{n,d}$ which he named *blenders*. What we present in this section is a selection of the content of Reznick's paper that will be useful in our study of the osculating conics. Specifically, we seek to better understand the space of *binary quartics*.

The general linear group $\mathrm{GL}(n;\mathbb{R})$ acts on the space of homogeneous polynomials $F_{n,d}$ by linear changes of coordinates. One way to interpret this action is to consider any element $p \in F_{n,d}$ as a function from $\mathbb{R}^n$ to $\mathbb{R}$, so the image of $p$ under the action of $M \in \mathrm{GL}(n;\mathbb{R})$ is the element of $F_{n,d}$ associated to the function $p \circ M$. More concretely, if $M = (m_{ij})_{n\times n}$, then $M.p \in F_{n,d}$ is given by

$$(M.p)(x_1, \dots, x_n) = p(l_1, \dots l_n), \quad \text{where } l_i(x_1, \dots, x_n) = \sum_{j=1}^{n} m_{ij} x_j.$$

Let us consider some particular structures in $F_{n,d}$ that are invariant under the linear change of variables. Throughout this thesis, the word *cone* refers to a set $C$ that satisfies the following property: if $x \in C$, then $\forall \lambda \geq 0$, $\lambda x \in C$. As one may observe, a cone is a set comprised of half-lines stemming from the origin $0 \in F_{n,d}$, which is the *vertex* of the cone. We may now define the sets that Reznick named *blenders*.

**Definition** (Blender)**.** A closed convex cone $C \subseteq F_{n,d}$ is called a *blender* if for every $p \in C$, we have $p \circ M \in C$ for any matrix $M \in \mathrm{GL}(n;\mathbb{R})$.

The assumption of being closed implies that $p \circ M \in C$ for any $n \times n$ matrix $M \in M(n;\mathbb{R})$, since $\mathrm{GL}(n;\mathbb{R})$ is a dense subset of $M(n;\mathbb{R})$. Some fundamental examples of blenders are the cone of nonnegative polynomials $P_{n,d}$, that of sums of squares $\Sigma_{n,d}$, and that of sums of powers of linear forms $Q_{n,d}$.

$$\begin{aligned}
P_{n,2r} &:= \Big\{ p \in F_{n,2r} : u \in \mathbb{R}^n \Rightarrow p(u) \geq 0 \Big\} \\
\Sigma_{n,2r} &:= \Big\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} h_k^2,\ h_k \in F_{n,r} \Big\} \\
Q_{n,2r} &:= \Big\{ p \in F_{n,2r} : p = \sum_{k=1}^{s} (\alpha_{k1} x_1 + \dots + \alpha_{kn} x_n)^{2r},\ \alpha_{kj} \in \mathbb{R} \Big\}
\end{aligned}$$

In the case of binary quadratic forms, $n = 2$ and $d = 2$, all these blenders coincide, and they contain the positive forms with nonpositive discriminant. However, this is not the case anymore if we increase the degree. When the degree $d$ is odd, all blenders are either $\{0\}$ or the whole space $F_{n,d}$. So the non-trivial blenders only occur for even-degree forms.

Hilbert proved in 1888 a result about the blenders $P_{n,d}$ and $\Sigma_{n,d}$. He showed that all positive homogeneous polynomials in $n$ variables and of degree $d$ can be written as a sum of squares if and only if $n = 2$ or $d = 2$, or ($n = 3$ and $d = 4$). In other words, he proved that the inclusion $\Sigma_{n,d} \subseteq P_{n,d}$ is strict except in these three cases. His proof is non-constructive, however. So the following first example of a positive homogeneous polynomial that cannot be written as a sum of squares was found only in 1967 by Motzkin [Motzkin]: $p(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ is nonnegative but cannot be given as a sum of squares.

### 2.2.2

### Multinomial notation and the inner product

Here we introduce the usual multinomial notation. Let $i = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, with all $i_j \geq 0$, and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We denote the monomials in a contracted way as $x^i := \prod_{k=1}^n x_k^{i_k}$. The degree of the monomial of index $i$ is $|i| := \sum_{k=1}^n i_k$. And the multinomial coefficients are given by $c(i) := \binom{|i|}{i_1,\ldots,i_n} = \frac{|i|!}{\prod\limits_{k=1}^{n} i_k!}$.

In the case of binary forms, where $n = 2$, we have the usual binomial coefficient $c(i) = \binom{i_1+i_2}{i_1}$. Let $d \geq 1$, $d \in \mathbb{Z}$, we denote $I(n, d) := \{i \in \mathbb{Z}_+^n \mid |i| = d\}$ the set of indices of fixed degree $d$. We may specify the cardinality of this set as $N(n, d) := |I(n, d)| = \binom{n+d-1}{n-1}$. A homogeneous polynomial in $n$ variables and of degree $d$, $p(x_1, \ldots, x_n) \in F_{n,d}$ is presented as:

$$p(x_1, \ldots, x_n) = \sum_{i \in I(n,d)} c(i)\, a(p\,; i)\, x^i, \quad \text{where } a(p\,; i) \in \mathbb{R}.$$

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, we may define a homogeneous polynomial $(\alpha\, \bullet)^d$ by:

$$(\alpha\, \bullet)^d(x) := (\alpha \bullet x)^d = \Big(\sum_{j=1}^n \alpha_j\, x_j\Big)^d = \sum_{j=1}^n c(i)\, \alpha^i\, x^i.$$

We introduce an inner product in $F_{n,d}$ given by:

$$[\, p\,, q\,] := \sum c(i)\, a(p\,; i)\, a(q\,; i).$$

We can identify $p \in F_{n,d}$ with $\left(c(i)^{1/2}a(p\,;i)\right)_{i\in I(n,d)} \in \mathbb{R}^{N(n,d)}$ to obtain an isomorphism between these two vector spaces, so that each coordinate contains the information about one of the polynomial's coefficients. Notice that the inner product above is then the Euclidean inner product in $\mathbb{R}^{N(n,d)}$ with this identification. It is symmetric and it may be used to obtain any coefficient of $p$, as for any $i \in \mathcal{I}(n,d)$ we have that $[p, x^i] = a(p; i)$. Moreover, a direct computation gives us another useful result: Let $p \in F_{n,d}$ and $\alpha \in \mathbb{R}^n$, then

$$[\,p\,,(\alpha\,\bullet)^d\,] = \sum c(i)\,a(p\,;i)\,\alpha^i = p(\alpha).$$

Thus, the inner product of any polynomial $p$ with another of the form $(\alpha\,\bullet)^d$ just consists of evaluating $p$ at $\alpha$.

### 2.2.3

### First properties of blenders

In this section, we show that if the degree $d$ is odd, then there are only trivial blenders, while if $d$ is even, then $Q_{n,d}$ is the smallest non-trivial blender and $P_{n,d}$ is the largest non-trivial blender. To achieve this, we need some propositions, as Reznick presents in his article [Reznick]. Let us denote the closed orbit of $p \in F_{n,d}$ by $[[p]] := \{p \circ M \mid M \in M(n;\mathbb{R})\}$. If $p = q \circ M$ for an invertible matrix $M \in \mathrm{GL}(n;\mathbb{R})$, we write $p \sim q$.

*Proposition* 2.2.1.

i. If $p \in F_{n,d}$ and $d$ is odd, then $p \sim \lambda p, \forall \lambda \in \mathbb{R}^*$.

ii. If $p \in F_{n,d}$ and $d$ is even, then $p \sim \lambda p, \forall \lambda \in \mathbb{R}^*_+$.

iii. If $u, \alpha \in \mathbb{R}^n$, then $\forall p \in F_{n,d}$, there exists a singular matrix $M \in M_n(\mathbb{R})$ such that $p \circ M = p(u)(\alpha\,\bullet)^d$.

*Proof.* The first two statements are obtained by employing a multiple of the identity matrix. For the third one, take the matrix $M$ given by $M_{ij} = u_i\alpha_j$ , $1 \le i,j \le n$. Then:

$$M(x) = \left( u_1 \sum_{j=1}^{n} \alpha_j x_j\,,\ \dots\ ,\, u_n \sum_{j=1}^{n} \alpha_j x_j \right) = (\alpha \bullet x)\,.\,(u_1, \dots, u_n)$$

$$p \circ M(x) = (\alpha \bullet x)^d\,.\,p(u_1, \dots, u_n)$$

□

The intersection, the Minkowski sum and the product set of two blenders are also blenders, so given two blenders $B_1$ and $B_2$, the sets $B_1 \cap B_2$, $B_1 + B_2 := \{p_1 + p_2 : p_i \in B_i\}$ and $B_1 * B_2 := \{\sum_{k=1}^{s} p_{1,k} p_{2,k} : p_{i,k} \in B_i, s \in \mathbb{N}\}$ are also blenders. The set $\mathcal{B}_{n,d}$ of all blenders in $F_{n,d}$ does not form a chain, that is, it may be the case that for $B_1, B_2 \in \mathcal{B}_{n,d}$ we have $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$. Reznick's paper presents an example for $(n, d) = (2, 8)$.

*Proposition* 2.2.2. Let $S \subset \mathbb{R}^n$ be a set with non-empty interior. Then $F_{n,d}$ is generated as a vector space by $\{(\alpha \,\bullet)^d \mid \alpha \in S\}$.

*Proof.* Let $U$ be the subspace of $F_{n,d}$ generated by $\{(\alpha \,\bullet)^d \mid \alpha \in S\}$ and consider $q \in U^{\perp}$. This means that $q(\alpha) = [\, q \,, (\alpha \,\bullet)^d \,] = 0$, $\forall \alpha \in S$. Since $q$ is a polynomial that vanishes on an open set, then $q \equiv 0$. Therefore, $U^{\perp} = \{0\}$ and $U = (U^{\perp})^{\perp} = \{0\}^{\perp} = F_{n,d}$. □

*Proposition* 2.2.3. Suppose $B$ is a blender and that there are $p, q \in B$ and $u, v \in \mathbb{R}^n$ such that $p(u) > 0 > q(v)$. Then $B = F_{n,d}$.

*Proof.* Proposition 2.2.1 implies that, $\forall \alpha \in \mathbb{R}^n$, $p(u)(\alpha \,\bullet)^d$ and $q(v)(\alpha \,\bullet)^d$ are in $B$. Therefore, proposition 2.2.2 gives us that $F_{n,d} \subseteq B$, because with the positive value $p(u)$ and the negative value $q(v)$, we have in $B$ the subspace spanned by $\{(\alpha \,\bullet)^d \mid \alpha \in \mathbb{R}^n\}$, which is the whole $F_{n,d}$. □

*Proposition* 2.2.4. If there exists a non-trivial blender $B \subseteq F_{n,d}$, then $d = 2r$ is even, and for an appropriate choice of sign, $Q_{n,d} \subseteq \pm B \subseteq P_{n,d}$.

*Proof.* If $B \neq \{0\}$, there exists $p \in B$ and $u \in \mathbb{R}^n$ such that $p(u) \neq 0$. If $d$ is odd, then $p(-u) = -p(u)$ and due to proposition 2.2.3, we have that $B = F_{n,d}$. If $d$ is even, by taking $-B$ if necessary, we may assume that $p(u) > 0$. Therefore, if $B \neq F_{n,d}$, then $\pm B \subseteq P_{n,d}$. Finally, since the finite sum of elements of a blender is also an element of the same blender, item *iii* of proposition 2.2.1 implies that $Q_{n,d} \subseteq \pm B$. □

### 2.2.4

### The dual cone

The inner product of $F_{n,d}$ allows us to define the dual cone of a set in the following manner.

**Definition** (Dual cone)**.** Let $A$ be a subset of $F_{n,d}$. We define its *dual cone* as:

$$A^* := \{y \in F_{n,d} \,|\, [x\,,y] \geq 0, \quad \forall x \in A\}.$$

This name is well justified because, as a direct consequence of this definition, $A^*$ is always a closed convex cone, and we have that $A \subseteq (A^*)^*$.

*Proposition* 2.2.5. Let $A \subseteq F_{n,d}$ be an arbitrary subset. Then $(A^*)^*$ is the closure of the smallest convex cone that contains $A$.

*Proof.* Let $\tilde{A}$ denote the closure of the smallest convex cone that contains $A$. Since the dual cone is always closed and convex and $A \subseteq (A^*)^*$, we have that $\tilde{A} \subseteq (A^*)^*$. It remains to prove that if $z \notin \tilde{A}$, then $z \notin (A^*)^*$. In order to do so, one just need to apply the Hyperplane Separation Theorem (a geometric form of the Hahn-Banach theorem). Since $\tilde{A}$ is a closed convex set, and $\{z\}$ is a compact convex set, then there exists a hyperplane that strictly separates them. In other words, there exist $v \in F_{n,d}$ and $c \in \mathbb{R}$ such that:

$$\forall x \in \tilde{A}, \quad [\,x\,,v\,] > c \quad \text{and} \quad [\,z\,,v\,] < c.$$

As $\tilde{A}$ contains the origin, then $c < 0$. Next, we show that $\forall x \in \tilde{A}$, $[\,x\,,v\,] \geq 0$. Suppose, for the sake of contradiction, that there exists an $x \in \tilde{A}$ such that $[\,x\,,v\,] = -\varepsilon < 0$. Since $\tilde{A}$ is a cone, then $\lambda x \in \tilde{A}, \forall \lambda \geq 0$. We have that $[\,\lambda x\,,v\,] = -\lambda\varepsilon$. By taking a large enough $\lambda$, say $\lambda \geq -c/\varepsilon$, we have that $-\lambda\varepsilon \leq c$ and thus the contradiction $[\,\lambda x\,,v\,] \leq c$. Finally, since $\forall x \in \tilde{A}, [\,x\,,v\,] \geq 0$, we have by definition that $v \in A^*$. Since $[\,z\,,v\,] < c < 0$, we conclude that $z \notin (A^*)^*$. □

As a corollary, if $C$ is already a closed convex cone, then $C = (C^*)^*$. In this case, one may say that the cones $C$ and $C^*$ are duals. Reznick also proves that the dual of a blender is also a blender. This is a consequence of the following proposition.

*Proposition* 2.2.6. For every $p, q \in F_{n,d}$ and $M \in M(n;\mathbb{R})$, it holds that $[\, p \circ M\,, q\,] = [\, p\,, q \circ M^{\mathsf{T}}\,]$.

*Proof.* By proposition 2.2.2, it suffices to show that the equality is true for linear forms raised to the $d$-th power. We have that:

$$\begin{aligned}[\, p \circ M\,, q\,] &= [\,(\alpha M \bullet)^d\,, (\beta \bullet)^d\,] = (\alpha M \beta^{\mathsf{T}})^d \\ &= (\alpha(\beta M^{\mathsf{T}})^{\mathsf{T}})^d = [\,(\alpha \bullet)^d\,, (\beta M^{\mathsf{T}} \bullet)^d\,] = [\, p\,, q \circ M^{\mathsf{T}}\,].\end{aligned}$$

□

**Definition** (Extremal point)**.** Let $C$ be a closed convex cone. We say that $u \in C$ is an *extremal point* of $C$ if $u = v_1 + v_2$, with $v_i \in C$ implies that $v_i = \lambda_i u$, $\lambda_i \geq 0$.

In other words, $u$ cannot be written as a sum of two elements of $C$ that are not multiples. We use $\mathcal{E}(C)$ to denote the set of extremal points of $C$, which consists of a set of half-lines originating from the origin. The set of extremal points is important because it provides the essential elements to reconstruct our cone $C$. A proof of the following proposition can be found in Rockafellar's book on convex analysis [Rockafellar].

*Proposition* 2.2.7. Let $C$ be a closed convex cone and suppose that it does not contain any line. Then $C$ coincides with the convex hull of $\mathcal{E}(C)$.

Moreover, $\mathcal{E}(C)$ is the minimal set with this property, because if we remove any half-line from it, its elements cannot be obtained as a convex combination of the other points in $C$. Notice also that all the blenders that interest us do not contain lines due to Proposition 2.2.3. Points in the interior of the cone may easily be written as a convex combination of other elements in their neighbourhood, so $\mathcal{E}(C)$ is contained in the boundary of the cone. Fortunately, the inner product gives us a practical way to distinguish whether a point is in the interior or on the boundary of $C$.

*Proposition* 2.2.8. Let $C$ be a closed convex cone. Then $u \in C^{o}$ if and only if $\forall v \in C^* \setminus \{0\}$, $[\, u\,, v\,] > 0$. Equivalently, $u \in \partial C$ if and only if there exists $v \in C^* \setminus \{0\}$, such that $[\, u\,, v\,] = 0$.

*Proof.* Take $u \in C$ such that for a given $v \in C^* \setminus \{0\}$, it holds that $[\, u\,, v\,] = 0$. Then $[\, u - \varepsilon v\,, v\,] = -\varepsilon[\, v\,, v\,] < 0$, so $u - \varepsilon v \notin C$ for all $\varepsilon > 0$, hence $u \notin C^{o}$. Conversely, if $[\, u\,, v\,] > 0$ for all $v \in C^* \setminus \{0\}$, then $[\, u\,, w\,] \geq \delta > 0$ on the compact set $\{w \in C^* : ||w|| = 1\}$. Then, by linearity, $[\,u\,,v\,] \geq \delta||v||$ for all $v \in C^*$. Therefore, if we take any element $z$ close enough to $u$, where $||u-z|| < \delta$, then for any $v \in C^*$ we have $[\,z\,,v\,] = [\,u\,,v\,] - [\,u-z\,,v\,] \geq \delta||v|| - ||u-z||.||v|| \geq 0$, so $z \in (C^*)^* = C$ and thus $u \in C^{o}$. $\square$

*Proposition* 2.2.9. $P_{n,2r}$ and $Q_{n,2r}$ are dual blenders.

*Proof.* We have that $p \in Q^*_{n,2r}$ if and only if for any $s \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_s \geq 0$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{R}^n$,

$$0 \leq \left[p\ ,\ \sum_{k=1}^{s} \lambda_k (\alpha_k \bullet)^{2r}\right] = \sum_{k=1}^{s} \lambda_k p(\alpha_k).$$

This is true if and only if $p(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}^n$, which is equivalent to $p \in P_{n,2r}$. $\square$

### 2.2.5

### Convex forms $K_{n,2r}$

A blender we have not yet presented is the one of convex homogeneous polynomials $K_{n,2r}$.

$$K_{n,2r} := \Big\{ p \in F_{n,2r} :\ p\colon \mathbb{R}^n \to \mathbb{R} \text{ is a convex function} \Big\}.$$

We may define what a convex form $p(x_1, \ldots, x_n)$ is in two equivalent ways:

i. The homogeneous polynomial $p$ is convex if $\forall u, v \in \mathbb{R}^n$ and $\lambda \in [0,1]$, then $p(\lambda u + (1-\lambda)v) \leq \lambda p(u) + (1-\lambda)p(v)$.

ii. Take $u, v \in \mathbb{R}^n$, we define the *Hessian* of $p$ in $u$ evaluated at $v$ as: $\mathrm{Hes}(p\,; u, v) := \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 p(u)}{\partial x_i \partial x_j} v_i v_j$. The form $p$ is convex if $\forall u, v \in \mathbb{R}^n, \mathrm{Hes}(p\,; u, v) \geq 0$.

Notice that $\mathrm{Hes}(p\,; u, u) = 2r(2r-1)p(u)$, and therefore every convex form is automatically nonnegative, that is, $K_{n,2r} \subseteq P_{n,2r}$ (as we have seen, this is a necessary property to be a blender 2.2.3).

The properties of being nonnegative and being a sum of squares are preserved by the processes of homogenization and dehomogenization, but this is no longer true for convexity. For example, $t^4 + 12t^2 + 1$ is convex, but its homogenized form $p(x, y) = x^4 + 12x^2y^2 + y^4$ has for its Hessian the expression

$Hes(p;(1,1),(v_1,v_2)) = 36v_1^2+96v_1v_2+36v_2^2$, which attains negative values as $Hes(p;(1,1),(1,-1)) = 36-96+36 = -24$, thus $p(x,y)$ is not convex.

Let us now focus only on the case of two variables. Our goal is to show that for low degrees, the blender of convex forms coincides with the smallest blender, that of sums of maximal powers of linear forms. This holds only for the degrees $2r = 2$ and $2r = 4$.

**Binary quadratic forms**

We will show that $P_{2,2} = Q_{2,2}$, and therefore all blenders are the same set. To verify this, it is enough to factor the forms. Consider $p(x,y) \in F_{2,2}$. Being a homogeneous polynomial, $p$ vanishes on lines through the origin. We need to distinguish between the cases of real and complex roots.

If $p$ has a real root, it vanishes on a line $l : ax+by = 0$. In this case, we can factor $p(x,y)$ by a power of this linear form $(ax+by)$. If we assume that $p(x,y) \in P_{2,2}$, its roots must be of even multiplicity, and since $p$ is of degree 2, then $p(x,y) = \lambda(ax+by)^2$, $\lambda \geq 0$. Therefore, $p(x,y) \in Q_{2,2}$ (Note that in this case $p$ is an extremal point).

If $p$, a polynomial with real coefficients, vanishes on a complex line $l : x-(a+ib)y = 0$, it necessarily also vanishes on the conjugate line $\bar{l} : x-(a-ib)y = 0$. Thus, we may factor $p$ as:

$$(x-(a+ib)y)(x-(a-ib)y) = \Big((x-ay)+iby\Big)\Big((x-ay)-iby\Big) = (x-ay)^2+b^2y^2.$$

Therefore $p(x,y) \in Q_{2,2}$ (Note that in this case $p$ is not an extremal point).

We conclude that $P_{2,2} = Q_{2,2}$ and hence all blenders coincide. In fact, this set is the circular cone whose extremal set consists of forms with double roots and is described by the equation $\Delta = 0$, where $\Delta$ is the *discriminant*.

**Binary quartic forms**

In $F_{2,4}$, the blenders $Q_{2,4}$ and $P_{2,4}$ are already distinct. Knowing that $Q_{2,4} = (P_{2,4})^*$, our strategy will be to show that every convex form belongs to $(P_{2,4})^*$.

In general, the extremal elements of $P_{2,2r}$ are forms with only double roots (which may eventually coincide resulting in roots of higher multiplicities of even order). In other words, these forms can be written as $p(x,y) = \Big(g(x,y)\Big)^2$, $g \in F_{2,r}$ where all roots of $g$ are real. To be convinced of this, take $p(x,y) = \lambda\prod_{j=1}^{r}(a_jx+b_jy)^2$, a form of this nature. If we write

$p = p_1 + p_2,\ p_i \in P_{2,2r}$, then both $p_i$ must have the same roots as $p$ and therefore we can factor them: $p_i(x,y) = \lambda_i \prod_{j=1}^{r}(a_j x + b_j y)^2$. Since we have already reached the degree $2r$, we have that both $p_i$ are multiples of $p$ and therefore it is extremal.

On the other hand, if $p$ is not of this form, then it presents at least one pair of conjugate complex roots and we can factor it as follows: $p(x,y) = \big(x-(a+ib)y\big)\big(x-(a-ib)y\big)g(x,y),\ g \in P_{2,2r-2}$. With this, we have $p(x,y) = \big((x-ay)^2 + b^2y^2\big)g(x,y) = (x-ay)^2 g(x,y) + b^2y^2 g(x,y)$. Thus, we can write $p$ as the sum of two positive polynomials that are not its multiples, $p_1(x,y) = (x-ay)^2 g(x,y)$ and $p_2(x,y) = b^2y^2g(x,y)$, hence $p$ is not extremal. So in the case of degree 4, we have that $p \in \mathcal{E}(P_{2,4}) \iff p(x,y) = [(ax+by)(cx+dy)]^2$.

A polynomial $q$ belongs to $(P_{2,4})^*$ if and only if $\forall p \in P_{2,4}\ [\, q\,, p\,] \geq 0$. By linearity, it is enough to show this inequality for extremal elements. Additionally, since $\forall M \in M(n;\mathbb{R}),\ [\, q \circ M\,, p\,] = [\, q\,, p \circ M^{\mathsf{T}}\,]$, we may use the action of $\mathrm{SL}(2;\mathbb{R})$ in our analysis. Let $p_0(x,y) = x^2y^2$, we can always find a matrix $A_p \in \mathrm{SL}(2;\mathbb{R})$ such that $p = p_0 \circ A_p^{\mathsf{T}}$. Thus, the calculation becomes:

$$[\, q\,, p\,] = [\, q\,, p_0 \circ A_p^{\mathsf{T}}\,] = [\, q \circ A_p\,, p_0\,].$$

But taking the inner product with the polynomial $p_0$ consists of evaluating the coefficient of $q \circ A_p$ accompanying $x^2y^2$. Therefore, $q \in (P_{2,4})^* \iff \forall p \in P_{2,4},\ a\big(q \circ A_p; (2,2)\big) \geq 0$.

Assuming $q \in K_{2,4}$, we have that for an arbitrary $p$, the polynomial $q \circ A_p(x,y) = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$ is convex. If we evaluate its Hessian at $(u,v) = \big((0,1),(1,0)\big)$ we obtain:

$$\mathrm{Hes}\big(q \circ A_p\,; (0,1),(1,0)\big) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 12c & 12d \\ 12d & 12e \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 12c \geq 0.$$

The idea is that if we consider the dehomogenized polynomial (evaluating it on the line $y = 1$), we have a one-variable polynomial $q \circ A_p(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$. Since $(q \circ A_p)''(0) = 12c$, it is convex at this point if and only if $c \geq 0$. This corresponds to evaluating the Hessian of $q \circ A_p(x,y)$ at the point $u = (0,1)$ and in the direction of $v = (1,0)$. The hypothesis that $q \circ A_p(x,y)$ is convex is stronger, as it indicates that

$\forall u, v \in \mathbb{R}^2$, $\mathrm{Hes}(q \circ A_p\,; u, v) \geq 0$, so we have the desired result, after all $a\Big(q \circ A_p; (2,2)\Big) = c \geq 0$.

We have thus proven that $K_{2,2} = Q_{2,2}$ and $K_{2,4} = Q_{2,4}$. To show that we no longer have this identity when we increase the degree, let us explore other important families of blenders.

### 2.2.6

### The Waring blenders

The Waring blenders are a natural generalization of $Q_{n,2r}$ and $\Sigma_{n,2r}$. We define them as follows: whenever $r$ may be factored as $r = uv,\ u, v \in \mathbb{N}$, then we have the blender

$$W_{n,\,(u,2v)} := \Big\{p \in F_{n,2r} : p = \sum_{k=1}^{s} h_k^{2v},\ h_k \in F_{n,u}\Big\}.$$

One may confirm that this family includes the two blenders previously mentioned, as $Q_{n,2r} = W_{n,\,(1,2r)}$ and $\Sigma_{n,2r} = W_{n,\,(r,2)}$. We may generalize even further, obtaining the so-called *generalized Waring blenders*. These are defined as follows: if $r = \sum_{i=1}^{m} u_i v_i$, then we have the blender

$$W_{n,\,\{(u_1,2v_1),\ldots,(u_m,2v_m)\}} := \Big\{p \in F_{n,2r} : p = \sum_{k=1}^{s} h_{k,1}^{2v_1} \ldots h_{k,m}^{2v_m},\ h_{k,i} \in F_{n,u_i}\Big\}.$$

*Proposition* 2.2.10. If $\sum v_i = r$, then $W_{2,\{(1,2v_1),\ldots,(1,2v_m)\}} = P_{2,2r}$ if and only if $m = r$ and $v_i = 1$.

*Proof.* First, let us show that any $p \in P_{2,2r} = \Sigma_{2,2r}$ can be given as a sum of just two squares, that is, $p = f_1^2 + f_2^2$, where $f_i \in F_{2,r}$. Since $p$ is a binary and nonnegative, by applying the Fundamental Theorem of Algebra we find that $p$ vanishes on real lines with an even multiplicity or on pairs of complex conjugate lines, thus it splits as: $p(x,y) = \prod_j (a_j x + b_j y)^2 \prod_k (\alpha_k x - \beta_k y)(\overline{\alpha_k} x - \overline{\beta_k} y)$, with $a_j, b_j \in \mathbb{R}$ and $\alpha_j, \beta_j \in \mathbb{C}$. The first product of squares can be written as $g(x,y)^2$, while the second, being a product of conjugate pairs, can be given as $(h_1(x,y) + ih_2(x,y))(h_1(x,y) - ih_2(x,y)) = h_1(x,y)^2 + h_2(x,y)^2$. Therefore, we have that $p(x,y) = (gh_1)^2 + (gh_2)^2 = f_1^2 + f_2^2$.

We may also factor $f_i$ as a product of linear terms and positive definite quadratic terms, which in turn are themselves sums of squares.

$$f_i = \prod_j l_{1,j} \prod_k (l_{2,k}^2 + l_{3,k}^2).$$

Consequently, we may write $f_i^2$ in a way that shows that it belongs to $W_{2,\{(1,2),\ldots,(1,2)\}}$.

$$f_i^2 = \prod_j l_{1,j}^2 \prod_k (l_{2,k}^2 + l_{3,k}^2)^2 = \prod_j l_{1,j}^2 \prod_k \Big( (l_{2,k}^2 - l_{3,k}^2)^2 + (2 l_{2,k} l_{3,k})^2 \Big) \in W_{2,\{(1,2),\ldots,(1,2)\}}.$$

The other inclusion, $W_{2,\{(1,2),\ldots,(1,2)\}} \subseteq P_{2,2r}$, is provided by proposition 2.2.3, which states that all blenders are contained in $P_{2,2r}$.

Now suppose that $m < r$ and that the following element of $P_{2,2r}$ may be obtained as an element of $W_{2,\{(1,2v_1),\ldots,(1,2v_m)\}}$.

$$\prod_{l=1}^{r} (x - ly)^2 = \sum_{k=1}^{s} h_{k,1}^{2v_1} \ldots h_{k,m}^{2v_m}, \quad h_{k,i}(x,y) = a_{k,i}x + b_{k,i}y \in F_{2,1}.$$

Then for each $k$, we have

$$\prod_{l=1}^{r} (x - ly) \,\Bigg|\, \prod_{i=1}^{m} (a_{k,i}x + b_{k,i}y).$$

But since $m < r$, the right-hand side has a lower degree, so it could only be identically 0, which is a contradiction. $\square$

These blenders are useful to us, because $K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ are dual blenders, as we will show later. In order to do so, we introduce certain *differential operators.*

**Definition.** For $i \in I(n,d)$, $i = (i_1, \ldots, i_n)$, we define the differential operator:

$$D^i := \prod_{k=1}^{n} \Big(\frac{\partial}{\partial x_k}\Big)^{i_k}.$$

We also define for $f \in F_{n,d}$ the following operator:

$$f(D) := \sum_{i \in I(n,d)} c(i)\, a(f\,; i)\, D^i.$$

***Properties:***.

i. By the commutativity of the differentials:
$\forall i \in I(n,d), \forall j \in I(n,d'),\ D^i D^j = D^{i+j} = D^j D^i$.

ii. By a straightforward calculation, we get:
$\forall f \in F_{n,d}, \forall g \in F_{n,d'},\ (fg)(D) = f(D)g(D) = g(D)f(D)$.

iii. If $i, j \in I(n,d)$ and $i \neq j$, then
$D^i(x^j) = 0$ and $D^i(x^i) = \prod_{k=1}^n (i_k)! = d!/c(i)$.

The following propositions relate the differential operator with the inner product of $F_{n,d}$.

*Proposition* 2.2.11. If $p, q \in F_{n,d}$, then $p(D)(q) = d![\,p\,,q\,] = q(D)(p)$.

*Proof.* The symmetry of the expression $p(D)(q)$ is a consequence of the symmetry of the inner product, so we just have to show that $p(D)(q) = d![\,p\,,q\,]$.

$$\begin{aligned}
p(D)(q) &= \sum_{i\in I(n,d)} \left( c(i)\, a(p\,;i)\, D^i \left( \sum_{j\in I(n,d)} c(j)\, a(q\,;j)\, x^j \right)\right) \\
&= \sum_{i\in I(n,d)} \sum_{j\in I(n,d)} c(i)\, c(j)\, a(p\,;i)\, a(q\,;j)\, D^i(x^j) \\
&= \sum_{i\in I(n,d)} c(i)^2\, a(p\,;i)\, a(q\,;i)\, D^i(x^i) \\
&= \sum_{i\in I(n,d)} c(i)\, a(p\,;i)\, a(q\,;i)\, d! \;=\; d![\,p\,,q\,].
\end{aligned}$$

□

*Proposition* 2.2.12. Let $f, gh \in F_{n,d}$, with $g \in F_{n,d-k}$ and $h \in F_{n,k}$. Then $d![\,f\,,gh\,] = (d-k)![\,g\,,h(D)(f)\,]$.

*Proof.* Notice that both $g$ and $h(D)(f)$ are forms of degree $d-k$, so that their inner product is well defined. The proof is then a simple calculation; it suffices to use the previous proposition twice and the second property listed above:

$$\begin{aligned}
d![\,f\,,gh\,] = gh(D)(f) &= \big(g(D)h(D)\big)(f) \\
&= g(D)\big(h(D)(f)\big) = (d-k)![\,g\,,h(D)(f)\,].
\end{aligned}$$

□

Now, with the help of this differential operator language, we can prove that it is possible to write the Hessian in terms of the inner product.

**Lemma 2.2.13.** *Given $p \in F_{n,2r}$ and $u, v \in \mathbb{R}^n$, we have that:*

$$\mathrm{Hes}(p\,; u, v) = 2r\,(2r-1)\,[\,p\,, (u\,\bullet)^{(2r-2)}\,(v\,\bullet)^2\,].$$

*Proof.* Simply write the expression for the Hessian using the differential operator notation to be able to use its properties. By definition, $\mathrm{Hes}(p\,; u, v) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 p(u)}{\partial x_i \partial x_j} v_i v_j$. If we write $f = (v\,\bullet)^2 = (v_1 x_1 + \cdots + v_n x_n)^2$, then $f(D) = \sum_{i=1}^n \sum_{j=1}^n v_i v_j \frac{\partial^2}{\partial x_i \partial x_j}$. Thus, $f(D)(p) = \sum_{i=1}^n \sum_{j=1}^n v_i v_j \frac{\partial^2 p}{\partial x_i \partial x_j} \in F_{n,2r-2}$. We know that in general, $\forall p \in F_{n,2r}, \forall u \in \mathbb{R}^n\ [\,p\,, (u\,\bullet)^d\,] = p(u)$, hence:

$$[\,f(D)(p)\,, (u\,\bullet)^{(2r-2)}\,] = \sum_{i=1}^n \sum_{j=1}^n v_i v_j \frac{\partial^2 p(u)}{\partial x_i \partial x_j} = \mathrm{Hes}(p\,; u, v).$$

And by the preceding proposition, we obtain:

$$[\,f(D)(p)\,, (u\,\bullet)^{(2r-2)}\,] = \frac{2r!}{(2r-2)!}\,[\,p\,, (u\,\bullet)^{(2r-2)}\,f\,] = 2r\,(2r-1)\,[\,p\,, (u\,\bullet)^{(2r-2)}\,(v\,\bullet)^2\,].$$

$\square$

**Theorem 2.2.14.** *$K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ are dual blenders.*

*Proof.* Let $p \in F_{n,2r}$ and $u, v \in \mathbb{R}^n$. By the lemma above, we have that $\mathrm{Hes}(p\,; u, v) = 2r\,(2r-1)\,[\,p\,, (u\,\bullet)^{(2r-2)}\,(v\,\bullet)^2\,]$. We may see that $K_{n,2r}$ and $W_{n,\{(1,2r-2),(1,2)\}}$ are dual blenders, since:

$$\begin{aligned}\mathrm{Hes}(p\,; u, v) \geq 0 &\iff [\,p\,, (u\,\bullet)^{(2r-2)}\,(v\,\bullet)^2\,] \geq 0, \forall u, v \in \mathbb{R}^n \\ &\iff p \in (W_{n,\{(1,2r-2),(1,2)\}})^*.\end{aligned}$$

$\square$

Notice that this result provides another way to prove that $K_{2,4} = Q_{2,4}$. The elements of $W_{2,\{(1,2),(1,2)\}}$ are presented as sums of products of squares: $p = \sum_{k=1}^s h_{k,1}^2\, h_{k,2}^2\,,\ h_{k,i} \in F_{2,1}$. However, we have seen that the extremal points of $P_{2,4}$ are exactly of the form $p(x, y) = (ax + by)^2(cx + dy)^2$, so $P_{2,4} = W_{2,\{(1,2),(1,2)\}}$. Therefore, $Q_{2,4} = (P_{2,4})^* = (W_{2,\{(1,2),(1,2)\}})^* = K_{2,4}$.

Moreover, we can show that the equality no longer hold for higher degrees, since for $2r \geq 6$, we have that $W_{2,\{(1,2r-2),(1,2)\}} \subsetneq P_{2,2r}$, and thus $(P_{2,2r})^* \subsetneq (W_{2,\{(1,2r-2),(1,2)\}})^* \iff Q_{2,2r} \subsetneq K_{2,2r}$.

### 2.2.7

### Blenders of binary quartics

Recall that $\mathcal{B}_{n,d}$ denotes the set of all blenders of $F_{n,d}$. We will show that $\mathcal{B}_{2,4}$ is a 1-parameter family of nested blenders increasing from $Q_{2,4}$ to $P_{2,4}$. Let us first take care of some particular quartics, those $p \in P_{2,4}$ that are not strictly positive and are also not a fourth power of a linear form. Let $Z_{2,4}$ denote the set of such quartics. If $p \in Z_{2,4}$, then $p = l^2 h$, where $l$ is linear and $h$ is a nonnegative quadratic form that is not a multiple of $l$.

**Lemma 2.2.15.** *If $B \in \mathcal{B}_{2,4}$ and there is a nontrivial $p \in B \cap Z_{2,4}$, then $B = P_{2,4}$.*

*Proof.* We have that $p \sim q$, where $q(x,y) = x^2(ax^2 + 2bxy + cy^2) \in B$, with $ac - b^2 \geq 0$ and $c > 0$. We may rewrite $q$ to obtain:

$$x^2(ax^2 + 2bxy + cy^2) = x^2\left(\left(\frac{ac-b^2}{c}\right)x^2 + c\left(\frac{b}{c}x + y\right)^2\right) \sim x^2(dx^2 + cy^2),$$

with $d \geq 0$. Now taking $(x,y) \mapsto (\varepsilon x, \varepsilon^{-1} y)$, we have $\varepsilon^4 dx^4 + cx^2y^2 \in B$, so by taking $\varepsilon$ to zero, in the limit we get that $x^2y^2 \in B$ and thus $l_1^2 l_2^2 \in B$. Therefore, we have shown that $W_{2,\{(1,2),(1,2)\}} \subseteq B$, but $W_{2,\{(1,2),(1,2)\}} = P_{2,4}$ by proposition 2.2.10, which concludes the proof. $\square$

Notice the importance of using $\mathrm{GL}(2;\mathbb{R})$ in the proof above. If we used a compact group, such as $\mathrm{SO}(2;\mathbb{R})$, we would not have the same result. It would not be possible to obtain fourth powers of linear forms nor the square of an indefinite quadratic form. Let us now define a particular quartic that will be very important in the study of this section.

$$f_\lambda(x,y) := x^4 + 6\lambda x^2y^2 + y^4.$$

We also define another quartic in the same class.

$$g_\lambda(x,y) := f_\lambda(x+y, x-y) := (2+6\lambda)x^4 + (12-12\lambda)x^2y^2 + (2+6\lambda)y^4.$$

We will employ two rational functions that we define below:

$$\Theta(z) \coloneqq \frac{1-z}{1+3z}, \qquad \Upsilon(z) \coloneqq \frac{1+3z}{3-3z}.$$

Notice that $g_\lambda = (2+6\lambda)f_{\Theta(\lambda)}$, and so if $\lambda \neq -\frac{1}{3}$, then $f_\lambda \sim f_{\Theta(\lambda)}$.

The function $\Theta$ is an involution, so for every $z \in \mathbb{R}$, $\Theta(\Theta(z)) = z$. Furthermore, $\Theta(0) = 1$, $\Theta(1/3) = 1/3$ and $\Theta(-1/3) = \infty$. This last identity is related to the equivalence $(x^2 - y^2)^2 \sim x^2y^2$. In addition, $\Theta$ is a decreasing bijection between $[1/3, \infty)$ and $(-1/3, 1/3]$. $\Upsilon$ is also an involution, $\Upsilon(0) = -1/3$, $\Upsilon$ is a decreasing bijection from $[-1/3, 0]$ to itself.

One may observe that $f_\lambda$ is nonnegative if and only if $\lambda \in [-1/3, \infty)$, and it is strictly positive when $\lambda \in (-1/3, \infty)$. If $B \in \mathcal{B}_{2,4}$, then $f_\lambda \in B \iff f_{\Theta(\lambda)} \in B$.

Since every blender is convex by definition, if $-1/3 < \lambda \leq 1/3$, then $f_\lambda \in B \implies f_\mu \in B$ for $\mu \in [\lambda, \Theta(\lambda)]$.

It is a classical result that a 'generic' binary quartic can be put in the form of $f_\lambda$ for some $\lambda$ with an invertible linear change of variables. However, the coefficients of this transformation may not be real, and the result is not universal; for example, $x^4 \nsim f_\lambda$.

*Proposition* 2.2.16. If $p \in P_{2,4}$ is strictly positive, then $p \sim f_\lambda$ for some $\lambda \in (-1/3, 1/3]$.

*Proof.* If $p = g^2$, then $g$ is a quadratic form with no real root, hence we may take $g \sim x^2 + y^2$ and $p \sim f_{1/3}$.

On the other hand, if $p$ is not the square of a quadratic form, then it is the product of two distinct definite quadratic forms, thus we may write $p(x, y) = (x^2 + y^2)q(x, y)$, with $q(x, y) = ax^2 + 2bxy + cy^2$.

A rotation preserves $x^2+y^2$ and sends $q$ to $dx^2+ey^2$, with $d, e > 0$, $d \neq e$, hence $p \sim (x^2+y^2)(dx^2+ey^2)$. Now the transformation $(x, y) \mapsto (d^{-1/4}x, e^{-1/4}y)$ gives us $p \sim f_\mu$, where $\mu = \frac{1}{6}(\gamma + \gamma^{-1}) > \frac{1}{3}$ for $\gamma = \sqrt{d/e} \neq 1$. Therefore we finally have $p \sim f_{\Theta(\mu)}$ where $\Theta(\mu) \in (-1/3, 1/3)$. $\square$

### 2.2.8

#### Invariants of binary quartics

Suppose that $p(x,y) = \sum_{k=0}^{4} \binom{4}{k} a_k(p) x^{4-k} y^k$. Here we present the two fundamental invariants of $p$, simplifying the notation for clarity:

$$S(p) := a_0 a_4 - 4a_1 a_3 + 3a_2^2,$$

$$T(p) := a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 - a_2^3.$$

The invariant $T(p)$ is the determinant of the *catalecticant matrix* (or Hankel matrix) of $p$.

$$T(p) = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

By applying a change of variables $q(x,y) = p(ax+by, cx+dy)$, the invariants behave in the following manner:

$$S(q) = (ad-bc)^4 S(p), \qquad T(q) = (ad-bc)^6 T(p).$$

We may then define a new invariant $K(p) := \frac{T(p)}{S(p)^{3/2}}$, that is an *absolute invariant*, which means that if $p \sim q$ then $K(p) = K(q)$. The aforementioned invariants of the quartic $f_\lambda$ in normal form are: $S(f_\lambda) = 1+3\lambda^2$, $T(f_\lambda) = \lambda - \lambda^3$ and $K(f_\lambda) = \frac{\lambda-\lambda^3}{(1+3\lambda^2)^{3/2}}$. Let us define the function $\phi(\lambda) := \frac{\lambda-\lambda^3}{(1+3\lambda^2)^{3/2}}$ for later use.

*Proposition* 2.2.17. If $p$ is strictly positive, then $p \sim f_\lambda$, where $\lambda$ is the only solution in $(-1/3, 1/3]$ of the equation $K(p) = \phi(\lambda)$. If $p \in Z_{2,4}$, then $K(p) = \phi(-1/3)$.

*Proof.* One just has to verify that $\phi$ is strictly increasing in this interval, as its first derivative is positive in the interval $(-1/3, 1/3)$.

We have seen that if $p \in Z_{2,4}$, then $p \sim q$, where $q(x,y) = ax^4 + 6cx^2y^2$, with $c > 0$. In this case, $S(q) = 3c^2$ and $T(q) = -c^3$. Therefore $K(p) = K(q) = -3^{-3/2} = \phi(-1/3)$. □

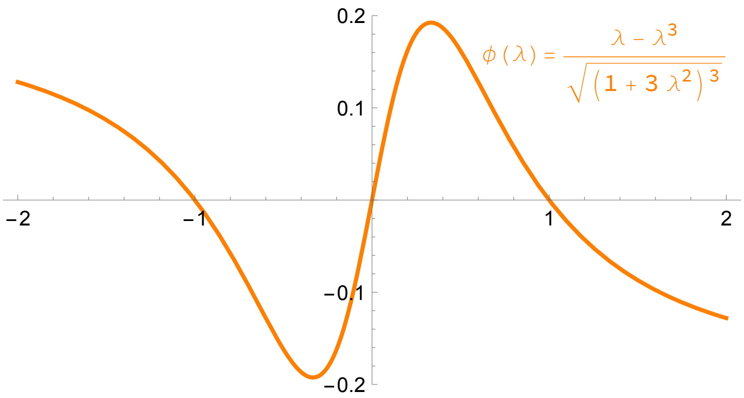


Figure 2.2: The graph of $\phi(\lambda)$, strictly increasing in the interval $[-1/3, 1/3]$.

Given a blender $B \in \mathcal{B}_{2,4}$, we define the set $\Delta(B) := \{\lambda \in \mathbb{R} : f_\lambda \in B\}$ of values of $\lambda \in \mathbb{R}$ such that the form $f_\lambda$ belongs to $B$. Next we show that $\Delta$ is always a particular type of interval.

*Proposition* 2.2.18. If $B \in \mathcal{B}_{2,4}$ is a nontrivial blender, then $\Delta(B) = [\tau, \Theta(\tau)]$ for some $\tau \in [-1/3, 0]$.

*Proof.* Firstly we point out that the set $\Delta(B)$ must be an interval because $B$ is a convex set, and it must be a closed interval because $B$ is a closed set.

For the blender $P_{2,4}$, we have that $\Delta(P_{2,4}) = [-1/3, +\infty)$, since $f_\lambda = x^4 + 6\lambda x^2y^2 + y^4 \geq (x^2 - y^2)^2$ for all values of $x$ and $y$ if and only if $\lambda \geq -1/3$. On the other hand, if $\lambda < -1/3$, then $f_\lambda(1,1) < 0$, and thus $f_\lambda \notin P_{2,4}$.

For the blender $Q_{2,4}$, we have that $\Delta(Q_{2,4}) = [0, 1]$. One way to prove this is by using the fact that $Q_{2,4} = K_{2,4}$, and checking that $f_\lambda$ is convex if and only if $\lambda \in [0, 1]$.

Finally, for any other blender $B$, let $\tau = \inf\{\lambda : f_\lambda \in B\}$. Since $Q_{2,4} \subsetneq B \subsetneq P_{2,4}$, then we know that $\tau \in (-1/3, 0)$. As $B$ is a closed set, we know that $f_\tau \in B$, and since $f_\tau \sim f_{\Theta(\tau)}$, then $f_{\Theta(\tau)} \in B$. Therefore $[\tau, \Theta(\tau)] \subseteq \Delta(B)$. If $\lambda < \tau$ then $f_\lambda \notin B$ by definition of $\tau$. And if $\lambda > \Theta(\tau)$ and $f_\lambda \in B$, then we would have that $f_{\Theta(\lambda)} \in B$, but $\Theta(\lambda) < \Theta(\Theta(\tau)) = \tau$, which is a contradiction. $\square$

We conclude with a full characterization of the nontrivial blenders of $F_{2,4}$. Let us first define a set comprised of the closed orbits of quartics in the normal form. Given a $\tau \in [-1/3, 0]$ we define

$$B_\tau := \bigcup_{\tau \le \lambda \le 1/3} [[f_\lambda]] = \{p : p \sim f_\lambda, \tau \le \lambda \le 1/3\} \cup \{(\alpha x + \beta y)^4 : \alpha, \beta \in \mathbb{R}\}.$$

Then it is true that if $B \in \mathcal{B}_{2,4}$ is a nontrivial blender, then $B = B_\tau$ for some $\tau \in [-1/3, 0]$ and $B_\tau^* = B_{\Upsilon(\tau)}$. The proof of this theorem may be found, once again, in Reznick's paper [Reznick]. In summary, the nontrivial blenders of binary quartics form a 1-parameter family of nested convex cones $B_\tau$, ranging from $B_0 = Q_{2,4}$ to $B_{-1/3} = P_{2,4}$.

## 2.3

### Cross-ratio of the roots of a positive convex quartic

In this section, we change our perspective in order to understand the positive quartics from the point of view of the relation of its four roots. We prove that any positive binary quartic $p(x, y)$ whose set of roots is $\{z_1, \overline{z_1}, z_2, \overline{z_2}\}$ with $z_1$ and $z_2$ in the upper half-plane of $\mathbb{C}$ is convex if and only if the cross-ratio of its roots given by $(z_1, z_2; \overline{z_2}, \overline{z_1})$ belongs to the interval $[1, 2]$.

## 2.3.1

### Positive binary quartics and their roots

Let $p(x, y)\colon \mathbb{R}^2 \to \mathbb{R}$ be a positive binary quartic. This means that it is a homogeneous polynomial in two real variables, $x$ and $y$, of degree 4, with real coefficients such that for every $(x, y) \neq 0$, the image $p(x, y) > 0$.

*Proposition* 2.3.1. Let $p(x, y)$ be a positive binary quartic such that the coefficient of $x^4$ is not 0. Then it can be factored as

$$p(x, y) = k(x - z_1 y)(x - z_2 y)(x - z_3 y)(x - z_4 y)$$

where $k \in \mathbb{R}_{>0}$ , and $z_j \in \mathbb{C}$ correspond to the roots of the *dehomogeneized* polynomial $p(x, 1) \in \mathbb{R}[x]$.

*Proof.* By the fundamental theorem of algebra, we may decompose

$$p(x, 1) = k(x - z_1)(x - z_2)(x - z_3)(x - z_4)$$

and then by the process of *homogeneization* we recover the factored form of the original polynomial $p(x, y)$. □

The condition that the coefficient of $x^4$ be not 0 will soon be dropped when we move our domain to the projective setting. Also, we will get rid of the number $k$ when we consider the action of $\mathrm{GL}\,(2;\mathbb{R})$ on $\mathbb{R}[x,y]$, since it will always be possible to set $k=1$, as we shall explain in the section 2.3.3.

The four roots $z_j$ of $p(x,1)$ determine completely their polynomial (up to a scalar multiple), so they must contain all information about $p(x,y)$. Notice that in the case we are considering, all roots $z_j$ are not real and come in conjugate pairs, since $p(x,y)$ is real and positive. It is then advantageous to change our mindset and shift to the complex plane by considering $p(x,y)\colon \mathbb{C}^2 \to \mathbb{C}$, so that we can suitably work with the roots, which we denote by $z_1, \overline{z_1}, z_2, \overline{z_2}$.

The zero set of $p(x,y)$ is actually a cone due to its homogeneity, that is, if $p(x_0,y_0) = 0$ , then for all $\lambda \in \mathbb{C}$, $p(\lambda x_0, \lambda y_0) = 0$. This fact allows us to think about the zeros of $p(x,y)$ as elements of $\mathbb{CP}^1$ instead of $\mathbb{C}^2$. Indeed, by definition, the whole fibre of $[x_0\colon y_0]$ is comprised of zeros of the polynomial and we may call the well-defined fraction $z_0 \coloneqq x_0/y_0 \in \mathbb{C} \sqcup \{\infty\}$ a root of $p(x,y)$.

Conversely, for the roots $z_j \in \mathbb{C}$, as presented in Proposition 2.3.1, we have corresponding sets $(\lambda z_j, \lambda) \subset \mathbb{C}^2$ of zeros. One can now naturally see how the assumption that the coefficient of $x^4$ be not 0 is artificial. This term vanishes if and only if one of the roots of $p(x,y)$ is $\infty$, in other words, $p(1,0)=0$. If one wishes to factor the polynomial in this case, for each factor where $z_j = \infty$ one should change the term $(x - z_j y)$ by $y$. For instance, if a single root is equal to $\infty$ we have

$$p(x,y) = k(x - z_1 y)(x - z_2 y)(x - z_3 y)y.$$

Nevertheless, this never happens in our subject, since we assume that $p(x,y)$ is positive, hence $p(1,0) \neq 0$.

### 2.3.2

### Cross-ratio of the roots

Given an ordered list of four complex numbers $(w_1, w_2, w_3, w_4)$, one defines the *cross-ratio*

$$(w_1, w_2; w_3, w_4) \coloneqq \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} \in \mathbb{C} \sqcup \{\infty\}.$$

This value is *not* invariant with respect to the permutation of its arguments. Indeed, if we name $\lambda \coloneqq (w_1, w_2; w_3, w_4)$, then the 24 permutations give rise to

6 possibly distinct values for the cross-ratio:

$$\lambda,\ \frac{1}{\lambda},\ 1-\lambda,\ \frac{1}{1-\lambda},\ \frac{\lambda-1}{\lambda},\ \frac{\lambda}{\lambda-1}.$$

For the purpose of analysing the properties of the polynomial $p(x, y)$, we shall compute the cross-ratio of its roots. Although they do not have any natural ordering, we do have the opportunity to list them in a particular fashion that will prevent any ambiguity about their cross-ratio. Consider $\mathcal{H} := \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, the upper half-plane of $\mathbb{C}$. We may name the roots so that $z_1$ and $z_2$ belong to $\mathcal{H}$. In this case, let $z_1 = a + ib$ and $z_2 = c + id$ with $a, b, c, d \in \mathbb{R}$ and $b, d > 0$. Then the cross-ratio $(z_1, z_2; \overline{z_2}, \overline{z_1})$ is a real number given by

$$(z_1, z_2; \overline{z_2}, \overline{z_1}) = \frac{(a-c)^2 + (b+d)^2}{4bd} \in \mathbb{R}.$$

Notice that this value is preserved if we interchange $a \leftrightarrow c$ and $b \leftrightarrow d$ due to the symmetry of the expression above. This means that it does not matter which root on $\mathcal{H}$ is denoted by $z_1$ and which is denoted by $z_2$, so the cross-ratio $(z_1, z_2; \overline{z_2}, \overline{z_1})$ of the roots of $p(x, y)$ is well defined.

With the cross-ratio of the roots of a quartic well defined, we may state the main theorem of this paper.

**Theorem 2.3.2.** *Let $p(x, y)$ be a positive binary quartic whose set of roots is $\{z_1, \overline{z_1}, z_2, \overline{z_2}\}$, with $z_1, z_2 \in \mathcal{H}$. Then $p(x, y)$ is convex if and only if the cross-ratio of the roots $(z_1, z_2; \overline{z_2}, \overline{z_1})$ belongs to the interval $[1, 2]$.*

The proof of theorem 2.3.2 will be provided in section 2.3.5.

*Remark.* The fact that the cross-ratio of the roots of a positive quartic is a real number is no coincidence. Indeed the cross-ratio is real if and only if the four points lie on a circle or line [Sarason], which is always the case for a pair of conjugate pairs.

### 2.3.3

#### Actions of linear groups on polynomials and its roots

We now consider the action of different linear groups on the set of positive binary quartics and on their roots, starting with the action of $\mathrm{GL}(2; \mathbb{R})$ given by *linear change of variables.* This means that for an element $A \in \mathrm{GL}(2; \mathbb{R})$, the polynomial $p(x, y)$ is mapped to $A \cdot p(x, y) := p(X, Y)$, where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

More explicitly, let us write it in terms of the coefficients of $A$.

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \begin{aligned} X &= \alpha x + \beta y \\ Y &= \gamma x + \delta y \end{aligned}$$

$$A \cdot p(x, y) = p(\alpha x + \beta y, \gamma x + \delta y)$$

By considering $p(x, y)$ in its factored form, one can see how this action affects the roots. Each factor undergoes the following transformation:

$$\begin{aligned} x - z_j y \mapsto X - z_j Y & \\ &= \alpha x + \beta y - z_j(\gamma x + \delta y) \\ &= (\alpha - z_j\gamma)x + (\beta - z_j\delta)y \\ &= (\alpha - z_j\gamma)\left(x + \frac{\beta - z_j\delta}{\alpha - z_j\gamma}y\right) \\ &= (\alpha - z_j\gamma)\left(x - \frac{\delta z_j - \beta}{-\gamma z_j + \alpha}y\right) \end{aligned}$$

Therefore, the whole polynomial $A \cdot p(x, y)$ is

$$A \cdot p(x, y) = k \prod_{j=1}^{4} (\alpha - z_j\gamma)\left(x - \frac{\delta z_j - \beta}{-\gamma z_j + \alpha}y\right) = p(\alpha, \gamma) \prod_{j=1}^{4} \left(x - \frac{\delta z_j - \beta}{-\gamma z_j + \alpha}y\right).$$

There are two facts that one may derive from this expression. Firstly, as stated in section 2.3.1, we can get rid of the real coefficient $k$ in the definition of $p(x, y)$ by picking a suitable $A \in \mathrm{GL}\,(2; \mathbb{R})$ such that $p(\alpha, \gamma) = 1$. Secondly, one can see that this action induces another action of $\mathrm{GL}\,(2; \mathbb{R})$, but on $\mathbb{C} \sqcup \{\infty\}$ by applying a *Möbius transformation* on the roots. Notice, however, that if one applies the action of $A$ on $p(x, y)$, then it is the Möbius transformation given by its inverse $A^{-1}$ that acts on the roots.

The full group of Möbius transformations is actually $\mathrm{PGL}\,(2; \mathbb{C})$. Since our goal is to determine under which conditions on the roots a positive binary quartic is convex, we may consider theses polynomials up to *positive* real multiples. A polynomial $p(x, y)$ is positive and convex if and only if $kp(x, y)$ is positive and convex for any $k > 0$. The action of $\mathrm{PGL}(2;\mathbb{R})$ is then well defined on the set of such classes of polynomials and also on $\mathbb{C} \sqcup \{\infty\}$, where the roots belong.

However, $\mathrm{PGL}(2;\mathbb{R})$ is still too big for our need. This group has 2 isomorphic connected components, one given by classes of matrices with positive determinant and the other given by those with negative determinant. The connected component of the identity is called $\mathrm{PSL}\,(2;\mathbb{R})$, and as a subgroup of $\mathrm{PGL}\,(2;\mathbb{C})$, it comprises Möbius transformations that preserve the upper half-plane $\mathcal{H}$.

Let us consider how these groups act on the roots of $p(x,y)$. If an element $A \in \mathrm{PGL}(2;\mathbb{R})$ maps a root $z \in \mathbb{C}$ to $w \in \mathbb{C}$, then it must map its conjugate pair $\overline{z}$ to $\overline{w}$, because all its coefficients are real. Since the ordering of the roots is irrelevant to the polynomial they define, it is enough to consider the action of $\mathrm{PSL}\,(2;\mathbb{R})$, which keeps $z_1$ and $z_2$ in $\mathcal{H}$.

Another crucial property for our investigation is the fact that any Möbius transformation $A \in \mathrm{PGL}\,(2;\mathbb{C})$ preserves the cross-ratio [Sarason]. That is, given 4 points $w_1, w_2, w_3, w_4 \in \mathbb{C} \sqcup \{\infty\}$ it holds that

$$(w_1, w_2; w_3, w_4) = (A \cdot w_1, A \cdot w_2; A \cdot w_3, A \cdot w_4)\,.$$

### 2.3.4

### Locus of the roots in $\mathcal{H}$ under $\mathrm{PSL}\,(2;\mathbb{R})$

In this section we study the transitivity of $\mathrm{PSL}\,(2;\mathbb{R})$ on $\mathcal{H}$ in order to understand what freedom we have with respect to the positioning of the roots of a positive quartic under such action. Our main goal is to prove the following lemma.

**Lemma 2.3.3.** *Let $p(x,y)$ be a positive binary quartic with distinct roots $\{z_1, z_2, \overline{z_2}, \overline{z_1}\}$ such that $z_1, z_2 \in \mathcal{H}$ and with cross-ratio $(z_1, z_2; \overline{z_2}, \overline{z_1}) = \lambda$. Then there exists a Möbius transformation in $\mathrm{PSL}\,(2;\mathbb{R})$ whose action maps $p(x,y)$ to a positive binary quartic with roots*

$$\begin{aligned} w_1 &= i, & w_2 &= \left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right) i, \\ \overline{w_1} &= -i, & \overline{w_2} &= -\left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right) i. \end{aligned}$$

Firstly, the well-known Fundamental Theorem of Möbius Geometry states that the action of $\mathrm{PGL}\,(2;\mathbb{C})$ on $\mathcal{H}$ is sharply 3-transitive [Sarason]. This means that given two triples $(z_1, z_2, z_3)$ and $(w_1, w_2, w_3)$ of distinct elements in $\mathbb{C} \sqcup \{\infty\}$, there exists an unique Möbius transformation $A \in \mathrm{PGL}\,(2; \mathbb{C})$ such that $A \cdot z_j = w_j$ for $j \in \{1, 2, 3\}$.

However, that is not the case for $\mathrm{PSL}\,(2; \mathbb{R})$. By dimension counting, one would expect that $\mathrm{PSL}\,(2; \mathbb{R})$ cannot even act 2-transitively on $\mathcal{H}$, because this would require 4 degrees of freedom, while $\mathrm{PSL}\,(2; \mathbb{R})$ is a 3-dimensional Lie group. This action is 1-transitive, so one can map the first root to the point $i$. But what can we say about the new position of the other root in $\mathcal{H}$? Because the cross-ratio is preserved by all Möbius transformations, one may use it to find out that the other root must lie on a particular circle, as we now show in detail.

Assume, as in section 2.3.2, that the four roots of the quartic $p(x, y)$ are $z_1 = a + ib$, $\overline{z_1} = a - ib$, $z_2 = c + id$ and $\overline{z_2} = c - id$, with $b, d > 0$, so that $z_1$ and $z_2$ belong to $\mathcal{H}$. Let $\lambda := (z_1, z_2; \overline{z_2}, \overline{z_1})$ be the cross-ratio of these roots. One may check that $\lambda \geq 1$, since $(a-c)^2 + (b+d)^2 \geq 4bd$ and $\lambda = \frac{(a-c)^2+(b+d)^2}{4bd}$. We will treat the special case where $\lambda = 1$ separately, since it corresponds to the case where $p(x, y)$ has a pair of double roots.

By applying the action of a suitable Möbius transformation $A \in \mathrm{PSL}\,(2; \mathbb{R})$, we map the four roots to:

$$A \cdot z_1 = i, \qquad A \cdot z_2 = x + iy, \qquad A \cdot \overline{z_2} = x - iy, \qquad A \cdot \overline{z_1} = -i.$$

Using the preserved cross-ratio $\lambda$, one can find the implicit equation that determines the possible values for the new coordinates $x$ and $y$ of the second root.

$$\lambda = \frac{x^2 + (y+1)^2}{4y} \quad \Leftrightarrow \quad x^2 + (y + 1 - 2\lambda)^2 = 4\lambda(\lambda - 1).$$

The equation above describes a circle of center $(0, 2\lambda - 1)$ and radius $2\sqrt{\lambda(\lambda - 1)}$. Thus we have shown that the image of the second root under the action of $A$ lies on this particular circle. On top of that, we have the freedom to place it anywhere on the circle thanks to the stabilizer of $i$ in $\mathrm{PSL}\,(2; \mathbb{R})$, denoted $\mathrm{Stab}(i)$. This subgroup happens to be the *projective special orthogonal group* $\mathrm{SO}\,(2; \mathbb{R})$.

$$\mathrm{Stab}(i) = \mathrm{SO}\,(2; \mathbb{R}) := \left\{ \left[ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \right] \in \mathrm{PSL}\,(2; \mathbb{R}) \;\middle|\; \theta \in {}^{\mathbb{R}}\!/_{\pi\mathbb{Z}} \right\}.$$

The $\mathrm{SO}\,(2; \mathbb{R})$ orbit of any point $z \in \mathcal{H} \setminus \{i\}$ is a circle whose center $c \in \mathcal{H}$

lies on the imaginary axis with $\mathrm{Im}(c) > 1$, which must therefore coincide with the circular locus available for the second root as previously described. Notice that the $\mathrm{SO}(2;\mathbb{R})$ orbit of the point $yi$, for $y > 0$, contains the point $i/y$. Consequently, we have the freedom to place the second root $z_2$ on the open segment between 0 and $i$, and we can express it explicitly in terms of the cross-ratio:

$$A \cdot z_2 = x + iy \quad \text{where} \quad x = 0 \quad \text{and} \quad y = 2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)} \in (0,1)\,.$$

The above is precisely the statement of lemma 5.1 we wished to prove. □

### 2.3.5

### Normal form and convexity

In the previous sections we have proved that, up to the action of a suitable Möbius transformation, one may assume that the four roots of a positive binary quartic $p(x,y)$ whose distinct roots yield a given cross-ratio $\lambda \in (1,\infty)$ are:

$$\begin{aligned} w_1 &= i, & w_2 &= \left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right) i, \\ \overline{w_1} &= -i, & \overline{w_2} &= -\left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right) i. \end{aligned}$$

Since we know these roots explicitly, we are able to present the expression for $p(x,y)$:

$$\begin{aligned} p(x,y) &= \left(x^2+y^2\right)\left(x^2 + \left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right)^2 y^2\right) \\ &= x^4 + \left(\left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right)^2 + 1\right) x^2y^2 + \left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right)^2 y^4. \end{aligned}$$

Let us define $\Lambda \coloneqq \left(2\lambda - 1 - 2\sqrt{\lambda\,(\lambda-1)}\right)^2$ in order to avoid having to write down the whole expression many times. Now in a more concise notation, the quartic $p(x,y)$ is given by the normal form:

$$\begin{aligned} p(x,y) &= \left(x^2+y^2\right)\left(x^2+\Lambda y^2\right) \\ &= x^4 + (\Lambda+1)\,x^2y^2 + \Lambda y^4. \end{aligned}$$

Considering $\Lambda$ as a function of $\lambda$, and recalling that $\lambda \in (1,\infty)$, one can verify that $\Lambda(\lambda)$ is a strictly decreasing function whose image is the interval $(0,1)$.

We are finally in a suitable setting to study the convexity of the quartic. It is convex if and only if its Hessian is positive semi-definite [Krantz]. A straightforward calculation yields

$$H(x,y) = \begin{pmatrix} 12x^2 + 2\,(\Lambda+1)\,y^2 & 4\,(\Lambda+1)\,xy \\ 4\,(\Lambda+1)\,xy & 12\Lambda y^2 + 2\,(\Lambda+1)\,x^2 \end{pmatrix}.$$

**Lemma 2.3.4.** *The Hessian $H(x,y)$ of the quartic in normal form $(x^2+y^2)\,(x^2+\Lambda y^2)$ is positive semi-definite if and only if $\Lambda \in [17-12\sqrt{2}, 1]$.*

*Proof.* A symmetric matrix is positive semi-definite if and only if all its principal minors are nonnegative [Prussing]. In our case, this amounts to checking the coefficients $H_{1,1}(x,y)$, $H_{2,2}(x,y)$ and the determinant of $H(x,y)$. Since $\Lambda$ is a square, it is always nonnegative, which implies that $H_{1,1}(x,y)$ and $H_{2,2}(x,y)$ are both nonnegative as well. For the determinant, we have

$$\det\left(H(x,y)\right) = 24\,(\Lambda+1)\,x^4 + \left(144\Lambda - 12\,(\Lambda+1)^2\right)x^2y^2 + 24\Lambda\,(\Lambda+1)\,y^4.$$

We need to find for which values of $\Lambda$ this determinant is nonnegative. Since its expression is a biquadratic polynomial in $x$ and $y$, that is, a quadratic polynomial in $x^2$ and $y^2$, let us change variables by taking $X = x^2$ and $Y = y^2$.

$$\det\left(H(X,Y)\right) = 24\,(\Lambda+1)\,X^2 + \left(144\Lambda - 12\,(\Lambda+1)^2\right)XY + 24\Lambda\,(\Lambda+1)\,Y^2.$$

Next, as $\det\left(H(X,0)\right) = 24\,(\Lambda+1)\,X^2 > 0$ for all $X \neq 0$, we may dehomogenize it by setting $Y = 1$ in order to obtain a univariate polynomial. Let us denote by $\Delta(\Lambda)$ the discriminant of $\det\left(H(X,1)\right)$. If $\Delta(\Lambda) \leq 0$, then the determinant is always nonnegative, because its leading coefficient $24\,(\Lambda+1)$ is positive for all values of $\Lambda$. On the other hand, if $\Delta(\Lambda) > 0$, then $\det\left(H(X,1)\right)$ has a pair of real roots, thus if at least one of them is positive, then there are values of $x \in \mathbb{R}$ such that $\det\left(H(x,1)\right) < 0$. We are going to show that this is exactly what happens; whenever $\Delta(\Lambda) > 0$, there are indeed positive roots for $\det\left(H(X,1)\right)$.

The discriminant of $\det\left(H(X,1)\right)$ is:

$$\Delta(\Lambda) = 144\left(1 - 36\Lambda + 70\Lambda^2 - 36\Lambda^3 + \Lambda^4\right).$$

The special case related to multiple roots of $p(x,y)$ gives $\lambda = \Lambda = 1$. One can verify that this value of $\Lambda$ yields a double root of the discriminant above. So one may divide it by $(\Lambda-1)^2$ to find the remaining roots.

$$\frac{\Delta(\Lambda)}{(\Lambda-1)^2}=\frac{144\,(1-36\Lambda+70\Lambda^2-36\Lambda^3+\Lambda^4)}{(\Lambda-1)^2}=144\left(1-34\Lambda+\Lambda^2\right).$$

The resulting quadratic polynomial has for its roots $17-12\sqrt{2}$ and $17+12\sqrt{2}$, the latter not belonging to $(0,1)$, the domain of $\Lambda$. Therefore, if $\Lambda\geq 17-12\sqrt{2}$, then $\Delta(\Lambda)\leq 0$, whereas if $\Lambda<17-12\sqrt{2}$, then $\Delta(\Lambda)>0$. In the case of positive discriminant, one may check that both roots of the determinant are positive by studying the signs of its coefficients. While the leading and the constant coefficients are always positive, the linear coefficient is negative for $\Lambda\in(0,17-12\sqrt{2})$. This implies that both roots have the same sign and must be positive. Therefore, $\det\left(H(x,y)\right)$ is nonnegative for every $(x,y)\in\mathbb{R}^2$ if and only if $\Lambda\in[17-12\sqrt{2},1]$. □

*Proof of Theorem 2.3.2.* Using Lemma 2.3.4 we have found the condition for the Hessian $H(x,y)$ to be positive semi-definite, which in turn is equivalent to $p(x,y)$ being convex. However, it was expressed in terms of $\Lambda$, while it would be more natural to present it in terms of the cross-ratio $\lambda$. Since $\Lambda(\lambda)$ is a strictly decreasing function, there is a unique value of $\lambda$ for which $\Lambda(\lambda)=17-12\sqrt{2}$, and it is $\lambda=2$.

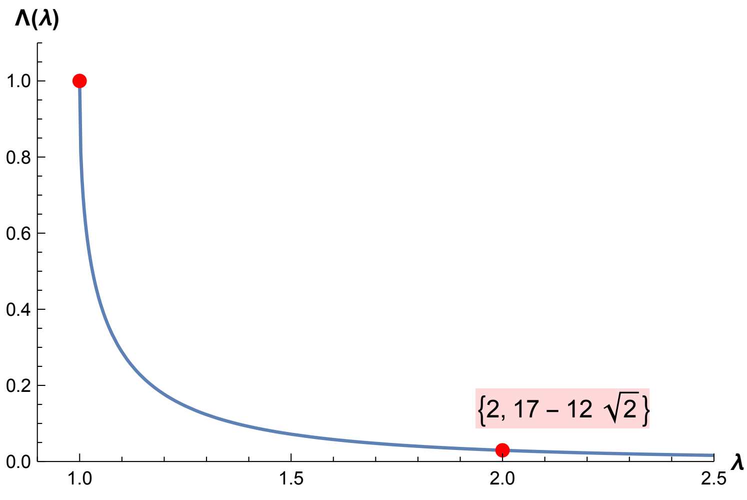


Figure 2.3: The graph of $\Lambda(\lambda)$.

In conclusion, we have proved that any positive binary quartic $p(x,y)$ whose set of roots is $\{z_1,\overline{z_1},z_2,\overline{z_2}\}$ with $z_1,z_2\in\mathcal{H}$ is convex if and only if the cross-ratio of the roots $(z_1,z_2;\overline{z_2},\overline{z_1})$ belongs to the interval $[1,2]$. The critical cases, which correspond to the boundary of this interval, also yield convex quartics, because the limit of convex functions is also convex. □

# 3

# Convexly nested Conics

In this chapter, we introduce the main tool we use to analyse the projective relative position of a pair of nondegenerate conics in the real projective plane. It consists of a equivalence class of binary quartics produced by injecting a quadratic parametrization of the first conic into the implicit equation of the second conic.

## 3.1

## Quadratic parametrization of a conic

Let $(t : w) \in \mathbb{RP}^1$ be a projective parameter. We say that a smooth map $(x_0(t : w) : x_1(t : w) : x_2(t : w)) : \mathbb{RP}^1 \to \mathbb{RP}^2$ is a projective parametrization of a curve on $\mathbb{RP}^2$. A *polynomial parametrization* is one such parametrization where all $x_i$ are polynomial in $(t, w)$. For this map to be well-defined, the polynomials $x_i$ must be homogeneous, because of the projective parameter and all of the same degree, because of the underlying projective plane. They must also never vanish simultaneously, since $(0 : 0 : 0)$ is not allowed in $\mathbb{RP}^2$. When they are all of degree 2, we say that $(x_0 : x_1 : x_2)$ is a *quadratic parametrization* of the curve.

**Lemma 3.1.1.** *Any irreducible conic in $\mathbb{RP}^2$ admits a 3-dimensional family of quadratic parametrizations, all related via linear change of parameters.*

*Proof.* The heart of the proof is that, on the one hand, the space of all quadratic parametrization is isomorphic to an open set of $\mathbb{RP}^8$, since any such map is given by the 9 coefficients of the quadratic polynomials, and any non-zero multiple of $(x_0, x_1, x_2)$ gives the same curve in $\mathbb{RP}^2$.

$$\begin{aligned} x_0(t, w) &= a_0t^2 + 2a_1tw + a_2w^2;\\ x_1(t, w) &= a_3t^2 + 2a_4tw + a_5w^2;\\ x_2(t, w) &= a_6t^2 + 2a_7tw + a_8w^2. \end{aligned}$$

On the other hand, the condition that such a map parametrizes a conic is given by 5 quadratic equations on these coefficients, so in the end there is a 3-dimensional space left of possible quadratic parametrizations of the conic.

In addition, the action of an element of $\mathrm{PGL}(2;\mathbb{R})$ by linear change of coordinates maps a quadratic parametrization into another, since the degrees are preserved. However, the curve on $\mathbb{RP}^2$ that both such parametrizations describe is the same, because they nullify the same implicit equation. We shall make good use of this 3-dimensional Lie group to generate the 3-dimensional space of quadratic parametrizations of a conic.

A projective parametrization of a conic $u$ is a triple $x_i \in F_{2,2}$, $i \in \{0,1,2\}$ such that, for every point $(x : y : z)$ on the conic, there exists $(t : w) \in \mathbb{RP}^1$ with $(x_0(t : w) : x_1(t : w) : x_2(t : w)) = (x : y : z)$, and such that for every $(t : w) \in \mathbb{RP}^1$, the point $(x_0(t : w) : x_1(t : w) : x_2(t : w))$ lies on the conic $u$. Therefore, $u(x_0, x_1, x_2)$ must be identically zero.

Without loss of generality, we may assume that the irreducible conic is given by $u = x^2 + y^2 - z^2$, by taking the action of a suitable element $A$ of $\mathrm{PGL}(3;\mathbb{R})$. Once we find all quadratic parametrizations of this conic $u$, we just have to apply the action of $A^{-1}$ to get all quadratic parametrizations of the original conic.

Any quadratic parametrization $(x_0 : x_1 : x_2)$ of $u$ satisfies $x_0^2 + x_1^2 - x_2^2 = 0$ for all values of $(t : w) \in \mathbb{RP}^1$. One may then see that $x_2$ must be a positive definite quadratic form, since if there existed $(t_0 : w_0)$ such that $x_2(t_0, w_0) = 0$, then we would also have that $x_0(t_0, w_0) = x_1(t_0, w_0) = 0$, which is not allowed. Since $x_2$ is positive definite, there is an element of $\mathrm{PGL}(2;\mathbb{R})$ that transforms $x_2(t, w)$ into $x_2(T, W) = T^2 + W^2$ by Sylvester's Law of Inertia. Notice that this step uses only 2 degrees of freedom available via the action of $\mathrm{PGL}(2;\mathbb{R})$, since the stabilizer of $T^2 + W^2$ is still a 1-dimensional subgroup whose elements are:

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Indeed, what we have accomplished is that $a_7 = 0$ and $a_6 = a_8$, which also indicates that only 2 degrees of freedom were necessary. So, after applying this linear change of coordinates, we have six real coefficients $b_i$ that determine the parametrization. Here we keep the lower case for simplicity.

$$\begin{aligned} x_0(t,w) &= b_1 t^2 + 2b_2 tw + b_3 w^2; \\ x_1(t,w) &= b_4 t^2 + 2b_5 tw + b_6 w^2; \\ x_2(t,w) &= t^2 + w^2. \end{aligned}$$

Next, by taking the suitable $R_\theta$ one may also set $b_2 = 0$. The action of $R_\theta$ on $x_0$ plays as follows:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} =$$

$$= \begin{pmatrix} b_1\cos^2(\theta) + 2b_2\cos(\theta)\sin(\theta) + b_3\sin^2(\theta) & (b_3-b_1)\cos(\theta)\sin(\theta) + b_2(\cos^2(\theta)-\sin^2(\theta)) \\ (b_3-b_1)\cos(\theta)\sin(\theta) + b_2(\cos^2(\theta)-\sin^2(\theta)) & b_1\sin^2(\theta) - 2b_2\cos(\theta)\sin(\theta) + b_3\cos^2(\theta) \end{pmatrix}$$

$$= \begin{pmatrix} b_1\cos^2(\theta) + 2b_2\cos(\theta)\sin(\theta) + b_3\sin^2(\theta) & \frac{b_3-b_1}{2}\sin(2\theta) + b_2\cos(2\theta) \\ \frac{b_3-b_1}{2}\sin(2\theta) + b_2\cos(2\theta) & b_1\sin^2(\theta) - 2b_2\cos(\theta)\sin(\theta) + b_3\cos^2(\theta) \end{pmatrix}.$$

Thus, the coefficient of $tw$ in $R_\theta \,.\, x_0(t,w)$ is $(b_3-b_1)\sin(2\theta)+2b_2\cos(2\theta)$, which vanishes for a value of $\theta$ that satisfies $\tan(2\theta) = \frac{2b_2}{b_1-b_3}$. If $b_1 = b_3$, it suffices to take $\theta = \pi/4$. Hence, by using the last degree of freedom available, we have five real coefficients $c_i$ such that:

$$\begin{aligned} x_0(t,w) &= c_1 t^2 + c_2 w^2; \\ x_1(t,w) &= c_3 t^2 + 2c_4 tw + c_5 w^2; \\ x_2(t,w) &= t^2 + w^2. \end{aligned}$$

Now, let us inject $x_i$ into the equation of $u$ and collect each monomial.

$$\begin{aligned} u(x_0,x_1,x_2) &= (c_1t^2 + c_2w^2)^2 + (c_3t^2 + 2c_4tw + c_5w^2)^2 - (t^2+w^2)^2 \\ &= (c_1^2 + c_3^2 - 1)t^4 + 4c_3c_4t^3w + 2(c_1c_2 + 2c_4^2 + c_3c_5 - 1)t^2w^2 \\ &\quad + 4c_4c_5tw^3 + (c_2^2 + c_5^2 - 1)w^4. \end{aligned}$$

In order for $(x_0 : x_1 : x_2)$ to be a parametrization of $u$, the quartic above must be identically zero, so we have the following system of equations:

$$\begin{cases} c_1^2 + c_3^2 = 1 \\ c_3 c_4 = 0 \\ c_1 c_2 + 2c_4^2 + c_3 c_5 = 1 \\ c_4 c_5 = 0 \\ c_2^2 + c_5^2 = 1 \end{cases}$$

If $c_4 = 0$, then there is no viable solution, as the system reduces to:

$$\begin{cases} c_1^2 + c_3^2 = 1 \\ c_1 c_2 + c_3 c_5 = 1 \\ c_2^2 + c_5^2 = 1 \end{cases}$$

Multiply the second equation by $-2$ and add all three equations to get:

$$(c_1 - c_2)^2 + (c_3 - c_5)^2 = 0.$$

So $c_1 = c_2 = \lambda$, $c_3 = c_5 = \mu$ and $\lambda^2 + \mu^2 = 1$. Then the quadratic parametrization for the conic would be $(\lambda(t^2 + w^2) : \mu(t^2 + w^2) : (t^2 + w^2)) = (\lambda : \mu : 1)$, which is a single point on $\mathbb{RP}^2$. Therefore, we do not get any suitable parametrization with $c_4 = 0$.

We consider the case were $c_4 \neq 0$ then. Immediately, we have that $c_3 = c_5 = 0$ and the system reduces to:

$$\begin{cases} c_1^2 = 1 \\ c_1 c_2 + 2c_4^2 = 1 \\ c_2^2 = 1 \end{cases}$$

We find that $c_1$ and $c_2$ could each be either 1 or $-1$. However, the second equation imposes that they must have opposite signs, because otherwise $c_3 = c_4 = c_5 = 0$, hence $x_1(t, w) = 0$ for all $(t, w) \in \mathbb{RP}^1$ and we do not have a suitable parametrization of the conic. Therefore, we have found 4 possible solutions for the system:

| Solution 1 | Solution 2 | Solution 3 | Solution 4 |
|---|---|---|---|
| $t^2 - w^2$ | $t^2 - w^2$ | $-t^2 + w^2$ | $-t^2 + w^2$ |
| $2tw$ | $-2tw$ | $2tw$ | $-2tw$ |
| $t^2 + w^2$ | $t^2 + w^2$ | $t^2 + w^2$ | $t^2 + w^2$ |

Nevertheless, they are all related by linear change of variables, since from the first solution we obtain the second by making $t \to -t$, $w \to w$; the third with $t \to w$, $w \to t$; and the fourth with $t \to -w$, $w \to t$. In conclusion, given any irreducible conic, in $\mathbb{RP}^2$, it has a unique quadratic parametrization up to linear change of parameters. Consequently, the conic admits a 3-dimensional family of quadratic parametrizations and they are all related via the $\mathrm{PGL}(2;\mathbb{R})$ action, belonging to a single orbit.

□

## 3.2

### Quartics that describe the relative position of two conics

Let $u$ and $v$ be a pair of conics, where $u$ is non-degenerate. We have seen in Section 3.1 that $u$ admits a unique quadratic parametrization up to the $\mathrm{PGL}(2;\mathbb{R})$ action as linear changes of the projective parameter $(t : w) \in \mathbb{RP}^1$. Let $(x_0(t : w) : x_1(t : w) : x_2(t : w))$ be one such parametrization for $u$. By injecting it into the implicit equation of $v$, one gets a *binary quartic* in $t$ and $w$, which we will call $\varphi(u, v)$. If $v$ is given by $v = ax^2 + by^2 + cz^2 + 2fxz + 2gyz + 2hxy$, then we may present $\varphi(u, v)$ more concretely as:

$$\varphi(u, v) = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

$$\varphi(u, v) = ax_0^2 + bx_1^2 + cx_2^2 + 2fx_0x_2 + 2gx_1x_2 + 2hx_0x_1.$$

Notice, however, that by taking two distinct quadratic parametrization for $u$, we do not obtain the same quartic when injecting them into the implicit equation of $v$. In fact, since these parametrizations must be related via an element of $\mathrm{PGL}(2;\mathbb{R})$, we get two quartics in the same $\mathrm{PGL}(2;\mathbb{R})$ orbit. On top of that, for any $\lambda \in \mathbb{R}^*$, one could take $\lambda v$ as the implicit equation of the second conic instead of $v$, which results in the quartic $\lambda\varphi(u, v)$ by linearity. This gives us the opportunity to specify the signature of $v$. Since it must be a non-degenerate conic in $\mathbb{RP}^2$, it can only have signature $(1,2)$ or $(2,1)$. As an arbitrary choice, let us only work with signature $(1,2)$. So, for example, we will come across the particular conic $\mathfrak{c} := -x^2 - y^2 + z^2$, whose zero set is the unitary circle centered at the origin when considering the affine chart $z = 1$.

Every non-degenerate conic divides $\mathbb{RP}^2$ into two disjoint components, one homeomorphic to a disc and the other homeomorphic to a Möbius band. In order to distinguish the two, one may simply check any given point not on the conic, because if the quadratic form has signature $(1,2)$, then every point in the region isomorphic to the disc has a positive value as its image, while the points in the other region have negative image.

The quartic $\varphi(u,v)$ is the object that we are going to use to study the relative position of $u$ with respect to $v$, so let us define it rigorously. Since we may consider any quadratic parametrization for $u$ and also any positive multiple $\lambda v$ for the second conic, ($\lambda > 0$ in order to preserve the signature of $v$) we cannot be too rigid with the definition of $\varphi(u,v)$. Instead of being a single specific quartic, it should be a whole $\mathrm{GL}\,(2;\mathbb{R})$ equivalence class.

**Definition.** Let $u$ and $v$ be two non-degenerate conics of $\mathbb{RP}^2$. Let $\rho(u) = (x_0(t:w) : x_1(t:w) : x_2(t:w))$ be any quadratic parametrization of $u$ and let $V$ be the symmetric matrix of signature $(1,2)$ associated to $v$. Then $\varphi(u,v)$ is the $\mathrm{GL}\,(2;\mathbb{R})$ orbit of the binary quartic given by $\rho(u)^{\mathsf{T}} V \rho(u)$.

We had to broaden the definition of $\varphi(u,v)$ in order to avoid any ambiguity in its construction, but we may always treat it as a binary quartic by picking an arbitrary representative of the class, as long as we work with its $\mathrm{GL}\,(2;\mathbb{R})$ invariant properties. We are allowed, for example, to consider its roots.

Notice that any intersection of the two curves, $u$ and $v$, must be related to a root of the associated quartic. In fact, by Bézout's theorem [Fulton] we know that they must intersect in 4 points in $\mathbb{CP}^2$, taking into account the multiplicity of the intersections, so even the non-real roots of $\varphi(u,v)$ are related to intersections of the conics, which just do not belong to $\mathbb{RP}^2$. We are, however, only interested in the real projective plane, so that we may define what it means for a conic to be *nested* inside another conic, a property that is invariant under real projective transformations.

**Definition.** We say that $u$ is *nested* with respect to $v$ if the zero set of $u$ is contained in the component of the complement of $v$ that is isomorphic to a disc.

A direct characterization of this property is the sign of the image of $\varphi(u,v)$.

As we have explained above, $u$ is nested with respect to $v$ if and only if the quartic $\varphi(u,v)(t,w)$ is strictly positive for all values of $(t:w) \in \mathbb{RP}^1$.

Next, we highlight the fact that $\varphi(\cdot,\cdot)$ is invariant under the action of $\mathrm{PGL}(3;\mathbb{R})$ as projective transformations on $\mathbb{RP}^2$. Indeed, if $A \in \mathrm{PGL}(3;\mathbb{R})$, then $A.(x_0(t:w) : x_1(t:w) : x_2(t:w))$ is a suitable quadratic parametrization of $A.u$, and the implicit equation of $A.v$ is obtained from $V$ via congruence with respect to $A^{-1}$. Thus one has:

$$\varphi(A.u, A.v) = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} A^\intercal (A^{-1})^\intercal \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix} A^{-1} A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \varphi(u,v).$$

The invariance of $\varphi(\cdot,\cdot)$ under $\mathrm{PGL}(3;\mathbb{R})$ establishes that the class of quartics $\varphi(u,v)$ gives us some projective information about the relative position of the conics $u$ and $v$.

It is also important to remark that, for a fixed first conic $u$, the class of quartics obtained via $\varphi(u,\cdot)$ is constant along *pencils through* $u$, with exception of $u$ itself, naturally. The class of $\varphi(u,u)$ is the class of the identically 0 polynomial, since the parametrization of $u$ stays on the zero-level curve of $u$ by definition. Now, given a distinct second conic $v \neq u$ and any $\lambda \in \mathbb{R}$, the class of $\varphi(u, \lambda u + v)$ is the same as that of $\varphi(u,v)$, because, by denoting the matrix forms of $u$ and $v$ by $U$ and $V$ respectively, one has by linearity that:

$$\begin{aligned} \varphi(u, \lambda u + v) &= \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} (\lambda U + V) \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} \lambda U \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} V \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

$$\varphi(u, \lambda u + v) = \lambda \varphi(u,u) + \varphi(u,v) = \varphi(u,v).$$

Notice, however, that $\lambda$ should actually be restricted to a smaller domain, because in the pencil $\lambda u + v$ there are conics of different signatures, as it always has at least one degenerate element. Let us analyse one concrete example to better understand what may happen.

Consider $u = -x^2 - y^2 + z^2$ and $v = -2x^2 - 2y^2 + z^2$. Using the parametrization $\rho(u) = (t^2 - w^2 : 2tw : t^2 + w^2)$ (which we explain shortly in the next section), we have that $\varphi(u, v) = -(t^2 + w^2)^2$. It stands to reason that $\varphi(u, v)$ is always negative, since in the affine chart $z = 1$ we see that $u$ is the unit circle while $v$ is the circle of radius $\sqrt{2}/2$ centered at the origin, so $u$ belongs to the component homeomorphic to the Möbius band. Algebraically, $\varphi(u, \lambda u + v)$ always results in this same exact quartic, as expected. However, if $\lambda = -1$, then $-u+v = -x^2-y^2$ is a degenerate conic. The same happens for $\lambda = -2$, as $-2u + v = -z^2$ is also degenerate. For $\lambda > -1$, the conic $\lambda u + v$ has signature $(1, 2)$ and everything is fine. But if $-2 < \lambda < -1$, the signature becomes $(0, 3)$ and there is no point of the conic in $\mathbb{RP}^2$, and $\varphi(u, \lambda u+v)$ is not defined. Finally, for $\lambda < -2$, the signature becomes $(2, 1)$, thus by definition of $\varphi$, we must consider $-\lambda u - v$ in the second term, and we get $\varphi(u, -\lambda u - v) = (t^2 + w^2)^2$. Again, it makes sense that the quartics obtained are positive, since $-\lambda u - v$ represent circles centered at the origin of radii greater than 1, so $u$ belongs to the component homeomorphic to the disc.

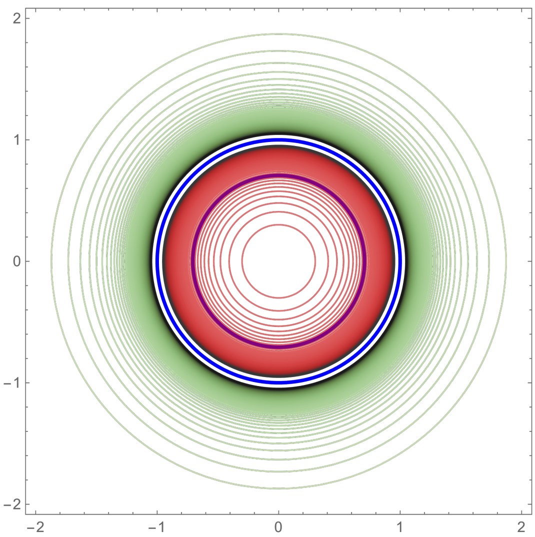


Figure 3.1: Conics of the pencil $\lambda u + v$ color-coded.
Those in green have $\lambda < -2$. The ones in red have $\lambda > -1$. The conic $u$ is in blue and $v$ is in purple.

One may remark that the choices we have made in the definition of $\varphi(\cdot, \cdot)$ are arbitrary. That is indeed the case, although it is also a strategic one. First we point out that it is not at all clear the relation between $\varphi(u, v)$ and $\varphi(v, u)$, so the order in which we consider the conics is important. Moreover, the proofs we present in Chapters 4, 5 and 6 rely on the analysis of the class of quartics $\varphi(u, v)$, namely its positivity, convexity and algebraic invariants. By demanding that the second conic has signature $(1, 2)$, these properties of the class of quartics are well suited to identify whether $u$ is nested with respect to $v$ and other stricter relations that we define further on.

## 3.3

### Anchoring the first conic

Since $\varphi(\cdot,\cdot)$ is invariant under the $\mathrm{PGL}(3;\mathbb{R})$ action by projective transformations, we may always take the non-degenerate conic $u$ to $\mathfrak{c} = -x^2 - y^2 + z^2$, which is the unitary circle centered at the origin when considering the affine chart $z = 1$. From now on, we will always consider $\mathfrak{c}$ as the first conic in the evaluation of $\varphi$, so we require one of its quadratic parametrizations. This can be achieved by a classical construction, which we explain below.

First we fix a line at infinity so that we can work with just two coordinates. Let us take the plane $z = 1$. We are going to obtain a quadratic rational parametrization on two coordinates, which we then turn into a quadratic parametrization by adding the third homogeneous coordinate and multiplying by the common denominator as it still describes the same curve on the projective plane.

Let us take the point $(1 : 0 : 1)$ on $\mathfrak{c}$. Consider the pencil of lines passing through that point and let it be linearly parametrized as a function of a parameter $t$, for example as $y = -t(x-1)$. For each fixed value of $t$, we can substitute the equation of its corresponding line in the equation of the conic to find the second intersection between these two curves.

$$\begin{cases} y = -t(x-1) \\ -x^2 - y^2 + 1 = 0 \end{cases} \qquad \Longrightarrow \qquad (-t^2-1)x^2 + 2t^2x + (-t^2+1) = 0.$$

For each fixed value of $t$, we may find two values of $x$ which are the roots of the expression above. They correspond to the $x$ coordinates of the intersections between the line and the conic. As we know that $x = 1$ is necessarily one of the solutions, we may factor it out and thus obtain a rational parametrization of the $x$ coordinate along the conic.

$$\frac{(-t^2-1)x^2 + 2t^2x + (-t^2+1)}{x-1} = x(-t^2-1) + (t^2-1),$$
$$x(-t^2-1) + (t^2-1) = 0 \implies x(t) = \frac{t^2-1}{t^2+1}.$$

Now, we just have to go back to the equation of the line in order to obtain the rational parametrization of the $y$ coordinate. Since it is an affine equation, we see that both coordinates have the same denominator in its rational parametrization, as we wanted.

$$y = -t(x-1) \implies y(t) = \frac{2t}{t^2+1}.$$

To conclude, we quit this affine plane in order to obtain a quadratic parametrization with a real parameter for our conic:

$$x(t) = t^2 - 1, \quad y(t) = 2t, \quad z(t) = t^2 + 1.$$

Notice that if we use a real parameter $t \in \mathbb{R}$, we fail to consider a particular line of the pencil. In the construction above, it is the tangent line to the conic, of equation $x = 1$, which can be obtained as a limit object if we make $t \to \pm\infty$. We can easily avoid this issue if we use a projective parameter $(t : w) \in \mathbb{RP}^1$ instead. To do so, one just homogenizes the expressions obtained at the end. Thus, we have found the following quadratic parametrization for $\mathfrak{c}$:

$$x(t,w) = t^2 - w^2 \ , \quad y(t,w) = 2tw \ , \quad z(t,w) = t^2 + w^2.$$

## 3.4

## Isomorphism between pencils through $\mathfrak{c}$ and $\mathbb{P}(F_{2,4})$

Now that we have decided to always work with $\mathfrak{c}$ as the first conic in the evaluation of $\varphi$ and that we have a quadratic parametrization for it, we move to the study of the map $\varphi(\mathfrak{c},\cdot)$. It takes a nondegenerate conic and returns a class of binary quartics up to the action of $\mathrm{GL}\,(2;\mathbb{R})$. In order to better understand its behaviour, let us first arbitrarily pin down a representative of the class by demanding that the quadratic parametrization used in the computation is $(t^2 - w^2 : 2tw : t^2 + w^2)$. This has the effect of removing the $\mathrm{PGL}(2;\mathbb{R})$ identification on the image. Now, for an arbitrary conic $v$ of signature $(1,2)$, we may compute the binary quartic $\varphi(\mathfrak{c}, v)$.

$$\varphi(\mathfrak{c}, v) = \begin{pmatrix} t^2 - w^2 & 2tw & t^2 + w^2 \end{pmatrix} \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix} \begin{pmatrix} t^2 - w^2 \\ 2tw \\ t^2 + w^2 \end{pmatrix}.$$

$$\varphi(\mathfrak{c}, v) = (a + 2f + c)t^4 + (4h + 4g)t^3w + (-2a + 4b + 2c)t^2w^2 + (-4h + 4g)tw^3 + (a - 2f + c)w^4.$$

Abstractly, if we provisionally lift the restriction about the signature, we have in our hands a linear map $L\colon \mathbb{R}^6 \to \mathbb{R}^5$ given by

$$\begin{pmatrix} a \\ b \\ c \\ f \\ g \\ h \end{pmatrix} \overset{L}{\mapsto} \begin{pmatrix} a + 2f + c \\ 4g + 4h \\ -2a + 4b + 2c \\ 4g - 4h \\ a - 2f + c \end{pmatrix}.$$

The kernel of this linear map is one dimensional and it is precisely the direction related to the conic $\mathfrak{c}$, with generating vector $a = -1, b = -1, c = 1, f = 0$, $g = 0, h = 0$. This implies that the map $L$ is surjective, so every binary quartic is attainable and corresponds to a single pencil of conics through $\mathfrak{c}$.

Let us use $\mathcal{F}_{\mathfrak{c}}$ to denote the space of pencils of conics containing $\mathfrak{c}$. Notice that it is isomorphic to $\mathbb{RP}^4$. The projectivization of the space of binary quartics is $\mathbb{P}(F_{2,4})$, which is also isomorphic to $\mathbb{RP}^4$. By passing to the quotient with respect to its kernel, the map $L$ establishes an isomorphism $\mathcal{F}_{\mathfrak{c}} \cong \mathbb{P}(F_{2,4})$.

The map $L$ also allows us to solve the inverse problem. Given a binary quartic $p(t, w) \in F_{2,4}$, one may find every conic $v$ such that $\varphi(\mathfrak{c}, v) = p(t, w)$. Due to the kernel, one may first set the coefficient $c = 0$. If $p(t, w) = \alpha t^4 + \beta t^3 w + \gamma t^2 w^2 + \delta t w^3 + \varepsilon w^4$, then the remaining coefficients of $v$ must be:

$$a = \frac{\alpha + \varepsilon}{2}; \quad b = \frac{\alpha + \gamma + \varepsilon}{4}; \quad f = \frac{\alpha - \varepsilon}{4}; \quad g = \frac{\beta + \delta}{8}; \quad h = \frac{\beta - \delta}{8}.$$

Finally, by adding back any scalar multiple $\lambda\mathfrak{c}$ to $v$, which equates to adding $-\lambda$ to the coefficients $a$ and $b$ while adding $\lambda$ to the coefficient $c$, one gets any conic of the pencil containing $\mathfrak{c}$ and $v$. To conclude, we check the signature of the quadratic form associated to $\lambda\mathfrak{c} + v$: If it is already $(1, 2)$ we are done. If it is $(2, 1)$, we have to multiply by $-1$, because only $\varphi(\mathfrak{c}, -\lambda\mathfrak{c} - v)$ is defined and it actually gives us the quartic $-p(t, w)$. If it is any other signature, then $\varphi(\mathfrak{c}, \lambda\mathfrak{c} + v)$ is not defined and this case does not belong to the scope of our research.

# 4

# Osculating Conics

The main goal of this chapter is to understand the relative position of two osculating conics of a smooth curve in the real projective plane in order to obtain a generalization of the Tait-Kneser theorem. It is already known that under certain hypotheses such conics are disjoint. This result was first published by Hayashi in 1926 [Hayashi]. Our approach will allow us to find a new proof of this fact and also a stronger result, showing that the osculating conics are in some sense "more than nested", that is, there is an even stronger condition on their relative position. We give it a new name by declaring that they are *convexly nested*.

**Definition.** Given two nondegenerate conics $u$ and $v$ in $\mathbb{RP}^2$, we say that $u$ is *convexly nested* with respect to $v$ if the binary quartic $\varphi(u,v) \in K_{2,4}$ is positive and convex.

With this definition at hand, we may now state one of the main theorems of this thesis.

**Theorem 4.0.1.** *Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{RP}^2$ be a smooth arc in the real projective plane with no inflections or sextactic points. Then the osculating conics of $\gamma$ are disjoint and convexly nested.*

A sextactic point is a point of the curve $\gamma$ where the order of contact between the curve and the osculating conic in higher than generically expected, that is, of order 6 or higher. It is the equivalent of a *vertex*, which is the point where the contact with the osculating circle is higher than expected, so 4 or higher. The proof of the theorem is provided throughout the remainder of this chapter.

## 4.1

## Osculating Conics

We shall begin by describing the space of conics of the real projective plane. Every conic is described implicitly by an algebraic expression of the form: $ax^2+by^2+cz^2+2fxz+2gyz+2hxy = 0$, where $x, y$ and $z$ are the homogeneous coordinates of $\mathbb{RP}^2$. This is a homogeneous polynomial of degree 2 in three variables, also called a *ternary quadratic form*. The set of all ternary quadratic forms is a 6-dimensional vector space denoted $F_{3,2}$.

However, although every non-null form gives rise to an algebraic curve on the *complex* projective plane, not every form corresponds to a non-empty curve in the *real* projective plane. For example, there are no points on $\mathbb{RP}^2$ whose homogeneous coordinates satisfy the expression $x^2+y^2+z^2=0$. Fortunately, it is possible to characterize which quadratic forms are associated to non-empty conics. One just has to study its associated symmetric matrix. Notice that the generic ternary quadratic form $ax^2+by^2+cz^2+2fxz+2gyz+2hxy=0$ can also be given by:

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

The curve described by this equation is in fact the set of isotropic vectors of the matrix above. Therefore the conic related to said matrix is present on $\mathbb{RP}^2$ if and only if it is not positive-definite or negative-definite. In fact, its signature reveals every information we need. Up to the action of the projective group $\mathrm{PGL}(3;\mathbb{R})$ we can classify the real projective conics and present their normal forms.

| **Classification of real projective conics** | | |
|---|---|---|
| Signature | Normal form | Type |
| $(3,0)$ or $(0,3)$ | $x^2+y^2+z^2=0$ | Empty irreducible |
| $(2,1)$ or $(1,2)$ | $x^2+y^2-z^2=0$ | Non-empty irreducible |
| $(2,0)$ or $(0,2)$ | $x^2+y^2=0$ | Imaginary line-pair |
| $(1,1)$ | $x^2-y^2=0$ | Real line-pair |
| $(1,0)$ or $(0,1)$ | $x^2=0$ | Repeated line |

The imaginary line-pair passes through a single point in $\mathbb{RP}^2$, so it does not interest us either. Since we will be looking for generic osculating conics, which are non-empty irreducible, we should only work on the open set of $F_{3,2}$ of the forms with signature $(2,1)$ or $(1,2)$. Notice, however, that since non-zero multiples of an expression describe the same curve, this is actually a fiber bundle over the true space of conics $\mathcal{C}$, which could be obtained through projectivization. Nevertheless, we are going to work directly on fiber bundles of base $\mathcal{C}$ throughout this chapter.

The total space we have in hand has two isomorphic connected components, characterized by the signature of their elements. The intersection of each fiber, which represents an unique conic, with one such connected component is comprised of a half-line starting from the origin. Moreover, the involution of multiplying the form by $-1$ swaps the two components of the fiber. Let us take a deeper look at what characterizes each component.

Take for example the conic given implicitly by the expression $u(x, y, z) = -x^2 - y^2 + z^2$. If we look at the affine chart given by $z = 1$, $u$ manifests itself as the unit circle centred at $(0 : 0 : 1)$. Any non-zero multiple of this expression $\lambda u$ describes the same curve, but if $\lambda$ is negative it changes the signature of the quadratic form. One can observe this difference by checking the sign of the values obtained in each component of the complement of the conic in $\mathbb{RP}^2$.

An irreducible conic divides the projective plane into two components, one homeomorphic to a disc and the other homeomorphic to a Möbius band. In our example, $u$ yields positive values on the disc (which we call from now on the "inside" of the circle) and negative values on the Möbius band (which we call the "outside" of the circle). In fact this is true for every quadratic form of signature $(1, 2)$, while the opposite holds for those of signature $(2, 1)$. Our goal is to study how the osculating conics evolve locally and since both connected components of the total space comprise the same information, specifically, both contain representative fibers for every irreducible conic, we may work on a single component. So we only consider conics given implicitly by real quadratic forms of signature $(1, 2)$, and this space shall be denoted by $\Omega$.

To summarize, we will consider the fiber bundle $(\Omega, \mathcal{C}, \pi, F)$ where the total space $\Omega$ is the open set of $F_{3,2}$ of ternary quadratic forms whose associated matrix has signature $(1, 2)$; the base space $\mathcal{C}$ is the set of non-degenerate conics of $\mathbb{RP}^2$; the projection $\pi : \Omega \to \mathcal{C}$ is the map that sends each form to its zero set; and each fiber $F$ is a half-line, so isomorphic to $\mathbb{R}$.

Working in this bundle will provide us with a degree of freedom that will be useful to prove the main theorem. Our strategy is to consider the curve $\Gamma : (-\varepsilon\,, \varepsilon) \to \Omega$ defined by the osculating conics of a smooth arc $\gamma : (-\varepsilon\,, \varepsilon) \to \mathbb{RP}^2$. We are going to describe a particularly convenient way to parametrize such curve $\Gamma$ that allows us to establish the result.

As a main tool, we use the map $\varphi$ defined in section 3.2. Given a pair of conics, this map produces a homogeneous polynomial of degree 4 in two variables, a binary quartic. The nature of this polynomial characterizes the relative position of the conics in $\mathbb{RP}^2$. We then use the concept of blenders introduced in section 2.2 to classify the relation between the conics. Once a suitable parametrization for $\Gamma$ has been found, we evaluate $\varphi\Big(\Gamma(t_1), \Gamma(t_2)\Big)$ and show that it always belongs to $K_{2,4} = Q_{2,4}$, showing that if there is no sextactic point or inflection, then the smaller osculating conic is convexly nested with respect to the bigger one, since the class of quartics that they generate is positive and convex. The geometric meaning of this property will be discussed in the next chapter.

## 4.2

### Parametrizing the path of osculating conics

Consider a smooth arc $\gamma : (-\varepsilon\,, \varepsilon) \to \mathbb{RP}^2$ with no inflexion point. The osculating conic to this curve at an arbitrary point $\gamma(s_0)$ is the only one that has a contact of order greater than or equal to 5 at precisely this point. We wish to determine the conic's implicit equation $u(x, y, z) = ax^2 + by^2 + cz^2 + 2fxz + 2gyz + 2hxy$, so we must obtain its 6 coefficients. Having a fifth order contact with the curve means that if we evaluate $u$ along $\gamma(s)$, not only do we get a zero at $s = s_0$, but the first four derivatives also vanish on this point. Therefore, this property is represented by the following system of equations:

$$\begin{cases} u\circ\gamma\,(s_0) = 0 \\ \frac{d}{ds}(u\circ\gamma)(s_0) = 0 \\ \frac{d^2}{ds^2}(u\circ\gamma)(s_0) = 0 \\ \frac{d^3}{ds^3}(u\circ\gamma)(s_0) = 0 \\ \frac{d^4}{ds^4}(u\circ\gamma)(s_0) = 0 \end{cases}$$

Notice that all of these equations are linear with respect to the coefficients $a, b, c, f, g, h$ that determine the conic. For a generic point of $\gamma$, we have 5 linearly independent equations which therefore provide us with a one-dimensional vector subspace of solutions within $F_{3,2}$. One such subspace contains exactly one fiber of our bundle $\Omega$, so this system of equations has effectively a unique solution. In order to have a concrete equation assigned to each osculating conic, we may introduce, without loss of generality, an additional condition setting one of the coefficients to be identically equal to 1, say $c = 1$. In this way, for every $s_0 \in (-\varepsilon\,, \varepsilon)$ we have an implicit equation $\Gamma(s_0) \in \Omega$ for the osculating conic of $\gamma$ at $\gamma(s_0)$, which gives us a parametrization of a *path of osculating conic* in our bundle $\Omega$.

However, we could have picked another representative for the osculating conic at each point and still have the same information. If we consider any smooth function with positive image $\lambda : (-\varepsilon\,, \varepsilon) \to \mathbb{R}_+^*$ and take the curve $\lambda(s)\Gamma(s) \subset \Omega$, then for every $s_0$ we still have an equation of the osculating conic at $\gamma(s_0)$. This step provides us with a degree of freedom of which we will make good use.

Our goal is to show that $\varphi\Big(\Gamma(0), \Gamma(s_0)\Big) \in K_{2,4}$ for an arbitrary $s_0 \in (0, \varepsilon)$. Our strategy will be to consider the whole path $\varphi\Big(\Gamma(0), \Gamma(s)\Big)$, letting $s$ vary from 0 to $s_0$. We know that it starts at the origin of the space of quartics, and by analysing the tangent vector $\frac{d}{ds}\varphi\Big(\Gamma(0), \Gamma(s)\Big)$ along this curve we can conclude in which region its other extremity lies.

Notice that since the first term we provide to $\varphi$ is constant, when we differentiate with respect to $s$ only the derivative of the second term, which provides the implicit equation, is important. Indeed, composing with a fixed function on the right is a linear operator, as $(\lambda f + g) \circ h = \lambda f \circ h + g \circ h$, for any $\lambda \in \mathbb{R}$ and any functions $f$, $g$ and $h$. Since $\varphi\Big(\Gamma(0), \Gamma(s)\Big) = \Gamma(s) \circ \rho\Big(\Gamma(0)\Big)$, then we have that $\frac{d}{ds}\varphi\Big(\Gamma(0), \Gamma(s)\Big) = \Big(\frac{d}{ds}\Gamma(s)\Big) \circ \rho\Big(\Gamma(0)\Big)$. Therefore, we may benefit from studying the tangent vector $\frac{d}{ds}\Gamma(s)$ in order to understand how the binary quartic of interest evolves.

### 4.2.1

### The tangent vector $\frac{d}{ds}\Gamma(s)$

Since $\Gamma(s) \subset \Omega \subset \mathbb{R}^6$ is a curve in a vector space, its tangent vector can also be interpreted as representing a conic. What can we know about said conic based simply on geometric concepts? By definition, the curve $\Gamma(s)$ consists of conics that have at all times a contact of order at least 5 with $\gamma$. We claim that, consequently, the tangent vector $\Gamma'(s)$ necessarily represents a conic whose contact with $\gamma$ is of order at least 4 at the corresponding point. Note that this property does not define it uniquely, in fact there is a one-dimensional *pencil of conics* that satisfies this condition.

The proof of this geometric statement is an exercise of differential calculus. In order to simplify the reading of the proof, let us adjust the notation. For a given $s \in (-\varepsilon\,, \varepsilon)$, $\Gamma(s)$ is the implicit equation of a conic, so it can be evaluated at any point of $\mathbb{RP}^2$. In order to determine the contact of this conic with a smooth curve $\gamma$ at the point $\gamma(r)$ we must consider the function $\Gamma(s) \circ \gamma(r)$, which we denote $F(s, r)$.

**Lemma 4.2.1.** *If for every $s \in (-\varepsilon\,, \varepsilon)$ the conic given implicitly by $\Gamma(s)$ has a contact of order at least 5 with the curve $\gamma : (-\varepsilon\,, \varepsilon) \to \mathbb{RP}^2$ at the point $\gamma(s)$, then the conic given by $\frac{d}{ds}\Gamma(s)$ has a contact of order at least 4 with $\gamma$ at the point $\gamma(s)$.*

*Proof.* For a fixed $s$, having a contact of order $n$ means that $\Gamma(s) \circ \gamma(r)$ as a function of $r$ has a root of order $n$ at $r = s$. Then, a contact of order 5 is expressed analytically by the following system of equations:

$$\begin{cases} \Gamma(s)\circ\gamma(s) = \ 0 \\ \frac{d}{dr}\Big(\Gamma(s)\circ\gamma\Big)(s) = 0 \\ \frac{d^2}{dr^2}(\Gamma(s)\circ\gamma)(s) = 0 \\ \frac{d^3}{dr^3}(\Gamma(s)\circ\gamma)(s) = 0 \\ \frac{d^4}{dr^4}(\Gamma(s)\circ\gamma)(s) = 0 \end{cases} \leftrightarrow \begin{cases} F(s,s) = 0 \\ \frac{d}{dr}F(s,s) = 0 \\ \frac{d^2}{dr^2}F(s,s) = 0 \\ \frac{d^3}{dr^3}F(s,s) = 0 \\ \frac{d^4}{dr^4}F(s,s) = 0 \end{cases}$$

The key point is that these equations hold for every $s \in (-\varepsilon\,, \varepsilon)$, and so we can differentiate them with respect to $s$. By doing it to an equation of the system and then considering the next one of higher degree, we are able to show an analogous expression for $\frac{d}{ds}\Gamma(s)$. Since we are able to carry this up to the fourth iteration, we can show that $\frac{d}{ds}\Gamma(s)$ has a contact of order at least 4 with $\gamma$ at the point $\gamma(s)$.

Given $k \in \mathbb{N}$, we have that $\frac{d^k}{dr^k}F(s,s) = D^k F_{(s,s)}.(e_r, \dots, e_r)$, where there are $k$ copies of $e_r$, the second vector of the standard basis. Now we take the derivative of this term with respect to $s$.

$$\begin{aligned} \frac{d}{ds}\Big(D^k F_{(s,s)}.(e_r, \dots, e_r)\Big)(s) &= D^{k+1}F_{(s,s)}.(e_r + e_s, e_r, \dots, e_r) \\ &= D^{k+1}F_{(s,s)}.(e_r, e_r, \dots, e_r) + D^{k+1}F_{(s,s)}.(e_s, e_r, \dots, e_r) \\ &= \frac{d^{k+1}}{dr^{k+1}}F(s,s) + \frac{d^k}{dr^k}\Big(\frac{d}{ds}F(s,r)\Big)(s,s). \end{aligned}$$

If the equation $\frac{d^k}{dr^k}F(s,s) = 0$ holds for all values of $s$, then its derivative presented above is also null. If in addition $\frac{d^{k+1}}{dr^{k+1}}F(s,s) = 0$, then the last term $\frac{d^k}{dr^k}\left(\frac{d}{ds}F\right)(s,s)$ vanishes as well. Let us rewrite it in terms of $\Gamma$ and $\gamma$ to see what it represents. Notice that the composition on the right by a fixed function $\gamma(r)$ is linear, hence $\frac{d}{ds}F(s,r) = \frac{d}{ds}\Big(\Gamma(s) \circ \gamma(r)\Big) = \frac{d}{ds}\Gamma(s) \circ \gamma(r)$. Therefore, the original system of equations for $\Gamma$ implies a corresponding system for $\frac{d}{ds}\Gamma$, which concludes the proof.

$$\begin{cases} \Gamma(s)\circ\gamma(s) = 0 \\ \frac{d}{dr}\Big(\Gamma(s)\circ\gamma\Big)(s) = 0 \\ \frac{d^2}{dr^2}(\Gamma(s)\circ\gamma)(s) = 0 \\ \frac{d^3}{dr^3}(\Gamma(s)\circ\gamma)(s) = 0 \\ \frac{d^4}{dr^4}(\Gamma(s)\circ\gamma)(s) = 0 \end{cases} \implies \begin{cases} \frac{d}{ds}\Gamma(s)\circ\gamma(s) = 0 \\ \frac{d}{dr}\Big(\frac{d}{ds}\Gamma(s)\circ\gamma\Big)(s) = 0 \\ \frac{d^2}{dr^2}(\frac{d}{ds}\Gamma(s)\circ\gamma)(s) = 0 \\ \frac{d^3}{dr^3}(\frac{d}{ds}\Gamma(s)\circ\gamma)(s) = 0 \end{cases}$$

□

It is worth mentioning that there are no restrictions on the tangent space, so the conic represented by the tangent vector $\frac{d}{ds}\Gamma(s)$ can be reducible, and in fact it is beneficial to do so. A reducible conic is a pair of lines; if in addition it has a contact of order at least 4 with a point of $\gamma$, which has no inflexions, then there is only one option: it must be the double tangent line at $\gamma(s)$.

It is at this point that we use the degree of freedom that we introduced earlier with the function $\lambda(s)$. As a reminder, we defined $\Gamma(s)$ point by point as the unique solution of a system of linear equations on the conic's coefficients, where one of such equations fixed the coefficient $c = 1$. At present, all we know is that $\frac{d}{ds}\Gamma(s)$ represents a conic with a fourth order contact with $\gamma$ at $\gamma(s)$. If we consider $\lambda(s)\Gamma(s)$ instead, by taking its derivative we obtain:

$$\frac{d}{ds}\lambda(s)\Gamma(s) = \lambda'(s)\Gamma(s) + \lambda(s)\Gamma'(s).$$

The expression above shows that, for a fixed $s$, $\frac{d}{ds}\lambda(s)\Gamma(s)$ is a linear combination of two conics $\Gamma(s)$ and $\Gamma'(s)$. This describes precisely the pencil of conics with a contact of order at least 4 at $\gamma(s)$, provided that $\Gamma'(s)$ is not a multiple of $\Gamma(s)$. The next lemma states that this problem only happens at sextactic points of the curve $\gamma$.

**Lemma 4.2.2.** *The conics given by* $\Gamma(s)$ *and* $\frac{d}{ds}\Gamma(s)$ *are the same if and only if* $\gamma(s)$ *is a sextactic point.*

*Proof.* As shown in the proof of lemma 4.2.1, since $\frac{d^4}{dr^4}\Big(\Gamma(s)\circ\gamma\Big)(s) = 0$ holds for every value of $s$, by taking the derivative of this expression we obtain the following equation.

$$\frac{d^5}{dr^5}\Big(\Gamma(s)\circ\gamma\Big)(s) + \frac{d^4}{dr^4}\left(\frac{d}{ds}\Gamma(s)\circ\gamma\right)(s) = 0.$$

Therefore, if one of the terms vanishes, so does the other. Suppose that for a fixed $s$ we have a sextactic point $\gamma(s)$. By definition, the contact of the osculating conic $\Gamma(s)$ at this point is of a greater order than expected, so at least 6. This means that $\frac{d^5}{dr^5}\Big(\Gamma(s)\circ\gamma\Big)(s) = 0$. On the other hand, since the osculating conic at a given point is unique, the conic given by $\Gamma(s)$ and $\frac{d}{ds}\Gamma(s)$ are the same if and only if $\frac{d}{ds}\Gamma(s)$ is itself the osculating conic and hence $\frac{d^4}{dr^4}\Big(\frac{d}{ds}\Gamma(s)\circ\gamma\Big)(s) = 0$. Consequently, those two properties are equivalent, which concludes the proof. $\square$

The lemma above provides us with the key element of the hypothesis of our main theorem. In order to follow the next steps, we are going to suppose that there are no sextactic points on the smooth arc $\gamma$. Under this condition, we may find a function $\lambda(s)$ such that for every $s$ the conic $\frac{d}{ds}\lambda(s)\Gamma(s)$ is the double tangent line. To do so, we must solve a differential equation. Consider a conic described by the usual implicit equation $u = ax^2 + by^2 + cz^2 + 2fxz + 2gyz + 2hxy$. As stated in section 4.1, it is reducible if and only if the determinant of its corresponding matrix vanishes.

$$\Delta(u) = \begin{vmatrix} a & h & f \\ h & b & g \\ f & g & c \end{vmatrix}.$$

The differential equation at hand is then $\Delta\Big(\lambda'(s)\Gamma(s) + \lambda(s)\Gamma'(s)\Big) = 0$. It has a unique solution if we add the initial condition $\lambda(0) = 1$. Therefore, we have a curve of osculating conics $\widetilde{\Gamma}(s) = \lambda(s)\Gamma(s) \subset \Omega$ parametrized in such a way that its tangent vector $\frac{d}{ds}\widetilde{\Gamma}(s)$ always corresponds to the the double tangent line at $\gamma(s)$.

## 4.3

### Osculating conics are disjoint and nested

In the last section we have achieved the desired parametrization for the path of osculating conics, which will be simply denoted by $\Gamma(s)$. Now we come back to the path on the space of quartics given by $\varphi\big(\Gamma(0), \Gamma(s)\big)$. We want to understand the nature of its tangent vector, which is given by

$$\frac{d}{ds}\varphi\Big(\Gamma(0), \Gamma(s)\Big) = \frac{d}{ds}\Gamma(s) \circ \rho\Big(\Gamma(0)\Big).$$

In order to understand the quartic that it represents, we must insert the polynomial parametrization of the initial conic $\rho\big(\Gamma(0)\big)$ into the implicit equation of the double tangent line to the other conic $\Gamma(s)$ at its point of contact with the original arc. In other words, the fundamental question is: does the osculating conic $\Gamma(0)$ intersect the tangent line to $\gamma$ in $\gamma(s)$ at a point in $\mathbb{RP}^2$?

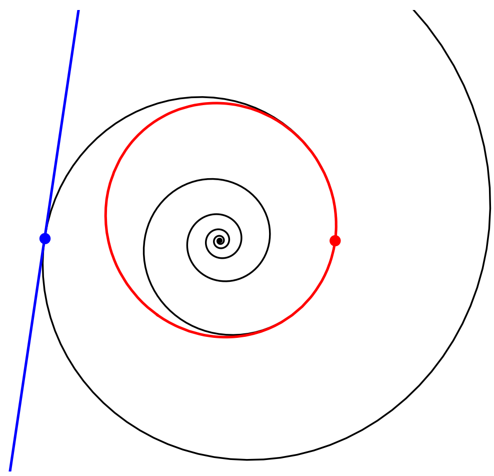

Figure 4.1: The tangent lines after the point of contact do not intersect the osculating conic at real points.

Let us start by analysing the tangent vector at the origin of the space of quartics $\frac{d}{ds}\varphi\big(\Gamma(0), \Gamma(s)\big)\Big|_{s=0}$. Since we are evaluating the derivative at $s = 0$, the referred line is tangent to the starting conic $\Gamma(0)$ and therefore the resulting quartic has a fourth root, say $[t : w] = [\alpha : \beta] \in \mathbb{RP}^1$. Thus, it is necessarily nonnegative or nonpositive. Geometrically, this indicates that in a neighbourhood of $s = 0$, outside the point of contact $\gamma(0)$, the conic $\Gamma(0)$ is either contained in the interior of $\Gamma(s)$ or in the exterior of $\Gamma(s)$. So up to changing the direction in which we traverse $\gamma$, we can assume that the resulting quartic is nonnegative. This means that the tangent vector is at the boundary of the $Q_{2,4}$ blender.

$$\frac{d}{ds}\varphi\Big(\Gamma(0),\Gamma(s)\Big)\Big|_{s=0} = (\beta t - \alpha w)^4 \in \partial Q_{2,4}.$$

This is relevant because it allows us to write the first term of the Taylor series of the curve $\varphi\Big(\Gamma(0),\Gamma(s)\Big)$.

$$\varphi\Big(\Gamma(0),\Gamma(s)\Big) = (\beta t - \alpha w)^4\, s + O(s^2).$$

Now consider any point on the initial conic $\Gamma(0)$ given by the coordinates $[t_0 : w_0]$. To find out whether this point belongs to another osculating conic, just evaluate $\varphi\Big(\Gamma(0),\Gamma(s)\Big)[t_0 : w_0]$. If this point is not the point of contact with $\gamma$, that is, if $[t_0 : w_0] \neq [\alpha : \beta]$, then for a sufficiently small $s > 0$ we can see that $\varphi\Big(\Gamma(0),\Gamma(s)\Big)[t_0 : w_0] > 0$ and therefore there is no intersection at this point.

In order to find out what happens at the point of contact, given by $[t_0 : w_0] = [\alpha : \beta]$, we need to consult some derivatives of higher order. Here we may apply once again the core idea of lemma 4.2.1 to figure out this information. Assuming we are not on a sextactic point, the first derivative $\frac{d}{ds}\Gamma(s)$ always represents a conic with a contact of order 4 with the arc $\gamma$. Consequently, the second derivative will be a conic with a contact of order 3, and so on. Since the conic $\Gamma(0)$ has a contact of order 5 with $\gamma$, we see that the contact between it and the second derivative $\frac{d^2}{ds^2}\Gamma(s)$ is of order 3 there too. In this way, we can write some more terms of the Taylor expansion of $\varphi\Big(\Gamma(0),\Gamma(s)\Big)$.

$$\begin{aligned}(\beta t - \alpha w)^4\, s + (\beta t - \alpha w)^3\, p_1(t,w)\,\frac{s^2}{2} + (\beta t - \alpha w)^2\, p_2(t,w)\,\frac{s^3}{6}\\ + (\beta t - \alpha w)\, p_3(t,w)\,\frac{s^4}{24} + p_4(t,w)\,\frac{s^5}{120} + O(s^6).\end{aligned}$$

Where $p_n(t,w)$ are homogeneous polynomials of degree $n$ and $[t_0 : w_0] = [\alpha : \beta]$ is not a root of any of them. So, when we evaluate at this point, we get:

$$\varphi\Big(\Gamma(0),\Gamma(s)\Big)[\alpha : \beta] = p_4(\alpha,\beta)\frac{s^5}{120} + O(s^6).$$

Since $p_4(\alpha, \beta) \neq 0$, in fact it must be positive by continuity, then for small values of $s > 0$ there is no intersection between $\Gamma(0)$ and $\Gamma(s)$ at the referred point. We have therefore proved that the osculating conics of a smooth arc on the real projective plane without sextactic points are locally disjoint. Moreover, since they evolve continuously, the only way that there can be no intersection is if they are nested. As this is a transitive property, we can extend it until we pass through a sextactic point.

We have thus provided a new proof for the already known result that the osculating conics are disjoint and nested. Next, we will use this fact to prove a new and stronger version of the theorem. We want to show that the path $\varphi\big(\Gamma(0), \Gamma(s)\big)$ is entirely contained in $Q_{2,4}$ and therefore the osculating conics are "more than nested", they are *convexly nested*.

## 4.4

### Osculating conics are convexly nested

To prove this new theorem we will use the same approach, the tangent vector analysis. So far, we have studied its contribution at the origin, now let us see the nature of the tangent vector at another point $\frac{d}{ds}\varphi\big(\Gamma(0), \Gamma(s)\big)\Big|_{s=s_0}$. We already know that the binary quartic that it represents is obtained by inserting the polynomial parametrization $\rho\big(\Gamma(0)\big)$ into the implicit equation of the double line $\frac{d}{ds}\Gamma(s)\Big|_{s=s_0}$ which is tangent to $\Gamma(s_0)$ at $\gamma(s_0)$. From the previous result, we can assume that the conic $\Gamma(0)$ is entirely contained in the inside of the conic $\Gamma(s_0)$, up to changing the direction in which we traverse the arc $\gamma$.

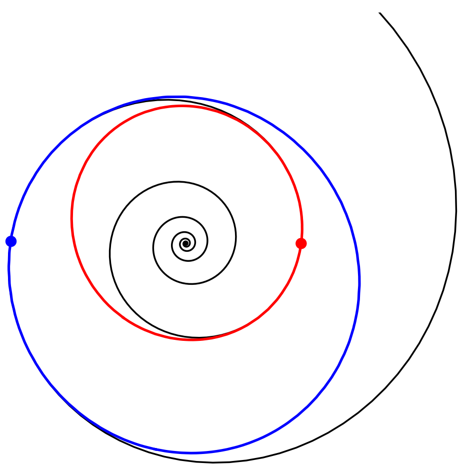

Figure 4.2: The osculating conics are nested.

Since $\Gamma(0)$ is contained within $\Gamma(s_0)$, there cannot be any intersection between the first conic and any line tangent to the second. This already shows us that

$\frac{d}{ds}\varphi\Big(\Gamma(0), \Gamma(s)\Big)\Big|_{s=s_0} \in P_{2,4}$ is a positive quartic, and with the help of a lemma about convex functions, we will show that it is also convex.

**Lemma 4.4.1.** *Let $f$ be a real convex function, and let $g$ be a real non-decreasing and convex function. Then the composite $g \circ f$ is also convex.*

*Proof.* Take $\lambda \in [0, 1]$ and $x, y$ in the domain of $f$. Then it holds that:

$$g\Big(f(\lambda x + (1-\lambda)y)\Big) \leq g\Big(\lambda f(x) + (1-\lambda)f(y)\Big) \leq \lambda g\Big(f(x)\Big) + (1-\lambda)g\Big(f(y)\Big).$$

Therefore $g \circ f$ is convex.

□

Let us now apply the lemma in our context. We have the following function to analyse:

$$\frac{d}{ds}\varphi\Big(\Gamma(0), \Gamma(s)\Big)\Big|_{s=s_0} = \frac{d}{ds}\Gamma(s)\Big|_{s=s_0} \circ \rho\Big(\Gamma(0)\Big).$$

As a binary quartic, consider it as a function from $\mathbb{R}^2$ to $\mathbb{R}$. We know that $\frac{d}{ds}\Gamma(s)\Big|_{s=s_0}$ is the implicit equation of a double line of the projective plane, so it is of the form:

$$\frac{d}{ds}\Gamma(s)\Big|_{s=s_0} = (Ax + By + Cz)^2, \quad A, B, C \in \mathbb{R}.$$

While $\rho\Big(\Gamma(0)\Big)$ is a polynomial parametrization of $\Gamma(0)$, which we can write as $\rho\Big(\Gamma(0)\Big) = \Big(x_0(t,w),\ x_1(t,w),\ x_2(t,w)\Big)$. In the end we have the expression:

$$\frac{d}{ds}\varphi\Big(\Gamma(0), \Gamma(s)\Big)\Big|_{s=s_0} = \Big(Ax_0(t,w) + Bx_1(t,w) + Cx_2(t,w)\Big)^2.$$

In order to apply lemma 4.4.1, we shall consider the map above as the composite of the following functions:

$$\begin{aligned} &g : [0, \infty) \to \mathbb{R} && f : \mathbb{R}^2 \to \mathbb{R} \\ &g(x) = x^2 && f(t,w) = Ax_0(t,w) + Bx_1(t,w) + Cx_2(t,w) \end{aligned}$$

The function $f$ is the binary quadratic form produced by inserting the polynomial parametrization of $\Gamma(0)$ into the implicit equation of the tangent line to $\Gamma(s_0)$ at $\gamma(s_0)$. Since these curves do not intersect, we know that $f$ has no real roots, so its discriminant is negative. Therefore, we can conclude that this function is positive and convex, up to taking $-f$ instead if necessary. Now both functions $g$ and $f$ follow the hypothesis of the lemma, and thus we conclude that $\frac{d}{ds}\varphi\Big(\Gamma(0),\Gamma(s)\Big)\Big|_{s=s_0}$ is a positive and convex quartic as we claimed.

There is only one step left to prove Theorem 4.0.1. The quartic that compares two osculating conics may be obtained by the following integral:

$$\varphi\Big(\Gamma(0),\Gamma(s_0)\Big) = \int_0^{s_0} \frac{d}{ds}\varphi\Big(\Gamma(0),\Gamma(s)\Big)\, ds.$$

A linear combination with positive coefficients of nonnegative and convex functions is still a nonnegative and convex function. This fact is illustrated by the convexity of the $Q_{2.4}$ blender. So, this property extends to integrals and therefore we can conclude that $\varphi\Big(\Gamma(0),\Gamma(s_0)\Big) \in Q_{2,4}$ is a positive and convex quartic. This concludes the proof that the osculating conics of a smooth arc on the real projective plane without inflections or sextactic points are disjoint and convexly nested.

# 5

# Characterizations of the Convex Nesting

In Chapter 4, we have proved that the osculating conics of a smooth curve with no inflections or sextactic points are convexly nested. We now dive further and show, with the help of *algebraic invariants of binary quartics*, that the condition on the relative position of two very close osculating conics is even stronger. But first, we introduce a technique to narrow down our moduli space of conics of signature $(1, 2)$. By making good use of the $\mathrm{PGL}(3; \mathbb{R})$ action, we may take any pair of conics to a normal form where one of them becomes $\mathfrak{c} = -x^2 - y^2 + z^2$ and the other is mapped to $-\alpha x^2 - \beta y^2 + z^2$. We call this procedure the *simultaneous diagonalization* of the pair of conics. Notice how it is consistent with respect to dimension counting. Each conic is given projectively by 5 degrees of freedom, while the group $\mathrm{PGL}(3; \mathbb{R})$ is 8−dimensional. So, in the end, only 2 parameters remain to be determined, which are $\alpha$ and $\beta$ in the expression of the second conic. We have thus managed to reduce the dimension of the moduli space to 2, making the ensuing analysis much simpler.

## 5.1

## Simultaneous diagonalization

In this section, we make full use of the projective group $\mathrm{PGL}(3; \mathbb{R})$ in order to map a pair of non-intersecting conics of $\mathbb{RP}^2$ to a particular configuration that facilitates our study of their relative position. The following lemma is found in [Ghys2] on page 22 and is attributed to Milnor.

**Lemma 5.1.1.** *Let $U$ and $V$ be two symmetric $n \times n$ matrices with $n \geq 3$. Assume that there is no vector $\mathbf{v} \in \mathbb{R}^n$ which is simultaneously isotropic for $U$ and $V$, that is such that $\mathbf{v}^\mathsf{T} U \mathbf{v} = \mathbf{v}^\mathsf{T} V \mathbf{v} = 0$. Then there is a basis in which both $U$ and $V$ are diagonal.*

In essence, this lemma states that if there is no common isotropic vector for $U$ and $V$, then there exists a matrix $A \in \mathrm{GL}(n; \mathbb{R})$ such that $A^\mathsf{T} U A$ and $A^\mathsf{T} V A$ are both diagonal matrices. Let us now apply this result in our setting of conics in $\mathbb{RP}^2$. A conic $u$ is a quadratic form, for which we may associate a $3 \times 3$ symmetric matrix $U$. A vector $\mathbf{v} = (x, y, z)$ is isotropic for $U$ if and only if the point $(x : y : z) \in \mathbb{RP}^2$ belongs to the conic $u$. Therefore, if a pair of conics $u$ and $v$ have no intersection in $\mathbb{RP}^2$, then there exists a projective transformation $A \in \mathrm{PGL}(3; \mathbb{R})$ that simultaneously diagonalizes the associated symmetric matrices.

In addition, by appropriately rescaling the $x$ and $y$ axis, we may map the conic $u$ to $\mathfrak{c}$, while $v$ is still represented by a diagonal matrix. If $v$ is non-degenerate, then the 3 coefficients present in the diagonal cannot be 0, and since every multiple $\lambda v$ represents the same conic, we may set the coefficient of $z^2$ to be 1. In summary, up to a projective transformation we have that $A.u = \mathfrak{c} = -x^2 - y^2 + z^2$ and $A.v = -\alpha x^2 - \beta y^2 + z^2$, with $\alpha \neq 0$ and $\beta \neq 0$. Moreover, since there can be no intersection between $\mathfrak{c}$ and $A.v$, there are only three different possibilities: i) Both $\alpha$ and $\beta$ are strictly greater than 1, in which case $\mathfrak{c}$ is in the component isomorphic to the Möbius band and thus $\varphi(\mathfrak{c}, v)$ is negative. ii) Both $\alpha$ and $\beta$ are between 1 and 0, in which case $\mathfrak{c}$ is in the component isomorphic to the disc and hence $\varphi(\mathfrak{c}, v)$ is positive. iii) Either $\alpha$ or $\beta$ is between 1 and 0 and the other is negative, in which case the signature of $v$ is $(2, 1)$, so we should instead consider $-v$ and, fittingly, $\mathfrak{c}$ is again in the component isomorphic to the Möbius band.

From the point of view of the group action, we have used 5 degrees of freedom of $\mathrm{PGL}(3; \mathbb{R})$ to map $u$ to $\mathfrak{c}$ and then the other 3 degrees of freedom of $\mathrm{PO}(2, 1)$ (the stabilizer of $\mathfrak{c}$, presented in more details in subsection 6.1.3) to constrain $v$ to a particular form, where it is no longer defined by 5 parameters, but only 2, namely $\alpha$ and $\beta$. Notice that in the affine chart $z = 1$ the conic $A.v$ is centered at $(0, 0)$ and its axes of symmetry coincide with the $x$ and $y$ axes.

A corresponding simplification also happens in the algebraic setting, where the $\mathrm{PGL}(2; \mathbb{R})$ action may take $\varphi(u, v)$ to a particular normal form. Assuming without loss of generality that $v = -\alpha x^2 - \beta y^2 + z^2$, we have that $\varphi(u, v) = (1 - \alpha)t^4 + (2\alpha - 4\beta + 2)t^2 w^2 + (1 - \alpha)w^4$. Since $\alpha \neq 1$, we may divide the quartic by $1 - \alpha$ and get a representative of the form:

$$\varphi(u, v) = t^4 + \tfrac{2\alpha - 4\beta + 2}{1 - \alpha} t^2 w^2 + w^4 = t^4 + 6\lambda t^2 w^2 + w^4, \quad \text{where } \lambda = \tfrac{\alpha - 2\beta + 1}{3(1 - \alpha)}. \tag{5.1}$$

The normal form $f_\lambda(t, w) = t^4 + 6\lambda t^2 w^2 + w^4$ is present in Bruce Reznick's article [Reznick]. There he explains that, generically, any binary quartic is equivalent under the PGL(3; $\mathbb{C}$) action to one such $f_\lambda$ for some $\lambda \in \mathbb{R}$. The term "generically" is necessary because some particular quartics, as $p(t, w) = t^4$, are not equivalent to any such $f_\lambda$. Also, we do not want to use complex numbers in the change of variable. Fortunately, in the context of the non-intersecting conics $\mathfrak{c}$ and $v$, we have already seen that the quartic $\varphi(\mathfrak{c}, v)$ is of the desired normal form and we even know how the parameter $\lambda$ is related to the coefficients of the conic $v$.

Let us recall some results from Reznick's paper that we have presented in section 2.2.7. Any strictly positive binary quartic is equivalent to $f_\lambda$ for some $\lambda$ in the interval $]-1/3, 1/3]$. Notice that when $\alpha = \beta$, we have the case in which $\lambda = 1/3$, and when $\beta \to 1$, then $\lambda \to -1/3$. The case where $\alpha > 1$, $\beta > 1$ and $\alpha \to 1$ at first glance would result in $\lambda \to \infty$, but we are soon going to see that this is equivalent to $\lambda \to -1/3$. The quartic $t^4 + 6\lambda t^2 w^2 + w^4$ is nonnegative if and only if $\lambda \geq -1/3$, because in this case, for any pair $(t, w)$, its image is greater than $t^4 - 2t^2w^2 + w^4 = (t^2 - w^2)^2 \geq 0$, and we have the opposite inequality if $\lambda < -1/3$, so the image of $(1, 1)$ is negative. Now, to see that we only need to consider $\lambda \leq 1/3$, let us define $g_\lambda(t, w) := f_\lambda(t+w, t-w)$. Then we have that:

$$
\begin{aligned}
g_\lambda(t, w) &= (t + w)^4 + 6\lambda(t + w)^2(t - w)^2 + (t - w)^4 \\
&= (1 + 6\lambda + 1)t^4 + (6 - 12\lambda + 6)t^2w^2 + (1 + 6\lambda + 1)w^4 \\
&= (2 + 6\lambda)\left(t^4 + 6\frac{1 - \lambda}{1 + 3\lambda}t^2w^2 + w^2\right).
\end{aligned}
$$

$$
g_\lambda(t, w) = (2 + 6\lambda)f_{\Theta(\lambda)}, \text{ where } \Theta(\lambda) = \frac{1 - \lambda}{1 + 3\lambda}.
$$

The calculation above shows that $f_\lambda \sim f_{\Theta(\lambda)}$, so it is useful to study this rational function $\Theta(\lambda)$. Some of its remarkable properties are: $\Theta$ is an involution, that is $\Theta(\Theta(\lambda)) = \lambda$; $\Theta(0) = 1$; $\Theta$ is injective; $\Theta(1/3) = 1/3$; $\lim_{\lambda \to -1/3^+} \Theta(\lambda) = +\infty$. This last property justifies our previous claim about the case where $\alpha > 1$, $\beta > 1$ and $\alpha \to 1$, as from the perspective of the quartics we have that $t^2w^2 \sim (t^2 - w^2)^2$. Notice, above all, that $\Theta$ is a 1-1 decreasing map from $[1/3, +\infty[$ to $]-1/3, 1/3]$, this explains why it is enough to concentrate our attention to the $]-1/3, 1/3]$ interval, as we do not get any new class after $\lambda = 1/3$.

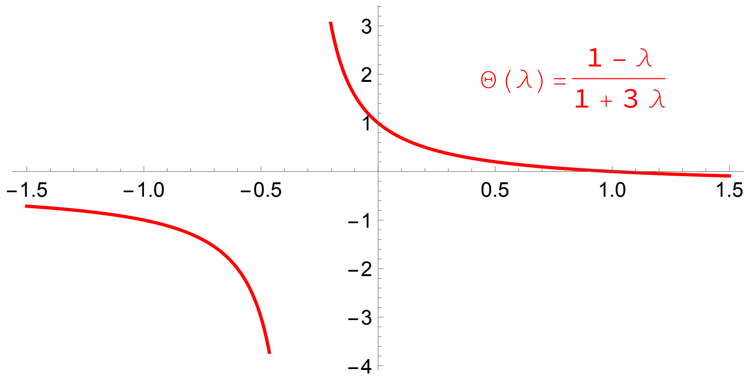


Figure 5.1: The graph of $\Theta(\lambda)$ with $\lambda \in [-1.5, 1.5]$.

To conclude his proof, Reznick presents an algebraic manipulation to show that any positive binary quartic can be put into the normal form $f_\lambda$. Since we have previously shown that the pencils of conic through an arbitrary non-degenerate conic are in bijection with the binary quartics, we have accomplished the same result as Reznick but with a geometric approach by applying Milnor's simultaneous diagonalization theorem to two non-intersecting conics.

Going back to the normal form $f_\lambda = t^4 + 6\lambda t^2 w^2 + w^4$, we wish to know for which values of $\lambda$ the quartic is nonnegative and convex. Reznick answers this question stating that $f_\lambda$ belongs to the blender $K_{2,4}$ if and only if $\lambda \in [0, 1]$. His prove consists in verifying for which values of $\lambda$ the *catalecticant* of $f_\lambda$ is positive semi-definite. But one may obtain the same result by studying the Hessian directly. The Hessian of $f_\lambda$ is:

$$H_{f_\lambda} = \begin{pmatrix} 12t^2 + 12\lambda w^2 & 24\lambda tw \\ 24\lambda tw & 12\lambda t^2 + 12w^2 \end{pmatrix}.$$

We already know that the quartic is nonnegative if and only if $\lambda \geq -1/3$. It is also convex if and only if the Hessian above is positive semi-definite, which is true if and only if its principal minors are all nonnegative. For $12t^2+12\lambda w^2$ and $12\lambda t^2 + 12w^2$ to be nonnegative, we just need $\lambda \geq 0$. The determinant of the Hessian is $144\lambda \left(t^4 + \frac{1-3\lambda^2}{\lambda} t^2 w^2 + w^4\right)$. From what we have already explained, the determinant is nonnegative for all values of $(t, w)$ if and only if $\frac{1-3\lambda^2}{\lambda} \geq -2$, which is true for $\lambda \leq -1/3$ or $0 < \lambda \leq 1$. When $\lambda = 0$, the Hessian is simply $144t^2w^2$, which is nonnegative. In conclusion, $f_\lambda$ is nonnegative and convex if and only if $\lambda \in [0, 1]$.

### 5.1.1

### Normal form and the cross-ratio of the roots of the quartic

Another possible approach to analyse the quartic $f_\lambda$ is by studying its roots and then applying what we have developed in section 2.3. Recall that a positive binary quartic is convex if and only if the cross-ratio of its roots belongs to the interval $[1, 2]$. We notice first that $f_\lambda = 0$ is a biquadratic equation, so we can easily compute its roots. They are $(t_j, 1)$, $j \in \{1, 2, 3, 4\}$ where

$$\begin{aligned} t_1 &= \sqrt{-3\lambda + \sqrt{9\lambda^2 - 1}}\,; & t_2 &= \sqrt{-3\lambda - \sqrt{9\lambda^2 - 1}}\,; \\ t_3 &= -\sqrt{-3\lambda - \sqrt{9\lambda^2 - 1}}\,; & t_4 &= -\sqrt{-3\lambda + \sqrt{9\lambda^2 - 1}}\,. \end{aligned}$$

We highlight that when $\lambda = \pm 1/3$, the term $9\lambda^2 - 1$ vanishes, so they correspond to quartics with a pair of double roots. If $\lambda = 1/3$, then we have the quartic $f_{1/3} = t^4 + 2t^2w^2 + w^4 = (t^2 + w^2)^2$, whose roots are $t_j = \pm i$. This corresponds to having $\alpha = \beta$, so $\mathfrak{c}$ and $v$ are concentric circles in the affine chart $z = 1$. If $\lambda = -1/3$, then we have $f_{-1/3} = t^4 - 2t^2w^2 + w^4 = (t^2 - w^2)^2$, whose roots are real $t_j = \pm 1$. This happens in the case $\beta = 1$ and geometrically, it means that $\mathfrak{c}$ and $v$ have two intersections where they are tangent. Notice that it is also the case for $\alpha = 1$, so these two circumstances are equivalent, as we have previously shown. The other way in which $f_\lambda$ could have double roots would be if $-3\lambda \pm \sqrt{9\lambda^2 - 1} = 0$. However, this is impossible, since this equation implies $9\lambda^2 = 9\lambda^2 - 1$.

Since we are focusing on the positive quartics, it is enough to consider $\lambda \in ]-1/3, 1/3[$ so that all four roots are not real and distinct. In order to apply what we have developed in section 2.3, we must identify the four roots as $z_1, \overline{z_1}, z_2$ and $\overline{z_2}$ with the condition that $z_1$ and $z_2$ belong to the upper half plane $\mathcal{H}$. By the definition of the principal branch of the square root for complex numbers, the roots $t_1$ and $t_2$ are the ones that belong to $\mathcal{H}$, so we define $z_1 \coloneqq t_1$ and $z_2 \coloneqq t_2$, recalling that the order of these two roots does not impact the cross-ratio.

$$(z_1, z_2; \overline{z_2}, \overline{z_1}) = \frac{(z_1 - \overline{z_2})(z_2 - \overline{z_1})}{(z_1 - \overline{z_1})(z_2 - \overline{z_2})}.$$

The four roots of $f_\lambda$ have a remarkable property: they form a rectangle in the complex plane. Since the quartic has only real coefficients, the roots come in conjugate pairs, and as they are not purely imaginary, one has necessarily that $t_3 = -z_1 = \overline{z_2}$ and $t_4 = -z_2 = \overline{z_1}$. Therefore, the cross-ratio we shall compute is $(z_1, z_2; \overline{z_2}, \overline{z_1}) = (t_1, t_2; t_3, t_4)$.

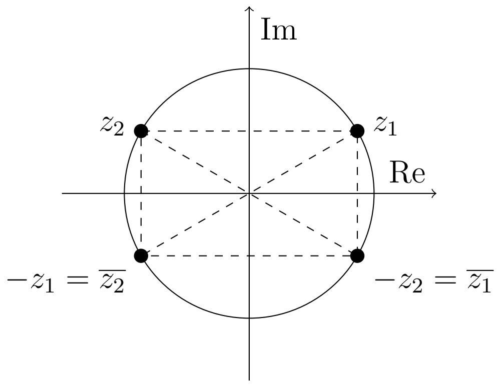


Figure 5.2: Rectangular arrangement of the roots of $f_\lambda$.

The particular arrangement of the roots simplifies the computation of the cross-ratio, because for the numerator we have $z_1 - \overline{z_2} = 2z_1$ and $z_2 - \overline{z_1} = 2z_2$. In addition, the arguments of $z_1$ and $z_2$ must add to $\pi$ due to the symmetry with respect to the imaginary axis, and all four roots have absolute value $|t_j| = 1$. This last fact can be obtained by noticing that the coefficients of $t^4$ and $w^4$ of the quartic $f_\lambda$ are always 1, so that the product of the roots must equal 1. Due to the rectangular symmetry of the roots, they must all have the same absolute value, which must thus be 1. Alternatively, one could just do a simple calculation, for example for $t_1$, it is the square root of a complex number of absolute value 1, because $t_1^2 = -3\lambda + i\sqrt{1 - 9\lambda^2}$, so $|t_1^2|^2 = 9\lambda^2 + 1 - 9\lambda^2 = 1$. Therefore, the numerator is $2z_1.2z_2 = -4$.

For the denominator, we have $z_1 - \overline{z_1} = z_1 + z_2$ and $z_2 - \overline{z_2} = z_2 + z_1$. The real parts of $z_1$ and $z_2$ are opposites, so they cancel out, while the imaginary parts are the same, so they add up and we get $(z_1 + z_2)^2 = (2i\,\mathrm{Im}(z_1))^2 = -4(\mathrm{Im}(z_1))^2$. Therefore, the cross-ratio is given by $(z_1, z_2; \overline{z_2}, \overline{z_1}) = 1/(\mathrm{Im}(z_1))^2$. To conclude, it remains for us to express the imaginary part of $z_1$ in terms of the parameter $\lambda$. As $z_1^2 = -3\lambda + \sqrt{9\lambda^2 - 1}$ and in general $z_1^2 = (\mathrm{Re}(z_1)^2 - \mathrm{Im}(z_1)^2) + 2i\,\mathrm{Re}(z_1)\,\mathrm{Im}(z_1)$, we have that:

$$\begin{cases} \mathrm{Re}(z_1)^2 - \mathrm{Im}(z_1)^2 = -3\lambda \, ; \\ 2i\,\mathrm{Re}(z_1)\,\mathrm{Im}(z_1) = \sqrt{9\lambda^2 - 1} \, . \end{cases}$$

We are interested in finding $(\mathrm{Im}(z_1))^2$, so let us isolate $\mathrm{Re}(z_1)$ in the second equation and substitute it in the first one.

$$2i\,\mathrm{Re}(z_1)\,\mathrm{Im}(z_1) = \sqrt{9\lambda^2-1} \iff \mathrm{Re}(z_1) = \frac{\sqrt{1-9\lambda^2}}{2\,\mathrm{Im}(z_1)}\ ;$$
$$\frac{1-9\lambda^2}{4(\mathrm{Im}(z_1))^2} - (\mathrm{Im}(z_1))^2 = -3\lambda \iff -4(\mathrm{Im}(z_1))^4 + 12\lambda(\mathrm{Im}(z_1))^2 + 1 - 9\lambda^2 = 0\ .$$

Denoting $(\mathrm{Im}(z_1))^2$ by $Y$, we see that it satisfies the quadratic equation $-4Y^2+12\lambda Y+1-9\lambda^2=0$, so $Y = \frac{-12\lambda\pm\sqrt{144\lambda^2+16(1-9\lambda^2)}}{-8} = \frac{-12\lambda\pm\sqrt{16}}{-8} = \frac{3\lambda\pm1}{2}$. Since $Y$ is the square of a real number, it must be positive, thus $Y = \frac{3\lambda+1}{2}$. In conclusion, the cross-ratio of the roots of $f_\lambda$ is $(t_1,t_2;t_3,t_4) = \frac{2}{3\lambda+1}$. This is a strictly decreasing function in the interval $]-1/3,1/3[$ and we attain the same conclusion about the convex quartics, because the cross-ratio is 2 when $\lambda=0$ and, although it is not defined for $\lambda = 1/3$ because there are double roots, its value converges to 1 as $\lambda$ goes to $1/3$. Therefore, for $\lambda$ in the interval $[0,1/3]$ we have positive convex quartics $f_\lambda$, and as $f_\lambda \sim f_{\Theta(\lambda)}$ under linear change of variables, we reach the claim that the quartic in normal form $f_\lambda$ is nonnegative and convex if and only if $\lambda\in[0,1]$.

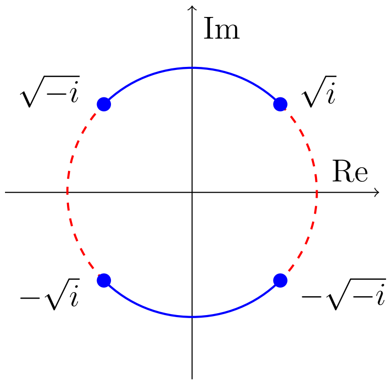


Figure 5.3: For $\lambda\in[-1/3,1/3]$, $f_\lambda$ is positive and convex if and only if its roots lie in the blue region.

For the sake of completeness, we also explain what happens to the roots of $f_\lambda$ for $\lambda\in[1/3,1]$. When $\lambda>1/3$, the term $9\lambda^2-1$ in the square root of all roots $t_j$ is positive, so $\sqrt{9\lambda^2-1}$ is a positive real number, but $-3\lambda+\sqrt{9\lambda^2-1}$ is still negative. Therefore, all four roots are purely imaginary, and as $\lambda$ ranges from $1/3$ to 1, the roots start as double roots at $i$ and $-i$, then they part ways going in opposite directions in the imaginary axis. At the end, when $\lambda=1$, the four roots reach:

$$t_1 = \sqrt{-3+2\sqrt{2}} = i\left(\sqrt{2}-1\right); \qquad t_2 = \sqrt{-3-2\sqrt{2}} = i\left(\sqrt{2}+1\right);$$
$$t_3 = -\sqrt{-3-2\sqrt{2}} = -i\left(\sqrt{2}+1\right); \quad t_4 = -\sqrt{-3+2\sqrt{2}} = -i\left(\sqrt{2}-1\right).$$

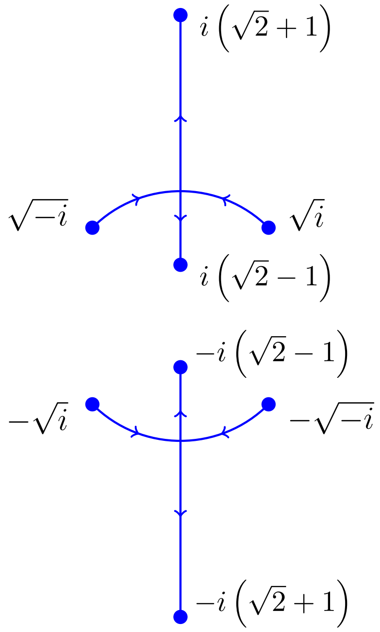


Figure 5.4: Paths of the roots of $f_\lambda$ as $\lambda$ goes from 0 to 1.

## 5.2

## Geometric characterization of the convex nesting

In the previous section, we have established an algebraic characterization for a conic $u$ to be convexly nested into another conic $v$, namely their associated quartic $\varphi(u,v)$ must be in the same class as $f_\lambda$ for some $\lambda \in [0,1]$. Now we move to the following geometric characterization, also making use of the simultaneous diagonalization.

**Theorem 5.2.1.** *Let $u$ be a non-degenerate conic in $\mathbb{RP}^2$. Then $u$ is convexly nested into another conic $v$ if and only if they do not intersect and, after realizing a simultaneous diagonalization taking $u$ to $-x^2-y^2+z^2$ and $v$ to $-\alpha x^2 - \beta y^2 + z^2$, the pair $(\alpha, \beta)$ belongs to the open square $]0,1[^2$ and satisfies $\frac{1}{2} \leq \frac{\beta-1}{\alpha-1} \leq 2$.*

*Proof.* As we have explained, if $u$ and $v$ do not intersect, it is always possible to find a projective transformation that maps $u$ to $-x^2-y^2+1$ and $v$ to $-\alpha x^2 - \beta y^2 + 1$. Moreover, the only case in which $u$ is in the component defined by $v$ that is homeomorphic to the disc is when both $\alpha$ and $\beta$ belong to the interval $]0,1[$, hence we have the first constraint on $v$ in order to have the convex nesting.

This process implies, as shown in equation (5.1), that the quartic $\varphi(u,v)$ is equivalent to $f_\lambda$ where $\lambda = \frac{\alpha-2\beta+1}{3(1-\alpha)}$. Therefore, we may use the algebraic result about $\lambda$ in order to find the condition with respect to $\alpha$ and $\beta$ for the conic $u$ to be convexly nested in $v$. Recall that the restriction is that $0 \leq \lambda \leq 1$.

First, $\lambda \geq 0$ gives us $\frac{\alpha-2\beta+1}{3(1-\alpha)} \geq 0$, which is equivalent to $\beta \leq \frac{\alpha+1}{2}$. Now the condition $\lambda \leq 1$ yields $\frac{\alpha-2\beta+1}{3(1-\alpha)} \leq 1$, which is equivalent to $\beta \geq 2\alpha - 1$. These restrictions are easier to understand if we plot the corresponding regions in the $(\alpha,\beta)$ plane.

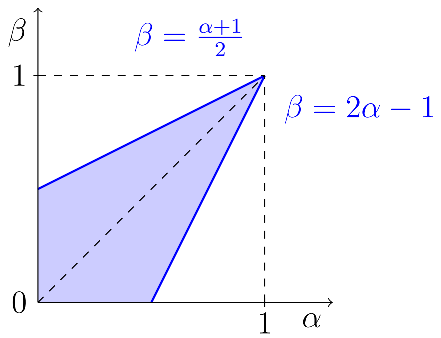


Figure 5.5: $u$ is convexly nested in $v$ for pairs $(\alpha,\beta)$ in the blue region.

By examining Figure 5.5, one may realize that $\beta(\alpha) = 2\alpha - 1$ is the inverse function of $\beta(\alpha) = \frac{\alpha+1}{2}$. This is to be expected, since up to a reflection in the $(x,y)$ plane, which is an element of $\mathrm{PO}\,(2,1)$, we may interchange the roles of $\alpha$ and $\beta$, and this results in reflecting the $(\alpha,\beta)$ plane with respect to the identity. The effect of the reflection also allows us to consider only the cases where $\beta \geq \alpha$, reducing the moduli space by half.

Notice that the set of conics into which $u$ is convexly nested is one side of a cone with vertex in $u$. The region of interest is given precisely by the inequalities of the claim, that is:

$$\frac{1}{2} \leq \frac{\beta - 1}{\alpha - 1} \leq 2. \tag{5.2}$$

$\square$

### 5.2.1

### Pencils of conics in the moduli space

In this subsection we investigate how the pencils of conics manifest themselves in the moduli space of simultaneously diagonalized conics. We claim that they are simply the lines joining the two points that represent the respective conics. To see this, let $u = -\alpha_u x^2 - \beta_u y^2 + z^2$ and $v = -\alpha_v x^2 - \beta_v y^2 + z^2$ be two distinct conics. An arbitrary conic in the pencil $\overline{uv}$ is given by:

$$\begin{aligned}\lambda u + \mu v &= -(\lambda\alpha_u + \mu\alpha_v)x^2 - (\lambda\beta_u + \mu\beta_v)y^2 + (\lambda + \mu)z^2 \\ &\sim -\frac{\lambda\alpha_u + \mu\alpha_v}{\lambda + \mu}x^2 - \frac{\lambda\beta_u + \mu\beta_v}{\lambda + \mu}y^2 + z^2.\end{aligned}$$

Therefore, the conic $\lambda u + \mu v$ is represented by the point $\left(\frac{\lambda\alpha_u+\mu\alpha_v}{\lambda+\mu}, \frac{\lambda\beta_u+\mu\beta_v}{\lambda+\mu}\right)$ in the moduli space. Let us show that this point belongs to the line that joins $u$ and $v$. By applying a translation by $(-\alpha_u, -\beta_u)$, which takes $u$ to the origin, we get

$$\frac{\lambda\alpha_u + \mu\alpha_v}{\lambda + \mu} - \alpha_u = \frac{\mu(\alpha_v - \alpha_u)}{\lambda + \mu}, \qquad \frac{\lambda\beta_u + \mu\beta_v}{\lambda + \mu} - \beta_u = \frac{\mu(\beta_v - \beta_u)}{\lambda + \mu}.$$

Notice that the vector $\left(\frac{\mu(\alpha_v-\alpha_u)}{\lambda+\mu}, \frac{\mu(\beta_v-\beta_u)}{\lambda+\mu}\right)$ is a multiple of $(\alpha_v - \alpha_u, \beta_v - \beta_u)$, which is the direction vector of the line through the points representing $u$ and $v$. Hence, for every $(\lambda : \mu) \in \mathbb{RP}^1$, the point representing $\lambda u + \mu v$ is obtained by adding to the point representing $u$ a multiple of the suitable direction vector, showing that the conics associated to the points in the line through $u$ and $v$ are precisely the pencil of conics $\overline{uv}$.

$$\left(\frac{\lambda\alpha_u + \mu\alpha_v}{\lambda + \mu}, \frac{\lambda\beta_u + \mu\beta_v}{\lambda + \mu}\right) = (\alpha_u, \beta_u) + \frac{\mu}{\lambda + \mu}\left(\alpha_v - \alpha_u, \beta_v - \beta_u\right).$$

Now that we know that lines in the moduli space of simultaneously diagonalized conics correspond to pencil of conics, we can interpret the cones of convex nesting in another way. By definition, the cone is a union of segments, all having one extremity at the vertex, which represents a given conic $u$. So the cone corresponds to a certain set of pencils containing $u$, and the quartic $\varphi(u, v)$ is constant when the second conic $v$ ranges along any segment coming from $u$.

## 5.3

### Algebraic invariants of binary quartics

The simultaneous diagonalization has managed to reduce the problem of understanding the projective relative position of two conics to a single parameter, as we only need to verify to which $\mathrm{PGL}(2;\mathbb{R})$ class the quartic $\varphi(u,v)$ belongs. Once it is put in the normal form $t^4 + 6\lambda t^2 w^2 + w^4$, we just have to check whether $\lambda \in [0,1]$ or not. However, it can be cumbersome to find the suitable linear change of variables that sends $\varphi(u,v)$ to the normal form. Fortunately, this is not necessary, as we may use *algebraic invariants* to determine the class of $\varphi(u,v)$. Here we employ the classical invariant theory, specifically targeting the binary quartics, which is our focus of interest. For a more detailed source on the subject, we refer to [Mukai].

As previously explained in Chapter 2, the set of real binary quartics $F_{2,4}$ is a 5-dimensional vector space, because 5 coefficients must be determined in order to define such a polynomial. We customarily include binomial coefficients in the presentation of the forms, so $p(t,w) = \alpha t^4 + 4\beta t^3 w + 6\gamma t^2 w^2 + 4\delta t w^3 + \varepsilon w^4$ is a generic binary quartic.

Consider now the $\mathrm{GL}(2;\mathbb{R})$ action on $F_{2,4}$ by linear change of coordinates. By changing $(t,w)$ for $(at+bw, ct+dw)$, we may get a new quartic whose coefficients are $\alpha', \beta', \gamma', \delta', \varepsilon'$. An *invariant of weight k* is a polynomial $P(\alpha,\beta,\gamma,\delta,\varepsilon)$ on the coefficients of the form such that, for any $A \in \mathrm{GL}(2;\mathbb{R})$, it holds that

$$P(\alpha',\beta',\gamma',\delta',\varepsilon') = \det(A)^k P(\alpha,\beta,\gamma,\delta,\varepsilon).$$

One may also consider *rational invariants* by admitting rational functions on the coefficients. We are soon going to do so in order to construct weight 0 invariants, the so called *absolute invariants*. It is crucial to our purposes that the invariants we use be absolute, because in our context of conics we deal with the $\mathrm{PGL}(2;\mathbb{R})$ action on $\mathbb{P}(F_{2,4})$, and only absolute invariants are well defined in this setting.

The binary quartics admit infinitely many invariants, but they are all generated from two basic invariants:

$$\begin{aligned} S &= \alpha\varepsilon - 4\beta\delta + 3\gamma^2 && \text{(weight 4);} \\ T &= \alpha\gamma\varepsilon + 2\beta\gamma\delta - \alpha\delta^2 - \beta^2\varepsilon - \gamma^3 && \text{(weight 6).} \end{aligned}$$

The *discriminant* $\Delta$, for example, is an invariant of weight 12 that vanishes if and only if the quartic has at least one multiple root. It is given in terms of $S$ and $T$ by:

$$\Delta = S^3 - 27T^2.$$

The most well-known absolute rational invariant is called the *J-invariant*. It is given by:

$$J = \frac{S^3}{\Delta} = \frac{S^3}{S^3 - 27T^2} = 1 + \frac{27T^2}{\Delta}.$$

Being an absolute invariant, it is natural to expect $J$ to be related to projective properties of the binary quartic, such as the cross-ratio $\xi$ of its roots. Indeed, Poston and Stewart present in their paper [Poston] the following formula that yields the $J$-invariant as a function of the cross-ratio of the roots $\xi$:

$$J(\xi) = \frac{4(\xi^2 - \xi + 1)^3}{27\xi^2(\xi - 1)^2}.$$

The function $J(\xi)$ has some interesting symmetries. A generic binary quartic has 4 distinct roots that do not have any particular ordering. As a consequence of this lack of ordering, one may obtain up to 6 different values for the cross-ratio by permuting the roots. In contrast, the quartic has a single well-defined value for its $J$-invariant. Thus the six values of the cross-ratio must result in the same $J$, more concretely:

$$J(\xi) = J(1-\xi) = J\left(\frac{1}{\xi}\right) = J\left(\frac{1}{1-\xi}\right) = J\left(\frac{\xi}{\xi-1}\right) = J\left(\frac{\xi-1}{\xi}\right).$$

Poston and Stewart's paper [Poston] focuses on the visualization of the $\mathrm{PGL}(2;\mathbb{C})$ orbits of $F_{2,4}$. They explain and illustrate the *foliation* of the space of binary quartics with respect to the $J$-invariant. Although of great help in understanding the subject, one must take care with the group at play. In our context, we are interested in the PGL$(2;\mathbb{R})$ action over $F_{2,4}$, which is more restricted than the PGL$(2;\mathbb{C})$ action, and thus yields a different set of orbits.

Since we have a normal form $f_\lambda = t^4 + 6\lambda t^2 w^2 + w^4$, with $\lambda > -1/3$ for the positive binary quartics, let us see how the $J$-invariant behaves as a function of the parameter $\lambda$. First, we obtain $S = 1 + 3\lambda^2$, $T = \lambda - \lambda^3$ and $\Delta = (1+3\lambda^2)^3 - 27(\lambda - \lambda^3)^2 = (1-9\lambda^2)^2$. Through a straight computation, we get:

$$J = \frac{S^3}{\Delta} = \frac{(1+3\lambda^2)^3}{(1-9\lambda^2)^2} = \frac{(1+3\lambda^2)^3}{(1+3\lambda)(1-3\lambda)}.$$

We have presented two ways to obtain the $J$-invariant, one using $\xi$, the cross-ratio of the roots, and the other via the parameter $\lambda$ of the normal form. Since the $J$ invariant is the same, this gives us an opportunity to verify the result we have obtained in section 5.1, where we proved that the cross-ratio of the roots of $f_\lambda$ is $\xi = \frac{2}{3\lambda+1}$. Indeed, if we substitute $\xi$ by $\frac{2}{3\lambda+1}$ in $\frac{4(\xi^2-\xi+1)^3}{27\xi^2(\xi-1)^2}$, we get the expected $\frac{(1+3\lambda^2)^3}{(1-9\lambda^2)^2}$.

From the formula above, one may observe that $J(\lambda)$ is an even function. If we use yet another formula for $J$, we get an equivalent algebraic expression that gives us some more information about $J(\lambda)$.

$$J = 1 + \frac{27T^2}{\Delta} = 1 + \frac{27(\lambda-\lambda^3)^2}{(1-9\lambda^2)^2}$$

$$\iff$$

$$\frac{J-1}{27} = \left(\frac{\lambda-\lambda^3}{1-9\lambda^2}\right)^2.$$

Here we see that the range of $J(\lambda)$ is $[1, +\infty]$ (considering that $J(-1/3) = J(1/3) = +\infty$), and that it attains its minimum at the three values of $\lambda$ for which the invariant $T$ is 0. They are $\lambda = -1$, $\lambda = 0$ and $\lambda = 1$. One may confirm all these properties in the graph of the function displayed below.

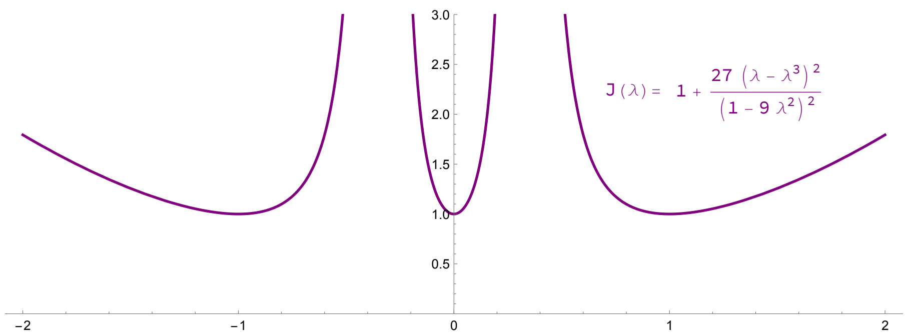


Figure 5.6: The graph of $J(\lambda)$ with $\lambda \in [-2, 2]$.

At this point we face a problem: more than one class of binary quartics share the same $J$-invariant. Indeed, for a regular value of $J$, there are 6 values of $\lambda$ that produce said image. As seen in section 5.1, these actually come in pairs, because $\lambda$ and $\Theta(\lambda)$ represent the same class. Moreover, it is enough to consider $\lambda \in ]-1/3, 1/3]$, since this interval corresponds to the *fundamental domain* for the $\mathrm{PGL}(2; \mathbb{R})$ action over the set of positive binary quartics, in other words, each orbit is associated to a single value of $\lambda$ in this interval. Nevertheless, the function $J(\lambda)$ is not injective in the interval $]-1/3, 1/3]$, so it is not enough to distinguish the orbits. Even worse, the same $J$ is associated to an orbit of convex quartics and to another of non-convex quartics. One possible solution would be to consider not only the invariant $J$, but also the sign of the invariant $T$, as it is positive for $\lambda \in ]0, 1/3]$ and negative for $\lambda \in [-1/3, 0[$. Reznick, however, preferred to use a different absolute invariant, one that contains a square root in its definition and which he named the $K$-invariant.

$$K := \frac{T}{S^{3/2}} = \frac{\lambda - \lambda^3}{\sqrt{(1+3\lambda^2)^3}}.$$

Reznick shows in his paper [Reznick] that $K(\lambda)$ is strictly increasing in the interval $[-1/3, 1/3]$, which implies that each class of positive binary quartics is associated to a single value of $K$. The range of $K(\lambda)$ is the interval $[-\sqrt{3}/9, \sqrt{3}/9]$, attaining its minimum at $\lambda = -1/3$ and its maximum at $\lambda = 1/3$. In addition, $K(\lambda)$ is an odd function, and any given positive quartic is convex if and only if its $K$-invariant is nonnegative. The graph below synthesizes and displays these informations.

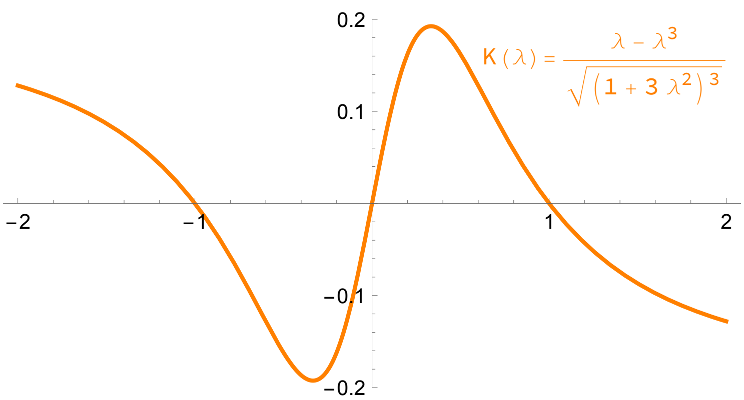


Figure 5.7: The graph of $K(\lambda)$ with $\lambda \in [-2, 2]$.

The relation between the invariants $K$ and $J$ is that $K^2 = \frac{1}{27}\left(1 - \frac{1}{J}\right) = \frac{T^2}{S^3}$.

With all this knowledge about the algebraic invariants of binary quartics and how they relate to the convexity of the forms, we shift our focus back to the osculating conics. In the next sections, we present a universal quadratic parametrization that suits any generic curve parametrized by the *projective length*, and use it to produce the first terms of the Taylor series of the algebraic invariants of the associated binary quartic. The concepts of projective length and projective curvature are better presented in Chapter 6. In what follows, we just use a particular parametrization that any smooth projective curve with no inflection or sextactic points admits.

## 5.4

## Algebraic invariants of quartics arising from osculating conics

In this section, we analyse the behaviour of a regular curve in $\mathbb{RP}^2$ away from an inflection or sextactic point. We prove that for a sufficiently small $\varepsilon > 0$, the $J$-invariant arising from the quartic related to the osculating conics at $s = 0$ and at $s = \varepsilon$ is close to $32/27$ and its $K$-invariant is close to $\sqrt{30}/72$. We reach these results by computing the first terms of the power series of these invariants:

**Theorem 5.4.1.** *For any regular curve, as long as there is no inflection or sextactic point, the power series of the $J$ and $K$ invariants are respectively $J(s) = \frac{32}{27} + \frac{8}{189}ks^2 + O(s^3)$ and $K(s) = \frac{\sqrt{30}}{72} + \frac{3\sqrt{30}}{2240}ks^2 + O(s^3)$, where $k$ is the projective curvature of the curve being analysed at $s = 0$.*

The proof of the theorem above will be given throughout this section.

### 5.4.1

### A general quadratic parametrization for osculating conics

Here we employ what may seem to be at first glance a specific condition on the parametrization of the curve, but, as we will show in section 6.1, it does not restrict the set of curves we are working with at all, because we may always find the suitable parametrization in the context of projective geometry. Consider a smooth curve $\mathbf{x}(s)$ satisfying the differential equation

$$\mathbf{x}'''(s) + 2k(s)\mathbf{x}'(s) + (k'(s) + 1)\mathbf{x}(s) = 0, \tag{5.3}$$

where $s$ is the projective length and $k$ is the projective curvature of the curve. Consider also the Frenet frame $\{\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2\}$ defined by:

$$\begin{cases} \mathbf{x}' = \mathbf{x}_1 \\ \mathbf{x}_1' = -k\mathbf{x} + \mathbf{x}_2 \\ \mathbf{x}_2' = -\mathbf{x} - k\mathbf{x}_1 \end{cases} \tag{5.4}$$

Let $u(s)$ denote the osculating conic of the curve at the point $\mathbf{x}(s)$. First we prove that the conic may be parametrized as a curve in $\mathbb{RP}^2$ by a particular quadratic expression involving the vectors of the Frenet Frame.

*Proposition* 5.4.2. The osculating conic $u(s_0)$ of $\mathbf{x}(s)$ at $s_0$ admits the parametrization $\mathbf{u}_{s_0} : \mathbb{RP}^1 \to \mathbb{RP}^2$ given by

$$\mathbf{u}_{s_0}(t, w) = \frac{1}{2}t^2\mathbf{x}_2(s_0) + tw\mathbf{x}_1(s_0) + w^2\mathbf{x}(s_0). \tag{5.5}$$

*Proof.* We need to show that the *contact* between the curves $\mathbf{x}(s)$ and $u(s_0)$ at their intersection is of order at least 5. To do so, let $U(s_0)$ be a $3 \times 3$ symmetric matrix such that the function $(x, y, z)U(s_0)(x, y, z)^\intercal = 0$ is the implicit equation of $u(s_0)$. Then, for all values of $(t, w) \in \mathbb{RP}^1$, we have that

$$\mathbf{u}_{s_0}(t, w)U(s_0)\mathbf{u}_{s_0}(t, w)^\intercal = 0. \tag{5.6}$$

In order to simplify the notation, we shall write: $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle := \mathbf{y}_1 U(s_0)\mathbf{y}_2^\intercal$. Also, since all calculations are at the value $s_0$, we omit it in the following expressions.

Taking $(t, w) = (0, 1)$ in equation (5.6) we obtain:

$$\langle \mathbf{x}, \mathbf{x} \rangle = 0. \tag{5.7}$$

Differentiating equation (5.6) with respect to $t$ and taking $(t, w) = (0, 1)$ we obtain:

$$\langle \mathbf{x}, \mathbf{x}_1 \rangle = 0. \tag{5.8}$$

Differentiating equation (5.6) two times with respect to $t$ and taking $(t, w) = (0, 1)$ we obtain:

$$\langle \mathbf{x}, \mathbf{x}_2 \rangle + \langle \mathbf{x}_1, \mathbf{x}_1 \rangle = 0. \tag{5.9}$$

Differentiating equation (5.6) three times with respect to $t$ and taking $(t, w) = (0, 1)$ we obtain:

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 0. \tag{5.10}$$

Finally differentiating equation (5.6) four times with respect to $t$ we obtain:

$$\langle \mathbf{x}_2, \mathbf{x}_2 \rangle = 0. \tag{5.11}$$

Now, consider the function $g(s) := \mathbf{x}(s)U(s_0)\mathbf{x}(s)^\intercal$. This function evaluates the contact of the conic $u(s_0)$ with the curve $\mathbf{x}(s)$ at the point $\mathbf{x}(s_0)$. More precisely, the *order* of its zero at $s = s_0$ indicates the order of contact between the two curves.

From equation (5.7), we know that $g(s_0) = 0$. Since $g'(s) = 2\langle \mathbf{x}(s), \mathbf{x}_1(s)\rangle$, from equation (5.8) we have that $g'(s_0) = 0$. The second derivative of $g$ is $g''(s) = 2\langle \mathbf{x}_1(s), \mathbf{x}_1(s)\rangle + 2\langle \mathbf{x}(s), \mathbf{x}_2(s) - k(s)\mathbf{x}(s)\rangle$. So, from equations (5.7) and (5.9), we have $g''(s_0) = 0$. The third derivative of $g$ is $g'''(s) = 6\langle \mathbf{x}_1(s), \mathbf{x}_2(s) - k(s)\mathbf{x}(s)\rangle + 2\langle \mathbf{x}(s), -(1 + k'(s))\mathbf{x}(s) - 2k(s)\mathbf{x}_1(s)\rangle$. Thus, from equations (5.7), (5.8) and (5.10), we obtain $g'''(s_0) = 0$. Finally, differentiating $g'''$ at $s_0$ directly, we get:

$$
\begin{aligned}
g^{iv}(s_0) &= 6\langle \mathbf{x}_2 - k\mathbf{x}, \mathbf{x}_2 - k\mathbf{x}\rangle + 6\langle \mathbf{x}_1, -\mathbf{x} - 2k\mathbf{x}_1 - k'\mathbf{x}\rangle \\
&\quad + 2\langle \mathbf{x}_1, -(1 + k')\mathbf{x} - 2k\mathbf{x}_1\rangle + 2\left\langle \mathbf{x}, (2k^2 - k'')\mathbf{x} - (1 + 3k')\mathbf{x}_1 - 2k\mathbf{x}_2\right\rangle.
\end{aligned}
$$

Using equations (5.7), (5.8), (5.9) and (5.11), we conclude that $g^{iv}(s_0) = 0$, thus proving that the order of contact between $\mathbf{x}(s)$ and $u(s_0)$ at $\mathbf{x}(s_0)$ is at least 5, which implies that $u(s_0)$ is indeed the osculating conic at the corresponding point. □

### 5.4.2

### The generic binary quartic of osculating conics and its invariants

With the quadratic parametrization found in subsection 5.4.1, we now compute the coefficients of the binary quartic related to the osculating conics at 0 and at an arbitrary $s$.

$$
\varphi(u(s), u(0))(t, w) = \mathbf{u}_s(t, w)U(0)\mathbf{u}_{s_0}(t, w)^\intercal. \tag{5.12}
$$

Since we have fixed the conic at 0 as the one that contributes with its implicit equation, we adapt the notation introduced in subsection 5.4.1 by setting $s_0 = 0$. In other words, we denote $\langle \mathbf{y}_1, \mathbf{y}_2\rangle = \mathbf{y}_1U(0)\mathbf{y}_2^\intercal$. Now we may use this notation and the quadratic parametrization 5.5 to express the coefficients of the quartic.

$$
\varphi(u(s), u(0))(t, w) = a(s)t^4 + 4b(s)t^3w + 6c(s)t^2w^2 + 4d(s)tw^3 + e(s)w^4, \tag{5.13}
$$

where

$$a(s) = \frac{1}{4}\langle \mathbf{x}_2(s), \mathbf{x}_2(s)\rangle\,, \quad d(s) = \frac{1}{2}\langle \mathbf{x}(s), \mathbf{x}_1(s)\rangle\,,$$
$$b(s) = \frac{1}{4}\langle \mathbf{x}_1(s), \mathbf{x}_2(s)\rangle\,, \quad e(s) = \langle \mathbf{x}(s), \mathbf{x}(s)\rangle\,,$$
$$c(s) = \frac{1}{6}\left(\langle \mathbf{x}(s), \mathbf{x}_2(s)\rangle + \langle \mathbf{x}_1(s), \mathbf{x}_1(s)\rangle\right).$$

Next, since we wish to determine the terms of the power series up to the order of $s^2$, we must compute three terms of the Taylor expansions of the coefficients at $s = 0$ starting from the first non-zero term. Notice that since $\varphi(u(0), u(0)) = 0$ is the identically zero form, every such expansions has zero constant term.

In the following computations, the definition of the Frenet Frame (5.4) and relations (5.7) through (5.11) are going to be very useful as we may write the derivatives of the coefficients of the quartic as a function of the other coefficients. In addition, let us define an auxiliary term $f(s) := -\frac{1}{2}\langle \mathbf{x}(s), \mathbf{x}_2(s)\rangle$ and a constant $\alpha := \langle \mathbf{x}_1(0), \mathbf{x}_1(0)\rangle$, which immediately implies that $\langle \mathbf{x}(0), \mathbf{x}_2(0)\rangle = -\alpha$. Now, a straightforward computation yields the following derivatives:

$$\begin{aligned}
a'(s) &= f(s) - 2k(s)b(s), & d'(s) &= 3c(s) - \frac{1}{2}k(s)e(s),\\
b'(s) &= a(s) - \frac{3}{2}k(s)c(s) - \frac{1}{2}d(s), & e'(s) &= 4d(s),\\
c'(s) &= 2b(s) - k(s)d(s) - \frac{1}{6}e(s), & f'(s) &= -2b(s) + k(s)d(s) + \frac{1}{2}e(s).
\end{aligned}$$

We have thus a first order linear system of ODEs, with the presence of the projective curvature $k(s)$ in its coefficients. By employing these relations recursively, we may obtain all the terms we seek. Since we know that $a(0) = b(0) = c(0) = d(0) = e(0) = 0$, the coefficients of the power series will not vanish only when the term $f(0) = \alpha/2$ is present. With this observation in mind, we may simplify the calculation by ignoring the terms that will not yield the term $f$ after the suitable number of differentiations. We shall denote this operation by the symbol $\sim$. In the results that follow, we denote the projective curvature at zero as $k$ for simplicity of notation.

**Lemma 5.4.3.** *The Taylor expansion of the coefficient $a(s)$ is*

$$a(s) = \frac{1}{2}\alpha s - \frac{1}{6}k\alpha s^3 + O(s^4).$$

*Proof.* We have $a' = f - 2kb$, so $a'(0) = f(0) = \frac{1}{2}\alpha$.

Next, $a'' = f' - 2k'b - 2kb' \sim -2kb' = -2k\left(a - \frac{3}{2}kc - \frac{1}{2}d\right) \sim -2ka$.
So $a''(0) = 0$.

Finally, $a''' \sim -2k'a - 2ka' \sim -2ka' = -2k\,(f - 2kb)$.
So $a'''(0) = -2k(0)f(0) = -k\alpha$. □

**Lemma 5.4.4.** *The Taylor expansion of the coefficient $b(s)$ is*

$$b(s) = \frac{1}{4}\alpha s^2 - \frac{5}{48}k\alpha s^4 + O(s^5).$$

*Proof.* We have $b' = a - \frac{3}{2}kc - \frac{1}{2}d$, so $b'(0) = 0$.

Next, $b'' = a' - \frac{3}{2}k'c - \frac{3}{2}kc' - \frac{1}{2}d' \sim a' - \frac{3}{2}kc' = f - 2kb - \frac{3}{2}k\left(2b - kd - \frac{1}{6}e\right)$
$\sim f - 5kb$. So $b''(0) = \frac{1}{2}\alpha$.

Next, $b''' \sim f' - 5k'b - 5kb' \sim -5kb' = -5k\left(a - \frac{3}{2}kc - \frac{1}{2}d\right) \sim -5ka$.
So $b'''(0) = 0$.

Finally, $b^{iv} \sim -5k'a - 5ka' \sim -5kf$. So $b^{iv}(0) = -\frac{5}{2}k\alpha$. □

**Lemma 5.4.5.** *The Taylor expansion of the coefficient $c(s)$ is*

$$c(s) = \frac{1}{6}\alpha s^3 - \frac{1}{15}k\alpha s^5 + O(s^6).$$

*Proof.* We have $c' = 2b - kd - \frac{1}{6}e \sim 2b - kd$, so $c'(0) = 0$.

Next, $c'' = 2b' - k'd - kd' \sim 2b' - kd' = 2\left(a - \frac{3}{2}kc - \frac{1}{2}d\right) - k\left(3c - \frac{1}{2}ke\right)$
$\sim 2a - 6kc$. So $c''(0) = 0$.

Next, $c''' \sim 2a' - 6k'c - 6kc' \sim 2a' - 6kc' = 2(f - 2kb) - 6k(2b - kd - \frac{1}{6}e)$
$\sim 2f - 16kb$. So $c'''(0) = \alpha$.

Next, $c^{iv} \sim 2f' - 16k'b - 16kb' \sim -16kb' = -16k\left(a - \frac{3}{2}kc - \frac{1}{2}d\right) \sim -16ka$.
So $c^{iv}(0) = 0$.

Finally, $c^{v} \sim -16k'a - 16ka' \sim -16kf$. So $c^{v}(0) = -8k\alpha$. □

**Lemma 5.4.6.** *The Taylor expansion of the coefficient $d(s)$ is*

$$d(s) = \frac{1}{8}\alpha s^4 - \frac{1}{24}k\alpha s^6 + O(s^7).$$

*Proof.* We have $d' = 3c - \frac{1}{2}ke$, so $d'(0) = 0$.

Next, $d'' = 3c' - \frac{1}{2}k'e - \frac{1}{2}ke' \sim 3c' - \frac{1}{2}ke' = 3\left(2b - kd - \frac{1}{6}e\right) - \frac{1}{2}k\,(4d) \sim 6b - 5kd$. So $d''(0) = 0$.

Next, $d''' \sim 6b' - 5k'd - 5kd' \sim 6b' - 5kd' = 6\left(a - \frac{3}{2}kc - \frac{1}{2}d\right) - 5k\left(3c - \frac{1}{2}ke\right)$ $\sim 6a - 24kc$. So $d'''(0) = 0$.

Next, $d^{iv} \sim 6a' - 24k'c - 24kc' \sim 6(f - 2kb) - 24k\left(2b - kd - \frac{1}{6}e\right) \sim 6f - 60kb$. So $d^{iv}(0) = 3\alpha$.

Next, $d^{v} \sim 6f' - 60k'b - 60kb' \sim -60kb' \sim -60ka$. So $d^{v}(0) = 0$.

Finally, $d^{vi} \sim -60kf$. So $d^{vi}(0) = -30k\alpha$. □

**Lemma 5.4.7.** *The Taylor expansion of the coefficient $e(s)$ is*

$$e(s) = \frac{1}{10}\alpha s^5 - \frac{1}{42}k\alpha s^7 + O(s^8).$$

*Proof.* Since $e' = 4d$, it is easy to find its power series from the one we already know for $d(s)$. We have that:

$$\begin{aligned} e'(0) &= 4d(0) = 0; \\ e''(0) &= 4d'(0) = 0; \\ e'''(0) &= 4d''(0) = 0; \\ e^{iv}(0) &= 4d'''(0) = 0; \\ e^{v}(0) &= 4d^{iv}(0) = 12\alpha; \\ e^{vi}(0) &= 4d^{v}(0) = 0; \\ e^{vii}(0) &= 4d^{vi}(0) = -120k\alpha. \end{aligned}$$

□

We may obtain the Taylor expansion of the invariants of the binary quartic $\varphi(u(s), u(0))$ now that we know the expansions around $s = 0$ of its coefficients. Recall that the invariants $S$ and $T$ are given by

$$S = ae - 4bd + 3c^2, \quad T = ace + 2bcd - ad^2 - b^2e - c^3.$$

Therefore, their respective Taylor expansions are:

$$S(s) = \frac{1}{120}\alpha^2 s^6 - \frac{1}{672}k\alpha^2 s^8 + O(s^9), \quad T(s) = \frac{1}{17280}\alpha^3 s^9 - \frac{1}{100800}k\alpha^3 s^{11} + O(s^{12}).$$

The invariant $\Delta$ is given by $\Delta = S^3 - 27T^2$. Hence, its expansion is

$$\Delta(s) = \frac{1}{2048000}\alpha^6 s^{18} - \frac{1}{3584000}k\alpha^6 s^{20} + O(s^{21}).$$

The $J$-invariant is defined as $J = S^3/\Delta$. We may compute the power series of this ratio, since we know the series of the numerator and denominator. By doing the calculation, a significant simplification plays out and we get the main result of this section, namely, the power series of $J(s)$.

$$J(s) = \frac{32}{27} + \frac{8}{189}ks^2 + O(s^3).$$

Even though the $J$-invariant is not defined for the identically zero form, we may observe the limit as the parameter $s$ goes to zero. It is clear that the limiting value is $32/27$. By doing a similar analysis in order to find the power series for the $K$-invariant we obtain:

$$K(s) = \frac{\sqrt{30}}{72} + \frac{3\sqrt{30}}{2240}ks^2 + O(s^3).$$

So the limiting value for the $K$-invariant is $\sqrt{30}/72$.

These results provide us with some new information about the osculating conics of a regular curve away from inflections and sextactic points. Not only are they convexly nested (with $J$-invariant in $[1, +\infty]$ for the associated binary quartic), but, at least locally, they yield quartics with $J$-invariant close to $32/27$. We remark that the value of the parameter $\lambda$ in the normal form that results in $J = 32/27$ is $\lambda = \frac{\sqrt{5}-2}{3}$, and the cross-ratio of the roots is $\xi = \frac{1+\sqrt{5}}{2}$, the *golden ratio*. Although we do not know an explanation for this phenomenon, the presence of the golden ration seems to indicate an important property. When the projective curvature is positive, $k > 0$, the osculating conics locally produce quartics with $J > 32/27$. We propose the term *harmonic nesting* to emphasise when this happens.

# 6

# Logarithmic Spirals

In this chapter, we are interested in solving the following reciprocal problem. Given two conics $u$ and $v$ in $\mathbb{RP}^2$, under which conditions there exists a curve $\gamma(s)$ with no inflections nor sextactic point that *joins* them, that is, a curve such that $u$ is the osculating conic at $\gamma(0)$ and $v$ is the osculating conic at $\gamma(s_0)$ for some value $s_0 \in \mathbb{R}$? In light of the result found in Section 5.4.2 about the algebraic invariants of the quartic related to the osculating conics, we should consider the values of the invariants of $\varphi(u, v)$ beforehand. We claim that, if the $J$-invariant of $\varphi(u, v)$ is greater than $32/27$, then by using *logarithmic spirals*, we can obtain a partial solution to this question. We will show that one can always find a logarithmic spiral with zero projective curvature that has $u$ and some conic of the pencil generated by $u$ and $v$ as its osculating conics. So let us first introduce this particular family of curves.

## 6.1

## Logarithmic spirals of zero projective curvature

Let us first define the *projective curvature* and then find all curves for which it is identically zero. The following explanation can be found in [Craizer].

Consider $x : (-\varepsilon, \varepsilon) \to \mathbb{RP}^2$, $\mathbf{x}(t) = (x(t) : y(t) : z(t))$ a smooth curve in $\mathbb{RP}^2$ and let us denote the determinant of a matrix whose columns are given by three vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ by $|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3|$. We say that a smooth curve is *strictly convex* when for all values of the parameter $t$, we have $|\mathbf{x}''(t), \mathbf{x}'(t), \mathbf{x}(t)| \neq 0$. Notice that this condition already implies that the curve $\mathbf{x}(t)$ has no inflection point. Let us assume that $|\mathbf{x}''(t), \mathbf{x}'(t), \mathbf{x}(t)| > 0$ for all values of $t$. Since this three vectors are always linearly independent, they form a continuous family of basis of $\mathbb{R}^3$. So we may write $\mathbf{x}'''(t)$ in this basis as

$$\mathbf{x}''' + p\mathbf{x}'' + q\mathbf{x}' + r\mathbf{x} = 0.$$

$$p = -\frac{|\mathbf{x}''', \mathbf{x}', \mathbf{x}|}{|\mathbf{x}'', \mathbf{x}', \mathbf{x}|}, \quad q = \frac{|\mathbf{x}''', \mathbf{x}'', \mathbf{x}|}{|\mathbf{x}'', \mathbf{x}', \mathbf{x}|}, \quad r = -\frac{|\mathbf{x}''', \mathbf{x}'', \mathbf{x}'|}{|\mathbf{x}'', \mathbf{x}', \mathbf{x}|}.$$

Now let us define a real function $H : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by making use of the coefficients $p(t)$, $q(t)$ and $r(t)$ above as $H(t) := r - \frac{1}{3}pq + \frac{2}{27}p^3 - \frac{1}{2}q' + \frac{1}{3}pp' + \frac{1}{6}p''$. Assuming that $H(t) \neq 0$ for all values of $t$, we may define the *projective length* $\sigma(t)$ of the curve $\mathbf{x}(t)$ by

$$\sigma(t) := \int_0^t \sqrt[3]{H(u)}du.$$

Let us assume that the curve $\mathbf{x}(\sigma)$ is parametrized by its projective length. In this case, the function $H(\sigma) \equiv 1$ is constant. Since our setting is the projective plane, for any non-vanishing real function $\lambda\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$, we have that $\mathbf{x}(\sigma)$ and $\lambda(\sigma)\mathbf{x}(\sigma)$ describe the same curve. By taking the suitable function $\lambda(\sigma) = \exp(\frac{1}{3}\int_0^\sigma p(t)dt)$, one may ensure that the coefficient $p(\sigma)$ is identically zero. In other words, we may assume that $\mathbf{x}$ satisfies the following system of equations for all values of $\sigma$.

$$\begin{cases} \mathbf{x}'''(\sigma) + q(\sigma)\mathbf{x}'(\sigma) + r(\sigma)\mathbf{x}(\sigma) = 0, \\ H(\sigma) = r(\sigma) - \dfrac{1}{2}q'(\sigma) = 1. \end{cases}$$

We define the *projective curvature* $k\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ as the function given by $k(\sigma) = \frac{1}{2}q(\sigma)$. Notice that in the parametrization we are considering, it also holds that $k'(\sigma) = r(\sigma) - 1$. With this new object in hand, we may define the *Frenet frame* of the curve $\mathbf{x}$. It is the frame $\{\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2\}$ defined by the equations:

$$\begin{cases} \dfrac{\partial \mathbf{x}}{\partial \sigma} = \mathbf{x}_1 \\ \dfrac{\partial \mathbf{x}_1}{\partial \sigma} = -k\mathbf{x} + \mathbf{x}_2 \\ \dfrac{\partial \mathbf{x}_2}{\partial \sigma} = -\mathbf{x} - k\mathbf{x}_1 \end{cases}$$

Now let us find which curves have zero projective curvature. The condition $k(\sigma) \equiv 0$ implies that $q(\sigma) \equiv 0$ and $r(\sigma) \equiv 1$. Then the differential equation for the curve $\mathbf{x}$ becomes $\mathbf{x}'''(\sigma) + \mathbf{x}(\sigma) = 0$. Considering the analogous differential equation in the setting of real functions, we have $y'''(x) + y(x) = 0$, which is a homogeneous linear differential equation of degree 3. The generic solution for this equation is:

$$y(x) = C_1 e^{-x} + C_2 e^{x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + C_3 e^{x/2} \sin\left(\frac{\sqrt{3}x}{2}\right), \quad C_1, C_2, C_3 \in \mathbb{R}.$$

If we set each term of the solution above as one of the functions for the homogeneous coordinates of the curve $\mathbf{x}(\sigma)$, we get a curve of zero projective curvature.

$$\mathbf{x}(\sigma) = \left(e^{\sigma/2} \cos\left(\frac{\sqrt{3}\sigma}{2}\right) : e^{\sigma/2} \sin\left(\frac{\sqrt{3}\sigma}{2}\right) : e^{-\sigma}\right).$$

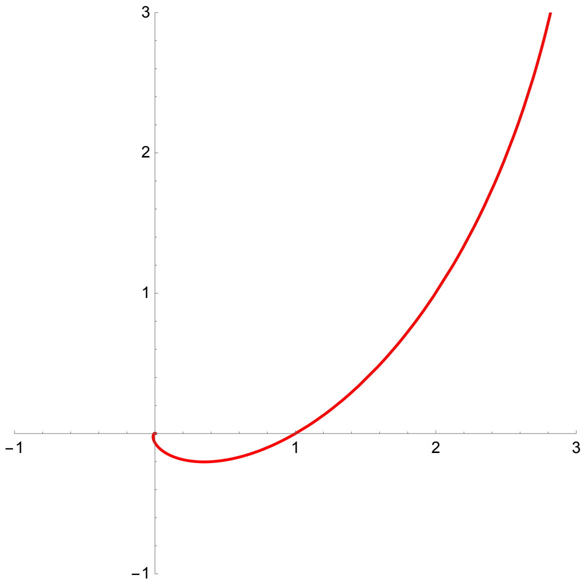


Figure 6.1: A logarithmic spiral with zero projective curvature.

We have found that a particular *logarithmic spiral* has zero projective curvature. However, since the entire setting is invariant under projective transformations, we can make good use of the projective group $\mathrm{PGL}(3;\mathbb{R})$ to obtain a larger class of solutions. Indeed, for any projective transformation $A \in \mathrm{PGL}(3;\mathbb{R})$, the curve $A.\mathbf{x}(\sigma)$ is also a solution for the differential equation, so its projective curvature is also $k(\sigma) \equiv 0$. Moreover, any solution of the differential equation can be obtained by one such projective transformation of $\mathbf{x}$. Therefore, there exists an 8-dimensional family of curves of zero projective curvature, all projectively equivalent to the logarithmic spiral described above. We will refer to any element of this set as a *zero projective curvature logarithmic spiral*.

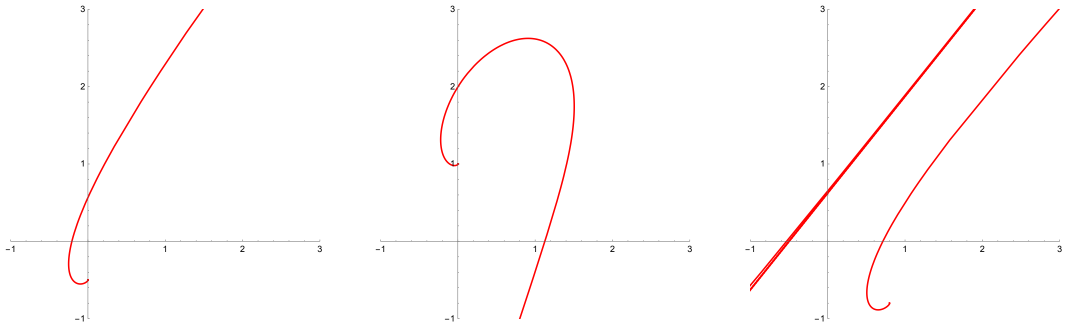

Figure 6.2: Some more logarithmic spirals of zero projective curvature.

From the point of view of projective geometry, they are all *the same* curve, just as all nondegenerate conics are the same curve. Nevertheless, since we want the logarithmic spiral to have a pair of specifically designated conics in the set of its osculating conics, we do get different paths of osculating conics for each representative of the class. There exists, however, a one-parameter subgroup of $\mathrm{PGL}(3;\mathbb{R})$ that does not change much this standard logarithmic spiral $\mathbf{x}(\sigma)$ due to the *self-similarity* that it displays. More specifically, there is a subgroup $G$ parametrized by $s \in \mathbb{R}$ that preserves the curve globally, only advancing the parameter of the curve by $s$. Let us take a close look at $\mathbf{x}(\sigma+s)$.

$$\mathbf{x}(\sigma+s) = \left(e^{\sigma/2+s/2}\cos\left(\tfrac{\sqrt{3}\sigma}{2}+\tfrac{\sqrt{3}s}{2}\right) \;:\; e^{\sigma/2+s/2}\sin\left(\tfrac{\sqrt{3}\sigma}{2}+\tfrac{\sqrt{3}s}{2}\right) \;:\; e^{-\sigma-s}\right)$$

$$= \left(e^{\sigma/2}\left(\cos\left(\tfrac{\sqrt{3}\sigma}{2}\right)\cos\left(\tfrac{\sqrt{3}s}{2}\right)-\sin\left(\tfrac{\sqrt{3}\sigma}{2}\right)\sin\left(\tfrac{\sqrt{3}s}{2}\right)\right) \;:\; e^{\sigma/2}\left(\sin\left(\tfrac{\sqrt{3}\sigma}{2}\right)\cos\left(\tfrac{\sqrt{3}s}{2}\right)+\cos\left(\tfrac{\sqrt{3}\sigma}{2}\right)\sin\left(\tfrac{\sqrt{3}s}{2}\right)\right) \;:\; e^{-\sigma-3s/2}\right).$$

$$\mathbf{x}(\sigma+s) = \begin{pmatrix} \cos\left(\frac{\sqrt{3}s}{2}\right) & -\sin\left(\frac{\sqrt{3}s}{2}\right) & 0 \\ \sin\left(\frac{\sqrt{3}s}{2}\right) & \cos\left(\frac{\sqrt{3}s}{2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-3s/2} \end{pmatrix} \mathbf{x}(\sigma).$$

The equality above shows that by applying an Euclidean rotation of $\frac{\sqrt{3}s}{2}$ in the $(x,y)$ plane and a homothety with respect to the origin of factor $e^{3s/2}$ to $\mathbf{x}(\sigma)$, one gets the same curve, but every point advances with respect to the parameter $\sigma$ by a fixed amount $s$. In terms of the $\mathrm{PGL}(3;\mathbb{R})$ action on the family of zero projective curvature logarithmic spirals, the subgroup $G$ allows us to move freely the point corresponding to $\sigma=0$. We will later make good use of this subgroup to move the point of contact of a given osculating conic.

### 6.1.1

#### Osculating conics of a logarithmic spiral

Let $\gamma\colon (-\varepsilon,\varepsilon) \to \mathbb{RP}^2$ be a smooth projective curve. For a given value $s_0 \in (-\varepsilon,\varepsilon)$, how can we determine the osculating conic at the point $\gamma(s_0)$? Consider $u(x,y,z) = ax^2+by^2+cz^2+2fxz+2gyz+2hxz$, the implicit equation of a generic conic. As we have discussed in section 4.2, for it to be the osculating conic of $\gamma$ at $\gamma(s_0)$, the composite $u \circ \gamma$ must have a zero of order at least 5 at that point. In other words, the following system of equations must be satisfied.

$$\begin{cases} u\circ\gamma\,(s_0) = 0 \\ \frac{d}{ds}(u\circ\gamma)(s_0) = 0 \\ \frac{d^2}{ds^2}(u\circ\gamma)(s_0) = 0 \\ \frac{d^3}{ds^3}(u\circ\gamma)(s_0) = 0 \\ \frac{d^4}{ds^4}(u\circ\gamma)(s_0) = 0 \end{cases}$$

We have 5 independent linear equations on the coefficients of $u$. Thus we have a one dimensional subspace of $\mathbb{R}^6$ of solutions, that is, a point of $\mathbb{RP}^5$ which corresponds to the unique osculating conic at $\gamma(s_0)$. One may think of $u(x,y,z)$ as a quadratic form, whose associated symmetric bilinear form $U(\mathbf{v}_1,\mathbf{v}_2)$ is given by the symmetric matrix

$$M_U = \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix}.$$

The evaluation of the bilinear form $U$ on two generic vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is $U(\mathbf{v}_1,\mathbf{v}_2) = \mathbf{v}_1^\intercal M_U \mathbf{v}_2$. One may recover the implicit equation of the conic by taking $u(x,y,z) = (x,y,z)^\intercal M_U (x,y,z)$. With the help of the bilinear form $U$, it is easy to compute the derivatives of $u \circ \gamma(s)$, since it can be rewritten as $u \circ \gamma(s) = U(\gamma(s),\gamma(s))$. Now the system of equations that defines the osculating conic becomes the one below. All expressions are evaluated at $s_0$, we have omitted it for simplicity.

$$\begin{cases} U(\gamma,\gamma) = 0 \\ \frac{d}{ds}U(\gamma,\gamma) = 2U(\gamma,\gamma') = 0 \\ \frac{d^2}{ds^2}U(\gamma,\gamma) = 2\left(U(\gamma',\gamma') + U(\gamma,\gamma'')\right) = 0 \\ \frac{d^3}{ds^3}U(\gamma,\gamma) = 2\left(3U(\gamma',\gamma'') + U(\gamma,\gamma''')\right) = 0 \\ \frac{d^4}{ds^4}U(\gamma,\gamma) = 2\left(3U(\gamma'',\gamma'') + 4U(\gamma',\gamma''') + U(\gamma,\gamma^{iv})\right) = 0 \end{cases}$$

In the case where $\gamma$ is a zero projective curvature logarithmic spiral, the system becomes even simpler, because $\gamma'''(s) = -\gamma(s)$ and $\gamma^{iv}(s) = -\gamma'(s)$ for every value of $s$. Consequently, the osculating conic at $\gamma(s_0)$ is the one whose coefficients satisfy:

$$\begin{cases} U(\gamma,\gamma) = 0 \\ U(\gamma,\gamma') = 0 \\ U(\gamma',\gamma') + U(\gamma,\gamma'') = 0 \\ U(\gamma',\gamma'') = 0 \\ U(\gamma'',\gamma'') = 0 \end{cases}$$

Let us now fix $\gamma(s) = \left(e^{s/2}\cos\left(\frac{\sqrt{3}s}{2}\right) : e^{s/2}\sin\left(\frac{\sqrt{3}s}{2}\right) : e^{-s}\right)$ and compute its osculating conic at $s = 0$. We have at this point: $\gamma(0) = (1 : 0 : 1)$, $\gamma'(0) = (1/2 : \sqrt{3}/2 : -1)$ and $\gamma''(0) = (-1/2 : \sqrt{3}/2 : 1)$. By applying these values in the osculating conic's system of equation above, we have:

$$\begin{cases} a + c + 2f = 0 \\ a - 2c - f + \sqrt{3}g + \sqrt{3}h = 0 \\ -a + 3b + 8c - 2f - 2\sqrt{3}g + 4\sqrt{3}h = 0 \\ -a + 3b - 4c + 4f = 0 \\ a + 3b + 4c - 4f + 4\sqrt{3}g - 2\sqrt{3}h = 0 \end{cases}$$

The system can be simplified and then it becomes easy to find a solution by setting $c = 1$.

$$\begin{cases} a + 5c = 0 \\ b + 3c = 0 \\ f - 2c = 0 \\ g - 2\sqrt{3}c = 0 \\ h - \sqrt{3}c = 0 \end{cases}$$

Therefore, the osculating conic to the logarithmic spiral at $s = 0$ is $-5x^2 - 3y^2 + z^2 + 4xz + 4\sqrt{3}yz + 2\sqrt{3}xy = 0$.

The calculation for the osculating conic at a generic point as a function of $s$ is more laborious, but it can be done by a mathematical computation program such as *Mathematica* or *Maple*. As in section 4.2.1, we may reparametrize the curve of osculating conics by multiplying all coefficients by a positive real function $\lambda(s)$ so that the tangent vector of this curve always points in the direction of a degenerate conic, specifically the double line tangent to the curve $\gamma$ at the point $\gamma(s)$. So first we set $c(s) = \lambda(s)$ and find the expressions for all coefficients in terms of $\lambda(s)$. Then, by solving the appropriate differential equation, we find that the suitable function is simply $\lambda(s) = e^{2s}$, and thus we obtain the coefficients of the osculating conics as a function of $s$. This computation was realized with the help of *Mathematica*.

$$\begin{aligned}
a(s) &= -4e^{-s} - \sqrt{3}e^{-s}\sin\left(\sqrt{3}s\right) - e^{-s}\cos\left(\sqrt{3}s\right) \\
b(s) &= -4e^{-s} + \sqrt{3}e^{-s}\sin\left(\sqrt{3}s\right) + e^{-s}\cos\left(\sqrt{3}s\right) \\
c(s) &= e^{2s} \\
f(s) &= -2\sqrt{3}e^{\frac{s}{2}}\sin\left(\frac{\sqrt{3}s}{2}\right) + 2e^{\frac{s}{2}}\cos\left(\frac{\sqrt{3}s}{2}\right) \\
g(s) &= 2e^{\frac{s}{2}}\sin\left(\frac{\sqrt{3}s}{2}\right) + 2\sqrt{3}e^{\frac{s}{2}}\cos\left(\frac{\sqrt{3}s}{2}\right) \\
h(s) &= -e^{-s}\sin\left(\sqrt{3}s\right) + \sqrt{3}e^{-s}\cos\left(\sqrt{3}s\right)
\end{aligned}$$

### 6.1.2

### A logarithmic spiral whose osculating conic at $s = 0$ is c

The logarithmic spiral $\gamma(s) = \left(e^{s/2}\cos\left(\frac{\sqrt{3}s}{2}\right) : e^{s/2}\sin\left(\frac{\sqrt{3}s}{2}\right) : e^{-s}\right)$ has $u = -5x^2 - 3y^2 + z^2 + 4xz + 4\sqrt{3}yz + 2\sqrt{3}xy$ as its osculating conic at $s = 0$. If we wish to find a logarithmic spiral whose osculating conic at $s = 0$ is

$\mathfrak{c} = -x^2 - y^2 + z^2$, we have to find a projective transformation $A \in \mathrm{PGL}(3;\mathbb{R})$ that takes $u$ to $\mathfrak{c}$, since this implies that $A.\gamma(s)$ is a suitable solution to the problem. One such projective transformation is

$$A = \begin{pmatrix} \frac{3\sqrt{2}}{2} & -\frac{\sqrt{6}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & -2\sqrt{2} \\ 0 & 0 & 3 \end{pmatrix}.$$

Indeed, the symmetric bilinear form associated to $A.u$ is given by the symmetric matrix $(A^{-1})^{\mathsf{T}} M_U A^{-1}$, where

$$A^{-1} = \begin{pmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{3} \\ -\frac{\sqrt{6}}{12} & \frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \qquad M_U = \begin{pmatrix} -5 & \sqrt{3} & 2 \\ \sqrt{3} & -3 & 2\sqrt{3} \\ 2 & 2\sqrt{3} & 1 \end{pmatrix}.$$

$$\begin{pmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{12} & 0 \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & 0 \\ \frac{1}{3} & \frac{\sqrt{3}}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -5 & \sqrt{3} & 2 \\ \sqrt{3} & -3 & 2\sqrt{3} \\ 2 & 2\sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{3} \\ -\frac{\sqrt{6}}{12} & \frac{\sqrt{6}}{4} & \frac{\sqrt{3}}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$(A^{-1})^{\mathsf{T}} M_U A^{-1} = M_C.$$

Since $M_C$ is the symmetric matrix of the symmetric bilinear form associated to $\mathfrak{c}$, we have that $A.u = \mathfrak{c}$. By applying the action of $A$ on the projective plane, we obtain a new logarithmic spiral $A.\gamma(s)$, which is parametrized by:

$$A.\gamma(s) = \left(e^{s/2}\left(\tfrac{3\sqrt{2}}{2}\cos\left(\tfrac{\sqrt{3}s}{2}\right) - \tfrac{\sqrt{6}}{2}\sin\left(\tfrac{\sqrt{3}s}{2}\right)\right) : e^{s/2}\left(\tfrac{\sqrt{2}}{2}\cos\left(\tfrac{\sqrt{3}s}{2}\right) + \tfrac{\sqrt{6}}{2}\sin\left(\tfrac{\sqrt{3}s}{2}\right)\right) - 2\sqrt{2}e^{-s} : 3e^{-s}\right).$$

In order to obtain the new coefficients of the generic osculating conic one has to compute the same congruence by $A^{-1}$, but with the matrix associated to the respective conic. In the end of the process, we get the following coefficients:

$$
\begin{aligned}
a(s) &= \frac{1}{3}e^{-s}\left(-2-\cos\left(\sqrt{3}s\right)\right)\\
b(s) &= e^{-s}\left(-2+\cos\left(\sqrt{3}s\right)\right)\\
c(s) &= \frac{1}{9}e^{-s}\left(-16+e^{3s}+16e^{\frac{3s}{2}}\cos\left(\frac{\sqrt{3}s}{2}\right)+8\cos\left(\sqrt{3}s\right)\right)\\
f(s) &= \frac{2\sqrt{6}}{9}e^{-s}\left(-e^{\frac{3s}{2}}\sin\left(\frac{\sqrt{3}s}{2}\right)-\sin\left(\sqrt{3}s\right)\right)\\
g(s) &= \frac{2\sqrt{2}}{3}e^{-s}\left(-2+e^{\frac{3s}{2}}\cos\left(\frac{\sqrt{3}s}{2}\right)+\cos\left(\sqrt{3}s\right)\right)\\
h(s) &= -\frac{\sqrt{3}}{3}e^{-s}\sin\left(\sqrt{3}s\right)
\end{aligned}
\tag{6.1}
$$

The projective transformation $A \in \mathrm{PGL}(3;\mathbb{R})$ presented in this section accomplishes the goal of providing a logarithmic spiral with $\mathfrak{c}$ as its osculating conic at $s = 0$. However, is not the only choice available. There is actually a 3-dimensional family of suitable projective transformations, since any element in $\mathrm{Stab}(\mathfrak{c})A$, the *right coset* of $A$ with respect to the *stabilizer* of $\mathfrak{c}$, would also serve the purpose.

Figure 6.3 below, for example, displays the conic $R.A.\gamma(s)$ with some of its osculating conics, where $R$ stands for the rotation by $\pi/4$. We applied this rotation in order to get a better illustration, in which $R.A.\gamma(0) = (1 : 0 : 1)$. Since $R \in \mathrm{Stab}(\mathfrak{c})$ preserves the conic $\mathfrak{c}$, the osculating conic at $s = 0$ is still $\mathfrak{c}$.

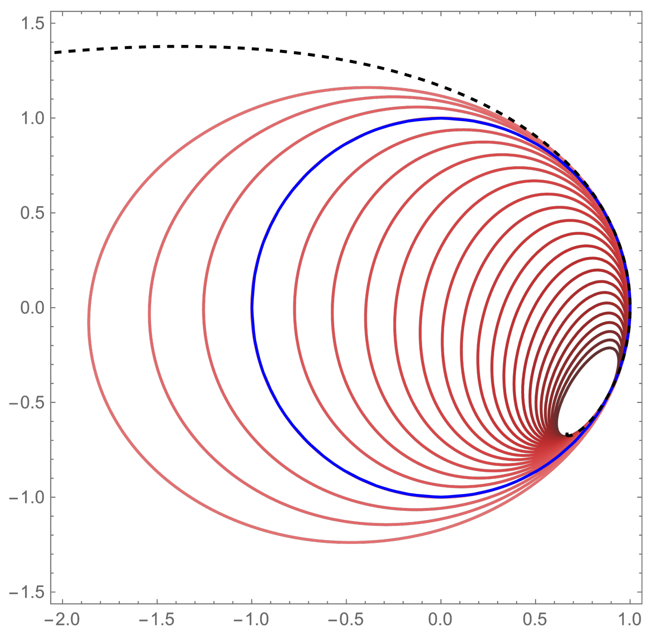


Figure 6.3: Logarithmic spiral with several osculating conics around $s = 0$. Notice the presence of $\mathfrak{c}$, highlighted in blue.

### 6.1.3

#### The stabilizer of $\mathfrak{c}$

In this section we present and briefly describe the stabilizer of $\mathfrak{c}$, namely $\mathrm{Stab}(\mathfrak{c}) = \mathrm{PO}\,(2,1) < \mathrm{PGL}(3;\mathbb{R})$. In order for a projective transformation $B$ to preserve the conic $\mathfrak{c}$, it must satisfy the relation: $B^\top M_C B = M_C$. The group of matrices that satisfy this condition is called the *orthogonal group with respect to the indefinite quadratic form* $x^2+y^2-z^2$ (or $-x^2-y^2+z^2$ equivalently) and is denoted by $\mathrm{O}(2,1)$. The group of projective transformations that preserve $\mathfrak{c}$ is thus $\mathrm{PO}\,(2,1)$, the quotient of $\mathrm{O}(2,1)$ by $\{\pm I\}$, as these are the only two scalar transformations in $\mathrm{O}(2,1)$. Notice that, since we are dealing with $3 \times 3$ matrices, the determinant of $-I$ is $-1$, so we can always take as the representative of a class in $\mathrm{PO}\,(2,1)$ a matrix with determinant equal to 1. In other words, $\mathrm{PO}\,(2,1)$ is isomorphic to $\mathrm{SO}(2,1)$, the subgroup of $\mathrm{O}(2,1)$ whose elements all have determinant equal to 1.

The group $\mathrm{SO}(2,1)$ has two connected components, which are isomorphic. The one that contains the identity only has transformations that preserve the orientation in both subspaces where the restriction of the quadratic form is definite. The other one reverses the orientation of both such subspaces. For example, the transformation

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

reverses the orientation of the $x, y$ subspace and also of the $z$ subspace. The isomorphism between the two connected components of $\mathrm{SO}(2,1)$ can be established via the product by this element $B$.

The connected component of the identity, denoted $\mathrm{SO}_0(2,1)$, is generated by 3 one-parameter groups, one of Euclidean rotations in the $x, y$ plane, and two of hyperbolic rotations, in the $x, z$ plane and in the $y, z$ plane.

$$R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad H_x(\tau) = \begin{pmatrix} \cosh(\tau) & 0 & \sinh(\tau) \\ 0 & 1 & 0 \\ \sinh(\tau) & 0 & \cosh(\tau) \end{pmatrix}; \quad H_y(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(\psi) & \sinh(\psi) \\ 0 & \sinh(\psi) & \cosh(\psi) \end{pmatrix}.$$

## 6.2

## Joining two harmonically nested conics with a logarithmic spiral

In this section, we state and then prove the main theorem of this chapter. It provides a sufficient condition for there to exist a zero projective curvature logarithmic spiral $\gamma(s)$ that joins two conics.

**Theorem 6.2.1.** *Let $u$ and $v$ be two distinct irreducible conics in $\mathbb{RP}^2$ without a pair of complex double intersections. If $u$ is harmonically nested with respect to $v$, then there exists a zero projective curvature logarithmic spiral $\gamma(s)$ such that $u$ is its osculating conic at $s = 0$ and some other conic of the pencil $\overline{uv}$ is its osculating conic at some other value $s_0 > 0$.*

The proof of the theorem revolves around the $K$-invariant. In the following subsection, we prove a lemma about the function $K\colon\, ]0, +\infty[\, \to\, ]\sqrt{30}/72, \sqrt{3}/9[$ that provides the $K$-invariant of the binary quartic $\varphi\left(\Gamma(0), \Gamma(s)\right)$ originating from a pair of osculating conics of a zero projective curvature logarithmic spiral.

### 6.2.1

### Algebraic invariants stemming from the logarithmic spiral

Let us prove that all but one class of binary quartics with $K$-invariant greater than $\sqrt{30}/72$ may be obtained from the osculating conics of a zero projective curvature logarithmic spiral.

**Lemma 6.2.2.** *Let $\gamma(s)$ be a zero projective curvature logarithmic spiral, and let $\Gamma(s)$ be the path of its osculating conics. Then the $K$-invariant of $\varphi\left(\Gamma(0), \Gamma(s)\right)$ attains every value in the open interval $]\sqrt{30}/72, \sqrt{3}/9[$.*

*Proof.* Let $K(s)$ be the function whose output is the $K$-invariant of the binary quartic $\varphi\left(\Gamma(0), \Gamma(s)\right)$. Since $K(s)$ is a continuous function, it is enough to show that $\lim_{s\to 0^+} K(s) = \sqrt{30}/72$ and $\lim_{s\to+\infty} K(s) = \sqrt{3}/9$.

For the infimum, we have proved in Theorem 5.4.1 that for any regular curve with no inflection or sextactic point, the limiting value for the $K$-invariant as $s$ goes to 0 is $\sqrt{30}/72$. As for the supremum, let us observe what happens to the osculating conics as $s$ goes to $+\infty$. Since the whole setting in invariant under the $\mathrm{PGL}(3;\mathbb{R})$ action on $\mathbb{RP}^2$, we may consider a particular zero projective curvature logarithmic spiral such as the one presented in 6.1.2. Notice that for this curve, $\Gamma(0) = \mathfrak{c}$ and that its osculating conics converge to the degenerate conic $\Gamma(+\infty) = z^2$ as $s$ goes to $+\infty$. This limiting osculating conic has two double complex intersections with $\mathfrak{c}$ in the line $z = 0$, at the so called *cyclic points* $(1 : i : 0)$ and $(1 : -i : 0)$. By having a pair of complex double intersections, the resulting quartic must be in the class of $t^4+2t^2w^2+w^4 = (t^2+w^2)^2$, which has the maximum value for the $K$-invariant, $\sqrt{3}/9$. This proves that indeed $\lim_{s\to+\infty} K(s) = \sqrt{3}/9$. $\square$

Let us illustrate the matter by making use of (6.1), the concrete parametrization of the path of osculating conics of the logarithmic spiral mentioned in the proof of the lemma. We employ *Mathematica* to compute the binary quartics $\varphi(\mathfrak{c}, \Gamma(s))$ and their algebraic invariants. Here we fix the first osculating conic $\mathfrak{c}$ and let the second osculating conic $\Gamma(s)$ evolve in terms of the parameter $s$ of the original curve. The figures below show the graphs of the functions $J(s)$ and $K(s)$, which yield the values of the $J$ and $K$ invariants respectively.

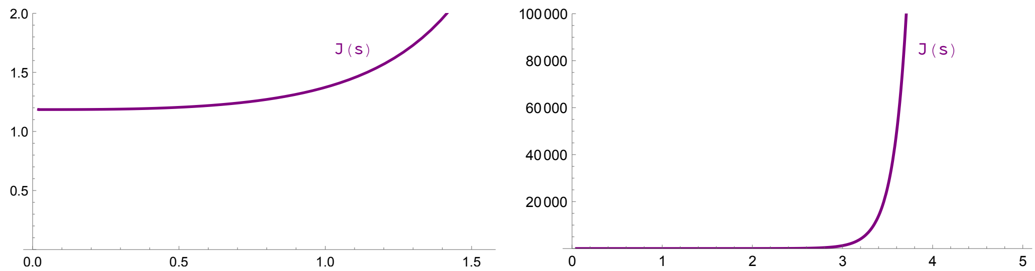


Figure 6.4: The graph of $J(s)$ of a logarithmic spiral, with $s \in [0, 1.5]$ on the left and $s \in [0, 5]$ on the right.

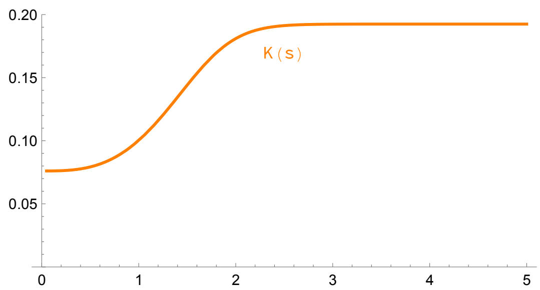


Figure 6.5: The graph of $K(s)$ of a logarithmic spiral with $s \in [0, 5]$.

*Mathematica* confirms that $\lim_{s\to0^+} J(s) = 32/27$ and $\lim_{s\to0^+} K(s) = \sqrt{30}/72$. Furthermore, it also validates that the limits when $s$ goes to infinity are $\lim_{s\to+\infty} J(s) = +\infty$ and $\lim_{s\to+\infty} K(s) = \sqrt{3}/9$.

We have thus shown that the path of osculating conics of the zero projective curvature logarithmic spiral produces binary quartics whose $K$-invariant may be any value in $]\sqrt{30}/72, \sqrt{3}/9[$. The absence of the supremum $\sqrt{3}/9$ corresponds to the unique class of quartics that cannot be obtained from the osculating conics, and this explains the hypothesis that the conics $u$ and $v$ cannot have a pair of complex double intersections.

We may also observe in the graphs in Figures 6.4 and 6.5 that both functions $J(s)$ and $K(s)$ seem to be strictly increasing, which would imply that the osculating conics of the logarithmic spiral are necessarily harmonically nested. If that is the case, then we would have a stronger result: two irreducible conics $u$ and $v$ with no complex double intersections may be joined by a zero projective curvature logarithmic spiral *if, and only if,* $u$ is harmonically nested with respect to $v$. Unfortunately, we do not yet have a proof for this fact. Actually, if we consider curves of constant *negative* projective curvature, we know that $K(s)$ cannot be injective since its power series begins with $K(s) = \frac{\sqrt{30}}{72} + \frac{3\sqrt{30}}{2240} ks^2 + O(s^3)$. We later examine the case of curves of constant projective curvature, other than zero, some more in section 6.3.

### 6.2.2

### Proof of the theorem

With the help of Lemma 6.2.2 above, let us prove Theorem 6.2.1.

*Proof.* Firstly, we observe that the claim is reasonable by dimension counting. We have 8 degrees of freedom from the projective group and 1 extra from the value of the parameter $s$, adding up to 9 degrees of freedom at our disposal. On the other hand, the coefficients of the conics $u$ and $v$ impose us 10 equations, but the liberty to take any other conic from the pencil $\overline{uv}$ relaxes the restriction by 1 degree, which brings us down to 9 conditions to be met. Thus, we may expect a discrete set of solutions to our problem.

Let $K_0$ be the value of the $K$-invariant of the quartic $\varphi(u, v)$. The hypothesis that $u$ is harmonically nested with respect to $v$ means that $K_0 > \sqrt{30}/72$, and the fact that they do not have a pair of double intersections prevents $K_0$ to be $\sqrt{3}/9$. We have shown in Lemma 6.2.2 that there is a zero projective curvature logarithmic spiral $\gamma_0(s)$ that has $\mathfrak{c}$ as its osculating conic at $s = 0$ and another conic $v_0$ at a certain $s = s_0$ such that the quartic $\varphi(\mathfrak{c}, v_0)$ has $K_0$ as its $K$-invariant.

Next, we apply the simultaneous diagonalization, as presented in 5.1, to both pairs $u$, $v$ and $\mathfrak{c}$, $v_0$. To be precise, let us denote by $A \in \mathrm{PO}\,(2,1)$ the projective transformation that performs the simultaneous diagonalization of $\mathfrak{c}$ and $v_0$. After its action, we obtain $A.\mathfrak{c} = \mathfrak{c}$ and $A.v_0 = v_1$, a new conic. A similar process plays out for the other pair. Let us denote by $B \in \mathrm{PGL}(3; \mathbb{R})$ the projective transformation that does the simultaneous diagonalization of $u$ and $v$. In the end, we get $B.u = \mathfrak{c}$ and $B.v = v_2$, yet another conic. As explained in Section 3.2, we have the following relations:

$$\varphi(\mathfrak{c}, v_0) = \varphi(A.\mathfrak{c}, A.v_0) = \varphi(\mathfrak{c}, v_1) \quad \text{and} \quad \varphi(u, v) = \varphi(B.u, B.v) = \varphi(\mathfrak{c}, v_2).$$

Since $\varphi(\mathfrak{c}, v_1)$ and $\varphi(\mathfrak{c}, v_2)$ both have the same $K$-invariant, the resulting parameter $\lambda \in ]\frac{\sqrt{5}-2}{3}, \frac{1}{3}[$ of both normal forms, as presented in 5.1, must be the same, so both quartics belong to the same $\mathrm{PGL}(2, \mathbb{R})$ orbit. In other words, $\varphi(\mathfrak{c}, v_1) = \varphi(\mathfrak{c}, v_2)$. Under the very restricted setting of the simultaneous diagonalization, this equality implies that $v_1$ and $v_2$ belong to the same pencil through $\mathfrak{c}$.

To conclude, we just have to compose the appropriate actions. Consider the zero projective curvature logarithmic spiral given by $\gamma(s) = B^{-1}.A.\gamma_0(s)$. Its osculating conic at $s = 0$ is $B^{-1}.A.\mathfrak{c} = u$, and the one at $s = s_0$ is $B^{-1}.A.v_0 = B^{-1}.v_1 = v_3$. Since the conic $v_1$ belongs to the pencil $\overline{\mathfrak{c}v_2}$, then $v_3$ belongs to the pencil $B^{-1}.\overline{\mathfrak{c}v_2} = \overline{uv}$. This shows that $\gamma(s)$ is a solution to the problem, thus concluding the proof.

□

### 6.2.3

#### No three osculating conics on the same pencil

We conclude this section by showing that the osculating conics of a zero projective curvature logarithmic spiral do not cross the same pencil of conics more than twice.

**Theorem 6.2.3.** *No pencil of conics contains more than two osculating conics of a zero projective curvature logarithmic spiral.*

*Proof.* Let us assume by contradiction that there exists such a zero projective curvature logarithmic spiral with three osculating conics that belong to the same pencil. First, up to a projective transformation, one may map the logarithmic spiral to the particular form whose osculating conics have their coefficients parametrized by (6.1). Also, as explained in the end of section 6.1, up to another projective transformation it is possible to set one of the three osculating conics at hand as the osculating conic at $s = 0$ specifically, so that it becomes $\mathfrak{c}$.

Now, the two other conics, let us denote them by $v$ and $w$, must have their coefficients given by the formulae above for some parameters $s_v$ and $s_w$. Also, since they are distinct and belong to the same pencil, there exists two non-zero real parameters $\alpha$ and $\beta$ such that $w = \alpha u + \beta v$. Therefore, the coefficients of $v$ and $w$ are related by the following 6 equations:

$$\begin{aligned} a_w &= -\alpha + \beta a_v & \qquad f_w &= \beta f_v \\ b_w &= -\alpha + \beta b_v & g_w &= \beta g_v \\ c_w &= \alpha + \beta c_v & h_w &= \beta h_v \end{aligned}$$

Beginning by the relation given by the coefficient $h(s)$, we have that

$$e^{-s_w} \sin(\sqrt{3}s_w) = \beta e^{-s_v} \sin(\sqrt{3}s_v).$$

Here we have to consider two cases, whether $\sin(\sqrt{3}s_v) = \sin(\sqrt{3}s_w) = 0$ or not. Let us treat first the case where the sines vanish, which means that $\sqrt{3}s_v$ and $\sqrt{3}s_w$ belong to $\pi\mathbb{Z}$. Subtracting the equation of the $a(s)$ coefficient from that of the $b(s)$, we get:

$$e^{-s_w}\left(1 - \cos(\sqrt{3}s_w)\right) = \beta e^{-s_v}\left(1 - \cos(\sqrt{3}s_v)\right).$$

Now there are again two cases to consider, whether $\cos(\sqrt{3}s_v) = \cos(\sqrt{3}s_w) = 1$ or $-1$. If they are both equal to $-1$, then $\beta = e^{s_v - s_w}$ and we will get to a contradiction quicker. Take now the relation given by the coefficient $f(s)$.

$$e^{-s_w}\left(e^{\frac{3s_w}{2}} \sin\left(\frac{\sqrt{3}s_w}{2}\right)\right) = \beta e^{-s_v}\left(e^{\frac{3s_v}{2}} \sin\left(\frac{\sqrt{3}s_v}{2}\right)\right).$$

In the case where $\sqrt{3}s_v$ and $\sqrt{3}s_w$ belong to $\pi + 2\pi\mathbb{Z}$, we have that $\beta = e^{s_v - s_w}$ and also $\sin\left(\frac{\sqrt{3}s_v}{2}\right)$ and $\sin\left(\frac{\sqrt{3}s_w}{2}\right)$ can only be $\pm 1$. In any case, we have a contradiction since $e^{\frac{3s_v}{2}} \neq \pm e^{\frac{3s_w}{2}}$. On the other hand, if $\sqrt{3}s_v$ and $\sqrt{3}s_w$ belong to $2\pi\mathbb{Z}$, then $\frac{\sqrt{3}s_v}{2}$ and $\frac{\sqrt{3}s_w}{2}$ belong to $\pi\mathbb{Z}$ and so the sines vanish and the equation of $f(s)$ is satisfied for any value of $\beta$.

Next, notice that $b(s)$ gives us that $\alpha = e^{-s_w} - \beta e^{-s_v}$. Substituting $\alpha$ in the relation given by $c(s)$ and joining with the equation of $g(s)$, we discover the necessary value of $\beta$. First, $g(s)$ gives us:

$$-e^{-s_w}+e^{\frac{s_w}{2}}\cos\left(\frac{\sqrt{3}s_w}{2}\right)=\beta\left(-e^{-s_v}+e^{\frac{s_v}{2}}\cos\left(\frac{\sqrt{3}s_v}{2}\right)\right).$$

While $c(s)$ yields:

$$\frac{1}{9}e^{-s_w}\left(-8+e^{3s_w}+16e^{\frac{3s_w}{2}}\cos\left(\frac{\sqrt{3}s_w}{2}\right)\right)=\alpha+\beta\frac{1}{9}e^{-s_v}\left(-8+e^{3s_v}+16e^{\frac{3s_v}{2}}\cos\left(\frac{\sqrt{3}s_v}{2}\right)\right)$$
$$\Longleftrightarrow$$
$$-e^{-s_w}+\frac{1}{9}e^{-s_w}\left(-8+e^{3s_w}+16e^{\frac{3s_w}{2}}\cos\left(\frac{\sqrt{3}s_w}{2}\right)\right)=-\beta e^{-s_v}+\beta\frac{1}{9}e^{-s_v}\left(-8+e^{3s_v}+16e^{\frac{3s_v}{2}}\cos\left(\frac{\sqrt{3}s_v}{2}\right)\right)$$
$$\Longleftrightarrow$$
$$-17e^{-s_w}+e^{2s_w}+16e^{\frac{s_w}{2}}\cos\left(\frac{\sqrt{3}s_w}{2}\right)=\beta\left(-17e^{-s_v}+e^{2s_v}+16e^{\frac{s_v}{2}}\cos\left(\frac{\sqrt{3}s_v}{2}\right)\right).$$

Joining the equations provided by $c(s)$ and $g(s)$, we obtain:

$$-e^{-s_w}+e^{2s_w}=\beta\left(-e^{-s_v}+e^{2s_v}\right)\iff\beta=\frac{-e^{-s_w}+e^{2s_w}}{-e^{-s_v}+e^{2s_v}}.$$

Finally, by substituting $\beta$ back in the relation we have obtained due to $g(s)$, we find a curious relation between $s_v$ and $s_w$.

$$-e^{-s_w}+e^{\frac{s_w}{2}}\cos\left(\frac{\sqrt{3}s_w}{2}\right)=\frac{-e^{-s_w}+e^{2s_w}}{-e^{-s_v}+e^{2s_v}}\left(-e^{-s_v}+e^{\frac{s_v}{2}}\cos\left(\frac{\sqrt{3}s_v}{2}\right)\right)$$
$$\Longleftrightarrow$$
$$\frac{-e^{-s_w}+e^{\frac{s_w}{2}}\cos\left(\frac{\sqrt{3}s_w}{2}\right)}{-e^{-s_w}+e^{2s_w}}=\frac{-e^{-s_v}+e^{\frac{s_v}{2}}\cos\left(\frac{\sqrt{3}s_v}{2}\right)}{-e^{-s_v}+e^{2s_v}}$$
$$\Longleftrightarrow$$
$$\frac{-e^{\frac{-3s_w}{2}}+\cos\left(\frac{\sqrt{3}s_w}{2}\right)}{-e^{\frac{-3s_w}{2}}+e^{\frac{3s_w}{2}}}=\frac{-e^{\frac{-3s_v}{2}}+\cos\left(\frac{\sqrt{3}s_v}{2}\right)}{-e^{\frac{-3s_v}{2}}+e^{\frac{3s_v}{2}}}.$$

This equation implies that $s_v=s_w$, because if we substitute $\frac{3s_v}{2}$ by $x$ and the cosine in the expression by either 1 or $-1$, the only two values it may assume, we get injective functions with disjoint ranges.

$$F(x)=\frac{-e^{-x}+1}{-e^{-x}+e^{x}}=\frac{1}{1+e^{x}},\quad\text{and}\quad G(x)=\frac{-e^{-x}-1}{-e^{-x}+e^{x}}=\frac{1}{1-e^{x}}.$$

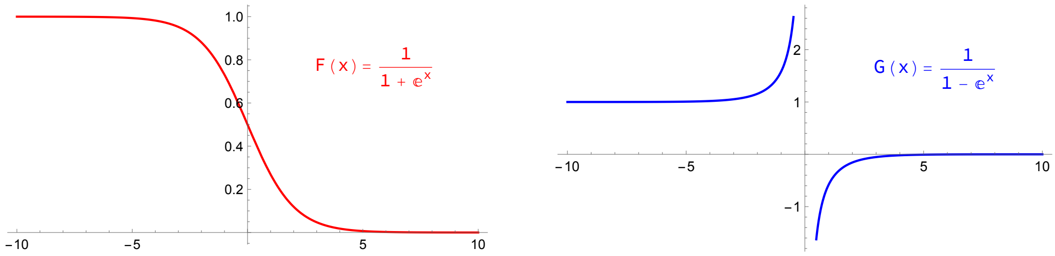


Figure 6.6: Graphs of the functions $F(x)$ and $G(x)$.

Indeed, both functions above are injective, the range of $F(x)$ is $]0,1[$, while the range of $G(x)$ is $]-\infty,0[\ \cup\ ]1,+\infty[$. Therefore, the relation we have obtained is true if and only if $s_v = s_w$, which concludes the case where $\sin(\sqrt{3}s_v) = \sin(\sqrt{3}s_w) = 0$.

Now let us consider the other case, when those sines do not vanish. The equation provided by the coefficient $h(s)$ gives us already the value of $\beta$ in terms of $s_v$ and $s_w$.

$$e^{-s_w}\sin(\sqrt{3}s_w) = \beta e^{-s_v}\sin(\sqrt{3}s_v) \iff \beta = e^{s_v - s_w}\frac{\sin(\sqrt{3}s_w)}{\sin(\sqrt{3}s_v)}.$$

Then, replacing $\beta$ in the relation provided by the difference of the equations of $a(s)$ and $b(s)$ yields a nice trigonometric identity. Since $\tan\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{\sin(x)}$, we have that

$$\begin{gathered} e^{-s_w}\left(1-\cos(\sqrt{3}s_w)\right) = \beta e^{-s_v}\left(1-\cos(\sqrt{3}s_v)\right) \\ \iff \\ \csc(\sqrt{3}s_w) - \cot(\sqrt{3}s_w) = \csc(\sqrt{3}s_v) - \cot(\sqrt{3}s_v) \\ \iff \\ \tan\left(\frac{\sqrt{3}s_w}{2}\right) = \tan\left(\frac{\sqrt{3}s_v}{2}\right). \end{gathered}$$

Since $\frac{\sqrt{3}s_w}{2}$ and $\frac{\sqrt{3}s_w}{2}$ must have the same tangent, then $\frac{\sqrt{3}s_v}{2} - \frac{\sqrt{3}s_w}{2} \in \pi\mathbb{Z}$ and hence $\sqrt{3}s_v - \sqrt{3}s_w \in 2\pi\mathbb{Z}$. This in turn implies that $\sin(\sqrt{3}s_v) = \sin(\sqrt{3}s_w)$, and so $\beta = e^{s_v - s_w}$.

To conclude, we just have to substitute this $\beta$ in the equation provided by the coefficient $b(s)$.

$$e^{-s_w}\left(-2+\cos(\sqrt{3}s_w)\right) = -\alpha + \beta e^{-s_v}\left(-2+\cos(\sqrt{3}s_v)\right)$$
$$\Longleftrightarrow$$
$$e^{-s_w}\left(-2+\cos(\sqrt{3}s_w)\right) = -\alpha + e^{-s_w}\left(-2+\cos(\sqrt{3}s_v)\right).$$

Since $\cos(\sqrt{3}s_v) = \cos(\sqrt{3}s_w)$, the equation above implies that $\alpha = 0$, which is a contradiction, since the 3 conics are supposed to be distinct. □

## 6.3

### Curves of constant projective curvature

For the sake of completeness and also in order to provide some more examples, let us present and discuss the case of curves that have constant projective curvature. As we have explained for the case where $k \equiv 0$ in Section 6.1, the curve $\mathbf{x}(\sigma)$ of constant projective curvature $k$ satisfies the following differential equation when parametrized by the projective length:

$$\mathbf{x}'''(\sigma) + 2k\mathbf{x}'(\sigma) + \mathbf{x}(\sigma) = 0.$$

We find the parametrization of such a curve by solving the analogous ODE: $y'''(x) + 2ky'(x) + y(x) = 0$. Notice that there are three possible behaviours depending on the value of $k$, because the *discriminant* of the characteristic polynomial $p(x) = x^3 + 2kx + 1$ is $\Delta = -27 - 32k^3$. The critical value is $k = \sqrt[3]{-27/32} \approx -0.945$, as $\Delta$ is positive if $k < \sqrt[3]{-27/32}$, it vanishes if $k = \sqrt[3]{-27/32}$ and it is negative if $k > \sqrt[3]{-27/32}$. In this last case, $p(x)$ has a single real root $r$ and a pair of complex conjugate roots $a + ib$ and $a - ib$. Also, since we know the coefficients of $p(x)$, we have three equations involving these roots:

$$\begin{cases} (a+ib) + (a-ib) + r = 0; \\ (a+ib)(a-ib) + (a+ib)r + (a-ib)r = 2k; \\ (a+ib)(a-ib)r = -1. \end{cases}$$

From these equations, we may express all variables in terms of $a$, as:

$$\begin{cases} r = -2a; \\ b = \sqrt{\dfrac{1}{2a} - a^2}; \\ k = \dfrac{1}{4a} - 2a^2. \end{cases}$$

Since we are working in the case where $k > \sqrt[3]{-27/32}$, the equation relating $k$ to $a$ indicates that we must consider only $0 < a < \sqrt[3]{1/2} \approx 0.794$. The square root in the relation between $b$ and $a$ also gives us the same restriction. Therefore, for any given $a$ in this interval, we have the generic solution of the ODE:

$$y(x) = C_1 e^{-2ax} + C_2 e^{ax} \cos\left(\sqrt{\tfrac{1}{2a} - a^2}\, x\right) + C_3 e^{ax} \sin\left(\sqrt{\tfrac{1}{2a} - a^2}\, x\right), \quad C_1, C_2, C_3 \in \mathbb{R}.$$

Again as in Section 6.1, if we set each term of the solution as one of the coordinates of $\mathbf{x}(\sigma)$ we get a curve of constant projective curvature $k = \frac{1}{4a} - 2a^2$.

$$\mathbf{x}(\sigma) = \left(e^{a\sigma} \cos\left(\sqrt{\frac{1}{2a} - a^2}\, \sigma\right) : e^{a\sigma} \sin\left(\sqrt{\frac{1}{2a} - a^2}\, \sigma\right) : e^{-2a\sigma}\right).$$

As one may notice, these are also logarithmic spirals. The case where $k = 0$ corresponds to $a = 1/2$.

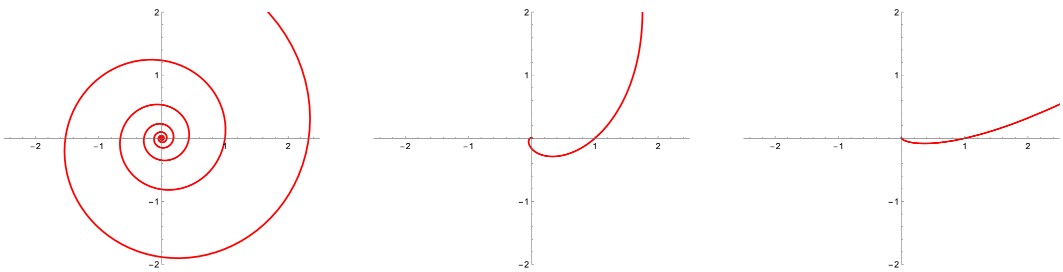


Figure 6.7: Some curves of constant projective curvature. Their respective values for $(a, k)$ are: $(0.1, 2.48)$, $(0.4, 0.305)$ and $(0.7, -0.622857)$

With the parametrization in hand, we may use *Mathematica* once again to compute the quadratic parametrizations and the implicit equations of the osculating conics. Then, we generate the binary quartics and analyse their algebraic invariants. As explained in Subsection 5.4.2, the power series of the

$J$-invariant around zero is $J(\sigma) = \frac{32}{27} + \frac{8}{189}k\sigma^2 + O(\sigma^3)$, while for the $K$-invariant we have $K(\sigma) = \frac{\sqrt{30}}{72} + \frac{3\sqrt{30}}{2240}k\sigma^2 + O(\sigma^3)$. Therefore, if the projective curvature is negative, then both of these invariants should attain values that are smaller than their limiting values at zero. This is indeed what we observe in their graphs.

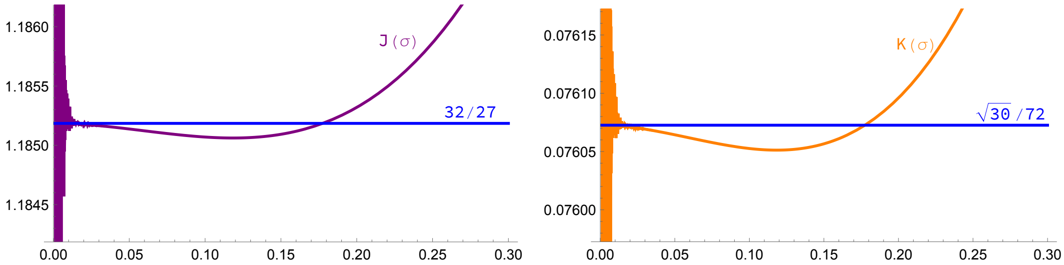


Figure 6.8: The graphs of $J(\sigma)$ and $K(\sigma)$ of a curve with constant projective curvature $k \approx -0.6$. There is considerable numerical instability around $\sigma = 0$.

# A

# Pencils of Conics

This appendix is a self-contained portion dedicated to an extensive study of the space of pencils of conics, both in the complex and real settings. The projective groups $\mathrm{PGL}(3;\mathbb{C})$ and $\mathrm{PGL}(3;\mathbb{R})$ act on the respective projective planes via projective transformations, which map conics into conics. By linearity, analogous actions are induced on the spaces of pencils of conics. We classify and describe in detail the orbits of such actions and we also analyse the stabilizer subgroups of a representative of each orbit that serve as *normal form*. Moreover, we consider also the case of *marked pencils of conics*, where either one or two conics of the pencil are highlighted. The basis of the classification that we present in this appendix can be found in [Persson]. We explain it in much further detail, provide figures and analyse the symmetries of each kind of pencil of conics.

## A.1

## The complex and real spaces of conics and their pencils

A conic in $\mathbb{CP}^2$ is given implicitly by a homogeneous equation of degree 2 in the homogeneous variables $x, y, z$. It is thus determined by 6 complex coefficients which cannot all vanish simultaneously. One may write it either as a straightforward equation or in matrix form:

$$ax^2 + by^2 + cz^2 + 2fxz + 2gyz + 2hxy = 0.$$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & h & f \\ h & b & g \\ f & g & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

Since multiples of a given expression represent the same curve, the space of conics is $\mathbb{CP}^5$, the complex projective space of dimension 5. We will call an element of this space a *real conic* when it admits an expression whose coefficients are all real. The subset of all real conics is then a copy of $\mathbb{RP}^5$ embedded in $\mathbb{CP}^5$.

The projective linear group $\mathrm{PGL}(3;\mathbb{C})$ acts on the projective plane via projective transformations. The fundamental theorem of projective geometry states that given two sets of four points of $\mathbb{CP}^2$ in general position, $\{P_1, P_2, P_3, P_4\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$, there exists a unique element of $\mathrm{PGL}(3;\mathbb{C})$ that maps $P_i$ to $Q_i$ for every $i \in \{1, 2, 3, 4\}$. In other words, the images of four points in general position fully determine a projective transformation. Moreover, this action transforms the algebraic curves of the projective plane while preserving their degree, so one such map sends a conic into another conic, therefore defining an action on the space of conics as well.

From the point of view of the implicit equations, the action of a projective transformation $M \in \mathrm{PGL}(3;\mathbb{C})$ on a conic is best understood in the matrix form. Let $U$ be a symmetric matrix associated to a conic, then the new conic obtained after the transformation $M$ has been applied to the projective plane is given by the congruent matrix $(M^{-1})^{\mathsf{T}} U M^{-1}$. This action has only 3 orbits, corresponding to the rank of the associated matrix: the *irreducible conics* have full rank, the *pairs of distinct lines* have rank 2, and the *double lines* have rank 1. Therefore, the determinant of the associated matrix is a very important quantity, it is called the *discriminant* and is denoted by $\Delta$. The subset given by the expression $\Delta = 0$ is an algebraic submanifold $\Sigma \subset \mathbb{CP}^5$ given by a cubic expression and it is called the *discriminant hypersurface*. Naturally, its complement is a dense open set which corresponds to the orbit of irreducible conics. Inside of $\Sigma$ itself there is a 2-dimensional submanifold $\Omega$ which corresponds to the double lines; hence it is another orbit of the action, the one of smallest dimension, while $\Sigma \setminus \Omega$ is the 4-dimensional orbit of distinct double lines.

The real case is similar, there is also a discriminant hypersurface $\Sigma \cap \mathbb{RP}^5 \subset \mathbb{CP}^5$ with its singular submanifold $\Omega \cap \mathbb{RP}^5$ which divide the conics into the same three major types. However, the action of the real projective linear group $\mathrm{PGL}(3;\mathbb{R})$ on the space of conics actually has 5 distinct orbits. Algebraically, this is due to *Sylvester's Law of Inertia*, which states that the action we have in hands preserves the *signature* of the symmetric matrices associated to the conics. Since we are dealing with implicit equations up to non-zero multiples, the signature of a conic is not well-defined, as a negative multiple interchanges the positive and negative *indices*, but one may still categorize these classes of equations into the following 5 types:

| **Classification of real projective conics** | | |
|---|---|---|
| Signature | Normal form | Type |
| $(3,0)$ or $(0,3)$ | $x^2+y^2+z^2=0$ | Empty irreducible |
| $(2,1)$ or $(1,2)$ | $x^2+y^2-z^2=0$ | Non-empty irreducible |
| $(2,0)$ or $(0,2)$ | $x^2+y^2=0$ | Imaginary line-pair |
| $(1,1)$ | $x^2-y^2=0$ | Real line-pair |
| $(1,0)$ or $(0,1)$ | $x^2=0$ | Repeated line |

As one can see, there are two kinds of irreducible conics, the *non-empty* ones, which manifest themselves in the real projective plane $\mathbb{RP}^2$ as a real curve, and the *empty* ones, that do not have a single point in $\mathbb{RP}^2$ as their associated matrices are *definite*. Similarly, there are two types of distinct double lines with real equations. They may either have two real factors and thus be made of a pair of real lines, or they might be composed of two conjugate complex factors, in which case it is a pair of imaginary lines that intersect at a single point in $\mathbb{RP}^2$.

Given two points $u$ and $v$ in $\mathbb{CP}^5$, each corresponding to a conic, one may consider the unique line that joins them, which may be parametrized by $\alpha u + \beta v$, where $[\alpha : \beta] \in \mathbb{CP}^1$. This set corresponds to a particular family of conics called a *complex pencil of conics*. Naturally, there is also the analogous concept for the real case: if $u$, $v \in \mathbb{RP}^5$ are two distinct elements, then $\alpha u+\beta v$ with $[\alpha : \beta] \in \mathbb{RP}^1$ is a *real pencil of conics*. The aforementioned actions of the corresponding projective linear groups map a pencil into another pencil, in other words, they act on the *space of lines* of $\mathbb{CP}^5$ or $\mathbb{RP}^5$.

A complex line of $\mathbb{CP}^5$ corresponds to a plane of $\mathbb{C}^6$, so the space of complex pencils of conics is the *Grassmannian* $\mathrm{Gr}(2;\mathbb{C}^6)$, while the space of real pencils of conics is $\mathrm{Gr}(2;\mathbb{R}^6)$. The transformations of the projective plane by the appropriate projective linear group give rise to actions on these spaces. The main goal of this appendix is to classify and to describe in detail their orbits.

**Theorem A.1.1** (Classification of complex pencils of conics)**.** *The action of* $\mathrm{PGL}(3;\mathbb{C})$ *on the space of complex pencils of conics has* 8 *orbits.*

$$\begin{array}{rr} (1,1,1,1) - \textit{dimension } 8 & (4) - \textit{dimension } 5 \\ (2,1,1) - \textit{dimension } 7 & (4^*) - \textit{dimension } 4 \\ (2,2) - \textit{dimension } 6 & (\infty,1) - \textit{dimension } 4 \\ (3,1) - \textit{dimension } 6 & (\infty) - \textit{dimension } 3 \end{array}$$

The names of the orbits will be explained as we describe them individually along the proof. Since $\mathrm{Gr}(2;\mathbb{C}^6)$ is an 8-dimensional complex manifold, one can see that this action admits one single generic orbit which is a dense open set of the space of complex pencils. We also describe the analogous result for the real setting.

**Theorem A.1.2** (Classification of real pencils of conics)**.** *The action of* $\mathrm{PGL}(3;\mathbb{R})$ *on the space of real pencils of conics has* 13 *orbits.*

$$\begin{array}{rrr} (1,1,1,1) - \textit{dimension } 8 & (2,2) - \textit{dimension } 6 & (4^*) - \textit{dimension } 4 \\ (1,1,1,\overline{1}) - \textit{dimension } 8 & (2,\overline{2}) - \textit{dimension } 6 & (4^{**}) - \textit{dimension } 4 \\ (1,\overline{1},1,\overline{1}) - \textit{dimension } 8 & (3,1) - \textit{dimension } 6 & (\infty,1) - \textit{dimension } 4 \end{array}$$

$$\begin{array}{rrr} (2,1,1) - \textit{dimension } 7 & (4) - \textit{dimension } 5 & (\infty) - \textit{dimension } 3 \\ (2,1,\overline{1}) - \textit{dimension } 7 & & \end{array}$$

The real orbits follow essentially the same division as the complex ones, but some cases are further subdivided. The generic orbit, for example, becomes three distinct classes. Every orbit is naturally a *homogeneous space* for the projective linear group $G = \mathrm{PGL}(3;k)$, for $k = \mathbb{C}$ or $\mathbb{R}$. One may choose a particular representative $\omega \in \mathrm{Gr}(2;k^6)$ as the *normal form* for each orbit and then look for its *stabilizer subgroup* $\mathrm{Stab}(\omega)$. The resulting coset space $G/\mathrm{Stab}(\omega)$ is homeomorphic to the orbit, and the sum of the dimensions of the orbit and the stabilizer must always be equal to 8, the dimension of the whole group due to the Orbit-Stabilizer Theorem. We shall pick a normal form and present its stabilizer for every type of pencil, obtaining larger subgroups as we pass through more degenerate cases.

Another interesting class of objects that one could consider are the *marked pencils of conics*, whose elements are pencils with a certain number of particular elements highlighted. Since any pencil is uniquely defined by a pair of distinct conics $\{u, v\} \subset k\mathbb{P}^5$, we are going to investigate the actions of the projective linear groups over the set of pencils marked with a single element and the set of pencils marked with two distinct elements.

## A.2

### The complex space of conics

In order to prove the theorem about the classification of the complex pencils of conics we describe firstly some fundamental aspects of the complex space of conics and its pencils, then we shall present each orbit individually and study them thoroughly. The space of conics itself is $\mathbb{CP}^5$ and a complex pencil of conic is a straight line in this space, which admits thus the structure of $\mathbb{CP}^1$. One possible way to determine such a pencil is to begin with two distinct conics $u, v \in \mathbb{CP}^5$ and consider the unique line containing these two points. In this case, we are going to call $u$ and $v$ the *generating conics* of the pencil and we may parametrize it by $\alpha u + \beta v$, where $[\alpha : \beta] \in \mathbb{CP}^1$. Notice that if one takes another pair of conics $u', v'$ in this same line, one obtains the same pencil, although it will be parametrized differently.

*Bézout's theorem* tells us that any two distinct conics $u, v$ with no common factor always intersect in 4 points on $\mathbb{CP}^2$ counting their multiplicities properly. Consequently, every element of a pencil must also pass through these same 4 *common points*, as they satisfy $\alpha u + \beta v = 0$ for all $[\alpha : \beta] \in \mathbb{CP}^1$. In the case where they are all distinct and in general position, we have another practical way of describing the pencil, it consists of all the conics which pass through those 4 points. It is worth mentioning that this case already encompasses a dense set among all possible pencils, with only the cases where at least three of the common points are collinear or when at least two of them coincide. But even in these situations, it is still possible to give geometric descriptions that characterize the conics of the pencil, because one just has to specify the *contact* between them in each of the common points, indicating, in particular, the *common tangent* when the contact is of order greater than 1.

Coming back to the algebraic expression $\alpha u + \beta v$ of a pencil, we may investigate its degenerate conics. These are easily spotted by their null discriminant $\Delta(\alpha u + \beta v) = 0$. As explained in Section A.1, the discriminant is the determinant of the associated symmetric matrix and it vanishes if and only if the conic is reducible. On top of that, one may also consider its first minors.

If all of them vanish alongside $\Delta$, then the matrix has rank 1 and thus the conic is a double line.

By evaluating the discriminant $\Delta(\alpha u + \beta v)$, we have in hand an expression on the projective parameters $[\alpha : \beta]$. Since it is given by the determinant of a matrix in which every coefficient is either linear in $\alpha$ and $\beta$ or null, the discriminant must be either a homogeneous polynomial of degree 3 in these variables or a constant $\Delta = 0$. This already shows us that a pencil is either completely degenerate containing only pairs of lines, or it has at least one and at most three degenerate conics. In addition, it can have at most two double lines, which correspond to simultaneous roots of $\Delta$ and the minors, all homogeneous polynomials of degree 2 or identically null.

With regard to the action of $\mathrm{PGL}(3;\mathbb{C})$ via projective transformations, the image of a pencil is determined by the images $u', v'$ of the generating conics $u, v$ respectively. Indeed, for any $M \in \mathrm{PGL}(3;\mathbb{C})$ we have by linearity that it maps an arbitrary element $\alpha u + \beta v$ of the first pencil to the element $\alpha u' + \beta v'$ of the second pencil, so from the point of view of their $\mathbb{CP}^1$ structure and considering the chosen parametrization, the map induced by $M$ is simply the identity.

Let us now suppose that $M$ is an element of the stabilizer subgroup of the pencil $\omega$ generated by $u, v$. In this case, how does $M$ act on it? We have a first fundamental lemma that answers this question.

**Lemma A.2.1.** *Let $\omega$ be the complex pencil of conics generated by two distinct conics $u, v \in \mathbb{CP}^5$.*
*Suppose that $M \in \mathrm{Stab}(\omega) \subset \mathrm{PGL}(3;\mathbb{C})$ is a projective transformation that preserves the pencil $\omega$. Then the action of $M$ induces a* projective transformation $\mathcal{M} \in \mathrm{PGL}(2;\mathbb{C})$ *on the $\mathbb{CP}^1$ structure of the pencil.*

*Proof.* Given that $M$ preserves the pencil, we know that the image of $u$ must stay on the same pencil, so we may write $M.u = au + bv$ for some $[a : b] \in \mathbb{CP}^1$. The same holds for $v$, $M.v = cu + dv$ for some $[c : d] \in \mathbb{CP}^1$. Now, for an arbitrary element of the pencil we have:

$$M.(\alpha u + \beta v) = \gamma u + \delta v, \qquad \mathcal{M}.[\alpha : \beta] = [\gamma : \delta].$$

To attain the result, we just have to apply the linearity of $M$.

$$M.(\alpha u+\beta v) = \alpha M.u+\beta M.v = \alpha(au+bv)+\beta(cu+dv) = (\alpha a+\beta c)u+(\alpha b+\beta d)v.$$

We have thus obtained $[\gamma : \delta]$ as a linear combination of $[\alpha : \beta]$, in other words:

$$\mathcal{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{PGL}(2;\mathbb{C}), \qquad \mathcal{M}.[\alpha : \beta] = [\gamma : \delta].$$

□

It will be very useful to bare in mind the fact that an element of the stabilizer subgroup of a pencil acts on it as a projective transformation. This implies, for example, that this action is completely determined by the image of 3 distinct elements.

Under the action of any element of $\mathrm{PGL}(3;\mathbb{C})$, the *configuration* of the common points is preserved, so it may be used to classify and name the 8 different orbits. In the following section we present every class, we find to each of them a normal form and its stabilizer subgroup and we further categorize the orbits of marked pencils starting from the generic orbit and moving progressively through increasingly degenerate cases.

## A.3

## Classification of complex pencils of conics

### A.3.1

### Complex $(1,1,1,1)$ Pencil

We begin by the case where the four common points are distinct and in general position. If we vary slightly their locations we still obtain the same type of configuration, thus this is a generic case, and we will soon see that this is the only generic type of complex pencil. Since every pair of conics in this pencil presents the same four *simple intersections*, we are going to name it $(1,1,1,1)$. By taking the action of an element of $\mathrm{PGL}(3;\mathbb{C})$, we may take these four points anywhere we want in $\mathbb{CP}^2$, as long as they stay in general position. Therefore, the family of $(1,1,1,1)$ pencils is a single orbit.

With regard to the dimension, since it is necessary to determine the position of four points in $\mathbb{CP}^2$, the set of pencils of the type $(1,1,1,1)$ constitutes a manifold of complex dimension 8. The space of all complex pencils of conics is the set of lines of $\mathbb{CP}^5$, which is also of dimension 8 because a line is determined by two points, but if one moves these points along the line, they still generate the same one; so we have $5 + 5 - 2 = 8$. It is natural that the submanifold of $(1, 1, 1, 1)$ pencils has the same dimension since it is a dense open set of the manifold of all lines of $\mathbb{CP}^5$.

In order to identify the degenerate conics that belong to this type of pencil, one just has to divide the common points into two pairs; each pair gives rise to a line, thus forming a pair of lines. Since one can do so in three different ways, there are three pairs of lines in the pencil, hence this is a case where the discriminant has three distinct roots.

**Normal form of the $(1, 1, 1, 1)$ pencil**

For the normal form, let us fix the common points as $P_1 = [1 : 0 : 1]$, $P_2 = [0 : 1 : 1]$, $P_3 = [-1 : 0 : 1]$ and $P_4 = [0 : -1 : 1]$. By doing so, the degenerate conics of the pencil are determined and may be given by the following expressions:

$$\begin{aligned} w_\infty &= \frac{1}{2}(-x + y + z)(x - y + z) = -\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 + \frac{1}{2}z^2, \\ w_0 &= \frac{1}{2}(x + y + z)(x + y - z) = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 - \frac{1}{2}z^2, \\ w_1 &= w_\infty + w_0 = 2xy. \end{aligned}$$

We may use these particular elements of the pencil to parametrize it. If we consider $\alpha w_\infty + \beta w_0$ with $[\alpha : \beta] \in \mathbb{CP}^1$, then the degenerate conics are given by the coordinates $[1 : 0]$, $[0 : 1]$ and $[1 : 1]$.

**Lemma A.3.1.** *The stabilizer subgroup of the normal form of the* $(1, 1, 1, 1)$ *pencil is isomorphic to the symmetric group* $S_4$.

*Proof.* This stabilizer must be isomorphic to $S_4$ because the only projective transformations that preserve the normal form are those that permute the four common points. $\square$

**Marked $(1, 1, 1, 1)$ pencil**

*Proposition* A.3.2. There are two kinds of orbits of $(1, 1, 1, 1)$ pencils marked with a single conic:

i. *The marked conic is irreducible:*
Infinitely many orbits of dimension 8 described by an invariant;

ii. *The marked conic is degenerate:*
A single orbit of dimension 8.

In addition, there are three kinds of orbits of $(1, 1, 1, 1)$ pencils marked with an unordered pair of conics.

i. *Both conics are irreducible:*
Infinitely many orbits of dimension 8 described by a pair of invariants;

ii. *A single conic is degenerate:*
Infinitely many orbits of dimension 8 described by an invariant;

iii. *Both conics are degenerate:*
A single orbit of dimension 8.

*Proof.* Consider a pair of irreducible conics $u$ and $v$ in a $(1, 1, 1, 1)$ pencil. When transforming the pencil in order to obtain the normal form, the generating conics $u$ and $v$ are sent naturally to two conics in the new pencil. The only flexibility that one still has in the choice of the new conics for $u$ and $v$ comes from the symmetries of the pencil resulting from the permutations of the common points. Some of these transformations act as the identity over the entire pencil preserving every element, while others act non trivially. The action induced on the pencil is entirely determined by the image of the three degenerate conics which are permuted following the common points. Let us describe them by the pair of common points that generate each of their lines.

$$\begin{aligned} w_\infty &= P_1P_4, P_2P_3, \\ w_0 &= P_1P_2, P_3P_4, \\ w_1 &= P_1P_3, P_2P_4. \end{aligned}$$

The transformations whose action on the pencil are trivial are those that preserve all three degenerate conics. One can verify that these are the *double transpositions*, which constitute a subgroup of order 4 isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, as the dihedral group $D_2$ and the Klein four-group $K_4$.

$$K_4 = \{(),\ (1, 4)(2, 3),\ (1, 2)(3, 4),\ (1, 3)(2, 4)\}.$$

Therefore, a marked $(1,1,1,1)$ pencil admits up to $24/4 = 6$ normal forms in the same orbit, although this number may be even lower if the marked conics are also preserved by some of these permutations, in order words, a marked $(1,1,1,1)$ pencil may have itself a non-trivial stabilizer. This fact has a natural connexion to the symmetries of the *cross-ratio*, which can also assume up to 6 different values by permuting the four points that define it. Indeed, the pencil generated by $u$ and $v$ has the structure of a projective line $\mathbb{CP}^1$ and once an order is chosen for the degenerate elements, each conic has well-defined coordinates $[\alpha : \beta] \in \mathbb{CP}^1$ that correspond to their cross-ratio with respect to those three special elements. This explains the classification of the marked $(1,1,1,1)$ pencils. □

Finally, we present a figure in the hope of illustrating this kind of pencil. Of course, we are unfortunately limited by the real picture, so every figure we display is just a portion of the complex pencil showing in fact a real pencil of conics contained in it. Here the pencil is parametrized by $\alpha w_\infty + \beta w_0$ with the following conics highlighted:

| Conic | Expression | Pencil's coordinates |
|---|---|---|
| $w_\infty$ | $-\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 + \frac{1}{2}z^2$ | $[1:0]$ |
| $w_0$ | $\frac{1}{2}x^2 + xy + \frac{1}{2}y^2 - \frac{1}{2}z^2$ | $[0:1]$ |
| $w_1$ | $2xy$ | $[1:1]$ |
| $u$ | $x^2 + y^2 - z^2$ | $[-1:1]$ |
| $v$ | $-x^2 + 4xy - y^2 + z^2$ | $[3:1]$ |

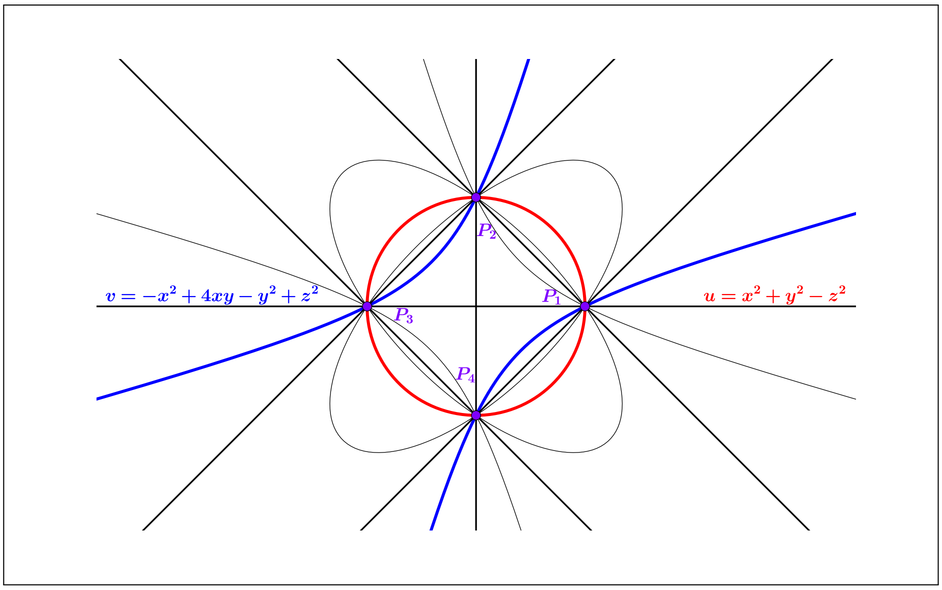


Figure A.1: Normal form of the $(1,1,1,1)$ pencil.

### A.3.2

### Complex $(2, 1, 1)$ Pencil

The second case happens if exactly two of the common points coincide, say at $P$. Let us call $Q$ and $Q'$ the other common points. In order to have a well-defined pencil, one has also to specify a line through $P$, which is going to be a *common tangent* to every nondegenerate conic of the pencil. If we want to avoid even more degenerate cases, it is also necessary to prohibit that the points $Q$ and $Q'$ belong to the common tangent and that $P$ belongs to the line $QQ'$. This characterizes the pencil of type $(2, 1, 1)$. Through the action of an element of $\mathrm{PGL}(3; \mathbb{C})$ we have full control on the location of the common points and of the common tangent, so this constitutes a single orbit.

The liberty to determine the position of three points in $\mathbb{CP}^2$ plus a line through one of them constitutes a manifold of complex dimension 7. Therefore, we have a submanifold of codimension 1 in the space of all pencils.

There are only two pairs of lines in a pencil of type $(2, 1, 1)$. Let $u$ be a irreducible conic of the pencil. For a pair of lines to have a contact with $u$ of order 2 in $P$, the first possibility is that one of the lines is tangent to $u$ at that point, and then the other line must be $QQ'$. The second possibility is to consider the pair $PQ, PQ'$. Thus the expression of the discriminant of this pencil has two distinct roots. Naturally, one of them must be a simple root, while the other is a double root. The multiple root is related to the pair of line that intersect at $P$. This can be verified by considering this pencil as a limit case of generic pencils.

**Normal form of the $(2, 1, 1)$ pencil**

Let us define the normal form of this orbit. One can find a projective transformation that sends the common points to $P = [0 : -1 : 1]$, $Q = [-1 : 0 : 1]$, $Q' = [1 : 0 : 1]$ and so that the common tangent through $P$ is given by $y+z = 0$. In this case, the degenerate conics of the pencil are:

$$w_\infty = (x-y-z)(x+y+z) = x^2-y^2-2yz-z^2, \qquad w_0 = 2y(y+z) = 2y^2+2yz.$$

This allows us once again to parametrize the pencil by $\alpha w_\infty + \beta w_0$, where $[\alpha : \beta] \in \mathbb{CP}^1$. Since $(2, 1, 1)$ is an orbit of codimension 1, we expect to find a stabilizer of dimension 1 as well. This greater flexibility can be thought in terms of the action induced on the $\mathbb{CP}^1$ structure of the pencil. A transformation of the stabilizer fixes the two degenerate conics, hence determining the position of two points, but a projective transformation on $\mathbb{CP}^1$ is only uniquely determined by the image of three elements. The freedom to choose the image of the third conic in the pencil manifests the continuous nature of the stabilizer. In other words, as a consequence of having just two degenerate conics, this kind of pencil has a larger set of symmetries in comparison to the generic case, instead of a finite set there is in fact a 1-parameter family of transformations that preserve the pencil.

**Lemma A.3.3.** *The stabilizer subgroup of the normal form of the* $(2,1,1)$ *pencil is generated by the* $1$*-parameter family of transformations* $M_t$*, with* $t \in \mathbb{C}^*$*, and the involution* $N$ *below.*

$$M_t = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t-1 & t \end{pmatrix}, \qquad N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* Firstly, let us consider the permutations of the common points. Since $P$ plays a different role of the remaining common points, it must be preserved by all symmetries, but we may permute $Q$ and $Q'$, which is precisely what the reflection $N : [x : y : z] \mapsto [-x : y : z]$ does. Furthermore, in order to preserve the pencil one should pay attention to the common tangent through $P$, which must also be preserved to safeguard the normal form. So if one considers an auxiliary point $P' = [1 : -1 : 1]$ on this line, any symmetry of the pencil must keep it on the same line while avoiding coinciding with $P$ and the collinearity with $Q$ and $Q'$. From the liberty to choose this image, we obtain the one-parameter family of symmetries.

$$\begin{aligned} &P \overset{M_t}{\longmapsto} P, \qquad && P' \overset{M_t}{\longmapsto} [t : -1 : 1] \text{ with } t \in \mathbb{C}^*, \\ &Q \overset{M_t}{\longmapsto} Q, \qquad && Q' \overset{M_t}{\longmapsto} Q'. \end{aligned}$$

$\square$

We draw attention to the fact that, differently from the generic case, every transformation of the stabilizer subgroup preserves $w_\infty$ and $w_0$ individually, so there can be no permutation of the two. There is a simple geometric reason for this behaviour; since $w_0$ is tangent to every irreducible conic of the pencil at $P$, while $w_\infty$ has two components that meet at this point, no projective transformation could interchange them because these properties are preserved.

**Marked $(2,1,1)$ pencil**

Let us now analyse how these transformations act on the degenerate conics at the level of their implicit equations. We recall that if one considers the matrix $U$ associated to a conic, then its image under the action of a transformation $M \in \mathrm{PGL}(3;\mathbb{R})$ is given by the congruent matrix $(M^{-1})^\intercal U M^{-1}$. Via a straightforward calculation we obtain:

$$M_t.w_\infty = \frac{1}{t^2} w_\infty, \qquad M_t.w_0 = \frac{1}{t} w_0,$$

$$N.w_\infty = w_\infty, \qquad N.w_0 = w_0.$$

This confirms algebraically that the stabilizer subgroup does not permute $w_\infty$ and $w_0$. On top of that, it allows us to understand how these transformations act on an arbitrary element of the pencil thanks to linearity.

$$M_t.(\alpha w_\infty + \beta w_0) = \frac{\alpha}{t^2}\, w_\infty + \frac{\beta}{t}\, w_0,$$

$$N.(\alpha w_\infty + \beta w_0) = \alpha w_\infty + \beta w_0.$$

As one can see, the element of coordinates $[\alpha : \beta] \in \mathbb{CP}^1$ of the pencil is sent to the element of coordinates $[\alpha : t\beta]$ by the transformation $M_t$, while the transformation $N$ acts trivially on the pencil, preserving every conic. Thus one can find a particular value of $t$ for which the transformation $M_t$ sends a marked irreducible conic $u$ to the conic of coordinates $[1 : 1]$ of the pencil, that is $u = w_\infty + w_0 = x^2 + y^2 - z^2$. However, one does not have any further liberty to control the image of a second marked conic $v$, after all, there are already three particular elements of the pencil determined, namely $u$ and the degenerate conics $w_\infty$, $w_0$. Again, since each of these have distinct geometric characteristics, they cannot be interchanged. With the image of $u$ fixed, there are only two symmetries left, the identity $M_1 = \mathrm{Id}$ and the reflection $N$, both acting trivially on the pencil. Let us summarize what we have found for the orbits of marked $(2,1,1)$ pencils.

*Proposition* A.3.4. There are three orbits of $(2, 1, 1)$ pencils marked with a single conic.

i. *The marked conic is irreducible:*
   A single orbit of dimension 8;

ii. *The conic is degenerate and contains the common tangent:*
   A single orbit of dimension 7;

iii. *The conic is degenerate and does not contain the common tangent:*
   A single orbit of dimension 7.

In addition, there are four types of orbits of $(2, 1, 1)$ pencils marked with an unordered pair of conics.

i. *Both marked conics are irreducible:*
   Infinitely many orbits of dimension 8, described by an invariant;

ii. *One is irreducible, the other is degenerate and contains the common tangent:*
   A single orbit of dimension 8;

iii. *One is irreducible, the other is degenerate and does not contain the common tangent:*
   A single orbit of dimension 8;

iv. *Both marked conics are degenerate:*
   A single orbit of dimension 7.

*Proof.* Given one such pencil, we may put it in the normal form and study where can the marked conics be. If there is a single marked conic, then it must become $w_0$ or $w_\infty$, then nothing can be done and these orbits have $< N, M_t >$ as their stabilizers, being two distinct orbits of dimension 7. On the other hand, if the marked conic is irreducible, then we have the freedom to move it along the pencil and define a normal form where it becomes $u = w_\infty + w_0 = x^2 + y^2 - z^2$, in which case the stabilizer is just $\{\mathrm{Id}, N\}$, and the orbit has dimension 8.

If there are two marked conics and one of them is degenerate, we fall back to the cases described in the last paragraph. If both marked conics are irreducible, then we can take one of them to the normal form, say $u = w_\infty + w_0 = x^2 + y^2 - z^2$. By doing so, this conic corresponds to the coordinates $[1 : 1]$ of the pencil, while the other marked conic $v$ is associated to the coordinates $[c : 1]$ for some $c \in \mathbb{C} \setminus \{0, 1\}$. Since we are considering it as an *unordered* pair, we could also switch roles by taking a projective transformation of the stabilizer of the unmarked pencil that sends $v$ to $[1:1]$. Since this transformation must preserve the degenerate conics $w_\infty$ and $w_0$, its action on the $\mathbb{CP}^1$ structure of the pencil is given simply by $[a:b] \mapsto [a/c:b]$ and hence it maps $u$ to $[1/c:1]$.

Therefore, each orbit of this kind could admit two possible normal forms in such a way that we may impose that one of the marked conics lies at $[1:1]$ and the other at $[c:1]$ with $|c| \leq 1$ and $c \neq 0$. Notice that at the border of the disc we must exclude $c = 1$ and identify conjugate points because they correspond to the same orbit. In particular, this shows us that the orbit given by $c = -1$ is somewhat special. We can say that this values $c$ is a projective invariant of two irreducible conics that intersect in the $(2,1,1)$ configuration, it characterizes the orbit that their marked pencil belongs to and it may be obtained by putting them in the normal form. This concludes the description of the *fundamental domain* of the action of $\mathrm{PGL}(3;\mathbb{C})$ on the space of $(2,1,1)$ pencils marked with two elements. $\square$

In the figure below, we highlight $u = x^2+y^2-z^2$, associated to the coordinates $[1:1]$ and $v = x^2 - yz - z^2$, associated to the coordinates $[1:1/2]$.

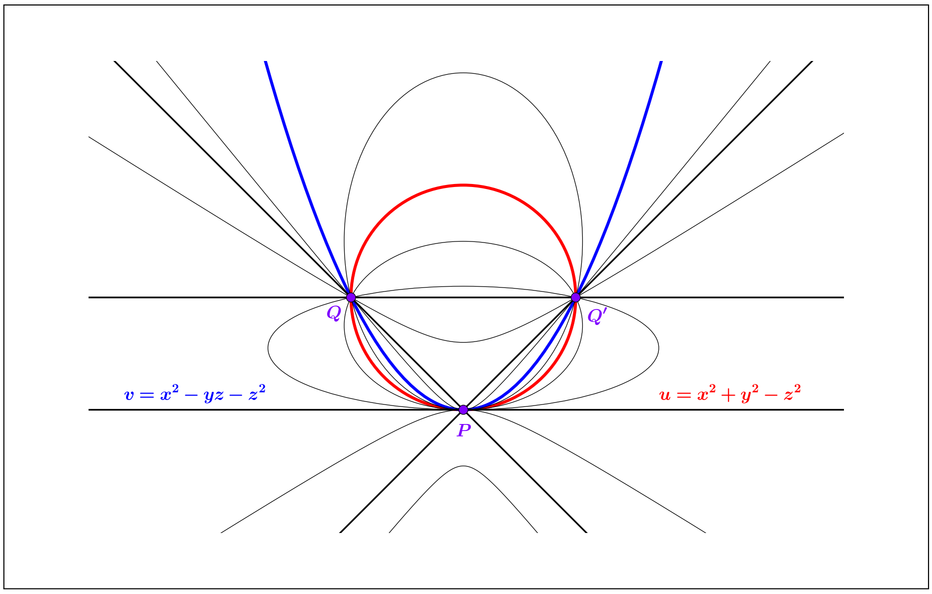


Figure A.2: Normal form of the $(2,1,1)$ pencil.

### A.3.3

#### Complex $(2,2)$ Pencil

In this case, the common points coincide in two pairs, in other words, every pair of conics of the pencil has two intersections of order 2, at $P$ and $Q$. In order to uniquely determine the pencil, one must once again indicate the common tangent through each of the common points and those lines must not intersect at $P$ or $Q$, otherwise we would have an even more degenerate case. Again, through the action of $\mathrm{PGL}(3;\mathbb{C})$ we have full control on the location of the common points and their common tangents, so we have indeed a single orbit.

The choice of two points in $\mathbb{CP}^2$ and a line through each one describes a manifold of complex dimension 6, so this is a case of codimension 2.

One can point out two and only two pairs of lines that belong to a pencil of type $(2,2)$. The first is comprised of the common tangent lines, while the second is the double line $PQ$, so the discriminant has two distinct roots. By analysing it as a limit case of generic configurations, one may conclude that the double line is associated to the double root of the discriminant.

**Normal form of the $(2,2)$ pencil**

One can find a projective transformation that sends the common points to $P = [0 : -1 : 1]$ and $Q = [0 : 1 : 1]$ while the common tangents become $y + z = 0$ and $y - z = 0$ respectively. In this case, the degenerate conics are given by:

$$w_\infty = x^2, \qquad w_0 = (y+z)(y-z) = y^2 - z^2.$$

We parametrize naturally the pencil by $\alpha w_\infty + \beta w_0$, where $[\alpha : \beta] \in \mathbb{CP}^1$. Once again, a projective transformation that brings the common points and tangents to the normal form is not uniquely determined, which means that we have a group of symmetries to study. This time it is even larger, as it contains a two-parameters family of transformations.

**Lemma A.3.5.** *The stabilizer subgroup of the normal form of the $(2,2)$ pencil is generated by the 2-parameter family of transformations $M_{t,s}$, with $t, s \in \mathbb{C}^*$, and the involution $N$ below.*

$$M_{t,s} = \begin{pmatrix} ts & 0 & 0 \\ 0 & \frac{t+s}{2} & \frac{t-s}{2} \\ 0 & \frac{t-s}{2} & \frac{t+s}{2} \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* Firstly, since $P$ and $Q$ play a similar role in this pencil being both points of contact of order 2, they may be permuted carrying their common tangent with them, which can be done by the reflection $N : [x : y : z] \mapsto [x : -y : z]$. On top of that, we look for the symmetries that preserve $P$, $Q$ and the common tangents. With the help of two auxiliary points $P' = [1 : -1 : 1]$ and $Q' = [1 : 1 : 1]$, each one belonging to one of the common tangents, and their respective images $[t : -1 : 1]$ and $[s : 1 : 1]$, with $s, t \in \mathbb{C}^*$ we uniquely determine the projective transformation $M_{t,s}$, as presented above. Notice that we deny points on the line at infinity in order to avoid the collinearity of three of these points.

$$\begin{aligned} P &\xmapsto{M_{t,s}} P, \qquad & P' &\xmapsto{M_{t,s}} [t : -1 : 1] \text{ with } t \in \mathbb{C}^*, \\ Q &\xmapsto{M_{t,s}} Q, \qquad & Q' &\xmapsto{M_{t,s}} [s : -1 : 1] \text{ with } s \in \mathbb{C}^*. \end{aligned}$$

□

**Marked $(2, 2)$ pencil**

In order to classify the orbits of the marked $(2, 2)$ pencils we should verify how the transformations of the stabilizer act on the degenerate conics.

$$M_{t,s}.w_\infty = \frac{1}{t^2 s^2}\, w_\infty, \qquad M_{t,s}.w_0 = \frac{1}{ts}\, w_0,$$

$$N.w_\infty = w_\infty, \qquad N.w_0 = w_0.$$

This actions allow us to choose freely the image of a marked irreducible conic, so we fix $u = w_\infty + w_0 = x^2 + y^2 - z^2$. With three conics settled, we do not have any liberty left to control the image of another marked conic $v$.

*Proposition* A.3.6. There are three orbits of $(2, 2)$ pencils marked with a single element.

i. *The marked conic is irreducible:*
A single orbit of dimension 7;

ii. *The marked conic is the pair of common tangents:*
A single orbit of dimension 6;

iii. *The marked conic is the double line:*
A single orbit of dimension 6.

In addition, there are four types of orbits of $(2,2)$ pencils marked with an unordered pair of conics.

i. *Both marked conics are irreducible:*
Infinitely many orbits of dimension 7 described by an invariant;

ii. *One is irreducible and the other is a double line:*
A single orbit of dimension 7;

iii. *One is irreducible and the other is the pair of common tangents:*
A single orbit of dimension 7;

iv. *Both marked conics are degenerate:*
A single orbit of dimension 6.

*Proof.* If $u$ is a degenerate conic, then it must be either $u = w_0$ or $u = w_\infty$ in the normal form, therefore every transformation of the subgroup $< N, M_{t,s} >$ preserves $u$ and so this is the stabilizer of the marked pencil, implying that those are two 6-dimensional orbits. However, if $u$ is irreducible, then we may define a normal form where $u = w_\infty + w_0 = x^2 + y^2 - z^2$, so it is associated to the coordinates $[1 : 1]$ of the pencil. By construction, any transformation $M_{t,s}$ preserves the degenerate conics, so we just have to check its action on the conic $u$.

$$M_{t,s}.u = M_{t,s}.(w_\infty + w_0) = \frac{1}{t^2 s^2}\, w_\infty + \frac{1}{ts}\, w_0.$$

Notice that the transformation $M_{t,s}$ maps $[1 : 1]$ to $[1 : ts]$. The image of three elements fully determine the action on the pencil, thus if $ts = 1$, then the symmetry $M_{t,s}$ acts trivially on the pencil, preserving every conic. With that, we have shown that the $(2,2)$ pencil has a one-parameter group of symmetries that preserve it globally. In other words, the stabilizer subgroup of this marked pencil is the one-dimensional group $< N, M_{t,1/t} >$. This fact implies that the $\mathrm{PGL}(3;\mathbb{C})$ orbit of marked $(2,2)$ pencils where the marked conic is irreducible has dimension 7.

Let us now move to pencils with two marked conics. The cases where at least one of them is degenerate correspond to the the orbits with a single marked conic explained above. If both are irreducible, we can take one of them to $u = w_\infty + w_0 = x^2 + y^2 - z^2$. As we have seen, the stabilizer of the pencil that also preserves $u$ is $< N, M_{t,1/t} >$, a 1-dimensional group that acts trivially on the whole pencil since it preserves three of its conics, namely $w_0, w_\infty$ and $u$. Therefore, the position of the other conic $v$ is set, the only freedom coming from the fact that the pair $\{u, v\}$ is unordered.

Indeed, we may do the same manipulation as in the proof of Proposition A.3.4 and establish that the coordinates of $v$ in the pencil are $[c : 1]$ for some $c \in \mathbb{C} \setminus \{0, 1\}$ with $|c| \leq 1$ and the same identifications hold. We can say that this value $c$ is a projective invariant of two irreducible conics that intersect in the $(2, 2)$ configuration, it characterizes the orbit that their marked pencil belongs to and it may be obtained by putting them in the normal form. Since the stabilizer has dimension 1, the orbit of the marked pencil must have dimension 7. This concludes the description of the *fundamental domain* of the action of $\mathrm{PGL}(3; \mathbb{C})$ on the space of $(2, 2)$ pencils marked with two elements. $\square$

For the first time, we have a continuous group of symmetries that preserve the entire pencil, that is, there exists a continuum of projective transformations that preserve the three particular conics $w_\infty$, $w_0$ and $u$, and hence every single conic in the pencil. In the following figure we highlight $u = x^2 + y^2 - z^2$ and an arbitrary conic $v = \frac{1}{2}w_\infty + w_0 = \frac{1}{2}x^2 + y^2 - z^2$.

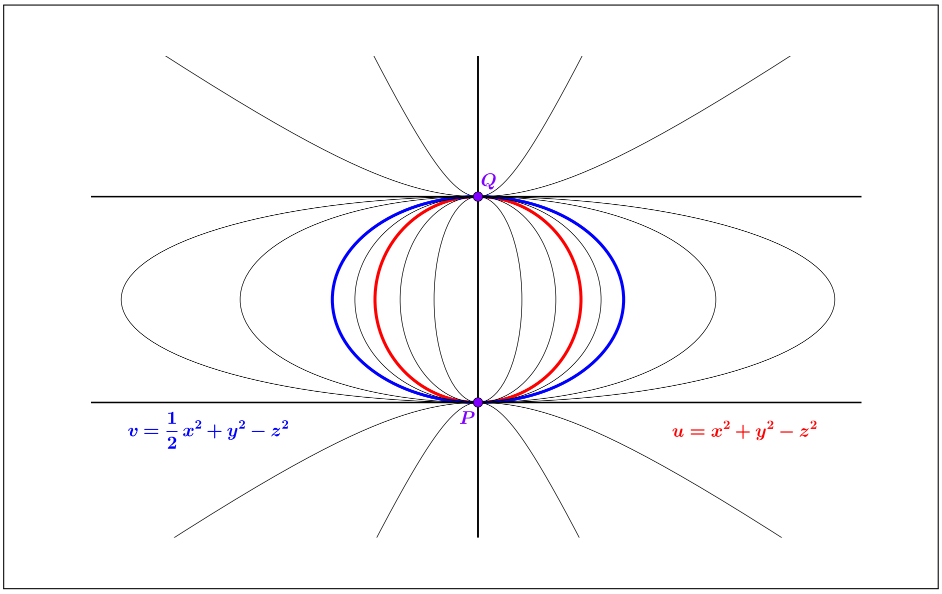


Figure A.3: Normal form of the $(2, 2)$ pencil.

### A.3.4

#### Complex $(3,1)$ Pencil

If three of the common points coincide at $P$ and the fourth one is a distinct point $Q$, then the conics of the pencil have a contact of order 3 at $P$ and order 1 at $Q$. In order to uniquely determine the pencil it is necessary to specify the common tangent at $P$, which must not contain $Q$. However, this is not enough, one must give more information about the contact at $P$. This information may be obtained by analysing the local development of the curves in a neighbourhood of the point.

Let $u(x,y,z)$ and $v(x,y,z)$ be two conics that intersect at $P=[a:b:c]\in\mathbb{CP}^2$. For us to study their contact at this point, we should firstly consider an appropriate affine chart. Suppose without loss of generality that the coordinate $c$ is not zero, so $P=[\,a/c:b/c:1]$. By taking a translation we can make the intersection happen at $P'=[0:0:1]$. Now the expressions of the conics are given by $u(x+a/c,y+b/c,z)$ and $v(x+a/c,y+b/c,z)$. Next, we consider the affine chart given by $z=1$, which provides us the equations of our conics in terms of $(x,y)$. Finally, for each curve we can give one of the coordinates as a function of the other in order to locally satisfy their implicit equations. As long as the tangent at $(0,0)$ is not vertical, we can take $y$ as a function of $x$. One may apply a rotation at this step if necessary. Then, to rate the contact between the curves we just have to calculate the successive derivatives of these functions and assess up to which order they coincide.

In order to better understand this analysis, we should work out a concrete example. We are going to study the contact of two conics of a $(3,1)$ pencil, but we must first know how to generate such a pencil. In terms of degenerate conics, one may identify only one pair of lines that has the appropriate contact with the other conics of the pencil at $P$ and $Q$. Let $u$ be a irreducible conic of the pencil. We notice that one of the lines has to be the tangent to $u$ at $P$, while the second one must be the line $PQ$. Therefore, this is a first case where the discriminant has a unique root of order 3.

Now we may study a concrete example. Consider the conic $u=x^2+(y-z)^2-z^2=x^2+y^2-2yz$ and let $P=[0:0:1]$ be the triple common point. The $(3,1)$ pencil would be determined if we fix the other common point $Q$ on $u$, but we may equivalently impose that the degenerate conic $v_a=y(x+ay)$, where $a\in\mathbb{C}$, is the other conic that generates the pencil. Notice that one of the lines is tangent to $u$ at $P$, and the other passes through $P$ and will intersect $u$ in another point which is thus the simple common point. In this way, we have the family of all type $(3,1)$ pencils that contain $u$ and whose triple common point is $P$. An arbitrary conic of this family is given by $w = \alpha u + \beta v_a$, where $[\alpha : \beta] \in \mathbb{CP}^1$. Let us begin by analysing the function $y(x)$ associated to the conic $u$ at a neighbourhood of $(0,0)$ by taking its derivatives.

$$\begin{aligned} u(x,y) &= x^2 + y^2 - 2y. \\ x^2 + y^2 - 2y = 0, &\qquad y(0) = 0; \\ 2x + 2yy' - 2y' = 0 &\implies y'(0) = 0; \\ 2 + 2\Big((y')^2 + yy''\Big) - 2y'' = 0 &\implies y''(0) = 1; \\ 2(3y'y'' + yy''') - 2y''' = 0 &\implies y'''(0) = 0. \end{aligned}$$

Next, we do the same for an arbitrary conic of the family we described above. The local parametrization is only relevant for the nondegenerate conics, so we suppose that $\alpha \neq 0$.

$$\begin{aligned} w = &\ \alpha u + \beta v_a, \\ w(x,y) = &\ \alpha x^2 + \beta xy + (\alpha + \beta a)y^2 - 2\alpha y. \\ \alpha x^2 + \beta xy + (\alpha + \beta a)y^2 - 2\alpha y = 0, &\qquad y(0) = 0; \\ 2\alpha x + \beta(y + xy') + 2(\alpha + \beta a)yy' - 2\alpha y' = 0 &\implies y'(0) = 0; \\ 2\alpha + \beta(2y' + xy'') + 2(\alpha + \beta a)\Big((y')^2 + yy''\Big) - 2\alpha y'' = 0 &\implies y''(0) = 1; \\ \beta(3y'' + xy''') + 2(\alpha + \beta a)\Big(3y'y'' + yy'''\Big) - 2\alpha y''' = 0 &\implies y'''(0) = \frac{3\beta}{2\alpha}. \end{aligned}$$

As one can see, any nondegenerate conic $w$ of this family has a contact of order at least 3 with $u$ at $P$ because the derivatives at $x = 0$ of their functions $y(x)$ coincide up to the second order. However, the third derivative is different for each conic of the pencil, depending on the projective parameter $[\alpha : \beta]$ which describes the pencil. Therefore, the contact at $P$ is exactly 3.

After this detailed reflection about the contact between curves, we may finally obtain the dimension of the submanifold of pencils of type $(3,1)$. One such pencil is given by the position of two points $P$ and $Q$ in $\mathbb{CP}^2$, by the direction of the common tangent at $P$ and by a value $k \in \mathbb{C}^*$ referring to the contact of order 3 at $P$. This is the common value of $y''(0)$ obtained in the analysis above and it is related to the common euclidean curvature of the curves in the given affine chart, so it must be non zero for there to be irreducible conics in the pencil. In this way, we have a manifold of complex dimension $2+2+1+1=6$, thus once again of codimension 2 in the space of all pencils.

**Normal form of the $(3,1)$ pencil**

Let us present a normal form for the $(3,1)$ pencil. We may take the common point of order 3 to $P=[0:-1:1]$ and the common point of order 1 to $Q=[0:1:1]$. We may also impose that the common tangent at $P$ is the line $y+z=0$ and that the curvature at $P$ is $k=1$, so that the pencil may be generated by the following conics:

$$w_\infty = x(y+z) = xy+xz, \qquad u = x^2+y^2-z^2.$$

We parametrize the pencil by $\alpha w_\infty + \beta u$, where $[\alpha:\beta]\in\mathbb{CP}^1$. A property that characterizes each conic of this normal form is the tangent line at the point $Q$. Indeed, consider the conic $v$ of coordinates $[\alpha:1]$, in other words, $v=\alpha w_\infty + u = x^2+y^2-z^2+\alpha xy+\alpha xz$. Its tangent line at $Q$ is given by $\alpha x + y - z = 0$, so for each conic of the pencil has a different tangent at that point.

We highlight that one must be careful, since only $w_\infty$ is intrinsically special in this pencil. There are projective transformations that preserve the pencil as a whole and that sends $u$ to another irreducible conic of the pencil, so the choice of $u$ to define the parametrization of the pencil is arbitrary. The stabilizer of the $(3,1)$ pencil is harder to describe, so we do it via another approach.

**Lemma A.3.7.** *The stabilizer subgroup of the normal form of the $(3,1)$ pencil is 2-dimensional, and each of its elements may be written in the form $M_t.Y_s$, with $t\in\mathbb{C}$ and $s\in\mathbb{C}^*$, where:*

$$M_t = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ t & 0 & 1 \end{pmatrix}, \qquad Y_s = \begin{pmatrix} 2s & 0 & 0 \\ 0 & s^2+1 & s^2-1 \\ 0 & s^2-1 & s^2+1 \end{pmatrix}.$$

*Proof.* In order to find the stabilizer of the normal form of the $(3,1)$ pencil, we take four points in general position: $P=[0:-1:1]$, $Q=[0:1:1]$, $P'=[1:0:0]$ and $Q'=[1:0:1]$. The transformations $Y_s$ all have three fixed points, namely $P$, $Q$ and $P'$. Moreover, they preserve the conic $u$ and have the role of moving the point $Q'$ to another point of $u$ of coordinates $[2s : s^2 - 1 : s^2 + 1]$. Notice that we avoid 0 and $\infty$ as values for $s$ because these values would map $Q'$ to $P$ and $Q$ respectively.

The transformation $M_t$ is also defined by the images of the same four points.

$$P \overset{M_t}{\longmapsto} P, \qquad P' \overset{M_t}{\longmapsto} [1 : -t : t],$$
$$Q \overset{M_t}{\longmapsto} Q, \qquad Q' \overset{M_t}{\longmapsto} [1 : -t : t + 1].$$

Geometrically, consider the tangents of $u$ at $P$ and $Q$. They intersect precisely at $P'$, so after the action of $M_t$, we get a new conic that passes through $P$ and $Q$, whose tangent at $P$ is still the line $y + z = 0$, since the image of $P'$ persists in that line, but with a new tangent at $Q$ determined by the image of $P'$. By the considerations presented thus far, the image of $u$ must already be a conic that has a contact of order 3 at $P$, that passes through $Q$ and that has the appropriate tangents at those points. However, it might not belong to the pencil in normal form, because the curvature at $P$ might not have been preserved by the transformation. In order to assure that $k = 1$, we must consider the image of the remaining point $Q'$.

As $Q'$ belongs to $u$, its image $[1 : -t : t + 1]$ belongs to the image of $u$ and it is the last information needed to determine it uniquely, as the projective transformation $M_t$ is fully defined by the image of 4 points. Since we already have that $M_t.u$ is an irreducible conic that is tangent to the line through $M_t.P'$ and $P$ at $P$ and to the line through $M_t.P'$ and $Q$ at $Q$, then it is completely determined by the location of a last point that it passes through, which we control via $M_t.Q'$. That is indeed the case because given two lines, one point in each line and a third point outside of both lines, then there exists a unique conic that passes through those three points and that is tangent to both lines in the prescribed points. We search for the location of $M_t.Q'$ on the line $x - y - z = 0$, because as $P$ belongs to this line, any irreducible conic intersect it at another single point. Therefore, we just have to find out through which other point of that line the conic of the $(3, 1)$ pencil in normal form that has the prescribed tangent at $Q$ passes. Here, it is the algebraic description of such conic that will lead us to the answer. One may verify that the conic $v = 2tw_\infty + u = 2tx(y + z) + x^2 + y^2 - z^2$ has for its tangent at $Q$ the line $2tx + y - z = 0$, which passes through the point $M_t.P' = [1 : -t, t]$. The intersections of $v$ with the line $x - y - z = 0$ are $P$ and $[1 : -t : t + 1]$, which is why it must be the image of $Q'$ in order to assure that the curvature of $M_t.u$ at $P$ satisfies $k = 1$.

□

**Marked $(3,1)$ pencil**

Since there is a single degenerate conic in a $(3,1)$ pencil, we have more freedom to pick the position of a couple of irreducible marked conics. This is not surprising, because the symmetries act as projective transformations on the $\mathbb{CP}^1$ structure of the pencil, so their actions are determined by the image of three elements.

*Proposition* A.3.8. There are two orbits of $(3,1)$ pencils marked with a single element:

i. *The marked conic is irreducible:* A single orbit of dimension 7;

ii. *The marked conic is degenerate:* A single orbit of dimension 6.

In addition, there are two orbits of $(3,1)$ pencils marked with an unordered pair of conics.

i. *Both marked conics are irreducible:* A single orbit of dimension 8;

ii. *One of the marked conics is degenerate:* A single orbit of dimension 7.

*Proof.* As we have already seen, we may take any $(3,1)$ pencil to its normal form, being parametrized by $\alpha w_\infty + \beta u$, where $w_\infty = xy + xz$ and $u = x^2 + y^2 - z^2$. If the marked conic is degenerate, then we have already spotted it. As the stabilizer of the normal form has dimension 2, then this orbit has dimension 6. On the other hand, if the marked conic is irreducible, then we may find a particular $t \in \mathbb{C}$ for which the transformation $M_t$ maps it to $u$. Then, the remaining transformations that preserve $u$ are only $Y_s$, so the stabilizer of this marked pencil has dimension 1, and hence the orbit is 7-dimensional.

Now consider that there are two marked conics in the pencil. If one is degenerate, we fall back to the previous case and reach a single 7-dimensional orbit. If both marked conics are irreducible, then we may first set one of them to $u$. Next, with the help of $Y_s$ we may still move the other conic to any desired location on the pencil, other than $w_\infty$ and $u$, of course. To see why, let us consider the action of $Y_s$ on these generating conics.

$$Y_s.w_\infty = \frac{1}{4s^3}\, w_\infty, \qquad Y_s.u = \frac{1}{4s^2}\, u.$$

Therefore, the conic of coordinates $[\alpha : \beta] \in \mathbb{CP}^1$ is sent by $Y_s$ to the element of coordinates $[\alpha : s\beta]$ for $s \in \mathbb{C}^*$. This allows us to place the second marked conic at any available position in the pencil and it also shows that the only transformation that preserves this marked pencil is the identity, so we have an orbit of dimension 8. $\square$

In the figure below, we highlight $u = x^2 + y^2 - z^2$ and the arbitrary conic $v = w_\infty + u = x^2 + y^2 - z^2 + xy + xz$.

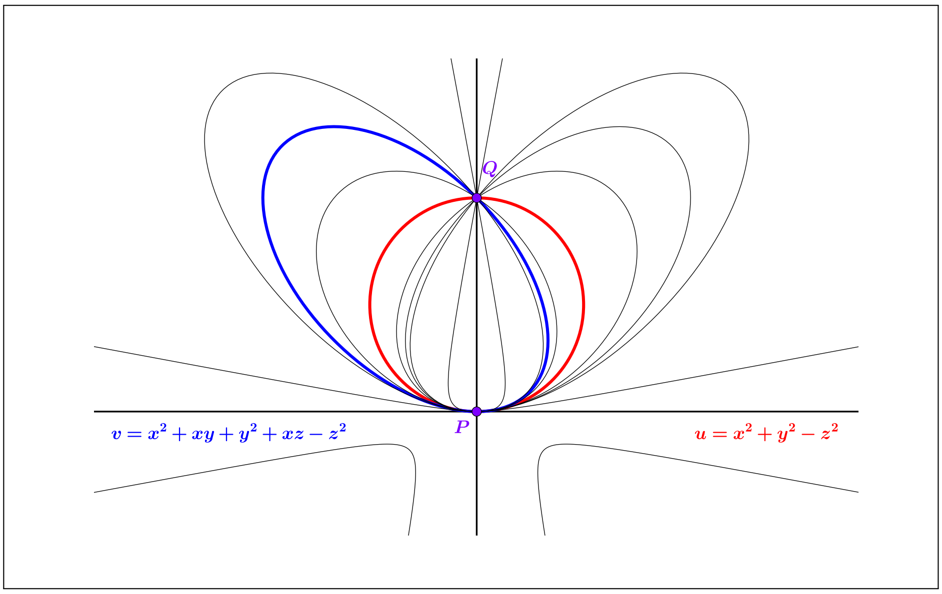


Figure A.4: Normal form of the $(3, 1)$ pencil.

### A.3.5

### Complex $(4)$ Pencil

There are two kinds of pencils where the four common points coincide in a single point $P \in \mathbb{CP}^2$. The first, which we name $(4)$, contains irreducible conics, while the second, named $(4^*)$, has only degenerate conics.

Every pair of conics of the $(4)$ pencil have a contact of order 4 at $P$. There is a single degenerate conic that satisfies this condition, the double common tangent to the other conics at $P$. Therefore, this is the second and last instance where the discriminant has a triple root.

Just as the last case, it is necessary to give additional information about the contact at $P$ in order to uniquely determine a pencil of type $(4)$. One must localize a point $P \in \mathbb{CP}^2$, give the common tangent through this point, and provide two values $k \in \mathbb{C}^*$ and $k' \in \mathbb{C}$ that describe respectively the common values of $y''(0)$ and $y'''(0)$ of every nondegenerate conic of the pencil. Therefore, the manifold of the (4) pencils is of complex dimension $2+1+1+1=5$ and codimension 3 in the space of all pencils.

**Normal form of the (4) pencil**

One may find a projective transformation that sends the common point to $P = [0 : -1 : 1]$ and the common tangent to $y + z = 0$. This conditions are very mild, as we still have a 3-dimensional family of conics that satisfy them. There is a group of projective transformations that acts transitively on this family, so we may impose that a particular conic $u = x^2 + y^2 - z^2$ belongs to the pencil in normal form. By doing so, we reach the normal form of the pencil, it is generated by the following conics:

$$w_\infty = (y+z)^2, \qquad u = x^2 + y^2 - z^2.$$

The demand that $u$ belongs to the pencil is equivalent to setting the two conditions on the curvature at the common point $P$, which in this case are $k = 1$ and $k' = 0$. Now every irreducible conic in the pencil has these characterizing quantities of the pencil. If we consider only the conics that abide by these values, we obtain a one-dimensional family of conics which is precisely the pencil we are after. It will be parametrized once again by $\alpha w_\infty + \beta u$ with $[\alpha : \beta] \in \mathbb{CP}^1$. Notice, however, that $u$ is an arbitrary irreducible conic of the pencil, so there are transformations of the stabilizer of the pencil that do not preserve $u$, mapping it instead to another irreducible conic of the pencil.

**Lemma A.3.9.** *The stabilizer subgroup of the normal form of the* (4) *pencil is* 3*-dimensional, and each of its elements may be written in the form* $M_t.N_{r,s}$*, with* $r, s, t \in \mathbb{C}$ *and* $r \neq s$*, where:*

$$M_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t+\frac{3}{2} & -t+\frac{1}{2} \\ 0 & t-\frac{1}{2} & t+\frac{1}{2} \end{pmatrix}, \qquad N_{r,s} = \begin{pmatrix} 2(s-r) & 2r & 2r \\ -2r(s-r) & (s-r)^2+1-r^2 & -(s-r)^2+1-r^2 \\ 2r(s-r) & -(s-r)^2+1+r^2 & (s-r)^2+1+r^2 \end{pmatrix}.$$

*Proof.* We obtain the stabilizer for the normal form of the (4) pencil in a similar fashion as we did for the (3, 1) pencil. Let us consider the same four points in general position: $P = [0 : -1 : 1]$, $Q = [0 : 1 : 1]$, $P' = [1 : 0 : 0]$

and $Q' = [1 : 0 : 1]$. The transformations $N_{r,s}$ belong to the stabilizer of the conic $u$, although they might move its individual points around. On the other hand, the transformations $M_t$ might not preserve the conic $u$ at all. Naturally, all transformation in the stabilizer has $P$ as a fixed point and must preserve the line $y + z = 0$, which is why the point $P'$ must be mapped to a point in this very line. We obtain the formula for $N_{r,s}$ by considering the image of the four points in question.

$$\begin{aligned} P &\xmapsto{N_{r,s}} P, & Q &\xmapsto{N_{r,s}} [2r : 1 - r^2 : 1 + r^2], \\ P' &\xmapsto{N_{r,s}} [1 : -r : r], & Q' &\xmapsto{N_{r,s}} [2s : 1 - s^2 : 1 + s^2]. \end{aligned}$$

As one may observe, both $Q$ and $Q'$ are mapped to points in $u$, their images being determined by the values of $r$ and $s$ respectively. Neither can be $\infty$, otherwise their image would be $P$, and they also cannot have the same image, which is why $r \neq s$. The image of $P'$ is determined by the same parameter $r$, and there is a good reason for it. Since $P'$ is a point on the tangent line to $u$ through $Q$, then its image must also be on the tangent line to the image of $u$ through the image of $Q$. The prescribes image of $P'$ is the one that results in the image of $u$ being itself. We can be sure that $N_{r,s}$ preserves $u$ in two ways. Algebraically, we may do the calculations and obtain $N_{r,s}.u = \frac{1}{4(s-r)^2} u$. Geometrically, we observe that the image of $u$ is a conic that must be tangent to the line through $N_{r,s}.P'$ and $P$ at $P$ and to the line through $N_{r,s}.P'$ and $N_{r,s}.Q$ at $N_{r,s}.Q$. Then it is uniquely determined by the position of an additional point that it passes through, and since $N_{r,s}.Q'$ belongs to $u$, then $N_{r,s}.u = u$.

The transformation $M_t$ is also defined by the images of the same four points.

$$\begin{aligned} P &\xmapsto{M_t} P, & Q &\xmapsto{M_t} [0 : 1 - t : t], \\ P' &\xmapsto{M_t} P', & Q' &\xmapsto{M_t} [1 : \tfrac{1}{2} - t : \tfrac{1}{2} + t]. \end{aligned}$$

It preserves both $P$ and $P'$. Consequently, the tangent line to the image of $u$ through the image of $Q$ must also pass through $P'$, since it did before the action of $M_t$. The restriction that the image of $Q$ belongs to the line $x = 0$ guarantees that the curvature of the new conic satisfies $k' = 0$ at $P$. This can be more clearly observed in an affine chart, since two points of an irreducible conic have parallel tangents if and only if the segment joining them passes through the center of the conic. In addition, the normal to a point in a conic passes through the center if and only if the point is a vertex, which means that $k' = 0$ at that point. This is precisely the situation for both $P$ and $M_t.Q$, so they are both vertices of the conic $M_t.u$.

The location of the final image $M_t.Q'$ is the single point on the line $x-y-z=0$ that results in the curvature of $M_t.u$ at $P$ being $k=1$. Since we already know that $M_t.u$ is tangent to the line $y+z=0$ at $P$ and to the line $ty-(1-t)z=0$ at $M_t.Q$, then it is uniquely determined by the position of $M_t.Q'$. We search for the appropriate location along the line $x-y-z=0$, because as $P$ already belongs to this line and to the image of $u$, there can only be one other point that belongs to both of these curves simultaneously. At this point, it is the algebraic description of $M_t.u$ that will lead us to the appropriate image of $Q'$. One may check that the only conic in the pencil in normal form that passes through $M_t.Q = [0 : 1-t : t]$ is $v = (2t-1)w_\infty + u = (2t-1)(y+z)^2 + x^2 + y^2 - z^2$. Finally, its intersections with the line $x-y-z$ are $P$ and $[1 : \frac{1}{2} - t : \frac{1}{2} + t]$, which is why it must be the image of $Q'$ in order to assure that the curvature of $M_t.u$ at $P$ satisfies $k=1$.

□

**Marked** (4) **pencil**

Once again, there is a single degenerate conic in a (4) pencil, so we have the freedom to pick the position of a couple of irreducible marked conics.

*Proposition* A.3.10. There are two orbits of (4) pencils marked with a single element:

i. *The marked conic is irreducible:* A single orbit of dimension 6;

ii. *The marked conic is degenerate:* A single orbit of dimension 5.

In addition, there are two orbits of (4) pencils marked with an unordered pair of conics.

i. *Both marked conics are irreducible:* A single orbit of dimension 7;

ii. *One of the conics is degenerate:* A single orbit of dimension 6.

*Proof.* If we have a single marked conic in our (4) pencil, then either it is degenerate, so it must be the double line and it is preserved by the whole stabilizer of the normal form, resulting in a 5-dimensional orbit; or it is irreducible, in which case it can be sent to $u = x^2 + y^2 - z^2$ by applying the action of the suitable $M_t$ transformation. Then, the stabilizer of this marked pencil is given by the transformations $N_{r,s}$, a 2-dimensional Lie group, so we have an orbit of dimension 6.

If there are two marked conics on the pencil, then either one of them is degenerate and we fall back to the previous case of a 6-dimensional orbit; or both of them are irreducible, so we may take one of them to $u$ and then move the other one using the $N_{r,s}$ transformations. Algebraically, their actions on the generating conics of the pencil are:

$$N_{r,s}.w_\infty = \frac{1}{4}\,w_\infty, \qquad N_{r,s}.u = \frac{1}{4(s-r)^2}\,u.$$

The different multiples obtained above allow us to manipulate the pencil, as the element of coordinates $[\alpha : \beta]$ is sent to the one of coordinates $[(s-r)^2\alpha : \beta]$. Here we notice a difference between the real and the complex settings. In the real case, the sign of the ratio of the coefficients is preserved, while in the complex one we have full freedom. We may then settle $v$ at the coordinates $[1 : 1]$ and obtain $v = x^2 + 2y^2 + 2yz$.

Finally, we describe the group of symmetries that preserve $w_\infty$, $u$ and $v$. Since it already has three elements determined, every transformation of this group acts trivially on the pencil preserving every conic. We already know how the transformations $N_{r,s}$ act algebraically on $w_\infty$ and on $u$, so it is easy to find the necessary relation on the parameters in order to obtain a symmetry of the marked pencil, we just need $(s-r)^2 = 1$, in other words, $s - r = \pm 1$. The fact that there are two possible values for $s - r$ corresponds to the presence of the involution $N_{0,-1} : [x : y : z] \mapsto [-x : y : z]$ in the group. In the end, we have proved that the group of transformations that preserve the (4) pencil marked with two irreducible conics is $< N_{0,-1}, N_{r,r+1} >$, a 1-parameter family with two connected components. Therefore, the orbit of such marked pencil has dimension 7.

$$N_{r,r+1} = \begin{pmatrix} 2 & 2r & 2r \\ -2r & 2-r^2 & -r^2 \\ 2r & r^2 & 2+r^2 \end{pmatrix}.$$

□

In the following figure we highlight $u = x^2 + y^2 - z^2$ and the conic $v = x^2 + 2y^2 + 2yz$ corresponds to the parameter $[1 : 1]$ of the pencil.

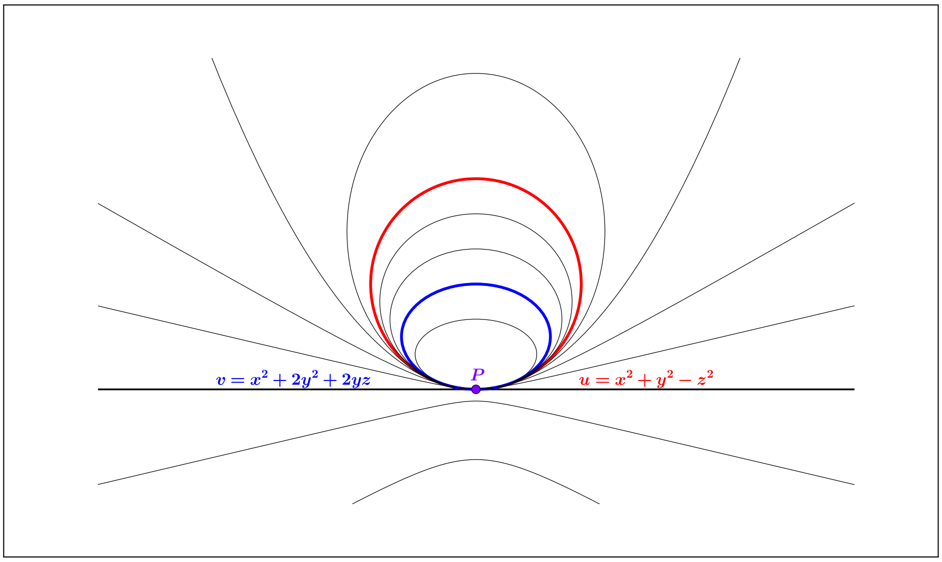


Figure A.5: Normal form of the (4) pencil.

### A.3.6

### Complex $(4^*)$ Pencil

The other possibility of pencil where the four common points meet is the one that contains two degenerate conics with a contact of order 4 at a point $P$. For example, two pairs of lines such that all four lines intersect simultaneously at $P$. Alternatively, one may consider two double lines that intersect at $P$. In the complex case which we study here, these two descriptions provide a same type of pencil. To see this, let us consider the following example. Let $u = xy$ and $v = (x + ay)(x + by)$, with $a, b \in \mathbb{C}^*$ and $a \neq b$, be two degenerate conics composed by a pair of distinct lines each. They have an intersection of order 4 at $P = [0 : 0 : 1]$, so we consider their pencil

$$\alpha u + \beta v = \alpha xy + \beta(x + ay)(x + by) = \beta x^2 + (\alpha + (a + b)\beta)xy + ab\beta y^2.$$

To find the double lines in this pencil one has to evaluate the minor $\delta$, which is a homogeneous polynomial of degree 2 in $\alpha, \beta$. Since its roots correspond to the double lines we can already tell that this pencil has at most two of them. In fact this is the only type of pencil that can have more than one double line, since the presence of two such conics implies the configuration $(4^*)$ for the common points.

$$\begin{aligned}\delta &= ab\beta^2 - \frac{1}{4}\Big(\alpha^2 + 2(a+b)\alpha\beta + (a+b)^2\beta^2\Big) \\ &= -\frac{1}{4}\Big(\alpha^2 + 2(a+b)\alpha\beta + (a-b)^2\beta^2\Big).\end{aligned} \tag{A.1}$$

Now we evaluate the discriminant of the quadratic polynomial $\alpha^2 + 2(a+b)\alpha\beta + (a-b)^2\beta^2$ to obtain $16ab$. Therefore, since $a$ and $b$ are non zero, $\delta$ has two distinct roots and thus the pencil has two distinct double lines. Notice that this will result in two different kinds of pencil in the real setting depending on the sign of $16ab$, because if it is negative, then there will be no double line in the real pencil.

Having always two double lines in the $(4^*)$ pencil, we can easily calculate the dimension of this family. The pencil is determined by two lines in $\mathbb{CP}^2$, which by duality corresponds to picking two points of $\mathbb{CP}^2$. So the family of $(4^*)$ pencils is a manifold of complex dimension 4 and codimension 4.

As we have seen in the previous type of pencil, an irreducible conic cannot have a contact of order 4 at $P$ with more than one degenerate conic. This implies that every conic in this kind of pencil must be degenerate, and so it is the first type of the list whose discriminant is identically zero.

**Normal form of the $(4^*)$ pencil**

One can find a projective transformation that sends the single common point to $P = [0:0:1]$ and the defining lines of the pencil to $x = 0$ and $y = 0$. Thus the double lines are given by:

$$w_\infty = x^2, \qquad w_0 = y^2.$$

We may then parametrize the pencil as $\alpha w_\infty + \beta w_0$ with $[\alpha : \beta] \in \mathbb{CP}^1$.

**Lemma A.3.11.** *The stabilizer subgroup of the normal form of the $(4^*)$ pencil is generated by the involution $N$ and the 4-parameter family of transformations $M_{a,b,r,s}$, with $a, b, r, s \in \mathbb{C}$, $a \neq 0$ and $r + as - b \neq 0$ as below.*

$$M_{a,b,r,s} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ b - as & b - r & r + as - b \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* For a projective transformation to preserve the normal form of the $(4^*)$ pencil, it must either preserve $w_\infty$ and $w_0$ or interchange them. The involution $N$ is responsible for the case where the double lines are swapped. For the case where they are preserved, we take three auxiliary points once again, so consider $P' = [1:1:1]$, $Q = [1:0:1]$ and $Q' = [0:1:1]$. Since the lines $x = 0$ and $y = 0$ are preserved, $Q$ and $Q'$ must stay on them and so their images are $[1:0:r]$ and $[0:1:s]$ respectively, with $r, s \in \mathbb{C}$. As for $P'$, it is free to move around $\mathbb{CP}^2$, we just have to take care to avoid the collinearity of any three image points. Its image is defined by two new parameters $[1:a:b]$. The location of these four images uniquely define the transformation $M_{a,b,r,s}$ presented in the statement of the lemma. The determinant of the matrix indicates the conditions on the four parameters so that there is no alignment of the images. $\det(M) = a(r + as - b) \neq 0$, so $a \in \mathbb{C}^*$ and $b, r, s \in \mathbb{C}$ with $r + as - b \neq 0$. □

**Marked $(4^*)$ pencil**

There are two special conics in the $(4^*)$ pencil, the double lines. We have the freedom to interchange them and to freely place a third conic anywhere on the pencil.

*Proposition* A.3.12. There are two orbits of $(4^*)$ pencils marked with a single element:

i. *The marked conic is not a double line:* A single orbit of dimension 5;

ii. *The marked conic is a double line:* A single orbit of dimension 4.

In addition, there are three orbits of $(4^*)$ pencils marked with an unordered pair of conics.

i. *Both marked conics are not double lines:*
Infinitely many orbits of dimension 5 described by an invariant;

ii. *A single marked conic is a double line:* A single orbit of dimension 5.

iii. *Both marked conics are double lines:* A single orbit of dimension 4.

*Proof.* If the marked conic is a double line, we may easily put the pencil in normal form, sending it to $w_\infty$. Since there is a 4-dimensional stabilizer, this orbit has also dimension 4. If the marked conic is not a double line, then first we put the pencil in normal form and then we consider the action of $M_{a,b,r,s}$ on the generating conics.

$$M_{a,b,r,s}.w_\infty = w_\infty, \qquad M_{a,b,r,s}.w_0 = \frac{1}{a^2}\,w_0.$$

Despite having so many degrees of freedom, we see that only one of the four parameters matters in terms of the action produced on the pencil. We may use it to transform the pencil, noticing just as in the (4) pencil case that there is a difference between the real and the complex setting due to the even power on the resulting multiple. Indeed, the element of coordinates $[\alpha : \beta]$ is mapped to $[a^2\alpha : \beta]$, so the sign of the ratio between the coefficients is preserved if we only consider real projective transformations. In the complex case, on the other hand, we have complete freedom to place the marked conic anywhere in the pencil, except for the double lines, of course. We may set it at $u = w_\infty - w_0 = x^2 - y^2$, for example. What is left is a stabilizer of dimension 3 with four connected components, so the orbit has dimension 5. When $a = \pm 1$, the transformations $M_{\pm 1,b,r,s}$ preserve the whole pencil. We highlight the involution $M_{-1,1,1,-1} : [x : y : z] \mapsto [x : -y : z]$, which together with the involution $N : [x : y : z] \mapsto [y : x : z]$ are responsible for the four connected components of the stabilizer of the (4*) pencil marked with a conic that is not a double line. The action of $N$ on the pencil is also clear, it maps the conic of coordinates $[\alpha : \beta]$ to the one with coordinates $[\beta : \alpha]$. This shows us that the two fixed conics under the action of $N$ are $w_\infty + w_0 = x^2 + y^2$ and $w_\infty - w_0 = x^2 - y^2$.

Next, we consider the cases where there are two marked conics. If at least one of them is a double line, we fall back to the previous cases of a single marked conic. If both are not a double line, then we first put the pencil in normal form and take one of the marked conics to $u = x^2 - y^2$. Now, we have the 3-dimensional stabilizer that preserves $\{w_\infty, w_0, u\}$ and hence the whole pencil if the double lines are not swapped. On the other case, we just get the action of $N$, so if the other marked conic has coordinates $[c : 1]$ for some $c \in \mathbb{C} \setminus \{0, 1\}$, then it is mapped to the conic in the pencil of coordinates $[1 : c] = [1/c : 1]$. Curiously, this has the same effect as changing which of the unordered marked conics is set to $u$, as first explained in Proposition A.3.4 for the marked $(2, 1, 1)$ pencils. Therefore, there are infinitely many orbits of dimension 5 for the (4*) pencils with two marked conics that are not double lines, each orbit being characterized by an invariant $c \in \mathbb{C} \setminus \{0, 1\}$ with $|c| \leq 1$ and identified conjugate pairs in the boundary. □

In the following figure, we highlight $u = x^2 - y^2$ and an arbitrary conic $v = x^2 - 4y^2$ which is associated to the parameter $[1 : -4]$ of the pencil.

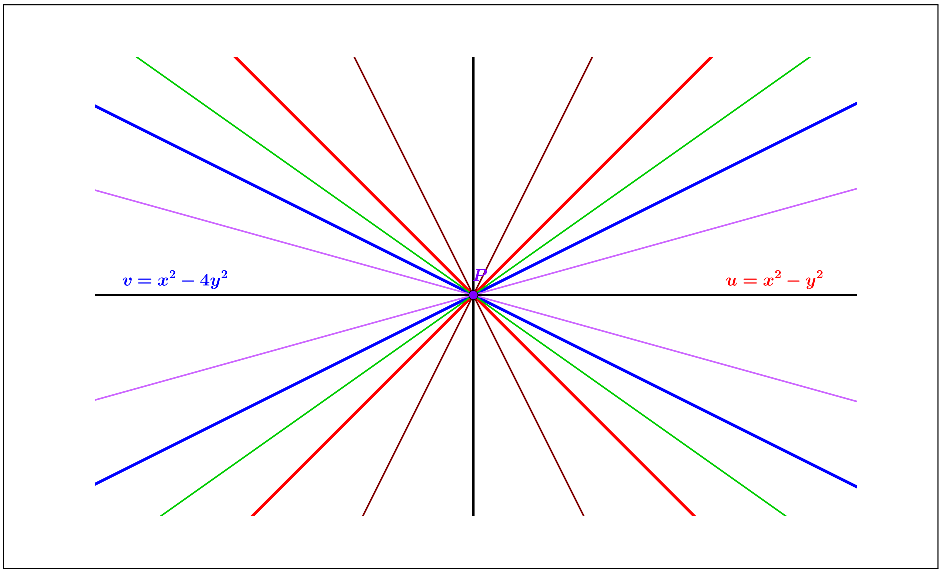


Figure A.6: Normal form of the $(4^*)$ pencil.

### A.3.7

### Complex $(\infty, 1)$ Pencil

At this point, there are only the cases of collinearity of the common points left to consider. If this happens, then the line $\ell$ through the collinear common points is necessarily a component of every conic of the pencil, being thus a *common line*. This already implies that every conic must be degenerate, so the discriminant is identically zero once again.

If the fourth common point $P$ does not belong to the common line $\ell$, then the other line that constitutes each conic of the pencil must pass through $P$. Therefore, this pencil is essentially the pencil of lines through $P$ accompanied by the common line $\ell$. Since there are infinite many common points over the common line plus an additional common point outside of $\ell$ of simple intersection between the conics, we call this type of pencil $(\infty, 1)$. As the second line of each pair must pass by $P$, which does not belong to $\ell$, this pencil has no double line.

A pencil of this kind is simply determined by a common line $\ell$ and a common point $P$, so it is also of dimension 4 and codimension 4 in the space of all pencils.

**Normal form of the $(\infty, 1)$ pencil**

One can find a projective transformation that sends the common point to $P = [0 : -1 : 1]$ and the common line to $y = 0$. For the first time, we do not have any distinguished conic to elect as the basis for the parametrization of the pencil, so we provisionally use two arbitrary conics $u$ and $v$. Each of them is comprised of the line $\ell$ and a line through $P$, so let $Q = [1 : 0 : 1]$ be the intersection of the components of $u$ and $Q' = [-1 : 0 : 1]$ be the intersection of the components of $v$. With that, the generating conics of the pencil are given by:

$$u = y(x - y - z), \qquad v = y(x + y + z).$$

The pencil is parametrized by $\alpha u + \beta v$ with $[\alpha : \beta] \in \mathbb{CP}^1$. There is no intrinsically relevant element of the pencil to position in the normal form, so it will be rather flexible.

**Lemma A.3.13.** *The stabilizer subgroup of the normal form of the* $(\infty, 1)$ *pencil is generated by the 4-parameter family of transformations* $M_{a,b,r,s}$*, with* $a, b \in \mathbb{C}$*,* $r, s \in \mathbb{C} \cup \{\infty\}$*,* $r \neq s$*,* $a - br - r \neq 0$ *and* $a - bs - s \neq 0$ *as below.*

$$M_{a,b,r,s} = \begin{pmatrix} \frac{r(a-bs-s)+s(a-br-r)}{r-s} & a & a \\ 0 & 1 & 0 \\ \frac{(a-bs-s)+(a-br-r)}{r-s} & b & b+1 \end{pmatrix}.$$

*Proof.* Any symmetry of the normal form of the $(\infty, 1)$ pencil must preserve the point $P$ and keep the points $Q$ and $Q'$ on the line $y = 0$. In order to uniquely define a projective transformation, we need a fourth point in general position, so let us take $P' = [0 : 1 : 0]$ and map it to an arbitrary point $[a : 1 : b] \in \mathbb{CP}^2$, only taking care to avoid the collinearity of any three of their images. In short, the transformation $M_{a,b,r,s}$ is defined by:

$$P \xmapsto{M_{a,b,r,s}} P, \qquad Q \xmapsto{M_{a,b,r,s}} [r : 0 : 1],$$
$$P' \xmapsto{M_{a,b,r,s}} [a : 1 : b], \qquad Q' \xmapsto{M_{a,b,r,s}} [s : 0 : 1].$$

We allow $r, s \in \mathbb{C} \cup \{\infty\}$ since there is no problem if one of them is mapped to $[1 : 0 : 0]$, we just have to guarantee that their images are distinct, so $r \neq s$. The transformation $M_{a,b,r,s}$ is then given by the matrix in the statement of the lemma. Its determinant is $\det(M_{a,b,r,s}) = -2\frac{(a-br-r)(a-bs-s)}{r-s}$, which provides us with good insight into the conditions on the parameters in order to avoid collinearity of the images. Indeed, the line through $P$ and $P'$ contains $Q$ if and only if $a - br - r = 0$, and it contains $Q'$ if and only if $a - bs - s = 0$, so these two identities must be avoided.

□

**Marked $(\infty, 1)$ pencil**

There is no special conic in the $(\infty, 1)$ pencil, so we may place two marked conics wherever we desire in the pencil with room to spare, as we could even choose the location of a third marked conic if necessary.

*Proposition* A.3.14. The orbits of $(\infty, 1)$ pencils with increasing number of marked conics are:

i. *A single marked conic:* A single orbit of dimension 5;

ii. *An unordered pair of conics:* A single orbit of dimension 6;

iii. *An unordered triplet of conics:* A single orbit of dimension 7;

iv. *An unordered quartet of conics:*
Infinitelly many orbits of dimension 7 described by an invariant.

*Proof.* First we take the pencil to its normal form and apply a suitable transformation $M_{0,0,r,-1}$ that takes the marked conic to $u = y(x - y - z)$. Now the group that preserves the pencil globally and that fixes $u$ is 3-dimensional and given by $M_{a,b,1,s}$, thus the orbit has dimension 5.

If we have a second marked conic, we just have to apply the suitable transformation $M_{0,0,1,s}$ that takes it to $v = y(x+y+z)$, and we have left a 2-dimensional stabilizer generated by $M_{a,b,1,-1}$, thus the orbit has dimension 6.

$$M_{a,b,1,-1} = \begin{pmatrix} b+1 & a & a \\ 0 & 1 & 0 \\ a & b & b+1 \end{pmatrix}.$$

We can place a third conic anywhere on the pencil, because the actions of $M_{a,b,1,-1}$ on $u$ and $v$ result in:

$$M_{a,b,1,-1}.u = \frac{1}{-a+b+1}\,u, \qquad M_{a,b,1,-1}.v = \frac{1}{a+b+1}\,v.$$

Since the resulting multiples are distinct if $a \neq 0$, we may move the third marked conic along the pencil. The ensuing stabilizer is $M_{0,b,1,-1}$, which preserves every conic in the pencil, so this orbit has dimension 7.

Finally, the position of the fourth marked conic is already determined, since all transformations that preserve the other three must preserve every conic in the pencil. However, we could permute the marked conics around, since the set is unordered. By permuting the location of the first three marked conics there are up to 6 possible locations for the fourth conic. This invariant is no other than the cross-ratio of the four lines through $P$ that are components of the marked conics. $\square$

In the following figure, we highlight the common line $\ell$, given by $y = 0$, and two arbitrary conics $u = y(x - y - z)$ and $v = y(x + y + z)$.

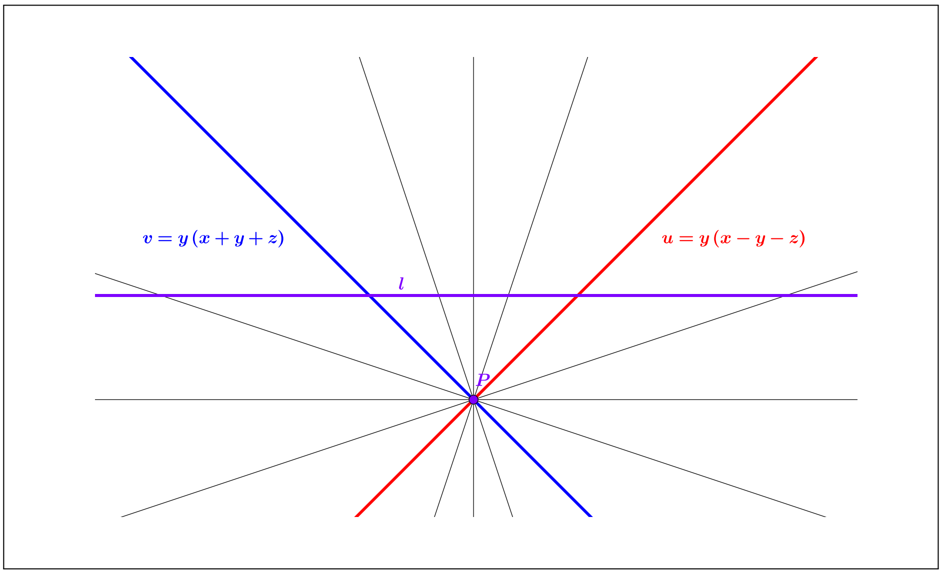


Figure A.7: Normal form of the $(\infty, 1)$ pencil.

### A.3.8

### Complex $(\infty)$ Pencil

Finally we arrive at last type of complex pencil of conics, in which the four common points are aligned. This configuration yields once again a common line $\ell$ that imposes all the conics of the pencil to be degenerate. However, since there is no common point not on $\ell$, the second factor of each conic is completely free. This is a problem, because the family of conics that follow this description is of complex dimension 2, so one must narrow the requirements further down in order to obtain a pencil.

Let us analyse one such pencil to obtain the additional geometric condition we are after. Consider two degenerate conics with a common factor referring to the common line $\ell$. In order for them to generate a pencil of type $(\infty)$, and not of type $(\infty, 1)$, the other factors must produce lines that intersect over a point $P$ on $\ell$. Consequently, the pencil generated by these conics is essentially the pencil of lines through $P$ accompanied by the common line $\ell$, which contains $P$. Notice that the double line given by two copies of $\ell$ belongs to this pencil, independently of the particular point $P$. This description corresponds to the degenerate case where either $a$ or $b$ is equal to zero in the example presented for the pencil of type $(4^*)$ (A.1). One can verify that in this case the minor $\delta$ has a double root which is associated to the double line $\ell$, the only double line in this pencil.

Therefore, a pencil of type $(\infty)$ is determined by a common line and an additional point on it. This shows that the submanifold of such kind of pencil is of complex dimension $2 + 1 = 3$, and of codimension 5, being the most degenerate type of complex pencil of conics.

**Normal form of the $(\infty)$ pencil**

The $(\infty)$ pencil is the one with the least amount of restriction with respect to its normal form. The best one can do is to fix the common line to $y = 0$ and the common point at $P = [0 : 0 : 1]$. There is only one distinguished conic in this pencil, the double line, so in order to be able to parametrize the pencil, we provisionally pick an arbitrary conic $u$ to serve as the second generating conic.

$$w_\infty = y^2, \qquad u = y(x - y).$$

The pencil is then parametrized by $\alpha w_\infty + \beta u$ with $[\alpha : \beta] \in \mathbb{CP}^1$.

**Lemma A.3.15.** *The stabilizer subgroup of the normal form of the $(\infty)$ pencil is given by the 5-parameter family of transformations $M_{a,b,c,d,t}$, with $a, b, c, d, t \in \mathbb{C}$, $a - c \neq 0$ and $t(a - c) - (b - d) \neq 0$ as below.*

$$M_{a,b,c,d,t} = \begin{pmatrix} a - c & a + c & 0 \\ 0 & 2 & 0 \\ b - d & b + d & t(a - c) - (b - d) \end{pmatrix}.$$

*Proof.* A projective transformation preserves the normal form of the $(\infty)$ pencil if and only if it preserves the line $y = 0$ and has $P$ as a fixed point. In order to obtain a uniquely defined transformation that meet these conditions, let us consider its action over the auxiliary points $P' = [1 : 0 : 1]$, $Q = [1 : 1 : 0]$ and $Q' = [-1 : 1 : 0]$ as follows:

$$P \xmapsto{M_{a,b,c,d,t}} P, \qquad Q \xmapsto{M_{a,b,c,d,t}} [a : 1 : b],$$
$$P' \xmapsto{M_{a,b,c,d,t}} [1 : 0 : t], \qquad Q' \xmapsto{M_{a,b,c,d,t}} [c : 1 : d].$$

The transformation defined by these mappings is the one in the statement of the lemma. The determinant of its matrix is $\det(M_{a,b,c,d,t}) = 2(a-c)(t(a-c)-(b-d))$, and it once again gives us the relations we should avoid to prevent the collinearity of the images, as $a-c=0$ if and only if the images of $Q$ and $Q'$ are aligned with $P$, and $t(a-c)-(b-d)=0$ if and only if they are aligned with the image of $P'$. □

**Marked $(\infty)$ pencil**

The double line is the unique special conic in the $(\infty)$ pencil, so we may place up to two marked conics anywhere along the pencil.

*Proposition* A.3.16. There are two orbits of $(\infty)$ pencils marked with a single element:

i. *The marked conic is not a double line:* A single orbit of dimension 4;

ii. *The marked conic is a double line:* A single orbit of dimension 3.

In addition, there are also two orbits of $(\infty)$ pencils marked with an unordered pair of conics.

i. *Both marked conics are not double lines:* A single orbit of dimension 5;

ii. *One of the marked conics is a double line:* A single orbit of dimension 4.

*Proof.* If the marked conic is a double line, then it becomes $w_\infty = y^2$ when the pencil is put in normal form. The stabilizer of the pencil is 5-dimensional given by $M_{a,b,c,d,t}$, so this orbit has dimension 3. If the marked conic is not a double line, then one may use a projective transformation to map it to $u = y(x-y)$. The action of a generic transformation of the stabilizer of the normal form on this conic yields:

$$M_{a,b,c,d,t}.u = \frac{1-a}{2(a-c)}w_\infty + \frac{1}{2(a-c)}u.$$

Here we see that the condition for one such transformation to preserve $u$ is $a = 1$. This may also be observed geometrically, since for the line $x - y = 0$ to be preserved the image of $Q$ must stay on that line, which happens if and only if $a = 1$. Thus the group that preserves the pencil marked with $u$ is 4 dimensional, given by $M_{1,b,c,d,t}$, and hence this orbit has dimension 4.

If there are two marked conics, there are also two cases. If one of them is the double line, we fall back to the previous case and obtain an orbit of dimension 4. On the other hand, if both are not the double line, then we may map one of them to $u$ and then map the other to $v = y(x + y)$. It is possible to do so because the action of $M_{1,b,c,d,t}$ on $w_\infty$ and $u$ results in distinct multiples:

$$M_{1,b,c,d,t}.w_\infty = \frac{1}{4}w_\infty, \qquad M_{1,b,c,d,t}.u = \frac{1}{2(1-c)}u.$$

After the marked conics are set to $u$ and $v$, the remaining stabilizer that preserves them and also $w_\infty$ is a 3-dimensional Lie group generated by $M_{1,b,-1,d,t}$ and the involution $N : [x : y : z] \mapsto [-x : y : z]$ that interchanges $u$ and $v$. Therefore, this orbit has dimension 5. □

In the following figure, we highlight the common line $\ell$, given by $y = 0$, and two arbitrary conics $u = y(x - y)$ and $v = y(x + y)$.

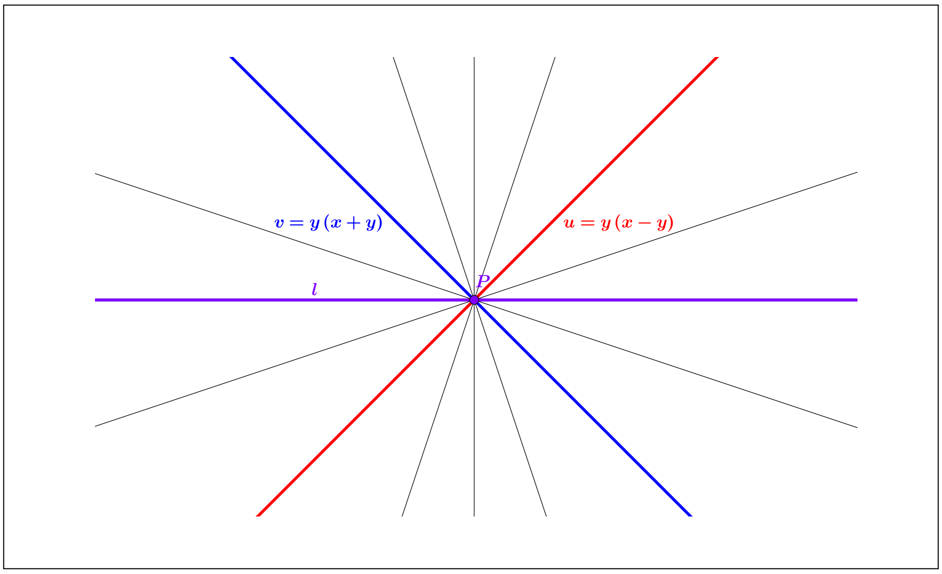


Figure A.8: Normal form of the $(\infty)$ pencil.

## A.4

### The real space of conics

Now we move to the study of the real pencils of conics. As explained in the A.1, they correspond to the lines of the space of real conics $\mathbb{RP}^5$ and are parametrized by $\alpha u + \beta v$ where $u$ and $v$ are distinct real conics and $[\alpha : \beta] \in \mathbb{RP}^1$. As the discriminant polynomial $\Delta(\alpha u + \beta v)$ is real, all of its non real roots come in conjugate pairs. Because of that, at least one of the roots is real, so any pencil has at least one degenerate conic. Furthermore, if the expression has a multiple root, all the roots are necessarily real, and if two real roots are known, then all roots are necessarily real. However, a real root indicates that for certain real values of $[\alpha : \beta]$ we get a pair of lines, but it can be a *pair of complex lines*, whose intersection is its only point present in $\mathbb{RP}^2$, that is, a point that admits real coordinates. In addition, there might be portions of the pencil where the conics are not present at all in $\mathbb{RP}^2$, because as one passes over a degenerate element, the conics may have a different signature, and the definite quadratic forms, of signature $(3, 0)$ or $(0, 3)$, do not have any real point. Indeed, one can think of the cubic hypersurface in the space of conics given by the expression $\Delta = 0$ aptly called the *discriminant.* This submanifold divides the space into many connected components, and the signature of the conics is constant in each of them. Just to be clear, since we are working in the projectivized setting, the signature is not exactly well-defined, since the multiplication by $-1$ inverses it, but we can still make the distinction between the regions where the signature is $(3, 0)$ or $(0, 3)$ and those where it is $(2, 1)$ or $(1, 2)$.

At this point it is important to explain in detail what we refer to as *real line* and *complex line.* Basically we refer to the coefficients of its expression; if it is possible to write it using only real terms, then it is a real line, otherwise we call it a complex line. However, the geometric interpretation deserves a more detailed explanation. An implicit equation, of real coefficients or not, manifests itself partially over $\mathbb{RP}^2$ in the visual manner that we are used to, but it has also hidden portions outside the real plane. A real line manifests itself in $\mathbb{RP}^2$ indeed as a line, that is, as a one dimensional submanifold. A complex line on the other hand, meets $\mathbb{RP}^2$ at a single point. Let us explain this phenomenon. Since its equation is linear, its zero set is a copy of $\mathbb{CP}^1$ contained in $\mathbb{CP}^2$. Thinking of it in terms of real dimensions, we have a submanifold of dimension 2 inside a manifold of dimension 4. There is also the real projective plane $\mathbb{RP}^2$ inside of $\mathbb{CP}^2$, which is another submanifold of dimension 2, but unlike the previous one, it is not algebraic. Since they have complementary dimensions, the intersection of an arbitrary line with the real projective plane is, in general, a submanifold of dimension 0, thus a set of discrete points. Notice, however, that if there is more than one intersection, the submanifold of dimension 1 joining two of these points must belong in its entirety to the intersection due to the linearity of the expression. In this case, we know how to obtain its implicit equation that has only real coefficients, so it is a real line. Therefore, there are two options, an arbitrary line meets $\mathbb{RP}^2$ either at a single point, in which case it is a complex line, or they meet over a one dimensional submanifold and we have a real line.

Coming back to the pencils of conics, we may once again think of them in terms of the common points of intersections. If there are four real points, we can easily tell the type of the pencil by looking closely at their configuration. All of the 8 cases listed in the previous section, about complex pencils of conics, have an analogous real version, but some of them are subdivided due to the existence of *complex common points*. They necessarily come in conjugate pairs, because as we are dealing with real conics, they are all symmetric with respect to the complex conjugation. We will list all 13 types of real pencils, present a *normal form* for each of them and analyse their stabilizer subgroups, including the cases of pencils marked with one or two conics. Some types of real pencils are directly analogous to the complex case, so their description will be brief as we refer to the analysis in their corresponding complex version. On the other hand, we expand on the idiosyncratic properties that some real pencils display.

The group of real projective transformations $\mathrm{PGL}(3;\mathbb{R})$ acts on the space of real conics $\mathbb{RP}^5$ by sending lines into lines and preserving the type of pencil that they correspond to. If we also want to work with the implicit equations and make the distinction between different multiples of a same expression, then we may consider the action of $\mathrm{GL}(3;\mathbb{R})/\{\mathrm{Id}, -\mathrm{Id}\}$ on $\mathbb{R}^6$. The illustrative figures that accompany each type of pencil show the affine chart of $\mathbb{RP}^2$ given by the condition $z = 1$, setting the line $z = 0$ as the line at infinity. Whenever possible, we will prioritize the presence of circles on this chart to obtain the normal form of each pencil.

# A.5

## Classification of real pencils of conics

### A.5.1

#### $(1, 1, 1, 1)$ Pencil

The first type of pencil happens when its conics intersect in four distinct real points in general position. In this type of pencil there are three degenerate conics, all pairs of real lines. They are obtained by dividing the common points into two pairs, which can be done in three different ways. The submanifold of this kind of pencil is of real dimension 8, because one such pencil is determined by the location of the four common points. This is an open submanifold of the manifold of all real pencils of conics, which is the space of lines of $\mathbb{RP}^5$.

This type of pencil is completely analogous to the complex $(1, 1, 1, 1)$ pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.1. In normal form, the degenerate conics are given by:

$$
\begin{aligned}
w_\infty &= \frac{1}{2}(-x+y+z)(x-y+z) = -\frac{1}{2}x^2 + xy - \frac{1}{2}y^2 + \frac{1}{2}z^2, \\
w_0 &= \frac{1}{2}(x+y+z)(x+y-z) = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 - \frac{1}{2}z^2, \\
w_1 &= w_\infty + w_0 = 2xy.
\end{aligned}
$$

The stabilizer of the normal form is isomorphic to the symmetric group $S_4$, as it corresponds to the permutations of the common points. The classification of the marked pencils is also the same as displayed in A.3.2.

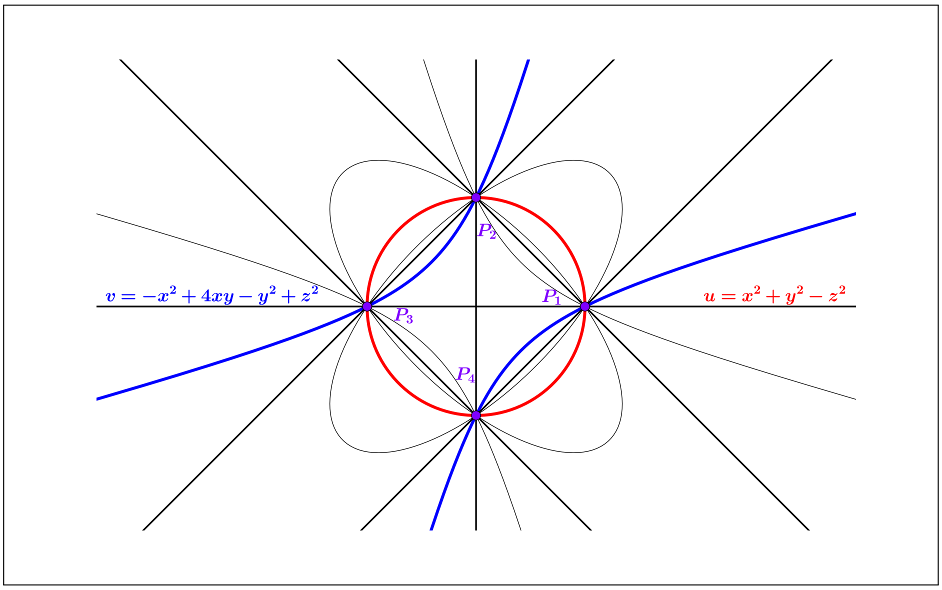


Figure A.9: Normal form of the $(1, 1, 1, 1)$ pencil.

### A.5.2

#### $(1,1,1,\overline{1})$ Pencil

We consider now the first subdivision originating from the existence of common points outside of the real projective plane. This type of pencil is thus the same as the $(1,1,1,1)$ pencil in the complex setting, but it has a distinct structure in the real point of view. It is the case in which the conics intersect in two real points $P$ and $P'$ and in a pair of complex conjugate points $Q$ and $\overline{Q}$. As a set of the space of real pencils, this is once again an open submanifold of dimension 8. The pencil is determined by the position of the real common points $P$ and $P'$, which contribute with 4 degrees of freedom, plus the position of complex common points $Q$ and $\overline{Q}$. The choice of a point in $\mathbb{CP}^2$ outside of the real plane yields 4 more real degrees of freedom. Notice however that the position of $Q$ and $\overline{Q}$ are intertwined, since one determines uniquely the other. Therefore, we have a total of 8 degrees of freedom to define the pencil.

Just as in the $(1,1,1,1)$ pencil, three pairs of lines are obtained by dividing the common points into two pairs: $PP', Q\overline{Q}$ / $PQ, P'\overline{Q}$ / $P\overline{Q}, P'Q$. However, while the first pair has a real expression, the other two only admit complex implicit equations. In other words, the discriminant of this pencil has one real root and two complex roots which are thus not attained in the real pencil. This makes of this pencil a peculiar type, because in every other case where there are irreducible conics, all the degenerate elements of the complex pencil also belong to the real pencil.

Let us study the line $Q\overline{Q}$ that joins two conjugate points from up close. We affirmed that it is in fact a real line, let us show why that is the case. The simplest explanation is that its equation is one of the factors of a real degenerate conic whose other factor is real (the line that joins $P$ and $P'$), thus its expression must also be real. We may also prove this fact in general, without needing the context of the conic equation. Since the line passes through $Q$ and $\overline{Q}$, its conjugate line also passes through these same points and therefore must be the same line. Being invariant under the complex conjugation, it must be a real line. A third way to prove it consists in showing explicitly that this line has more than one real point. If it contains two conjugate points $Q = [a_x + i\,b_x : a_y + i\,b_y : a_z + i\,b_z]$ and $\overline{Q} = [a_x - i\,b_x : a_y - i\,b_y : a_z - i\,b_z]$ then we may parametrize its complex version by $\alpha Q + \beta\,\overline{Q}$, where $[\alpha : \beta] \in \mathbb{CP}^1$ is a complex projective parameter. By taking $[\alpha : \beta] = [1 : 1]$ we obtain a first real point in this line $Q + \overline{Q} = [2a_x : 2a_y : 2a_z]$. On the other hand, if $[\alpha : \beta] = [1 : -1]$, we have a second real point $Q - \overline{Q} = [2i\, b_x : 2i\, b_y : 2i\, b_z] = [2b_x : 2b_y : 2b_z]$. Notice that these are indeed two different points of $\mathbb{RP}^2$, otherwise we would have:

$$Q = [a_x + i\, a_x : a_y + i\, a_y : a_z + i\, a_z] = [a_x(1+i) : a_y(1+i) : a_z(1+i)] = [a_x : a_y : a_z],$$

thus $Q$ would be a real point. As we have seen in A.4, if a line intersects $\mathbb{RP}^2$ in more than one point, then it must be a real line.

The other two pairs of lines are of a different nature, they comprise of two complex lines that are not conjugates. A first line of the pair, for instance $PQ$, cannot be real, otherwise it would also contain the point $\overline{Q}$ and we would have thus a more degenerate type of pencil. Therefore, $PQ$ is a complex line that intersects $\mathbb{RP}^2$ in $P$, and the other line of the pair is also complex for the same reason and intersects the real projective plane at a different point $P'$. This implies that those two lines are not a conjugate pair, so the implicit equation of the pair cannot be real and hence it does not belong to the real pencil.

**Normal form of the $(1, 1, 1, \overline{1})$ pencil**

Concerning the normal form, we have for the first time a pair of complex common points, which allow us to prioritize the presence of circles in the chosen affine chart. Indeed, one can find a projective transformation that sends the common points to $P = [0 : -1 : 1]$, $P' = [0 : 1 : 1]$, $Q = [1 : i : 0]$, $\overline{Q} = [1 : -i : 0]$. These complex points are called the *cyclic points*, because a real irreducible conic passes through them if and only if it is a circle in the affine chart $z = 1$. Therefore, by fixing them as common points, all the nondegenerate conics of the pencil become circles and we obtain a familiar *pencil of circles*. The pairs of lines through the common points are given implicitly by:

$$\begin{aligned}
&PQ, P'\overline{Q} \leftrightarrow -\frac{1}{2}(-ix + y + z)(ix + y - z) = -\frac{1}{2}x^2 - \frac{1}{2}y^2 - ixz + \frac{1}{2}z^2, \\
&P\overline{Q}, P'Q \leftrightarrow \frac{1}{2}(ix + y + z)(-ix + y - z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - ixz - \frac{1}{2}z^2, \\
&PP', Q\overline{Q} \leftrightarrow 2xz.
\end{aligned}$$

As expected, only the last element of this list belongs to the real $(1, 1, 1, \overline{1})$ pencil. Let us call it $w_\infty = 2xz$. Notice that the conic $u = x^2 + y^2 - z^2$ also belongs to this pencil, so we may parametrize the pencil by $\alpha w_\infty + \beta u$, $[\alpha : \beta] \in \mathbb{RP}^1$.

**Lemma A.5.1.** *The stabilizer subgroup of the normal form of the* $(1, 1, 1, \overline{1})$ *pencil is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

*Proof.* Once we have the image of four points determined, there is little flexibility left to transform the pencil. The only thing we can do is to permute those points as we have seen for the $(1, 1, 1, 1)$ pencil. However, as we should only consider *real* projective transformations in this setting, the real common points cannot be interchanged with the complex ones. Therefore, there are just four possible symmetries, given by:

$$\begin{aligned} ()\colon (x, y, z) &\mapsto (x, y, z), \\ (PP')\colon (x, y, z) &\mapsto (-x, -y, z), \\ (Q\overline{Q})\colon (x, y, z) &\mapsto (-x, y, z), \\ (PP')(Q\overline{Q})\colon (x, y, z) &\mapsto (x, -y, z). \end{aligned}$$

This group of symmetries is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

**Marked** $(1, 1, 1, \overline{1})$ **pencil**

*Proposition* A.5.2. There are two kinds of orbits of $(1, 1, 1, \overline{1})$ pencils marked with a single conic:

i. *The marked conic is irreducible:*
   Infinitely many orbits of dimension 8 described by an invariant;

ii. *The marked conic is degenerate:*
   A single orbit of dimension 8.

In addition, there are also two kinds of orbits of $(1, 1, 1, \overline{1})$ pencils marked with an unordered pair of conics.

i. *Both conics are irreducible:*
   Infinitely many orbits of dimension 8 described by a pair of invariants;

ii. *A single conic is degenerate:*
   Infinitely many orbits of dimension 8 described by an invariant;

*Proof.* If the marked conic is degenerate, then when taken to the normal form it must become $w_\infty$. The discrete stabilizer indicates that this orbit has dimension 8. If the marked conic is irreducible, then all we can do is apply the transformations of the stabilizer. Notice that two of them, namely $() : (x, y, z) \mapsto (x, y, z)$ and $(PP')(Q\overline{Q}) : (x, y, z) \mapsto (x, -y, z)$, act trivially on the pencil, that is, they preserve all conics. We have explained this phenomenon in the study of the $(1,1,1,1)$ pencil A.3.1; the double transpositions of the common points act as the identity on the pencil. Consequently, the only freedom we have to move the marked conic is via the transformation $(Q\overline{Q})\colon (x,y,z) \mapsto (-x,y,z)$. We observe that the conic $u = x^2+y^2-z^2$ is special in this pencil, since it is preserved by every symmetry of the pencil. Indeed, the action of $(Q\overline{Q})$ yields:

$$(Q\overline{Q}).w_\infty = -w_\infty, \qquad (Q\overline{Q}).u = u.$$

Therefore, the conic $v = \alpha w_\infty + \beta u$ of parameter $[\alpha : \beta] \in \mathbb{RP}^1$ is mapped to the one given by $[-\alpha : \beta]$. So up to applying this action, we may take the marked conic to the one whose parameter is $[c : 1]$ with $c \geq 0$. This invariant characterizes each of the infinitely many 8-dimensional orbits of this kind of marked pencil.

If there are two marked conics and one of them is degenerate, we fall back to the previous case. If both are irreducible, then the pair is uniquely characterized by a pair of invariants $(c,c') \in \mathbb{R}^2$, with $c \neq c'$, as one conic has parameter $[c : 1]$ and the other, $[c' : 1]$. The best we can do with the stabilizer of the pencil is guarantee that $c + c' \geq 0$, observing that the equality holds if and only if the conics are symmetric with respect to the action of $(Q\overline{Q})$. $\square$

In the figure below we highlight the conic $u$ and the conic $v = x^2+y^2+2xz-z^2 = (x+z)^2+y^2-2z^2$, whose parameter is $[1 : 1]$.

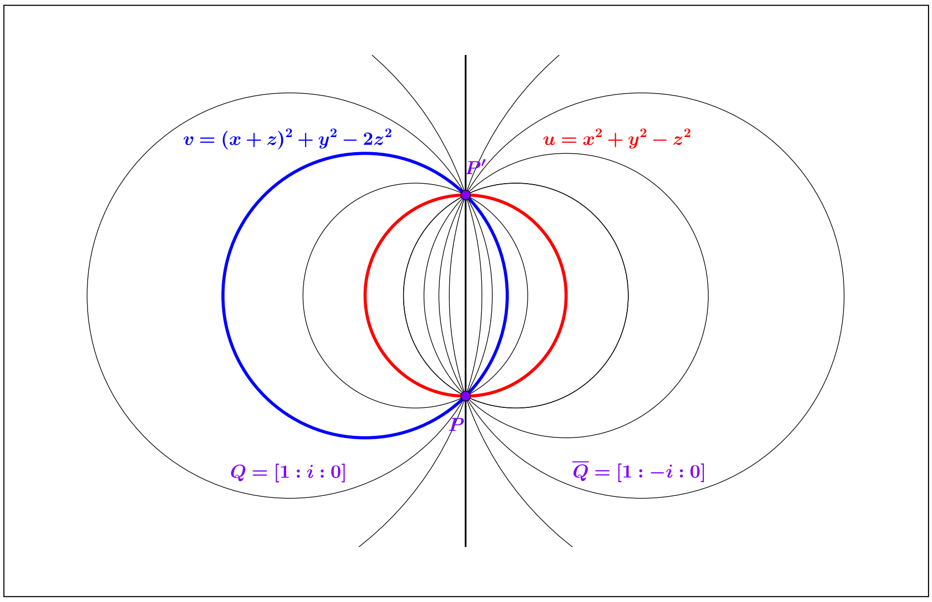


Figure A.10: Normal form of the $(1,1,1,\overline{1})$ pencil.

### A.5.3

#### $(1, \overline{1}, 1, \overline{1})$ **Pencil**

The third and last type of pencil where the four common points are distinct and in general position happens when the conics of the pencil have four distinct complex intersection, necessarily composed of two conjugate pairs $P, \overline{P}, Q, \overline{Q}$. Once again this family constitutes an open submanifold of dimension 8 of the set of all real pencils. Such a pencil is defined by the location of $P$ and $Q$, since they automatically determine the position of the other common points, their conjugates. As they are points outside of $\mathbb{RP}^2$, each one contributes with 4 real degrees of freedom, adding up to give the 8 dimensions of the submanifold.

A $(1, \overline{1}, 1, \overline{1})$ pencil has three degenerate conics: a pair of real lines $P\overline{P}$, $Q\overline{Q}$ and two pairs of complex lines $PQ$, $\overline{PQ}$ / $P\overline{Q}$, $\overline{P}Q$. Concerning the first pair, we have previously shown that a line containing conjugate points, as does $P\overline{P}$, is a real line. For the other pairs, one can easily see that they are conjugate pairs of lines, after all, the conjugate line to $PQ$ passes through $\overline{P}$ and $\overline{Q}$, thus it can only be the line $\overline{PQ}$. Because they are conjugate pairs of lines, they admit real implicit equations and each pair meets $\mathbb{RP}^2$ at a single point, precisely in the intersection of their components.

**Normal form of the $(1, \overline{1}, 1, \overline{1})$ pencil**

With a real projective transformation, one can send the common points to $P = [0 : i : 1]$,$\overline{P} = [0 : -i : 1]$, $Q = [1 : i : 0]$, $\overline{Q} = [1 : -i : 0]$. So once again we obtain a pencil of circles. The degenerate conics of the normal form are:

$$\begin{aligned}
PQ, \overline{PQ} &\leftrightarrow w_\infty = \frac{1}{2}(x + iy + z)(-x + iy - z) = -\frac{1}{2}x^2 - \frac{1}{2}y^2 - xz - \frac{1}{2}z^2, \\
P\overline{Q}, \overline{P}Q &\leftrightarrow w_0 = -\frac{1}{2}(-x + iy + z)(x + iy - z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - xz + \frac{1}{2}z^2, \\
P\overline{P}, Q\overline{Q} &\leftrightarrow w_1 = w_\infty + w_0 = -2xz.
\end{aligned}$$

We parametrize the pencil as $\alpha w_\infty + \beta w_0$, $[\alpha : \beta] \in \mathbb{RP}^1$.

**Lemma A.5.3.** *The stabilizer subgroup of the normal form of the* $(1, \overline{1}, 1, \overline{1})$ *pencil is isomorphic to the dihedral group* $D_4$.

*Proof.* The only projective transformations that preserve the pencil are those that permute the four common points. However, real transformations send conjugate pairs of points simultaneously to a new pair of conjugate points, so there are even fewer permutations available. Another way of formulating this interdiction is to think in terms of permutations of the degenerate conics, after all, the pair of real lines cannot become one of the pairs of complex lines via a real transformation. Therefore, only 8 permutations remain, 4 of which act as the identity on the pencil, for they preserve the three degenerate conics and thus all conics of the pencil.

$$\begin{aligned} ()\colon (x,y,z) &\mapsto (x,y,z), & (P\overline{P})\colon (x,y,z) &\mapsto (-x,-y,z),\\ (P\overline{P})(Q\overline{Q})\colon (x,y,z) &\mapsto (x,-y,z), & (Q\overline{Q})\colon (x,y,z) &\mapsto (-x,y,z),\\ (PQ)(\overline{PQ})\colon (x,y,z) &\mapsto (z,y,x), & (P\overline{Q}\overline{P}Q)\colon (x,y,z) &\mapsto (-z,y,x),\\ (P\overline{Q})(\overline{P}Q)\colon (x,y,z) &\mapsto (z,-y,x), & (PQ\overline{PQ})\colon (x,y,z) &\mapsto (-z,-y,x). \end{aligned}$$

In order to visualize the isomorphism with $D_4$, picture the four common points displayed in a square where conjugate points are in opposite corners. The transformation that preserves the pencil are in bijection with the elements of the dihedral group of that square and the group operations are respected, so we have a group isomorphism. Notice that this group is not abelian.

Moreover, the stabilizer is generated by two transformations, the involution $I = (Q\overline{Q})\colon (x,y,z) \mapsto (-x,y,z)$ and the symmetry of order 4 given by $R = (P\overline{Q}\overline{P}Q)\colon (x,y,z) \mapsto (-z,y,x)$. □

**Marked** $(1,\overline{1},1,\overline{1})$ **pencil**

*Proposition* A.5.4. There are three kinds of orbits of $(1,\overline{1},1,\overline{1})$ pencils marked with a single conic:

i. *The marked conic is irreducible:*
   Infinitely many orbits of dimension 8 described by an invariant;

ii. *The marked conic is a pair of complex lines:*
   A single orbit of dimension 8;

iii. *The marked conic is a pair of real lines:*
   A single orbit of dimension 8.

In addition, there are five kinds of orbits of $(1,\overline{1},1,\overline{1})$ pencils marked with an unordered pair of conics.

i. *Both marked conics are irreducible:*
   Infinitely many orbits of dimension 8 described by a pair of invariants;

ii. *An irreducible and a pair of complex lines:*
   Infinitely many orbits of dimension 8 described by an invariant;

iii. *An irreducible and a pair of real lines:*
Infinitely many orbits of dimension 8 described by an invariant;

iv. *Two pairs of complex lines:*
A single orbit of dimension 8;

v. *A pair of complex lines and a pair of real lines:*
A single orbit of dimension 8.

*Proof.* Consider first that the pencil has a single marked conic. When taken to the normal form, if the marked conic is a pair of real lines, then the whole stabilizer of the pencil preserves this marked conic, resulting in an orbit of dimension 8. If it is a pair of complex lines, then it could occupy two different spots on the pencil, either $w_\infty$ or $w_0$. Naturally, the involution $I$ does not preserve this marked pencil as it interchanges these pairs of complex lines. The transformation $R$ also swaps these two conics, so the stabilizer of this marked pencil is $< R.I, R^2 >$, being generated by these two involutions, so it is isomorphic to $D_2$. In any case, we get again a single orbit of dimension 8.

If the marked conic is irreducible, then not much can be done, as the 4 transformations that do not preserve the whole pencil must act in the same way on it. Indeed, since there is a copy of $D_2$ that acts trivially, then the group of remaining actions must be $D_4/D_2 \cong \mathbb{Z}_2$. Algebraically, what these transformations do is $w_\infty \mapsto -w_0$, $w_0 \mapsto -w_\infty$ and $w_1 \mapsto -w_1$. So the arbitrary conic of the pencil $u = \alpha w_\infty + \beta w_0$ is mapped to the conic given by the parameter $[\beta : \alpha]$. This allows us to map the marked conic to the one whose parameter is $[c : 1]$ with $|c| \leq 1$ and $c \notin \{0, 1\}$. This value $c$ cannot be 0 or 1 because it would result in degenerate conics, $w_0$ and $w_1$ respectively. Another conic stands out, when $c = -1$ we get $v = -w_\infty + w_0 = x^2 + y^2 + z^2$, which is preserved by the whole stabilizer of the unmarked pencil. Therefore, we have an 8-dimensional orbit for each value of $c \in [-1, 0) \cup (0, 1)$, observing that the signature of the conic is $(2, 1)$ when $c > 0$, so the conic is present in $\mathbb{RP}^2$, and it is $(3, 0)$ when $c < 0$, so there is no point of the conic in $\mathbb{RP}^2$.

Now consider there are two marked conics. If one of them is a pair of real lines, then we fall back to previous cases explained above. If both are pairs of complex lines, then the whole stabilizer of the unmarked pencil preserves this marked pencil too and we get the 8-dimensional orbit. If one is irreducible and the other is a pair of complex lines, then we get two copies of the orbits characterized by the invariant $c$ presented in the last paragraph due to the location of the marked degenerate conic. Finally, if both conics are irreducible, then we need two invariants to define an orbit. The first is the same $c \in [-1, 0) \cup (0, 1)$ already explained, while the other we have no further control over. The other marked conic is given by $[d : 1]$ with $d \in \mathbb{R}$ and $d \notin \{0, 1, c\}$. This concludes the description of these 8-dimensional orbits given by two invariants. $\square$

In the figure below we highlight two symmetric conics with respect to the line $x = 0$, which are permuted by this non trivial symmetry of the pencil. They are designated by the parameters $[\frac{1}{3} : 1]$ and $[3 : 1]$.

$$u = \frac{1}{3}w_\infty + w_0 = \frac{1}{3}\Big((x-2z)^2 + y^2 - 3z^2\Big),$$
$$v = 3w_\infty + w_0 = -\Big((x+2z)^2 + y^2 - 3z^2\Big).$$

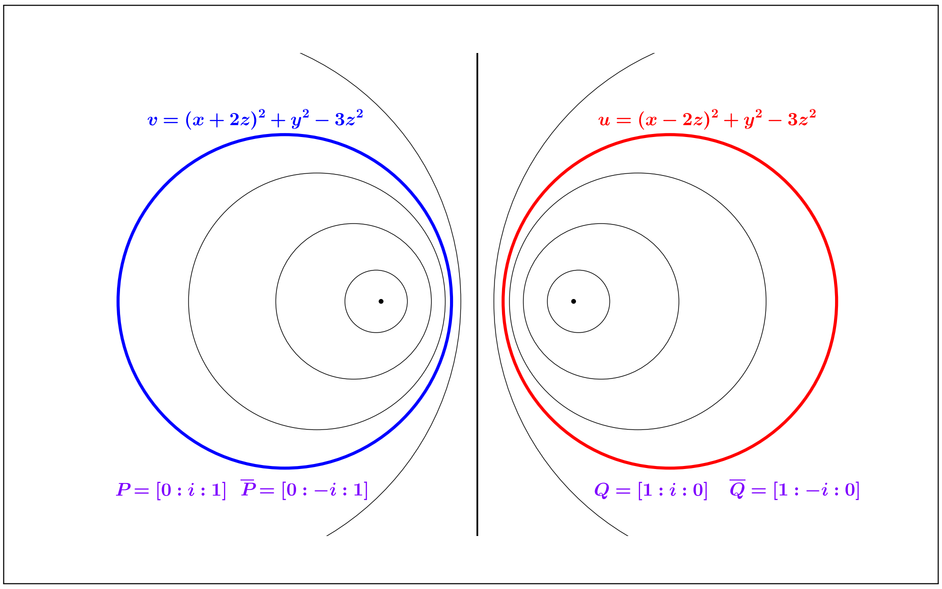


Figure A.11: Normal form of the $(1, \overline{1}, 1, \overline{1})$ pencil.

### A.5.4

### $(2, 1, 1)$ Pencil

Let us now consider the case where the conics have a contact of order 2 at a real point $P$ and two simple intersections at two other real points $Q$ and $Q'$. In order to uniquely determine the pencil, it is necessary to indicate a real line through $P$ which will be the common tangent of the irreducible conics of the pencil. In this way, there are 7 degrees of freedom to define a pencil of this family, so it constitutes a submanifold of codimension 1 in the space of all real pencils of conics.

The nature of this type of pencil is completely analogous to the complex $(2,1,1)$ pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.2.

In normal form, the degenerate conics are given by:

$$w_\infty = (x-y-z)(x+y+z) = x^2-y^2-2yz-z^2, \qquad w_0 = 2y(y+z) = 2y^2+2yz.$$

The stabilizer subgroup of the normal form is generated by the 1-parameter family of transformations $M_t$, with $t \in \mathbb{R}^*$ and the involution $N$ below.

$$M_t = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t-1 & t \end{pmatrix}, \qquad N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The classification of the marked pencils is also the same as displayed in A.3.4. Notice that the invariant $c$ must now be in $[-1,0)\cup(0,1)$. The ensuing marked conic has signature $(2,1)$ when $c>0$, and signature $(1,2)$ when $c<0$.

In the figure below, we highlight a particular conic $v$ just to show that in the chosen affine chart the normal form contains a parabola.

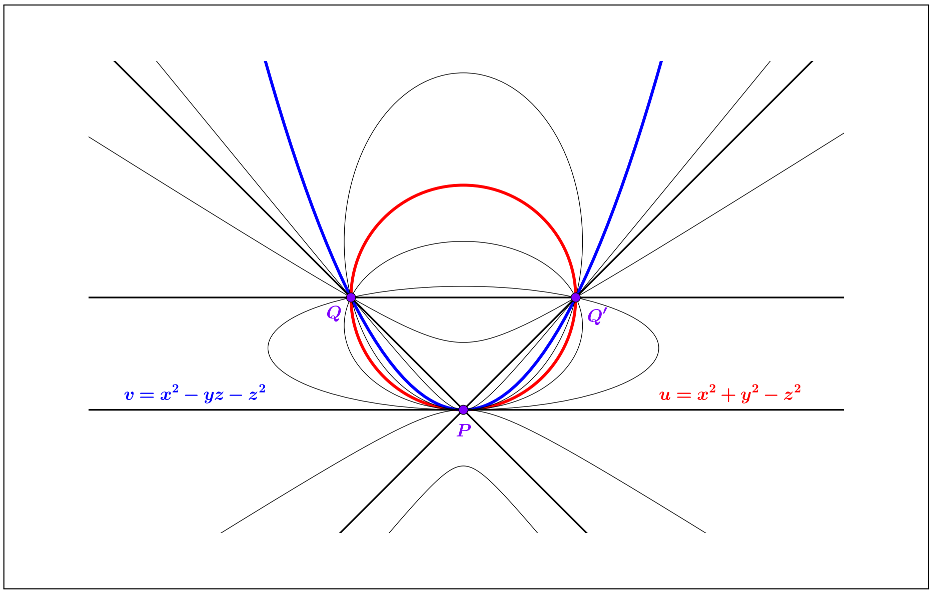


Figure A.12: Normal form of the $(2,1,1)$ pencil.

### A.5.5

$(2, 1, \overline{1})$ **Pencil**

Next, we have the case where the conics have a contact of order 2 at a real point $P$ and two conjugate complex intersections $Q$ and $\overline{Q}$. This kind of pencil is very similar to the previous one, belonging to the same class in the complex point of view. This is also a 7 dimensional family, so of codimension 1, because there are 2 degrees of freedom to determine $P$, 1 to choose the common tangent through $P$ and 4 to fix the pair of conjugate points $Q$ and $\overline{Q}$.

In the pencil there are once again two degenerate conics, but this time one is a pair of real lines and the other is a pair of complex lines. The complex pair corresponds to the lines $PQ$ and $P\overline{Q}$, which naturally intersect at $P$, bearing in mind that any conjugate pair of lines meets $\mathbb{RP}^2$ at a single point. Meanwhile, the real pair of lines consists of the common tangent of the pencil paired with the real line whose equation vanishes over the complex points $Q$ and $\overline{Q}$.

**Normal form of the $(2, 1, \overline{1})$ pencil**

In terms of the normal form, we will make good use of the complex common points to obtain a pencil of circles in the chosen affine chart, as we have done previously. One can find a projective transformation that sends the common points to $P = [0 : 0 : 1]$, $Q = [1 : i : 0]$, $\overline{Q} = [1 : -i : 0]$ and that maps the common tangent to the line $x = 0$. In this way, the degenerate conics become:

$$w_\infty = (x + iy)(x - iy) = x^2 + y^2, \qquad w_0 = -2xz.$$

Just as in the $(2, 1, 1)$ pencil, any symmetry of the pencil must preserve these degenerate conics, which have distinct characteristics.

**Lemma A.5.5.** *The stabilizer subgroup of the normal form of the* $(2, 1, \overline{1})$ *pencil is generated by the* 1*-parameter family of transformations* $M_t$*, with* $t \in \mathbb{R}^*$*, and the involution* $N$ *below.*

$$M_t = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* Firstly, we may permute the complex points, an action provided by the involution $N : (x, y, z) \mapsto (x, -y, z)$. Besides, there is a 1-parameter family of symmetries that preserve all three common points. One such transformation

$M_t$ may by obtained by considering an auxiliary point $P' = [0 : 1 : 1]$ on the common tangent. Its image $[0 : t : 1]$, with $t \in \mathbb{R}^*$, must stay on this line and one should avoid the collinearity with $Q$ and $\overline{Q}$, which is why it cannot be on the line at infinity. Notice that $M_t$ are simply homotheties of center $(0, 0)$ and ratio $t$ in the affine chart $z = 1$. The group of all symmetries of the pencil is thus generated by $N$ and the family $M_t$. □

**Marked $(2, 1, \overline{1})$ pencil**

*Proposition* A.5.6. There are three kinds of orbits of $(2, 1, \overline{1})$ pencils marked with a single conic:

i. *The marked conic is irreducible:* A single orbit of dimension 8;

ii. *The marked conic is a pair of complex lines:* A single orbit of dimension 7;

iii. *The marked conic is a pair of real lines:* A single orbit of dimension 7.

In addition, there are four kinds of orbits of $(2, 1, \overline{1})$ pencils marked with an unordered pair of conics.

i. *Both marked conics are irreducible:*
Infinitely many orbits of dimension 8 described by an invariant;

ii. *An irreducible and a pair of complex lines:*
A single orbit of dimension 8;

iii. *An irreducible and a pair of real lines:*
A single orbit of dimension 8;

iv. *Both marked conics are degenerate:*
A single orbit of dimension 7.

*Proof.* Consider the pencil marked with a single conic. When taken to the normal form, if it is degenerate then the whole stabilizer of the unmarked pencil also preserves the marked conic, because the pair of real lines and the pair of complex lines cannot be interchanged by a real projective transformation. Therefore, we reach orbits of dimension 7. If the marked conic is irreducible, then we may map it anywhere we want inside the pencil due to the distinct multiples that arise from the actions of the symmetries of the pencil on its degenerate elements.

$$M_t w_\infty = \frac{1}{t^2} w_\infty, \qquad M_t w_0 = \frac{1}{t} w_0,$$
$$N w_\infty = w_\infty, \qquad N w_0 = w_0.$$

One may notice that the behaviour is exactly the same as in the $(2,1,1)$ pencil. After we set the marked conic at $u = w_\infty + w_0 = x^2 - 2xz + y^2 = (x-z)^2 + y^2 - z^2$, the stabilizer of this marked pencil is only $\{\mathrm{Id}, N\}$, so we get an orbit of dimension 8.

Now let us take a pencil marked with two conics. If at least one of them is degenerate, we fall back to one of the previous cases. If both are irreducible, we first set one to $u$. At this point, we already have 3 determined conics of distinct nature in the pencil, so there is no liberty to move any other element around. So the other marked conic $v$ is defined by its parameter $[c : 1]$, with $c \in \mathbb{R}$ and $c \notin \{0, 1\}$. We have thus an orbit of dimension 8 for each such value of the invariant $c$. □

In the figure below we highlight $u$ and the conic which is symmetric to it with respect to the common tangent, that is $v = M_{-1}.u = (x-z)^2 + y^2 - z^2$.

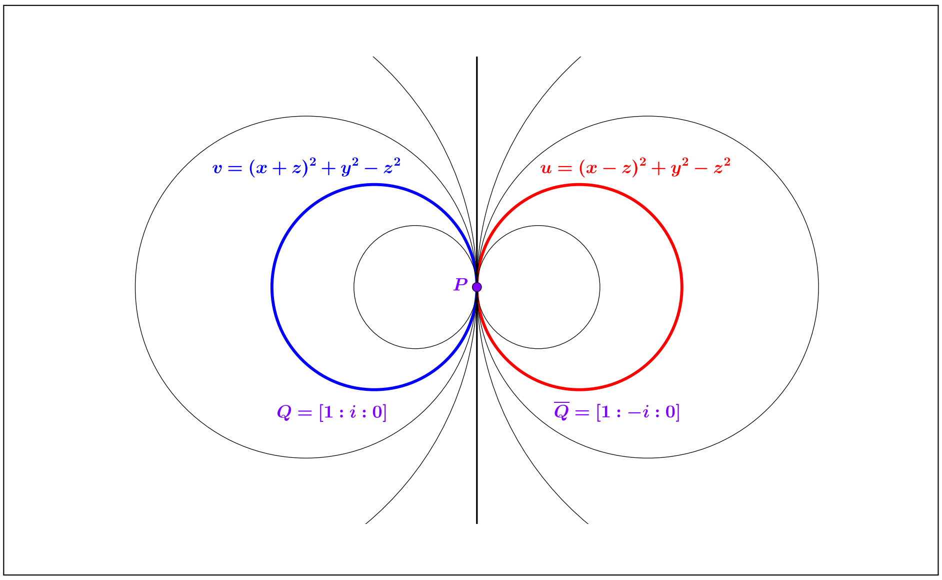


Figure A.13: Normal form of the $(2, 1, \overline{1})$ pencil.

### A.5.6

#### $(2, 2)$ Pencil

Let us now consider the case where the conics have a contact of order 2 at two real points $P$ and $Q$. This family of pencils constitutes a 6 dimensional submanifold, so of codimension 2, because in order to uniquely determine such a pencil one must pick the points $P$ and $Q$, each contributing with two degrees of freedom, as well as indicate the common tangent at each of them, each providing an additional degree of freedom adding up to $2+2+1+1=6$.

The nature of this type of pencil is completely analogous to the complex $(2,2)$ pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.3. In normal form, the degenerate conics are given by:

$$w_\infty = x^2, \qquad w_0 = (y+z)(y-z) = y^2 - z^2.$$

The stabilizer subgroup of the normal form of the $(2,2)$ pencil is generated by the 2-parameter family of transformations $M_{t,s}$, with $t, s \in R^*$, and the involution $N$ below.

$$M_{t,s} = \begin{pmatrix} ts & 0 & 0 \\ 0 & \frac{t+s}{2} & \frac{t-s}{2} \\ 0 & \frac{t-s}{2} & \frac{t+s}{2} \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The classification of the marked pencils is also the same as displayed in A.3.6. Notice that the invariant $c$ must again be in $[-1,0) \cup (0,1)$. The resulting marked conic has signature $(2,1)$ when $c>0$, and signature $(1,2)$ when $c<0$.

In the following figure we highlight an arbitrary conic of the pencil $v = \frac{1}{2}w_\infty + w_0 = \frac{1}{2}x^2 + y^2 - z^2$.

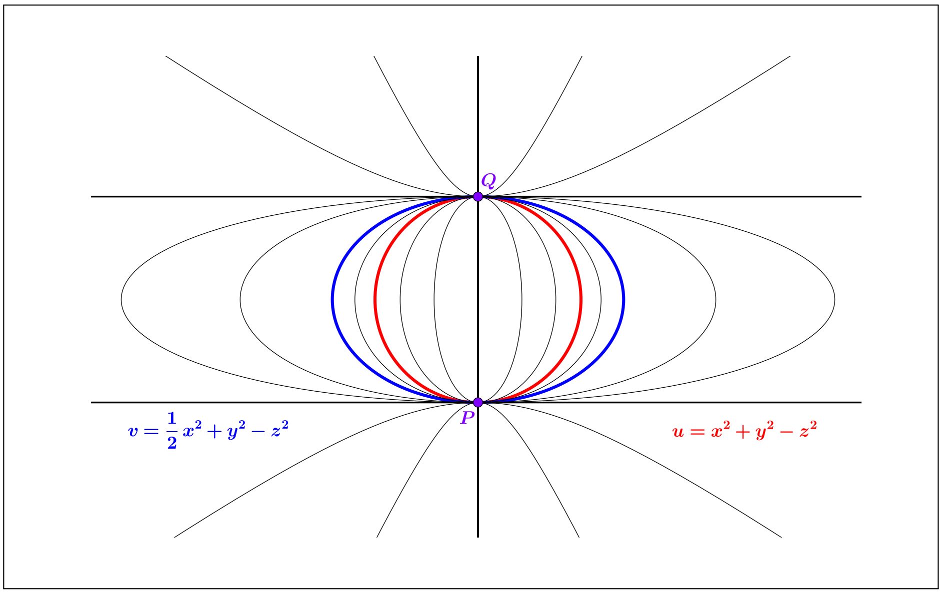


Figure A.14: Normal form of the $(2,2)$ pencil.

## A.5.7

### $(2, \overline{2})$ Pencil

Now we have the case where the conics have a contact of order 2 at two complex points, necessarily conjugates, $P$ and $\overline{P}$. Just as in the previous case, this family constitutes a submanifold of dimension 6 and codimension 2, because one such pencil is determined by the position of $P$ and by the common tangent through it. Since $P$ is a complex point, it contributes with four degrees of freedom. As for the common tangent, it is also complex, so it adds two more degrees of freedom. Once defined, the other common point $\overline{P}$ and its common tangent are already uniquely determined, for they must be their conjugate pairs. This fact is a consequence of a more general result that we prove in the following lemma.

**Lemma A.5.7.** *Let $\gamma$ be a real algebraic curve in $\mathbb{CP}^2$. If $P$ is a point of $\gamma$, and $\ell$ is a tangent line to $\gamma$ at $P$, then $\overline{P}$ is also a point of $\gamma$ and $\bar{l}$ is a tangent line to $\gamma$ at $\overline{P}$.*

*Proof.* One way to verify that a line $\ell$ is tangent to $\gamma$ at $P$ is to parametrize it as $\alpha P + \beta Q$ where $[\alpha : \beta] \in \mathbb{CP}^1$ and $Q$ is any other point of $\ell$, then inject this parametrization into the implicit equation of the curve and check that the multiplicity of the root corresponding to the point $P$ is of order at least 2. Let us consider next the conjugate line $\bar{l}$ of parametrization $\overline{\alpha P + \beta Q}$ and evaluate it in the implicit equation of the same curve. We obtain the conjugate expression of the one produced by $\ell$, because the equation of $\gamma$ is real. Therefore, there is a root associated to the point $\overline{P}$ and it is of the same order as the one obtained for $P$, so at least 2 by the hypothesis. We have thus shown that the line $\bar{l}$ is tangent to $\gamma$ at $\overline{P}$. □

This type of pencil has two degenerate conics, the double line $P\overline{P}$, which is a real line, and the pair of complex lines given by the common tangents at $P$ and $\overline{P}$. We have just shown that this lines are necessarily conjugate, so they meet at a single point of $\mathbb{RP}^2$ and the pair has a real implicit equation. The double line once again corresponds to the double root of the discriminant.

#### Normal form of the $(2, \overline{2})$ pencil

In terms of the normal form, we can send the common points to the cyclic points $P = [1 : i : 0]$, $\overline{P} = [1 : -i : 0]$ to obtain a pencil of circles. Furthermore, we can impose that the common tangents meet at the real point $[0 : 0 : 1]$. The centre of a conic in an affine chart is given precisely by the intersection of its tangents at its points contained in the line at infinity, in our case $z = 0$. Consequently, all the irreducible conics of this pencil that are present in $\mathbb{RP}^2$ are concentric circles. This remark gives us many details about the pencil already, but we shall proceed without making use of it and we are going to arrive at the same conclusion by analysing the group of symmetries of the normal form. The degenerate conics of the pencil are:

$$w_\infty = -z^2, \qquad w_0 = (x+iy)(x-iy) = x^2+y^2.$$

**Lemma A.5.8.** *The stabilizer subgroup of the normal form of the* $(2,\overline{2})$ *pencil is the* 2*-parameter family of transformations* $M_{t,s}$*, with* $(t,s) \in \mathbb{R}^2 \setminus \{(0,0)\}$*, and the involution* $N$ *below.*

$$M_{t,s} = \begin{pmatrix} t & -s & 0 \\ s & t & 0 \\ 0 & 0 & t^2+s^2 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* Firstly, we point out the transformation that permutes the common points and common tangents given by $N : (x,y,z) \mapsto (x,-y,z)$. Other than that, we can present a 2-parameter family of transformations $M_{t,s}$ that preserve the common points and common tangents. They may be obtained by using two conjugate auxiliary points $P' = [1 : i : 1]$, $\overline{P'} = [1 : -i : 1]$ and their respective images $[1 : i : t+si]$ and $[1 : -i : t-si]$, with $(t,s) \in \mathbb{R}^2 \setminus \{(0,0)\}$, that remain on the common tangents. Notice that since we are considering only real transformations, the image of $\overline{P'}$ is completely determined by the image of $P'$, so by considering the images of $P$, $\overline{P}$, $P'$ and $\overline{P'}$, we uniquely determine the transformation $M_{t,s}$ as presented in the statement of the lemma. □

### Marked $(2,\overline{2})$ pencil

*Proposition* A.5.9. There are four kinds of orbits of $(2,\overline{2})$ pencils marked with a single conic:

i. *The marked conic is irreducible and present in* $\mathbb{RP}^2$*:*
   A single orbit of dimension 7;

ii. *The marked conic is irreducible and not present in* $\mathbb{RP}^2$*:*
   A single orbit of dimension 7;

iii. *The marked conic is a pair of complex lines:*
A single orbit of dimension 6;

iv. *The marked conic a double line:*
A single orbit of dimension 6.

In addition, there are eight kinds of orbits of $(2, \overline{2})$ pencils marked with an unordered pair of conics.

i. *Both marked conics are irreducible and present in* $\mathbb{RP}^2$*:*
Infinitely many orbits of dimension 7 described by an invariant;

ii. *Both marked conics are irreducible and not present in* $\mathbb{RP}^2$*:*
Infinitely many orbits of dimension 7 described by an invariant;

iii. *Both marked conics are irreducible and only one is present in* $\mathbb{RP}^2$*:*
Infinitely many orbits of dimension 7 described by an invariant;

iv. *A pair of complex lines and one irreducible present in* $\mathbb{RP}^2$*:*
A single orbit of dimension 7;

v. *A pair of complex lines and one irreducible not present in* $\mathbb{RP}^2$*:*
A single orbit of dimension 7;

vi. *A double line and one irreducible present in* $\mathbb{RP}^2$*:*
A single orbit of dimension 7;

vii. *A double line and one irreducible not present in* $\mathbb{RP}^2$*:*
A single orbit of dimension 7;

viii. *A double line and a pair of complex lines:*
A single orbit of dimension 6.

*Proof.* Consider first a pencil with one marked conic. If it is degenerate, then the stabilizer of the unmarked pencil preserves it, so we get 6-dimensional orbits. Next, we evaluate how the elements of the stabilizer act on the degenerate conics of the pencil:

$$M_{t,s}.w_\infty = \frac{1}{(t^2+s^2)^2}\, w_\infty, \qquad M_{t,s}.w_0 = \frac{1}{t^2+s^2}\, w_0.$$

The different multiples obtained allow us to manipulate the pencil while preserving the degenerate conics. The arbitrary conic associated to the parameter $[\alpha : \beta]$ is sent to the one given by $[\alpha : (t^2+s^2)\beta]$. Notice however that there is something remarkable in this pencil, because unlike all previous cases, the factor $(t^2 + s^2)$ is always positive and thus it preserves the sign of the ratio between the coefficients of the projective parameter. This means that the pencil has two distinct intervals delimited by the degenerate conics and no real transformation can make a conic go from one interval to the other. This fact is related to the different signatures of the conics of the pencil. If the coefficients $\alpha$ and $\beta$ have the same sign, then the conic obtained has signature $(2, 1)$ or $(1, 2)$. Whereas if $\alpha$ and $\beta$ have opposite signs, then the conic has signature $(3, 0)$ or $(0, 3)$ and thus it is not present at all in the real projective plane. It is clear that it is impossible for a real transformation to relate conics of these two distinct natures, after all, its action is given by a matrix congruence, which always preserves the signature.

If the marked conic is present in $\mathbb{RP}^2$, then we may map it to $u = w_\infty + w_0 = x^2 + y^2 - z^2$. Otherwise, it may be mapped to $v = -w_\infty + w_0 = x^2 + y^2 + z^2$. In any case, there is still a 1-parameter group with two connected components that acts trivially on the entire pencil. First, we notice that the transformation $N$ belongs to this group. It is responsible for the existence of the two components of this stabilizer, one that preserves the orientation of the conics and the other that inverts it. Secondly, we observe that the condition on $(t, s)$ for the transformation $M_{t,s}$ to act trivially on the pencil is $t^2 + s^2 = 1$. This implies that the group of symmetries of the $(2, \overline{2})$ pencil marked with an irreducible conic is isomorphic to a pair of circles $\mathbb{S}^1 \sqcup \mathbb{S}^1$, because it is generated by $N$ and the family $M_{\cos(\theta),\sin(\theta)}$. This orbit is then 7-dimensional.

$$M_{\cos(\theta),\sin(\theta)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One may recognize the rotations around the point $(0, 0)$ in the affine chart considered. They preserve every conic of the pencil, which must therefore be concentric circles.

Now we move to the pencils marked with an unordered pair of conics. If at least one of them is degenerate, we fall back to one of the previous cases explained above. If both are irreducible, there are still three cases to consider due to the different possible signatures. If at least one of the conics is present in $\mathbb{RP}^2$, we set it to $u = x^2 + y^2 - z^2$. At this point, the location of the other conic is completely determined, say at $[c : 1]$, with $c \in \mathbb{R} \setminus \{0, 1\}$. This invariant determines the orbit, and since the stabilizer is still $< M_{\cos(\theta),\sin(\theta)}, N >$, it is 7-dimensional.

If both marked conics are not present in $\mathbb{RP}^2$, then we may set one of them to $v = x^2 + y^2 + z^2$ and the other will go to $[c : 1]$, with $c < 0$ and $c \neq -1$. This invariant once again determines the orbit, and since the stabilizer is $< M_{\cos(\theta),\sin(\theta)}, N >$, it is 7-dimensional. $\square$

In the following figure we highlight also the conic associated to the parameter $[4 : 1]$, that is $v = x^2 + y^2 - 4z^2$.

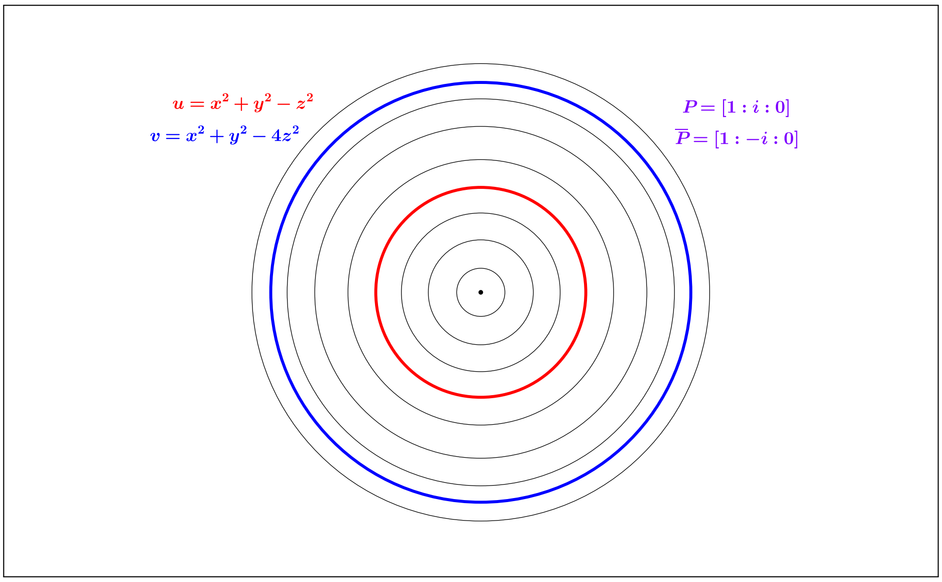


Figure A.15: Normal form of the $(2, \overline{2})$ pencil.

### A.5.8

#### $(3, 1)$ **Pencil**

This is the case where the conics have a contact of order 3 at a real point $P$ and also intersect at another real point $Q$. As explained in Subsection A.3.4, in order to uniquely determine a pencil of this family one must specify the location of the common points $P$ and $Q$, the common tangent through $P$ and yet another information about the intersection at $P$ referring to the contact of order 3. To do so, we fix an affine chart, say $z = 1$, then describe the variable $y$ as a function of $x$ and evaluate the second derivative at the point $P$. Every irreducible conic of the pencil share the same value for the second derivative at this point, which means that they have the same *curvature* there. This quantity is not a projective invariant, but, just as the common tangent, under the action of a projective transformation every conic of the pencil keep sharing a common value for the second derivative of the function $y(x)$ in the corresponding point of contact. Since we consider here real equations, this value is necessarily real and cannot be zero $k \in \mathbb{R}^*$. Therefore, the family of $(3,1)$ pencils constitute a submanifold of dimension 6 and codimension 2 in the space of all real pencils.

The nature of this type of pencil is completely analogous to the complex $(3,1)$ pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.4.

In normal form, the degenerate conic of this pencil is given by:

$$w_\infty = x(y+z) = xy + xz.$$

The stabilizer subgroup of the normal form of the $(3,1)$ pencil is 2-dimensional, and each of its elements may be written in the form $M_t.Y_s$, with $t \in \mathbb{R}$ and $s \in \mathbb{R}^*$, where:

$$M_t = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ t & 0 & 1 \end{pmatrix}, \qquad Y_s = \begin{pmatrix} 2s & 0 & 0 \\ 0 & s^2+1 & s^2-1 \\ 0 & s^2-1 & s^2+1 \end{pmatrix}.$$

The classification of the marked pencils is also the same as displayed in A.3.8.

In the figure below, we highlight $u = x^2 + y^2 - z^2$ and the arbitrary conic $v = w_\infty + u = x^2 + y^2 - z^2 + xy + xz$.

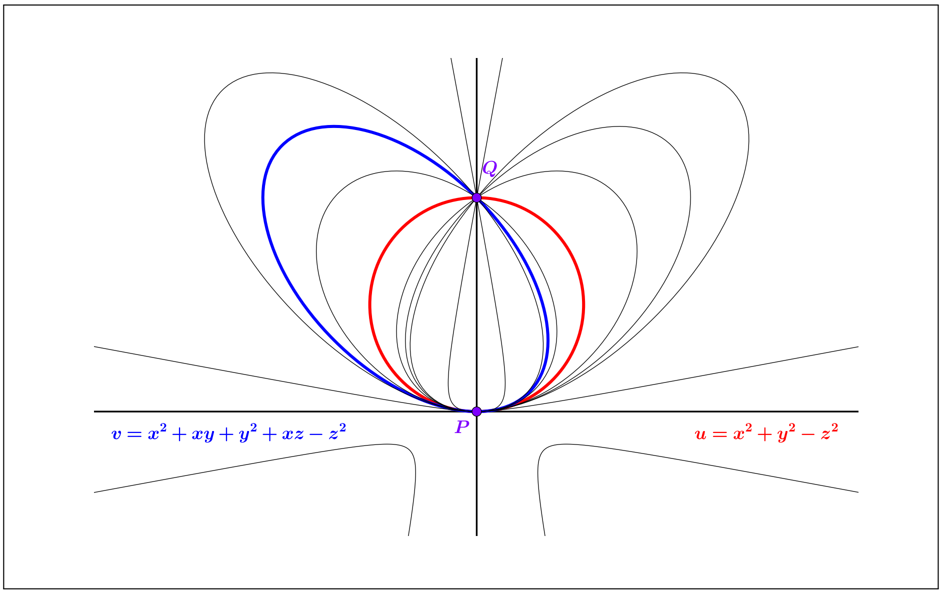


Figure A.16: Normal form of the $(3,1)$ pencil.

### A.5.9

#### (4) **Pencil**

We arrive at the cases where the conics have a contact of order 4 at a real point $P$. In the real setting, there are three types of pencil that meet this condition. We begin with the one that has irreducible conics in the pencil. Just as in the previous case, in order to determine a pencil of this family one must specify the location of $P$, the common tangent and give further information about the contact at $P$. In this case, one needs two real values, $k \in \mathbb{R}^*$ and $k' \in \mathbb{R}$, for the second and third derivatives respectively on the point referring to $P$ in the local parametrizations $y(x)$ of the irreducible conics of the pencil. Therefore, this family is a submanifold of dimension 5 and codimension 3 of the space of all real pencils of conics.

The nature of this type of pencil is completely analogous to the complex (4) pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.5.

In normal form, the degenerate conic is given by:

$$w_\infty = (y + z)^2.$$

The stabilizer subgroup of the normal form of the (4) pencil is 3-dimensional, and each of its elements may be written in the form $M_t.N_{r,s}$, with $r, s, t \in \mathbb{R}$ and $r \neq s$, where:

$$M_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t + \frac{3}{2} & -t + \frac{1}{2} \\ 0 & t - \frac{1}{2} & t + \frac{1}{2} \end{pmatrix}, \qquad N_{r,s} = \begin{pmatrix} 2(s-r) & 2r & 2r \\ -2r(s-r) & (s-r)^2 + 1 - r^2 & -(s-r)^2 + 1 - r^2 \\ 2r(s-r) & -(s-r)^2 + 1 + r^2 & (s-r)^2 + 1 + r^2 \end{pmatrix}.$$

The classification of the marked pencils is similar to the one displayed in A.3.10, but there is a caveat due to the fact that we are no longer working over the field $\mathbb{C}$. This pencil has only one degenerate conic, so we use an arbitrary conic $u = x^2 + y^2 - z^2$ to parametrize it. The transformations $N_{r,s}$ preserve this conic $u$, indeed its action on the pencil is defined by:

$$N_{r,s}.w_\infty = \frac{1}{4}\, w_\infty, \qquad N_{r,s}.u = \frac{1}{4(s-r)^2}\, u.$$

The different multiples obtained above allow us to manipulate the pencil, as the element of coordinates $[\alpha : \beta]$ is sent to the one of coordinates $[(s-r)^2\alpha : \beta]$. Since we are working over $\mathbb{R}$, the value $(s-r)^2$ is always nonnegative, so this could appear to be an obstacle to the placement of a second marked conic, but that is not the case because the pair of marked conic is unordered. Indeed, the geometric explanation for this phenomenon is that, since $N_{r,s}$ preserves $u$, then if the other conic $v$ is contained in the region bounded by $u$ that is homeomorphic to the disc, its image must also remain in this same region. Whereas, if it belongs to the region homeomorphic to the Möbius band, then its image must again stay in the same region. That is why there is an algebraic distinction between the two cases. Nevertheless, since the pair in unordered, we may always pick them in such a way that $v$ is contained in the region homeomorphic to the disc. In other words, we may always place the marked conics at $u = x^2 + y^2 - z^2$ and $v = w_\infty + u = x^2 + 2y^2 + 2yz$ and get a single 7-dimensional orbit.

In the following figure we highlight both of these conics.

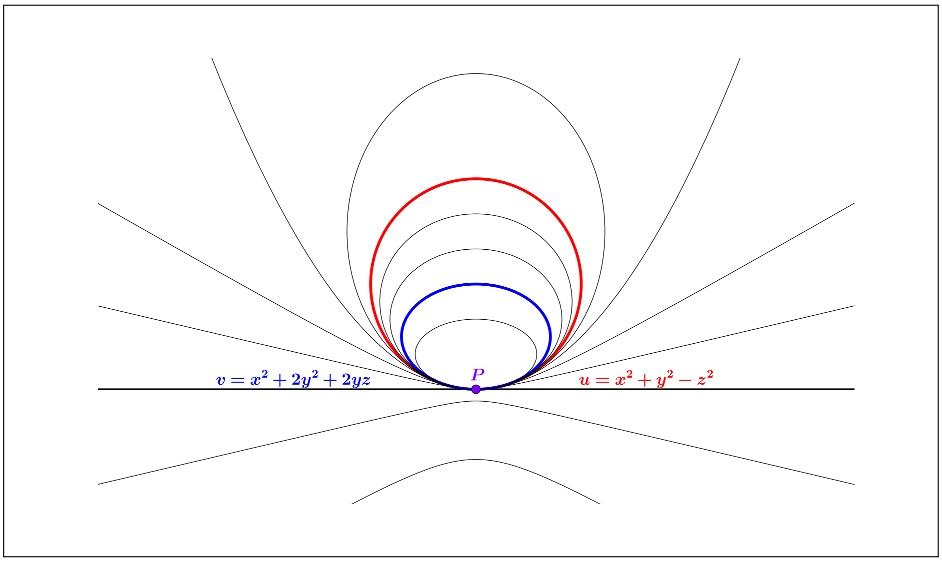


Figure A.17: Normal form of the (4) pencil.

### A.5.10

### (4*) **Pencil**

From now on, there are only the completely degenerate pencils left, that is, those where all the elements are pairs of lines. As we have seen in the complex setting, there exists a kind of pencil where the four common points coincide but there are no irreducible conics due to the presence of more than one degenerate element. We have named it the (4*) pencil, and there is naturally a corresponding real pencil, but it is actually subdivided into two distinct types, which we are going to call $(4^*)$ and $(4^{**})$. What distinguishes them is the presence or absence of double lines. We have shown that the complex $(4^*)$ pencil always has two double lines which emerge as roots of the first minor $\delta$. In the real setting on the other hand, these roots could either be distinct and real or a complex conjugate pair. So we name the prior case the $(4^*)$ pencil, and the latter the $(4^{**})$ pencil.

Having two double lines, the $(4^*)$ pencil is uniquely determined by them. By the duality of the projective plane, this corresponds to picking two points of $\mathbb{RP}^2$, and therefore this family is a submanifold of dimension 4 and codimension 4 in the space of all real pencils. Notice also that it is contained inside the discriminant hypersurface.

The nature of this type of pencil is closely related to the complex $(4^*)$ pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.6.

In normal form, the double lines are given by:

$$w_\infty = x^2, \qquad w_0 = y^2.$$

The stabilizer subgroup of the normal form of the $(4^*)$ pencil is generated by the involution $N$ and the 4-parameter family of transformations $M_{a,b,r,s}$, with $a, b, r, s \in \mathbb{R}$, $a \neq 0$ and $r + as - b \neq 0$ as below.

$$M_{a,b,r,s} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ b - as & b - r & r + as - b \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The classification of the marked pencils is similar to the one displayed in A.3.12, but there is a caveat due to the fact that we are no longer working over the field $\mathbb{C}$. Consider the actions of $M_{a,b,r,s}$ on the double lines that define the pencil.

$$M_{a,b,r,s}.w_\infty = w_\infty, \qquad M_{a,b,r,s}.w_0 = \frac{1}{a^2}\, w_0.$$

Despite having so many degrees of freedom, we see that only one of the four parameters matters in terms of the action produced on the pencil. We may use it to transform the pencil, but we do not have complete control over it due to the even power on the resulting multiple. Indeed, the element of coordinates $[\alpha : \beta]$ is mapped to $[a^2\alpha : \beta]$, so the sign of the ratio between the coefficients is preserved. This means that the pencil is segmented by $w_\infty$ and $w_0$ into two distinguished intervals with different signatures. In this case it is quite clear, if $\alpha$ and $\beta$ have the same sign, then the resulting conic $w = \alpha w_\infty + \beta w_0 = \alpha x^2 + \beta y^2$ has signature $(2, 0)$ or $(0, 2)$, and hence it is a pair of complex lines that meet at $P = [0 : 0 : 1]$, whereas if they have opposite signs, then we obtain conics of signature $(1, 1)$, which are pairs of real lines meeting also at $P$. Due to this difference with respect to the complex $(4^*)$ pencil, we now present and prove the orbits of marked real $(4^*)$ pencils.

**Marked real $(4^*)$ pencil**

*Proposition* A.5.10. There are three orbits of $(4^*)$ pencils marked with a single element:

i. *The marked conic is a pair of real lines:* A single orbit of dimension 5;

ii. *The marked conic is a pair of complex lines:* A single orbit of dimension 5;

iii. *The marked conic is a double line:* A single orbit of dimension 4.

In addition, there are six orbits of $(4^*)$ pencils marked with an unordered pair of conics.

i. *The marked conics are two pairs of real lines:*
Infinitely many orbits of dimension 5 described by an invariant;

ii. *A real and a complex pair of lines:*
Infinitely many orbits of dimension 5 described by an invariant;

iii. *Two pairs of complex lines:*
Infinitely many orbits of dimension 5 described by an invariant;

iv. *A double line and a pair of real lines:*
A single orbit of dimension 5;

v. *A double line and a pair of complex lines:*
A single orbit of dimension 5;

vi. *Two double lines:*
A single orbit of dimension 4.

*Proof.* Let us begin with the pencil with a single marked conic. If it is a double line, we may put the pencil in normal form so that it becomes $w_\infty$. The 4-dimensional group of transformations $M_{a,b,r,s}$ preserves it, so this is a

4-dimensional orbit. If the marked conic is a real pair of lines, we may use a suitable $M_{a,1,1,1}$ to place it as $u = w_\infty - w_0 = x^2 - y^2$, whereas if it is a complex pair of lines, we may put it as $u' = w_\infty + w_0 = x^2 + y^2$. In either case, the stabilizer of these marked pencils is $< M_{\pm 1,b,r,s}, N >$, so the orbit is 5-dimensional. Notice that $N$ is present in this stabilizer because $u$ and $u'$ are very particular elements of the pencil. Indeed, the action of $N$ interchanges $w_\infty$ and $w_0$, so the conic of parameter $[\alpha : \beta]$ is mapped to the one of parameter $[\beta : \alpha]$. The only conics preserved by such action are precisely $u$ and $u'$.

Now consider a pencil with a couple of unordered marked conics. If at least one of them is a double line, we fall back to a previous case explained above. If at least one of them is a real double line, we may set it to $u$ and then the only freedom we have left to move the other marked conic in the pencil is due to the action of $N$. This allows us to place it at the conic of parameter $[1 : c]$, with $c \in (-1, 0) \cup (0, 1]$. The other conic is a real pair of lines if $c \in (-1, 0)$ and it is a complex pair of lines if $c \in (0, 1]$. Lastly, if both marked conics are pairs of complex lines, then we set one to $u'$ and the other is given by $[1 : c']$, with $c' \in (0, 1)$. In all these cases, the stabilizer has dimension 3, so for each value of $c$ or $c'$ we have a 5-dimensional orbit. $\square$

In the following figure, we highlight an arbitrary conic $v = x^2 - 4y^2$ which is associated to the parameter $[1 : -4]$ of the pencil.

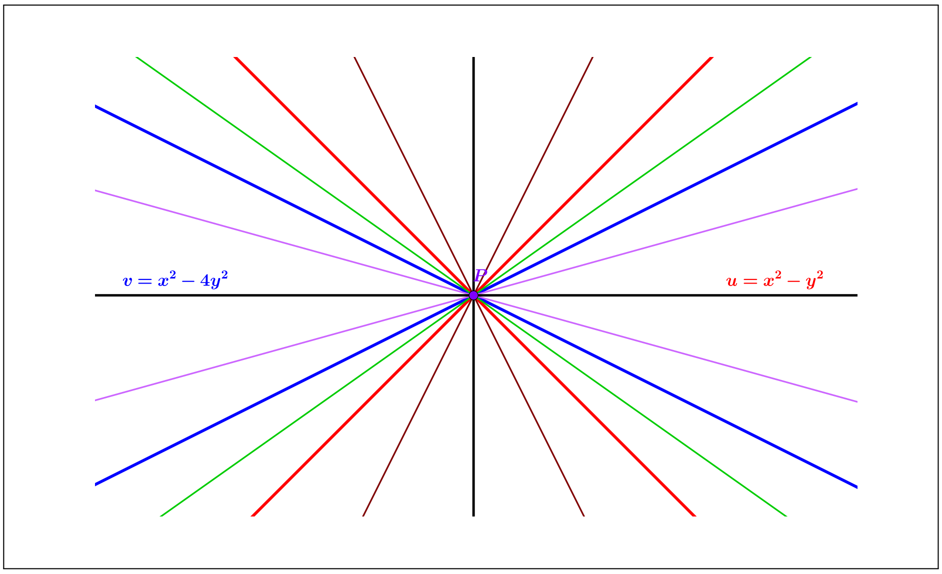


Figure A.18: Normal form of the $(4^*)$ pencil.

### A.5.11

#### $(4^{**})$ Pencil

As we have previously explained, the complex $(4^*)$ pencil has two distinct versions in the real setting depending whether its double lines are real or complex. In the $(4^{**})$ they are a conjugate pair of double lines, which therefore have a contact of order 4 at a real point. Although these complex lines are not present in the real pencil itself, they may be used to define it, just as the complex common points were used in some previous types of pencil. One may think of the real pencil $(4^{**})$ as a particular $\mathbb{RP}^1$ where all the conics are real inside of the complex pencil, isomorphic to $\mathbb{CP}^1$, generated by these two elements.

In order to determine a $(4^{**})$ pencil, one just needs to pick a complex line, which in pair with its conjugate will be the two double lines of the pencil. This is equivalent to choosing a point of $\mathbb{CP}^2$ by duality, but avoiding pure real points. In any case, this gives rise to a submanifold of real dimension 4 and codimension 4 in the space of all real pencils of conics.

**Normal form of the $(4^{**})$ pencil**

We are going to use the "virtual" double lines to define the normal form of this pencil. One can find a projective transformation that sends the common point to $P = [0 : 0 : 1]$ and these special lines to $x + iy = 0$ and $x - iy = 0$. Thus the double lines are given by:

$$w_\infty = \frac{1}{2}(x+iy)^2 = \frac{1}{2}(x^2+2ixy-y^2), \qquad w_0 = \frac{1}{2}(x-iy)^2 = \frac{1}{2}(x^2-2ixy-y^2).$$

Naturally, these conics cannot belong to the real pencil that we are studying, but we can still use them to parametrize it in a clever way. Let us consider for a moment the complex pencil generated by $w_\infty$ and $w_0$.

$$(a+ib)w_\infty + (c+id)w_0 = \frac{1}{2}\Big(a+c+i(b+d)\Big)(x^2-y^2) + i\Big(a-c+i(b-d)\Big)xy.$$

If we want to find the real conics of this pencil, we just have to make sure that all terms accompanying the imaginary unit $i$ vanish. This simply means $b+d=0$ and $a-c=0$. Therefore we obtain:

$$(a+ib)w_\infty + (a-ib)w_0 = a(x^2-y^2) - 2bxy.$$

With that, we have a real pencil parametrized in a peculiar way: $(\alpha+i\beta)w_\infty + (\alpha-i\beta)w_0$ with $[\alpha:\beta] \in \mathbb{RP}^1$.

**Lemma A.5.11.** *The stabilizer subgroup of the normal form of the* $(4^{**})$ *pencil is generated by the involution* $N$ *and the* 4*-parameter family of transformations* $M_{a,b,\theta,r}$*, with* $a,b,r \in \mathbb{R}$*,* $r \geq 0$*,* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ *and* $r - a\cos(\theta) - b\sin(\theta) \neq 0$ *presented below.*

$$M_{a,b,\theta,r} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ a\cos(\theta)+b\sin(\theta) & b\cos(\theta)-a\sin(\theta) & r-a\cos(\theta)-b\sin(\theta) \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* We obtain the expression of $M_{a,b,\theta,r}$ by once again considering the image of 4 points in general position. Naturally any transformation of the stabilizer must preserve $P$. This time we will need two complex auxiliary points, the first $Q = [1:i:0]$ which belongs to the line $x+iy=0$ must stay on this very line, so it is mapped to $[1:i:a+ib]$, where $a,b \in \mathbb{R}$. This condition automatically implies that its conjugate point $\overline{Q} = [1:-i:0]$ is sent to $[1:-i:a-ib]$, so this gives us an extra condition for our transformation. Finally, we consider $P' = [1:0:1]$ whose image $[\cos(\theta):\sin(\theta):r]$, with $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $r \geq 0$, is relatively free, we just have to avoid a possible alignment or coincidence of the images. These obstacles are detected by the determinant $\det(M_{a,b,\theta,r}) = r - a\cos(\theta) - b\sin(\theta)$, which explains the condition in the statement of the lemma. The images of $P$, $P'$, $Q$ and $Q'$ uniquely determine the transformation $M_{a,b,\theta,r}$.

The involution $N$ corresponds to the interchange of the "virtual" double lines that defined the pencil, as $N.w_\infty = w_0$ and $N.w_0 = w_\infty$. □

**Marked** $(4^{**})$ **pencil**

*Proposition* A.5.12. There is only one orbit of $(4^{**})$ pencils marked with a single conic:

i. *The marked conic is a pair of real lines:* A single orbit of dimension 5.

In addition, there is only one kind of orbit of $(4^{**})$ pencils marked with an unordered pair of conics.

i. *Both marked conics are pairs of real lines:*
Infinitely many orbits of dimension 5 described by an invariant.

*Proof.* The fact that the ($4^{**}$) pencil has only degenerate conics and no double line implies that every conic is of the same nature, there is no distinguished element on the pencil. When put in the normal form, the appearance in the affine chart $z = 1$ is that of real pairs of perpendicular lines meeting at $(0, 0)$. In order to evaluate the freedom we have to move elements in the pencil, let us check the action of the transformations $M_{a,b,\theta,r}$ on $w_\infty$ and $w_0$.

$$M_{a,b,\theta,r}.w_\infty = \frac{1}{\Big(\cos(\theta)+i\sin(\theta)\Big)^2} w_\infty, \qquad M_{a,b,\theta,r}.w_0 = \frac{1}{\Big(\cos(\theta)-i\sin(\theta)\Big)^2} w_0.$$

This time the conclusion is not so straightforward since $w_\infty$ and $w_0$ do not belong to the pencil, so we should see how the transformations act on an arbitrary element $(\alpha+i\beta)w_\infty+(\alpha-i\beta)w_0$ associated do the parameter $[\alpha : \beta]$ instead. A careful calculation leads us to:

$$\begin{aligned} M_{a,b,\theta,r}.\Big((\alpha + i\beta)w_\infty + (\alpha - i\beta)w_0\Big) = \Big((\cos(2\theta)\alpha + \sin(2\theta)\beta) + i(\cos(2\theta)\beta - \sin(2\theta)\alpha)\Big)w_\infty \\ +\Big((\cos(2\theta)\alpha + \sin(2\theta)\beta) - i(\cos(2\theta)\beta - \sin(2\theta)\alpha)\Big)w_0. \end{aligned}$$

In other words, if we consider the action of $M_{a,b,\theta,r}$ on the pencil's parameter, it maps the conic given by $[\alpha : \beta]$ to the one associated to $[\cos(2\theta)\alpha + \sin(2\theta)\beta : \cos(2\theta)\beta - \sin(2\theta)\alpha]$. This shows us firstly that only the parameter $\theta$ is relevant in regard to the transformation of the pencil, indeed it corresponds to the rotation of the pair of lines around the origin in the aforementioned affine chart. It also allows us to take a marked conic to an arbitrary element, say $u = w_\infty + w_0 = x^2 - y^2$. Indeed, to send the conic of parameter $[\alpha : \beta]$ to $u$, which has parameter $[1 : 0]$, one just has to solve the trigonometric equation for $\theta$: $\cos(2\theta)\beta - \sin(2\theta)\alpha = 0 \iff \cot(2\theta) = \alpha/\beta$. This shows that there is only one orbit of ($4^{**}$) pencils with one marked conic. Next, we show that the stabilizer of the pencil marked with $u$ has dimension 3, resulting in a 5-dimensional orbit.

In order to find the group of symmetries of the pencil marked with $u$ we look for the transformations $M_{a,b,\theta,r}$ that preserve $u$. Its image under this action has parameter $[\cos(2\theta) : -\sin(2\theta)]$, so we find a condition for $\theta$: it can only take four values $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$. This leaves us with a group comprised apparently of four components of 3-parameter families of transformations.

$$M_{a,b,0,r} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & r-a \end{pmatrix}, \qquad M_{a,b,\pi/2,r} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ b & -a & r-b \end{pmatrix},$$
$$M_{a,b,\pi,r} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -a & -b & r+a \end{pmatrix}, \qquad M_{a,b,3\pi/2,r} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -b & a & r+b \end{pmatrix}.$$

However, since we are dealing with elements of $\mathrm{PGL}(3;\mathbb{R})$, there are some special identifications when $r = 0$, because $M_{a,b,0,0} = M_{a,b,\pi,0}$ and $M_{a,b,\pi/2,0} = M_{a,b,3\pi/2,0}$. This means that we may move continuously from any $M_{a,b,0,r}$ to any $M_{a',b',\pi,r'}$. In any case, these transformation all act trivially on the pencil, so they do not give us any liberty to move a second marked conic. Geometrically, we see in the affine chart $z = 1$ that all conics of the pencil are invariant under a rotation by $\pi/2$ around $(0,0)$, which justifies our statement that the pair of lines are orthogonal at the origin. The transformation $M_{0,0,\pi/2,0} : (x,y,z) \mapsto (-y,x,z)$ realizes this rotation, so let us name it $R$.

We still have, however, the involution $N$. It also preserves $u$, so it must be considered in the stabilizer of the marked pencil. Its action on the pencil is given by $[\alpha : \beta] \mapsto [\alpha : -\beta]$, thus it has two fixed points, namely the conics whose parameters are $[1:0]$ and $[0:1]$. Therefore, if we have a second marked conic in the pencil whose parameter is $[c:1]$, we may use the action of $N$ to assure that $c \geq 0$. If $c = 0$, so that the second marked conic is $v = 2xy$, then the stabilizer of this pencil marked with two conics is $< M_{a,b,0,r}, R, N >$. Otherwise, the stabilizer does not contain $N$, so it is $< M_{a,b,0,r}, R >$. In either case it is a 3-dimensional stabilizer, so we get for each value of $c \geq 0$ a 5-dimensional orbit. □

In the following figure we highlight $u = x^2 - y^2$ and $v = 2xy$, which are invariant under the action of $N$.

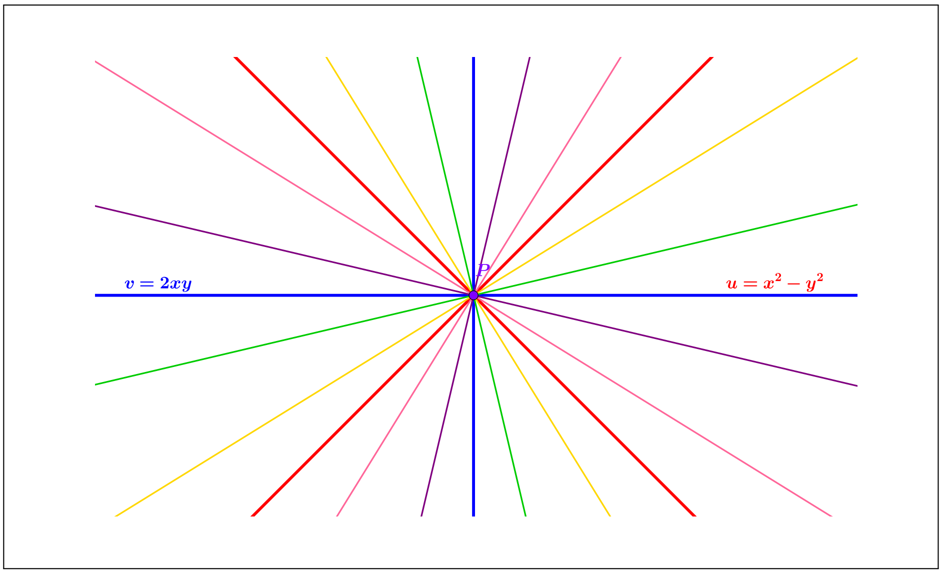


Figure A.19: Normal form of the $(4^{**})$ pencil.

### A.5.12

#### $(\infty, 1)$ Pencil

We approach the end of the list of real pencils, having to consider only the two remaining cases where the common points are not in general position, that is, at least three of them are aligned. If that is the case, then every conic of the pencil is necessarily degenerate and one of its factors correspond to the line that contains those collinear points. Therefore, these types of pencil actually contain a *common line* $\ell$. The only distinction left is whether the fourth common point $P$ belongs also to this line or not. First we consider the case where it is not on the common line. It forces every conic of the pencil to pass through it, which has to be done by the other factor of the conics and thus the pencil is essentially the pencil of lines through $P$ accompanied by the common line $\ell$. Notice that there can be no double lines in this pencil. Since there are infinitely many common point on the common line plus an additional one elsewhere, we will name this pencil $(\infty, 1)$.

A pencil of this family is uniquely determined by a line in $\mathbb{RP}^2$ and a point which does not belong to it. Therefore it constitutes a submanifold of dimension 4 and codimension 4 in the space of all real pencils of conics.

The nature of this type of pencil is completely analogous to the complex $(\infty, 1)$ pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.7.

In normal form, the pencil may be parametrized using the following arbitrary conics:

$$u = y(x - y - z), \qquad v = y(x + y + z).$$

The stabilizer subgroup of the normal form of the $(\infty, 1)$ pencil is generated by the 4-parameter family of transformations $M_{a,b,r,s}$, with $a, b \in \mathbb{R}$, $r, s \in \mathbb{R} \cup \{\infty\}$, $r \neq s$, $a - br - r \neq 0$ and $a - bs - s \neq 0$ as below.

$$M_{a,b,r,s} = \begin{pmatrix} \frac{r(a-bs-s)+s(a-br-r)}{r-s} & a & a \\ 0 & 1 & 0 \\ \frac{(a-bs-s)+(a-br-r)}{r-s} & b & b+1 \end{pmatrix}.$$

The classification of the marked pencils is also the same as displayed in A.3.14.

In the following figure, we highlight the common line $\ell$, given by $y = 0$, and two arbitrary conics $u = y(x - y - z)$ and $v = y(x + y + z)$.

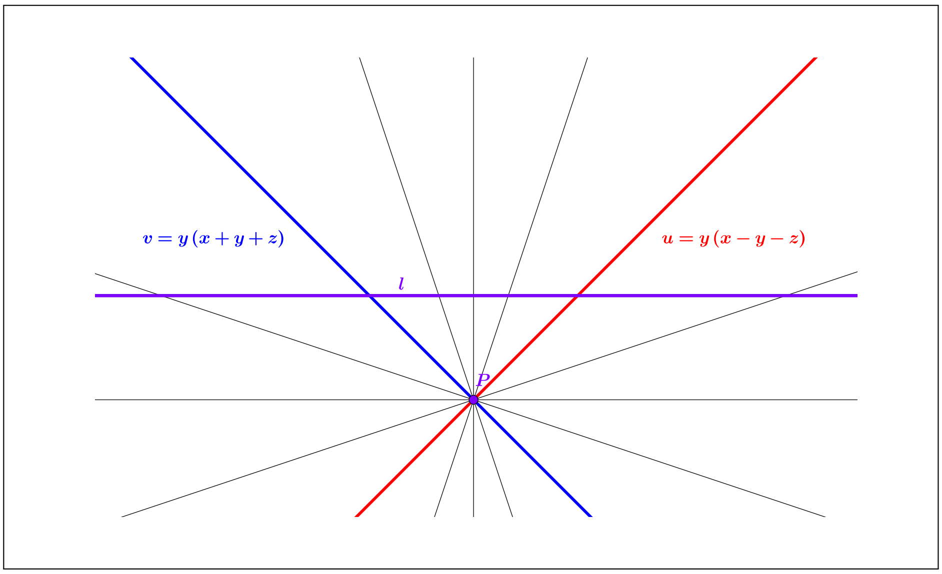


Figure A.20: Normal form of the $(\infty, 1)$ pencil.

### A.5.13

#### $(\infty)$ **Pencil**

We have finally arrived at the last type of real pencil of conics, which happens when all four common points are aligned. Just as in the previous case, this obliges all the conics of the pencil to be degenerate and to have a common factor referring to the common line $\ell$. Unlike the $(\infty, 1)$ pencil, this one does not have an additional common point not on the common line, so we will name it simply $(\infty)$. The other components of the conics of the pencil must intersect at a real point $P$, and since there can not be a common point out of $\ell$, this intersection must be a point of the common line. So there is in fact an additional particular point that characterizes the pencil, it is just hidden in $\ell$. As a consequence, every other element of the pencil is a pair of lines where one is $\ell$ and the other passes through $P$. This pencil is once again essentially the pencil of lines through $P$ accompanied by the common line. Notice, however, that there is a double line in this pencil, which corresponds to a double root of the minor $\delta$.

To uniquely determine a pencil of this family one must pick a line in $\mathbb{RP}^2$ and a point on that line. Therefore, it constitutes a submanifold of real dimension 3 and codimension 5 in the space of all real pencils of conics. This is thus the most degenerate type of pencil.

The nature of this type of pencil is completely analogous to the complex $(\infty)$ pencil, one just has to change the field $\mathbb{C}$ for $\mathbb{R}$ in the analysis presented in Subsection A.3.8.

In normal form, the pencil is parametrized by the following double line $w_\infty$ and pair of lines $u$:

$$w_\infty = y^2, \qquad u = y(x - y).$$

The stabilizer subgroup of the normal form of the $(\infty)$ pencil is given by the 5-parameter family of transformations $M_{a,b,c,d,t}$, with $a, b, c, d, t \in \mathbb{R}$, $a - c \neq 0$ and $t(a - c) - (b - d) \neq 0$ as below.

$$M_{a,b,c,d,t} = \begin{pmatrix} a - c & a + c & 0 \\ 0 & 2 & 0 \\ b - d & b + d & t(a - c) - (b - d) \end{pmatrix}.$$

The classification of the marked pencils is also the same as displayed in A.3.16.

In the following figure, we highlight the common line $\ell$, given by $y = 0$, and two arbitrary conics $u = y(x - y)$ and $v = y(x + y)$.

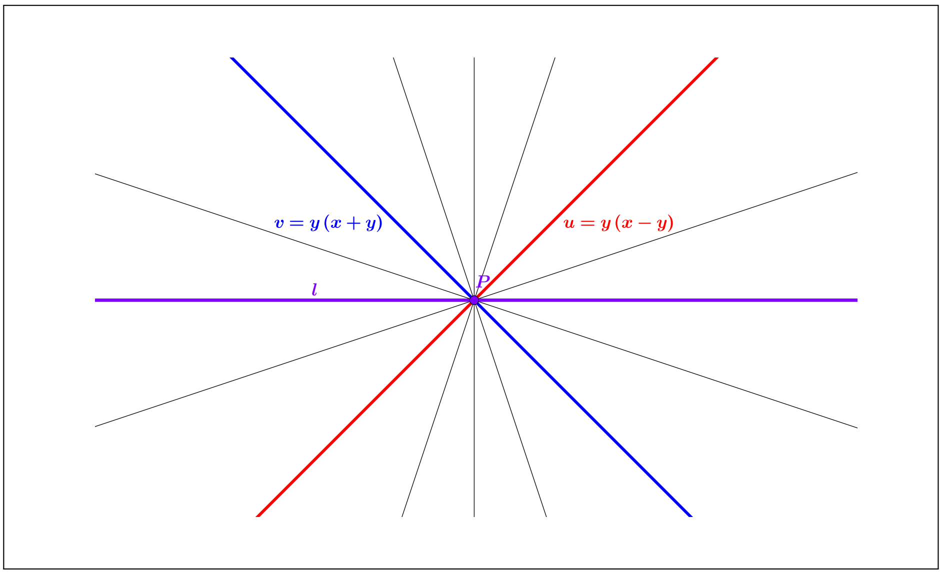


Figure A.21: Normal form of the $(\infty)$ pencil.

This concludes our detailed study and classification of the real pencils of conics.

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