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**Homotopy Type of Intersections of Real
Bruhat Cells in Dimension 6**

Tese de Doutorado

Thesis presented to the Programa de Pós-graduação em Matemática, do Departamento de Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor : Prof. Nicolau Corção Saldanha
Co-advisor: Profa. Emília Carolina Santana Teixeira Alves

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To my father, my greatest supporter
even in heaven.

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Abstract

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In this work, we investigate the arbitrary intersection of real Bruhat cells. Such objects have attracted interest from various authors, particularly due to their appearance in different contexts: such as in Kazhdan-Lusztig theory and in the study of locally convex curves. We study the homotopy type of the intersection of two real Bruhat cells. This homotopy type is the same as that of an explicit submanifold of the group of real lower triangular matrices with diagonal entries equal to 1. For $(n + 1) \times (n + 1)$ matrices with $n \leq 4$, these submanifolds are the disjoint union of contractible connected components. Our focus is on such intersections for 6×6 real matrices. For this, we study the connected components of Bruhat cells for permutations $\sigma \in S_6$ with at most 12 inversions. We make use of the structure of the dual CW complexes associated with these components. We show that for permutations with at most 12 inversions, with the exception of $\sigma = [563412]$, all connected components are contractible. Furthermore, for $\sigma = [563412]$, we identify new non-contractible connected components with the homotopy type of the circle.

Keywords

Permutation group; Clifford algebra; Spin group; Bruhat cell; Homotopy type.

Resumo

Leal, Giovanna Luisa Coelho; Saldanha, Nicolau Corção; Alves, Emília Carolina Santana Teixeira. **Tipo homotópico de interseções entre células de Bruhat reais em dimensão 6**. Rio de Janeiro, 2025. 251p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Neste trabalho, investigamos a interseção arbitrária de células de Bruhat reais. Tais objetos têm despertado o interesse de diversos autores, em especial devido à sua aparição em diferentes contextos: como na teoria de Kazhdan-Lusztig e no estudo de curvas localmente convexas. Estudamos o tipo homotópico da interseção de duas células de Bruhat reais. Este tipo homotópico é o mesmo que o de uma subvariedade explícita do grupo de matrizes triangulares inferiores reais com entradas diagonais iguais a 1. Para matrizes $(n+1) \times (n+1)$ com $n \leq 4$, estas subvariedades são a união disjunta de componentes conexas contráteis. Nosso foco é em tais interseções para matrizes reais 6×6 . Para tal, analisamos as componentes conexas das células de Bruhat para permutações $\sigma \in S_6$ com no máximo 12 inversões. Utilizamos a estrutura dos CW complexos duais associados a estas componentes. Mostramos que para permutações com no máximo 12 inversões, com exceção de $\sigma = [563412]$, todas as componentes conexas são contráteis. Além disso, para $\sigma = [563412]$, identificamos novas componentes conexas não contráteis, com o tipo homotópico do círculo.

Palavras-chave

Grupo das Permutações; Álgebra de Clifford; Grupo Spin; Célula de Bruhat; Tipo homotópico.

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Future is gonna be okay.

Agust D, *D-Day*.

1

Introduction

Over the past century, Bruhat cell decompositions have been important to mathematics, particularly in the study of Grassmannians and flag spaces, and have become standard tools in fields like topology, enumerative geometry, representation theory and the study of locally convex curves. Despite their long-standing importance, the topological study of Bruhat cell intersections, whether in pairs or more complex collections, remains relatively underexplored. These intersections naturally arise in various mathematical areas, including singularity theory, Kazhdan-Lusztig theory, and matroid theory. However, detailed topological results on these intersections are still scarce (see [9]).

One notable exception to this lack of topological insight is the problem of counting connected components in pairwise intersections of big Bruhat cells over the real numbers. Significant advances were made in this area during the late 1990s, with key contributions found in works such as [8], [3], and [10]. Essentially, this problem reduces to counting the orbits of a specific finite group of symplectic transvections acting on a finite-dimensional vector space over the finite field $\mathbb{F}_2$ ([11]).

We examine the intersections between a top-dimensional cell and a cell corresponding to a different basis. These intersections can naturally be identified with a subset of the lower nilpotent group Lo_{n+1}^1 .

For a permutation $\sigma \in S_{n+1}$, let P_σ be the permutation matrix. Let Lo_{n+1}^1 be the group of real lower triangular matrices with diagonal entries equal to 1. Following the Bruhat decomposition, partition Lo_{n+1}^1 into subsets BL_σ for $\sigma \in S_{n+1}$:

$$\text{BL}_\sigma = \{L \in \text{Lo}_{n+1}^1 \mid \exists U_0, U_1 \in \text{Up}_{n+1}, L = U_0 P_\sigma U_1\},$$

where Up_{n+1} is the upper triangular matrix group.

The intersection of two opposite big Bruhat cells in Flag_{n+1} is homeomorphic to BL_η , where $\eta \in S_{n+1}$ is the longest element. The number of connected components of BL_η is 2, 6, 20, and 52 for $n = 1, 2, 3, 4$, respectively. For $n \geq 5$, the number of connected components stabilizes and is given by $3 \cdot 2^n$. This stabilization can be explained by the fact that, for $n \geq 5$, it is possible to embed

the E_6 lattice into a particular lattice that emerges in this problem ([8]).

The relative positions of two big Bruhat cells in Flag_{n+1} correspond bijectively to the elements of S_{n+1} . In particular, opposite big Bruhat cells are associated with the top permutation $\eta \in S_{n+1}$. The study of the number of connected components in the intersection of two big cells for a given relative position σ was initiated in [8]. For any specific $\sigma \in S_{n+1}$, the number of connected components can be determined based on the results from [7]. However, to the best of our knowledge, no closed formula has been founded.

In [1], Alves and Saldanha introduce useful tools for studying the homotopy type of these intersections. They apply these tools to prove the following theorem:

Theorem 1. (*E. Alves, N. Saldanha - IMRN, 2022*) Consider $\sigma \in S_{n+1}$ and $\text{BL}_\sigma \subset \text{Lo}_{n+1}^1$.

1. For $n \leq 4$, every connected component of every set BL_σ is contractible.
2. For $n = 5$ and $\sigma = [563412] \in S_6$, there exist connected components of BL_σ , which are homotopically equivalent to $\mathbb{S}^1$.
3. For $n \geq 5$, there exist connected components of BL_η , which have even Euler characteristic.

Our aim is to extend this construction to the case $n = 5$. Specifically, we examine the connected components of the set BL_σ , for $\sigma \in S_6$. The main result of this thesis is the following:

Theorem 2. Consider $\sigma \in S_6$ and $\text{BL}_\sigma \subset \text{Lo}_6^1$.

1. For $\text{inv}(\sigma) \leq 11$, every component of every set BL_σ is contractible;
2. For $\text{inv}(\sigma) = 12$, except for $\sigma = [563412]$, every component of every set BL_σ is contractible;
3. For $\sigma = [563412]$ there exist 100 connected components, where exactly 24 are homotopically equivalent to $\mathbb{S}^1$, 4 are inconclusive with the Euler characteristic equal to 1 and the others 72 are contractible.

According to Theorem 2 in [1], for $\sigma \in S_{n+1}$ there exist a finite CW complex BLC_σ homotopically equivalent to BL_σ . In particular, the connected components of BLC_σ correspond precisely to those of BL_σ .

Therefore, to determine the homotopy type of BL_σ , for $\sigma \in S_6$, we classify the permutations by their number of inversions and study the connected components of BLC_σ . The maximum number of inversions is 15. Our study

covers the case up to $\text{inv}(\sigma) = 12$. For $\text{inv}(\sigma) \geq 13$ there are 20 permutations. Analyzing the components using our current method becomes increasingly challenging as the dimension of the ancestries grows.

Visualizing cells of dimension greater than four is particularly difficult, and we believe that continuing this work will require additional tools and techniques.

Nevertheless, some conclusions can still be drawn about these permutations. For $\sigma \in S_6$ with $\text{inv}(\sigma) = 13$, the connected components have Euler characteristics of either 1 or 0. In the latter case, the components are homotopically equivalent to S^1 . The same holds for $\sigma \in S_6$ with $\text{inv}(\sigma) = 14$.

It is well known from [1] that for the permutation $\sigma = \eta \in S_6$, the only one with $\text{inv}(\sigma) = 15$, there exists a connected component with the Euler characteristic equal to 2, consequently non-contractible. The remaining connected components have Euler characteristic equal to 1.

Here is an overview of this thesis:

In Chapter 2, we introduce some concepts relevant to this work, including the wiring diagram, which will be used extensively throughout.

Chapter 3, provides a brief overview of matrix groups such as Quat_{n+1} , Spin_{n+1} and $\tilde{B}_{n+1}^+$. In addition, we present a summary of the Clifford Algebra Cl_{n+1}^0 .

In Chapter 4, we introduce two key concepts essential to this work: preancestry and ancestry. Understanding these concepts in the context of the wiring diagram is fundamental to the development of this research.

In Chapter 5, we study Bruhat cells and their properties, including proving the previously mentioned diffeomorphisms. Additionally, we examine certain properties of totally positive matrices and their relationship to the Bruhat cells.

Chapter 6 presents the stratification BLS_e and its corresponding strata, beginning with ancestries of dimension 0 and extending to the generalized concept.

In Chapter 7, we investigate the CW complex BLC_σ and its gluing maps. Furthermore, we provide a formula for the Euler characteristic of BL_z as presented in [1], and conclude with the presentation of Theorem 1.

In Chapter 8, we present several wiring diagram decompositions. Next, we introduce some lemmas that will help in studying the CW complexes BLC_σ for permutations that can be decomposed in specific ways.

Chapters 9 through 14 present the connected components of BL_σ for $\sigma \in S_6$, with $\text{inv}(\sigma) \leq 11$. For these permutations, all connected components of BL_σ are contractible.

In Chapters 15 and 16, we study the connected components of BL_σ for $\sigma \in S_6$, with $\text{inv}(\sigma) = 12$. Chapter 15 addresses nearly all permutations for which all connected components of BL_σ are contractible. Chapter 16 focuses on the permutation $\sigma = [563412]$, with each connected component of this permutation examined in a separate section. Section 16.2 illustrates the connected component presented in the second item of Theorem 1, illustrated below in Figure 1.1. Furthermore, we provide a concrete method using matrices to understand the curve that forms $\mathbb{S}^1$. Section 16.3 introduces a new non-contractible connected component of BL_σ , constructed step by step. The remaining connected components are presented in the following sections. Furthermore, in this chapter, we present our result.

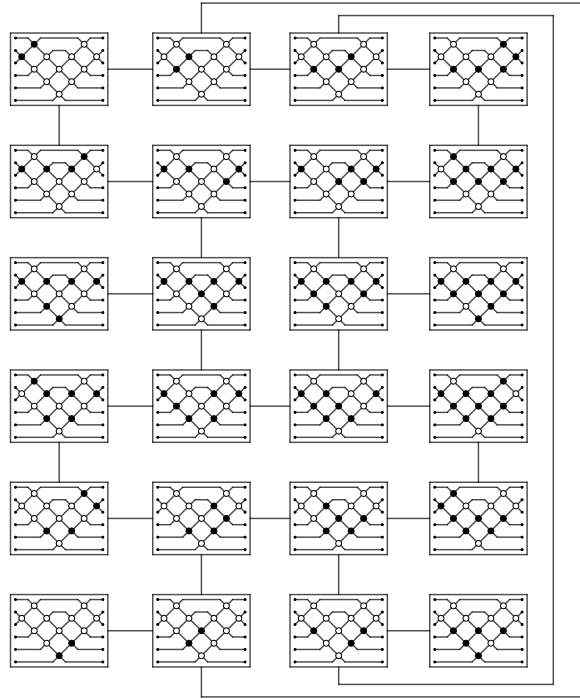


Figure 1.1: Connected component homotopically equivalent to $\mathbb{S}^1$.

To conclude, Chapter 17 presents information on the Euler characteristics of the connected components for permutations $\sigma \in S_6$ with $\text{inv}(\sigma) \geq 13$.

2

The Symmetric Group

In this chapter, we review key concepts and properties of the permutation group S_{n+1} . The first section provides an overview of the definition and some fundamental concepts. Following that, we explore the set of signed permutations, introducing additional important sets. Finally, Section 3 focuses on the Bruhat order, an essential concept in this work.

2.1

Permutations

There are several ways to represent a permutation $\sigma \in S_{n+1}$, a common one is given by $\sigma = [1^\sigma 2^\sigma 3^\sigma 4^\sigma] \in S_4$. Another way is by using Coxeter-Weyl generators $a_i = (i, i+1)$, with $i \in \llbracket n \rrbracket = \{1, \dots, n\}$. Using this notation, a permutation can be written as a product of these transpositions. For instance, $\sigma = [4321] = a_1 a_2 a_1 a_3 a_2 a_1$. This representation is referred to as a **word** for the permutation.

Definition 2.1.1. The set of pairs (i, j) that are inversions of σ is given by

$$\text{Inv}(\sigma) = \{(i, j) \in \llbracket n+1 \rrbracket^2 \mid (i < j) \wedge (i^\sigma > j^\sigma)\}.$$

Additionally, $\text{inv}(\sigma) = \text{card}\{\text{Inv}(\sigma)\}$.

A set $I \subseteq \{(i, j) \in \llbracket n+1 \rrbracket^2 \mid i < j\}$ is the set of inversions of a permutation $\sigma \in S_{n+1}$, if and only if $\forall i, j, k \in \llbracket n+1 \rrbracket$ with $i < j < k$, the following conditions are satisfied:

1. if $(i, j), (j, k) \in I$ then $(i, k) \in I$;
2. if $(i, j), (j, k) \notin I$ then $(i, k) \notin I$.

Also, if $\rho = \sigma\eta$ then $\text{Inv}(\sigma) \sqcup \text{Inv}(\rho) = \text{Inv}(\eta)$.

Definition 2.1.2. A **reduced word** for a permutation $\sigma \in S_{n+1}$ is an expression of σ as a product of generators $a_i = (i, i+1)$, where the number of generators is minimal and equal to $\text{inv}(\sigma)$.

To obtain the reduced word, we consider certain properties of the generators:

1. $a_i a_i = e$, where e is the identity permutation;
2. $a_j a_i = a_i a_j$, for $|i - j| \neq 1$;
3. $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}$.

There may be more than one reduced word for a given permutation, but all reduced words are related through a sequence of moves based on the properties above.

There is a unique permutation $\eta = a_1 a_2 a_1 a_3 a_2 a_1 \dots a_n a_{n-1} \dots a_2 a_1$, known as the **top permutation**, where the length of its reduced word is $\text{inv}(\eta) = \frac{n(n+1)}{2}$, the largest possible value.

Definition 2.1.3. Given $\sigma_0 \in S_j$ and $\sigma_1 \in S_k$, define $\sigma = \sigma_0 \oplus \sigma_1 \in S_{j+k}$, such that

$$i^\sigma = \begin{cases} i^{\sigma_0}, & i \leq j, \\ (i - j)^{\sigma_1} + j, & i > j. \end{cases} \quad (2-1)$$

Example 2.1.1. Let $\sigma_0 = [231] = a_2 a_1 \in S_3$ and $\sigma_1 = [312] = a_1 a_2 \in S_3$. Then $\sigma = \sigma_0 \oplus \sigma_1 = [231645] = a_2 a_1 a_4 a_5 \in S_6$. $\diamond$

Definition 2.1.4. Let $\sigma \in S_{n+1}$. The permutation matrix P_σ is defined by $e_k^T P_\sigma = e_k^T \sigma$, where e_k^T is the k -th standard basis row vector.

Example 2.1.2. For $\eta \in S_{n+1}$, the permutation matrix is:

$$P_\eta = \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix}.$$

$\diamond$

Remark 2.1.5. If $\sigma = \sigma_0 \oplus \sigma_1$, then $P_\sigma = P_{\sigma_0} \oplus P_{\sigma_1}$, i.e., the matrix P_σ has two diagonal blocks P_{σ_0} and P_{σ_1} , and is zero elsewhere. $\diamond$

Example 2.1.3. Consider $\sigma_0 = [21] \in S_2$, $\sigma_1 = [312] \in S_3$ and $\sigma = \sigma_0 \oplus \sigma_1 = [21534] \in S_5$. We have

$$P_{\sigma_0} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad P_{\sigma_1} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \text{ and then}$$

$$P_\sigma = \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

$\diamond$

2.2

Wiring Diagram

Reduced words for a permutation σ can be represented using a diagram. There are multiple ways to interpret this diagram. In our approach, each point represents a number, starting from 1 at the top and ending at $n + 1$ at the bottom, with the permutation being read by mapping the points on the left side to the points on the right. Each crossing in the diagram corresponds to a generator a_i , read from left to right. Moreover, from top to bottom, the space between two adjacent points corresponds to a single generator, starting from a_1 up to a_n . This representation helps us identify reduced words for permutations.

Example 2.2.1. Consider $n = 2$ and $\eta = [321]$. We construct the diagram of η by marking the points as described above. In this diagram, we map the first point on the left to the last point on the right, and follow the permutation for the other points accordingly. Thus, we obtain the following diagram:

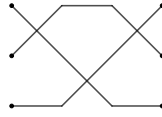


Figure 2.1: Wiring diagram of $\eta \in S_3$.

Now, we need to read the diagram. As described, the generators are read from top to bottom and from left to right. Therefore, the reduced word for η is given by

$$\eta = a_1 a_2 a_1.$$

Notice that $\eta = [321]$ has two different reduced words, namely $\eta = a_1 a_2 a_1$ and $\eta = a_2 a_1 a_2$. $\diamond$

The inversion $a_i = (i, i + 1)$ appears on the wiring diagram at height $i + \frac{1}{2}$.

Definition 2.2.1. The horizontal row between the starting points of two adjacent wires at height $i + \frac{1}{2}$ is called r_i .

The row r_i does not appear explicitly in the wiring diagram.

Definition 2.2.2. A **region** is a bounded connected component of the complement in the plane of the union of the wires in a wiring diagram.

A region of a wiring diagram has vertices k_1 and k_2 on the row i_{k_1} , along with all vertices k where $k_1 < k < k_2$ and $|i_k - i_{k_1}| = 1$.

Figure 2.2 shows the rows and provides an example of a region. Note that this wiring diagram contains two regions, although we are explicitly showing only one.

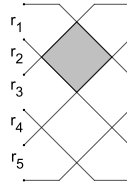


Figure 2.2: Example of a region in the wiring diagram of the permutation $\sigma = a_2a_1a_4a_3a_2a_5a_4 \in S_6$.

The following concept is closely related to the wiring diagram.

Definition 2.2.3. A permutation $\sigma \in S_{n+1}$ **blocks at** j , $1 \leq j \leq n$, if and only if $i \leq j$ implies $i^\sigma \leq j$. Equivalently, σ blocks at j if and only if a_j does not appear in a reduced word for σ . Let $\text{Block}(\sigma)$ be the set of j such that σ blocks at j and $b = \text{block}(\sigma) = |\text{Block}(\sigma)|$. A permutation σ does not block if $\text{block}(\sigma) = 0$.

Example 2.2.2. Let $\sigma = [231645] = a_2a_1a_4a_5 \in S_6$.

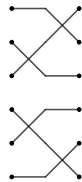


Figure 2.3: Wiring diagram of $\sigma = a_2a_1a_4a_5 \in S_6$.

The permutation σ blocks at 3. Notice that when a permutation blocks, it is easy to write it as a sum, in this case $\sigma = \sigma_0 \oplus \sigma_1$, where $\sigma_0 = [231] \in S_3$ and $\sigma_1 = [312] \in S_3$. $\diamond$

2.3

Signed Permutations

In the previous section, we associated a permutation $\sigma \in S_{n+1}$ with an $(n+1) \times (n+1)$ matrix, denoted by P_σ . In this section, we explore another type of matrix associated with permutations and study the corresponding groups of matrices.

Let B_{n+1} be the group of signed permutation matrices, which are orthogonal matrices P such that there exists a permutation $\sigma \in S_{n+1}$, where

$$e_i^T P = \pm e_{i\sigma}^T, \quad \forall i \in \llbracket n+1 \rrbracket.$$

Example 2.3.1. Consider the matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

Since $e_1^T P = e_3^T$, $e_2^T P = -e_2^T$, and $e_3^T P = -e_1^T$, there exists a permutation $\sigma \in S_3$ associated with P , and $\sigma = [321] = a_1 a_2 a_1$. Hence, $P \in B_3$. $\diamond$

The intersection of B_{n+1} with the group of orthogonal matrices with determinant equal to 1 is defined as $B_{n+1}^+ = B_{n+1} \cap \text{SO}_{n+1}$.

Additionally, the normal subgroup $\text{Diag}_{n+1}^+ \subset B_{n+1}^+$ is defined consisting of permutation matrices that are diagonal and orthogonal with determinant 1. This subgroup is isomorphic to $\{\pm 1\}^n$.

The map $\phi : B_{n+1}^+ \rightarrow S_{n+1}$ given by $P \mapsto \sigma_P$ is a surjective homomorphism, with kernel Diag_{n+1}^+ . Therefore, since Diag_{n+1}^+ is a normal subgroup, then

$$\frac{B_{n+1}^+}{\text{Diag}_{n+1}^+} \approx S_{n+1}.$$

By organizing the signs into a diagonal matrix, this isomorphism intuitively indicates that if we “forget” the signs, we are left with a permutation.

Thus, we have seen that every permutation $\sigma \in S_{n+1}$ corresponds to a matrix $P_\sigma \in B_{n+1}^+$, where $\sigma_P = \sigma$.

2.4

Bruhat Order

The Bruhat order is another key concept in this work. There are several types of Bruhat order, we use two of them.

Definition 2.4.1. Given $\sigma_0, \sigma_1 \in S_{n+1}$, we write $\sigma_0 \triangleleft \sigma_1$ if and only if there are reduced words $\sigma_1 = a_{i_1} a_{i_2} \dots a_{i_l}$ and $\sigma_0 = a_{i_1} a_{i_2} \dots a_{i_{k-1}} a_{i_{k+1}} \dots a_{i_l}$.

Example 2.4.1. Let $\sigma_0 = a_1 a_2$, $\sigma_1 = a_1 a_2 a_1 \in S_3$. It is easy to see that $\sigma_0 = a_1 a_2 \triangleleft a_1 a_2 a_1 = \sigma_1$. $\diamond$

Definition 2.4.2. (Strong Bruhat order) Given $\sigma_0, \sigma_1 \in S_{n+1}$, we write $\sigma_0 < \sigma_1$ if and only if there is a reduced word for σ_0 in terms of the Coxeter generators a_j that is a subexpression of a reduced word for σ_1 .

We have $\sigma_0 \leq \sigma_k$, with $k = \text{inv}(\sigma_k) - \text{inv}(\sigma_0) \geq 0$, if and only if there are $\sigma_1 \dots \sigma_{k-1}$ such that $\sigma_0 \triangleleft \sigma_1 \triangleleft \dots \triangleleft \sigma_{k-1} \triangleleft \sigma_k$.

Example 2.4.2. Let $\sigma_0 = e, \sigma_1 = a_1 a_2 a_1 \in S_3$. The sequence $e \triangleleft a_1 \triangleleft a_1 a_2 \triangleleft a_1 a_2 a_1$, shows that $e \leq a_1 a_2 a_1$.

On the other hand $a_1 \not\leq a_2, a_2 \not\leq a_1$ and $e \not\triangleleft a_1 a_2 a_1$. $\diamond$

3

Matrix Groups

In this chapter, we study examples of real matrices groups, with a particular focus on Quat_{n+1} , Spin_{n+1} , and $\tilde{B}_{n+1}^+$.

More precisely, a matrix group G is a subgroup of the group of invertible real matrices $\text{Gl}_{n+1} \subset \mathbb{R}^{(n+1) \times (n+1)}$. The group Spin_{n+1} is a smooth manifold and therefore a Lie group. In contrast, the groups Quat_{n+1} and $\tilde{B}_{n+1}^+$ are finite groups. The detailed constructions presented in this chapter, along with the matrix representations of the generators, are outlined in [6].

3.1

The Group Quat_{n+1}

In this section, we define the group Quat_{n+1} by its generators $\hat{a}_i$.

Definition 3.1.1. The group Quat_{n+1} is generated by the elements $\pm\hat{a}_1, \dots, \pm\hat{a}_n$ that satisfy the following relations:

- (i) $\hat{a}_i^2 = -1$;
- (ii) $\hat{a}_i\hat{a}_j = \hat{a}_j\hat{a}_i$ if $|i - j| \neq 1$;
- (iii) $\hat{a}_i\hat{a}_j = -\hat{a}_j\hat{a}_i$ if $|i - j| = 1$.

Therefore, the elements of this finite group can be listed, with the cardinality given by $|\text{Quat}_{n+1}| = 2^{n+1}$,

$$\text{Quat}_{n+1} = \{\pm 1, \pm\hat{a}_1, \pm\hat{a}_2, \pm\hat{a}_1\hat{a}_2, \pm\hat{a}_3, \pm\hat{a}_1\hat{a}_3, \pm\hat{a}_2\hat{a}_3, \pm\hat{a}_1\hat{a}_2\hat{a}_3, \dots, \pm\hat{a}_1 \dots \hat{a}_n\}.$$

The group Quat_{n+1} can be regarded as a group of $2^n \times 2^n$ real matrices by interpreting its generators as matrices.

Note that each $\hat{a}_i$ is an antisymmetric matrix. Additionally, each block of the matrix $\hat{a}_i$ has determinant 1. Consequently, $\hat{a}_i$ has determinant 1.

Since each element $q \in \text{Quat}_{n+1}$ is a product of generators $\hat{a}_i$, such that

$$q = \pm\hat{a}_1^{\varepsilon_1} \dots \hat{a}_n^{\varepsilon_n} \in \text{Quat}_{n+1}$$

with $\varepsilon_k \in \{0, 1\}$, it follows $\det(q) = 1$.

The matrices $\hat{a}_i$, each have exactly one non-zero entry per column, and this entry is either 1 or -1 . Furthermore, since $\det(\hat{a}_i) = 1$, $\hat{a}_i$ is a signed permutation matrix. Therefore, $\hat{a}_i \in B_{2^n}^+$ for each $i \in \llbracket n \rrbracket$.

Example 3.1.1. For $n = 2$, the matrices $\hat{a}_i$ with $i \in \{1, 2\}$ are $2^2 \times 2^2$ matrices of the following form:

$$\hat{a}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{e} \quad \hat{a}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

◇

3.2

Clifford Algebra

In this section, we explore the matrix algebra generated by the elements $\hat{a}_i$, which is called the even Clifford algebra, denoted by Cl_{n+1}^0 . In the previous section, we saw that Quat_{n+1} is a finite group and precisely identified its elements.

Note that $\text{Quat}_{n+1} = \text{HQuat}_{n+1} \sqcup (-\text{HQuat}_{n+1})$, where HQuat_{n+1} consists of elements appearing with a positive sign in Quat_{n+1} . Furthermore, observe that $\text{HQuat}_{n+1} \subset \text{Quat}_{n+1}$ is not a subgroup, since $\hat{a}_i^2 = -1 \notin \text{HQuat}_{n+1}$.

Definition 3.2.1. Cl_{n+1}^0 is an associative algebra with unity over $\mathbb{R}$, which is a vector space of dimension 2^n , with an orthonormal basis HQuat_{n+1} .

Therefore, the Clifford algebra Cl_{n+1}^0 is generated by the elements $\hat{a}_i$, which satisfy the relations previously seen in the definition of the generators of Quat_{n+1} . Additionally, as a vector space, it is endowed with an inner product defined by

$$\langle z_1, z_2 \rangle = 2^{-n} \text{Trace}(z_1 z_2^T).$$

For n small enough, Clifford algebras are well-known algebras.

Example 3.2.1. For $n = 1$, Cl_2^0 is a 2-dimensional algebra over $\mathbb{R}$ with basis $\{1, \hat{a}_1\}$, where $\hat{a}_1$ are 2×2 matrices satisfying the condition $\hat{a}_1^2 = -1$.

Therefore, the elements of Cl_2^0 are of the form $u + v\hat{a}_1$, where $u, v \in \mathbb{R}$. From the previous section, we know that

$$\hat{a}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that this is the matrix form of $\mathbf{i} \in \mathbb{C}$. Therefore, the elements of Cl_2^0 are given by $u + v\hat{a}_1$, where $\hat{a}_1 = \mathbf{i}$, which means $\text{Cl}_2^0 = \mathbb{C}$. $\diamond$

Describing the generators of Clifford algebras for $n \geq 2$ becomes a relatively laborious and extensive task, as the dimension grows exponentially.

Classifications for Clifford algebras can be found in [12].

3.3

One Parameter Subgroups

A **one-parameter subgroup** of a group G is a continuous homomorphism from $\mathbb{R}$ as an additive group to the group G .

Define the one-parameter subgroup $\alpha_i^{\text{SO}} : \mathbb{R} \rightarrow \text{SO}_{n+1}$ by

$$\alpha_i^{\text{SO}}(\theta) = \begin{pmatrix} I_1 & & \\ & \cos(\theta) & -\sin(\theta) \\ & \sin(\theta) & \cos(\theta) \\ & & & I_2 \end{pmatrix},$$

where $I_1 \in \mathbb{R}^{(i-1) \times (i-1)}$ and $I_2 \in \mathbb{R}^{(n-i) \times (n-i)}$ are identity matrices.

From the generators $\hat{a}_i \in \text{Quat}_{n+1}$, one can define the one-parameter subgroups of the group SO_{2^n}

$$\alpha_i^{\text{Spin}} : \mathbb{R} \rightarrow \text{SO}_{2^n}, \quad \alpha_i^{\text{Spin}}(\theta) = \exp\left(\theta \frac{\hat{a}_i}{2}\right).$$

Since $\hat{a}_i$ are matrices with zero diagonals, it is not difficult to see that $\exp(\theta \frac{\hat{a}_i}{2})$ is a matrix where the diagonal entries are $\cos(\frac{\theta}{2})$, and the non-zero entries in the positions of $\hat{a}_i$ are $\sin(\frac{\theta}{2})$.

Therefore, the elements $\alpha_i^{\text{Spin}}(\theta)$ are $2^n \times 2^n$ matrices defined as

$$\alpha_i^{\text{Spin}}(\theta) = \exp\left(\theta \frac{\hat{a}_i}{2}\right) = \cos\left(\frac{\theta}{2}\right) + \hat{a}_i \sin\left(\frac{\theta}{2}\right).$$

For simplicity, $\alpha_i^{\text{Spin}}(\theta)$ will be denoted as $\alpha_i(\theta)$.

Note that the elements $\alpha_i(\theta)$ are block orthogonal matrices, with each matrix having identical diagonal elements. Additionally, the determinant of each block is 1, hence the determinant of the entire matrix is also 1.

3.4

The Group Spin_{n+1}

Having defined $\alpha_i(\theta)$, the next step is to consider the group generated by these elements.

Definition 3.4.1. The group generated by the elements $\alpha_i(\theta)$, where $\theta \in \mathbb{R}$ and $i \in \llbracket n \rrbracket = \{1, 2, \dots, n\}$, is defined as Spin_{n+1} .

Since Spin_{n+1} is defined by its generators $\alpha_i(\theta)$, which are expressed as matrices, the group can be seen as a matrix group. By adjusting the codomain of $\alpha_i(\theta)$, we obtain $\alpha_i = \alpha_i^{\text{Spin}} : \mathbb{R} \rightarrow \text{Spin}_{n+1}$.

Given that $\alpha_i(\pi) = \hat{a}_i$, it follows that $\text{Quat}_{n+1} \subset \text{Spin}_{n+1} \subset \text{Cl}_{n+1}^0$. From $\alpha_i(\pi) = \hat{a}_i$, we can define in Cl_{n+1}^0 the elements

$$\mathfrak{a}_i^{\text{Spin}} = \frac{1}{2}\hat{a}_i, \quad i \in \llbracket n \rrbracket.$$

Let $\mathfrak{spin}_{n+1} \subset \text{Cl}_{n+1}^0$ be the Lie algebra generated by the elements $\mathfrak{a}_i^{\text{Spin}}$. There exists an isomorphism between $\mathfrak{spin}_{n+1}$ and $\mathfrak{so}_{n+1}$ as Lie algebras, thus the dimension of $\mathfrak{spin}_{n+1}$ is given by $\frac{n(n+1)}{2}$, which is the dimension of $\mathfrak{so}_{n+1}$. Therefore, the group Spin_{n+1} has the same dimension.

Remark 3.4.2. Multiplication by an element of the group Spin_{n+1} defines a linear transformation of the Clifford algebra Cl_{n+1}^0 on itself. The basis HQuat_{n+1} allows us to express this linear transformation as a $2^n \times 2^n$ real matrix. $\diamond$

3.5

The Group $\tilde{\text{B}}_{n+1}^+$

Having defined the groups Quat_{n+1} and Spin_{n+1} along with their generators, we can now define the finite group $\tilde{\text{B}}_{n+1}^+ \subset \text{Spin}_{n+1}$.

Let us define the elements $\acute{a}_i$ and $\grave{a}_i$, such that $(\acute{a}_i)^{-1} = \grave{a}_i$ as follows:

$$\acute{a}_i = \alpha_i\left(\frac{\pi}{2}\right) = \frac{1 + \hat{a}_i}{\sqrt{2}}, \quad \grave{a}_i = \alpha_i\left(\frac{-\pi}{2}\right) = \frac{1 - \hat{a}_i}{\sqrt{2}},$$

where $\acute{a}_i, \grave{a}_i \in \text{Spin}_{n+1} \subset \text{Cl}_{n+1}^0$. Note that $\hat{a}_i = \acute{a}_i^2$ and $\hat{a}_i^2 = \acute{a}_i^4 = -1$.

Definition 3.5.1. The group generated by the elements $\{\acute{a}_1, \dots, \acute{a}_n\}$ is defined as $\tilde{\text{B}}_{n+1}^+ \subset \text{Spin}_{n+1}$.

Since both Quat_{n+1} and Spin_{n+1} are regarded as matrix groups, $\tilde{\text{B}}_{n+1}^+$ can also naturally be viewed as a matrix group.

Note that the matrices are orthogonal, with determinant 1. Additionally, the elements on each diagonal are equal, and if $i \neq j$ we have $a_{ij} = -a_{ji}$.

Let us examine some properties that illustrate how the elements $\hat{a}_i$, $\acute{a}_i$ and $\grave{a}_i$ interact with one another.

For $\acute{a}_i \in \tilde{\text{B}}_{n+1}^+$ e $\hat{a}_i \in \text{Quat}_{n+1}$, the following identities hold:

- (i) For all $i \in \llbracket n-1 \rrbracket$, we have
$$\begin{cases} \acute{a}_i \acute{a}_{i+1} \acute{a}_i = \acute{a}_{i+1} \acute{a}_i \acute{a}_{i+1} \\ (\acute{a}_i)^{-1} \acute{a}_{i+1} (\acute{a}_i)^{-1} = \acute{a}_{i+1} (\acute{a}_i)^{-1} \acute{a}_{i+1} \end{cases} ;$$
- (ii) If $|i-j| \neq 1 \implies \begin{cases} \acute{a}_j \acute{a}_i = \acute{a}_i \acute{a}_j \\ \hat{a}_j \acute{a}_i = \acute{a}_i \hat{a}_j \end{cases} ;$
- (iii) If $|i-j| = 1 \implies \begin{cases} \hat{a}_j \acute{a}_i = (\acute{a}_i)^{-1} \hat{a}_j \\ \hat{a}_j \hat{a}_i = -\hat{a}_i \hat{a}_j \end{cases} .$

The acute and grave maps are defined using reduced words and the elements $\acute{a}_i, \grave{a}_i \in \tilde{B}_{n+1}^+ \subset \text{Cl}_{n+1}^0$.

Definition 3.5.2. Let $\sigma \in S_{n+1}$, such that $\sigma = a_{i_1} \dots a_{i_l}$ is a reduced word. Let $\grave{a}_i = (\acute{a}_i)^{-1}$. Define the following maps:

- (i) *acute* : $S_{n+1} \rightarrow \tilde{B}_{n+1}^+$, given by $\text{acute}(\sigma) = \acute{\sigma} = \acute{a}_{i_1} \dots \acute{a}_{i_l}$;
- (ii) *grave* : $S_{n+1} \rightarrow \tilde{B}_{n+1}^+$, given by $\text{grave}(\sigma) = \grave{\sigma} = \grave{a}_{i_1} \dots \grave{a}_{i_l}$.

At first glance, the definition seems to depend on the chosen reduced word. Lemma 3.2 in [4] shows that the maps are well-defined and thus there is no such dependence.

Example 3.5.1. Let $\sigma = \eta = [654321] = a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1$. Recall that $\acute{a}_i = \frac{1+\hat{a}_i}{\sqrt{2}}$, then $\acute{\eta} = \acute{a}_1 \acute{a}_2 \acute{a}_1 \acute{a}_3 \acute{a}_2 \acute{a}_1 \acute{a}_4 \acute{a}_3 \acute{a}_2 \acute{a}_1 \acute{a}_5 \acute{a}_4 \acute{a}_3 \acute{a}_2 \acute{a}_1$. So

$$\acute{\eta} = \left(\frac{1+\hat{a}_1}{\sqrt{2}} \right) \dots \left(\frac{1+\hat{a}_1}{\sqrt{2}} \right).$$

Thus, keeping in mind the relationships between $\hat{a}_i$, $\acute{a}_i$, and $\grave{a}_i$, after some work we conclude that

$$\acute{\eta} = \frac{1}{2\sqrt{2}} (1 + \hat{a}_3 - \hat{a}_2 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_1 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5).$$

◇

3.6

The Homomorphism $\Pi : \text{Spin}_{n+1} \rightarrow \text{SO}_{n+1}$

The group Spin_{n+1} , as previously defined, is also recognized as the double cover of SO_{n+1} . As Lie algebras, there is a unique homomorphism between $\mathfrak{spin}_{n+1}$ and $\mathfrak{so}_{n+1}$. Additionally, Spin_{n+1} is simply connected and there exists a unique homomorphism $\Pi : \text{Spin}_{n+1} \rightarrow \text{SO}_{n+1}$, such that $\alpha_i(\theta) \mapsto \alpha_i^{\text{SO}}(\theta)$. In other words, the map is defined by:

$$\Pi : \text{Spin}_{n+1} \rightarrow \text{SO}_{n+1}$$

$$\alpha_i(\theta) \mapsto \begin{pmatrix} I_1 & & \\ & \text{Rot}(\theta) & \\ & & I_2 \end{pmatrix},$$

where $I_1 \in \mathbb{R}^{(i-1) \times (i-1)}$ and $I_2 \in \mathbb{R}^{(n-i) \times (n-i)}$ are identity matrices. Moreover, $\text{Rot}(\theta)$ is the 2×2 rotation matrix given by

$$\text{Rot}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Note that $\Pi(\hat{a}_i)$ is a diagonal matrix with determinant 1. Furthermore, $\Pi(\hat{a}_i)$ is a permutation matrix, also with determinant 1.

Recall that $\text{Quat}_{n+1} \subset \text{Spin}_{n+1}$ is generated by $\hat{a}_i$, and $\tilde{\text{B}}_{n+1}^+ \subset \text{Spin}_{n+1}$ is generated by $\hat{a}_i$. Therefore,

$$\Pi[\text{Quat}_{n+1}] \subset \text{Diag}_{n+1}^+ \text{ and } \Pi[\tilde{\text{B}}_{n+1}^+] \subset \text{B}_{n+1}^+.$$

Since the reverse inclusions are also valid, it follows that

$$\Pi[\text{Quat}_{n+1}] = \text{Diag}_{n+1}^+ \text{ and } \Pi[\tilde{\text{B}}_{n+1}^+] = \text{B}_{n+1}^+.$$

It has already been established that $\phi : \text{B}_{n+1}^+ \rightarrow \text{S}_{n+1}$ it is a surjective homomorphism, whose kernel is Diag_{n+1}^+ . Thus,

$$\frac{\text{B}_{n+1}^+}{\text{Diag}_{n+1}^+} \approx \text{S}_{n+1}.$$

Furthermore, $\phi \circ \Pi = \sigma$, where

$$\sigma : \tilde{\text{B}}_{n+1}^+ \rightarrow \text{S}_{n+1}$$

$$z \mapsto \sigma_z$$

it is a homomorphism, whose kernel is Quat_{n+1} .

The map $\Pi : \text{Spin}_{n+1} \rightarrow \text{SO}_{n+1}$ provides the following exact sequences, i.e., chained homomorphisms where the image of the predecessor is the kernel of the successor:

- (i) $1 \rightarrow \text{Quat}_{n+1} \hookrightarrow \tilde{\text{B}}_{n+1}^+ \xrightarrow{\sigma} S_{n+1} \rightarrow 1;$
- (ii) $1 \rightarrow \{\pm 1\} \hookrightarrow \text{Quat}_{n+1} \xrightarrow{\Pi} \text{Diag}_{n+1}^+ \rightarrow 1,$

where $\tilde{\text{B}}_{n+1}^+ = \Pi^{-1}[\text{B}_{n+1}^+]$ and $\text{Quat}_{n+1} = \Pi^{-1}[\text{Diag}_{n+1}^+]$.

Example 3.6.1. Let $z = \acute{a}_1 \acute{a}_3 \acute{a}_2 \in \tilde{\text{B}}_4^+$, then

$$\Pi(z) = \Pi(\acute{a}_1 \acute{a}_3 \acute{a}_2) = \Pi(\acute{a}_1) \Pi(\acute{a}_3) \Pi(\acute{a}_2)$$

$$\Pi(z) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

with $\phi(\Pi(z)) = [3142] = a_1 a_3 a_2$. $\diamond$

With the homomorphism well understood, from now on, we omit ϕ and simply refer to it as $\sigma = \Pi(z)$.

The group Diag_{n+1} acts by conjugations on SO_{n+1} . The quotient $\mathcal{E}_n = \frac{\text{Diag}_{n+1}}{\pm I}$ is inherently isomorphic to $\{\pm 1\}^{\llbracket n \rrbracket}$: a matrix $D \in \text{Diag}_{n+1}$ corresponds to $E \in \mathcal{E}_n = \{\pm 1\}^{\llbracket n \rrbracket}$, with $E_i = D_{i,i} D_{i+1,i+1}$.

Furthermore, the group $\mathcal{E}_n$ also acts by automorphisms on SO_{n+1} . This action can be lifted to Spin_{n+1} and then extended to Cl_{n+1}^0 . Specifically, each element $E \in \mathcal{E}_n$ defines automorphisms of Spin_{n+1} and Cl_{n+1}^0 through the following relations:

$$(\alpha_i(\theta))^E = \alpha_i(E_i \theta), \quad (\hat{a}_i)^E = E_i \hat{a}_i.$$

3.7

The Real Part

In this section, we explore various results regarding the real part of an element $z \in \text{Cl}_{n+1}^0$. The proofs of the results in this section can be found in [1] and [6].

An element of Cl_{n+1}^0 can be written as a linear combination of elements from $\text{Quat}_{n+1} \subset \text{Spin}_{n+1} \subset \text{Cl}_{n+1}^0$. Therefore, any element $z \in \text{Cl}_{n+1}^0$ can be expressed as

$$z = \sum_{q \in \text{HQuat}_{n+1}} c_q q, \quad \text{with } c_q \in \mathbb{R}.$$

Definition 3.7.1. The real part of $z \in \text{Cl}_{n+1}^0$ is defined by

$$\Re(z) = 2^{-n} \text{Trace}(z) = \langle z, 1 \rangle.$$

Thus, for $z = \sum_{q \in \text{HQuat}_{n+1}} c_q q \in \text{Cl}_{n+1}^0$ the real part is the independent coefficient $\Re(z) = c_1$.

Let us see a result that relates the real part of $z \in \text{Spin}_{n+1} \subset \text{Cl}_{n+1}^0$, with the eigenvalues of the matrix $\Pi(z) \in \text{SO}_{n+1}$.

Fact 3.7.1. For $z \in \text{Spin}_{n+1} \subset \text{Cl}_{n+1}^0$, let $Q = \Pi(z) \in \text{SO}_{n+1}$ such that the eigenvalues are $\exp(\pm\theta_1 i), \dots, \exp(\pm\theta_k i), 1, \dots, 1$. Then

$$\Re(z) = \pm \cos\left(\frac{\theta_1}{2}\right) \dots \cos\left(\frac{\theta_k}{2}\right).$$

In particular, $\Re(z) = 0$ if and only if, -1 is an eigenvalue of Q .

Example 3.7.1. Let $z \in \text{Spin}_4$, such that $\{\exp(\pm\frac{\pi}{2}i), \exp(\pm\frac{\pi}{3}i)\}$ is the set of eigenvalues of $Q = \Pi(z) \in \text{SO}_4$. Thus, we can assume

$$Q = \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & & \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & & \\ & & \cos(\frac{\pi}{3}) & -\sin(\frac{\pi}{3}) \\ & & \sin(\frac{\pi}{3}) & \cos(\frac{\pi}{3}) \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ & & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

and $z = \alpha_1(\frac{\pi}{2})\alpha_3(\frac{\pi}{3})$. Therefore, by Fact 3.7.1,

$$\Re(z) = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{4}.$$

◇

The previous result shows that $\Re(z)$ can be computed using information about $Q = \Pi(z) \in \text{SO}_{n+1}$. Next, we focus on how $\Re(z)$ can be computed based on information about $Q \in \text{B}_{n+1}^+$.

Definition 3.7.2. A matrix $Q \in \text{B}_{n+1}^+$ is said to be an even-length cycle if there exist indices $i_1, \dots, i_k$ such that

- (i) $(e_{i_k})^T Q = -(e_{i_1})^T$,
- (ii) $(e_{i_j})^T Q = (e_{i_{j+1}})^T$ for $1 \leq j < k$,
- (iii) $(e_j)^T Q = (e_j)^T$ for $j > k$.

If the length is odd, $(e_{i_k})^T Q = (e_{i_1})^T$, and (ii) and (iii) are still valid.

Fact 3.7.2. Let $z_0 \in \tilde{B}_{n+1}^+$, such that $\Pi(z_0) = Q_0 \in B_{n+1}^+$ is a cycle of length k . Then $\Re(z_0) = \pm 2^{\frac{-k+1}{2}}$.

Example 3.7.2. Let $z = \acute{a}_3 \acute{a}_2 \in \tilde{B}_4^+$. Thus, $\sigma = \phi \circ \Pi(z) = a_3 a_2 = (234) \in S_4$, such that $\Pi(z) = Q \in B_4^+$ is the permutation matrix of σ , given by

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since σ is a cycle of length 3, the eigenvalues of Q are: $\exp(\pm \frac{2\pi i}{3})$ and 1. Thus, by the previous result

$$\Re(z) = \pm \cos\left(\frac{\pi}{3}\right) = \pm \frac{1}{2} = \pm 2^{\frac{-k+1}{2}}, \quad \text{with } k = 3.$$

Notice that, through manual computation, we obtain

$$z = \acute{a}_3 \acute{a}_2 = \left(\frac{1 + \hat{a}_3}{\sqrt{2}}\right) \left(\frac{1 + \hat{a}_2}{\sqrt{2}}\right) = \frac{1}{2}(1 + \hat{a}_2 + \hat{a}_3 + \hat{a}_3 \hat{a}_2).$$

Therefore, $\Re(z) = \frac{1}{2}$. ◇

Another way to compute the real part, in the case of an element in Spin_{n+1} of a specific type, is by using the number of cycles of a permutation in S_{n+1} .

Recall the exact sequences:

$$1 \rightarrow \text{Quat}_{n+1} \hookrightarrow \tilde{B}_{n+1}^+ \xrightarrow{\sigma} S_{n+1} \rightarrow 1;$$

$$1 \rightarrow \{\pm 1\} \hookrightarrow \text{Quat}_{n+1} \xrightarrow{\Pi} \text{Diag}_{n+1}^+ \rightarrow 1,$$

where $\Pi : \text{Spin}_{n+1} \rightarrow \text{SO}_{n+1}$ and $\phi \circ \Pi(z) = \sigma$, with $\phi : B_{n+1}^+ \rightarrow S_{n+1}$.

We have $\Pi^{-1}[\{\sigma\}] = \acute{\sigma} \text{Quat}_{n+1} \subset \tilde{B}_{n+1}^+$, which implies $\Pi[\acute{\sigma} \text{Quat}_{n+1}] \subset \Pi[\tilde{B}_{n+1}^+] = B_{n+1}^+$. From the first exact sequence above, it follows that for any $\sigma \in S_{n+1}$, the set $\acute{\sigma} \text{Quat}_{n+1}$ is a coset.

Definition 3.7.3. The subgroup $H_{\text{Diag}, X} \leq \text{Diag}_{n+1}^+$ with index $2^{|X|-1}$ consists of matrices $E \in \text{Diag}_{n+1}^+$ such that, if $A = \{i_1, \dots, i_k\} \in X$, then the product $E_{i_1 i_1} \dots E_{i_k i_k} = 1$.

Example 3.7.3. Let $n = 4$ and $X = \{\{1, 3\}, \{2, 4, 5\}\}$. Let $E \in \text{Diag}_5^+$ be the

matrix defined by

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $E_{2,2}E_{4,4}E_{5,5} = (-1).(-1).1 = 1$ e $E_{1,1}E_{3,3} = 1.1 = 1$ with $A = \{2, 4, 5\}, B = \{1, 3\} \in X$, then $E \in H_{\text{Diag}, X}$. $\diamond$

Let $H_X = \Pi^{-1}[H_{\text{Diag}, X}] \leq \text{Quat}_{n+1}$, where $\Pi : \text{Quat}_{n+1} \rightarrow \text{Diag}_{n+1}^+$ is the restriction of $\Pi : \text{Spin}_{n+1} \rightarrow \text{SO}_{n+1}$.

For a permutation $\sigma \in S_{n+1}$, consider X_σ the partition of $\llbracket n+1 \rrbracket$ into cycles of σ . Let $H_\sigma = H_{X_\sigma} \leq \text{Quat}_{n+1}$. It follows that $|H_\sigma| = 2^{n+2-c}$, where c is the number of cycles of σ .

Example 3.7.4. Let $\sigma = (15)(234) \in S_5$. Thus, $X_\sigma = \{\{1, 5\}, \{2, 3, 4\}\}$. Moreover, $|H_\sigma| = 2^{4+2-2} = 16$.

By a simple computation, we can see that the subgroup $H_{\text{Diag}, X_\sigma}$ is generated by

$$\text{diag}(-1, 1, 1, 1, -1), \text{diag}(1, -1, -1, 1, 1), \text{diag}(1, 1, -1, -1, 1) \in \text{Diag}_5^+.$$

Lifting to H_σ , we have the generators $\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4, \hat{a}_2, \hat{a}_3 \in \text{Quat}_5$, then

$$H_\sigma = \{\pm 1, \pm \hat{a}_2, \pm \hat{a}_3, \pm \hat{a}_2\hat{a}_3, \pm \hat{a}_1\hat{a}_4, \pm \hat{a}_1\hat{a}_2\hat{a}_4, \pm \hat{a}_1\hat{a}_3\hat{a}_4, \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\}.$$

Note that $|H_\sigma| = 16$, as expected. $\diamond$

Fact 3.7.3. Consider $\sigma \in S_{n+1}$, written as a product of disjoint cycles, such that c is the number of cycles. Choose $z_0 \in \sigma \text{Quat}_{n+1}$, such that $\Re(z_0) > 0$. For $q \in \text{Quat}_{n+1}$, we have

$$|\Re(qz_0)| = |\Re(z_0q)| = \begin{cases} 2^{-\frac{(n+1-c)}{2}}, & q \in H_\sigma, \\ 0, & q \notin H_\sigma. \end{cases}$$

There are 2^{n+1-c} values of $q \in \text{Quat}_{n+1}$, such that $\Re(qz_0) > 0$ (similarly for $\Re(z_0q)$). Furthermore, if z_0 is expanded in the canonical basis as $z_0 = \sum_{p \in H\text{Quat}_{n+1}} c_p p$, then $c_p \neq 0$, if and only if $p \in H_\sigma$.

Example 3.7.5. Let $\sigma = (13)(24) = a_2a_1a_3a_2 \in S_4$. Then, making use of the known relations for $\hat{a}_i$, we have:

$$\sigma = \frac{\hat{a}_1 + \hat{a}_2 + \hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3}{2}.$$

Moreover, $H_{\text{Diag}, X_\sigma}$ is generated by

$$\text{diag}(1, -1, 1, -1), \text{diag}(-1, 1, -1, 1) \in \text{Diag}_4^+.$$

Then, H_σ is generated by $\hat{a}_1\hat{a}_2, \hat{a}_2\hat{a}_3 \in \text{Quat}_4$, thus

$$H_\sigma = \{\pm 1, \pm \hat{a}_1\hat{a}_2, \pm \hat{a}_1\hat{a}_3, \pm \hat{a}_2\hat{a}_3\}.$$

Let us choose $q_0 = -\hat{a}_3 \in \text{Quat}_4$. We have,

$$z_0 = -\hat{a}_3\sigma = \frac{1 - \hat{a}_1\hat{a}_3 + \hat{a}_2\hat{a}_3 + \hat{a}_1\hat{a}_2}{2}.$$

Therefore, $\Re(z_0) = \frac{1}{2} > 0$.

We can see that the terms of z_0 match the elements of H_σ . ◇

4

Preancestries and Ancestries

In this chapter, we introduce two key concepts for this work: preancestry and ancestry. Given a permutation, a preancestry is a sequence of elements in S_{n+1} , and an ancestry is a sequence of elements in $\tilde{B}_{n+1}^+$.

These two concepts guide the direction of this work. In the upcoming chapters, their influence on our study will be further explored and better understood.

See [1] for the proofs of the results of this chapter.

4.1

Preancestries

A preancestry for a permutation is directly connected to the reduced word. It can be represented by a sequence consisting of ± 2 and 0.

Definition 4.1.1. Let $\sigma = a_{i_1}a_{i_2}\dots a_{i_l} \in S_{n+1}$ be a reduced word. A **preancestry** is a sequence $(\rho_k)_{0 \leq k \leq l}$ of permutations with the following properties:

1. $\rho_0 = \rho_l = \eta$;
2. for all $k \in \llbracket l \rrbracket$, either $\rho_k = \rho_{k-1}$ or $\rho_k = \rho_{k-1}a_{i_k}$;
3. for all $k \in \llbracket l \rrbracket$, if $\rho_{k-1}a_{i_k} > \rho_{k-1}$ then $\rho_k = \rho_{k-1}a_{i_k}$.

Example 4.1.1. Let $\sigma = a_1a_2a_1 \in S_3$. Then

$$(\rho_0 = \eta, \rho_1 = \rho_0, \rho_2 = \rho_1, \rho_3 = \rho_2) = (\eta, \eta, \eta, \eta)$$

is a preancestry sequence. This is just one example of a preancestry, specifically the trivial one, but it is not the only possible preancestry. The following sequence also defines a valid one:

$$(\rho_0 = \eta, \rho_1 = \eta a_1, \rho_2 = \rho_1, \rho_3 = \rho_2 a_1) = (\eta, a_1 a_2, a_1 a_2, \eta).$$

◇

It is generally more practical in this work to represent a preancestry (ρ_k) using a sequence of ± 2 and 0:

$$\varepsilon_0 : \llbracket n \rrbracket \rightarrow \{-2, 0, +2\}$$

$$\varepsilon_0(k) = \begin{cases} -2, & \rho_k = \rho_{k-1}a_{i_k} < \rho_{k-1}, \\ 0, & \rho_k = \rho_{k-1}, \\ 2, & \rho_k = \rho_{k-1}a_{i_k} > \rho_{k-1}. \end{cases}$$

Therefore, a sequence ε_0 is considered a preancestry if the sequence of permutations $(\rho_k)_{0 \leq k \leq l}$ defined below, satisfies the conditions for a preancestry:

$$\rho_0 = \eta, \quad \rho_k = \begin{cases} \rho_{k-1}a_{i_k}, & \varepsilon_0(k) \neq 0, \\ \rho_{k-1}, & \varepsilon_0(k) = 0. \end{cases}$$

It should be noted that in any preancestry ε_0 , the count of $k \in \llbracket l \rrbracket$ such that $\varepsilon_0(k) = -2$ equals the count of $k \in \llbracket l \rrbracket$ such that $\varepsilon_0(k) = +2$.

Definition 4.1.2. The dimension $d = \dim(\varepsilon_0)$ of a preancestry is determined by the number of occurrences of $+2$ (or -2) in the sequence.

In the wiring diagram for σ , a preancestry ε_0 is represented using diamonds to indicate the values in the sequence: a black diamond denotes -2 , and a white diamond denotes $+2$. If $\varepsilon_0(k) = 0$, the space remains empty.

Example 4.1.1 presents two preancestries. The preancestry with dimension 0 is depicted in the wiring diagram by leaving the inversions empty, while the preancestry with dimension 1 is represented by marking a black diamond for the first inversion and a white diamond for the second. Figure 4.1 illustrates these diagrams.



Figure 4.1: Preancestries of dimension 0 and 1, respectively, $\varepsilon_0 = (0, 0, 0)$ and $\varepsilon_0 = (-2, 0, +2)$.

Example 4.1.2. Consider the reduced word for $\sigma = a_2a_3a_2a_1a_2a_4a_3a_2 \in S_5$. The sequences and the wiring diagram, below represents a preancestry of dimension 1

$$(\eta, \eta, \eta a_2, \eta a_2, \eta a_2, \eta a_2, \eta a_2, \eta, \eta) = (0, -2, 0, 0, 0, 0, 2, 0).$$

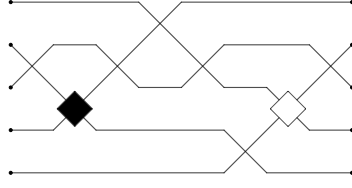


Figure 4.2: Preancestry of dimension 1, $\varepsilon = (0, -2, 0, 0, 0, 0, +2, 0)$.

The next sequences and wiring diagram, represents a preancestry of dimension 2

$$(\eta, \eta a_2, \eta a_2 a_3, \eta a_2 a_3, \eta a_2 a_3, \eta a_2 a_3, \eta a_2 a_3, \eta a_2, \eta) = (-2, -2, 0, 0, 0, 0, +2, +2).$$

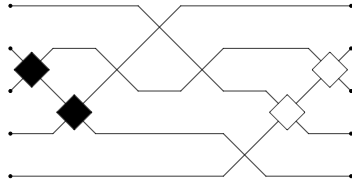


Figure 4.3: Preancestry of dimension 2, $\varepsilon = (-2, -2, 0, 0, 0, 0, +2, +2)$.

◇

Fact 4.1.1. *Consider a permutation $\sigma \in S_{n+1}$. The number of preancestries per dimension does not depend on the choice of the reduced word.*

Counting preancestries is a task that becomes progressively more challenging. We have methods to count them for dimensions 0, 1, and 2. However, for higher dimensions, we require a different approach.

There exists one preancestry with dimension 0, identified by the absence of marked vertices.

Preancestries with dimension 1 are straightforward to categorize: we simply mark two adjacent intersections along the same row. In essence, a 1-dimensional preancestry corresponds to a bounded section of the wiring diagram complement, with the two marked intersections representing its left and right extremes. The count of 1-dimensional preancestries is $l - n + b$ where $l = \text{inv}(\sigma)$ and $b = \text{block}(\sigma)$.

Figure 4.2 shows a preancestry of dimension 1. Note that for $\sigma = a_2 a_3 a_2 a_1 a_2 a_4 a_3 a_2 \in S_6$ there are $l - n + b = 8 - 4 + 0 = 4$ preancestries with this dimension.

For dimension 2, the scenario becomes slightly more intricate. Consider a preancestry ε_0 of dimension 2, and let $k_1 < k_2 < k_3 < k_4$ such that $|\varepsilon_0(k_i)| = 2$. In this case, $\varepsilon_0(k_1) = -2$ and $\varepsilon_0(k_4) = +2$. If $\varepsilon_0(k_2) = +2$, then $\varepsilon_0(k_3) = -2$, $i_{k_1} = i_{k_2}$, and $i_{k_3} = i_{k_4}$. In this scenario, intersections k_1 and k_2 are consecutive on row i_{k_1} , and intersections k_3 and k_4 are consecutive on row i_{k_3} . If $\varepsilon_0(k_2) = -2$ and $|i_{k_1} - i_{k_2}| > 1$, we also observe two pairs of consecutive intersections on two rows. In both cases, the preancestry is classified as type I.

The figure below illustrates a preancestry of this type.

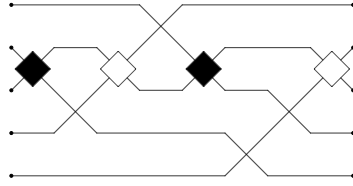


Figure 4.4: Preancestry of dimension 2, $\varepsilon = (-2, 0, 2, 0, -2, 0, 0, 2)$.

If $\varepsilon_0(k_2) = -2$ and $\varepsilon_0(k_3) = +2$, with $|i_{k_1} - i_{k_2}| = 1$, then ε_0 belongs to type II. Here, $i_{k_1} = i_{k_4}$ and $i_{k_2} = i_{k_3}$, and intersections k_2 and k_3 are consecutive on row i_{k_2} . Intersection k_1 is the last on row i_{k_1} before k_2 , while intersection k_4 is the first on row i_{k_1} after k_3 . There exists no limit to the number of intersections on row i_{k_1} between k_2 and k_3 . Figure 4.3 shows a preancestry of dimension 2 and type II.

The subword comprised of all marked letters has value 1. Additionally, the subword consisting of unmarked letters contains valuable information, as highlighted by the following result. This result helps us estimate the maximum dimension of a preancestry.

Fact 4.1.2. Consider $\sigma \in S_{n+1}$ and a fixed reduced word of length $l = \text{inv}(\sigma)$. Consider a preancestry ε_0 of dimension $d = \dim(\varepsilon_0)$. There are $\delta = l - 2d$ unmarked crossings $k_1, \dots, k_\delta$. Assume that the unmarked crossing k_j is $(i_{j,0}, i_{j,1}) \in \text{Inv}(\sigma)$. We then have

$$\sigma = (i_{\delta,0}i_{\delta,1}) \dots (i_{1,0}i_{1,1}).$$

If $c = \text{nc}(\sigma)$ is the number of cycles then $2d \leq l + c - n - 1$.

Fact 4.1.3. For $n \geq 2$, let $\eta \in S_{n+1}$ be the top permutation. The largest possible dimension among all preancestries is

$$d_{\max} = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Furthermore, there exists a unique preancestry of dimension d_{max} .

4.2

Ancestries

An ancestry is closely related to a preancestry. In terms of a wiring diagram, it identifies the inversions that the preancestry does not mark, with circles, either black or white. In terms of a sequence, it assigns -1 or 1 to the zeros of ε_0 .

Definition 4.2.1. Let $\sigma = a_{i_1} \dots a_{i_l} \in S_{n+1}$ be a fixed reduced word. An **ancestry** is a sequence $(\varrho)_{0 \leq k \leq l}$ of elements of $\tilde{B}_{n+1}^+$ such that:

1. $\varrho_0 = \dot{\eta}$, $\varrho_l \in \dot{\eta} \text{Quat}_{n+1}$;
2. for all k , we have $\varrho_k = \varrho_{k-1}$ or $\varrho_k = \varrho_{k-1} \acute{a}_{i_k}$ or $\varrho_k = \varrho_{k-1} \hat{a}_{i_k}$;
3. the sequence (ρ_k) defined by $\rho_k = \Pi_{\tilde{B}_{n+1}^+, S_{n+1}}(\varrho_k)$ is a preancestry.

The final condition can be restated as follows: if $\Pi(\varrho_{k-1}) < \Pi(\varrho_{k-1})a_{i_k}$, it implies that $\varrho_k = \varrho_{k-1} \acute{a}_{i_k}$, for all k .

The ancestry (ϱ_k) corresponds to the preancestry (ρ_k) .

Example 4.2.1. Consider $\sigma = a_1 a_2 a_1 \in S_3$, from Example 4.1.1. The sequences below represent some ancestries for this permutation:

$$(\dot{\eta}, \dot{\eta}, \dot{\eta}, \dot{\eta}), (\dot{\eta}, \dot{\eta} \acute{a}_1 \hat{a}_2, \dot{\eta} \hat{a}_2, \dot{\eta} \hat{a}_1).$$

◇

There are three additional sequences that represent an ancestry. Two consist of integers, and the other comprises elements of Quat_{n+1} . The first is defined recursively by

$$\xi : \llbracket l \rrbracket \rightarrow \{0, 1, 2\}$$

$$\varrho_k = \varrho_{k-1} (\acute{a}_{i_k})^{\xi(k)}.$$

It follows that $\varrho_k = \varrho_0 \cdot (\acute{a}_{i_1})^{\xi(1)} \dots (\acute{a}_{i_k})^{\xi(k)}$, where $\varrho_0 = \dot{\eta}$.¹

The sequence of elements $(q_k)_{0 \leq k \leq l}$ is defined as follows:

$$\dot{\rho}_k q_k = \varrho_k, \quad \text{with } q_k \in \text{Quat}_{n+1},$$

so that, in particular $q_0 = 1$ and $q_l = \dot{\eta} \varrho_l$.

¹In the reference [1], the equation is missing the term ϱ_0 .

The last sequence is the one used most frequently. Given an ancestry, we define a sequence:

$$\varepsilon : \llbracket l \rrbracket \rightarrow \{\pm 1, \pm 2\}$$

$$\varepsilon(k) = \begin{cases} -2, & \xi(k) = 1, \quad \rho_k < \rho_{k-1}, \\ +2, & \xi(k) = 1, \quad \rho_k > \rho_{k-1}, \\ (1 - \xi(k))[\hat{a}_{i_k}, q_{k-1}], & \xi(k) \neq 1. \end{cases}$$

It is possible to recover ξ and (ϱ_k) from ε by:

$$\xi(k) = \begin{cases} 0, & \varepsilon(k) = [\hat{a}_{i_k}, q_{k-1}], \\ 2, & \varepsilon(k) = -[\hat{a}_{i_k}, q_{k-1}], \\ 1, & |\varepsilon(k)| = 2. \end{cases} \quad \varrho_k = \varrho_{k-1}(\acute{a}_{i_k})^{\xi(k)};$$

Notation 4.2.1. Here, $[\hat{a}_{i_k}, q_{k-1}] = (\hat{a}_{i_k})^{-1} q_{k-1}^{-1} \hat{a}_{i_k} q_{k-1} \in \{\pm 1\}$ represents the commutator.

Furthermore, $q_k = (\rho_k)^{-1} \varrho_k = (\acute{a}_{i_k})^{-\text{sign}(\varepsilon(k))} q_{k-1} \acute{a}_{i_k}$.

Given the reduced word, each of the sequences (ϱ_k) , ξ , and ε allows us to obtain (q_k) and the other two sequences. With the preancestry and (q_k) , the three sequences mentioned above can also be derived. Therefore, these three sequences are considered alternative descriptions of an ancestry.

Example 4.2.2. Consider $\sigma = a_1 a_2 a_1 \in S_3$, the permutation from the previous example. Let us find the sequences above. Given $\varepsilon_1 = (+1, +1, +1)$, we get

$$q_{k_1} = (1, 1, 1, 1), \quad \xi_1 = (0, 0, 0), \quad \varrho_{k_1} = (\acute{\eta}, \acute{\eta}, \acute{\eta}, \acute{\eta}).$$

Given $\varepsilon_2 = (-2, +1, +2)$, we get

$$q_{k_2} = (1, \hat{a}_1, \hat{a}_1 \hat{a}_2, \hat{a}_2), \quad \xi_2 = (1, 2, 1), \quad \varrho_{k_2} = (\acute{\eta}, \acute{\eta} \acute{a}_1 \hat{a}_2, \acute{\eta} \hat{a}_2, \acute{\eta} \hat{a}_1).$$

◇

Definition 4.2.2. Let $\sigma \in S_{n+1}$. For an ancestry ε , define $P(\varepsilon) = \acute{\sigma}(q_l)^{-1}$.

From this definition it follows that $\varrho_l = \acute{\eta}(P(\varepsilon))^{-1} \acute{\sigma}$.

Fact 4.2.1. Let $\sigma \in S_{n+1}$. Given an ancestry ε , we have

$$P(\varepsilon) = (\acute{a}_{i_1})^{\text{sign}(\varepsilon(1))} \dots (\acute{a}_{i_l})^{\text{sign}(\varepsilon(l))}.$$

Example 4.2.3. From Example 4.2.2, we have the sequences q_k , ξ , and ε for the two ancestries. For the first $P(\varepsilon_1) = \acute{a}_1 \acute{a}_2 \acute{a}_1$, and for the second $P(\varepsilon_2) = \acute{a}_1 \acute{a}_2 \acute{a}_1$.

◇

Definition 4.2.3. The dimension $d = \dim(\varepsilon)$ of an ancestry is determined by the number of occurrences of $+2$ (or -2) in the sequence. This dimension is the same as that of the associated preancestry.

In the wiring diagram, an ancestry is represented by the sequence ε . Similar to a preancestry, where -2 and $+2$ are represented by a black and a white diamond, respectively, in an ancestry, -1 and $+1$ are represented by a black and a white disk, respectively. From now on, we will also represent the sequence ε of an ancestry using black and white disks and diamonds. For instance, $\varepsilon = (-2, +1, +2)$ is written as $(\blacklozenge \bullet \diamond)$.

Example 4.2.4. For $\sigma = a_1 a_2 a_1 \in S_3$, the wiring diagrams for the ancestries from the previous example are illustrated as follows:



Figure 4.5: Ancestries $\varepsilon_1 = (+1, +1, +1) = (\circ \circ \circ)$ and $\varepsilon_2 = (-2, +1, +2) = (\blacklozenge \bullet \diamond)$ with dimension 0 and 1, respectively.

◇

Definition 4.2.4. If vertices k_1 and k_2 define a region and have opposite signs, we can change the signs along the boundary of this region. This operation is called a **click**.

Figure 4.6 shows a diagram before and after a click in the upper region. The diagram on the left has ancestry $\varepsilon_1 = (\bullet \bullet \bullet \bullet \circ \circ \bullet)$, and on the right, $\varepsilon_2 = (\circ \circ \bullet \circ \bullet \circ \bullet)$.

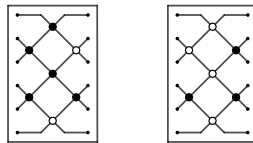


Figure 4.6: Example of a click in the wiring diagram of the permutation $\sigma = a_2 a_1 a_4 a_3 a_2 a_5 a_4 \in S_6$.

4.3

Counting Ancestries

In this section, we will examine how to compute the number of ancestries ε associated with a preancestry ε_0 such that $P(\varepsilon) = z$, for a given $z \in \text{Quat}_{n+1}$.

Definition 4.3.1. For a given preancestry ε_0 and an element $z \in \acute{\sigma} \text{Quat}_{n+1}$, define $L_{\varepsilon_0}(z)$ as the set of ancestries ε associated with ε_0 such that $P(\varepsilon) = z$. The cardinality of $L_{\varepsilon_0}(z)$ is denoted by $\text{NL}_{\varepsilon_0}(z)$.

Fact 4.3.1. For any $z \in \acute{\sigma} \text{Quat}_{n+1}$, we have

$$\text{NL}_{\varepsilon_0}(z) - \text{NL}_{\varepsilon_0}(-z) = 2^{\frac{l-2d}{2}} \Re(z). \quad (4-1)$$

Recall that X_σ is the partition of $\llbracket n+1 \rrbracket$ into cycles of σ .

Definition 4.3.2. Given a preancestry ε_0 , define a partition X_{ε_0} as a refinement of X_σ . The partition X_{ε_0} is the most refined partition that satisfies the following condition: for each k where $\varepsilon_0(k) = 0$ and the k -th crossing is (i_0, i_1) , the pair $\{i_0, i_1\}$ must be contained in some set A in X_{ε_0} .

Let $H_{\varepsilon_0} = H_{X_{\varepsilon_0}} \leq \text{Quat}_{n+1}$. It follows from Fact 4.1.2 that $H_\sigma \leq H_{\varepsilon_0}$.

Example 4.3.1. Consider $\sigma = (15)(234) = a_1a_3a_2a_1a_4a_3a_2a_1a_4 \in S_5$. From Example 3.7.4, we have $X_\sigma = \{\{1, 5\}, \{2, 3, 4\}\}$ and

$$H_\sigma = \{\pm 1, \pm \hat{a}_2, \pm \hat{a}_3, \pm \hat{a}_2\hat{a}_3, \pm \hat{a}_1\hat{a}_4, \pm \hat{a}_1\hat{a}_2\hat{a}_4, \pm \hat{a}_1\hat{a}_3\hat{a}_4, \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\}.$$

There is only one preancestry with the maximum dimension $d = 3$, given by $(-2, 0, -2, 0, -2, 0, +2, +2, +2)$. Additionally, there are five preancestries of dimension 2 and five of dimension 1.

For $\varepsilon_0 = (-2, 0, -2, 0, -2, 0, +2, +2, +2)$, the unmarked crossings, where $\varepsilon_0(k) = 0$ occur at $(1, 5)$, $(2, 4)$ and $(3, 4)$. Thus $X_{\varepsilon_0} = X_\sigma$, which implies $H_{\varepsilon_0} = H_\sigma$.

In the case of $\varepsilon_0 = (-2, -2, 0, 2, 0, 2, 0, 0, 0)$, the unmarked crossings, where $\varepsilon_0(k) = 0$ are $(1, 3)$, $(1, 4)$, $(2, 5)$, $(3, 5)$ and $(4, 5)$, which leads to $X_{\varepsilon_0} = \{1, 2, 3, 4, 5\}$ implying that $H_{\varepsilon_0} = \text{Quat}_5$.

The remaining four preancestries of dimension 2, as well as all preancestries of dimensions 1 and 0, have the same X_{ε_0} and H_{ε_0} . $\diamond$

The next result is an important result for this work. Together with Fact 4.3.1, it provides a method for counting the ancestries associated with a preancestry for a permutation.

Fact 4.3.2. Consider a preancestry ε_0 and the subgroup $H_{\varepsilon_0} \leq \text{Quat}_{n+1}$. Choose $z_0 \in \acute{o} \text{Quat}_{n+1}$ with $\Re(z_0) > 0$. For $z = qz_0$, we have

$$\text{NL}_{\varepsilon_0}(z) + \text{NL}_{\varepsilon_0}(-z) = \begin{cases} 2^{l-2d+1}/|H_{\varepsilon_0}|, & q \in H_{\varepsilon_0}, \\ 0, & q \notin H_{\varepsilon_0}. \end{cases} \quad (4-2)$$

The following equation provides the number of ancestries with $d = 0$:

$$\text{NL}(z) = 2^{l-n+b-1} + 2^{\frac{l}{2}-1} \Re(z), \quad (4-3)$$

where $\Re(z) = 2^{-n} \text{Trace}(z) = \langle z, 1 \rangle$.

Example 4.3.2. Consider $\sigma = (15)(234) = a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_4 \in S_5$. From the previous example, we know the value of $|H_{\varepsilon_0}|$. Let

$$z = \acute{o} = \frac{-\hat{a}_1 - \hat{a}_1 \hat{a}_2 + \hat{a}_1 \hat{a}_3 - \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_2 \hat{a}_4 - \hat{a}_3 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4}{2\sqrt{2}}.$$

Note that $\Re(z) = 0$. For ε_0 of dimension 0, it follows that

$$\text{NL}(z) = 2^{9-4-1} + 2^{\frac{9}{2}-1} \Re(z) = 2^4 = 16.$$

For ε_0 with dimension 1, we have

$$\text{NL}_{\varepsilon_0}(z) - \text{NL}_{\varepsilon_0}(-z) = 2^{\frac{9-2}{2}} \Re(z) = 0,$$

$$\text{NL}_{\varepsilon_0}(z) + \text{NL}_{\varepsilon_0}(-z) = \frac{2^{9-2+1}}{2^5} = \frac{2^8}{2^5} = 8.$$

Thus, $2 \cdot \text{NL}_{\varepsilon}(z) = 8$, so $\text{NL}_{\varepsilon}(z) = 4$.

As seen in the previous example, each of the five preancestries of dimension 1 has the same H_{ε_0} , resulting in $4 \times 5 = 20$ ancestries ε with $\dim(\varepsilon) = 1$.

For ε_0 with dimension 2, we have

$$\text{NL}_{\varepsilon_0}(z) - \text{NL}_{\varepsilon_0}(-z) = 2^{\frac{9-4}{2}} \Re(z) = 0,$$

$$\text{NL}_{\varepsilon_0}(z) + \text{NL}_{\varepsilon_0}(-z) = \frac{2^{9-4+1}}{2^5} = \frac{2^6}{2^5} = 2.$$

Thus, $2 \cdot \text{NL}_{\varepsilon}(z) = 2$, so $\text{NL}_{\varepsilon}(z) = 1$.

As noted above, all six preancestries of dimension 2 share the same H_{ε_0} , resulting in $1 \times 6 = 6$ ancestries ε with $\dim(\varepsilon) = 2$.

For ε_0 with dimension 3, we have

$$\text{NL}_{\varepsilon_0}(z) - \text{NL}_{\varepsilon_0}(-z) = 2^{\frac{9-6}{2}} \Re(z) = 0,$$

$$\text{NL}_{\varepsilon_0}(z) + \text{NL}_{\varepsilon_0}(-z) = \frac{2^{9-6+1}}{2^4} = \frac{2^4}{2^4} = 1.$$

Therefore, $2 \cdot \text{NL}_{\varepsilon}(z) = 1$, giving $\text{NL}_{\varepsilon}(z) = \frac{1}{2} < 1$, which results in $\text{NL}_{\varepsilon}(z) = 0$.

◇

4.4

Thin Ancestries

An ancestry of dimension 0 for a permutation $\sigma \in S_{n+1}$ can be of two types: thin or thick. In this section, we will focus on thin ancestry.

Definition 4.4.1. Consider a permutation $\sigma \in S_{n+1}$ and its reduced word. An ancestry of dimension 0 is called **thin** if, whenever $i_{k_0} = i_{k_1}$, it follows that $\varepsilon(k_0) = \varepsilon(k_1)$. Otherwise, the ancestry is called **thick**.

In the wiring diagram, an ancestry is thin when the inversions in the same row have the same sign.

Consequently, there are 2^{n-b} thin ancestries, where $b = \text{block}(\sigma)$. We assume for now that σ does not block, that is, $b = 0$.

We already know how to count the ancestries that satisfy $P(\varepsilon) = z$ for each dimension. Now, the task is to determine how many of these ancestries are thin. This number will be denoted by $\text{NL}_{\text{thin}}(z)$.

Let ε_0 be the empty preancestry and consider a fixed element $z \in \sigma \text{Quat}_{n+1}$. By definition, there are $\text{NL}_{\varepsilon_0}(z)$ ancestries ε corresponding to ε_0 and satisfying $P(\varepsilon) = z$.

From the previous chapter, it follows that the group $\mathcal{E}_n$ acts by automorphisms on SO_{n+1} , Spin_{n+1} and Cl_{n+1}^0 .

Consider $\sigma \in S_{n+1}$, $z_0 \in \sigma \text{Quat}_{n+1}$, and $Q_0 = \Pi(z_0) \in \text{B}_{n+1}^+$. For an element to belong to the same orbit as Q_0 , it must preserve the cycle structure. Consequently, the orbit $\mathcal{O}_{Q_0}$ of Q_0 under the action of $\mathcal{E}_n$ on SO_{n+1} has a cardinality of 2^{n-c+1} .

Regarding the action of $\mathcal{E}_n$ on σQuat_{n+1} , there are two possibilities for the size of the orbit $\mathcal{O}_{z_0}$. If there exists $E \in \mathcal{E}_n$ such that $z_0^E = -z_0$, we set $c_{\text{anti}}(z_0) = 1$; otherwise, $c_{\text{anti}}(z_0) = 0$.

If $\Re(z) = 0$, we can always find a $E \in \mathcal{E}_n$ such that $z^E = -z$, implying $c_{\text{anti}} = 1$. Conversely, if $\Re(z) \neq 0$, there are no $E \in \mathcal{E}_n$ such that $z^E = -z$, leading to $c_{\text{anti}} = 0$ (see [1]).

- If $c_{\text{anti}}(z_0) = 1$, the orbit is $\Pi^{-1}[\mathcal{O}_{Q_0}]$, with cardinality 2^{n-c+2} .
- If $c_{\text{anti}}(z_0) = 0$, the orbits $\mathcal{O}_{z_0}$ and $\mathcal{O}_{-z_0}$ are disjoint, each with cardinality 2^{n-c+1} , and their union is $\Pi^{-1}[\mathcal{O}_{Q_0}]$; in this case we say the orbits split.

For $z \in \text{Spin}_{n+1}$, define $\mathcal{E}_z \subseteq \mathcal{E}_n$ as the isotropy group of z , i.e.,

$$\mathcal{E}_z = \{E \in \mathcal{E}_n \mid z^E = z\}.$$

For thin ancestries, we focus on the group where $z = \sigma$, i.e., $\mathcal{E}_\sigma \leq \mathcal{E}_n$.

Fact 4.4.1. *Given $\sigma \in S_{n+1}$, let $c = \text{nc}(\sigma)$ be the number of cycles of σ . We have $|\mathcal{E}_\sigma| = 2^{\tilde{c}}$ where $\tilde{c} \in \mathbb{Z}$, $c - 2 \leq \tilde{c} \leq c$.*

The value of $c - \tilde{c} \in \{0, 1, 2\}$ can be deduced by following the proof. However, it does not appear to have a simple formula.

Definition 4.4.2. Let $\sigma^{\mathcal{E}_n}$ denote the orbit of σ under the action of $\mathcal{E}_n$:

$$\sigma^{\mathcal{E}_n} = \{\sigma^E, E \in \mathcal{E}_n\}.$$

The next result follows straightforwardly.

Fact 4.4.2. *Let $\tilde{c}$ be such that $|\mathcal{E}_\sigma| = 2^{\tilde{c}}$. For $z \in \sigma \text{Quat}_{n+1}$, we have*

$$\text{NL}_{\text{thin}}(z) = \begin{cases} 2^{n-\tilde{c}}, & z \in \sigma^{\mathcal{E}_n}, \\ 0, & z \notin \sigma^{\mathcal{E}_n}. \end{cases}$$

Furthermore, $|\sigma^{\mathcal{E}_n}| = 2^{\tilde{c}}$.

Example 4.4.1. Let $\sigma = (15)(26)(3)(4) = a_2a_1a_3a_2a_4a_3a_2a_1a_5a_4a_3a_2 \in S_6$. It follows that

$$\sigma = \frac{1}{2}(-\hat{a}_1 - \hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5) \in \tilde{B}_6^+,$$

then we have,

$$\sigma^{\mathcal{E}_n} = \left\{ \frac{\pm \hat{a}_1 \pm \hat{a}_2\hat{a}_3\hat{a}_4 \pm \hat{a}_5 \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5}{2} \right\},$$

where the signs must be such that there is an odd number of equal signs.

Let $\sigma = z$. Since the real part is $\Re(z) = 0$, there exists $E \in \mathcal{E}_n$ such that $z = \sigma^E = -\sigma = -z$. Therefore $c_{\text{anti}} = 1$.

The size of the orbit $\sigma^{\mathcal{E}_n}$ is given by

$$|\sigma^{\mathcal{E}_n}| = 2^{n-c+2} = 2^{\tilde{c}}.$$

Given $n = 5$ and $c = 4$:

$$|\sigma^{\mathcal{E}_n}| = 2^{5-4+2} = 8 = 2^3.$$

Hence, $\tilde{c} = 3$.

The number of ancestries of dimension 0 is given by:

$$\text{NL}_{\varepsilon_0}(z) = 2^{l-n-1} + 2^{\frac{l}{2}-1} \mathfrak{R}(z).$$

Given $l = 12, n = 5$ and $\mathfrak{R}(z) = 0$:

$$\text{NL}_{\varepsilon_0}(z) = 2^{12-5-1} = 2^6 = 64.$$

If $z \in \sigma^{\mathcal{E}_n}$, then

$$\text{NL}_{thin}(z) = 2^{n-\tilde{c}} = 2^{5-3} = 2^2 = 4$$

Therefore, for $z \in \sigma^{\mathcal{E}_n}$, there are 64 ancestries of dimension 0 labeled by ε_0 , and among these, 4 are classified as thin.

This demonstrates that while there are 64 possible ancestries of dimension 0 for the given permutation, only 4 of them are thin, highlighting the relative rarity of thin ancestries in this context. $\diamond$

Fact 4.4.3. Consider $\sigma \in S_{n+1}$ which does not block, and let ε_0 be the empty preancestry. If $l = \text{inv}(\sigma) > 2n + 2$ then for all $z \in \sigma \text{Quat}_{n+1}$ we have $\text{NL}_{\varepsilon_0}(z) > \text{NL}_{thin}(z)$.

5 Bruhat Cells

In this chapter, we introduce Bruhat cells in the matrix groups GL_{n+1} and SO_{n+1} . This will eventually lead us to the Bruhat cell in $Spin_{n+1}$, which is the central object of our study.

Following this introduction, we present key results from [4] concerning Bru_z , which will provide useful information for working with elements of Bru_z .

5.1 Bruhat Cells in $Spin_{n+1}$

First, we define the Bruhat decomposition for the sets GL_{n+1} and SO_{n+1} (see [13]).

Definition 5.1.1. The Bruhat decomposition of a matrix $M \in GL_{n+1}$ is given by the following:

For every $M \in GL_{n+1}$, there exists a unique permutation $\sigma \in S_{n+1}$ and matrices $U_0, U_1 \in Up_{n+1}$ such that

$$M = U_0 P_\sigma U_1.$$

Note that since $\sigma \in S_{n+1}$ is determined uniquely, the permutation matrix P_σ is as well. However, U_0 and $U_1 \in Up_{n+1}$ are not.

After decomposing each matrix in GL_{n+1} , we obtain the partition of the real general linear group into double cosets of Up_{n+1} ,

$$GL_{n+1} = \bigsqcup_{\sigma \in S_{n+1}} Up_{n+1} P_\sigma Up_{n+1}.$$

Definition 5.1.2. For $\sigma \in S_{n+1}$, define the Bruhat cell of σ in GL_{n+1} as

$$Bru_\sigma^{GL} = \{M \in GL_{n+1} \mid \exists U_0, U_1 \in Up_{n+1}, M = U_0 P_\sigma U_1\} \subset GL_{n+1}.$$

Bruhat cells can also be defined for other matrix groups. In our case, we consider SO_{n+1} and Lo_{n+1}^1 , the latter of which will be explored in detail later in this work.

Definition 5.1.3. For $\sigma \in S_{n+1}$, define the Bruhat cell of σ in SO_{n+1} as

$$\text{Bru}_\sigma^{\text{SO}} = \{Q \in SO_{n+1} \mid \exists U_0, U_1 \in \text{Up}_{n+1}, Q = U_0 P_\sigma U_1\} \subset SO_{n+1}.$$

The Bruhat decomposition of SO_{n+1} is known as Bruhat stratification with signs and is given by

$$SO_{n+1} = \bigsqcup_{P \in B_{n+1}^+} \text{Bru}_P, \quad \text{Bru}_P = (\text{Up}_{n+1}^+ P \text{Up}_{n+1}^+) \cap SO_{n+1}, \quad P \in B_{n+1}^+,$$

where $B_{n+1}^+ = B_{n+1} \cap SO_{n+1}$.

Recall the homomorphism introduced in Section 3.6:

$$\Pi : \text{Spin}_{n+1} \rightarrow SO_{n+1}.$$

Let $\text{Bru}_\sigma = \Pi^{-1}[\text{Bru}_\sigma^{\text{SO}}] \subset \text{Spin}_{n+1}$. This set has 2^{n+1} connected components, each one containing an element $z \in \sigma \text{Quat}_{n+1}$.

For $z \in \tilde{B}_{n+1}^+$, let Bru_z be the connected component of $\text{Bru}_{\Pi(z)}$ containing z , where $\sigma = \Pi(z)$. We have

$$\text{Bru}_\sigma = \bigsqcup_{z \in \sigma \text{Quat}_{n+1}} \text{Bru}_z.$$

The set Bru_z is a smooth contractible submanifold of Spin_{n+1} of dimension $l = \text{inv}(\sigma)$ and is referred to as a signed Bruhat cell. The Bruhat stratification of Spin_{n+1} is given by:

$$\text{Spin}_{n+1} = \bigsqcup_{z \in \tilde{B}_{n+1}^+} \text{Bru}_z.$$

The union of signed Bruhat cells Bru_z with $z \in \tilde{B}_{n+1}^+$ such that $\Pi(z) = P_\sigma \in SO_{n+1}$ is the unsigned Bruhat cell $\text{Bru}_\sigma \subset \text{Spin}_{n+1}$, where $\sigma \in S_{n+1}$. Each connected component of an unsigned Bruhat cell contains exactly one element $z \in \tilde{B}_{n+1}^+ \subset \text{Spin}_{n+1}$.

In [4], several important results regarding Bru_z are discussed, which are pertinent to our work. We outline these results without providing their proofs.

Fact 5.1.1. *Given reduced words $a_{i_1} \dots a_{i_k} < a_{i_1} \dots a_{i_k} a_j$ for consecutive permutations in S_{n+1} and signs $\varepsilon_1, \dots, \varepsilon_k, \varepsilon \in \{\pm 1\}$, set $z_1 = (a_{i_1})^{\varepsilon_1} \dots (a_{i_k})^{\varepsilon_k}, z_0 = z_1 (a_j)^\varepsilon \in \tilde{B}_{n+1}^+$. Given $q \in \text{Quat}_{n+1}$, the map*

$$\Phi : \text{Bru}_{qz_1} \times (0, \pi) \rightarrow \text{Bru}_{qz_0}$$

$$\Phi(z, \theta) = z\alpha(\varepsilon\theta),$$

is a diffeomorphism.

Fact 5.1.2. In the conditions of the Fact 5.1.1, i.e., with $z_1 = (\acute{a}_{i_1})^{\varepsilon_1} \dots (\acute{a}_{i_k})^{\varepsilon_k}$, $z_0 = z_1(\acute{a}_j)^{\varepsilon} \in \tilde{B}_{n+1}^+$ and $q \in \text{Quat}_{n+1}$, we have the inclusion $\overline{\text{Bru}_{qz_1}} \subset \overline{\text{Bru}_{qz_0}}$.

Fact 5.1.3. Given $q \in \text{Quat}_{n+1}$, a reduced word $a_{i_1} \dots a_{i_k} \in S_{n+1}$, and signs $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$, the map

$$\Psi : (0, \pi)^k \rightarrow \text{Bru}_{q(\acute{a}_{i_1})^{\varepsilon_1} \dots (\acute{a}_{i_k})^{\varepsilon_k}}$$

$$\Psi(\theta_1, \dots, \theta_k) = q\alpha_{i_1}(\varepsilon_1\theta_1) \dots \alpha_{i_k}(\varepsilon_k\theta_k)$$

is a diffeomorphism.

Fact 5.1.4. Consider $\sigma_0, \sigma_1 \in S_{n+1}$, $\sigma = \sigma_0\sigma_1$. If $\text{inv}(\sigma) = \text{inv}(\sigma_0) + \text{inv}(\sigma_1)$ then $\text{Bru}_{\acute{\sigma}_0} \text{Bru}_{\acute{\sigma}_1} = \text{Bru}_{\acute{\sigma}}$. Moreover, the map

$$\text{Bru}_{\acute{\sigma}_0} \times \text{Bru}_{\acute{\sigma}_1} \rightarrow \text{Bru}_{\acute{\sigma}}$$

$$(z_0, z_1) \rightarrow z_0 z_1$$

is a diffeomorphism.

These results provide a parameterization for $\text{Bru}_{\acute{\sigma}}$ in terms of α_{i_k} and θ_k

$$\text{Bru}_{\acute{\sigma}} = \{\alpha_{i_1}(\theta_1) \dots \alpha_{i_k}(\theta_k); \quad \theta_i \in (0, \pi)\},$$

where $\sigma = a_{i_1} \dots a_{i_k} \in S_{n+1}$.

Example 5.1.1. Consider $\eta = a_1 a_2 a_1 \in S_3$. Then

$$\text{Bru}_{\acute{\eta}} = \{\alpha_1(\theta_1)\alpha_2(\theta_2)\alpha_1(\theta_3); \quad \theta_i \in (0, \pi)\}.$$

Let $z \in \text{Bru}_{\acute{\eta}}$ with $\theta_1 = \theta_2 = \theta_3 = \frac{\pi}{2}$, then

$$z = \alpha_1\left(\frac{\pi}{2}\right)\alpha_2\left(\frac{\pi}{2}\right)\alpha_1\left(\frac{\pi}{2}\right) = \acute{a}_1\acute{a}_2\acute{a}_1 = \acute{\eta}.$$

◇

The strong Bruhat order, as defined in Definition 2.4.2, can be expressed as follows:

$$\sigma_0 \leq \sigma_1 \iff \text{Bru}_{\sigma_0} \subseteq \overline{\text{Bru}_{\sigma_1}}.$$

The results above also provide insights into the behavior of elements within Bruhat cells. Specifically:

- If $z \in \text{Bru}_{z_0}$ and $\sigma_0 = \Pi(z_0) \triangleleft \sigma_1 = \sigma_0 a_i$, then $z\alpha_i(\theta) \in \text{Bru}_{z_0 a_i}$ for $\theta \in (0, \pi)$.
- If $z \in \text{Bru}_{z_0}$ and $\sigma_1 \triangleleft \sigma_0 = \Pi(z_0) = \sigma_1 a_i$, then there exists $\theta = \Theta_i(z) \in (0, \pi)$ such that $z\alpha_i(-\theta) \in \text{Bru}_{z_1}$, where $z_0 = z_1 a_i$. Additionally, $z\alpha_i(\tilde{\theta}) \in \text{Bru}_{z_1}$ for all $\tilde{\theta} \in (-\theta, \pi - \theta)$.

A partial order on $\tilde{B}_{n+1}^+$, called the lifted Bruhat order, is defined as follows:

$$z_0 \leq z_1 \iff \text{Bru}_{z_0} \subseteq \overline{\text{Bru}_{z_1}}.$$

It is evident that $z_0 \leq z_1$ implies $\Pi(z_0) \leq \Pi(z_1)$, but the converse does not necessarily hold.

Notice that $z_0 \leq z_1$ and $\Pi(z_0) = \Pi(z_1)$ implies $z_0 = z_1$.

5.2

The Upper Set

Having defined the lifted Bruhat order, we can also establish a partial order on the set of ancestries for a given permutation.

Definition 5.2.1. Given two ancestries ε and $\tilde{\varepsilon}$, let (ϱ_k) and $(\tilde{\varrho}_k)$ be the sequences in Definition 4.2.1. We define a partial order on ancestries as follows:

$$\varepsilon \leq \tilde{\varepsilon} \iff (\forall k, \varrho_k \leq \tilde{\varrho}_k).$$

The fact that this is a partial order is straightforward.

If $\varepsilon \leq \tilde{\varepsilon}$ then, $\varrho_k \leq \tilde{\varrho}_k$, and thus $\Pi(\varrho_k) \leq \Pi(\tilde{\varrho}_k)$. Additionally, $\Pi(\varrho_l) = \eta = \Pi(\tilde{\varrho}_l)$, then $\varrho_l = \tilde{\varrho}_l$. Therefore, $P(\varepsilon) = \sigma \varrho_l^{-1} = \sigma(\dot{\eta} \varrho_l)^{-1} = \sigma(\dot{\eta} \tilde{\varrho}_l)^{-1} = P(\tilde{\varepsilon}) = \sigma \tilde{\varrho}_l^{-1} = P(\tilde{\varepsilon})$.

Thus, $\varepsilon \leq \tilde{\varepsilon}$ implies $P(\varepsilon) = P(\tilde{\varepsilon})$.

Definition 5.2.2. A set U of ancestries is an **upper set** if for any $\varepsilon \in U$ and $\varepsilon \leq \tilde{\varepsilon}$ it follows that $\tilde{\varepsilon} \in U$. The upper set generated by ε is denoted by $U_\varepsilon = \{\tilde{\varepsilon} \mid \varepsilon \leq \tilde{\varepsilon}\}$.

For an ancestry of dimension 0, there is no ancestry $\tilde{\varepsilon}$ such that $\tilde{\varepsilon} \leq \varepsilon$, meaning that ε is $\leq$ -maximal.

For an ancestry ε with $\dim(\varepsilon) > 0$, we define $\tilde{\varepsilon}$ setting $\tilde{\varepsilon}(k) = \text{sign}(\varepsilon(k))$. This ensures that $\tilde{\varepsilon} \leq \varepsilon$. In a wiring diagram, the ancestry $\tilde{\varepsilon}$ is obtained by replacing each diamond with a disk of the same color.

When $\dim(\varepsilon) = 1$, the upper set U_ε generated by ε includes ε itself and two ancestries of dimension 0. One is $\tilde{\varepsilon} = \text{sign}(\varepsilon)$, where the two diamonds are replaced by disks of the same color. The second is obtained from $\tilde{\varepsilon}$ performing a click in the region corresponding to ε .

Example 5.2.1. For $\sigma = [321] = a_1 a_2 a_1 \in S_3$, Figure 5.1 shows an ancestry of dimension 1, $\varepsilon = (-2, +1, +2)$, and the upper set generated by it.

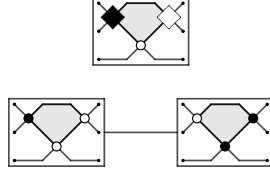


Figure 5.1: Upper set of $\varepsilon = (\blacklozenge \circ \diamond)$.

The upper set consists of two ancestries of dimension 0 and one of dimension 1: $U_\varepsilon = \{(\blacklozenge \circ \diamond), (\bullet \circ \diamond), (\circ \bullet \bullet)\}$. $\diamond$

In the figure, the upper set is depicted by an edge connecting two ancestries of dimension 0. This edge represents the ancestry of dimension 1, which is shown above the edge.

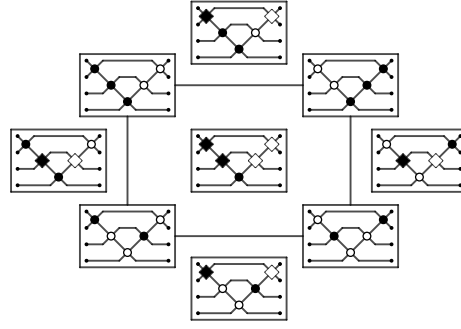
An ancestry, denoted by ε , of dimension 0 can be illustrated on a diagram for $\sigma \in S_{n+1}$ by indicating a sign at each intersection, as previously established. The edges are then constructed as follows:

When a click can be performed in a region, we generate an ancestry of dimension 1 represented by an edge, connecting two ancestries of dimension 0: one with the same signs as the ancestry of dimension 1, and the other with signs altered by the click.

For ancestries where $\dim(\varepsilon) > 1$, the description of the upper set U_ε generated by ε is more complex.

Let ε be an ancestry of dimension 2, type I. The set U_ε contains 4 elements of dimension 0, 4 elements of dimension 1 and one element of dimension 2, which is ε .

Example 5.2.2. Let $\sigma = [4231] = a_1 a_2 a_3 a_2 a_1 \in S_4$ and $\varepsilon = (\blacklozenge \blacklozenge \bullet \diamond \diamond)$ an ancestry of dimension 2. Figure 5.2 shows the upper set generated by ε .

Figure 5.2: Upper set of $\varepsilon = (\blacklozenge \blacklozenge \bullet \lozenge \lozenge)$.

Notice that the upper set contains exactly four ancestries of dimension 0, four ancestries of dimension 1, and one ancestry of dimension 2. $\diamond$

If ε is a type II ancestry of dimension 2, the structure of U_ε becomes more intricate.

5.3

Bruhat Cells in Lo_{n+1}^1

Following the Bruhat decomposition, we can partition Lo_{n+1}^1 into subsets BL_σ for $\sigma \in S_{n+1}$:

$$\text{BL}_\sigma = \{L \in \text{Lo}_{n+1}^1 \mid \exists U_0, U_1 \in \text{Up}_{n+1}, L = U_0 P_\sigma U_1\}.$$

Therefore,

$$\text{Lo}_{n+1}^1 = \bigsqcup_{\sigma \in S_{n+1}} \text{BL}_\sigma.$$

Let $\text{Up}_{n+1}^+ \subset \text{Up}_{n+1}$ be the group of upper triangular matrices with positive diagonal entries.

For a matrix $L \in \text{Lo}_{n+1}^1$, perform the usual QR factorization:

$$L = QR, \quad Q \in \text{SO}_{n+1}, \quad R \in \text{Up}_{n+1}^+.$$

Next, we focus on the orthogonal component of the matrix L , specifically $Q \in \text{SO}_{n+1}$. This component defines a smooth map:

$$\mathbf{Q}_{\text{SO}} : \text{Lo}_{n+1}^1 \rightarrow \text{SO}_{n+1}; \quad \mathbf{Q}_{\text{SO}}(L) = Q.$$

Lift this map to define

$$\mathbf{Q} : \text{Lo}_{n+1}^1 \rightarrow \text{Spin}_{n+1}, \quad \text{with } \mathbf{Q}(I) = 1.$$

The set $\mathcal{U}_1 = \mathbf{Q}[\mathrm{Lo}_{n+1}^1] \subset \mathrm{Spin}_{n+1}$ is an open contractible neighborhood of $1 \in \mathrm{Spin}_{n+1}$. We have $\mathcal{U}_1 = \dot{\eta} \mathrm{Bru}_{\dot{\eta}}$. In other words, $\mathcal{U}_1$ is a top-dimensional Bruhat cell for the basis described by $\dot{\eta}$, which is, up to signs, $e_{n+1}, e_n, \dots, e_2, e_1$.

The inverse map

$$\mathbf{L} = \mathbf{Q}^{-1} : \mathcal{U}_1 \rightarrow \mathrm{Lo}_{n+1}^1,$$

is also a smooth diffeomorphism and corresponds to the LU factorization.

Now we are ready to define the main object of study in this work: the set BL_z , which plays a central role in the analysis of the associated CW complexes. After introducing its definition, we will show that BL_z is diffeomorphic to the intersection of two Bruhat cells for different bases in Spin_{n+1} .

Definition 5.3.1. For $z \in \tilde{\mathbf{B}}_{n+1}^+$, define

$$\mathrm{BL}_z = \mathbf{Q}^{-1}[\mathrm{Bru}_z] = \mathbf{Q}^{-1}[\mathrm{Bru}_z \cap \dot{\eta} \mathrm{Bru}_{\dot{\eta}}] \subseteq \mathrm{Lo}_{n+1}^1.$$

Therefore, we can partition BL_{σ} into 2^{n+1} subsets which are both open and closed

$$\mathrm{BL}_{\sigma} = \bigsqcup_{\dot{\sigma} \in \mathrm{Quat}_{n+1}} \mathrm{BL}_z.$$

Recall that $\mathrm{Inv}(\sigma) = \{(i, j) \in \llbracket n+1 \rrbracket^2 \mid (i < j) \wedge (i^{\sigma} > j^{\sigma})\}$, and $\mathrm{Inv}(\eta\sigma) = \mathrm{Inv}(\eta) \setminus \mathrm{Inv}(\sigma)$.

Definition 5.3.2. Let $\sigma \in \mathbf{S}_{n+1}$, define

$$\mathrm{Lo}_{\sigma} = \{L \in \mathrm{Lo}_{n+1}^1 \mid i > j, L_{i,j} \neq 0 \rightarrow (j, i) \in \mathrm{Inv}(\sigma)\}.$$

Example 5.3.1. Let $\sigma = [312] \in \mathbf{S}_3$, $\mathrm{Inv}(\sigma) = \{(1, 2), (1, 3)\}$. Then,

$$\mathrm{Lo}_{\sigma} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

◇

Lemma 5.3.1. Consider $\sigma \in \mathbf{S}_{n+1}$. Then

(a) Lo_{σ} is a subgroup of Lo_{n+1}^1 ;

(b) The map

$$\phi : \mathrm{Lo}_{\sigma} \times \mathrm{Lo}_{\sigma\eta} \rightarrow \mathrm{Lo}_{n+1}^1$$

$$(L_0, L_1) \mapsto L_0 L_1$$

is a diffeomorphism.

Proof.

- (a) Let $L_\alpha, L_\beta \in \text{Lo}_\sigma$. Let us check that for $i_2 > i_0$, $(L_\alpha L_\beta)_{i_2, i_0} \neq 0$ implies $(i_0, i_2) \in \text{Inv}(\sigma)$.

We have,

$$(L_\alpha L_\beta)_{i_2, i_0} \neq 0,$$

which implies

$$\sum_{i_0 \leq i_1 \leq i_2} (L_\alpha)_{i_2, i_1} (L_\beta)_{i_1, i_0} \neq 0.$$

Thus, there exists i_1 such that

$$(L_\alpha)_{i_2, i_1} \neq 0, \quad (L_\beta)_{i_1, i_0} \neq 0.$$

Consider the following cases:

- If $i_1 = i_0$, then $(L_\alpha)_{i_2, i_0} \neq 0$, implying $(i_0, i_2) \in \text{Inv}(\sigma)$;
- If $i_1 = i_2$, then $(L_\beta)_{i_2, i_0} \neq 0$, implying $(i_0, i_2) \in \text{Inv}(\sigma)$;
- If $i_0 < i_1 < i_2$, then $(L_\alpha)_{i_2, i_1} \neq 0 \neq (L_\beta)_{i_1, i_0}$.

Thus, $(i_0, i_1), (i_1, i_2) \in \text{Inv}(\sigma)$, which implies $(i_0, i_2) \in \text{Inv}(\sigma)$.

Therefore, we conclude that $\text{Lo}_\sigma \leq \text{Lo}_{n+1}^1$.

- (b) Let us construct the inverse map.

Given $L \in \text{Lo}_{n+1}^1$, our aim is to find $L_0 \in \text{Lo}_\sigma$ and $L_1 \in \text{Lo}_{\sigma\eta}$.

We work inductively on the entries (i, j) . Proceed with $i = j + t$ where $i - j = t$ is increasing.

For $t = 1$, i.e. $i = j + 1$, we have $(j, i) \in \text{Inv}(\sigma)$ or $(j, i) \in \text{Inv}(\sigma\eta)$.

Therefore,

$$L_{i,j} = (L_0)_{i,j} + (L_1)_{i,j},$$

with either $(L_0)_{i,j} = 0$, or $(L_1)_{i,j} = 0$.

Inductive step: $t > 1$.

Assume that for all pairs (i, j) where $i = j + k$ and $k < t$, the entries $L_{i,j}$ can be decomposed as described. Now consider $i = j + t$.

We have:

$$\begin{aligned} L_{i,j} &= \sum_{j \leq k \leq i} (L_0)_{i,k} (L_1)_{k,j} \\ &= (L_0)_{i,j} (L_1)_{j,j} + \left(\sum_{j < k < i} (L_0)_{i,k} (L_1)_{k,j} \right) + (L_0)_{i,i} (L_1)_{i,j} \\ &= (L_0)_{i,j} + \left(\sum_{j < k < i} (L_0)_{i,k} (L_1)_{k,j} \right) + (L_1)_{i,j}. \end{aligned}$$

From the induction hypothesis, we have already dealt with

$$\sum_{j < k < i} (L_0)_{i,k} (L_1)_{k,j},$$

and either

$$(L_0)_{i,j} = 0, \text{ or } (L_1)_{i,j} = 0.$$

Therefore, we conclude that there exists a well-defined inverse map ϕ^{-1} .

Hence, ϕ is a diffeomorphism. $\square$

Let $\sigma_0 \in S_{n+1}$, $z_0 = \sigma_0 q_0 \in \tilde{B}_{n+1}^+$, $q_0 \in \text{Quat}_{n+1}$ and $Q_0 = \Pi(z_0) \in \text{SO}_{n+1}$.

The maps $\mathbf{Q}$ and $\mathbf{L}$ yield the following diffeomorphisms:

$$\mathcal{U}_{z_0} \approx z_0 \text{Lo}_{n+1}^1, \quad \text{Bru}_{z_0} \approx z_0 \text{Lo}_{\sigma_0^{-1}}.$$

Also, the map

$$\psi : z_0 \text{Lo}_{\sigma_0^{-1}} \times \text{Lo}_{\sigma_0^{-1}\eta} \rightarrow z_0 \text{Lo}_{n+1}^1, \quad \psi(z_0 L_a, L_b) = z_0 L_b L_a \quad (5-1)$$

is a diffeomorphism, as can be seen from $\text{Lo}_{n+1}^1 = \text{Lo}_{\sigma_0^{-1}} \text{Lo}_{\sigma_0^{-1}\eta} = \text{Lo}_{\sigma_0^{-1}\eta} \text{Lo}_{\sigma_0^{-1}}$.

Lemma 5.3.2. *Let $L_a \in \text{Lo}_{\sigma_0^{-1}}$, $L_b \in \text{Lo}_{\sigma_0^{-1}\eta}$ and $q \in \text{Quat}_{n+1}$. Then, $z_0 L_a \in \mathcal{U}_q$ if and only if $\psi(z_0 L_a, L_b) \in \mathcal{U}_q$.*

Proof. We have $\psi(z_0 L_a, L_b) = z_0 L_b L_a = \tilde{L}_b z_0 L_a$, and we want to show that

$$z_0 L_a \in \mathcal{U}_q \iff \tilde{L}_b z_0 L_a \in \mathcal{U}_q.$$

Consider that if $L_a = LU$, then $\tilde{L}_b z_0 L_a = \tilde{L}_b LU$. Therefore, $z_0 L_a \in \mathcal{U}_q$, then $\tilde{L}_b z_0 L_a \in \mathcal{U}_q$ as well. $\square$

Proposition 5.3.3. $\mathcal{U}_{z_0} \cap \mathcal{U}_q \approx (\text{Bru}_{z_0} \cap \mathcal{U}_q) \times \mathbb{R}^{n-l}$.

Proof. We have

$$\mathcal{U}_{z_0} \approx z_0 \text{Lo}_{n+1}^1, \quad \text{Bru}_{z_0} \approx z_0 \text{Lo}_{\sigma_0^{-1}}, \quad \mathbb{R}^{n-l} \approx \text{Lo}_{\sigma_0^{-1}\eta}.$$

Therefore,

$$\mathcal{U}_{z_0} \cap \mathcal{U}_q \subseteq z_0 \text{Lo}_{n+1}^1, \quad (\text{Bru}_{z_0} \cap \mathcal{U}_q) \times \mathbb{R}^{n-l} \subseteq z_0 \text{Lo}_{\sigma_0^{-1}} \times \text{Lo}_{\sigma_0^{-1}\eta}.$$

We know that ψ (as in equation (5-1)) is a diffeomorphism. Furthermore, by the previous lemma, we can apply ψ to obtain the desired local structure, completing the proof. $\square$

Recall that

$$\text{Bru}_{z_0} \cap \mathcal{U}_q = \text{Bru}_{q^{-1}z_0} \cap \mathcal{U}_1, \quad \mathbf{Q}^{-1}[\text{Bru}_{q^{-1}z_0} \cap \mathcal{U}_1] = \text{BL}_{q^{-1}z_0}.$$

Therefore, the set BL_z is diffeomorphic to $\text{Bru}_z \cap (\dot{\eta} \text{Bru}_{\dot{\eta}}) = \text{Bru}_z \cap \mathcal{U}_1$, the intersection of two Bruhat cells for different bases in Spin_{n+1} .

Example 5.3.2. Let $\eta = [321] = a_1 a_2 a_1 \in S_3$. From Chapter 3, we have

$$\dot{\eta} = \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}, \quad \dot{\eta} \text{Quat}_3 = \left\{ \frac{\pm 1 \pm \hat{a}_1 \hat{a}_2}{\sqrt{2}}, \frac{\pm \hat{a}_1 \pm \hat{a}_2}{\sqrt{2}} \right\},$$

with signs assigned arbitrarily. Then $|\dot{\eta} \text{Quat}_3| = 8$. We have

$$\text{Lo}_3^1 = \left\{ L = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

For a matrix $L \in \text{Lo}_3^1$ to be in BL_η it must satisfy the minor determinants conditions, so it follows that $\text{BL}_\eta = \{L \mid z \neq 0, z \neq xy\} \subset \text{Lo}_3^1$. After a computation we get

$$\begin{aligned} \text{BL}_{\frac{1-\hat{a}_1\hat{a}_2}{\sqrt{2}}} &= \{L \mid z > \max\{0, xy\}\}, & \text{BL}_{\frac{1+\hat{a}_1\hat{a}_2}{\sqrt{2}}} &= \{L \mid z < \min\{0, xy\}\}, \\ \text{BL}_{\frac{\hat{a}_1+\hat{a}_2}{\sqrt{2}}} &= \{L \mid x > 0, 0 < z < xy\}, & \text{BL}_{\frac{\hat{a}_1-\hat{a}_2}{\sqrt{2}}} &= \{L \mid x > 0, xy < z < 0\}, \\ \text{BL}_{\frac{-\hat{a}_1-\hat{a}_2}{\sqrt{2}}} &= \{L \mid x < 0, 0 < z < xy\}, & \text{BL}_{\frac{-\hat{a}_1+\hat{a}_2}{\sqrt{2}}} &= \{L \mid x < 0, xy < z < 0\}, \\ \text{BL}_{\frac{-1+\hat{a}_1\hat{a}_2}{\sqrt{2}}} &= \text{BL}_{\frac{-1-\hat{a}_1\hat{a}_2}{\sqrt{2}}} = \emptyset. \end{aligned}$$

◇

5.4

The Set of Totally Positive Matrices Pos_σ

In this section, we study how the positive matrices behave in the set BL_σ .

Recall $[[n+1]] = \{1, 2, \dots, n+1\}$. Let $[[n+1]]^{(k)}$ be the set of subsets $\mathbf{i} \subseteq [[n+1]]$, with $\text{card}(\mathbf{i}) = k$. For $\mathbf{i}_0, \mathbf{i}_1 \in [[n+1]]^{(k)}$, where $\mathbf{i}_j = \{i_{j1} < i_{j2} < \dots < i_{jk}\}$, write:

$$\mathbf{i}_0 \leq \mathbf{i}_1 \iff i_{11} \leq i_{01}, i_{12} \leq i_{02}, \dots, i_{1k} \leq i_{0k}.$$

Definition 5.4.1. A matrix $L \in \text{Lo}_{n+1}^1$ is **totally positive** if for all $k \in$

$[[n+1]]$ and for all indices $\mathbf{i}_0 \geq \mathbf{i}_1 \in [[n+1]]^{(k)}$,

$$\mathbf{i}_0 \geq \mathbf{i}_1 \implies \det(L_{\mathbf{i}_0, \mathbf{i}_1}) > 0.$$

Let $\text{Pos}_\eta \subset \text{Lo}_{n+1}^1$ be the set of totally positive matrices.

Let $\mathfrak{lo}_{n+1}^1$ denote the Lie algebra of Lo_{n+1}^1 , consisting of strictly lower triangular matrices. For $\mathfrak{l}_i \in \mathfrak{lo}_{n+1}^1$ let $\mathfrak{l}_i$ be the matrix whose only nonzero entry is $(\mathfrak{l}_i)_{(i+1, i)} = 1$.

Let λ_i be the corresponding one-parameter subgroup:

$$\lambda_i : \mathbb{R} \rightarrow \text{Lo}_{n+1}^1, \quad \lambda_i(t) = \exp(t\mathfrak{l}_i) = I + t\mathfrak{l}_i.$$

The group $\mathcal{E}_n = \{\pm 1\}^{[[n]]}$ acts by automorphisms on Lo_{n+1}^1 as follows:

$$(\lambda_i(t))^E = \lambda_i(E_i t).$$

This action modifies the entries of the matrices in Lo_{n+1}^1 according to the signs specified by $E \in \mathcal{E}_n$.

Hence, we have $\mathbf{Q}(L)^E = (\mathbf{Q}(L))^E$ for all $L \in \text{Lo}_{n+1}^1$ and $E \in \mathcal{E}_n$. Additionally, $\mathbf{L}(z^E) = (\mathbf{L}(z))^E$ for all $z \in \mathcal{U}_1$ and $E \in \mathcal{E}_n$. For $z \in \tilde{\text{B}}_{n+1}^+$ and $E \in \mathcal{E}_n$, we have $(\text{BL}_z)^E = \text{BL}_{z^E}$. In particular, the sets BL_z and BL_{z^E} are diffeomorphic via the map $L \mapsto L^E$.

To determine the homotopy type of BL_σ , we decompose σQuat_{n+1} into $\mathcal{E}_n$ -orbits. For each orbit, we select a representative z and determine the homotopy type of BL_z .

Example 5.4.1. For $\eta \in \text{S}_3$, the orbits of ηQuat_3 are:

$$\begin{aligned} \mathcal{O}_{\frac{1+\hat{a}_1\hat{a}_2}{\sqrt{2}}} &= \left\{ \frac{1 \pm \hat{a}_1\hat{a}_2}{\sqrt{2}} \right\}, & \mathcal{O}_{\frac{-1+\hat{a}_1\hat{a}_2}{\sqrt{2}}} &= \left\{ \frac{-1 \pm \hat{a}_1\hat{a}_2}{\sqrt{2}} \right\}, \\ \mathcal{O}_{\frac{\hat{a}_1+\hat{a}_2}{\sqrt{2}}} &= \left\{ \frac{\pm\hat{a}_1 \pm \hat{a}_2}{\sqrt{2}} \right\}. \end{aligned}$$

From Example 5.3.2, we can see that for $z \in \mathcal{O}_{\frac{1+\hat{a}_1\hat{a}_2}{\sqrt{2}}}$ the sets BL_z are diffeomorphic. The same holds for $z \in \mathcal{O}_{\frac{\hat{a}_1+\hat{a}_2}{\sqrt{2}}}$. For $z \in \mathcal{O}_{\frac{-1+\hat{a}_1\hat{a}_2}{\sqrt{2}}}$ the sets are empty. $\diamond$

For any reduced word $\eta = a_{i_1} a_{i_2} \dots a_{i_l}$ where $l = \text{inv}(\eta)$, the map

$$(0, \infty)^l \rightarrow \text{Pos}_\eta$$

$$(t_1, t_2, \dots, t_l) \mapsto \lambda_{i_1}(t_1) \lambda_{i_2}(t_2) \dots \lambda_{i_l}(t_l)$$

is a diffeomorphism.

In other words, a matrix $L \in \text{Lo}_{n+1}^1$ is totally positive if and only if there exist positive numbers $t_1, \dots, t_l$ such that

$$L = \lambda_{i_1}(t_1) \dots \lambda_{i_l}(t_l).$$

The set Pos_η of totally positive matrices is an open semigroup and contractible connected component of BL_η .

Moreover, the closure $\overline{\text{Pos}_\eta}$ has a stratification given by:

$$\overline{\text{Pos}_\eta} = \{L \in \text{Lo}_{n+1}^1 \mid \forall \mathbf{i}_0, \mathbf{i}_1, ((\mathbf{i}_0 \geq \mathbf{i}_1) \rightarrow (\det(L_{\mathbf{i}_0, \mathbf{i}_1}) \geq 0))\} = \bigsqcup_{\sigma \in \mathcal{S}_{n+1}} \text{Pos}_\sigma.$$

Here, $\text{Pos}_\sigma \subset \text{Lo}_{n+1}^1$ is a smooth manifold of dimension $\text{inv}(\sigma)$. If $\sigma = a_{i_1} \dots a_{i_l}$ is a reduced word, with $l = \text{inv}(\sigma)$, then the map

$$(0, \infty)^l \rightarrow \text{Pos}_\sigma, \quad (t_1, t_2, \dots, t_l) \mapsto \lambda_{i_1}(t_1) \lambda_{i_2}(t_2) \dots \lambda_{i_l}(t_l)$$

is a diffeomorphism. Similarly, if $\sigma_1 \triangleleft \sigma_0 = \sigma_1 a_{i_l}$, then the map

$$\text{Pos}_{\sigma_1} \times (0, \infty) \rightarrow \text{Pos}_{\sigma_0}, \quad (L, t_k) \mapsto L \lambda_{i_l}(t_l)$$

is a diffeomorphism.

In other words, we have $L \in \text{Pos}_\sigma$ if and only if there exist positive numbers $t_1, \dots, t_l$ such that

$$L = \lambda_{i_1}(t_1) \dots \lambda_{i_l}(t_l).$$

The set BL_σ is also a contractible connected component of BL_σ .

Note that different reduced words result in distinct diffeomorphisms, but they map to the same set Pos_σ .

Example 5.4.2. For $n = 2$ and

$$L(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix},$$

we have

$$\text{Pos}_{a_1 a_2} = \{L(x, y, 0) \mid x, y > 0\}, \quad \text{Pos}_{a_2 a_1} = \{L(x, y, xy) \mid x, y > 0\},$$

$$\text{Pos}_{a_1 a_2 a_1} = \text{Pos}_\eta = \{L(x, y, z) \mid x, y > 0; 0 < z < xy\}.$$

Therefore, from Example 5.3.2, $\text{BL}_{\eta} = \text{Pos}_{\eta}$. $\diamond$

As we can see, for any $\sigma \in S_{n+1}$, we have $\text{Pos}_{\sigma} \subseteq \text{BL}_{\sigma}$. However, as n increases and for most permutations σ , Pos_{σ} constitutes a small connected component of the much larger set BL_{σ} .

We now present several results from [4] that establish connections between the set of positive matrices Pos_{σ} and the set of interest Bru_z .

Fact 5.4.1. *Consider $\sigma \in S_{n+1}$. Then $\mathbf{Q}[\text{Pos}_{\sigma}] \subset \text{Bru}_{\sigma}$. Furthermore, if $\sigma \neq e$ then σ does not belong to $\mathbf{Q}[\text{Pos}_{\sigma}]$.*

Fact 5.4.2. *Consider $\sigma_{k-1} \triangleleft \sigma_k = \sigma_{k-1}a_{i_k} \in S_{n+1}$. Consider $z_{k-1} \in \text{Bru}_{\sigma_{k-1}}$ and $z_k \in \text{Bru}_{\sigma_k}$, $z_k = z_{k-1}\alpha_{i_k}(\theta_k)$, $\theta_k \in (0, \pi)$. If $z_k \in \mathbf{Q}[\text{Pos}_{\sigma}]$ then $z_{k-1} \in \mathbf{Q}[\text{Pos}_{\sigma_{k-1}}]$ and $z_{k-1}\alpha_{i_k}(\theta) \in \mathbf{Q}[\text{Pos}_{\sigma_k}]$ for all $\theta \in (0, \theta_k]$.*

Fact 5.4.3. *Let $\sigma = a_{i_1} \dots a_{i_k} \in S_{n+1}$ be a reduced word. Let $t_1, \dots, t_k \in \mathbb{R} \setminus \{0\}$; for $1 \leq i \leq k$, let $\varepsilon_i = \text{sign}(t_i) \in \{\pm 1\}$. Let*

$$L = \lambda_{i_1}(t_1) \dots \lambda_{i_k}(t_k), \quad z = (a_{i_1})^{\varepsilon_1} \dots (a_{i_k})^{\varepsilon_k} \in \tilde{\text{B}}_{n+1}^+,$$

then $L \in \mathbf{Q}^{-1}[\text{Bru}_z]$.

These results will be useful in the following chapters.

6

The Stratification BLS_ε

In this chapter, we study the stratification BLS_ε and its strata. First, we examine some examples of BLS_ε for ancestries of dimension 0. After that, we generalize the concept and explore some properties of the structure.

6.1

The Strata BLS_ε With $\dim(\varepsilon) = 0$

Consider a permutation $\sigma \in S_{n+1}$ and a reduced word $a_{i_1} \dots a_{i_l}$. For an ancestry ε with $\dim(\varepsilon) = 0$, define

$$\text{BLS}_\varepsilon = \{\lambda_{i_1}(t_1) \dots \lambda_{i_l}(t_l) \mid t_k \in \mathbb{R} \setminus \{0\}, \text{sign}(t_k) = \varepsilon_k\} \subset \text{BL}_\sigma. \quad (6-1)$$

From Fact 5.4.3, it follows that

$$\text{BLS}_\varepsilon \subseteq \text{BL}_z, \quad z = P(\varepsilon) = (\acute{a}_{i_1})^{\varepsilon(1)} \dots (\acute{a}_{i_l})^{\varepsilon(l)} \in \acute{\sigma} \text{Quat}_{n+1}.$$

The subsets $\text{BLS}_\varepsilon \subset \text{BL}_\sigma$ are open, and the union over all ancestries of dimension 0 is open and dense.

If ε is a thin ancestry, the corresponding subset BLS_ε is also labeled thin. Notice that $\varepsilon = (+1, +1, \dots, +1)$ is thin, with $P(\varepsilon) = \acute{\sigma}$ and $\text{BLS}_\varepsilon = \text{Pos}_\sigma \subseteq \text{BL}_{\acute{\sigma}}$, which is a contractible connected component.

In a more general scenario, for any thin ancestry ε , there exists a corresponding $E \in \mathcal{E}_n$ such that $\varepsilon(k) = (\acute{a}_{i_k})^E$ for all k . This leads to $P(\varepsilon) = \acute{\sigma}^E$ and $\text{BLS}_\varepsilon = (\text{Pos}_\sigma)^E$. Consequently, $\text{BLS}_\varepsilon \subseteq \text{BL}_{\acute{\sigma}^E}$ represents a contractible connected component. The set

$$\text{BL}_{z, \text{thick}} = \text{BL}_z \setminus \bigcup_{\varepsilon \text{ thin}} \text{BLS}_\varepsilon$$

is referred to as the thick part of BL_z .

Example 6.1.1. Let $\sigma = \eta = a_1 a_2 a_1 \in S_3$. Figure 4.1 shows the two possible preancestries. We reproduce the figure below for clarity.



Figure 6.1: Preancestries of dimension 0 and 1, respectively, $\varepsilon_0 = (0, 0, 0)$ and $\varepsilon_0 = (-2, 0, +2)$.

The eight ancestries of dimension 0 are $(\pm 1, \pm 1, \pm 1)$; the two ancestries of dimension 1 are $(-2, \pm 1, +2)$.

From Example 5.3.2, we have:

$$\text{Lo}_3^1 = \left\{ L = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}, \quad \text{BL}_\eta = \{L \mid z \neq 0, z \neq xy\} \subset \text{Lo}_3^1.$$

A computation yields:

$$\lambda_1(t_1)\lambda_2(t_2)\lambda_1(t_3) = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix}.$$

If $L \in \text{BLS}_\varepsilon$, we write $L = \lambda_1(t_1)\lambda_2(t_2)\lambda_1(t_3)$, where $\text{sign}(t_k) = \varepsilon_k$.

For $\varepsilon = (+1, +1, +1)$, we have $x = t_1 + t_3 > 0$, $y = t_2 > 0$, $z = t_2 t_3 > 0$, with $xy = (t_1 + t_3)t_2 = t_1 t_2 + t_2 t_3$. Thus, $0 < z < xy$. Consequently,

$$\text{BLS}_{(+1, +1, +1)} = \{L \mid x > 0, 0 < z < xy\} = \text{BL}_{\frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}}.$$

Through similar computations, we obtain

$$\text{BL}_{\frac{-\hat{a}_1 + \hat{a}_2}{\sqrt{2}}} = \{L \mid x < 0, xy < z < 0\}, \quad \text{BL}_{\frac{\hat{a}_1 - \hat{a}_2}{\sqrt{2}}} = \{L \mid x > 0, xy < z < 0\},$$

$$\text{BL}_{\frac{-\hat{a}_1 - \hat{a}_2}{\sqrt{2}}} = \{L \mid x < 0, 0 < z < xy\}, \quad \text{BL}_{\frac{-1 + \hat{a}_1 \hat{a}_2}{\sqrt{2}}} = \text{BL}_{\frac{-1 - \hat{a}_1 \hat{a}_2}{\sqrt{2}}} = \emptyset.$$

Additionally,

$$\text{BLS}_{(-1, +1, +1)} = \{L \mid z > \max\{0, xy\}, y > 0\},$$

$$\text{BLS}_{(+1, -1, -1)} = \{L \mid z > \max\{0, xy\}, y < 0\}.$$

Let $z_0 = \frac{1 - \hat{a}_1 \hat{a}_2}{\sqrt{2}}$. Note that $P(-1, +1, +1) = P(+1, -1, -1) = P(-2, +1, +2) = z_0$. These are the only ancestries ε for which $P(\varepsilon) = z_0$. As we will see later,

$$\text{BLS}_{(-2, +1, +2)} = \{L \mid y = 0, z > 0\}.$$

Then,

$$\text{BL}_{z_0} = \text{BLS}_{(-1,+1,+1)} \sqcup \text{BLS}_{(-2,+1,+2)} \sqcup \text{BLS}_{(+1,-1,-1)},$$

$$\text{BL}_{\frac{1-\hat{a}_1\hat{a}_2}{\sqrt{2}}} = \{L \mid z > \max\{0, xy\}\}.$$

A similar decomposition applies to

$$\text{BL}_{\frac{1+\hat{a}_1\hat{a}_2}{\sqrt{2}}} = \{L \mid z < \min\{0, xy\}\}.$$

Notice that the six non empty sets are contractible.

Recall that η can also be expressed as the reduced word $\eta = a_2 a_1 a_2$. The interpretation of the ancestry **differs** depending on the reduced word used, but the homotopy type remains the same. $\diamond$

Referencing [8], it is established that BL_η comprises $3 \cdot 2^n$ connected components. Additionally, [1] provides an efficient enumeration of these connected components.

Fact 6.1.1. *For $n \geq 5$, the $3 \cdot 2^n$ connected components of BL_η are*

$$\text{Pos}_\eta^E, E \in \mathcal{E}_n, \quad \text{BL}_{z, \text{thick}}, z \in \dot{\eta} \text{Quat}_{n+1}.$$

The first list are the 2^n thin connected components; the second are the 2^{n+1} thick connected components.

6.2

The Stratification BLS_ε

In this section, we show how to determine the ancestry for a given $L \in \text{BL}_\sigma$, where $\sigma = a_{i_1} \dots a_{i_l} \in \text{S}_{n+1}$ is a fixed word, with $l = \text{inv}(\sigma)$. Subsequently, we present the stratifications of BL_σ and BL_z in terms of BLS_ε .

Let $\tilde{z}_l = \mathbf{Q}(L)$. Choose $q_l \in \text{Quat}_{n+1}$ such that $z_l = \tilde{z}_l q_l \in \text{Bru}_\sigma$. Define the sequence recursively as follows:

$$\sigma_0 = 1, \quad \sigma_1 = a_{i_1}, \quad \sigma_k = \sigma_{k-1} a_{i_k} = a_{i_1} \dots a_{i_k},$$

so that $\sigma = \sigma_l$. According to Theorem 5.1.1, we have well-defined sequences $(\theta_k)_{0 \leq k \leq l}$ and $(z_k)_{0 \leq k \leq l}$, with $z_0 = 1 \in \text{Spin}_{n+1}$, such that

$$z_k = z_{k-1} \alpha_{i_k}(\theta_k) \in \text{Bru}_{\sigma_k}, \quad \theta_k \in (0, \pi).$$

Choose $(\varrho_k) \in \tilde{\text{B}}_{n+1}^+$ such that $z_k \in \dot{\eta} \text{Bru}_{\varrho_k}$. The sequence (ϱ_k) represents the desired ancestry. The corresponding preancestry is given by (ρ_k) , where $\rho_k = \Pi_{\tilde{\text{B}}_{n+1}^+, \text{S}_{n+1}}(\varrho_k)$ so that $z_k \in \dot{\eta} \text{Bru}_{\rho_k}$.

Given an ancestry ε , define $\text{BLS}_\varepsilon \subset \text{BL}_\sigma$ as the set of matrices L with ancestry ε . In Equation 6-1, we explicitly define BLS_ε for $\dim(\varepsilon) = 0$.

Now that we have identified the desired sequences (ρ_k) and (ϱ_k) , let us verify that they indeed represent preancestry and ancestry. We have $\rho_0 = \eta$ and $\varrho_0 = \dot{\eta}$, since $z_0 = 1 \in \dot{\eta} \text{Bru}_{\dot{\eta}} \subset \dot{\eta} \text{Bru}_\eta$.

- If $z_{k-1} \in \dot{\eta} \text{Bru}_{\varrho_{k-1}}$ and $\rho_{k-1} < \rho_{k-1} a_{i_k}$, then $z_{k-1} \alpha_{i_k}(\theta) \in \dot{\eta} \text{Bru}_{\varrho_{k-1} \hat{a}_{i_k}}$ for all $\theta \in (0, \pi)$. This implies $\varrho_k = \varrho_{k-1} \hat{a}_{i_k}$ and $\rho_k = \rho_{k-1} a_{i_k}$.
- If $z_{k-1} \in \dot{\eta} \text{Bru}_{\varrho_{k-1}}$ and $\rho_{k-1} > \rho_{k-1} a_{i_k}$, then $z_{k-1} \alpha_{i_k}(\theta)$ belongs to one of the following three sets, for $\theta \in (0, \pi)$:

$$\dot{\eta} \text{Bru}_{\varrho_{k-1}}, \quad \dot{\eta} \text{Bru}_{\varrho_{k-1} \hat{a}_{i_k}}, \quad \dot{\eta} \text{Bru}_{\varrho_{k-1} \hat{a}_{i_k}}.$$

This implies ϱ_k can be one of

$$\varrho_{k-1}, \quad \varrho_{k-1} \hat{a}_{i_k}, \quad \varrho_{k-1} \hat{a}_{i_k}.$$

Finally, $\tilde{z}_k \in \mathcal{U}_1$ implies $z_k \in \text{Bru}_\eta$ and $\varrho_l \in \dot{\eta} \text{Quat}_{n+1}$.

Therefore, we conclude that

$$\text{BL}_\sigma = \bigsqcup_{\varepsilon} \text{BLS}_\varepsilon, \quad \text{BL}_z = \bigsqcup_{P(\varepsilon)=z} \text{BLS}_\varepsilon,$$

where ε varies over the ancestries.

In Definition 4.3.1 of Chapter 4, we define $\text{NL}_{\varepsilon_0}(z)$ as the cardinality of the set of ancestries ε associated with a preancestry ε_0 such that $P(\varepsilon) = z$. It follows from the definition of BLS_ε and the equation on the right above that $\text{BLS}_\varepsilon \subset \text{BL}_{P(\varepsilon)}$. Thus, for any preancestry ε_0 and any $z \in \dot{\sigma} \text{Quat}_{n+1}$, we have $\text{NL}_{\varepsilon_0}(z) = N_{\varepsilon_0}(z)$, where $N_{\varepsilon_0}(z)$ is the number of ancestries ε for which $\text{BLS}_\varepsilon \subset \text{BL}_z$.

Therefore, we have Theorem 4 from [1]:

Fact 6.2.1. *Consider a permutation $\sigma \in S_{n+1}$, a reduced word and a preancestry ε_0 . Let $z_0 \in \dot{\sigma} \text{Quat}_{n+1}$ be such that $\Re(z_0) > 0$.*

For any $z = qz_0 \in \dot{\sigma} \text{Quat}_{n+1}$, we have:

$$N_{\varepsilon_0}(z) - N_{\varepsilon_0}(-z) = 2^{\frac{l-2d}{2}} \Re(z);$$

$$N_{\varepsilon_0}(z) + N_{\varepsilon_0}(-z) = \begin{cases} 2^{l-2d+1} / |H_{\varepsilon_0}|, & q \in H_{\varepsilon_0}, \\ 0, & q \notin H_{\varepsilon_0}. \end{cases}$$

We can also use $\xi(k)$ to provide information about the size of θ_k . Consider $z_{k-1} \in \dot{\eta} \text{Bru}_{\varrho_{k-1}}$. The following cases arise:

1. If $\rho_{k-1} < \rho_{k-1}a_{i_k}$: for all $\theta \in (0, \pi)$, we have $z_{k-1}\alpha_{i_k}(\theta) \in \dot{\eta} \text{Bru}_{\varrho_{k-1}\hat{a}_{i_k}}$. In this case, we set $\xi(k) = 1$.
2. If $\rho_{k-1} > \rho_{k-1}a_{i_k}$: there exists a unique $\theta_\bullet \in (0, \pi)$ such that $z_{k-1}\alpha_{i_k}(\theta_\bullet) \in \dot{\eta} \text{Bru}_{\varrho_{k-1}\hat{a}_{i_k}}$. We then consider the following sub-cases based on the value of θ_k :
 - If $\theta_k < \theta_\bullet$: we have $z_k \in \dot{\eta} \text{Bru}_{\varrho_{k-1}}$, $\varrho_k = \varrho_{k-1}$ and $\xi(k) = 0$;
 - If $\theta_k > \theta_\bullet$: we have $z_k \in \dot{\eta} \text{Bru}_{\varrho_{k-1}\hat{a}_{i_k}}$, $\varrho_k = \varrho_{k-1}\hat{a}_{i_k}$ and $\xi(k) = 2$;
 - If $\theta_k = \theta_\bullet$: we have $z_k \in \dot{\eta} \text{Bru}_{\varrho_{k-1}\hat{a}_{i_k}}$, $\varrho_k = \varrho_{k-1}\hat{a}_{i_k}$ and $\xi(k) = 1$.

In summary, $\xi(k)$ provides the following information about θ_k :

- $\xi(k) = 0$ means that θ_k is small;
- $\xi(k) = 2$ means that θ_k is large;
- $\xi(k) = 1$ means that θ_k is just right.

Let us introduce some additional notation. Define

$$\mathcal{U}_1^\diamond = \bigsqcup_{\sigma \in \mathbb{S}_{n+1}} \dot{\eta} \text{Bru}_\sigma, \quad \mathcal{U}_1 \subset \mathcal{U}_1^\diamond \subset \overline{\mathcal{U}_1} \subset \text{Spin}_{n+1}.$$

The set $\mathcal{U}_1^\diamond$ is a fundamental domain for the action of Quat_{n+1} on Spin_{n+1} . Given any $z \in \text{Spin}_{n+1}$, there exists a unique $q \in \text{Quat}_{n+1}$ such that $zq \in \mathcal{U}_1^\diamond$. For each k , write $z_k = \tilde{z}_k q_k$ with $\tilde{z}_k \in \mathcal{U}_1^\diamond$ and $q_k \in \text{Quat}_{n+1}$. Consequently, we have $\tilde{z}_k \in \dot{\eta} \text{Bru}_{\rho_k}$.

The following results are the Lemmas 12.1 and 12.2 in [1].

Fact 6.2.2. *There exist unique $\tilde{\theta}_k \in (-\pi, 0) \cup (0, \pi)$ such that $\tilde{z}_k = \tilde{z}_{k-1}\alpha_{i_k}(\tilde{\theta}_k)$. Furthermore, for $s = [\hat{a}_{i_k}, q_{k-1}] \in \{\pm 1\}$ we have $\tilde{\theta}_k = s\theta_k$ or $\tilde{\theta}_k = s(\theta_k - \pi)$. In the first case, we have $q_k = q_{k-1}$; in the second case, $q_k = q_{k-1}\hat{a}_{i_k}$.*

We have already provided the interpretation of $\xi(k)$. Now, we are prepared to explain the meaning of $\varepsilon(k)$.

Fact 6.2.3. *We have $\text{sign}(\varepsilon(k)) = \text{sign}(\tilde{\theta}_k)$. Also, $\varepsilon(k) = -2$ if and only if $\rho_k < \rho_{k-1}$; $\varepsilon(k) = +2$ if and only if $\rho_k > \rho_{k-1}$.*

The above construction can be viewed as an extension of the method described in Section 6.1, which is based on the functions λ_i . This extension is necessary for cases where the construction in Lo_{n+1}^1 is not feasible. Since Lo_{n+1}^1

is unsuitable, we instead operate within the compact group Spin_{n+1} (or SO_{n+1}), using the functions α_i in place of λ_i and making the necessary adaptations.

Let us examine a step-by-step example to clarify.

Example 6.2.1. Let us consider $\sigma = \eta = a_1 a_2 a_1 \in S_3$, and

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

If $L_0 = \lambda_1(t_1)\lambda_2(t_2)\lambda_1(t_3)$, then

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t_1 + t_3 & 1 & 0 \\ t_2 t_3 & t_2 & 1 \end{pmatrix},$$

which implies $t_2 = 0$ and $t_2 t_3 = 1$. This is a contradiction. Thus, $L_0 \notin \text{BLS}_\varepsilon$ for any ε with $\dim(\varepsilon) = 0$. Next, applying the Gram-Schmidt process to L_0 yields

$$\Pi(\tilde{z}_3) = \mathbf{Q}(L_0) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Denote $\Pi(\tilde{z}_3)$ simply as $\tilde{z}_3$. We have $\tilde{z}_3 = \alpha_1(\tilde{\theta}_1)\alpha_2(\tilde{\theta}_2)\alpha_1(\tilde{\theta}_3)$. A computation yields that $\tilde{z}_1 = \alpha_1(-\frac{\pi}{2})$, $\tilde{z}_2 = \alpha_1(-\frac{\pi}{2})\alpha_2(\frac{\pi}{4})$ and $\tilde{z}_3 = \alpha_1(-\frac{\pi}{2})\alpha_2(\frac{\pi}{4})\alpha_1(\frac{\pi}{2})$, with $\rho_1 = \rho_2 = a_1 a_2$.

From the previous result, we already know the signs of the ancestry. Additionally, we know that in this case, the dimension of the ancestry must be 1, so $\varepsilon = (-2, +1, +2)$ and $L_0 \in \text{BLS}_{(-2, +1, +2)}$. $\diamond$

Now, we present some results from [1] that demonstrate the well-behaved nature of BLS_ε . More precisely, the results show that BLS_ε is a smooth submanifold diffeomorphic to $\mathbb{R}^{l-d}$. Furthermore, the union of all BLS_ε is an open subset of the larger space.

Fact 6.2.4. Consider a permutation and a reduced word $\sigma = a_{i_1} \dots a_{i_l} \in S_{n+1}$ and an ancestry ε . The subset $\text{BLS}_\varepsilon \subseteq \text{BL}_\sigma$ is a smooth submanifold of codimension $d = \dim(\varepsilon)$.

Fact 6.2.5. Consider a permutation and a reduced word $\sigma = a_{i_1} \dots a_{i_l} \in S_{n+1}$ and an ancestry ε with $d = \dim(\varepsilon)$. The smooth submanifold $\text{BLS}_\varepsilon \subset \text{BL}_\sigma$ is diffeomorphic to $\mathbb{R}^{l-d}$, where $l = \text{inv}(\sigma)$.

Fact 6.2.6. Let $\varepsilon, \tilde{\varepsilon}$ be ancestries. If $\text{BLS}_\varepsilon \cap \overline{\text{BLS}_{\tilde{\varepsilon}}} \neq \emptyset$ then $\varepsilon \preceq \tilde{\varepsilon}$.

Observe that in the previous result, we do not assert equivalence, nor do we state that any of the conditions above imply $\text{BLS}_\varepsilon \subseteq \overline{\text{BLS}_\varepsilon}$.

Fact 6.2.7. *If U_ε is an upper set of ancestries, then*

$$\bigcup_{\varepsilon \in U_\varepsilon} \text{BLS}_\varepsilon \subseteq \text{BL}_\sigma$$

is an open subset.

7

The CW Complex

In this chapter, for $\sigma \in S_{n+1}$ we introduce the CW complex BLC_σ associated with BL_σ . We then examine the Euler characteristic of BL_z , and investigate the glueing maps of the CW complexes. Finally, we present the homotopy type of BL_σ for $n \leq 4$.

7.1

The CW Complex BLC_σ

The concept behind the CW complex BLC_σ is that it behaves as a dual cell structure to the stratification. This type of construction, particularly under more favorable conditions, should be familiar with the Poincaré duality. As we have seen, we have sufficient conditions to implement a similar construction in our context.

Consider $\mathbb{S}_r^{k-1}$ and $\mathbb{D}_r^k$ as follow:

$$\mathbb{S}_r^{k-1} = \{v \in \mathbb{R}^k \mid |v| = r\}, \quad \mathbb{D}_r^k = \{v \in \mathbb{R}^k \mid |v| \leq r\}.$$

For a CW complex X , let $X^{[j]} \subseteq X$ denote the j -dimensional skeleton, which is the union of all cells of dimension at most j .

The following result is Lemma 14.1 from [1] and is a key concept concerning smooth manifolds, essential for the proof of Theorem 2 also in [1], which we will soon present.

Fact 7.1.1. *Let $M_0 \subset M_1$ be smooth manifolds of dimension l . Assume that $N_1 = M_1 \setminus M_0 \subset M_1$ is a smooth submanifold of codimension k , $0 < k \leq l$, and that N_1 is diffeomorphic to $\mathbb{R}^{l-k}$. Assume that X_0 is a finite CW complex and that $i_0 : X_0 \rightarrow M_0$ is a homotopy equivalence.*

There exists a map $\beta : \mathbb{S}^{k-1} \rightarrow X_0^{[k-1]}$ with the following properties. Let X_1 be obtained from X_0 by attaching a cell C_1 of dimension k with glueing map β . There exists a map $i_1 : X_1 \rightarrow M_1$ with $i_1|_{X_0} = i_0$ such that $i_1 : X_1 \rightarrow M_1$ is a homotopy equivalence.

Observe that since $M_0 \subset M_1$ is a submanifold of codimension 0, it follows that M_0 is an open subset of M_1 . Consequently, the subset $N_1 \subset M_1$ is closed. Additionally, if $k < l$ it follows that M_1 is not compact. The maps i_0 and i_1

can often be taken as inclusions in many examples, but this is not a strict requirement.

Now, we proceed to present Theorem 2 from [1] and its proof:

Fact 7.1.2. *For $\sigma \in S_{n+1}$, there exists a finite CW complex BLC_σ and a continuous map $i_\sigma : \text{BLC}_\sigma \rightarrow \text{BL}_\sigma$ with the following properties:*

1. *The map i_σ is a homotopy equivalence.*
2. *The cells BLC_ε of BLC_σ are labeled by ancestries ε . For each ancestry ε of dimension d , the cell BLC_ε has dimension d .*

Proof. Ancestries of dimension 0 are the maximal elements under the partial order $\preceq$. Let $\text{BL}_{\sigma;0} \subseteq \text{BL}_\sigma$ be the union of the open, disjoint, and contractible sets BLS_ε for ε an ancestry of dimension 0. The set $\text{BL}_{\sigma;0}$ is homotopically equivalent to a finite set with one vertex per ancestry, which is of course a CW complex of dimension 0. This is the basis of a recursive construction.

We can list the set of ancestries of positive dimension as $(\varepsilon_i)_{1 \leq i \leq N_\varepsilon}$ in such a way that $\varepsilon_j \preceq \varepsilon_i$ implies $j \geq i$. Define recursively the subsets $\text{BL}_{\sigma;i} = \text{BL}_{\sigma;i-1} \cup \text{BLS}_{\varepsilon_i} \subseteq \text{BL}_\sigma$. The family of sets $\text{BL}_{\sigma;i}$ defines a filtration:

$$\text{BL}_{\sigma;0} \subset \text{BL}_{\sigma;1} \subset \dots \subset \text{BL}_{\sigma;N_\varepsilon-1} \subset \text{BL}_{\sigma;N_\varepsilon} = \text{BL}_\sigma.$$

The partial order $\preceq$ and Fact 6.2.6 guarantee that $\text{BL}_{\sigma;i-1} \subset \text{BL}_{\sigma;i}$ is an open subset. Fact 6.2.4 tells us that $\text{BLS}_{\varepsilon_i} = \text{BL}_{\sigma;i} \setminus \text{BL}_{\sigma;i-1}$ is a smooth submanifold of codimension $d = \dim(\varepsilon_i)$ and Fact 6.2.5 tell us that $\text{BLS}_{\varepsilon_i}$ is diffeomorphic to $\mathbb{R}^{l-d}$. Notice that $\text{BLS}_{\varepsilon_i} \subset \text{BL}_{\sigma;i}$ is a closed subset. We may therefore apply Fact 7.1.1 to the pair $M_0 = \text{BL}_{\sigma;i-1} \subset \text{BL}_{\sigma;i} = M_1$, completing the recursive construction and the proof. $\square$

The proof of Fact 7.1.1, see [1], and Fact 7.1.2 provides us with instructions for the actual construction of the CW complex BLC_σ and the map i_σ . However, this construction of the CW complex and the glueing maps is not as straightforward as one might hope.

7.2

The Euler Characteristic

Fact 7.1.2 provides information about the CW complexes, while the following result from [1] offer a formula for the Euler characteristic.

Fact 7.2.1. *For $\sigma \in S_{n+1}$ and $z \in \sigma \text{Quat}_{n+1}$, we have*

$$\chi(\text{BL}_z) = \sum_{\varepsilon_0} (-1)^{\dim(\varepsilon_0)} N_{\varepsilon_0}(z).$$

The summation is taken over all preancestries ε_0 .

Fact 7.2.2. *Let $z_0 \in \dot{\eta} \text{Quat}_{n+1}$ be such that $\Re(z_0) > 0$. We have that $\chi(\text{BL}_{z_0})$ is odd and $\chi(\text{BL}_{-z_0})$ is even.*

Fact 7.2.3. *Consider $n \geq 5$ and $z_0 \in \dot{\eta} \text{Quat}_{n+1}$ with $\Re(z_0) > 0$. Then $\text{BL}_{-z_0, \text{thick}}$ is non empty, connected and its Euler characteristic $\chi(\text{BL}_{-z_0, \text{thick}})$ is even.*

Example 7.2.1. For $n = 5$, BLC_{-z_0} is connected and has: 480 vertices, 1120 cells of dimension 1, 864 cells of dimension 2, 228 cells of dimension 3, 6 cells of dimension 4 and no cells of higher codimension. It follows that $\chi(\text{BL}_{z_0}) = 480 - 1120 + 864 - 228 + 6 = 2$. In particular, BL_{-z_0} is not contractible. $\diamond$

7.3

The glueing Maps

The glueing maps for the CW complexes BLC_σ present challenges. To gain a better understanding, we examine several results from [1] that offer valuable insights.

In general, for an upper set U of ancestries, define

$$\text{BLS}_U = \bigcup_{\varepsilon \in U} \text{BLS}_\varepsilon \subseteq \text{BL}_\sigma, \quad \text{BLC}_U = \bigcup_{\varepsilon \in U} \text{BLC}_\varepsilon \subseteq \text{BLC}_\sigma.$$

According to Fact 6.2.7, $\bigcup_{\varepsilon \in U} \text{BLS}_\varepsilon = \text{BLS}_U \subseteq \text{BL}_\sigma$ is an open subset.

Fact 7.3.1. *Let U be an upper set of ancestries. The subset $\text{BLC}_U \subseteq \text{BLC}_\sigma$ is closed and a CW complex. The restriction $i_\sigma|_{\text{BLC}_U} : \text{BLC}_U \rightarrow \text{BLS}_U$ is a homotopy equivalence.*

Let $U_\varepsilon^* = U_\varepsilon \setminus \{\varepsilon\}$. It follows directly from the previous result that the image of the glueing map for BLC_ε is contained in $\text{BLC}_{U_\varepsilon^*}$.

Consider an ancestry ε with $\dim(\varepsilon) > 0$. Define two non empty subsets $U_\varepsilon^\pm \subset U_\varepsilon^*$. Denote the largest index k such that $\varepsilon(k) = -2$ by $k_\bullet$. It holds that $\varrho_{k_\bullet} = \varrho_{k_\bullet-1} \hat{a}_{i_k}$. Define $\varrho_{k_\bullet}^- = \varrho_{k_\bullet-1}$ and $\varrho_{k_\bullet}^+ = \varrho_{k_\bullet-1} \hat{a}_{i_k}$. For $\tilde{\varepsilon} \in U_\varepsilon^*$, let $(\tilde{\varrho}_k)_{0 \leq k \leq l}$ be defined as the standard. Then:

$$\tilde{\varepsilon} \in U_\varepsilon^\pm \iff \tilde{\varrho}_{k_\bullet} = \varrho_{k_\bullet}^\pm \text{ and } \tilde{\varrho}_k = \varrho_k \text{ for } 0 \leq k < k_\bullet.$$

These sets $U_\varepsilon^\pm$ are disjoint.

Example 7.3.1. For ε with $\dim(\varepsilon) = 1$, U_ε consists of three elements: ε itself and two ancestries with dimension 0. Consequently, the sets $U_\varepsilon^\pm$ each contain one element. Figure 5.1 shows an example.

For ε with $\dim(\varepsilon) = 2$ of type I, the sets $U_\varepsilon^\pm$ each contain one element. In an upcoming chapter, we explore several examples illustrating these upper sets, such as Figure 10.4, where U_ε^+ corresponds to the edge on the left and U_ε^- to the one on the right. $\diamond$

Following from that, we define the sets near BLS_ε as:

$$\text{BLS}_\varepsilon^\pm = \bigcup_{\tilde{\varepsilon} \in U_\varepsilon^\pm} \text{BLS}_{\tilde{\varepsilon}}.$$

Fact 7.3.2. Let ε be an ancestry of dimension $d = \dim(\varepsilon) > 0$. If W is a sufficiently thin open tubular neighborhood of BLS_ε then $(\text{BLS}_\varepsilon \cup \text{BLS}_\varepsilon^\pm) \cap W \subset W$ are smooth submanifolds with boundary. Both manifolds have codimension $d - 1$ and boundary equal to BLS_ε .

Let $W^* = W \setminus \text{BLS}_\varepsilon$. There exists a diffeomorphism $\Phi : \mathbb{S}^{d-1} \times (0, r) \times \mathbb{R}^{l-d} \rightarrow W^*$ such that

$$\Phi^{-1}[\text{BLS}_\varepsilon^+] = \{\mathbf{N}\} \times (0, r) \times \mathbb{R}^{l-d}, \quad \Phi^{-1}[\text{BLS}_\varepsilon^-] = \{\mathbf{S}\} \times (0, r) \times \mathbb{R}^{l-d},$$

where $\mathbf{N}, \mathbf{S} \in \mathbb{S}^{d-1}$ are the north and south poles.

This result describe the sets near BLS_ε .

Let M be a smooth manifold and $N \subset M$ be a transversally oriented submanifold of codimension k that is also a closed set. The intersection with N defines an element of $H^k(M; \mathbb{Z})$. The intersection with either $\text{BLS}_\varepsilon^\pm$ defines in W^* a generator of $H^{d-1}(W^*; \mathbb{Z}) \approx \mathbb{Z}$.

If the manifold $\text{BLS}_{U_\varepsilon^*}$ is homotopically equivalent to $\mathbb{S}^{d-1}$ and the intersection with $\text{BLS}_\varepsilon^\pm$ defines generators of $H^{d-1}(\text{BLS}_{U_\varepsilon^*}; \mathbb{Z}) \approx \mathbb{Z}$, the ancestry ε of dimension $d = \dim(\varepsilon) > 0$ is called **tame**. If these conditions are not satisfied, ε is classified as **wild**.

In terms of $\text{BLC}_{U_\varepsilon^*}$, the first condition states that $\text{BLC}_{U_\varepsilon^*}$ is homotopically equivalent to $\mathbb{S}^{d-1}$. The second condition asserts that we can construct cocycles $\omega_{\text{BLC}}^\pm \in \mathbb{Z}^{d-1}(\text{BLC}_{U_\varepsilon^*}; \mathbb{Z})$ by considering elements of U_ε^* of dimension $d - 1$ as cells of $\text{BLC}_{U_\varepsilon^*}$, these cocycles $\omega_{\text{BLC}}^\pm$ are generators of $H^{d-1} \approx \mathbb{Z}$.

Example 7.3.2. From Examples 5.2.1 and 5.2.2, we see that an ancestry ε with $d = \dim(\varepsilon) = 1$ or $d = 2$ of type I is tame. $\diamond$

For the case of tame ancestries, we refer to Lemma 16.6 from [1], which provides a method for obtaining the glueing map. Here, we outline the proof for better understanding.

Fact 7.3.3. *If ε is tame, then the glueing map $\beta : \mathbb{S}^{d-1} \rightarrow \text{BLC}_{U_\varepsilon^*}$ is a homotopy equivalence.*

Proof. Let $W^* = W \setminus \text{BLS}_\varepsilon$ as in Fact 7.3.2 and $\omega_{W^*}^\pm \in H^{d-1}(W^*; \mathbb{Z})$ be defined by intersection with $\text{BLS}_\varepsilon^\pm$. By the definition of tameness, each one serves as a generator. Consider the small transversal section $\alpha_1 : \mathbb{D}_{\frac{1}{2}}^d \rightarrow \text{BLS}_{U_\varepsilon}$ with $\alpha_1(0) = z_1 \in \text{BLS}_{U_\varepsilon} \setminus \text{BLS}_{U_\varepsilon^*}$, and the restriction $\beta_1 = \alpha_1|_{\mathbb{S}_{\frac{1}{2}}^{d-1}}$, where, ignoring the radius, $\beta_1 : \mathbb{S}^{d-1} \rightarrow W^*$. We have a pairing $H^{d-1}(W^*; \mathbb{Z}) \times \pi_{d-1}(W^*) \rightarrow \mathbb{Z}$. According to Fact 7.3.2, $|\omega_{W^*}^\pm \beta_1| = 1$.

Let $i : W^* \rightarrow \text{BLS}_{U_\varepsilon^*}$ denote the inclusion map. Define $\omega_{\text{BLS}}^\pm \in H^{d-1}(\text{BLS}_{U_\varepsilon^*}; \mathbb{Z})$ by their intersection with $\text{BLS}_\varepsilon^\pm$, as per the definition of tameness. Consider

$$i^* = H^{d-1}(i) : H^{d-1}(\text{BLS}_{U_\varepsilon^*}; \mathbb{Z}) \rightarrow H^{d-1}(W^*; \mathbb{Z});$$

we have $i^*(\omega_{\text{BLS}}^\pm) = \omega_{W^*}^\pm$. Thus, $\omega_{\text{BLS}}^\pm(i \circ \beta_1) = \omega_{W^*}^\pm \beta_1$ and $i \circ \beta_1$ is a generator of $\pi_{d-1}(\text{BLS}_{U_\varepsilon^*})$. From the proof of Fact 7.1.1, see [1], so is the glueing map β . The result follows. $\square$

From this result, Fact 7.3.1, and Fact 7.1.1, where $M_0 = \text{BLS}_{U_\varepsilon^*} \subset M_1 = \text{BLS}_{U_\varepsilon} = \text{BLS}_\varepsilon \cup \text{BLS}_{U_\varepsilon^*}$, we conclude that we can attach a cell BLC_ε of dimension $d = \dim(\varepsilon)$ to obtain $\text{BLC}_{U_\varepsilon}$.

Additionally, by the previous result and the examples in this section, we observe that the cells BLC_ε of dimension 1 in BLC_σ are edges joining the two vertices corresponding to the elements of dimension 0 in U_ε . If ε has dimension 2 type I, then BLC_ε fills in a square hole.

Up until the end of this work, we have not come across any wild ancestry. This does not imply that they do not exist; perhaps they appear in higher dimensions.

7.4

The Homotopy Type of BL_σ for $n \leq 4$

Several examples, combined with previously presented results, contribute to proving Theorems 1 and 3 in [1], as outlined below. In Section 16.2 of Chapter 16, we present the component referenced in item 2 of Fact 7.4.2. Refer to [1] and [2] for proofs and examples.

Fact 7.4.1. *For $n \leq 4$ and $z \in \tilde{B}_{n+1}^+$, each connected component $X \subseteq \text{BLC}_z$ collapses to a point.*

Fact 7.4.2. *Consider $\sigma \in S_{n+1}$ and $\text{BL}_\sigma \subset \text{Lo}_{n+1}^1$.*

1. *For $n \leq 4$, every connected component of every set BL_σ is contractible;*
2. *For $n = 5$ and $\sigma = 563412 \in S_6$, there exist connected components of BL_σ which are homotopically equivalent to the circle $\mathbb{S}^1$;*
3. *For $n \geq 5$, there exist connected components of BL_η which have even Euler characteristic.*

In the upcoming chapters, we construct BL_σ for $\sigma \in S_6$.

8

Wiring Diagram Decomposition

In this chapter, we explore methods for decomposing a wiring diagram, with a focus on block decomposition and split decomposition. We introduce and define three distinct types of splits applicable to a wiring diagram.

8.1

Block Decomposition

In this section, we explore how to decompose a wiring diagram based on the number of blocks. Recall that $\sigma \in S_{n+1}$ blocks at j if and only if a_j does not appear in a reduced word for σ .

Fact 8.1.1. *If $\sigma \in S_{n+1}$ blocks at j then there exist permutations $\sigma_0 \in S_j$ and $\sigma_1 \in S_{n+1-j}$ such that $\sigma = \sigma_0 \oplus \sigma_1$.*

Example 8.1.1. Let $\sigma = [231645] = a_2 a_1 a_4 a_5 \in S_6$.



Figure 8.1: Wiring diagram of $\sigma = a_2 a_1 a_4 a_5 \in S_6$.

Note that σ blocks at 3. This permutation can be represented as a sum of two permutations: $\sigma = \sigma_1 \oplus \sigma_2$, where $\sigma_1 = a_2 a_1 \in S_3$ and $\sigma_2 = a_1 a_2 \in S_3$. $\diamond$

Lemma 8.1.1. *Let $\sigma = a_{i_1} \dots a_{i_k} \in S_{n+1}$ be a reduced word. If σ blocks at j such that, $\sigma = \sigma_0 \oplus \sigma_1$ with $\sigma_0 \in S_j$ and $\sigma_1 \in S_{n+1-j}$ then $\text{BL}_\sigma = \text{BL}_{\sigma_0} \oplus \text{BL}_{\sigma_1}$.*

Proof. If $\sigma \in S_{n+1}$ blocks at j such that $\sigma = \sigma_0 \oplus \sigma_1$, with $\sigma_0 \in S_j$ and $\sigma_1 \in S_{n+1-j}$, then the permutation matrix P_σ has two diagonal blocks, P_{σ_0} and P_{σ_1} , such that $P_\sigma = P_{\sigma_0} \oplus P_{\sigma_1}$.

Let $L \in \text{Lo}_{n+1}^1$. Suppose that there exist $L_0 \in \text{BL}_{\sigma_0}$ and $L_1 \in \text{BL}_{\sigma_1}$, such that $L = L_0 \oplus L_1$. Therefore,

$$\begin{aligned}
L &= L_0 \oplus L_1 = (U_1 P_{\sigma_0} U_2) \oplus (U_3 P_{\sigma_1} U_4) \\
&= (U_1 \oplus U_3)(P_{\sigma_0} U_2 \oplus P_{\sigma_1} U_4) \\
&= (U_1 \oplus U_3)(P_{\sigma_0} \oplus P_{\sigma_1})(U_2 \oplus U_4).
\end{aligned}$$

Since, $U_1 \oplus U_3, U_2 \oplus U_4 \in \text{Up}_{n+1}$, and $P_{\sigma_0} \oplus P_{\sigma_1} = P_\sigma$, then

$$L = \tilde{U}_1 P_\sigma \tilde{U}_2,$$

where $\tilde{U}_1 = (U_1 \oplus U_3)$ and $\tilde{U}_2 = (U_2 \oplus U_4)$. Therefore, $L \in \text{BL}_\sigma$.

In conclusion, $L \in \text{BL}_\sigma$ if and only if there exist $L_0 \in \text{BL}_{\sigma_0}$ and $L_1 \in \text{BL}_{\sigma_1}$, such that $L = L_0 \oplus L_1$. $\square$

We have seen how to represent a permutation as a direct sum of smaller permutations, now we explore methods to decompose permutations in different ways.

8.2

Split Type 1

In this section, we explore the behavior of a wiring diagram when it can be decomposed in a way similar to a direct sum. This approach simplifies the analysis, as the permutation is associated with a sum of permutations that have already been studied.

Definition 8.2.1. If a curve can be traced in the wiring diagram from r_i to r_{i+1} , or from r_{i+1} to r_i , such that it transversely crosses only one wire, without passing through an inversion, then a **split type 1** can be performed on the diagram. This split is said to be performed at row r_i . The operation decomposes the diagram into two parts, resulting in permutations $\sigma_1 \in S_{i+1}$ and $\sigma_2 \in S_{n+1-i}$.

The permutations $\sigma_1 \in S_{j+1}$ and $\sigma_2 \in S_{n+1-j}$ are obtained by joining the wire that was cut with the dot that does not have a wire entering or leaving it. It is important to note that the resulting words are still reduced.

Definition 8.2.2. Let $\sigma = a_{i_1} \dots a_{i_l}$ be a reduced word for a permutation $\sigma \in S_{n+1}$. If a split type 1 can be performed at r_j , then:

- $\sigma_1 = a_{i_{k_1}} \dots a_{i_{k_m}} \in S_{j+1}, \forall i_{k_s} \leq j,$
- $\sigma_2 = a_{i_{k_1-j}} \dots a_{i_{k_n-j}} \in S_{n+1-j}, \forall i_{k_s} > j,$

where $k_s \leq k_{s+1}$, $m = \text{inv}(\sigma_1)$ and $n = \text{inv}(\sigma_2)$.

Example 8.2.1. Let $\sigma = [325614] = a_1a_4a_3a_2a_1a_5a_4 \in S_6$.

We can trace a curve on the diagram, crossing the fifth wire and separating it into two parts.

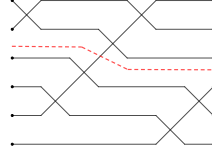


Figure 8.2: First step to apply the split type 1 on the wiring diagram of the permutation $\sigma = a_1a_4a_3a_2a_1a_5a_4 \in S_6$.

The upper part is essentially $\sigma_1 = a_1a_2a_1 \in S_3$, and the lower part is essentially $\sigma_2 = a_2a_1a_3a_2 \in S_4$.

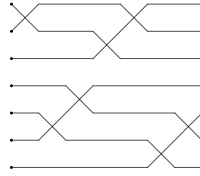


Figure 8.3: Result of apply the split type 1.

◇

Lemma 8.2.1. Let $\sigma = a_{i_1} \dots a_{i_l} \in S_{n+1}$ be a reduced word. If a split type 1 can be performed at $\sigma \in S_{n+1}$, resulting in permutations $\sigma_1 \in S_{i+1}$ and $\sigma_2 \in S_{n+1-i}$, then $\text{BLC}_\sigma = \text{BLC}_{\sigma_1} \times \text{BLC}_{\sigma_2}$.

Proof. When split type 1 is applied, the permutation is decomposed into two parts such that no region in one part has inversions lying on the boundary of a region in the other part. Consequently, performing a click in the region of σ corresponding to σ_1 does not affect the signs of the inversions associated with σ_2 .

This establishes a correspondence between 1-skeletons of the desired CW-complexes. In order to extend this correspondence to higher dimensional cells, it suffices to verify that a valid pattern of black and white diamonds (i.e., a preancestry) for the original permutation corresponds to a pair of such patterns for σ_1 and σ_2 .

This implies that the CW complex of $\sigma \in S_{n+1}$ is equivalent to that of $\sigma_1 \oplus \sigma_2 \in S_{n+2}$. By Lemma 8.1.1, we have $\text{BLC}_\sigma = \text{BLC}_{\sigma_1} \times \text{BLC}_{\sigma_2}$. $\square$

See Section 10.1 for a detailed application of the lemma.

8.3

Split Type 2

In this section, we introduce the concept of a tourist and examine how its presence enables us to decompose a wiring diagram. This decomposition simplifies the analysis similarly to the way split type 1 does.

Definition 8.3.1. Let $\sigma = a_{i_1} \dots a_{i_l}$ be a reduced word for a permutation $\sigma \in S_{n+1}$. A **split type 2** on a wiring diagram at inversion a_{i_k} is a decomposition of the diagram into two parts, which satisfies the following conditions:

1. For all $j \neq k$, $a_{i_k} \neq a_{i_j}$;
2. The remaining words in either S_{i_k+1} and S_{n+1-i_k} , or S_{i_k} and S_{n+2-i_k} , remain reduced.

We call the inversion a_{i_k} a **tourist**.

Note that the **tourist** is an inversion that does not impact the possibility of applying the click operation; it is only affected by the click. One could say that the inversion only observes what is happening, like a tourist.

The move involves separating the wiring diagram into two parts in such a way that the inversion a_{i_k} is in one of the two parts, the wires that were cut are then reconnected to the dots that do not have a wire entering or leaving. The other part is obtained by connecting the wires that we have cut.

A split type 2 can be performed at a_{i_k} by drawing a line at height $i_k + 1/4$ or $i_k + 3/4$. In the first case, the inversion will lie in the upper subdiagram; in the second case, it will lie in the lower subdiagram. In both scenarios, the split is said to occur at the inversion a_{i_k} .

When split type 2 is applied at a_{i_k} , at height $i_k + 1/4$, the resulting permutations are $\sigma_1 \in S_{i_k}$ and $\sigma_2 \in S_{n+2-i_k}$. In the other case, the resulting permutations are $\sigma_1 \in S_{i_k+1}$ and $\sigma_2 \in S_{n+1-i_k}$.

Remark 8.3.2. If the tourist is not at the boundary of any region, split type 1 can be applied. $\diamond$

Definition 8.3.3. Let $\sigma = a_{i_1} \dots a_{i_l}$ be a reduced word for a permutation $\sigma \in S_{n+1}$. If a split type 2 can be performed at a_j , then:

1. For $j + \frac{1}{4}$, the resulting permutations are:

- $\sigma_1 = a_{i_{k_1}} \dots a_{i_{k_m}} \in S_j$, $\forall i_{k_s} \leq j-1$,
- $\sigma_2 = a_{i_{k_1}-j-1} \dots a_{i_{k_n}-j-1} \in S_{n+2-j}$, $\forall i_{k_s} > j-1$,

where $k_s \leq k_{s+1}$, $m = \text{inv}(\sigma_1)$ and $n = \text{inv}(\sigma_2)$.

2. For $j + \frac{3}{4}$, the resulting permutations are:

- $\sigma_1 = a_{i_{k_1}} \dots a_{i_{k_m}} \in S_{j+1}$, $\forall i_{k_s} \leq j$,
- $\sigma_2 = a_{i_{k_1}-j} \dots a_{i_{k_n}-j} \in S_{n+1-j}$, $\forall i_{k_s} > j$,

where $k_s \leq k_{s+1}$, $m = \text{inv}(\sigma_1)$ and $n = \text{inv}(\sigma_2)$.

Example 8.3.1. Let $\sigma = a_2a_1a_3a_2a_1a_5a_4 \in S_6$ be a reduced word. The inversion a_3 is a candidate for applying the split type 2. To begin, we mark the red line in Figure 8.4, where the split type 2 is performed at the height $3 + \frac{3}{4}$.

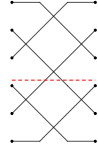


Figure 8.4: First step to apply the split type 2 on $\sigma = a_2a_1a_4a_3a_2a_5a_4 \in S_6$.

Next, we connect the wires to form the resulting diagrams, as shown in Figure 8.5. Note that $a_2a_1a_3a_2 \in S_4$ and $a_1a_2a_1 \in S_3$ are still reduced.

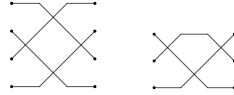


Figure 8.5: Resulting permutations: $\sigma_1 = a_2a_1a_3a_2 \in S_4$ and $\sigma_2 = a_1a_2a_1 \in S_3$.

We can also apply split type 2 at the tourists a_1, a_3 and a_5 . $\diamond$

The next example shows more types of tourists.

Example 8.3.2. The wiring diagram on the left of Figure 8.6 has three tourists: a_1, a_2 and a_3 . On the right, there is one tourist: a_1 .

Note that in both diagrams, one might think that a_5 is a tourist, but this is not the case, as if we apply the split type 2 at this inversion, the remaining word will not be reduced.



Figure 8.6: Wiring diagrams of the permutations $\sigma_1 = a_4a_5a_4a_3a_2a_1 \in S_6$ and $\sigma_2 = a_2a_3a_2a_4a_3a_2a_1a_5a_4a_3a_2 \in S_6$.

Furthermore, the tourists in the first diagram do not belong to the boundary of any region. Therefore, we can apply split type 1. $\diamond$

Lemma 8.3.1. *Let $\sigma = a_{i_1} \dots a_{i_l} \in S_{n+1}$ be a reduced word. If a_{i_k} is a tourist, then $\text{BLC}_\sigma = \text{BLC}_{\sigma_1} \times \text{BLC}_{\sigma_2}$, where $\sigma_1 \in S_{i_k+1}$ and $\sigma_2 \in S_{n+1-i_k}$ are the remaining permutations obtained by performing a split type 2 at a_{i_k} .*

The proof is similar to the proof of Lemma 8.2.1.

Proof. Since a_{i_k} is a tourist, there are no preancestries for $\sigma \in S_{n+1}$ with the inversion a_{i_k} marked. By applying split type 2, the permutation is decomposed into two parts such that row r_{i_k} contains only the inversion a_{i_k} . Consequently, a preancestry for the original permutation corresponds to a pair of preancestries for σ_1 and σ_2 . This implies that the CW complex of $\sigma \in S_{n+1}$ is equivalent to that of $\sigma_1 \oplus \sigma_2 \in S_{n+2}$. By Lemma 8.1.1, we have $\text{BLC}_\sigma = \text{BLC}_{\sigma_1} \times \text{BLC}_{\sigma_2}$. $\square$

See Section 10.2 for a detailed application of the lemma.

8.4

Split Type 3

In this section, we introduce the final method for decomposing a diagram. Unlike the previous methods, applying split type 3 does not result in the number of components of BL_σ being a simple product.

Definition 8.4.1. Consider a wiring diagram where we trace a curve starting in r_i at height $i + \frac{1}{2}$, that passes from r_i to r_{i-1} without crossing any wire. The curve then crosses a wire at height $i - \epsilon$, and moves up to height $i - \frac{1}{4}$. The curve moves horizontally at this height and then moves down, crossing another wire at height $i - \epsilon$. The curve then moves into r_i and continues at height $i + \frac{1}{2}$ until the end. In the process the curve crosses wires exactly twice. We assume there are only two crossings in r_{i-1} . If such a curve can be traced in the wiring diagram, we can perform **split type 3**. The operation decomposes the diagram into two parts, resulting in the permutations $\sigma_1 \in S_{i+1}$ and $\sigma_2 \in S_{n+2-i}$. In the crossed region, the wires are first reconnected by linking the left wire to the

right dot, and vice versa, creating an inversion a_i . The remaining wires are then connected by joining them at their nearest starting and ending points.

Definition 8.4.2. Let $\sigma = a_{i_1} \dots a_{i_l}$ be a reduced word for a permutation $\sigma \in S_{n+1}$. If a split type 3 can be performed at r_j , then:

- $\sigma_1 = a_{i_{k_1}} \dots a_{i_{k_m}} \in S_{j+1}, \quad \forall i_{k_s} \leq j,$
- $\sigma_2 = a_{i_{k_1}-j-1} \dots a_{i_{k_n}-j-1} \in S_{n+2-j}, \quad \forall i_{k_s} \geq j,$

where $k_s \leq k_{s+1}$, $m = \text{inv}(\sigma_1)$ and $n = \text{inv}(\sigma_2)$. In σ_1 , the subword $a_j \dots a_j$ will be represented by a single a_j , which is the new inversion introduced by the split.

Remark 8.4.3. In split types 1 and 2, the permutation is decomposed into two smaller permutations whose dimensions sum to $n + 2$. In split type 3, however, one additional inversion is generated in σ_1 . Here, the sum of the dimensions of σ_1 and σ_2 is $n + 3$. The sign of the additional inversion does not alter the homotopy type of the associated CW complex. Simply taking the direct sum would result in twice as many components, so this must be adjusted accordingly. $\diamond$

Example 8.4.1. Let $\sigma = a_1 a_2 a_3 a_2 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1 \in S_6$. In Figure 8.7, we trace a red curve that only crosses one region in the diagram, in accordance with the conditions outlined in the definition.

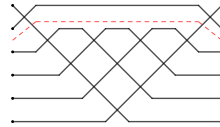
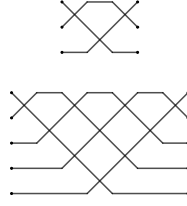


Figure 8.7: First step to perform a split type 3 on the diagram of $\sigma \in S_6$.

Now, we connect the wires that we cut in the upper part of the diagram to the dots representing 3 on both sides, creating an inversion a_3 in the diagram. The resulting permutation is $\eta \in S_3$.

After that, we connect the wires that we cut in the lower part of the wiring diagram to the dots representing 1 on both sides. The resulting permutation is $\eta \in S_5$.

Figure 8.8: Resulting permutations $\eta \in S_3$ and $\eta \in S_5$.

◇

Lemma 8.4.1. *Let $\sigma = a_{i_1} \dots a_{i_l} \in S_{n+1}$ be a reduced word. If a split type 3 can be performed at $\sigma \in S_{n+1}$, resulting in $\sigma_1 \in S_{j+2}$ and $\sigma_2 \in S_{n+1-j}$, then $\text{BLC}_\sigma \times \{\pm 1\} = \text{BLC}_{\sigma_1} \times \text{BLC}_{\sigma_2}$. In particular, the number of connected components in BLC_σ is half the product of the number of connected components in BLC_{σ_1} and BLC_{σ_2} .*

Proof. Applying split type 3 the permutation is decomposed into two parts: the upper part, which includes the new inversion a_j and is represented by $\sigma_1 \in S_{j+2}$, and the lower part. Performing a click operation in the region corresponding to σ_1 changes the signs of all inversions in r_j simultaneously. Consequently, this change of signs does not affect the possibility of performing a click in the regions corresponding to σ_2 . This establishes a correspondence between 1-skeletons of the desired CW-complexes.

The curve passes through exactly one region which is contained between the two only crossings in r_{i-1} . Thus, any preancestry with diamonds in r_{i-1} has only one possible way to be marked in this row. For ancestries of dimension greater than 1 that include diamonds in r_{i-1} , the possible positions for the diamonds in the remaining rows are not affected by the diamonds in r_{i-1} . Therefore, any preancestry in σ corresponds to a pair of preancestries in σ_1 and σ_2 .

If no click is performed in the region corresponding to σ_1 , the sign of a_j in σ_1 is $\circ$ (or $\bullet$); if a click is performed, the sign changes to $\bullet$ (or $\circ$). This results in two copies of the same component.

Therefore, $\text{BLC}_\sigma \times \{\pm 1\} = \text{BLC}_{\sigma_0}$, where $\sigma_0 = \sigma_1 \oplus \sigma_2 \in S_{n+3}$. By Lemma 8.1.1, it follows that $\text{BLC}_{\sigma_0} = \text{BLC}_{\sigma_1} \times \text{BLC}_{\sigma_2}$. Thus, the number of connected components of BLC_σ is half the product of the number of connected components of BLC_{σ_1} and BLC_{σ_2} . □

Example 8.4.2. Let $\sigma = a_1 a_2 a_3 a_2 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1 \in S_6$, as in the previous example. In the diagram of $\sigma_1 = a_1 a_2 a_1 \in S_3$ (Figure 8.8), the inversion a_2 is generated during the process of separating the diagrams. However, this inversion does not affect the overall analysis. Its effect is limited to changing the signs of r_1 in the diagram of $\sigma_2 = a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1 \in S_5$.

The CW complex is formed as the product of BLC_{σ_2} and the cells of BLC_{σ_1} disregarding the influence of a_2 . In this example, the cells correspond to two dots and one segment, represented as $(\bullet x \bullet)$, $(\circ x \circ)$, and $(\blacklozenge x \blacklozenge)$. Assigning either $\circ$ or $\bullet$ to the position x yields the same CW complex.

Therefore, only one possibility needs to be considered. Consequently, the number of connected components is half the product of the components.

From this analysis, it follows that BLC_{σ} contains $3 \times 52 = 156$ connected components, all of which are contractible. $\diamond$

In the following chapters, we examine the homotopy type of BL_{σ} with $\sigma \in S_6$ categorizing the analysis by the number of inversions. With our understanding of how to decompose a wiring diagram, we can now distinguish between permutations that can be reduced and those that cannot.

9

The Homotopy Type of BL_σ for $\text{inv}(\sigma) \leq 6$

For those $\sigma \in S_6$ with $\text{inv}(\sigma) \leq 6$, determining the homotopy type of BL_σ is relatively straightforward. In this chapter, we first focus on the cases where $\text{inv}(\sigma) \leq 4$, and then proceed to analyze those with $\text{inv}(\sigma) = 5$ and $\text{inv}(\sigma) = 6$.

9.1

The Homotopy Type of BL_σ for $\text{inv}(\sigma) \leq 4$

For $\sigma \in S_6$ with $\text{inv}(\sigma) \leq 4$, we have $\text{block}(\sigma) = |\text{Block}(\sigma)| = b \neq 0$ and the permutation can be expressed as a sum of well known permutations.

As stated in Definition 2.1.3, if $\sigma = \sigma_1 \oplus \sigma_2$, then $BL_\sigma = BL_{\sigma_1} \oplus BL_{\sigma_2}$. Since $\sigma_1 \in S_j$ and $\sigma_2 \in S_{6-j}$ with $j \leq 5$, both BL_{σ_1} and BL_{σ_2} are contractible. Consequently, the sum $BL_\sigma = BL_{\sigma_1} \oplus BL_{\sigma_2}$ is also contractible. The number of connected components is the product of the number of connected components of BL_{σ_1} and BL_{σ_2} .

Example 9.1.1. Example 8.1 presents the permutation $\sigma = [231645] = a_2a_1a_4a_5 \in S_6$. We can express σ as the sum of two permutations: $\sigma = \sigma_1 \oplus \sigma_2$, where $\sigma_1 = a_2a_1 \in S_3$ and $\sigma_2 = a_1a_2 \in S_3$. It is well known that both BL_{σ_1} and BL_{σ_2} each have 4 connected components, all contractible.

Therefore, BL_σ has 16 connected components, all of which are contractible. These connected components are thin ancestries representing points in the CW complex. $\diamond$

9.2

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 5$

The permutations will be categorized based on the number of blocks. For $\text{inv}(\sigma) = 5$ there are a total of 71 permutations distributed across the following cases:

1. There are 55 permutations with $b \neq 0$.

In this case, the permutation can be expressed as a sum of well-known permutations. Consequently, BL_σ is contractible.

2. There are 16 permutations with $b = 0$.

Since there are five rows and five inversions, it follows that there is exactly one inversion in each row. Consequently, the connected components are thin and therefore, contractible.

Example 9.2.1. Let $\sigma = [512364] = a_1a_2a_3a_5a_4 \in S_6$.

There are $2^5 = 32$ ancestries, all with dimension 0. Figure 9.1 shows one of these ancestries.

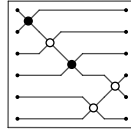


Figure 9.1: Thin component with $\varepsilon = (\bullet \circ \bullet \circ \circ)$.

Notice that each ancestry is thin, since there is only one inversion in each row. Therefore, BL_σ has 32 connected components, all of which are contractible. $\diamond$

Since BL_σ is contractible for both $b = 0$ and $b \neq 0$, it follows that BL_σ is contractible for all $\sigma \in S_6$ with $\text{inv}(\sigma) \leq 5$.

9.3

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 6$

For $\sigma \in S_6$ with $\text{inv}(\sigma) = 6$, there are 90 permutations distributed across the following cases:

1. There are 46 permutations with $b \neq 0$.

In this case, the permutation can be expressed as a sum of well-known permutations. Consequently, BL_σ is contractible.

2. There are 33 permutations that can be analyzed using the permutation $\sigma = a_1a_2a_1 \in S_3$.

Example 9.3.1. Let $\sigma_1 = [234651] = a_4a_5a_4a_3a_2a_1 \in S_6$.

The permutation has three tourists, a_3 , a_2 and a_1 , so we can use split type 1 to separate the wiring diagram into two parts. In this case we apply the

split at r_3 . These parts correspond to the permutations $\sigma_1 = a_3a_2a_1 \in S_4$ and $\sigma_2 = a_1a_2a_1 \in S_3$.

It is well-known that $\sigma_2 = a_1a_2a_1 \in S_3$ has 6 connected components, all of them contractible. The components consist of 4 thin and 2 thick. The permutation has 3 additional inversions, represented by $\sigma_1 = a_3a_2a_1 \in S_4$. These inversions are tourists, meaning that they do not affect the homotopy type of the connected components. They essentially contribute to the number of components. Figure 9.2 shows a thin connected component of BL_σ .

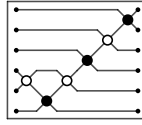


Figure 9.2: Thin component with ancestry $\varepsilon = (\circ \bullet \circ \bullet \circ \bullet)$.

Since the permutation has 5 rows, it is easy to see that BL_σ has 32 thin connected components.

Figure 9.3 shows the other type of connected component, with dimension 1. This component consists of two dots connected by an edge. Note that the component is generated by the part associated with $\sigma_2 = a_1a_2a_1 \in S_3$, the other part remains unchanged.

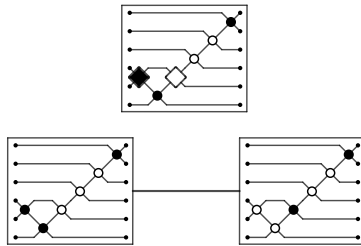


Figure 9.3: Connected component of dimension 1 with ancestry $\varepsilon = (\blacklozenge \blacklozenge \circ \bullet \bullet)$.

Consequently, BL_σ contains $2 \cdot 8 = 16$ connected components of this type. Therefore, BL_σ has $32 + 16 = 6 \cdot 2^3 = 48$ connected components, all of them contractible. $\diamond$

3. There are 11 permutations that can be analyzed using the permutation $\sigma = a_2 a_1 a_3 a_2 \in S_4$.

This case is similar to the previous one. The permutation has two tourists, and we can apply split type 1 to solve it.

Therefore, BL_σ has $12 \cdot 2^2 = 48$ connected components, all of them contractible.

As a result, for all $\sigma \in S_6$ with $\text{inv}(\sigma) = 6$, BL_σ is contractible. Hence, BL_σ is contractible for all $\sigma \in S_6$ with $\text{inv}(\sigma) \leq 6$.

10

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 7$

For $\text{inv}(\sigma) = 7$, there are 101 permutations distributed across the following cases. In the first case, the permutation is blocked. For cases 2 to 10, split type 1 is applied and for case 11, split type 3 is applied.

1. There are 32 permutations with $b \neq 0$.

In this case, the permutation can be expressed as a sum of well-known permutations. Consequently, BL_σ is contractible.

2. There are 12 permutations that can be analyzed using the permutation $\sigma_1 = a_1a_2a_3a_2a_1 \in S_4$.

The permutation $\sigma \in S_6$ has two tourists to which we apply split type 1.

It is well known that $\sigma_1 = a_1a_2a_3a_2a_1 \in S_4$ has 18 connected components, all of which are contractible.

Therefore, BL_σ has $18 \cdot 2^2 = 72$ connected components, all of them contractible.

3. There are 12 permutations that can be analyzed using the permutation $\sigma_2 = a_2a_1a_3a_2a_1 \in S_4$.

The permutation has two tourists to which we apply split type 1.

It is well known that $\sigma_2 = a_2a_1a_3a_2a_1 \in S_4$ has 16 connected components, all of which are contractible.

Therefore, BL_σ has $16 \cdot 2^2 = 64$ connected components, all of them contractible.

4. There are 12 permutations that can be analyzed using the permutation $\sigma_3 = a_1a_2a_1a_3a_2 \in S_4$.

The permutation has two tourists to which we apply split type 1.

Therefore, BL_σ has 64 connected components, all of them contractible.

5. There are 4 permutations that can be analyzed using the permutation $\sigma_4 = a_2a_1a_3a_2a_4a_3 \in S_5$.

The permutation has one tourist and we apply split type 1.

Therefore, BL_σ has 64 connected components, all of them contractible.

6. There are 4 permutations that can be analyzed using the permutation $\sigma_5 = a_2a_1a_3a_4a_3a_2 \in S_5$.

The permutation has one tourist to which we apply split type 1.

Therefore, BL_σ has 72 connected components, all of them contractible.

7. There are 4 permutations that can be analyzed using the permutation $\sigma_6 = a_3a_2a_1a_4a_3a_2 \in S_5$.

The permutation has one tourist to which we apply split type 1.

Therefore, BL_σ has 64 connected components, all of them contractible.

8. There are 4 permutations that can be analyzed using the permutation $\sigma_7 = a_1a_3a_2a_1a_4a_3 \in S_5$.

The permutation has one tourist to which we apply split type 1.

Therefore, BL_σ has 72 connected components, all of them contractible.

9. There are 12 permutations that can be analyzed using the permutation $\sigma_8 = a_1a_2a_1 \in S_3$.

Example 10.0.1. Let $\sigma = [324651] = a_1a_4a_5a_4a_3a_2a_1 \in S_6$.

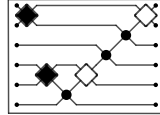


Figure 10.1: Ancestry $\varepsilon = (\blacklozenge \blacklozenge \bullet \diamond \bullet \bullet \diamond)$ of dimension 2.

Note that we can apply split type 1 at row 2, crossing the sixth wire. Or at row 3, crossing the same wire.

It is well known that $\sigma_8 = a_1a_2a_1 \in S_3$ has 6 contractible connected components. In this case, we have 2 copies of the same permutation and an additional inversion that does not alter the homotopy type of the components.

Furthermore, BL_{σ_8} only has components of dimension 0 and 1, whereas $\sigma \in S_6$ has ancestries of dimension 2, Figure 10.1 shows one of these ancestries. The connected components will be the product of those with lower dimension. The next example will explain this in details.

Therefore, BL_σ has $6 \cdot 6 \cdot 2 = 72$ connected components, all of them contractible. $\diamond$

10. There are 4 permutations that can be studied through the sum of $\sigma_9 = a_1 a_2 a_1 \in S_3$ and $\sigma_{10} = a_2 a_1 a_3 a_2 \in S_4$. This case will be detailed in Section 10.1.
11. The permutation $\sigma = a_2 a_1 a_4 a_3 a_2 a_5 a_4 \in S_6$ will be studied in detail in Section 10.2.

10.1

Case 10

Let $\sigma = [325614] = a_1 a_4 a_3 a_2 a_1 a_5 a_4 \in S_6$. As shown in Example 8.2.1, we can apply split type 1 to decompose σ into $\sigma_1 = a_1 a_2 a_1 \in S_3$ and $\sigma_2 = a_2 a_1 a_3 a_2 \in S_4$.

It is well known that BL_{σ_1} and BL_{σ_2} have 6 and 12 connected components, respectively, all of them contractible. Therefore, BL_σ has $6 \cdot 12 = 72$ connected components, all contractible.

First, let us analyze the case without applying split type 1.

There exist $2^5 = 32$ thin ancestries, resulting in 32 components similar to the one shown in Figure 10.2. Consequently, BL_σ has 32 thin connected components, all contractible.

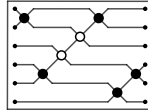


Figure 10.2: Thin component with ancestry $\varepsilon = (\bullet \bullet \circ \bullet \bullet \bullet)$.

For dimension 1, there are 2 possible positions for the diamonds, as shown in Figure 10.3. For each position, the rows that do not have diamonds have only one sign. This yields $2^4 = 16$ copies for each position.

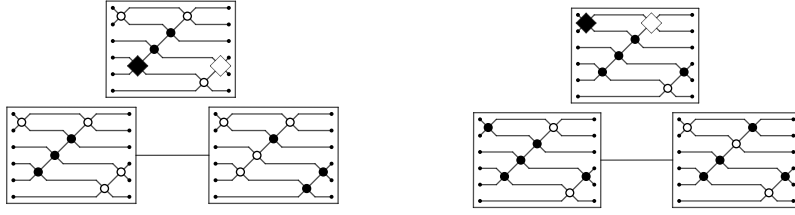


Figure 10.3: CW complexes with 1-dimensional ancestries $\varepsilon_1 = (\circ \blacklozenge \bullet \bullet \circ \circ \blacklozenge)$ and $\varepsilon_2 = (\blacklozenge \bullet \bullet \bullet \circ \blacklozenge \bullet)$, respectively.

Thus, BL_σ has a total of 32 connected components of this type, all contractible.

For dimension 2, there is one possible position for the diamonds, as shown in Figure 10.4. In this case, each row that does not have diamonds takes one sign. As a result, we have $2^3 = 8$ copies.

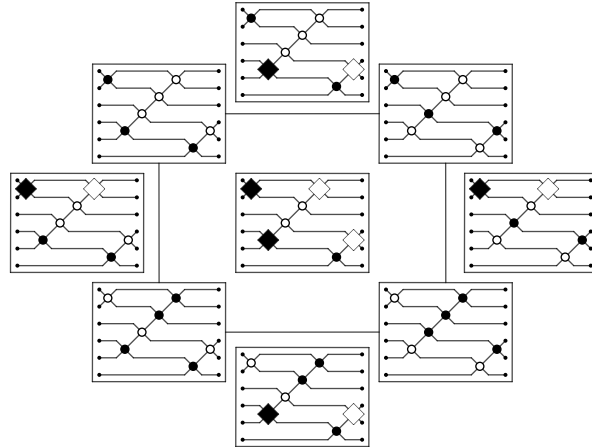


Figure 10.4: CW complex with the 2-dimensional ancestry $\varepsilon = (\blacklozenge \blacklozenge \circ \circ \blacklozenge \bullet \blacklozenge)$.

Therefore, there are 8 connected components of this type in BL_σ , all contractible.

Summing up, BL_σ has 72 connected components, all of them are contractible.

When analyzing through the split, there is a difference in the consideration of ancestries.

For $\sigma_1 = a_1 a_2 a_1 \in S_3$ and $\sigma_2 = a_2 a_1 a_3 a_2 \in S_4$, the permutations do not have ancestries of dimension 2. However, for $\sigma \in S_6$, ancestries of dimension 2 appear. This occurs because the CW complex of $\sigma \in S_6$ is the product of the CW complexes of $\sigma_1 \in S_3$ and $\sigma_2 \in S_4$. These ancestries appear when we take

the sum, altering the structure of the CW complex. However, the homotopy type is preserved.

It is well known that BL_{σ_1} has two connected components of dimension 1 and four thin ones. Furthermore, BL_{σ_2} has four connected components of dimension 1 and eight thin ones. The connected components of dimension 1 are shown in Figure 10.5.

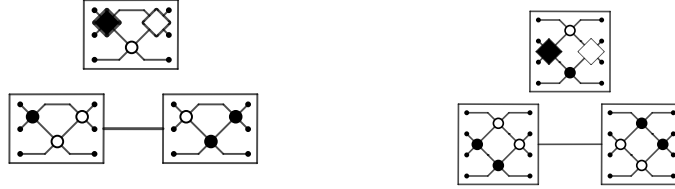


Figure 10.5: CW complexes of dimension 1 of BL_{σ_1} and BL_{σ_2} , respectively.

One can see that the product of these components yields the component of dimension 2 in Figure 10.4.

The other three permutations are

$$\sigma_1 = a_2a_1a_4a_5a_4a_3a_2, \quad \sigma_2 = a_2a_1a_3a_2a_4a_5a_4,$$

$$\sigma_3 = a_1a_2a_1a_4a_3a_5a_4.$$

They are all expressed with the same two permutations.

10.2

Case 11

Let $\sigma = [351624] = a_2a_1a_4a_3a_2a_5a_4 \in S_6$ be a fixed a reduced word. For this permutation, there are three possible approaches: the first is applying clicks, the second is using the orbits, and the third is applying split type 3.

10.2.1

First Approach

For $\sigma = a_2a_1a_3a_2a_1a_4a_3 \in S_6$, the maximal dimension for the ancestries is 2. Let us understand what happens for each possible ancestry.

In dimension 0, an ancestry can be either thin or thick. The latter only appears in CW complexes of dimension greater than 0.

There exist $2^5 = 32$ thin ancestries, resulting in 32 components similar to the one shown in Figure 10.6. Consequently, BL_σ has 32 thin connected components, all contractible.

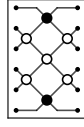


Figure 10.6: Component of dimension 0 with ancestry $\varepsilon_0 = (\circ \bullet \circ \circ \circ \bullet \circ)$.

In dimension 1, there are two possible positions for the diamonds. For each position, when rows without diamonds have only one sign, a component is formed as shown in Figure 10.7. This yields $2^4 = 16$ copies for each position.

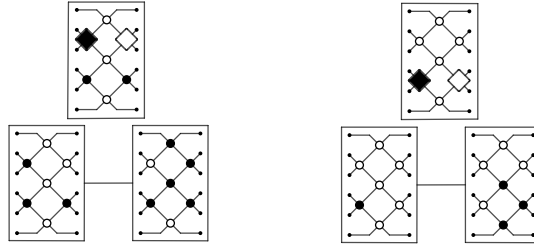


Figure 10.7: CW complexes of dimension 1, with ancestries $\varepsilon_1 = (\blacklozenge \circ \circ \circ \blacklozenge \circ \circ)$ and $\varepsilon_2 = (\circ \circ \blacklozenge \circ \circ \circ \blacklozenge)$, respectively.

Thus, BL_σ has a total of 32 connected components of these types, all contractible.

The remaining ancestries of dimension 1 appear in CW complexes of higher dimensions.

In dimension 2, there is only one possible position for the diamonds, as shown in Figure 10.8. In this case, each row without diamonds takes one sign.

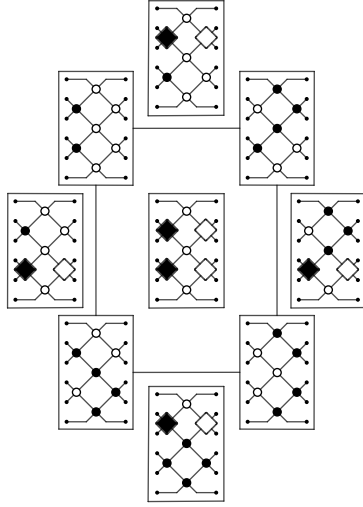


Figure 10.8: CW complex with 2-dimensional ancestry $\varepsilon_3 = (\blacklozenge \circ \blacklozenge \circ \diamond \circ \diamond \circ \diamond)$.

Therefore, there are $2^3 = 8$ connected components of this type in BL_σ , all of them contractible.

Summing up, BL_σ has $32 + 32 + 8 = 72$ connected components, all of them contractible.

10.2.2

Second Approach

For $\sigma = a_2 a_1 a_4 a_3 a_2 a_5 a_4 \in S_6$, it follows that

$$\sigma = \frac{1}{2\sqrt{2}}(\hat{a}_3 - \hat{a}_1 \hat{a}_2 \hat{a}_3 + \hat{a}_1 \hat{a}_4 + \hat{a}_2 \hat{a}_4 + \hat{a}_1 \hat{a}_5 + \hat{a}_2 \hat{a}_5 - \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5).$$

The set σQuat_6 has 5 orbits of sizes 16, 8, 16, 16, 8:

$$\mathcal{O}_\sigma = \left\{ \frac{\pm \hat{a}_3 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \pm \hat{a}_1 \hat{a}_4 \pm \hat{a}_2 \hat{a}_4 \pm \hat{a}_1 \hat{a}_5 \pm \hat{a}_2 \hat{a}_5 \pm \hat{a}_3 \hat{a}_4 \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5}{2\sqrt{2}} \right\},$$

$$\mathcal{O}_{\hat{a}_3 \sigma} = \left\{ \frac{-1 \pm \hat{a}_1 \hat{a}_2 \pm \hat{a}_1 \hat{a}_3 \hat{a}_4 \pm \hat{a}_2 \hat{a}_3 \hat{a}_4 \pm \hat{a}_4 \hat{a}_5 \pm \hat{a}_1 \hat{a}_3 \hat{a}_5 \pm \hat{a}_2 \hat{a}_3 \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5}{2\sqrt{2}} \right\},$$

$$\mathcal{O}_{\hat{a}_1 \sigma} = \left\{ \frac{\pm \hat{a}_1 \hat{a}_3 \pm \hat{a}_2 \hat{a}_3 \pm \hat{a}_4 \pm \hat{a}_1 \hat{a}_2 \hat{a}_4 \pm \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_5 \pm \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 \pm \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5}{2\sqrt{2}} \right\},$$

$$\mathcal{O}_{\hat{a}_4 \sigma} = \left\{ \frac{\pm \hat{a}_1 \pm \hat{a}_2 \pm \hat{a}_3 \hat{a}_4 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \pm \hat{a}_3 \hat{a}_5 \pm \hat{a}_1 \hat{a}_4 \hat{a}_5 \pm \hat{a}_2 \hat{a}_4 \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5}{2\sqrt{2}} \right\},$$

$$\mathcal{O}_{\hat{a}_1 \hat{a}_4 \sigma} = \left\{ \frac{1 \pm \hat{a}_1 \hat{a}_2 \pm \hat{a}_1 \hat{a}_3 \hat{a}_4 \pm \hat{a}_2 \hat{a}_3 \hat{a}_4 \pm \hat{a}_4 \hat{a}_5 \pm \hat{a}_1 \hat{a}_3 \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 \pm \hat{a}_2 \hat{a}_3 \hat{a}_5}{2\sqrt{2}} \right\}.$$

In the expressions within the Clifford algebra notation, the signs must be such that there is an even number of equal signs.

The elements $z \in \sigma \text{Quat}_6$ have $\Re(z) \in \{-\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}\}$. Using the Formula 4-3 for the number of ancestries of dimension 0 for a given $z \in \sigma \text{Quat}_6$, it follows that $N(z) \in \{0, -2, 4\}$.

- (i) If $\Re(z) = -\frac{1}{2\sqrt{2}} < 0$, then $N(z) = 0$, and thus the corresponding set BL_z is empty. Therefore, for each $z \in \mathcal{O}_{\hat{a}_3\sigma}$ the set BL_z is empty.
- (ii) If $z_0 = \sigma$, then $\Re(z_0) = 0$ and $N(z_0) = N_{\text{thin}}(z_0) = 2$. Thus, for each $z \in \mathcal{O}_\sigma$ the set BL_z has 2 contractible thin connected components. This component is illustrated in Figure 10.6.

Hence, this yields 32 connected components of BL_σ , all contractible.

The CW complex BLC_{z_0} is represented by two dots.

- (iii) Let $z = \hat{a}_2\hat{a}_1\hat{a}_4\hat{a}_3\hat{a}_2\hat{a}_5\hat{a}_4$. Then $\Re(z) = 0$, $N(z) = 2$ and there is no thin ancestry. By Formulas 4-1 and 4-2, it follows that for dimension 1, $N(z) = 1$. The component is shown in Figure 10.7.

Therefore, for each $z \in \mathcal{O}_{\hat{a}_1\sigma}$ the set BL_z has one connected component, which is contractible. The same applies to $z \in \mathcal{O}_{\hat{a}_4\sigma}$, resulting in 32 connected components of BL_σ , all of which are contractible.

The CW complex BLC_z is represented by two vertices and one edge.

- (iv) Let $z = \hat{a}_2\hat{a}_1\hat{a}_4\hat{a}_3\hat{a}_2\hat{a}_5\hat{a}_4$. In this case, $\Re(z) = \frac{1}{2\sqrt{2}} > 0$ and $N(z) = 4$. By Formulas 4-1 and 4-2, it follows that for dimension 1, $N(z) = 4$ (two for each preancestry of dimension 1). Additionally, for dimension 2, $N(z) = 1$. The component is shown in Figure 10.8.

Therefore, for each $z \in \mathcal{O}_{\hat{a}_1\hat{a}_4\sigma}$, the set BL_z has one connected component, which is contractible. Thus, we have 8 connected components of BL_σ , all of which are contractible.

The CW complex BLC_z consists of one connected component, with 4 vertices connected by 4 edges.

In summary, BL_σ has 72 connected components, all contractible.

10.2.3

Third Approach

Note that the permutation has three tourists, a_1, a_3 and a_5 . The split type 2 can be applied to any of them. As in Example 8.3.1, consider a_3 . One can observe that inversion $a_3 = (3, 4)$ does not affect the click operation, it only gets affected. This means that the inversion does not significantly change the analysis.

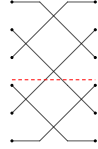


Figure 10.9: The wiring diagram of the permutation $\sigma = a_2a_1a_4a_3a_2a_5a_4 \in S_6$.

Note that the upper part is equivalent to $\sigma_1 = a_2a_1a_3a_2 \in S_4$, and the lower part is equivalent to $\sigma_2 = a_1a_2a_1 \in S_3$. Now, apply Lemma 8.3.1.

Then $BLC_\sigma = BLC_{\sigma_1} \times BLC_{\sigma_2}$ and, consequently, BL_σ has $12 \cdot 6 = 72$ connected components, all contractible.

Furthermore, it is important to note that the CW complexes of $BL_{\sigma_1} \in S_4$ and $BL_{\sigma_2} \in S_3$ consist only of 0-cells and 1-cells. Their product generates the 2-cell in $BL_\sigma \in S_6$.

The permutations $\sigma_1 \in S_3$ and $\sigma_2 \in S_4$ are the same as in Section 10.1. Therefore, we have already seen that the product of the components of dimension 1 yields the component of dimension 2.

A closer examination reveals that cases 10 and 11 are fundamentally the same. This occurs because of the presence of tourists.

As a result, BL_σ is contractible for all $\sigma \in S_6$ with $\text{inv}(\sigma) = 7$.

11

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 8$

For $\text{inv}(\sigma) = 8$, there are 101 permutations distributed across the following cases. In the first case, the permutation is blocked. For cases 2 to 13, split type 1 is applied. For case 14, split type 1 or 2 is applied, and for case 15, split type 2 is applied. Consequently, for all cases, BL_σ is contractible.

1. There are 18 permutations with $b \neq 0$;
2. There are 12 permutations that can be analyzed using the permutation $\sigma_1 = a_1a_2a_1a_3a_2a_1 \in S_4$;
3. There are 4 permutations that can be analyzed using the permutation $\sigma_2 = a_3a_2a_1a_4a_3a_2a_1 \in S_5$;
4. There are 4 permutations that can be analyzed using the permutation $\sigma_3 = a_2a_1a_3a_4a_3a_2a_1 \in S_5$;
5. There are 4 permutations that can be analyzed using the permutation $\sigma_4 = a_2a_3a_2a_1a_4a_3a_2 \in S_5$;
6. There are 4 permutations that can be analyzed using the permutation $\sigma_5 = a_1a_3a_2a_4a_3a_2a_1 \in S_5$;
7. There are 5 permutations that can be analyzed using the permutation $\sigma_6 = a_2a_1a_3a_2a_4a_3a_1 \in S_5$;
8. There are 4 permutations that can be analyzed using the permutation $\sigma_7 = a_1a_2a_3a_4a_3a_2a_1 \in S_5$;
9. There are 5 permutations that can be analyzed using the permutation $\sigma_8 = a_1a_2a_3a_2a_1a_4a_3 \in S_5$;
10. There are 4 permutations that can be analyzed using the permutation $\sigma_9 = a_1a_2a_1a_3a_4a_3a_2 \in S_5$;
11. There are 3 permutations that can be analyzed using the permutation $\sigma_{10} = a_1a_2a_1a_3a_2a_4a_3 \in S_5$;

12. There are 4 permutations that can be analyzed using the permutation $\sigma_{11} = a_1 a_3 a_2 a_1 a_4 a_3 a_2 \in S_5$;
13. There are 3 permutations that can be analyzed using the permutation $\sigma_{12} = a_2 a_1 a_3 a_2 a_4 a_3 a_2 \in S_5$;
14. There are 18 permutations that can be studied through the sum of two permutations one in S_3 and the other in S_4 ;
- The permutation in S_3 is the same for all the 18 permutations, $\sigma_1 = a_1 a_2 a_1 \in S_3$. The permutations in S_4 are

$$\sigma_1 = a_2 a_1 a_3 a_2 a_1, \quad \sigma_2 = a_1 a_2 a_3 a_2 a_1 \quad \text{or} \quad \sigma_3 = a_1 a_2 a_1 a_3 a_2 \in S_4;$$

15. There are 9 permutations that we can apply split type 2. In some of them, we can also apply split type 3. In the following section, we explore an example.

11.1

Case 15

Let $\sigma = [361452] = a_2 a_1 a_3 a_4 a_5 a_4 a_3 a_2 \in S_6$ be a reduced word.

11.1.1

First Approach

There exist $2^5 = 32$ thin ancestries similar to Figure 11.1. Therefore, BL_σ has 32 thin connected components, all contractible.

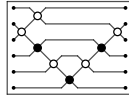


Figure 11.1: Thin component with ancestry $\varepsilon_1 = (\circ \circ \bullet \circ \bullet \circ \bullet \circ)$.

For dimension 1, there are three possible positions for the diamonds, r_2, r_3 and r_4 . If the remaining rows have equal signs, each one have the same CW complex structure. In Figure 11.2, we can see an example. In these cases, we have 16 copies for each position.

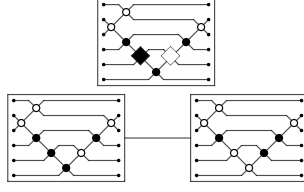


Figure 11.2: CW complex of dimension 1 with ancestry $\varepsilon_2 = (\circ \circ \bullet \blacklozenge \bullet \diamond \bullet \circ)$.

If the remaining rows consist of one with equal signs and the other with opposite signs, the ancestries will appear in the CW complex of dimension 2. If they all have opposite signs, the ancestry will be part of the CW complex of dimension 3, which we discuss below.

Therefore, BL_σ has a total of 48 connected components of these types, all contractible.

For dimension 2, the diamonds can be positioned in three ways: in r_2 and r_3 ; r_3 and r_4 ; or r_2 and r_4 . For each position, there are two possibilities for the row that does not have diamonds and contains more than one inversion, either having equal or opposite signs.

If the signs are equal, we have an example in Figure 11.3. In these cases, there are 8 copies for each position.

If the signs are opposite, they will appear in a CW complex of dimension 3, which will be discussed next.

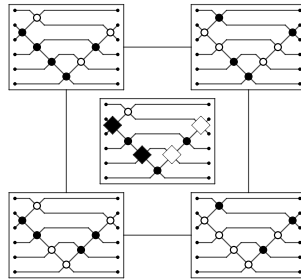


Figure 11.3: CW complex of dimension 2 with ancestry $\varepsilon_3 = (\blacklozenge \circ \bullet \blacklozenge \bullet \diamond \bullet \circ)$.

Therefore, BL_σ has a total of 24 connected components of these types, all contractible.

For dimension 3, the diamonds have only one position, resulting in 4 copies of a solid cube. In Figure 11.5, we have an example represented as a cube planar projection. The 3-dimensional cell in Figure 11.4 completely fills the cube. Note that the faces of the cube correspond to the 2-dimensional ancestries mentioned above.

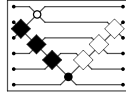
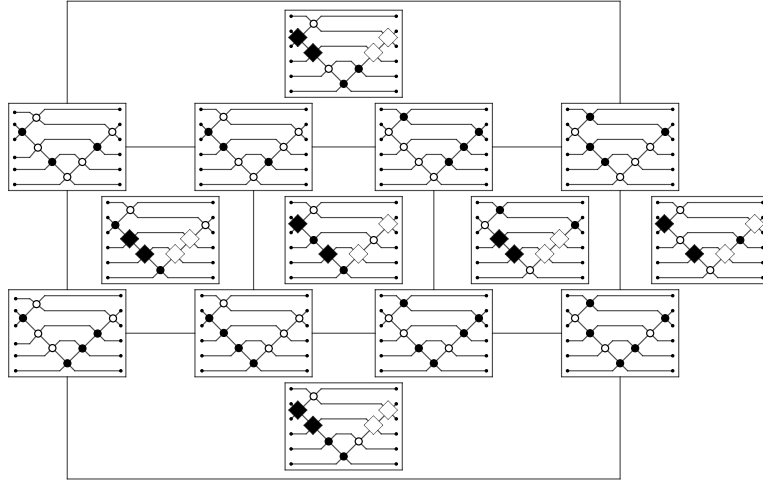
Figure 11.4: Ancestry of dimension 3 that fills the cube $\varepsilon_4 = (\blacklozenge \circ \blacklozenge \blacklozenge \bullet \diamond \diamond \diamond)$.

Figure 11.5: The cube that represents the CW complex of dimension 3.

Therefore, BL_σ has 4 connected components of dimension 3, all of them contractible.

Summing up, BL_σ has 108 connected components, all contractible.

11.1.2

Second Approach

In this subsection, we assume that the components are already known and we now examine which orbit corresponds to each CW complex we drew in the previous section.

The analysis can also be done without relying on the CW complexes mentioned above; this would require additional calculations involving the number of higher-dimensional ancestries.

For $\sigma = [361452] = a_2a_1a_3a_4a_5a_4a_3a_2 \in S_6$, it follows that

$$\acute{\sigma} = \frac{1}{2}(\hat{a}_1\hat{a}_4 + \hat{a}_2\hat{a}_4 + \hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5) \in \tilde{B}_6^+.$$

The set σQuat_6 consists of 9 orbits, with the first 7 having size 8, and the last two having size 4:

$$\begin{aligned}\mathcal{O}_\sigma &= \left\{ \frac{\pm \hat{a}_1 \hat{a}_4 \pm \hat{a}_2 \hat{a}_4 \pm \hat{a}_3 \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1 \sigma} &= \left\{ \frac{\pm \hat{a}_4 \pm \hat{a}_1 \hat{a}_2 \hat{a}_4 \pm \hat{a}_1 \hat{a}_3 \hat{a}_5 \pm \hat{a}_2 \hat{a}_3 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_3 \sigma} &= \left\{ \frac{\pm \hat{a}_1 \hat{a}_3 \hat{a}_4 \pm \hat{a}_2 \hat{a}_3 \hat{a}_4 \pm \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1 \hat{a}_3 \sigma} &= \left\{ \frac{\pm \hat{a}_3 \hat{a}_4 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \pm \hat{a}_1 \hat{a}_5 \pm \hat{a}_2 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_3 \hat{a}_4 \sigma} &= \left\{ \frac{\pm \hat{a}_1 \hat{a}_3 \pm \hat{a}_2 \hat{a}_3 \pm \hat{a}_4 \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_4 \sigma} &= \left\{ \frac{\pm \hat{a}_1 \pm \hat{a}_2 \pm \hat{a}_3 \hat{a}_4 \hat{a}_5 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1 \hat{a}_3 \hat{a}_4 \sigma} &= \left\{ \frac{\pm \hat{a}_3 \pm \hat{a}_1 \hat{a}_2 \hat{a}_3 \pm \hat{a}_1 \hat{a}_4 \hat{a}_5 \pm \hat{a}_2 \hat{a}_4 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1 \hat{a}_4 \sigma} &= \left\{ \frac{1 \pm \hat{a}_1 \hat{a}_2 \pm \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 \pm \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 \sigma} &= \left\{ \frac{-1 \pm \hat{a}_1 \hat{a}_2 \pm \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 \pm \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5}{2} \right\}.\end{aligned}$$

In the expressions within the Clifford algebra notation, the signs must be such that there is an odd number of equal signs.

The elements $z \in \sigma \text{Quat}_6$ have $\Re(z) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. Using the Formula 4-3 of the number of ancestries of dimension 0 for a given $z \in \sigma \text{Quat}_6$, it follows that $N(z) \in \{0, 4, 8\}$.

- (i) For $z = \sigma$, we have $\Re(z) = 0$ and $N(z) = 4 = N_{\text{thin}}(z)$. Therefore, for each $z \in \mathcal{O}_\sigma$ the set BL_z has 4 connected components, all contractible.
- (ii) For $z \in \sigma \text{Quat}_6$ with $\Re(z) = -\frac{1}{2}$ we have $N(z) = 0$. Therefore, for each $z \in \mathcal{O}_{\hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 \sigma}$ the set BL_z is empty.
- (iii) For $z \in \mathcal{O}_{\hat{a}_1 \sigma}$, we have $\Re(z) = 0$, $N(z) = 4$ and no thin ancestry. By Formulas 4-1 and 4-2, it follows that for dimension 1, $N(z) = 2$. Therefore, for each $z \in \mathcal{O}_{\hat{a}_1 \sigma}$ the set BL_z has 2 connected components, that are contractible, so we have 16 connected components of BL_σ .

The CW complex BLC_z is the one in Figure 11.2.

The same applies to $z \in \mathcal{O}_{\hat{a}_3 \sigma}, \mathcal{O}_{\hat{a}_1 \hat{a}_3 \hat{a}_4 \sigma}$, then BL_z has 16 connected components for each orbit. Summing up, BL_σ has 48 connected components of these types.

- (iv) For $z \in \mathcal{O}_{\hat{a}_1\hat{a}_3}$, we have $\mathfrak{R}(z) = 0$, $N(z) = 4$ and no ancestry thin. By Formulas 4-1 and 4-2, it follows that for dimension 1, $N(z) = 4$. Therefore, for each $z \in \mathcal{O}_{\hat{a}_3\hat{\sigma}}$ the set BL_z has 1 connected component, that are contractible, so we have 8 connected components of BL_σ .

The CW complex BLC_z is the one in Figure 11.3.

The same is applied to $z \in \mathcal{O}_{\hat{a}_4}, \mathcal{O}_{\hat{a}_3\hat{a}_4}$, then BL_z has 8 connected components for each orbit. Summing up, BL_σ has 24 connected components of these types.

- (v) If we have $\mathfrak{R}(z) = \frac{1}{2}$, then $N(z) = 8$ so that the corresponding set BL_z has 1 connected component, that is contractible. By Formulas 4-1 and 4-2, it follows that for dimension 1, $N(z) = 12$. Therefore, for each $z \in \mathcal{O}_{\hat{a}_1\hat{a}_4\hat{\sigma}}$ the corresponding sets BL_z are contractible, so we have 32 connected components of BL_σ , all of them contractible.

The CW complex BLC_z is the cube in Figure 11.5.

11.1.3

Third Approach

For this permutation, we can apply split type 2 at a_1 , resulting in $\sigma_1 = a_1a_2a_3a_4a_3a_2a_1 \in S_5$. Alternatively, we can also apply split type 3 at r_3 or r_4 , leading to $\sigma_3 = a_2a_1a_3a_2 \in S_4$ and $\sigma_4 = a_1a_2a_3a_2a_1 \in S_4$, or $\sigma_5 = a_2a_1a_3a_4a_3a_2 \in S_5$ and $\sigma_6 = a_2a_1a_2 \in S_3$, respectively.

The approach involving split type 1 is the most straightforward. Since we already know the connected components of $\sigma_1 \in S_5$, the only change when transitioning to $\sigma \in S_6$ is the increase in the number of components.

As a result, BL_σ is contractible for all $\sigma \in S_6$ with $\text{inv}(\sigma) = 8$.

12

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 9$

For $\text{inv}(\sigma) = 9$, we have 90 permutations distributed in the following cases. In the first case, the permutation is blocked. For cases 2 to 10, split type 1 is applied. For case 11, split type 1 or 2 is applied. For case 12, split type 2 is applied, and for case 13 split type 3 is applied. Consequently, for cases 1 to 13, BL_σ is contractible. Case 14 will be studied separately.

1. There are 8 permutations with $b \neq 0$;
2. There are 4 permutations that can be analyzed using the permutation $\sigma_1 = a_2a_3a_2a_1a_4a_3a_2a_1 \in S_5$;
3. There are 4 permutations that can be analyzed using the permutation $\sigma_2 = a_1a_3a_2a_1a_4a_3a_2a_1 \in S_5$;
4. There are 4 permutations that can be analyzed using the permutation $\sigma_3 = a_2a_1a_3a_2a_4a_3a_2a_1 \in S_5$;
5. There are 4 permutations that can be analyzed using the permutation $\sigma_4 = a_2a_1a_3a_2a_1a_4a_3a_2 \in S_5$;
6. There are 4 permutations that can be analyzed using the permutation $\sigma_5 = a_1a_2a_3a_2a_4a_3a_2a_1 \in S_5$;
7. There are 4 permutations that can be analyzed using the permutation $\sigma_6 = a_1a_2a_1a_3a_4a_3a_2a_1 \in S_5$;
8. There are 4 permutations that can be analyzed using the permutation $\sigma_7 = a_1a_2a_3a_2a_1a_4a_3a_2 \in S_5$;
9. There are 4 permutations that can be analyzed using the permutation $\sigma_8 = a_1a_2a_1a_3a_2a_4a_3a_2 \in S_5$;
10. There are 4 permutations that can be analyzed using the permutation $\sigma_9 = a_1a_2a_1a_3a_2a_1a_4a_3 \in S_5$;
11. There are 7 permutations that can be studied through the sum of two permutations one in S_3 and the other in S_4 ;

The permutations in S_3 e S_4 are the same for all four cases. Specifically, $\sigma_1 = a_1 a_2 a_1 \in S_3$ and $\sigma_2 = a_1 a_2 a_1 a_3 a_2 a_1 \in S_4$;

12. There are 12 permutations that we can apply split type 2;
13. There are 19 permutations that we can apply split type 3;
14. There are 8 permutations that needs to be studied separately.

12.1

Case 14

For $\sigma = [651234] = a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1 \in S_6$ it follows that

$$\begin{aligned} \hat{\sigma} = \frac{1}{4\sqrt{2}} & (-1 - \hat{a}_1 - \hat{a}_2 - \hat{a}_1 \hat{a}_2 - \hat{a}_3 + \hat{a}_1 \hat{a}_3 - \hat{a}_2 \hat{a}_3 + \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_1 \hat{a}_4 \\ & + \hat{a}_2 \hat{a}_4 - \hat{a}_1 \hat{a}_2 \hat{a}_4 - \hat{a}_3 \hat{a}_4 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_2 \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_5 + \hat{a}_1 \hat{a}_5 + \hat{a}_2 \hat{a}_5 \\ & - \hat{a}_1 \hat{a}_2 \hat{a}_5 + \hat{a}_3 \hat{a}_5 + \hat{a}_1 \hat{a}_3 \hat{a}_5 - \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_4 \hat{a}_5 - \hat{a}_2 \hat{a}_4 \hat{a}_5 \\ & - \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 + \hat{a}_3 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are four possible positions for the diamonds. The component will be determined by the rows that do not have diamonds. If the rows r_1 or r_4 has opposite signs and the remaining rows have equal signs, we obtain the CW complex in Figure 12.1. This results in 32 copies.

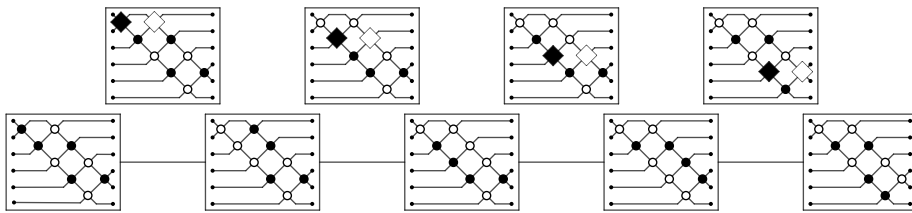


Figure 12.1: CW complex of dimension 1.

Therefore, BL_σ has a total of 32 connected components of this type, all contractible.

The ancestries of dimension 1 that appear in components of this type can be categorized as follows: those with diamonds in r_1 or r_4 , where the remaining rows have equal signs; those with diamonds in r_2 , where r_3 has opposite signs, and r_1 and r_4 have equal signs; or those with diamonds in r_3 , where r_2 has

opposite signs, while r_1 and r_4 have equal signs. The remaining ancestries of dimension 1 appear in the 2-dimensional CW complex.

For dimension 2, there are three possible positions for the diamonds that will appear together. This results in 32 components similar to Figure 12.2.

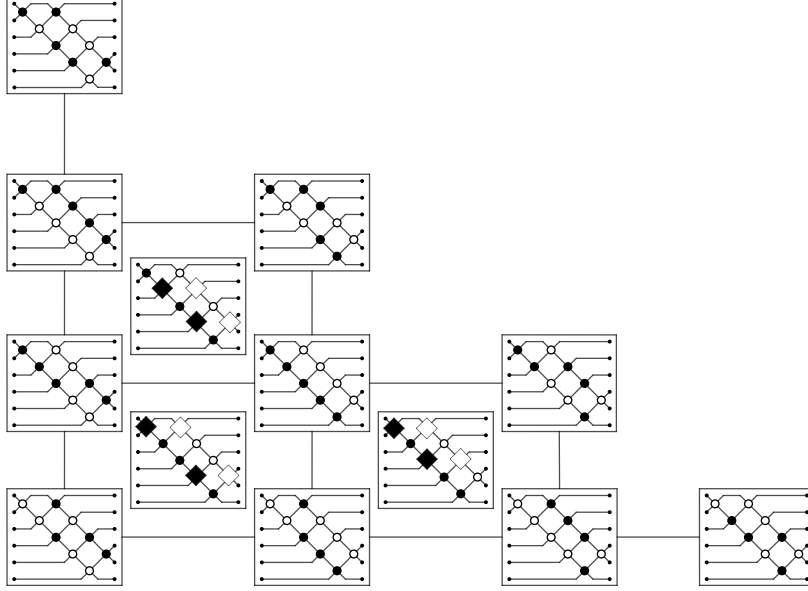


Figure 12.2: CW complex of dimension 2 with ancestries $\varepsilon_2 = (\blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet)$, $\varepsilon_3 = (\blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge)$ and $\varepsilon_4 = (\bullet \bullet \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge)$ of dimension 2.

Therefore, BL_σ has a total of 32 connected components of this type, all contractible. Summing up, BL_σ has 96 connected components, all of them are contractible.

The other 7 permutations have a CW complex similar to the one described. They are,

$$\sigma_1 = a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1, \quad \sigma_2 = a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4,$$

$$\sigma_3 = a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2, \quad \sigma_4 = a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3,$$

$$\sigma_5 = a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4, \quad \sigma_6 = a_1 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3,$$

$$\sigma_7 = a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_5 a_4 \in S_6.$$

As a result, BL_σ is contractible for all $\sigma \in S_6$ with $\text{inv}(\sigma) = 9$.

13

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 10$

For $\text{inv}(\sigma) = 10$, we have 71 permutations distributed in the following cases. In the first case, the permutation is blocked. for cases 2 to 5, split type 1 is applied. For case 6, split type 2 is applied, and for case 7 split type 3 is applied. Consequently, for cases 1 to 7, BL_σ is contractible. Case 8 will be studied separately.

1. There are 2 permutations with $b \neq 0$;
2. There are 4 permutations that can be analyzed using the permutation $\sigma_1 = a_2a_1a_3a_2a_1a_4a_3a_2a_1 \in S_5$;
3. There are 4 permutations that can be analyzed using the permutation $\sigma_2 = a_1a_2a_3a_2a_1a_4a_3a_2a_1 \in S_5$;
4. There are 4 permutations that can be analyzed using the permutation $\sigma_3 = a_1a_2a_1a_3a_2a_4a_3a_2a_1 \in S_5$;
5. There are 4 permutations that can be analyzed using the permutation $\sigma_4 = a_1a_2a_1a_3a_2a_1a_4a_3a_2 \in S_5$;
6. There are 12 permutations that we can apply split type 2;
7. There are 21 permutations that we can apply split type 3;
8. There are 20 permutations that needs to be studied separately.

13.1

Case 8

These 20 permutations can be classified into two types of CW complexes, which will be analyzed individually.

Type 13.1. For $\sigma = [346521] = a_3a_4a_3a_2a_1a_5a_4a_3a_2a_1 \in S_6$ it follows that

$$\begin{aligned} \dot{\sigma} = & \frac{1}{4}(-\hat{a}_1 - \hat{a}_2 - \hat{a}_3 + \hat{a}_1\hat{a}_2\hat{a}_3 + \hat{a}_1\hat{a}_4 + \hat{a}_2\hat{a}_4 - \hat{a}_3\hat{a}_4 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 \\ & - \hat{a}_1\hat{a}_2\hat{a}_5 + \hat{a}_1\hat{a}_3\hat{a}_5 - \hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_4\hat{a}_5 - \hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Hence, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are five possible positions for the diamonds. Analyzing these positions, we generate all connected components. If r_1 or r_4 have opposite signs, and the remaining rows have equal signs, we have the CW complex in Figure 13.1. This results in 32 copies.

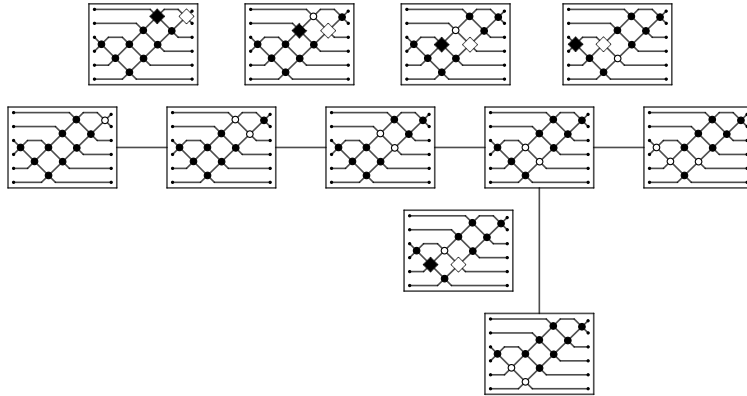


Figure 13.1: CW complex of dimension 1.

The remaining ancestries of dimension 1 appear in higher-dimensional CW complexes.

If r_2 has opposite signs and the remaining rows have the same signs, we have the CW complex in Figure 13.2. This results in 32 copies.

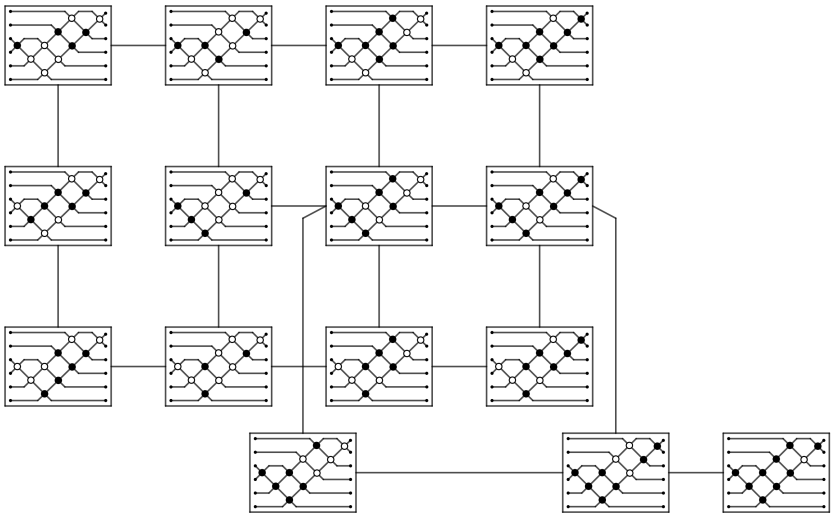


Figure 13.2: CW complex of dimension 2 with five 2-cells.

The remaining ancestries of dimension 2 appear in the 3-dimensional CW complex.

If the signs in r_3 are $(\bullet \bullet \circ)$ or $(\circ \circ \bullet)$, and the remaining rows have equal signs, we obtain the CW complex shown in Figure 13.3. This results in 16 copies.

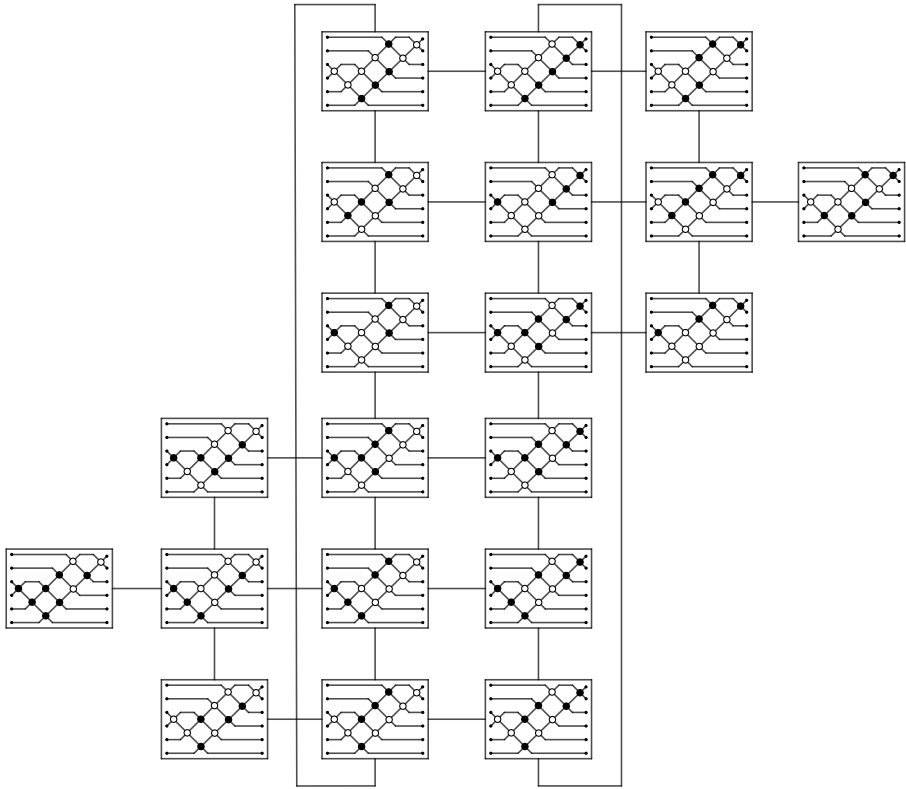


Figure 13.3: CW complex of dimension 3.

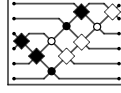


Figure 13.4: 3-cell with ancestry $\varepsilon_3 = (\blacklozenge \blacklozenge \circ \bullet \blacklozenge \bullet \diamond \bullet \diamond \circ \diamond)$.

In this CW complex, there are twelve 2-cells that fill the squares and hexagons, along with a 3-cell (Figure 13.4) that completely fills the prism. The structure resembles a prism, with 2-cells acting as “wings” attached to it. These wings, in turn, have attached 1-cells that resemble antennas.

These possible positions for the squares yield all the connected components of BL_σ . Therefore, BL_σ has a total of 112 connected components, all contractible.

There are 11 permutations that have a CW complex similar to the one described. They are,

$$\sigma_1 = a_3a_2a_1a_4a_3a_5a_4a_3a_2a_1, \quad \sigma_2 = a_1a_4a_3a_2a_1a_5a_4a_3a_2a_1,$$

$$\sigma_3 = a_1a_3a_4a_3a_2a_1a_5a_4a_3a_2, \quad \sigma_4 = a_2a_1a_3a_4a_3a_2a_1a_5a_4a_3,$$

$$\sigma_5 = a_1a_3a_2a_1a_4a_3a_5a_4a_3a_2, \quad \sigma_6 = a_1a_3a_2a_1a_4a_3a_2a_1a_5a_4,$$

$$\sigma_7 = a_1a_2a_1a_4a_3a_2a_1a_5a_4a_3, \quad \sigma_8 = a_2a_1a_3a_2a_1a_4a_3a_5a_4a_3,$$

$$\sigma_9 = a_1a_2a_1a_3a_4a_3a_2a_5a_4a_3, \quad \sigma_{10} = a_1a_2a_1a_3a_2a_4a_3a_2a_5a_4,$$

$$\sigma_{11} = a_1a_2a_1a_3a_2a_1a_4a_3a_5a_4 \in S_6.$$

Type 13.2. For $\sigma = [354621] = a_2a_4a_3a_2a_1a_5a_4a_3a_2a_1 \in S_6$ it follows that

$$\begin{aligned} \acute{\sigma} = \frac{1}{4} & (-\hat{a}_1 - \hat{a}_2 + \hat{a}_1\hat{a}_3 - \hat{a}_2\hat{a}_3 - \hat{a}_4 - \hat{a}_1\hat{a}_2\hat{a}_4 - \hat{a}_3\hat{a}_4 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 \\ & - \hat{a}_1\hat{a}_2\hat{a}_5 + \hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_4\hat{a}_5 - \hat{a}_2\hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Thus, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are five possible positions for the diamonds. If the diamonds are in r_1 , and the remaining rows have equal signs, we have the CW complex in Figure 13.5. This results in 32 copies.

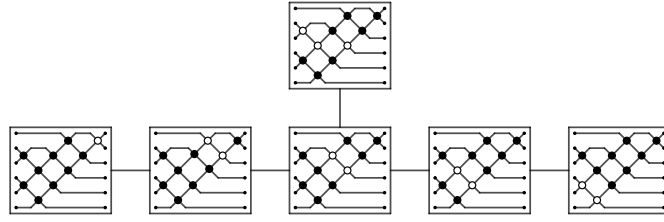


Figure 13.5: CW complex of dimension 1.

If the diamonds are in r_3 , and the remaining rows have equal signs, we have the CW complex in Figure 13.6. This results in 32 copies.

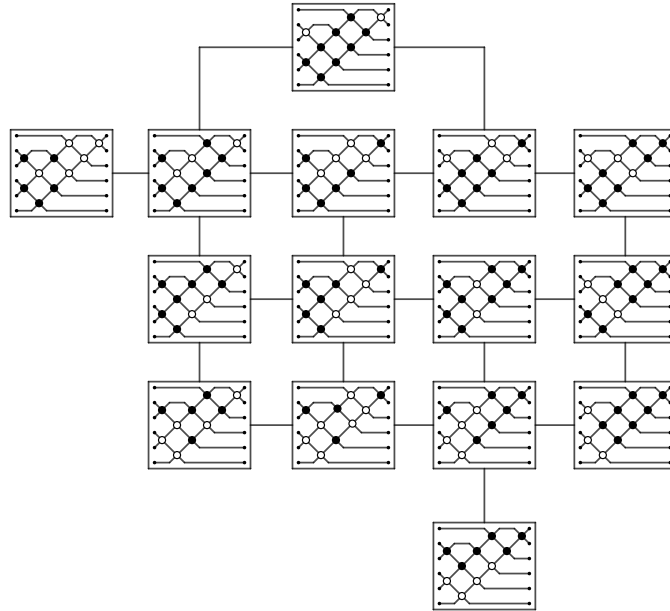


Figure 13.6: CW complex of dimension 2, with 6 ancestries of dimension 2, for example $\varepsilon_2 = (\blacklozenge \bullet \blacklozenge \bullet \circ \bullet \circ \diamond \diamond \bullet)$.

The remaining ancestries of dimensions 1 and 2 appear in the 3-dimensional CW complex. Therefore, BL_σ has a total of 64 connected of these types, all contractible.

For dimension 3, there is only one possible position for the diamonds. Figure 13.7 depicts the CW complex that has a 3-cell, this cell completely fills the cube in the CW complex. This results in 16 copies.

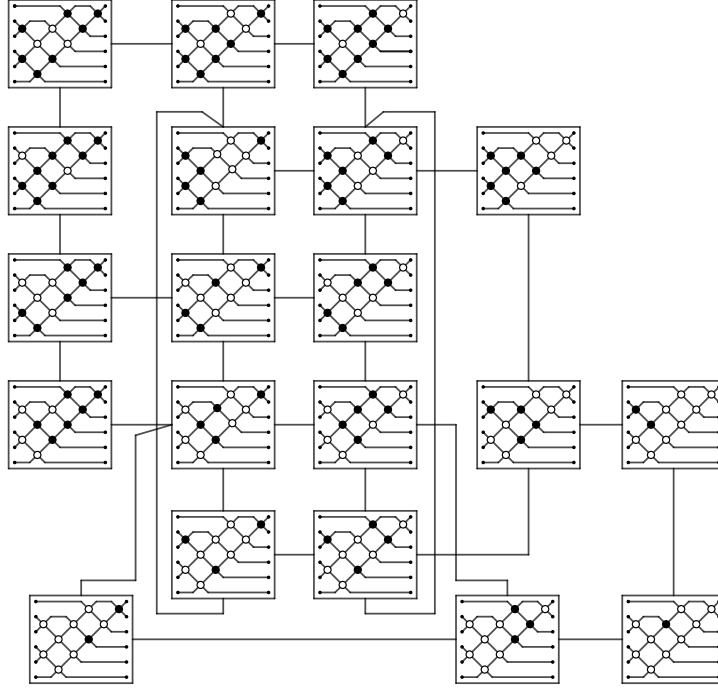


Figure 13.7: CW complex of dimension 3 with ancestry $\varepsilon_3 = (\blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge)$.

In this CW we have twelve 2-cells that fill the squares and the hexagons, and one 3-cell that fills the cube. The 2-cells that are not part of the cube are attached to it.

Therefore, BL_σ has 16 connected components of this type, all contractible. In summary BL_σ , it has a total of 112 connected components, all contractible.

There are 7 permutations that have a CW complex similar to the one described. They are,

$$\sigma_1 = a_2 a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4, \quad \sigma_2 = a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1,$$

$$\sigma_3 = a_1 a_2 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2, \quad \sigma_4 = a_2 a_1 a_3 a_2 a_4 a_3 a_2 a_1 a_5 a_4,$$

$$\sigma_5 = a_1 a_2 a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4, \quad \sigma_6 = a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_5 a_4,$$

$$\sigma_7 = a_1 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 a_2 \in S_6.$$

As a result, BL_σ is contractible for all $\sigma \in S_6$ with $\text{inv}(\sigma) = 10$.

14

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 11$

For $\text{inv}(\sigma) = 11$, we have 49 permutations distributed across the following cases:

1. There are 4 permutations that can be analyzed using permutation $\sigma_1 = a_1a_2a_1a_3a_2a_1a_4a_3a_2a_1 \in S_5$, here we apply split type 1 or 2;
2. There are 3 permutations that we can apply split type 2;
3. There are 9 permutations that we can apply split type 3;
4. There are 33 permutations that needs to be studied separately.

14.1

Case 4

These 33 permutations can be classified into seven types of CW complexes, which will be analyzed individually.

Type 14.1. For $\sigma = [356421] = a_3a_2a_4a_3a_2a_1a_5a_4a_3a_2a_1 \in S_6$ it follows that

$$\sigma = \frac{1}{2\sqrt{2}}(-\hat{a}_1 - \hat{a}_2 - \hat{a}_3\hat{a}_4 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_5 - \hat{a}_1\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5).$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are six possible positions for the diamonds. If the diamonds are in r_1 or r_4 and the remaining rows have equal signs, we have the CW complex in Figure 14.1. This results in 32 copies.

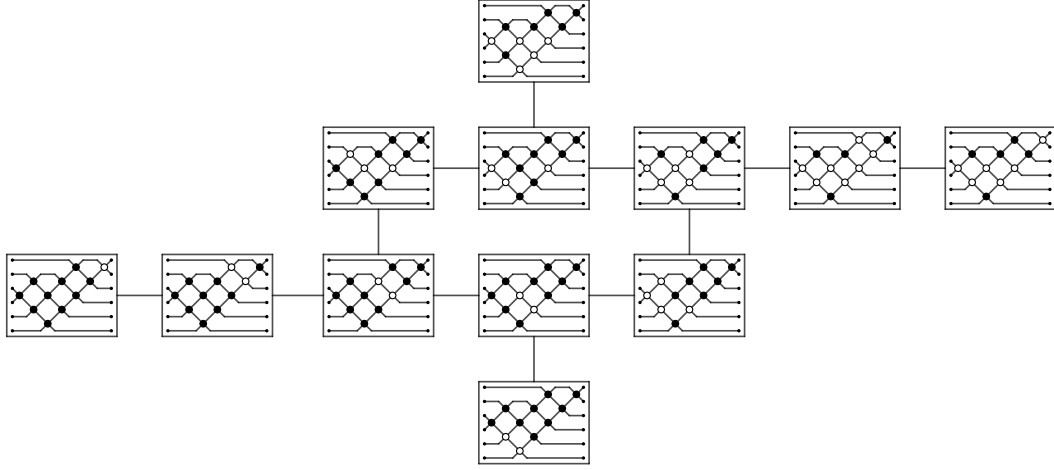


Figure 14.1: CW complex of dimension 2 with ancestry $\varepsilon_1 = (\blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet)$.

If the diamonds are in r_3 , with signs $(\bullet \bullet \bullet)$ or $(\bullet \circ \circ)$, and the remaining rows have equal signs, we obtain the CW complex depicted in Figure 14.2. This results in 16 copies.

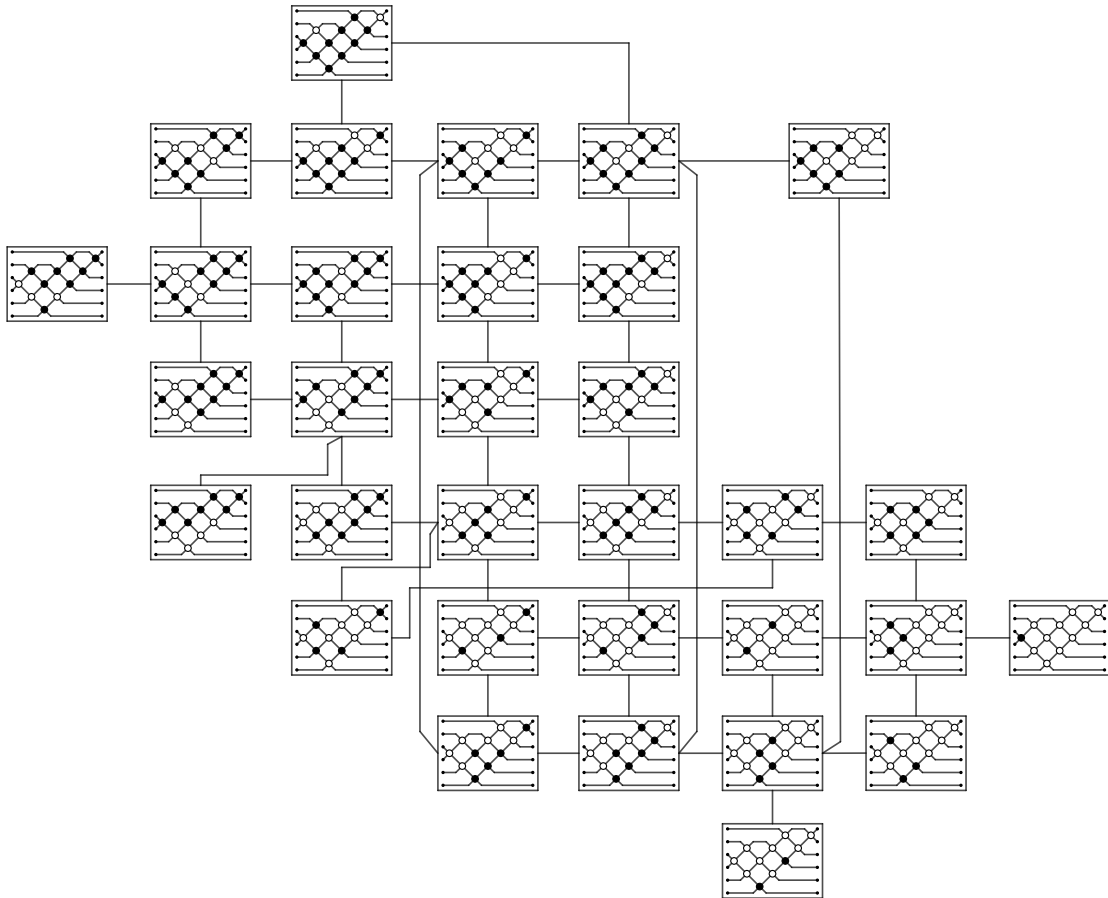


Figure 14.2: CW complex of dimension 3 with ancestry $\varepsilon_2 = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet)$.

Note that in this CW complex, there is a 3-cell that fills the prism in the center of the figure. The ancestries $\varepsilon_3 = (\bullet \bullet \bullet \bullet \bullet \circ \bullet \circ \circ \bullet)$ and $\varepsilon_4 = (\circ \circ \bullet \bullet \bullet \circ \circ \bullet \circ \circ \bullet)$ are the vertices on the upper left and lower right corners of the prism.

This CW complex comprises one 3-cell and ten 2-cells attached to it. Additionally, it includes four 1-cells and four 0-cells attached to 2-cells. This structure resembles a solid prism with wings, some of which have antennas. However, none of these alter the homotopy type of the component.

If the diamonds are in r_2 with signs $(\bullet \circ \circ)$, and the remaining rows have equal signs, we have the CW complex shown in Figure 14.3. This results in 16 copies.

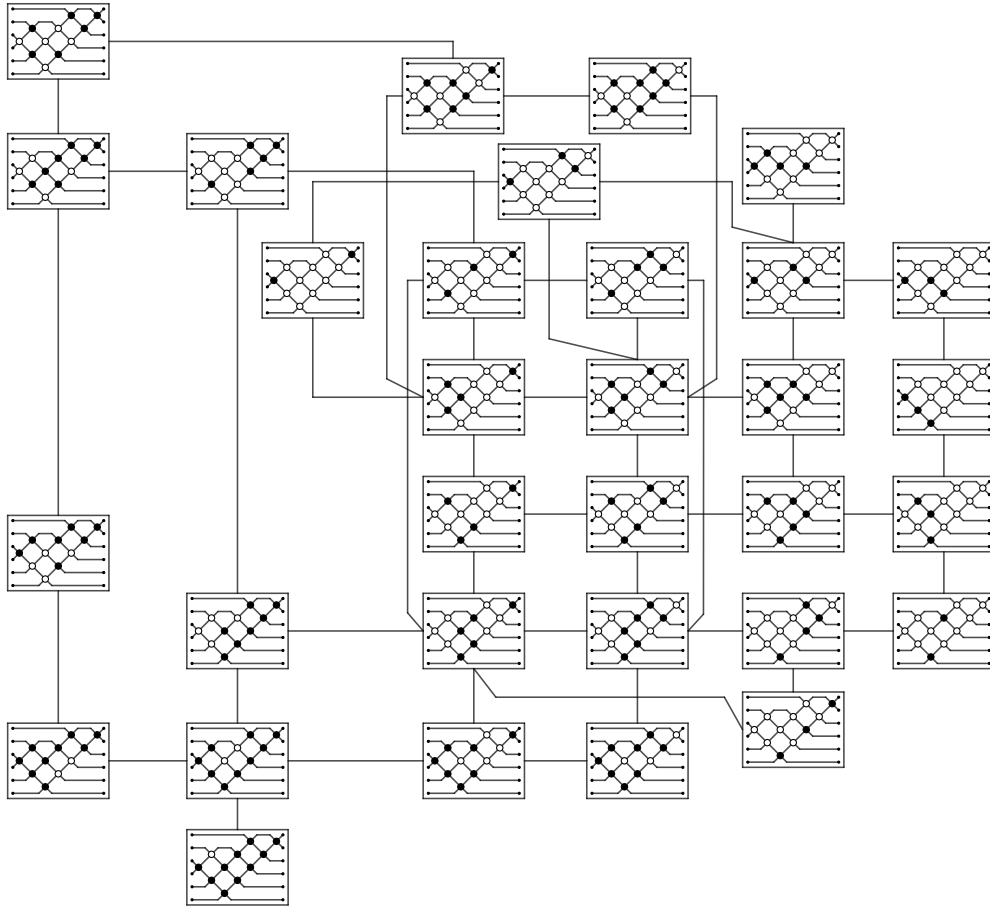


Figure 14.3: CW complex of dimension 3 with ancestry $\varepsilon_5 = (\circ \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \circ \circ \bullet \circ \blacklozenge)$.

Note that we have a 3-cell in this CW complex, this cell fills the cube in the CW completely. The ancestries $\varepsilon_6 = (\circ \circ \bullet \circ \bullet \circ \circ \circ \circ \bullet)$ and $\varepsilon_7 = (\circ \circ \circ \bullet \bullet \bullet \bullet \circ \bullet \circ)$ are the vertices on the upper left and lower right corners of the prism.

This CW complex comprises one 3-cell and twelve 2-cells attached to it. Additionally, it includes one 1-cell with one 0-cell attached to cells of dimension 2. This structure resembles a solid cube with wings and antennas. However, none of these alter the homotopy type of the component.

If r_2 is $(\circ \circ \bullet)$ and the remaining rows have equal signs, we have the CW complex in Figure 14.4. This results in 16 copies.

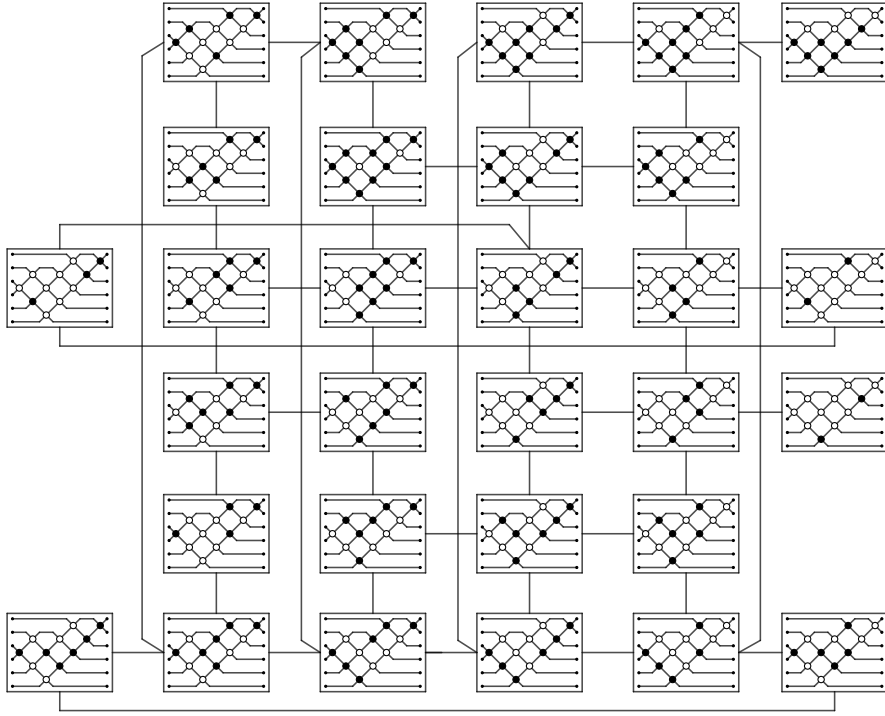


Figure 14.4: CW complex with three 3-dimensional ancestries.

This CW complex comprises three 3-cells with ancestries $\varepsilon_8 = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \diamond \diamond \circ \circ)$, $\varepsilon_9 = (\blacklozenge \blacklozenge \circ \circ \diamond \blacklozenge \bullet \bullet \diamond \circ \diamond)$ and $\varepsilon_{10} = (\blacklozenge \blacklozenge \blacklozenge \diamond \bullet \bullet \circ \diamond \diamond \bullet)$. The three 3-cells are attached through one 2-cell for each pair of 3-cells. Additionally, there are four 2-cells and two 1-cells, each with a 0-cell attached.

The ancestries $\varepsilon_{11} = (\bullet \bullet \circ \circ \circ \bullet \circ \bullet \circ \bullet)$ and $\varepsilon_{12} = (\bullet \circ \bullet \bullet \bullet \bullet \circ \bullet \bullet)$ are the vertices on the upper left and lower right corners of the first “parallelepiped”. The ancestries $\varepsilon_{13} = (\bullet \bullet \bullet \bullet \circ \bullet \bullet \circ \circ \bullet)$ and $\varepsilon_{14} = (\bullet \circ \bullet \circ \circ \circ \bullet \bullet \bullet)$ are vertices of the other “parallelepiped”. The second “parallelepiped” attaches to the previous one through the 2-cell with ancestry $\varepsilon_{15} = (\blacklozenge \blacklozenge \bullet \bullet \diamond \bullet \bullet \circ \diamond \bullet \bullet)$.

The ancestries $\varepsilon_{16} = (\bullet \circ \bullet \bullet \bullet \bullet \circ \bullet \bullet \bullet)$ and $\varepsilon_{17} = (\bullet \circ \bullet \circ \circ \bullet \bullet \circ \bullet \circ \bullet)$ are vertices of the prism. This prism attaches to the second “parallelepiped” through the 2-cell with ancestry $\varepsilon_{18} = (\blacklozenge \blacklozenge \bullet \circ \diamond \circ \bullet \bullet \diamond \bullet \bullet)$.

Therefore, BL_σ has 64 connected components of these types, all contractible.

The remaining ancestries appear in the 4-dimensional CW complex.

For dimension 4, the permutation has only one possible position for the diamonds. We will see that the CW complex is contractible in two ways: by analyzing the CW complex and by considering collapses.

Let us construct this CW complex step by step:

Step 1: There is a 4-cell in Figure 14.5, which comprises eight 3-cells. Horizontally, four 3-cells are attached, two prisms and two “parallelepipeds” yielding a solid torus. Vertically, the structure is similar, with two cubes and two prisms. Then, we have two solid tori attached such that every 3-cell in one solid torus is glued to every 3-cell in the other solid torus.

Therefore, by the known decomposition of a $\mathbb{S}^3$ into two solid tori (see [5]), we obtain a $\mathbb{S}^3$. Finally, a 4-cell with ancestry $\varepsilon_{19} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \bullet \blacklozenge \diamond \circ \blacklozenge)$ is attached, resulting in a $\mathbb{D}^4$.

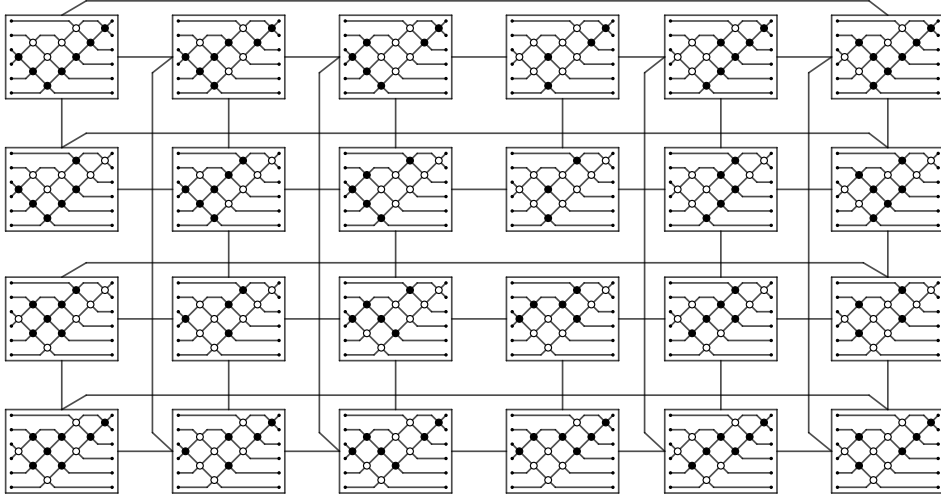


Figure 14.5: First step of the CW complex, with ancestry of dimension 4.

Step 2: Attach one 3-cell to the previous 4-cell.

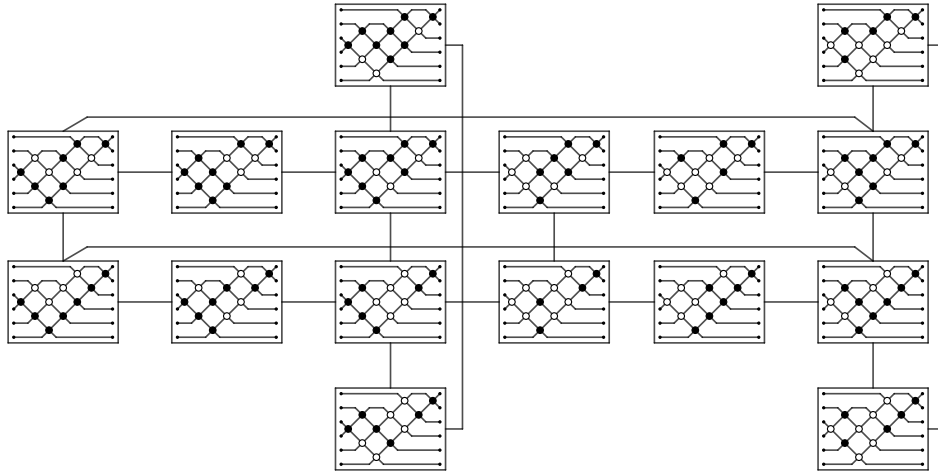


Figure 14.6: Second step of the CW complex, with 3-dimensional ancestry $\varepsilon_{20} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet)$.

This cell is attached to Figure 14.5 through a 2-cell with ancestry $\varepsilon_{21} = (\blacklozenge \blacklozenge \bullet \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet)$. Note that the 3-cell has two 2-cells attached like wings, these cells are also attached to Figure 14.5.

Step 3: Attach another 3-cell to the previous 4-cell.

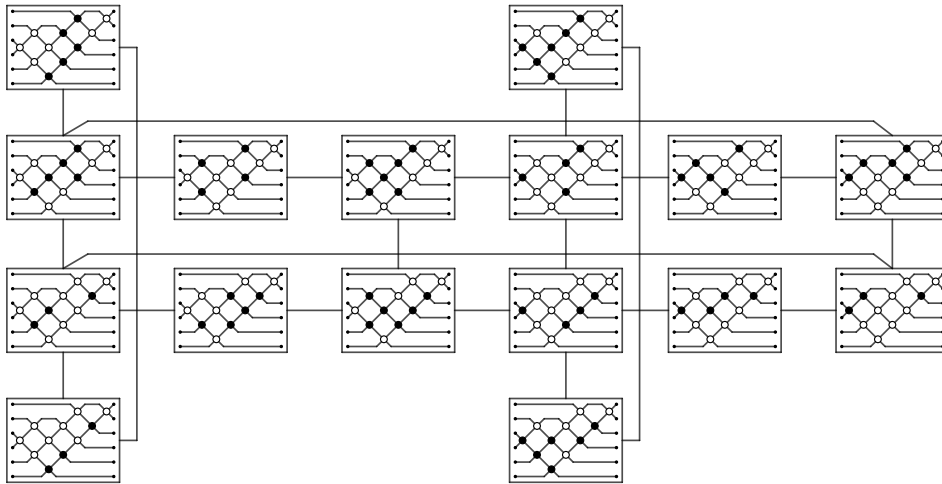


Figure 14.7: Third step of the CW complex, with 3-dimensional ancestry $\varepsilon_{22} = (\blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \circ)$.

This cell is attached to Figure 14.5 through a 2-cell with ancestry $\varepsilon_{23} = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \circ)$. Note that the 3-cell has two 2-cells attached like wings, these cells are also attached to Figure 14.5.

Therefore, BL_σ has 8 connected components of this type, all contractible.

Using collapses to analyze this problem, we can start with the initial CW complex and apply collapses to simplify it. First of all, we remove the cell of the higher dimension, in this case, 4.

$$(\diamond \diamond \diamond \circ \diamond \diamond \circ \diamond \diamond \diamond \diamond), (\diamond \diamond \bullet \diamond \diamond \bullet \bullet \diamond \diamond \bullet)$$

After that, we remove the cells of dimension 3.

$$\begin{aligned} &(\diamond \diamond \circ \diamond \diamond \bullet \circ \bullet \diamond \diamond \circ), (\diamond \diamond \circ \circ \diamond \circ \circ \circ \diamond \bullet \circ) \\ &(\diamond \diamond \bullet \circ \diamond \diamond \bullet \circ \diamond \diamond \circ), (\diamond \diamond \bullet \circ \diamond \circ \circ \circ \diamond \bullet \bullet) \\ &(\diamond \diamond \diamond \circ \diamond \bullet \bullet \diamond \diamond \circ \circ), (\diamond \diamond \bullet \circ \diamond \bullet \bullet \circ \diamond \circ \circ) \\ &(\diamond \diamond \diamond \circ \diamond \bullet \bullet \diamond \diamond \circ \circ), (\diamond \diamond \circ \bullet \diamond \bullet \bullet \diamond \circ \circ) \\ &(\diamond \diamond \circ \bullet \diamond \diamond \circ \bullet \diamond \diamond \circ), (\diamond \diamond \circ \bullet \diamond \circ \circ \bullet \diamond \bullet \bullet) \\ &(\circ \diamond \diamond \circ \diamond \diamond \circ \diamond \bullet \circ \diamond), (\circ \diamond \circ \bullet \diamond \diamond \bullet \bullet \bullet \circ \diamond) \\ &(\diamond \bullet \diamond \circ \circ \diamond \bullet \diamond \diamond \circ \diamond), (\circ \circ \circ \diamond \circ \diamond \bullet \circ \diamond \circ \diamond) \\ &(\bullet \diamond \diamond \circ \diamond \diamond \bullet \diamond \circ \circ \diamond), (\bullet \diamond \bullet \circ \diamond \diamond \bullet \circ \circ \circ \diamond) \\ &(\diamond \circ \diamond \circ \circ \diamond \circ \diamond \diamond \circ \diamond), (\bullet \circ \bullet \diamond \bullet \bullet \bullet \circ \diamond \circ \diamond) \end{aligned}$$

Now, we continue with a long sequence of more 72 collapses, ending with a point. In this case

$$(\bullet \circ \bullet \circ \bullet \bullet \bullet \bullet \circ \circ \bullet).$$

Therefore, BL_σ has 8 connected components of this type, all contractible. Summing up, BL_σ has a total of 104 connected components, all contractible.

There are 3 permutations that have a CW complex similar to the one described. They are,

$$\sigma_1 = a_1 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2, \quad \sigma_2 = a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4,$$

$$\sigma_3 = a_1 a_2 a_1 a_3 a_2 a_4 a_3 a_2 a_5 a_4 a_3 \in S_6.$$

Type 14.2. For $\sigma = [364521] = a_2 a_3 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1 \in S_6$, it follows that

$$\begin{aligned} \sigma = \frac{1}{4\sqrt{2}} &(1 - \hat{a}_1 - \hat{a}_2 + \hat{a}_1 \hat{a}_2 - \hat{a}_3 + \hat{a}_1 \hat{a}_3 - \hat{a}_2 \hat{a}_3 + \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_1 \hat{a}_4 + \hat{a}_2 \hat{a}_4 \\ &- \hat{a}_1 \hat{a}_2 \hat{a}_4 - \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_3 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_5 - \hat{a}_1 \hat{a}_5 - \hat{a}_2 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_5 \\ &+ \hat{a}_3 \hat{a}_5 + \hat{a}_1 \hat{a}_3 \hat{a}_5 - \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_4 \hat{a}_5 - \hat{a}_2 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 \\ &- \hat{a}_3 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are six possible positions for the diamonds. If r_1 has opposite signs while the remaining rows have equal signs, this configuration results in 32 copies of the CW complex shown in Figure 14.8.

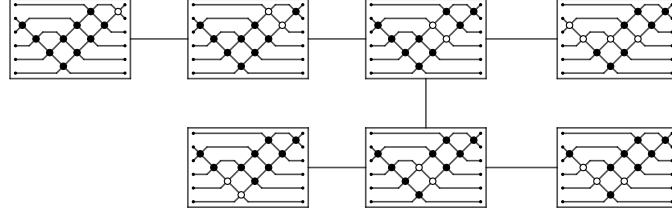


Figure 14.8: CW complex of dimension 1.

If r_2 has signs $(\bullet \bullet \circ)$ and the remaining rows have equal signs, we have the CW complex shown in Figure 14.9. This results in 32 copies.

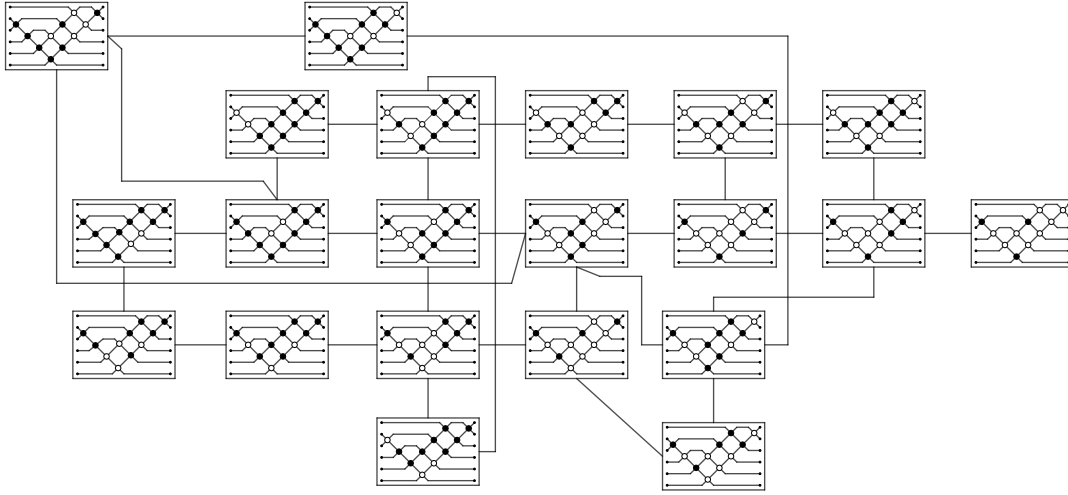


Figure 14.9: CW complex with ten 2-cells.

Therefore, BL_σ has 64 connected components of these types, all contractible.

The remaining ancestries of dimensions 1 and 2 appear in higher-dimensional CW complexes.

In dimension 3, there are four possible positions for the diamonds, all of which are illustrated together in Figure 14.10. In this CW complex, some 3-cells have 2-cells attached to them, resembling wings.

Let us see that the CW complex is contractible in two ways: by analyzing the CW complex and by considering collapses.

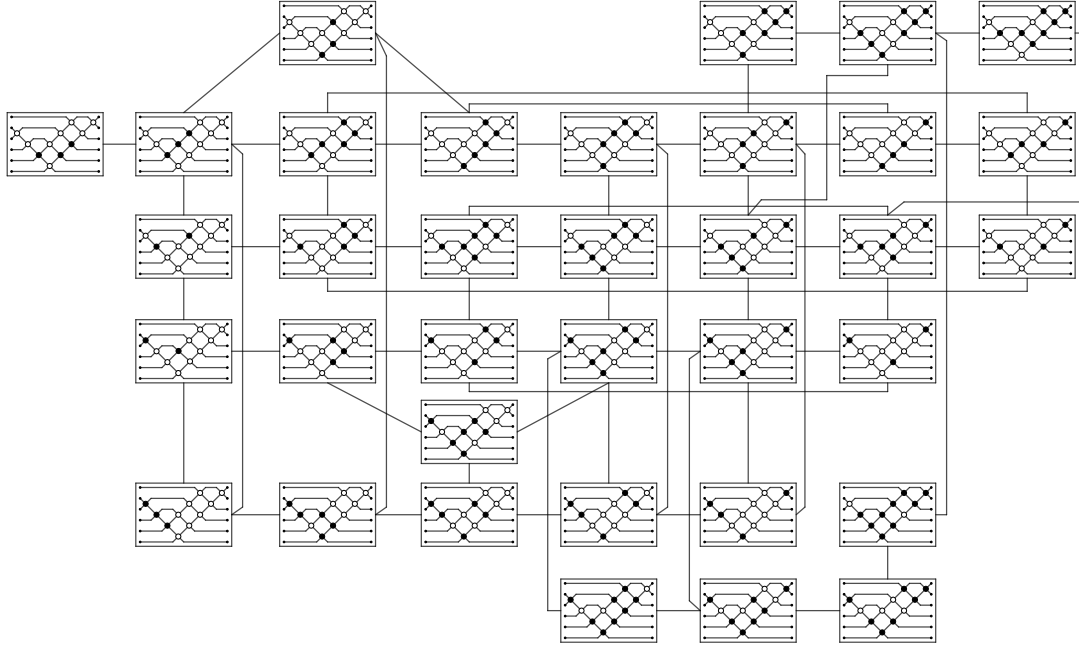


Figure 14.10: CW complex with three 3-cells.

The 3-cells fill a vertically convex solid with 12 faces, along with a cube extending from left to right. Horizontally, from top to bottom, the 3-cells fill a prism and a cube.

Considering collapses, begin by removing the 3-dimensional cells:

$$\begin{aligned}
 &(\blacklozenge \blacklozenge \blacklozenge \circ \circ \circ \circ \diamond \diamond \diamond \circ), (\circ \circ \blacklozenge \blacklozenge \blacklozenge \circ \circ \diamond \circ \diamond \circ) \\
 &(\circ \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \diamond \diamond \diamond \circ), (\circ \blacklozenge \circ \diamond \circ \diamond \circ \circ \blacklozenge \blacklozenge) \\
 &(\blacklozenge \blacklozenge \circ \diamond \diamond \blacklozenge \blacklozenge \circ \circ \circ \diamond), (\blacklozenge \blacklozenge \circ \diamond \diamond \blacklozenge \blacklozenge \circ \circ \circ \circ) \\
 &(\blacklozenge \circ \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \diamond \circ \circ \diamond), (\blacklozenge \circ \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \diamond \circ \circ \circ)
 \end{aligned}$$

After that, we continue to remove the cells with a long sequence of more 61 collapses until we finish with one point:

$$(\circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet).$$

Therefore, BL_σ has 32 connected components of this type, all contractible. Summing up, BL_σ has a total of 128 connected components, all of them contractible.

There are 11 permutations that have a CW complex similar to the one described. They are,

$$\sigma_1 = a_2 a_3 a_2 a_1 a_4 a_3 a_5 a_4 a_3 a_2 a_1, \quad \sigma_2 = a_2 a_1 a_3 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1,$$

$$\sigma_3 = a_1 a_2 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1, \quad \sigma_4 = a_1 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1,$$

$$\begin{aligned}
\sigma_5 &= a_2a_1a_3a_2a_4a_3a_5a_4a_3a_2a_1, & \sigma_6 &= a_1a_2a_3a_4a_3a_2a_1a_5a_4a_3a_2, \\
\sigma_7 &= a_1a_2a_3a_2a_1a_4a_3a_5a_4a_3a_2, & \sigma_8 &= a_1a_2a_3a_2a_1a_4a_3a_2a_1a_5a_4, \\
\sigma_9 &= a_1a_2a_1a_3a_4a_3a_2a_5a_4a_3a_2, & \sigma_{10} &= a_1a_2a_1a_3a_2a_4a_3a_5a_4a_3a_2, \\
\sigma_{11} &= a_1a_2a_1a_3a_2a_4a_3a_2a_1a_5a_4 \in S_6.
\end{aligned}$$

Type 14.3. For $\sigma = [436521] = a_1a_3a_4a_3a_2a_1a_5a_4a_3a_2a_1 \in S_6$, it follows that

$$\begin{aligned}
\acute{\sigma} = \frac{1}{2\sqrt{2}} & (1 - \hat{a}_1 - \hat{a}_2 - \hat{a}_1\hat{a}_2 - \hat{a}_3 - \hat{a}_1\hat{a}_3 - \hat{a}_2\hat{a}_3 + \hat{a}_1\hat{a}_2\hat{a}_3 - \hat{a}_4 + \hat{a}_1\hat{a}_4 + \hat{a}_2\hat{a}_4 \\
& + \hat{a}_1\hat{a}_2\hat{a}_4 - \hat{a}_3\hat{a}_4 - \hat{a}_1\hat{a}_3\hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 - \hat{a}_1\hat{a}_5 + \hat{a}_2\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_5 \\
& - \hat{a}_3\hat{a}_5 + \hat{a}_1\hat{a}_3\hat{a}_5 - \hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_4\hat{a}_5 \\
& + \hat{a}_3\hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5).
\end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are six possible positions for the diamonds. If the diamonds are in r_4 and the remaining rows have equal signs, we have the CW complex in Figure 14.11. This results in 32 copies. Therefore, BL_σ has 32 contractible connected components of this type.

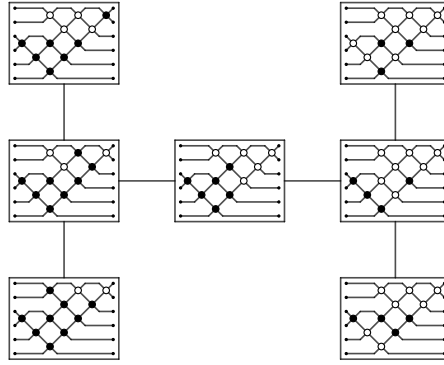


Figure 14.11: CW complex of dimension 1.

The remaining ancestries of dimension 1 appear in higher-dimensional CW complexes.

For dimension 2, we have ten possible positions for the diamonds. If r_1 or r_3 has signs $(\circ \bullet \circ)$, and the remaining rows have the same signs, we have the CW complex in Figure 14.12. This results in 32 copies. Therefore, BL_σ has 32 contractible connected components of this type.

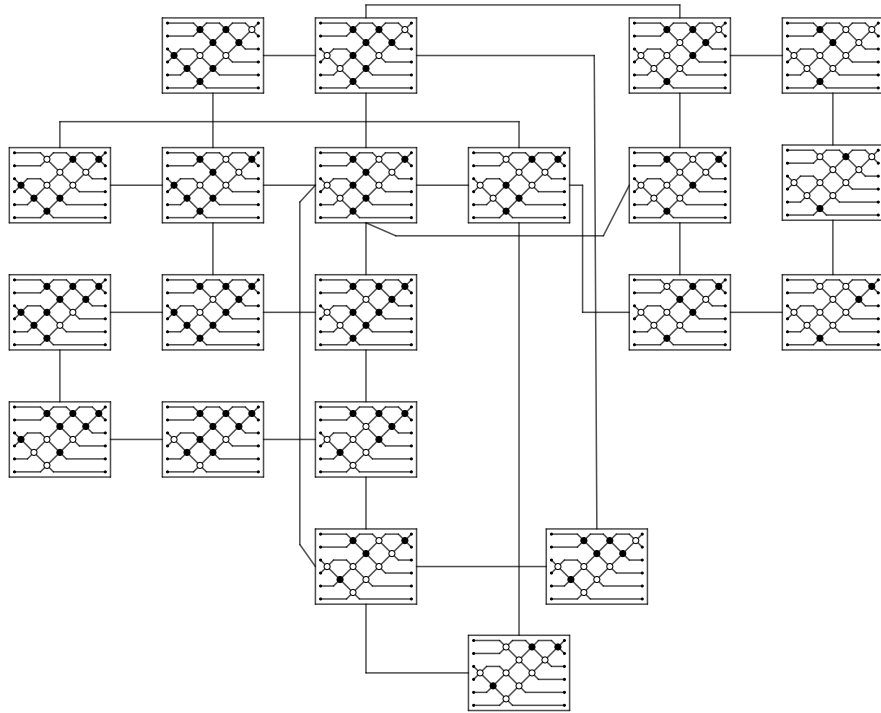


Figure 14.12: CW complex with ten 2-cells.

The remaining ancestries of dimension 2 appear in a higher-dimensional CW complex.

For dimension 3, we have four possible positions for the diamonds and they will appear together. In dimension 1, if the diamonds are in r_2 , and the other rows have equal signs, we obtain the CW complex in Figure 14.13, which results in 32 copies. The cells of dimension 3 fill the four prisms completely.

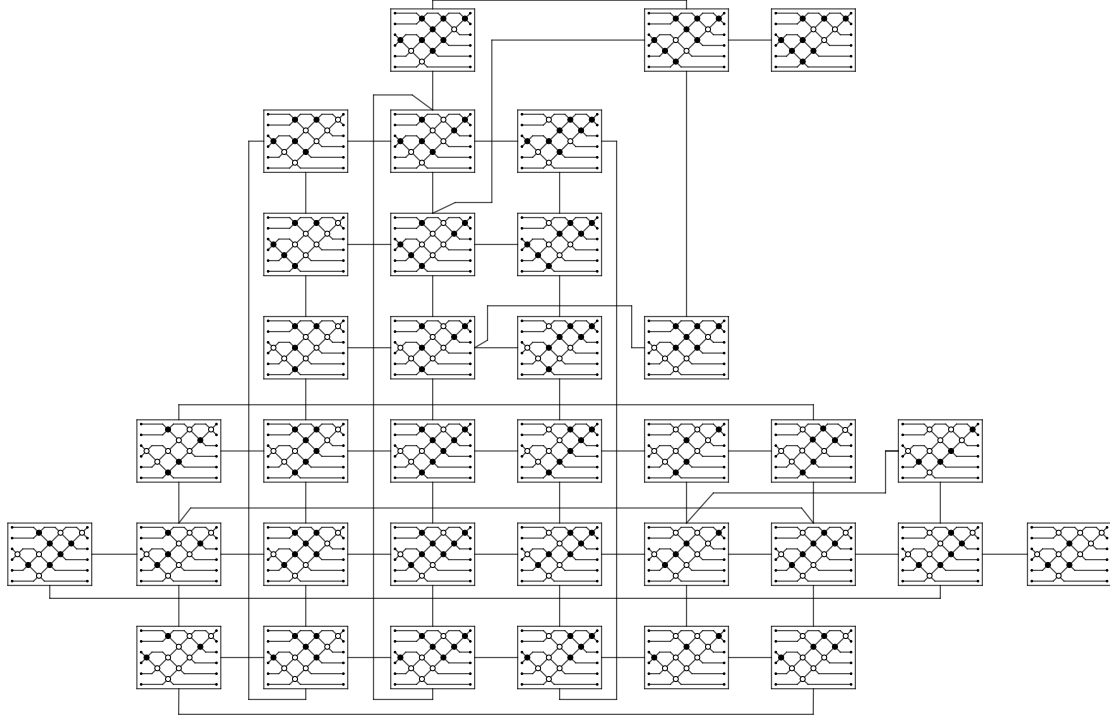


Figure 14.13: CW complex of dimension 3.

The following pairs of ancestries represent the vertices at the upper left and lower right corners of the four prisms:

$$\varepsilon_3 = (\bullet \circ \circ \circ \circ \circ \bullet \bullet \bullet \bullet \circ), \varepsilon_4 = (\circ \circ \bullet \bullet \bullet \bullet \circ \circ \circ \bullet \circ);$$

$$\varepsilon_5 = (\bullet \circ \bullet \bullet \circ \circ \circ \circ \bullet \circ), \varepsilon_6 = (\circ \bullet \circ \circ \bullet \bullet \circ \circ \bullet \circ);$$

$$\varepsilon_7 = (\bullet \bullet \circ \bullet \circ \bullet \circ \bullet \circ \circ \circ), \varepsilon_8 = (\bullet \bullet \circ \circ \bullet \circ \circ \circ \bullet \bullet);$$

$$\varepsilon_9 = (\bullet \bullet \circ \bullet \circ \circ \circ \bullet \circ \bullet \bullet), \varepsilon_{10} = (\circ \bullet \circ \circ \circ \bullet \circ \circ \bullet \bullet).$$

Horizontally, from left to right, the prisms are $\varepsilon_{11} = (\bullet \blacklozenge \blacklozenge \circ \circ \blacklozenge \bullet \blacklozenge \blacklozenge \circ \circ \blacklozenge)$ and $\varepsilon_{12} = (\blacklozenge \blacklozenge \circ \circ \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet)$. Vertically, from left to right, the prisms are $\varepsilon_{13} = (\blacklozenge \circ \blacklozenge \blacklozenge \bullet \circ \blacklozenge \bullet \blacklozenge \blacklozenge)$ and $\varepsilon_{14} = (\blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \bullet \circ \circ \bullet \blacklozenge \blacklozenge)$.

Therefore, BL_σ has 32 contractible connected components of this type. Summing up, BL_σ has a total of 128 connected components, all of them contractible.

Three permutations share a CW complex similar to the one described. They are:

$$\sigma_1 = a_1 a_3 a_2 a_1 a_4 a_3 a_5 a_2 a_3 a_2 a_1, \quad \sigma_2 = a_1 a_2 a_1 a_3 a_4 a_3 a_2 a_1 a_5 a_4 a_3,$$

$$\sigma_3 = a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_5 a_4 a_3 \in S_6.$$

Type 14.4. For $\sigma = [453621] = a_2a_1a_4a_3a_2a_1a_5a_4a_3a_2a_1 \in S_6$ it follows that

$$\sigma = \frac{1}{2\sqrt{2}}(-\hat{a}_1 - \hat{a}_2\hat{a}_3 - \hat{a}_4 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5).$$

There exist $2^5 = 32$ thin ancestries, resulting in 32 contractible connected components in BL_σ .

For dimension 1, there are six possible positions for the diamonds. If the diamonds are in r_4 , and the remaining rows have equal signs, we have the CW complex in Figure 14.14. This results in 16 copies. Hence, BL_σ has 16 contractible connected components of this type.

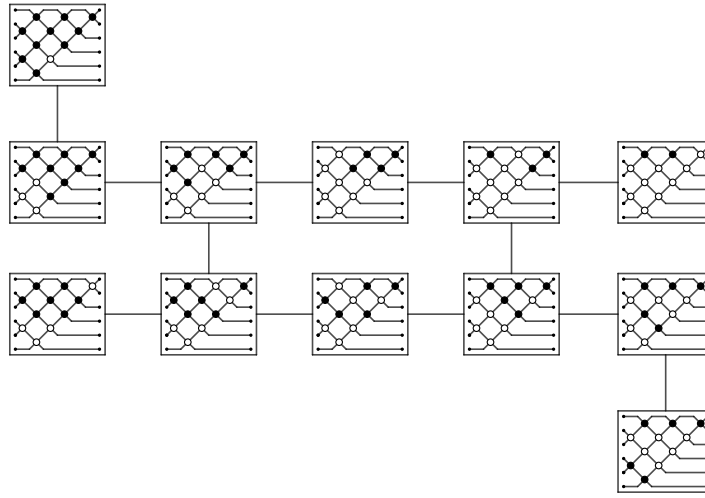


Figure 14.14: CW complex of dimension 2.

If r_1 has signs $(\bullet \circ \circ)$ and the other rows have equal signs, we obtain the CW complex shown in Figure 14.15. This results in 16 copies.

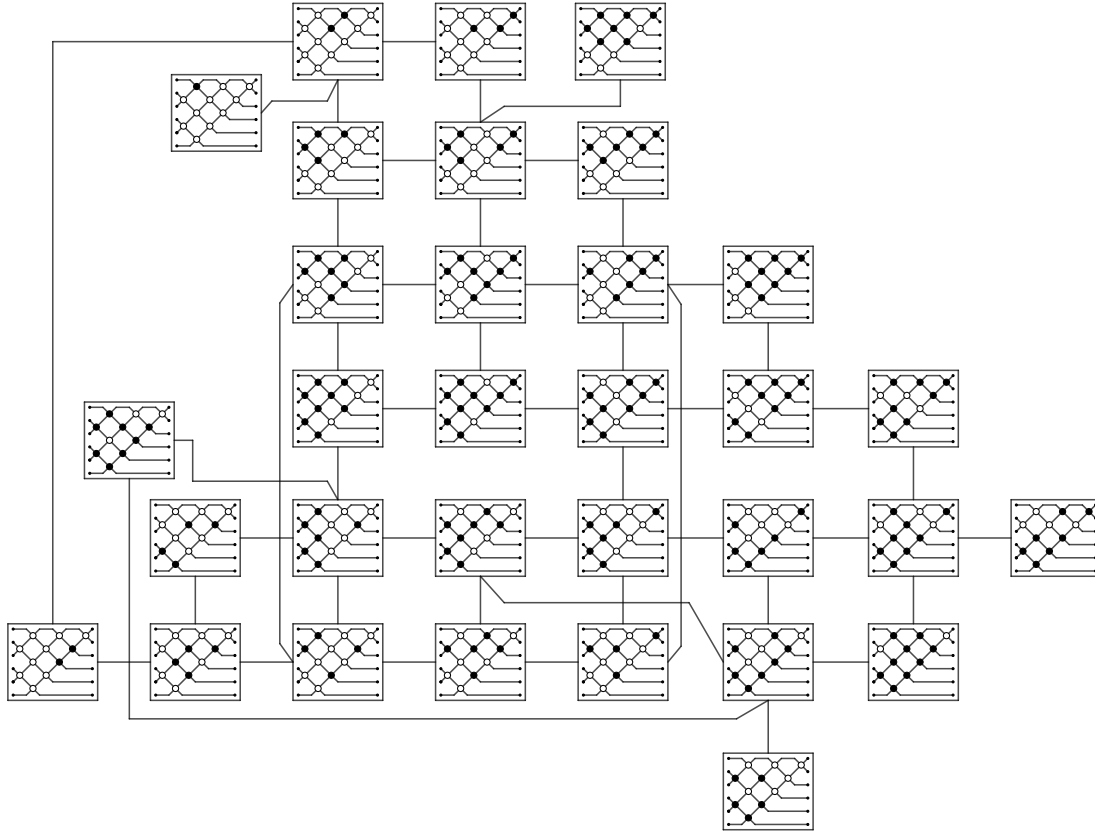


Figure 14.15: CW complex of dimension 3.

The cell with ancestry $\varepsilon_2 = (\bullet \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \blacklozenge)$ of dimension 3 completely fills the prism. The ancestries $\varepsilon_3 = (\bullet \bullet \circ \circ \bullet \bullet \circ \bullet \bullet \circ \circ)$ and $\varepsilon_4 = (\bullet \circ \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet)$ are the vertices on the upper left and lower right corners of the prism.

This CW complex comprises one 3-cell and ten 2-cells attached to the previous one. Additionally, there are four 1-cells and four 0-cells attached.

If r_2 has signs $(\bullet \circ \circ)$ and the remaining rows have equal signs, we have the CW complex shown in Figure 14.16. This results in 16 copies.

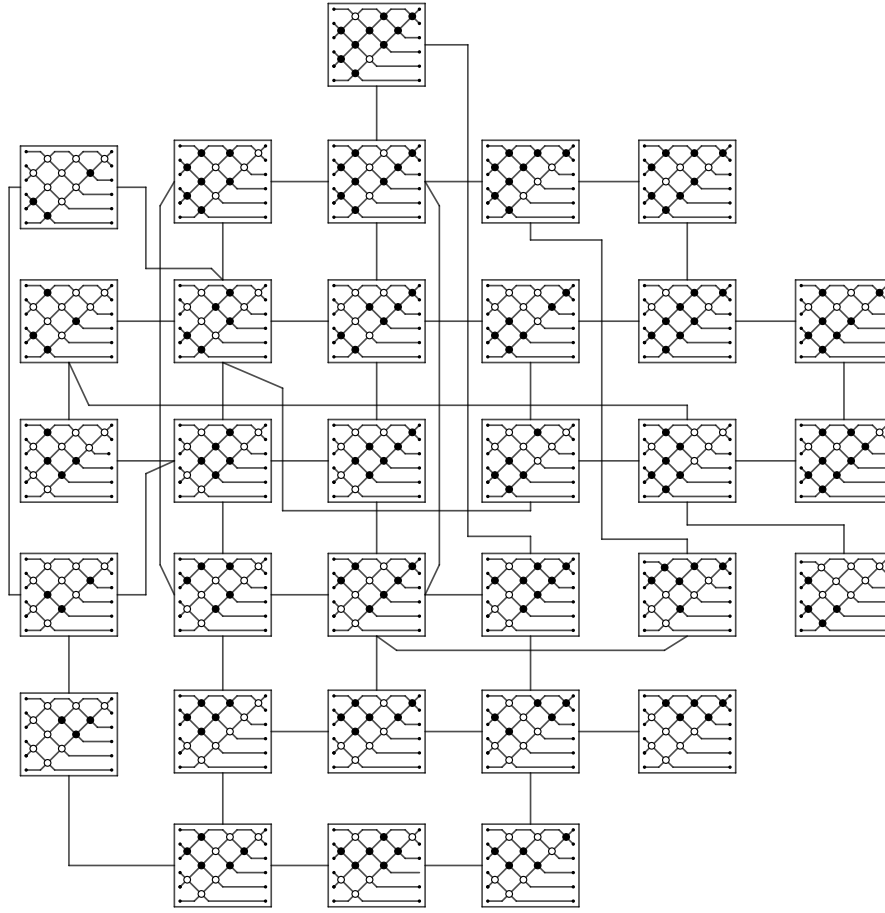


Figure 14.16: CW complex of dimension 3.

The cell with ancestry $\varepsilon_5 = (\diamond \bullet \diamond \bullet \diamond \bullet \diamond \bullet \circ \diamond)$ of dimension 3 fills the cube completely. The ancestries $\varepsilon_6 = (\bullet \bullet \bullet \bullet \circ \bullet \bullet \circ \bullet \circ \circ)$ and $\varepsilon_7 = (\bullet \bullet \circ \circ \circ \circ \bullet \bullet \bullet \bullet)$ are the vertices on the upper left and lower right corners of the prism.

This CW complex comprises one 3-cell and twelve 2-cells attached to the previous one. Additionally, there are two 1-cells and two 0-cells attached.

If the diamonds are in r_3 and the remaining rows have equal signs, we have the CW complex in Figure 14.17. This results in 16 copies.

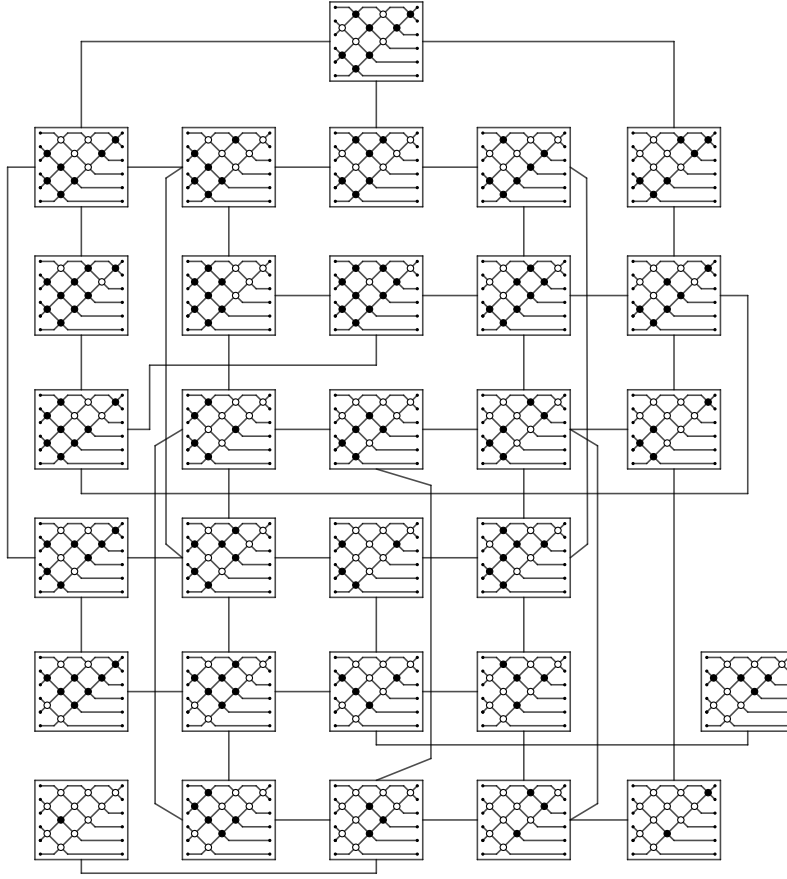


Figure 14.17: CW complex of dimension 3.

There are three cells of dimension 3 in this CW complex. They fill two prisms and one “parallelepiped” completely. The “parallelepiped”, at the top of Figure 14.17, is $\varepsilon_8 = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ)$. The first prism, below the previous “parallelepiped”, is $\varepsilon_9 = (\blacklozenge \blacklozenge \blacklozenge \circ \circ \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \circ)$. The second prism, which is harder to visualize in the figure due to its position on top of the “parallelepiped”, is $\varepsilon_{10} = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \blacklozenge)$.

The following pairs of ancestries represent the vertices at the upper left and lower right corners of the prisms and the “parallelepiped”, respectively:

$$\varepsilon_{11} = (\bullet \bullet \bullet \circ \circ \bullet \circ \bullet \circ \circ), \varepsilon_{12} = (\circ \circ \circ \circ \bullet \circ \bullet \circ \bullet \circ);$$

$$\varepsilon_{13} = (\bullet \circ \bullet \bullet \circ \circ \bullet \bullet \circ \bullet \bullet), \varepsilon_{14} = (\circ \circ \bullet \bullet \circ \circ \bullet \circ \circ \bullet \bullet);$$

$$\varepsilon_{15} = (\bullet \circ \bullet \bullet \circ \bullet \bullet \bullet \circ \circ \circ), \varepsilon_{16} = (\circ \bullet \bullet \bullet \bullet \circ \bullet \circ \circ \bullet \circ).$$

Thus, BL_σ has 48 connected components of these types, all contractible.

The remaining ancestries of dimensions 1, 2 and 3 will appear in a 4-dimensional CW complex.

For dimension 4, there is only one possible position for the diamonds. This CW complex consists of ten cells of dimension 3, with two of each possible type. Let us construct step by step.

Step 1: First, there are eight 3-cells that yields the 4-cell in Figure 14.18 with ancestry $\varepsilon_{17} = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$, which as we saw before, represents a $\mathbb{D}^4$.

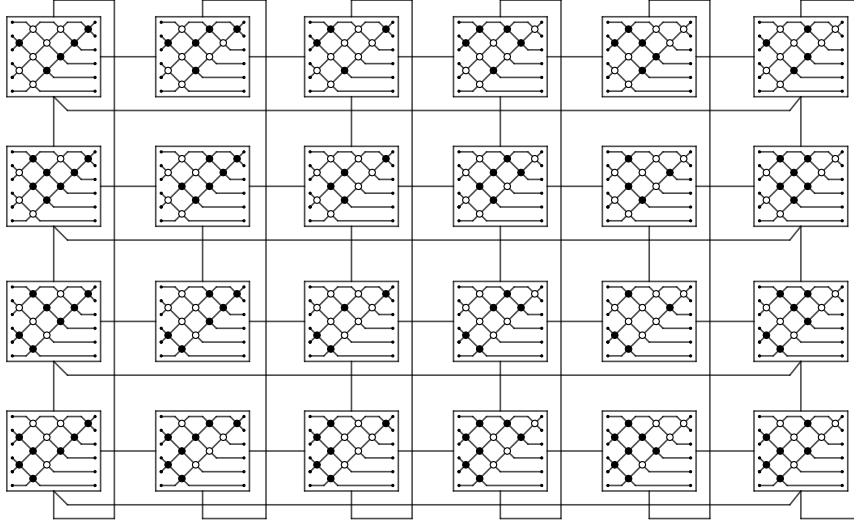


Figure 14.18: First part of the CW complex of dimension 4.

Step 2: Attach to this 4-cell the 3-cell in Figure 14.19 with ancestry $\varepsilon_{18} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet)$. The 3-cell has two 2-cells attached. The attachment occurs through the 2-cell with ancestry $\varepsilon_{20} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet)$,

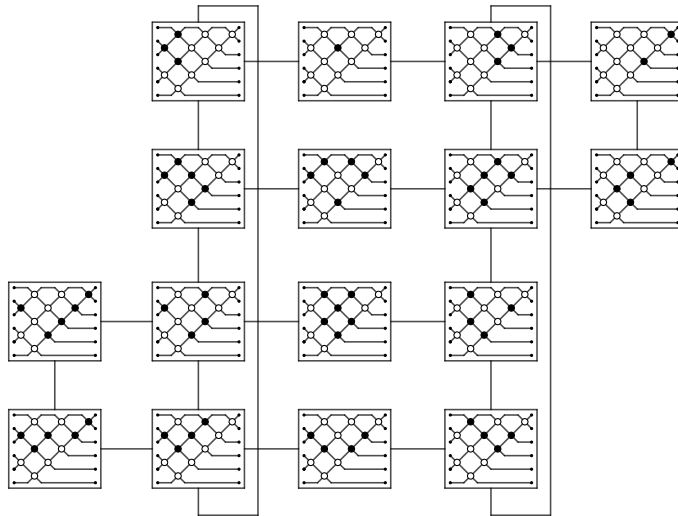


Figure 14.19: 3-cell with ancestry $\varepsilon_{18} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet)$.

Step 3: Attach to this 4-cell the 3-cell in Figure 14.20 with ancestry $\varepsilon_{19} = (\blacklozenge \blacklozenge \circ \blacklozenge \circ \diamond \circ \circ \diamond \circ)$. The 3-cell has two 2-cells attached. The attachment occurs through the 2-cell with ancestry $\varepsilon_{21} = (\blacklozenge \blacklozenge \bullet \bullet \circ \diamond \bullet \circ \circ \diamond \bullet)$.

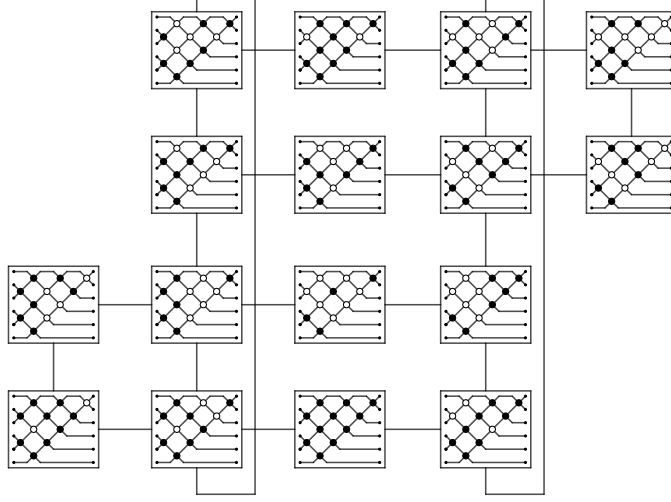


Figure 14.20: 3-cell with ancestry $\varepsilon_{19} = (\blacklozenge \blacklozenge \circ \blacklozenge \circ \diamond \circ \circ \diamond \circ)$.

Therefore, BL_σ has 8 connected components of this type, all contractible. Summing up, BL_σ has a total of 104 connected components, all of them contractible.

There are 3 permutations that have a CW complex similar to the one described. They are,

$$\sigma_1 = a_1 a_3 a_2 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2, \quad \sigma_2 = a_2 a_1 a_3 a_2 a_4 a_3 a_2 a_1 a_5 a_4 a_3,$$

$$\sigma_3 = a_1 a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4 \in S_6.$$

Type 14.5. For $\sigma = [456231] = a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1 \in S_6$, it follows that

$$\begin{aligned} \sigma = \frac{1}{4\sqrt{2}} & (-1 - \hat{a}_1 - \hat{a}_2 + \hat{a}_1 \hat{a}_2 + \hat{a}_3 - \hat{a}_1 \hat{a}_3 - \hat{a}_2 \hat{a}_3 - \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_1 \hat{a}_4 \\ & + \hat{a}_2 \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_4 - \hat{a}_3 \hat{a}_4 - \hat{a}_1 \hat{a}_3 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_5 - \hat{a}_1 \hat{a}_5 \\ & + \hat{a}_2 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_5 - \hat{a}_3 \hat{a}_5 + \hat{a}_1 \hat{a}_3 \hat{a}_5 - \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 + \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_4 \hat{a}_5 \\ & + \hat{a}_2 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 - \hat{a}_3 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are six possible positions for the diamonds. If the diamonds are in r_4 , and the remaining rows have equal signs, we have the

CW complex in Figure 14.21. This results in 32 copies. Therefore, BL_σ has 32 connected components of this type, all contractible.

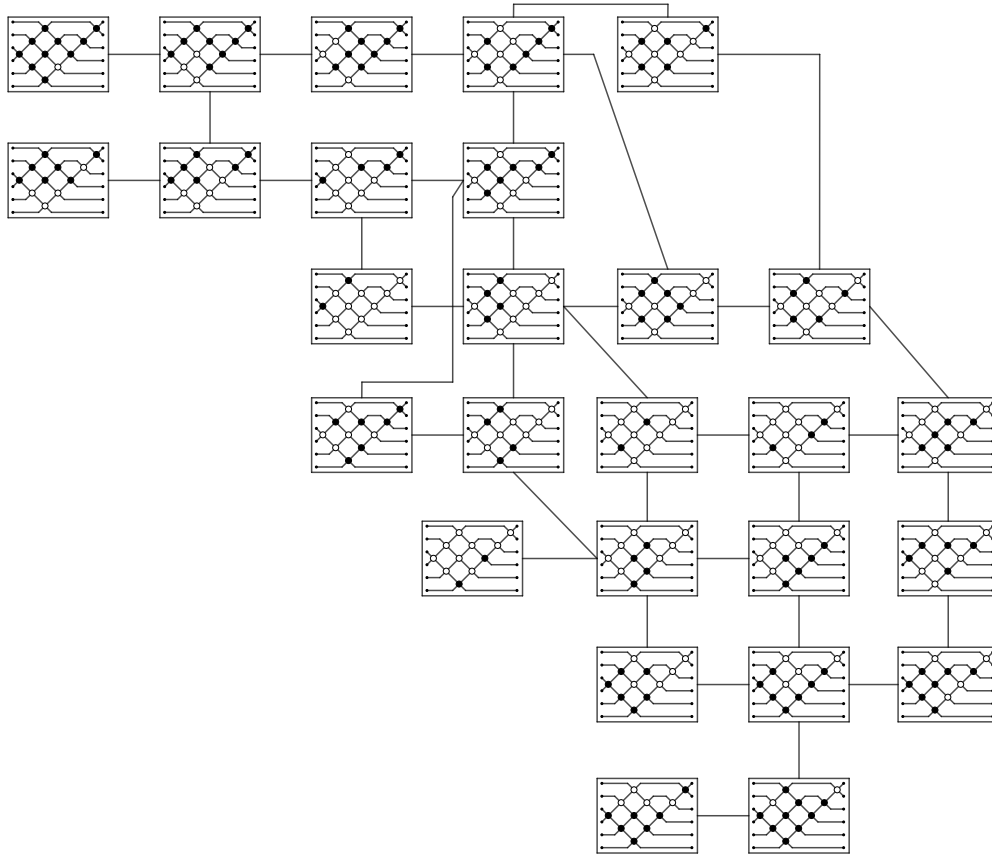


Figure 14.21: CW complex of dimension 2 with ten ancestries of dimension 2.

The remaining ancestries of dimensions 1 and 2 appear in higher-dimensional CW complexes.

For dimension 3, there are five possible positions for the diamonds, as shown in Figure 14.22.

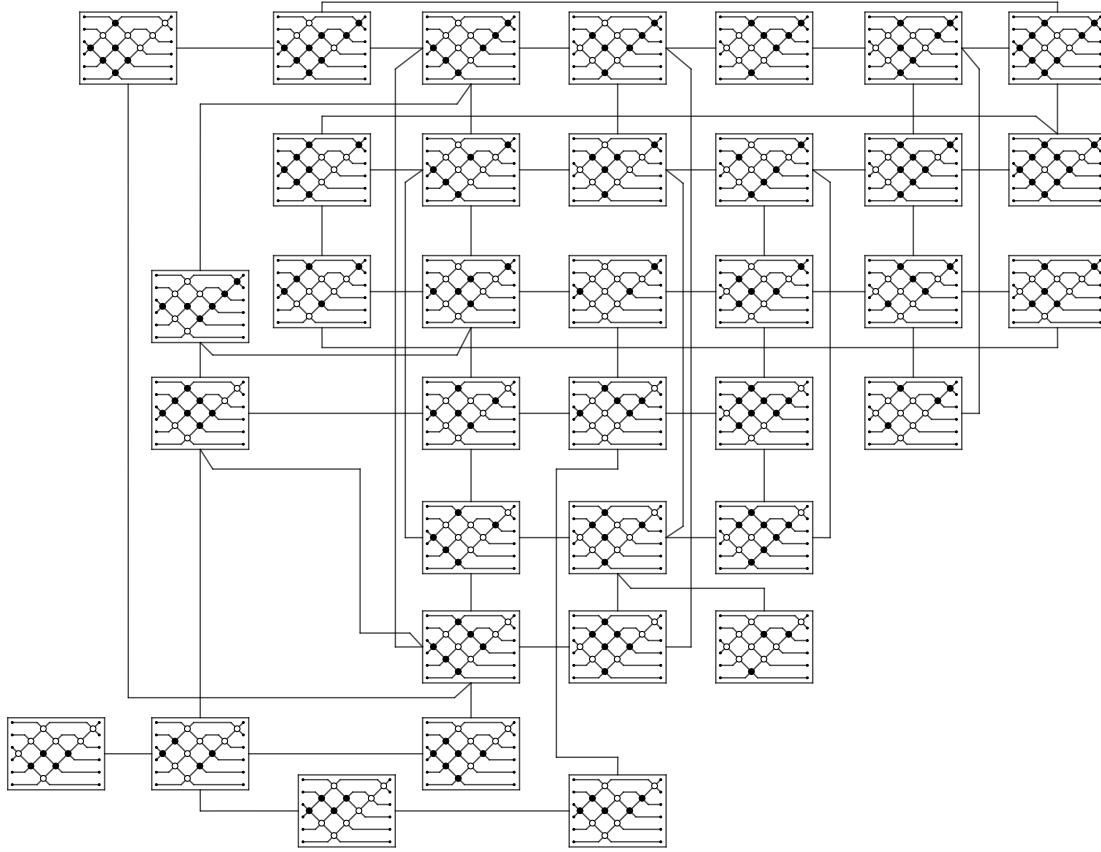


Figure 14.22: CW complex with five 3-cells.

There are five cells of dimension 3, filling two “parallelepipeds”, two cubes, and one prism completely. Additionally, there are some cells of lower dimension attached.

Horizontally, at the top of the CW complex, we have the two “parallelepiped”: $\varepsilon_1 = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \circ \bullet \circ \blacklozenge \blacklozenge \bullet)$ and $\varepsilon_2 = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ \bullet)$. Vertically, from left to right, we have the two cubes: $\varepsilon_3 = (\blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge)$ and $\varepsilon_4 = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet)$. Furthermore, in the center of Figure 14.22, we have the prism: $\varepsilon_5 = (\bullet \circ \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$.

Let us see that is contractible thinking about collapses. First of all, we remove the cells of dimension 3:

$$\begin{aligned}
 &(\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \blacklozenge \circ \bullet), (\bullet \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet) \\
 &(\blacklozenge \circ \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \blacklozenge), (\bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge) \\
 &(\blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet), (\bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet) \\
 &(\blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge), (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge) \\
 &(\bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge), (\bullet \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet).
 \end{aligned}$$

After that, we continue to remove the cells with a long sequence off more 60 collapses until we finish with the point

$$(\bullet \circ \circ \circ \bullet \circ \bullet \bullet \circ \circ \circ \bullet).$$

Therefore, BL_σ has 32 copies of this type, all contractible. Summing up, BL_σ has a total of 96 connected components, all of them contractible.

There are 3 permutations that have a CW complex similar to the one described. They are,

$$\sigma_1 = a_2 a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3, \quad \sigma_2 = a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 a_2,$$

$$\sigma_3 = a_1 a_2 a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 \in S_6.$$

Type 14.6. For $\sigma = [456312] = a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 \in S_6$ we have

$$\begin{aligned} \acute{\sigma} = \frac{1}{4\sqrt{2}} & (-1 - \hat{a}_1 - \hat{a}_2 - \hat{a}_1 \hat{a}_2 - \hat{a}_3 - \hat{a}_1 \hat{a}_3 - \hat{a}_2 \hat{a}_3 - \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_1 \hat{a}_4 - \hat{a}_2 \hat{a}_4 \\ & + \hat{a}_1 \hat{a}_2 \hat{a}_4 - \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_3 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_5 + \hat{a}_1 \hat{a}_5 + \hat{a}_2 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_5 \\ & - \hat{a}_3 \hat{a}_5 + \hat{a}_1 \hat{a}_3 \hat{a}_5 \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 \\ & - \hat{a}_3 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are six possible positions for the diamonds. If the diamonds are in r_1 and the remaining rows have equal signs, we have the CW complex in Figure 14.23. This results in 32 copies. Therefore, BL_σ has 32 contractible connected components of this type.

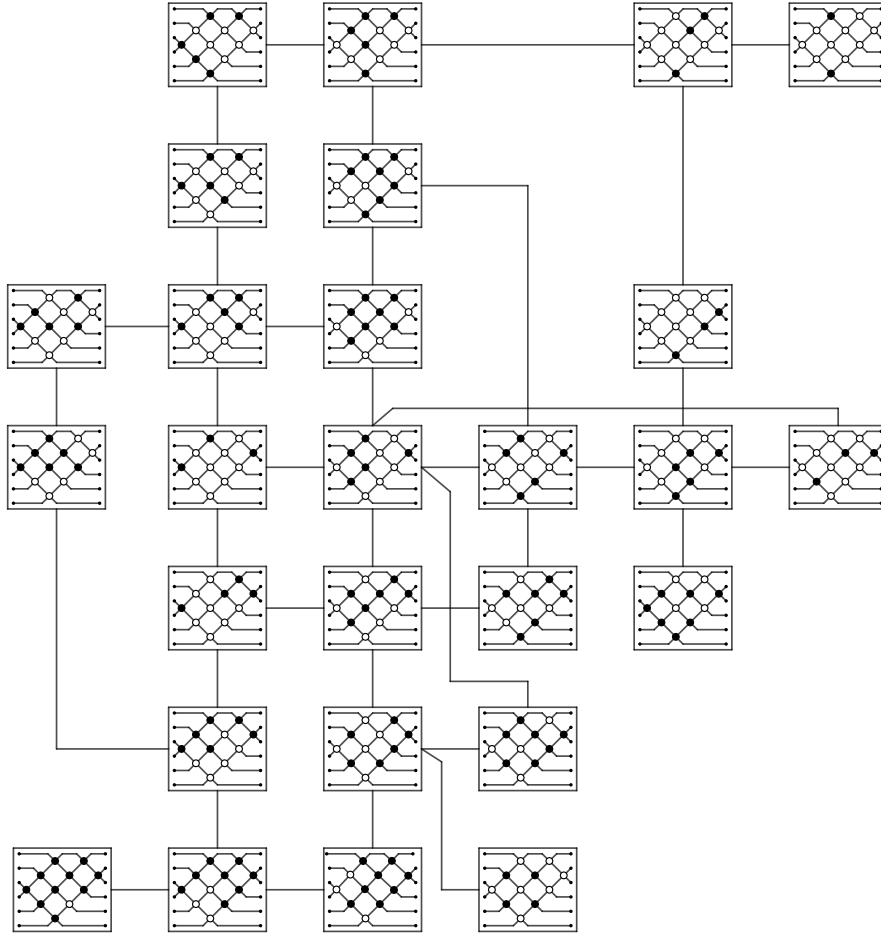


Figure 14.23: CW complex of dimension 2 with ten ancestors of dimension 2.

The remaining ancestors of dimensions 1 and 2 appear in higher-dimensional CW complexes.

For dimension 3, there are five possible positions for the diamonds, which appear together, as shown in Figure 14.24.

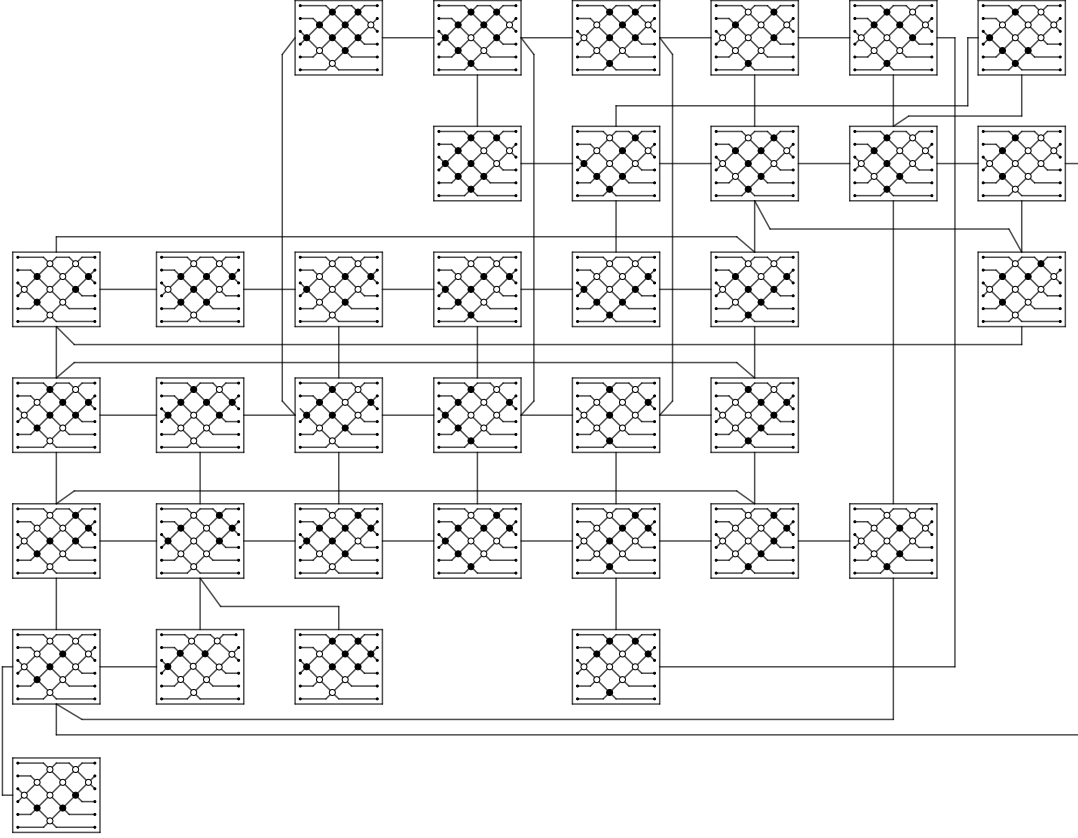


Figure 14.24: CW complex with five 3-cells.

There are five cells of dimension 3, filling three “parallelepipeds” and two prisms completely. Additionally, there are some cells of lower dimension attached.

Horizontally, in the center of Figure 14.24 we have the one “parallelepiped” and one prism: $\varepsilon_1 = (\blacklozenge \blacklozenge \bullet \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \bullet)$ and $\varepsilon_2 = (\blacklozenge \circ \blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \blacklozenge \circ)$. Vertically, from left to right, we have two “parallelepipeds”: $\varepsilon_3 = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$ and $\varepsilon_4 = (\circ \blacklozenge \blacklozenge \circ \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge)$. Furthermore, the last one is more challenging to spot in the Figure 14.24, it is the prism: $\varepsilon_5 = (\circ \blacklozenge \blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \blacklozenge \circ)$.

Let us see that it is contractible by considering collapses. First, we remove the cells of dimension 3:

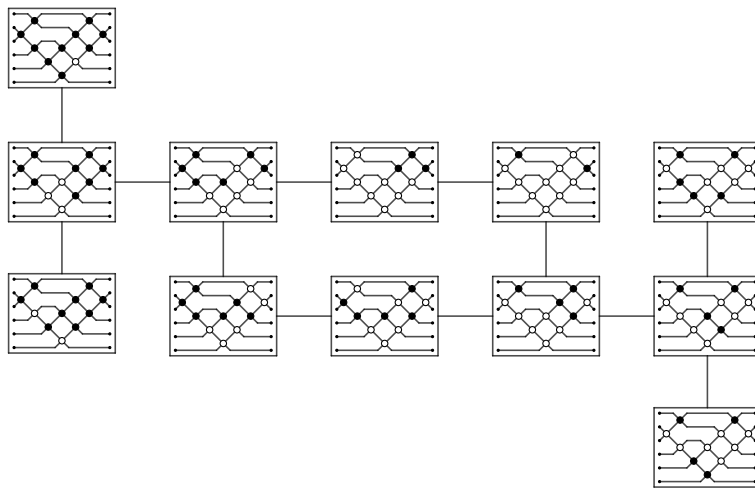
$$\begin{aligned}
 &(\circ \blacklozenge \blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \blacklozenge \circ), (\circ \bullet \circ \blacklozenge \circ \bullet \blacklozenge \bullet \circ \blacklozenge \circ \blacklozenge) \\
 &(\blacklozenge \circ \blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \blacklozenge \circ), (\blacklozenge \bullet \bullet \bullet \blacklozenge \circ \blacklozenge \bullet \circ \bullet \bullet) \\
 &(\blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge), (\blacklozenge \circ \circ \bullet \blacklozenge \bullet \bullet \bullet \bullet \circ \blacklozenge) \\
 &(\blacklozenge \bullet \bullet \bullet \circ \blacklozenge \circ \bullet \blacklozenge \bullet \bullet), (\bullet \blacklozenge \bullet \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \bullet \bullet) \\
 &(\circ \blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \blacklozenge), (\circ \bullet \blacklozenge \circ \blacklozenge \circ \bullet \bullet \bullet \blacklozenge \circ).
 \end{aligned}$$

$$(\circ \circ \bullet \circ \bullet \bullet \bullet \bullet \circ \bullet \circ).$$

The permutation $\sigma_1 = a_2a_1a_3a_2a_1a_4a_3a_2a_5a_4a_3 \in S_6$ has a CW complex similar to the one described.

$$\sigma = \frac{1}{2\sqrt{2}}(-\hat{a}_1 - \hat{a}_2\hat{a}_3 + \hat{a}_2\hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5).$$

For dimension 1, there are six possible positions for the diamonds. If r_4 has opposite signs and the remaining rows have equal signs, we have the CW complex in Figure 14.25. This results in 16 copies. Therefore, BL_σ has 16 connected components of this type, all contractible.



If r_1 has opposite signs and the remaining rows have equal signs, we have the CW complex Figure 14.26. This results in 16 copies.

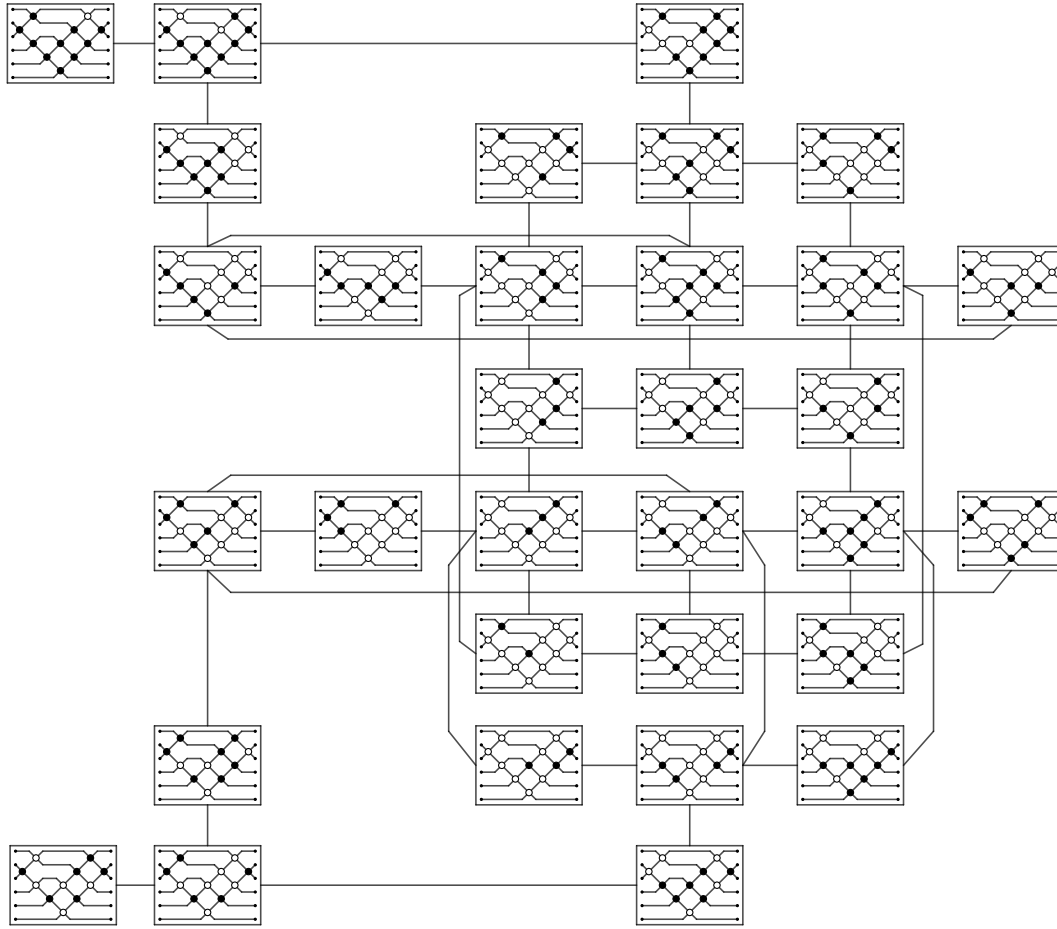


Figure 14.26: CW complex of dimension 3.

This CW complex has one 3-cell with ancestry $\varepsilon_2 = (\circ \blacklozenge \blacklozenge \circ \circ \diamond \circ \diamond \circ \circ)$ and ten 2-cells attached to the previous one. Additionally, there are two 1-cells with two 0-cells attached. The cell of dimension 3 fills the prism completely. The ancestries $\varepsilon_3 = (\circ \bullet \circ \circ \circ \circ \bullet \circ \circ \bullet \bullet \circ)$ and $\varepsilon_4 = (\circ \bullet \bullet \circ \bullet \circ \circ \bullet \bullet \circ \circ)$ represent the vertices at the upper left and lower right corners of the prism. Therefore, BL_σ has 16 connected components of this type, all contractible.

If r_2 has signs $(\bullet \circ \circ)$ or $(\circ \circ \bullet)$, and the remaining rows have equal signs we have the CW complex in Figure 14.27. This results in 16 copies.

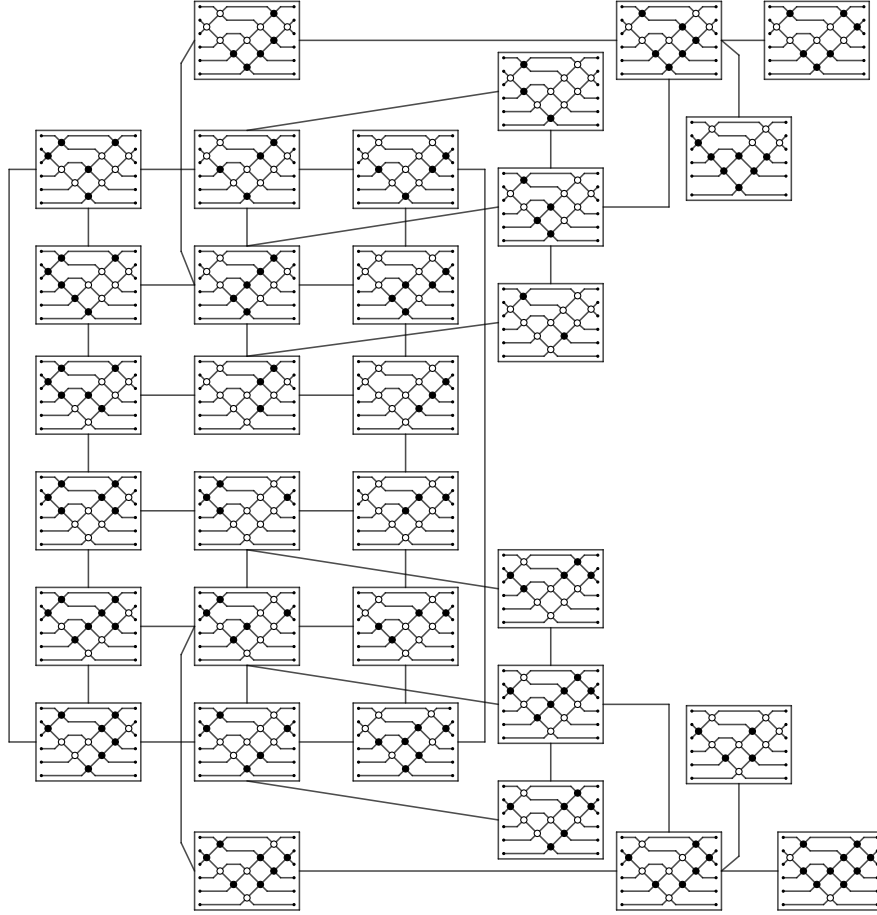


Figure 14.27: CW complex of dimension 3 with $\varepsilon_5 = (\blacklozenge \bullet \blacklozenge \blacklozenge \circ \circ \bullet \bullet \blacklozenge \blacklozenge \blacklozenge)$.

This CW complex has one cell of dimension 3 that completely fills the solid, and some cells with lower dimension attached, these cells do not alter the homotopy type of the component. Therefore, BL_σ has 16 connected components of this type, all contractible.

The remaining ancestries of dimensions 1 and 2 appear in a higher-dimensional CW complex.

Thinking over dimension 3, if the diamonds are in r_1, r_2 and r_4 , while the remaining rows have equal signs, we have the CW complex in Figure 14.28. This results in 16 copies.

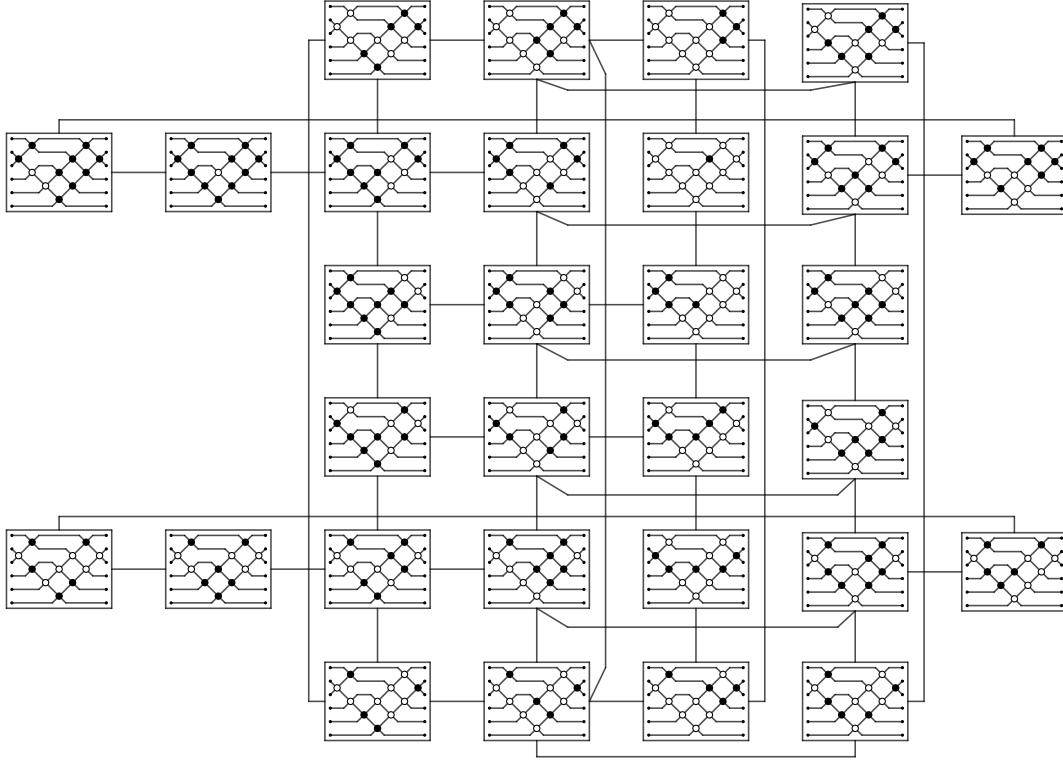


Figure 14.28: CW complex of dimension 3.

This CW has three cells of dimension 3, that fills two prisms and one “parallelepiped” completely, $\varepsilon_7 = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge)$, $\varepsilon_8 = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge)$ and $\varepsilon_9 = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge)$, respectively. Additionally, there are two 2-cells attached like wings. Therefore, BL_σ has 16 contractible connected components of this type.

The following pairs of ancestries represent the vertices at the upper left and lower right corners of the prisms and the “parallelepiped”, respectively:

$$\varepsilon_{10} = (\circ \circ \circ \bullet \bullet \bullet \bullet \bullet \circ \circ \bullet), \varepsilon_{11} = (\circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \bullet \bullet);$$

$$\varepsilon_{12} = (\circ \circ \circ \bullet \bullet \bullet \bullet \bullet \circ \bullet \bullet), \varepsilon_{13} = (\circ \bullet \bullet \bullet \bullet \circ \circ \bullet \bullet \bullet \bullet);$$

$$\varepsilon_{14} = (\circ \circ \circ \bullet \bullet \bullet \bullet \bullet \circ \bullet \bullet), \varepsilon_{15} = (\circ \bullet \circ \circ \bullet \bullet \circ \bullet \bullet \bullet \bullet).$$

The remaining ancestries of dimension 3 appear in the higher-dimensional CW complex.

For dimension 4, we have only one possible position for the diamonds.

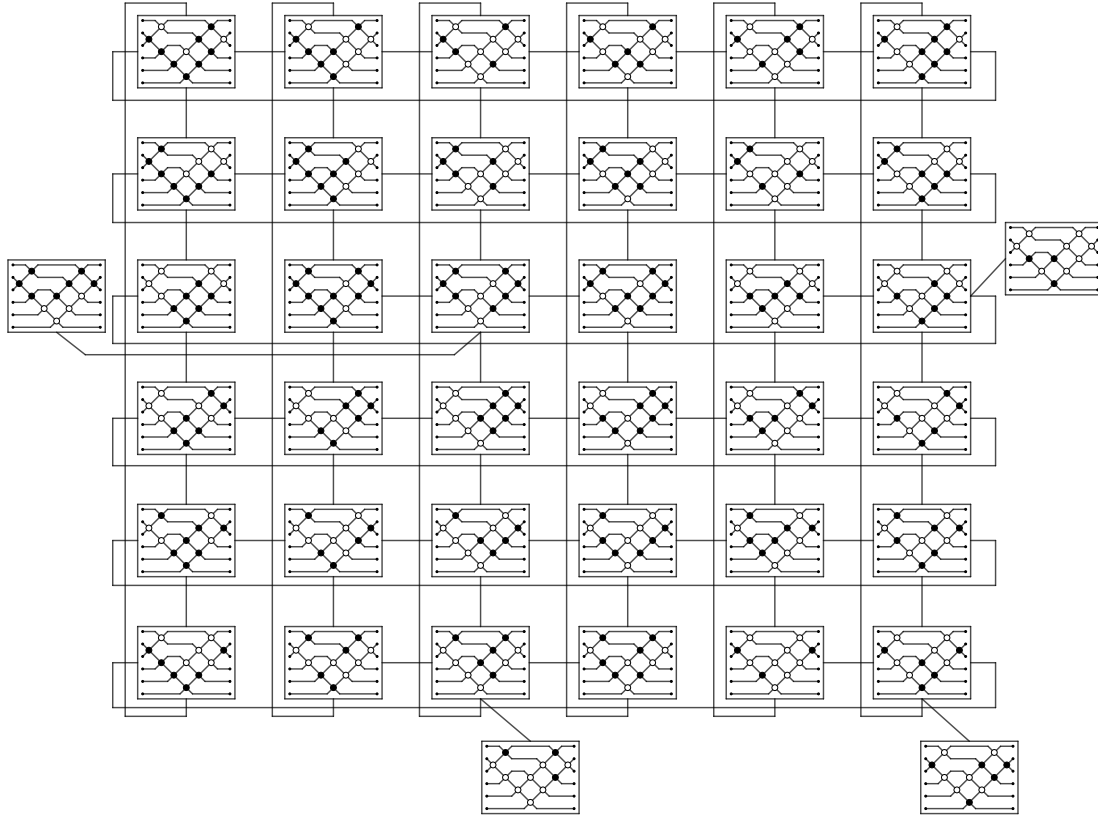


Figure 14.29: CW complex of dimension 4 with $\varepsilon_{13} = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \blacklozenge)$.

First, we collapse the four 1-cells that appear as “antennas” in the CW complex. Then, the CW complex will have a familiar structure. Vertically, this CW complex has six 3-cells, which are 4 prisms and 2 “parallelepipeds”, which attach along a solid torus. Horizontally, this CW complex has four 3-cells, which are 2 solids with 12 faces and 2 prisms, that glue along a solid torus as well. Thus, we have a $\mathbb{S}^3$ and finally, we glue a 4-cell that leads to a $\mathbb{D}^4$.

Therefore, BL_σ has 8 contractible connected components of this type. Summing up, BL_σ has a total of 104 connected components, all of them contractible.

There are 2 permutations that have a CW complex similar to the one described. They are

$$\sigma_1 = a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3, \quad \sigma_2 = a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_5 a_4 a_3 a_2 \in S_6.$$

As a result, BL_σ is contractible for all $\sigma \in S_6$ with $\text{inv}(\sigma) = 11$.

15

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 12$

For $\text{inv}(\sigma) = 12$, the 29 permutations are distributed in two cases:

1. There are 2 permutations that we can apply split type 3;
2. There are 27 permutations that needs to be studied separately.

15.1

Case 2

These 27 permutations can be classified into nine distinct types of CW complexes. The last type will be examined in detail in the next chapter.

Type 15.1. For $\sigma = [365421] = a_2a_3a_2a_4a_3a_2a_1a_5a_4a_3a_2a_1 \in S_6$ it follows that

$$\begin{aligned} \sigma = \frac{1}{4\sqrt{2}} & (1 - \hat{a}_1 - \hat{a}_2 + \hat{a}_1\hat{a}_2 - \hat{a}_3\hat{a}_4 + \hat{a}_1\hat{a}_3\hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 \\ & - \hat{a}_1\hat{a}_5 - \hat{a}_2\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_5 - \hat{a}_3\hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If the diamonds are in r_1 and the remaining rows have equal signs, we have the CW complex in Figure 15.1. This results in 32 copies. Therefore, BL_σ has 32 connected components of this type, all contractible.

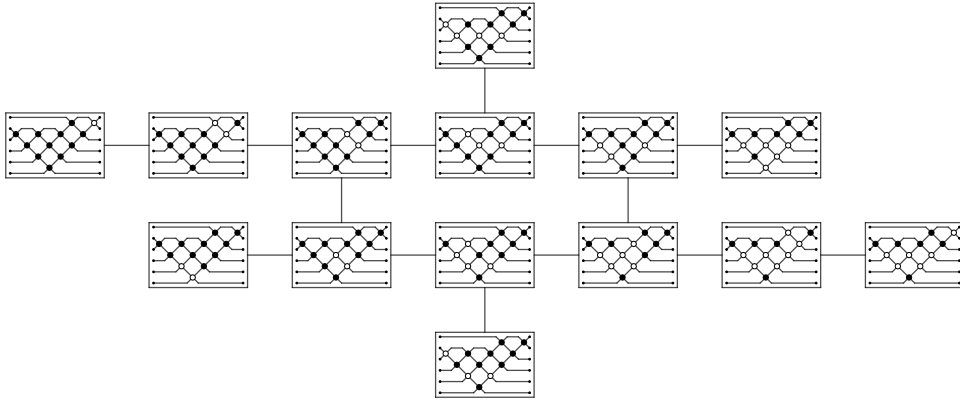


Figure 15.1: CW complex with ancestry $\varepsilon_1 = (\bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet)$.

If the diamonds are in r_2 with signs $(\bullet \bullet \bullet \circ)$, and the remaining rows have equal signs, a CW complex with 4 cells of dimension 3 is obtained. This results in 16 copies. The construction of this CW complex will be analyzed by attaching the 3-cells one by one.

Step 1: Begin with a 3-cell that fills the prism in Figure 15.2.

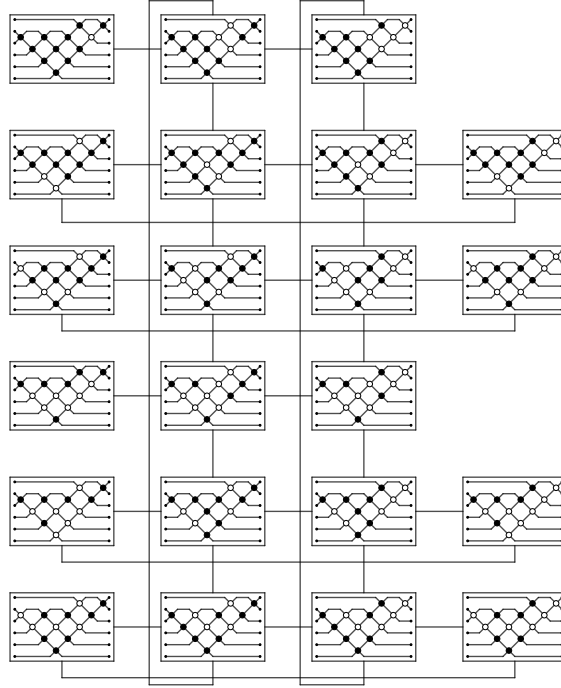


Figure 15.2: First part of the CW complex of dimension 3.

The cell with ancestry $\varepsilon_2 = (\bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \circ \blacklozenge)$ fills the prism. Note that there are four 2-cells attached like wings in the first part of the CW. Additionally, two vertices with only one edge each are also attached to this part.

Step 2: Attach the next 3-cell that fills a “parallelepiped” with $\varepsilon_3 = (\bullet \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \circ)$. Similar to the previous one, this part has four 2-cells attached as wings.

This part attaches to the first one through the hexagon on the left side of the 3-cell in Figure 15.3, with $\varepsilon_4 = (\bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \circ \circ)$. The left vertices of the wings in Figure 15.3 are attached to the right vertices of the wings in Figure 15.2.

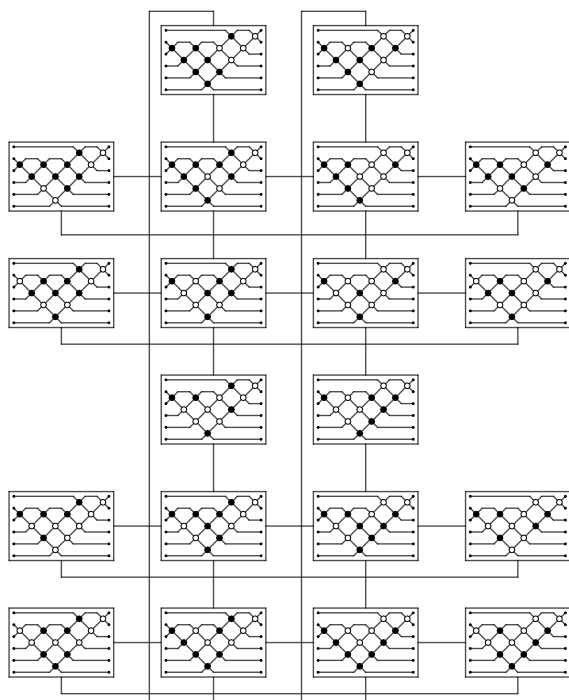


Figure 15.3: Second part of the CW complex of dimension 3.

Step 3: The third part includes a 3-cell that fills another “parallelepiped”.

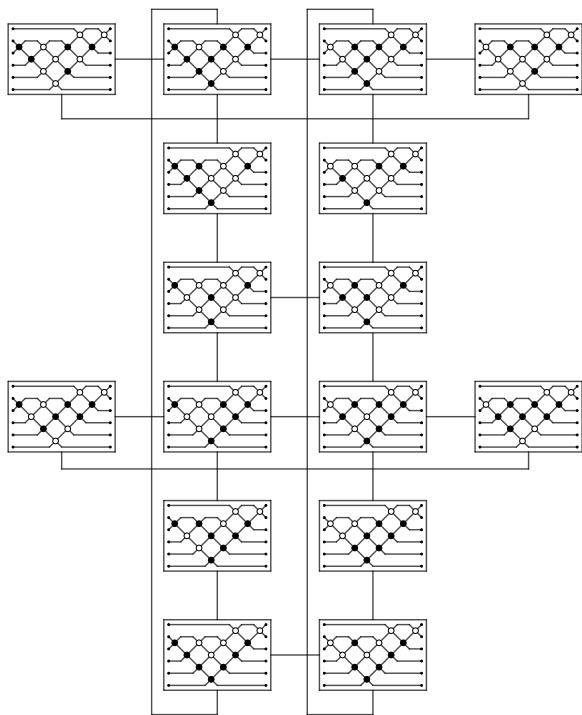


Figure 15.4: Third part of the CW complex of dimension 3.

The cell with ancestry $\varepsilon_5 = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \diamond \circ \bullet \diamond \bullet \circ)$ fills the “parallelepiped”. Similar to the previous one, it has 2-cells that appear as wings, in this case, there are two of them.

In this part, we attach the cell in Figure 15.4 to the cell in the second part through the hexagon on the left side of the cell with ancestry $\varepsilon_6 = (\bullet \blacklozenge \blacklozenge \bullet \circ \diamond \circ \bullet \circ \diamond \bullet \circ)$. The left vertices of the wings in Figure 15.4 are attached to right vertices of the wings in Figure 15.3.

Step 4: To complete the attachment, we glue the last 3-cell, which fills the third “parallelepiped”, and Attachment occurs similar to the previous case, through the left hexagon in Figure 15.5 with ancestry $\varepsilon_7 = (\bullet \blacklozenge \blacklozenge \bullet \diamond \circ \bullet \diamond \bullet \circ)$. Additionally, two edges of Figure 15.5 are attached to the wings in Figure 15.4.

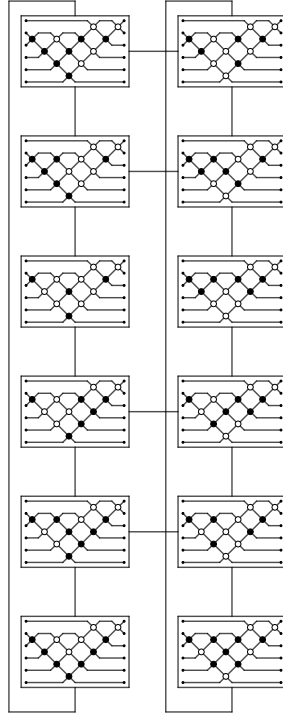


Figure 15.5: Fourth part of the CW complex of dimension 3 with ancestry $\varepsilon_8 = (\bullet \blacklozenge \blacklozenge \blacklozenge \circ \diamond \circ \bullet \diamond \diamond \bullet \circ)$.

Upon completing all attachments, we have a contractible component. This CW complex comprises three 3-cells attached side by side, with an additional 3-cell attached between two of them, sharing one 2-cell in common. Furthermore, cells of lower dimension are also attached to these 3-cells, without altering the homotopy type of the component. Therefore, BL_σ has 16 connected components of this type, all contractible.

If the diamonds are in r_3 with signs $(\bullet \circ \circ)$, and the other rows have equal signs, we obtain a CW complex with seven 3-cells. This results in 32 copies. The construction of the CW is a bit confusing, so we need a step by step construction.

Step 1: Start with a 3-cell that fills a convex solid with eighteen faces in Figure 15.6. Note that there are three vertices, each with only one edge attached to this part.

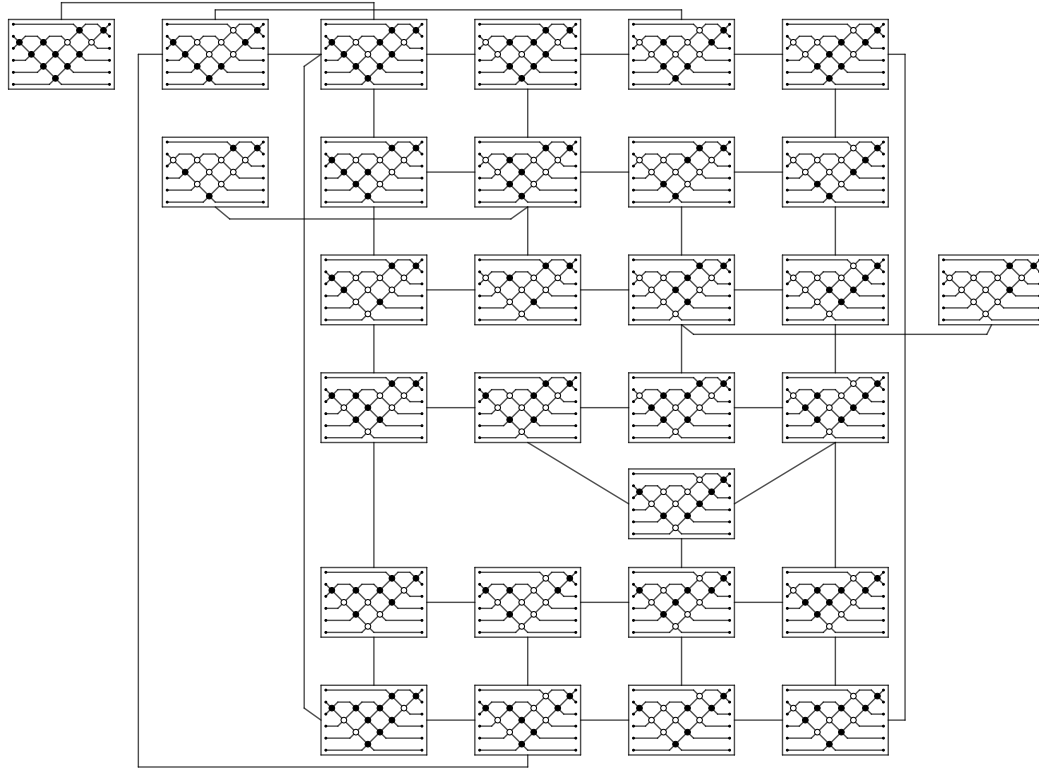


Figure 15.6: First part of the CW complex of dimension 3 with ancestry $\varepsilon_9 = (\blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \bullet \bullet \diamond \diamond \diamond \bullet)$.

Step 2: The second part of this CW complex consists of a 3-cell with ancestry $\varepsilon_{10} = (\circ \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \circ \diamond \diamond \circ \diamond)$ that fills the prism shown in Figure 15.7. Attachment occurs through the hexagon on the right side of Figure 15.6 to the hexagon on the left side of Figure 15.7 with ancestry $\varepsilon_{11} = (\circ \blacklozenge \bullet \bullet \bullet \bullet \circ \circ \diamond \bullet \bullet)$.

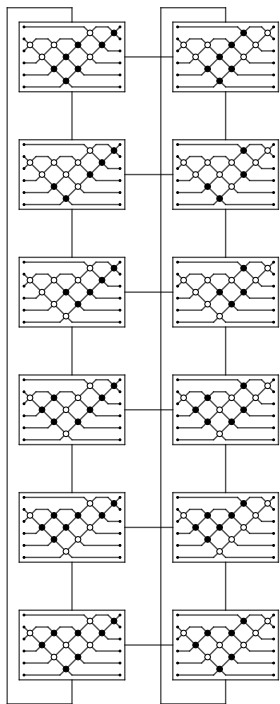


Figure 15.7: Second part of the CW complex of dimension 3.

Step 3: Following attach the 3-cell illustrated in Figure 15.8.

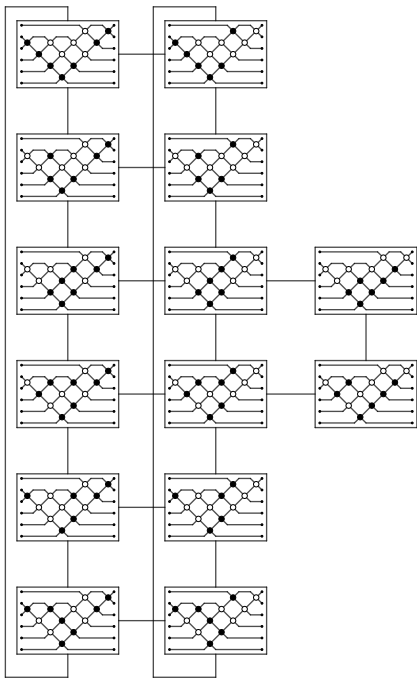


Figure 15.8: Third part of the CW complex of dimension 3.

The 3-cell with ancestry $\varepsilon_{12} = (\diamond \diamond \circ \bullet \diamond \diamond \bullet \bullet \circ \circ \diamond)$ fills the prism. Notice the presence of a 2-cell attached to the prism, resembling a wing.

The attachment to Figure 15.8 occurs through the square in the center and the square in the previous Figure 15.7, with ancestry $\varepsilon_{13} = (\circ \diamond \bullet \circ \diamond \bullet \bullet \bullet \circ \circ \diamond)$. This part also attaches to Figure 15.6 through a 2-cell with ancestry $\varepsilon_{14} = (\diamond \diamond \circ \bullet \diamond \diamond \circ \bullet \bullet \circ \bullet \bullet)$ that fills the hexagon on the left side of this figure.

Step 4: The forth part is a 3-cell that fills the cube in Figure 15.9. Attachment occurs through the square in the center of the cube to the square at the bottom of Figure 15.7, with ancestry $\varepsilon_{15} = (\circ \bullet \bullet \diamond \bullet \bullet \diamond \circ \diamond \circ \circ \diamond)$.

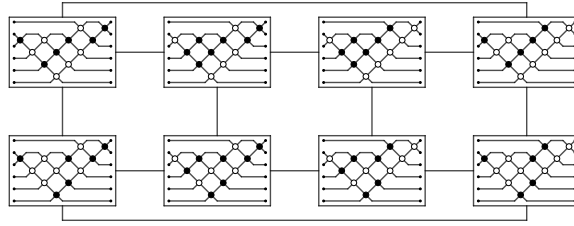


Figure 15.9: Fourth part of the CW complex of dimension 3 with ancestry $\varepsilon_{16} = (\diamond \circ \diamond \diamond \bullet \bullet \bullet \diamond \circ \diamond \circ \circ \diamond)$.

This part attaches to Figure 15.6 through a 2-cell with ancestry $\varepsilon_{17} = (\diamond \circ \diamond \diamond \bullet \bullet \bullet \circ \diamond \circ \bullet \bullet)$ that fills the square on the left side of this figure. Furthermore, this part also attaches to Figure 15.8 through a 2-cell that fills the square on the bottom of this figure, with ancestry $\varepsilon_{18} = (\diamond \circ \diamond \circ \circ \bullet \bullet \bullet \circ \circ \diamond)$.

Step 5: The fifth component of the CW complex in Figure 15.10 comprises a 3-cell with ancestry $\varepsilon_{19} = (\bullet \circ \diamond \diamond \circ \diamond \diamond \circ \diamond \circ \circ \diamond)$ that fills the cube and five 2-cells that are attached to the 3-cell.

The attachment to Figure 15.9 occurs through the square with ancestry $\varepsilon_{20} = (\diamond \circ \diamond \bullet \bullet \bullet \diamond \circ \circ \circ \diamond)$. This part is attached to Figure 15.6 through a 2-cell, with ancestry $\varepsilon_{21} = (\bullet \circ \diamond \diamond \circ \diamond \circ \circ \diamond \circ \bullet \bullet)$. Furthermore, this part also attaches to Figure 15.8 through a 2-cell, with ancestry $\varepsilon_{22} = (\bullet \circ \diamond \circ \bullet \diamond \bullet \bullet \bullet \circ \circ \diamond)$.

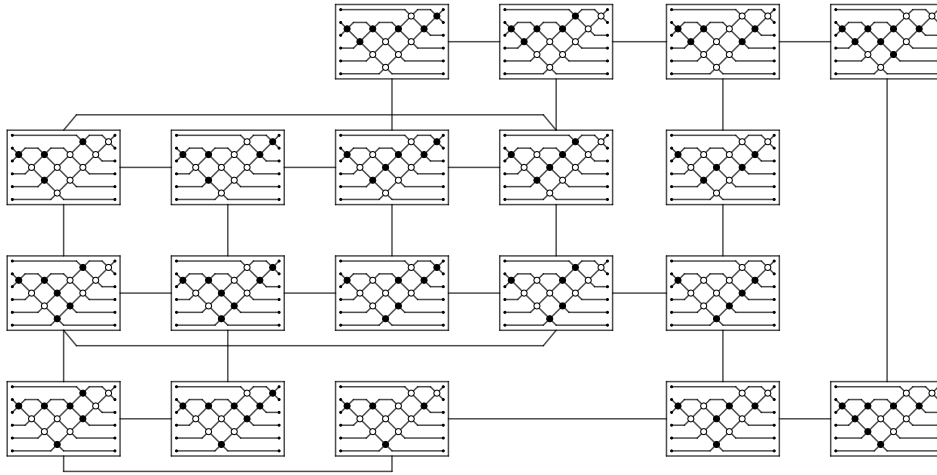


Figure 15.10: Fifth part of the CW complex of dimension 3.

Step 6: The sixth part is similar to the previous one, consisting of one 3-cell that fills the cube and two 2-cells attached.

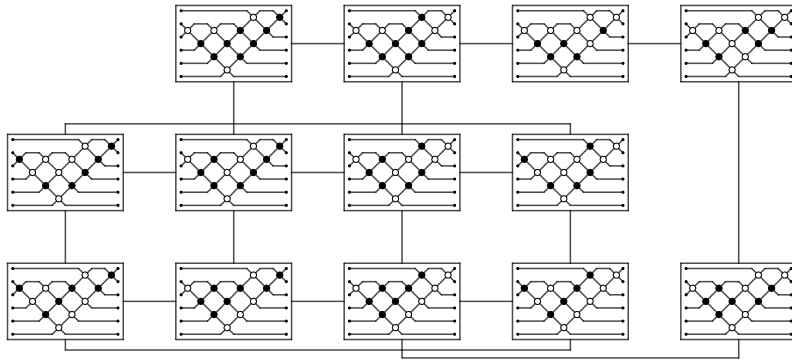


Figure 15.11: Sixth part of the CW complex of dimension 3 with ancestry $\varepsilon_{23} = (\blacklozenge \circ \diamond \bullet \blacklozenge \bullet \blacklozenge \circ \circ \diamond \circ \diamond)$.

Attachment occurs at the square at the bottom of the cube to the one in Figure 15.9, with ancestry $\varepsilon_{24} = (\blacklozenge \circ \diamond \bullet \bullet \bullet \blacklozenge \circ \circ \circ \diamond)$. This part attaches to Figure 15.6 through a 2-cell, with ancestry $\varepsilon_{25} = (\blacklozenge \circ \diamond \bullet \blacklozenge \bullet \circ \circ \circ \diamond \bullet \bullet)$. Furthermore, this part also attaches to Figure 15.7 through a 2-cell, with ancestry $\varepsilon_{26} = (\circ \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge \circ \circ \diamond \circ \diamond)$.

Step 7: To complete the CW complex, the seventh part includes the last 3-cell that fills the cube. Additionally, this part has three 2-cells attached to the 3-cell.

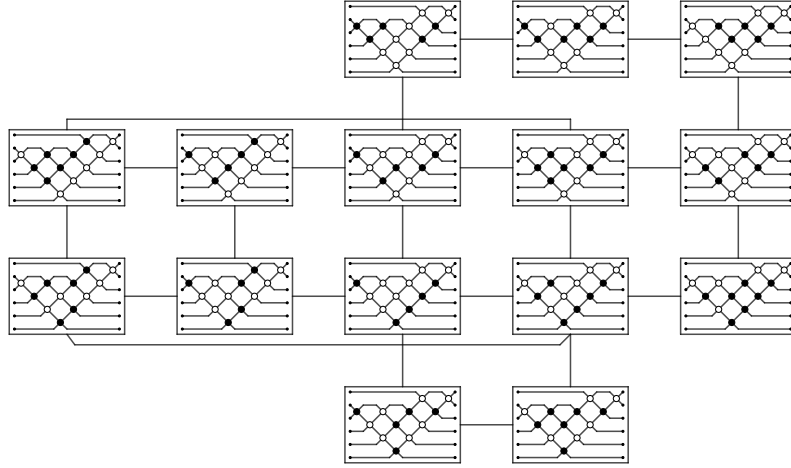


Figure 15.12: Seventh part of the CW complex of dimension 3 with ancestry $\varepsilon_{27} = (\diamond \circ \diamond \bullet \bullet \bullet \diamond \bullet \circ \diamond \circ \diamond \circ)$.

Attachment occurs in the square on the left side of the cube to the square on the right side of the cube in Figure 15.9, with ancestry $\varepsilon_{28} = (\diamond \circ \diamond \bullet \bullet \bullet \circ \diamond \circ \circ \circ)$.

Similar to the previous case, this CW complex has seven 3-cells attached, along with some cells of lower dimension that do not alter the homotopy type. Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 32 connected components of this type, all contractible.

The remaining possible positions for the diamonds in dimensions 1, 2 and 3 appear in cells of higher dimensions.

For dimension 4, there are three possible positions for the diamonds, all of which are depicted together in a CW complex. Constructing this CW complex requires some careful steps. Let us proceed with its construction step by step.

Step 1: Start with a 4-cell with ten 3-cells.

Vertically, this CW complex consists of six 3-cells filling “parallelepipeds”, which are attached along a solid torus. Horizontally, the CW complex comprises four 3-cells, consisting of two convex solids as seen in Figure 15.6 and two cubes, also attached along a solid torus. As a result, we obtain an $\mathbb{S}^3$, and finally a cell of dimension 4 is attached, resulting in a $\mathbb{D}^4$.

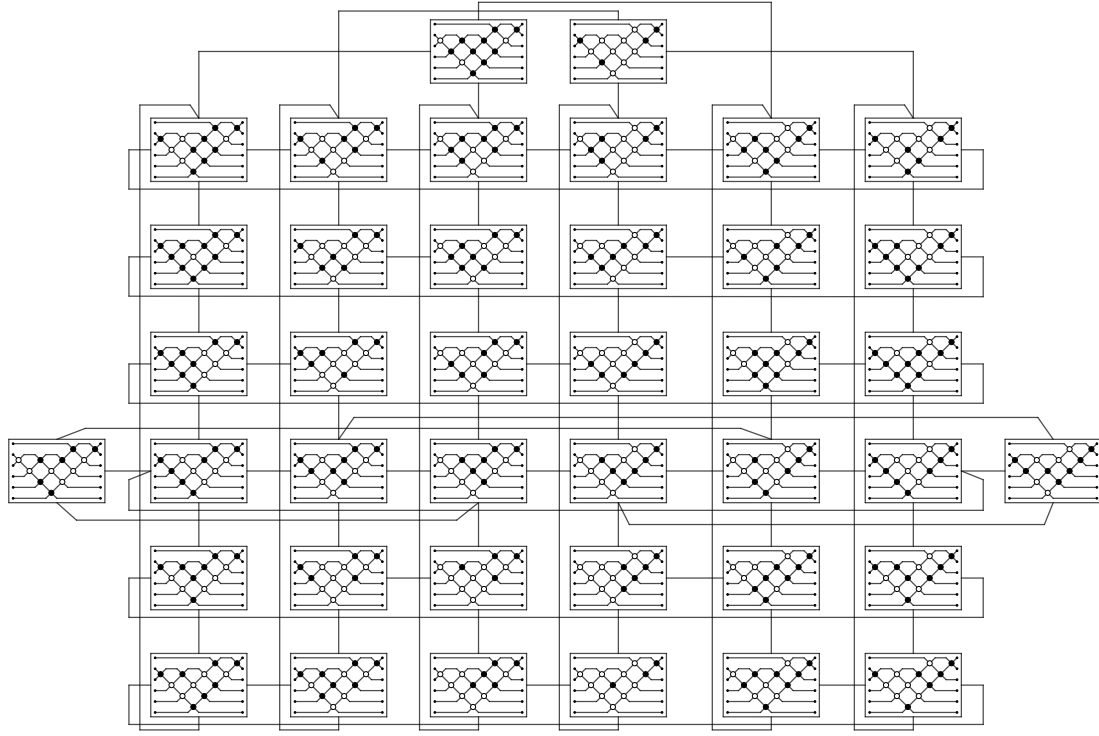


Figure 15.13: First part of the CW complex of dimension 4 with ancestry $\varepsilon_{29} = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet)$.

Step 2: The second part consists of another $\mathbb{D}^4$, with ancestry $\varepsilon_{30} = (\circ \blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \circ \blacklozenge)$. The cell has eight 3-cells. The attachment is made through a 3-cell that fills the vertical “parallelepiped” in the center of Figure 15.14, attaching it to the fourth vertical “parallelepiped” in Figure 15.13 with ancestry $\varepsilon_{31} = (\circ \blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \circ \circ \blacklozenge \bullet \bullet)$.

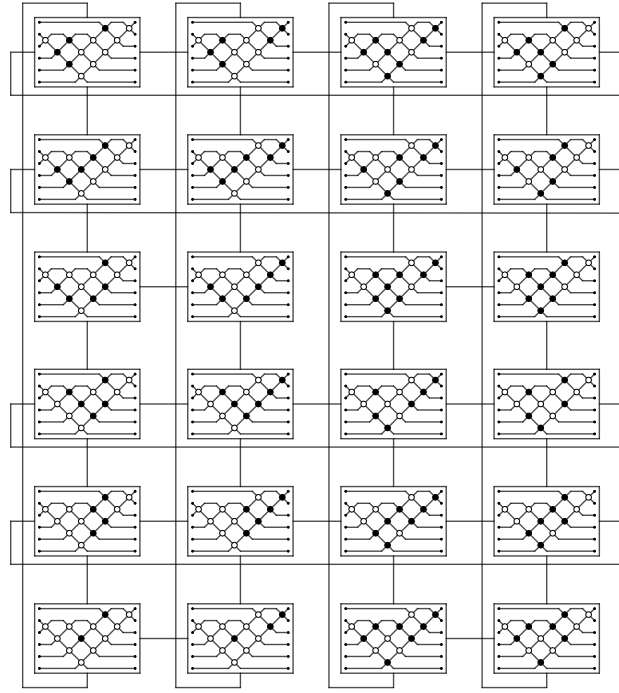


Figure 15.14: Second part of the CW complex of dimension 4.

Step 3: Attach a 4-cell similar to the previous one, with 3-cells attached.

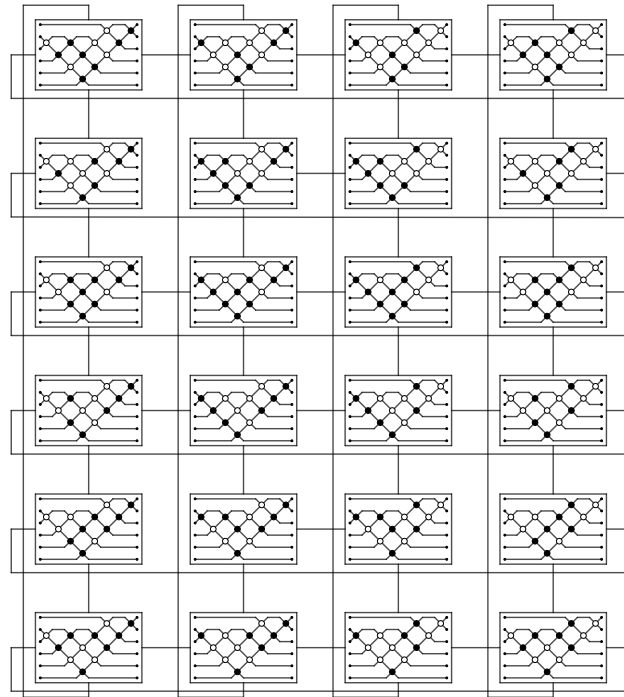


Figure 15.15: Third part of the CW complex of dimension 4 with ancestry $\varepsilon_{32} = (\blacklozenge \blacklozenge \blacklozenge \bullet \diamond \diamond \blacklozenge \circ \bullet \diamond \circ \diamond)$.

To complete the attachment, we have three more 3-cells.

Step 4: The fourth part involves attaching the cube in Figure 15.16 to all the 4-cells. In Figure 15.16, the square on the right side of the cube with ancestry $\varepsilon_{33} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet)$ is attached to Figure 15.13. The upper square of the cube with ancestry $\varepsilon_{34} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge)$ is attached to Figure 15.14. Finally, the square in the center of the cube with ancestry $\varepsilon_{35} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge)$ attaches to Figure 15.15.

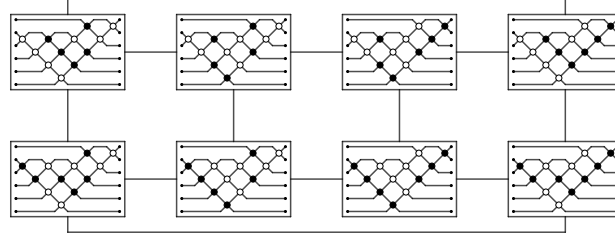


Figure 15.16: Fourth part of the CW complex of dimension 3 with ancestry $\varepsilon_{36} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge)$.

Step 5: For the fifth part, attach another cube from Figure 15.17 to all the 4-cells. The square in the center of the cube with ancestry $\varepsilon_{37} = (\blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet)$ attaches to Figure 15.13. The square at the bottom of the cube with ancestry $\varepsilon_{38} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge)$ attaches to Figure 15.14. Finally, the square on the right side of the cube with ancestries $\varepsilon_{39} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge)$ attaches to Figure 15.15.

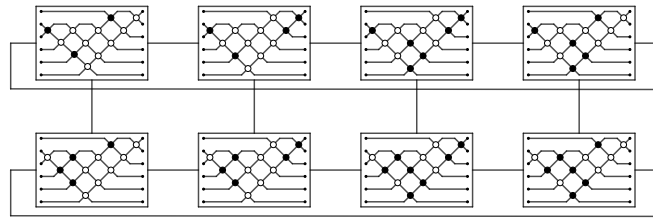


Figure 15.17: Fifth part of the CW complex of dimension 3 with ancestry $\varepsilon_{40} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge)$.

Step 6: For the sixth and last part, attach a “parallelepiped” to two 4-cells. The hexagon at the bottom of Figure 15.18 with ancestries $\varepsilon_{41} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet)$ attaches to Figures 15.15 and 15.14. Note that Figure 15.18 has four 2-cells that appear like wings. They do not alter the homotopy type.

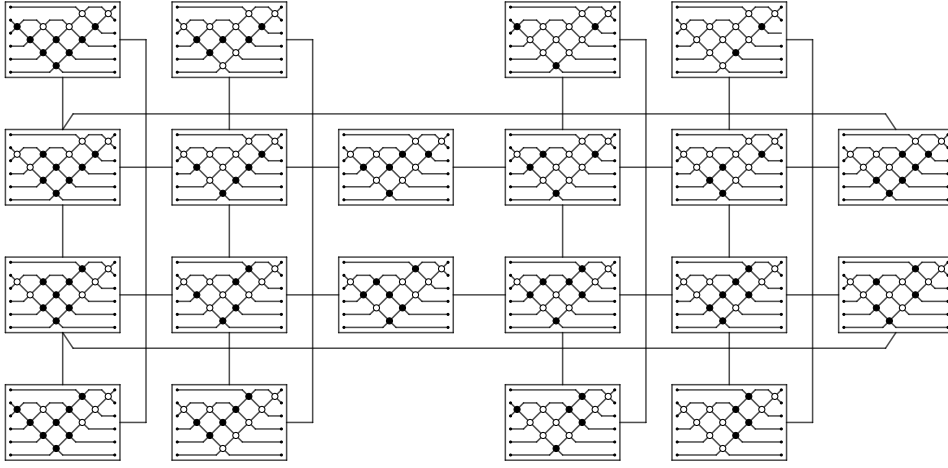


Figure 15.18: Sixth part of the CW complex of dimension 3 with ancestry $\varepsilon_{42} = (\circ \blacklozenge \blacklozenge \blacklozenge \circ \diamond \blacklozenge \circ \diamond \diamond \circ \diamond)$.

The CW complex comprises three 4-cells, each resembling a $\mathbb{D}^4$, attached through a cell of dimension 3. Specifically, each pair of 4-cells is joined by a 3-cell, and all three 4-cells share one common 2-cell. In addition, two 3-cells are attached to each of the three 4-cells and finally one 3-cell is connected to only two of the 4-cells. After completing all these attachments, the resulting component is contractible.

Therefore, BL_σ has 16 connected components of this type, all contractible. In summary, BL_σ has a total of 112 connected components, all contractible.

The permutations

$$\sigma_1 = a_1 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1, \quad \sigma_2 = a_2 a_1 a_3 a_2 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1 \in S_6$$

have a CW complex structure similar to the one described.

Type 15.2. For $\sigma = [456321] = a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1 \in S_6$ it follows that

$$\begin{aligned} \sigma = \frac{1}{4\sqrt{2}} & (-\hat{a}_1 - \hat{a}_2 - \hat{a}_1 \hat{a}_3 - \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_4 - \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_5 \\ & - \hat{a}_1 \hat{a}_2 \hat{a}_5 - \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist 32 thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If the diamonds are r_4 and the remaining rows have equal signs, the resulting CW complex will be constructed in two steps. This results in 16 copies.

Step 1: The first part includes a cell of dimension 3 that fills a cube in Figure 15.19.

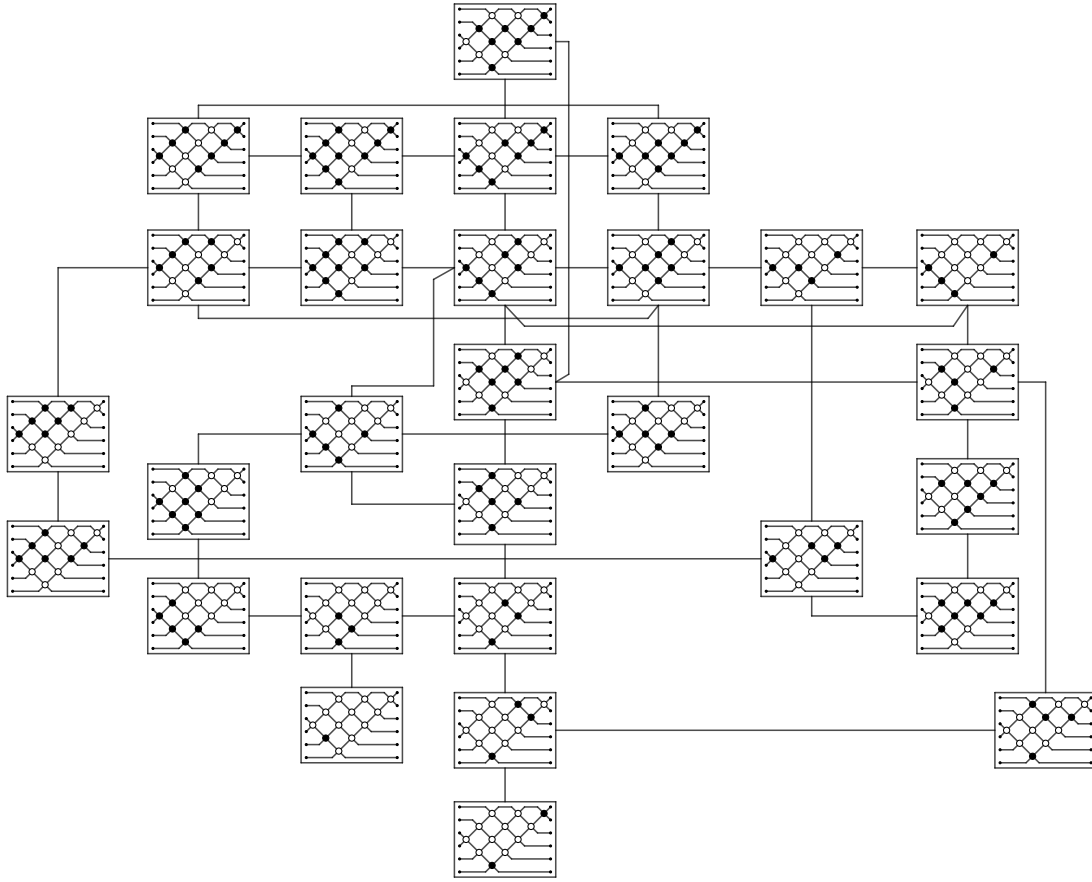


Figure 15.19: First part of the CW complex of dimension 3 with ancestry $\varepsilon_1 = (\bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge)$.

Step 2: Now, attach Figure 15.20 to the previous one through five 0-cells and four 1-cells. The second part consists of attaching lower-dimensional cells, which can be easily collapsed. Once all attachments are completed, the component remains contractible. Therefore, BL_σ has a total of 16 connected components of this type, all contractible.

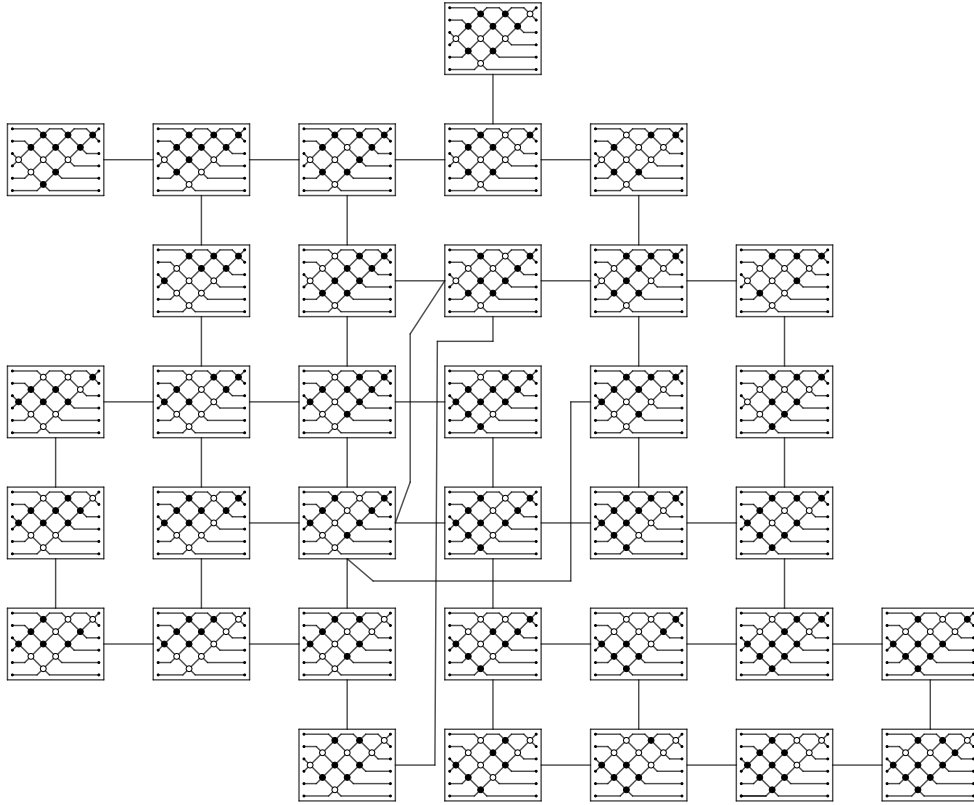


Figure 15.20: Second part of the CW complex with 14 2-cells.

If the diamonds are in r_2 with signs $(\bullet \circ \circ)$ and the remaining rows have identical signs, the resulting CW complex consists of ten 3-cells and must be constructed step by step. This results in 32 copies.

Step 1: Begin with the first part of the CW complex in Figure 15.21, which consists of three 3-cells that fill three prisms with ancestries $\varepsilon_2 = (\bullet \circ \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \blacklozenge)$, $\varepsilon_3 = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge)$, and $\varepsilon_4 = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$. Vertically, there are two 3-cells, while horizontally, there is one. Note that there are five 2-cells and four 1-cells attached to the 3-cells in Figure 15.21.

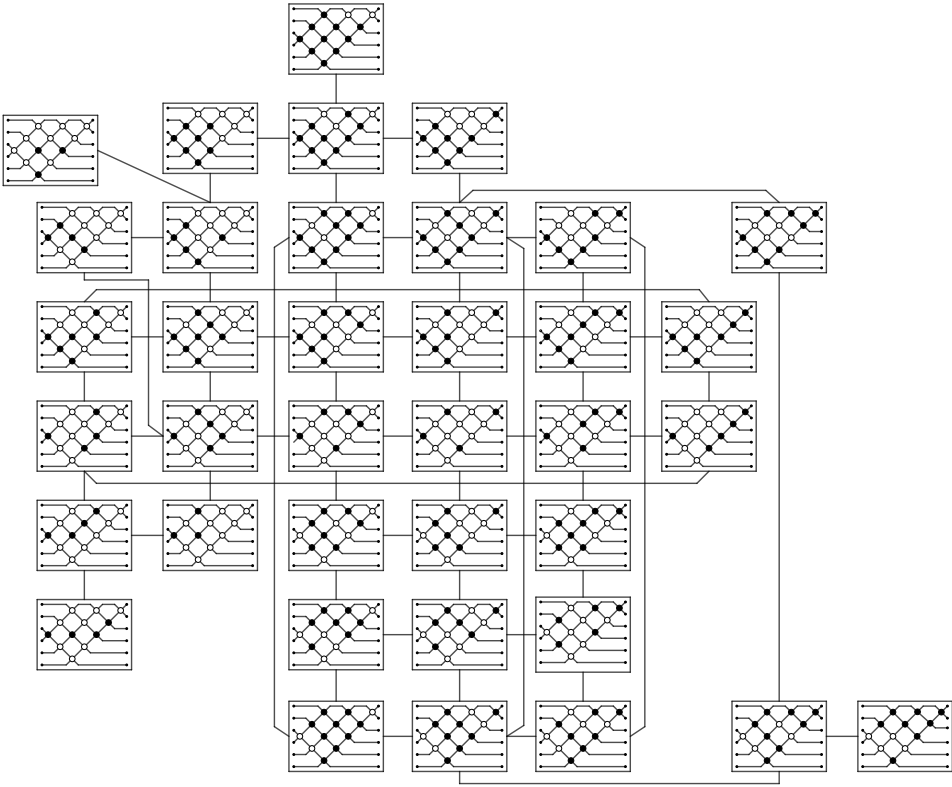


Figure 15.21: First part of the CW complex with three 3-cells.

Step 2: The second part consists of attach a 3-cell to Figure 15.21.

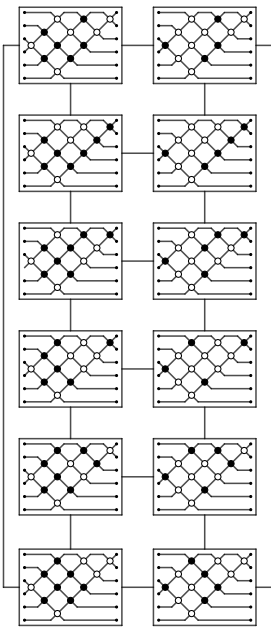


Figure 15.22: Second part of the CW complex.

The cell fills the prism in Figure 15.22, with ancestry $\varepsilon_5 = (\blacklozenge \circ \blacklozenge \circ \blacklozenge \blacklozenge \circ \circ \bullet \bullet \blacklozenge \blacklozenge)$. Attachment occurs through the square with ancestry $\varepsilon_6 = (\blacklozenge \circ \bullet \bullet \blacklozenge \circ \blacklozenge \circ \bullet \bullet \bullet \blacklozenge)$.

Step 3: The third part consists of attaching the 3-cell, that fills the “parallelepiped” from Figure 15.23, to Figure 15.21. Attachment occurs through the hexagon on the left of the previous figure, with ancestry $\varepsilon_8 = (\blacklozenge \circ \circ \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \circ \bullet)$.

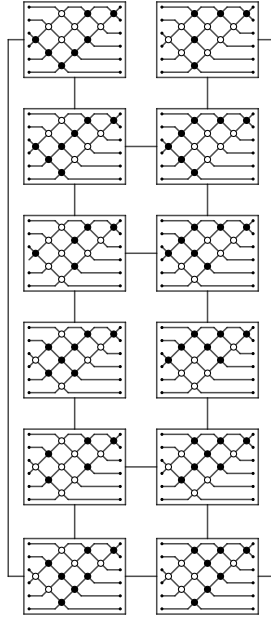


Figure 15.23: Third part of the CW complex of dimension 3 with ancestry $\varepsilon_7 = (\blacklozenge \bullet \bullet \blacklozenge \bullet \circ \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ \bullet)$.

Step 4: The fourth part consists of attaching two more 3-cells. The left is attached through the square with ancestry $\varepsilon_9 = (\circ \blacklozenge \circ \blacklozenge \circ \blacklozenge \bullet \circ \blacklozenge \bullet \bullet \bullet)$ to Figure 15.23. This cell is also attached to Figure 15.21 through the square with ancestry $\varepsilon_{10} = (\circ \bullet \blacklozenge \blacklozenge \circ \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \bullet)$. The right is attached through the hexagon with ancestry $\varepsilon_{11} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \circ \bullet)$ to the Figure 15.23. This cell is also attached to Figure 15.22 through the square with ancestry $\varepsilon_{12} = (\blacklozenge \circ \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet)$.

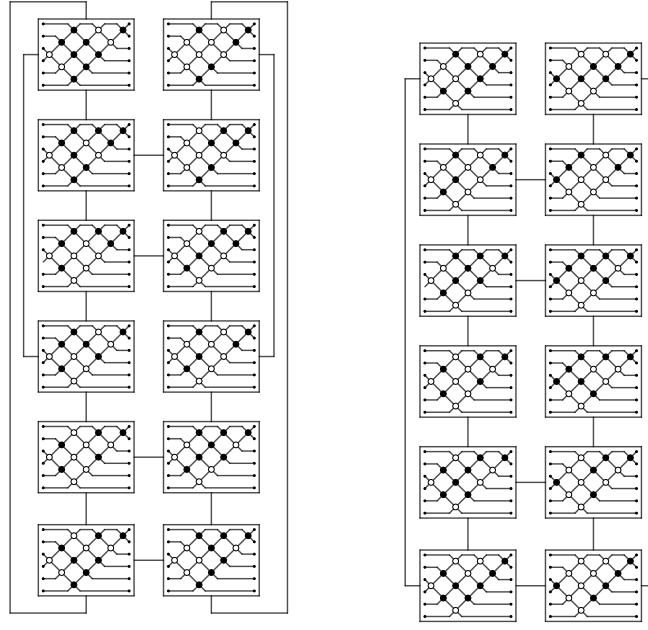


Figure 15.24: Fourth part of the CW complex of dimension 3 with ancestries $\varepsilon_{13} = (\circ \blacklozenge \blacklozenge \blacklozenge \circ \bullet \diamond \circ \diamond \bullet \diamond \bullet)$ and $\varepsilon_{14} = (\blacklozenge \blacklozenge \bullet \circ \blacklozenge \diamond \bullet \circ \bullet \circ \diamond \bullet)$.

Step 5: The fifth part consists of attaching the “parallelepiped” in Figure 15.25. Note that in the cell in Figure 15.25, there is one 2-cell attached like a wing.

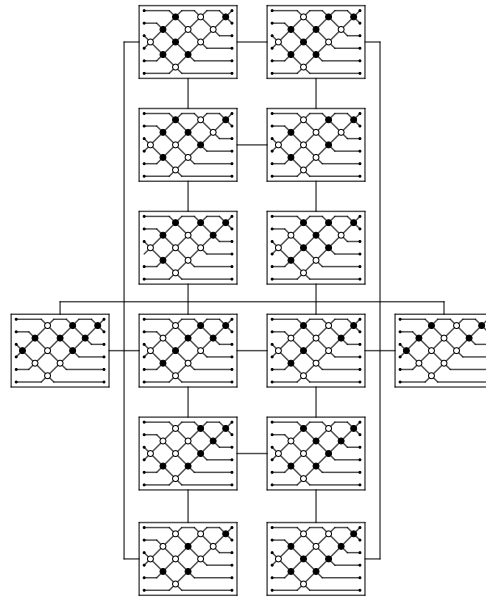


Figure 15.25: Fifth part of the CW complex of dimension 3 with ancestry $\varepsilon_{15} = (\circ \blacklozenge \blacklozenge \bullet \blacklozenge \circ \diamond \circ \bullet \diamond \bullet)$.

Attachment occurs through the hexagon with ancestry $\varepsilon_{16} = (\circ \blacklozenge \circ \bullet \blacklozenge \bullet \bullet \bullet \circ \blacklozenge \bullet)$ to the right cell in Figure 15.24, and the hexagon in the center with ancestry $\varepsilon_{17} = (\circ \blacklozenge \blacklozenge \bullet \bullet \bullet \circ \blacklozenge \circ \bullet \bullet \blacklozenge \bullet)$ to the left cell in Figure 15.24. This cell is also attached to Figure 15.21 through the square with ancestry $\varepsilon_{18} = (\circ \bullet \blacklozenge \bullet \blacklozenge \circ \blacklozenge \circ \bullet \bullet \blacklozenge \circ \bullet)$.

Step 6: The sixth part consists of attaching the “parallelepiped” in Figure 15.26. The attachment to Figure 15.25 occurs through the hexagon with ancestry $\varepsilon_{20} = (\circ \blacklozenge \blacklozenge \bullet \bullet \bullet \circ \blacklozenge \circ \bullet \bullet \blacklozenge \bullet)$. This cell is also attached to Figure 15.22 through the hexagon in the right with ancestry $\varepsilon_{21} = (\circ \bullet \blacklozenge \bullet \bullet \blacklozenge \circ \bullet \bullet \bullet \blacklozenge \blacklozenge)$.

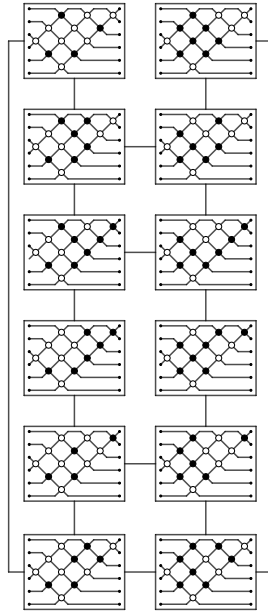


Figure 15.26: Sixth part of the CW complex of dimension 3 with ancestry $\varepsilon_{19} = (\circ \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \blacklozenge)$.

Step 7: The seventh and final part consists of attaching the prism in Figure 15.27. Attachment occurs through the square with ancestry $\varepsilon_{23} = (\circ \blacklozenge \circ \bullet \bullet \bullet \blacklozenge \blacklozenge \circ \bullet \bullet \bullet \circ \blacklozenge)$ to Figure 15.26. This cell is also attached to Figure 15.22 through the square with ancestry $\varepsilon_{24} = (\blacklozenge \circ \circ \circ \blacklozenge \circ \blacklozenge \circ \bullet \bullet \bullet \circ \blacklozenge)$. Note that in the cell on Figure 15.27, there is one 2-cell attached like a wing.

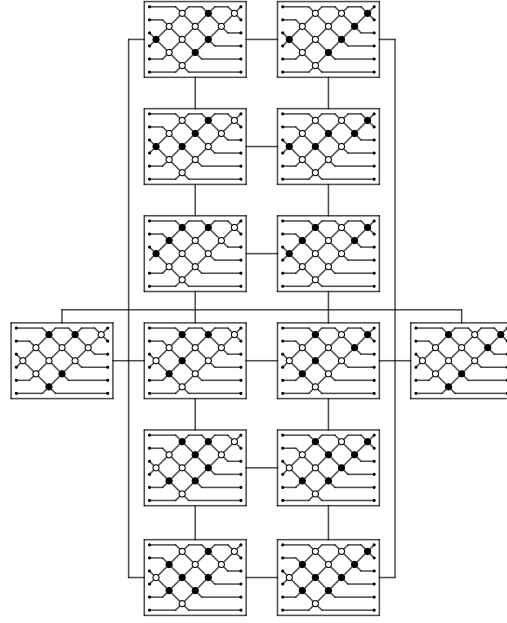


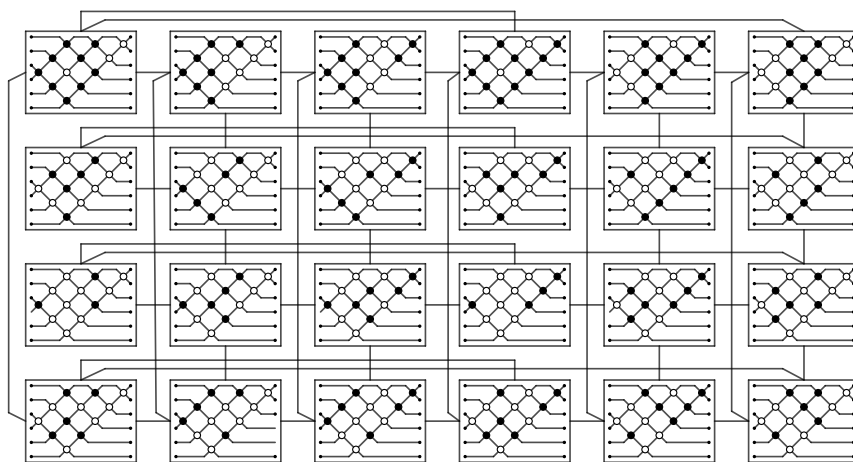
Figure 15.27: Seventh part of the CW complex of dimension 3 with ancestry $\varepsilon_{22} = (\blacklozenge \blacklozenge \bullet \circ \circ \blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \circ \blacklozenge)$.

Upon completing all attachments, we have a contractible CW complex. Therefore, BL_σ has a total of 32 connected components of this type, all contractible.

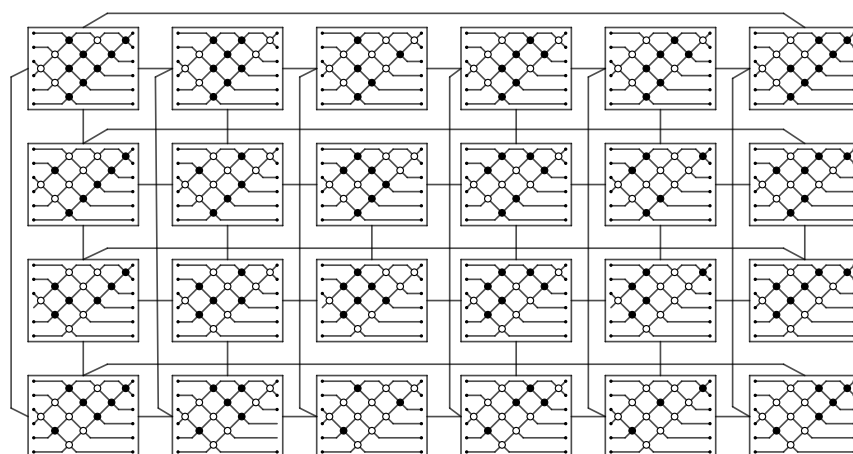
The remaining ancestries of dimensions 1, 2 and 3 appear in higher-dimensional CW complexes.

For dimension 4, there are two possible positions for the diamonds, which appear together in a CW complex that will also be constructed step by step. This results in 16 copies.

Step 1: Start with the cell of dimension 4 in Figure 15.28, which is a $\mathbb{D}^4$, as we saw before.


$$\varepsilon_{25} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \diamond \blacklozenge \bullet \diamond \diamond \circ \diamond).$$

Step 2: Now, attach the second cell of dimension 4, which is also a $\mathbb{D}^4$.


$$\varepsilon_{26} = (\circ \blacklozenge \blacklozenge \blacklozenge \bullet \blacklozenge \diamond \circ \diamond \bullet \diamond \diamond).$$

first cube in Figure 15.29 with ancestry $\varepsilon_{27} = (\bullet \blacklozenge \bullet \blacklozenge \bullet \diamond \blacklozenge \bullet \diamond \circ \circ \diamond)$.

Step 3: The third part of the CW complex consists of attaching two 3-cells to the two previous 4-cells.

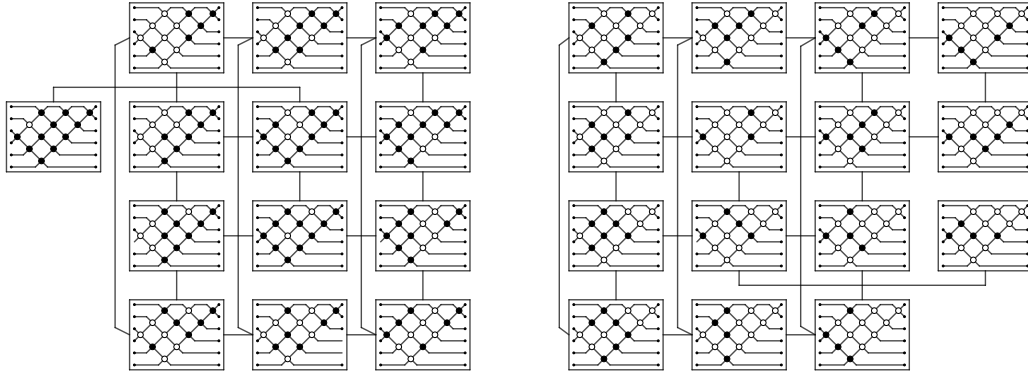


Figure 15.30: Third part of the CW complex of dimension 3 with ancestries $\varepsilon_{28} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \circ \blacklozenge \bullet \blacklozenge \bullet \bullet)$ and $\varepsilon_{29} = (\blacklozenge \circ \blacklozenge \bullet \bullet \circ \blacklozenge \bullet \blacklozenge \bullet \circ \circ)$.

The cell on the left of Figure 15.30 attaches to Figure 15.28 through the hexagon with ancestry $\varepsilon_{30} = (\blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \circ \bullet \blacklozenge \bullet \bullet)$. The attachment to Figure 15.29 is through the square with ancestry $\varepsilon_{31} = (\circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \bullet)$. The cell on the right of Figure 15.30 attaches to Figure 15.28 through the hexagon with ancestry $\varepsilon_{32} = (\blacklozenge \circ \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet \circ \circ)$. The attachment to Figure 15.29 is through the square with ancestry $\varepsilon_{33} = (\circ \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \circ \circ)$.

Step 4: The fourth part of the CW complex consists of attaching two 3-cells to Figure 15.28.

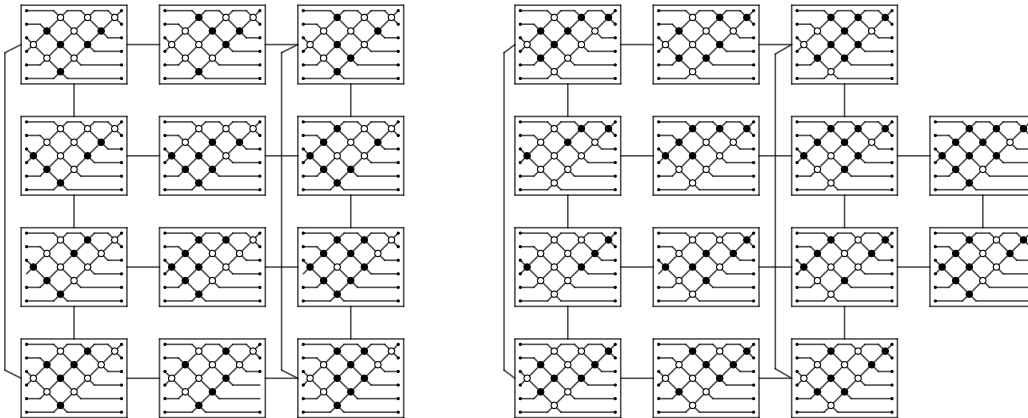


Figure 15.31: Fourth part of the CW complex of dimension 3 with ancestries $\varepsilon_{34} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \circ \blacklozenge \bullet \circ)$ and $\varepsilon_{35} = (\blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \bullet \bullet \bullet \circ \bullet \bullet \blacklozenge \bullet)$.

The cell on the left of Figure 15.30 attaches to Figure 15.28 through the hexagon with ancestry $\varepsilon_{36} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet)$. The cell on the right of Figure 15.30 attaches to Figure 15.28 through the hexagon with ancestry $\varepsilon_{37} = (\blacklozenge \blacklozenge \bullet \bullet \circ \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet)$.

Step 5: The fifth part of the CW complex consists of attaching two 3-cells to the two previous 4-cells.

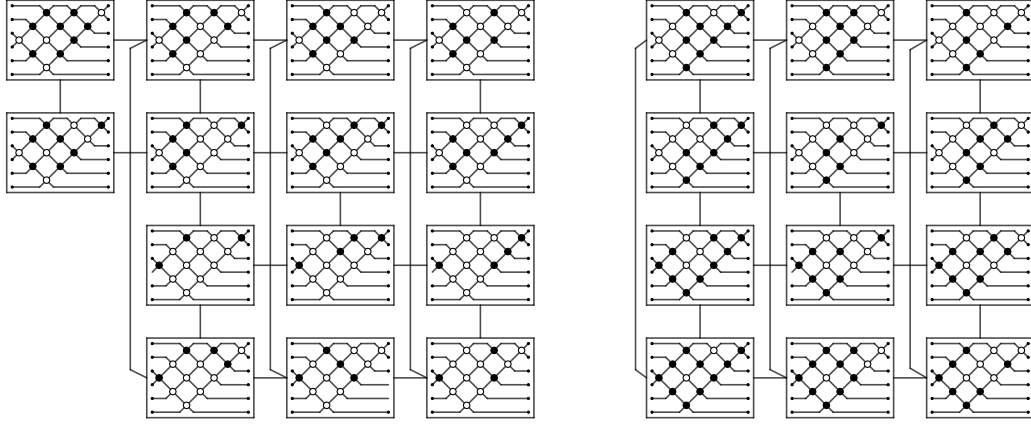


Figure 15.32: Fifth part of the CW complex of dimension 3 with ancestries $\varepsilon_{38} = (\diamond \circ \diamond \circ \diamond \diamond \circ \circ \circ \bullet \diamond \diamond)$ and $\varepsilon_{39} = (\diamond \bullet \diamond \bullet \diamond \diamond \bullet \bullet \bullet \bullet \diamond \diamond)$.

The cell on the left of Figure 15.32 attaches to Figure 15.28 through the square with ancestry $\varepsilon_{40} = (\diamond \circ \circ \circ \diamond \diamond \circ \circ \bullet \circ \diamond)$. The attachment to Figure 15.29 is through the hexagon with ancestry $\varepsilon_{41} = (\circ \bullet \diamond \bullet \bullet \diamond \circ \circ \circ \bullet \diamond \diamond)$. Furthermore, this cell is attached to the right cell in Figure 15.30 and to the right cell Figure 15.31. The attachment to the cell in Figure 15.30 is through the square with ancestry $\varepsilon_{42} = (\diamond \bullet \diamond \bullet \diamond \bullet \diamond \bullet \bullet \bullet \bullet)$. The attachment to the cell in Figure 15.31 is through the square with ancestry $\varepsilon_{43} = (\diamond \bullet \circ \circ \diamond \diamond \circ \bullet \circ \circ \diamond \bullet)$.

The cell on the right of Figure 15.32 attaches to Figure 15.28 through the square with ancestry $\varepsilon_{44} = (\diamond \bullet \bullet \bullet \diamond \bullet \diamond \bullet \bullet \bullet \circ \diamond)$. The attachment to Figure 15.29 is through the hexagon with ancestry $\varepsilon_{45} = (\circ \circ \diamond \circ \bullet \diamond \bullet \bullet \bullet \bullet \diamond \diamond)$. Moreover, this cell is attached to the left cell in Figure 15.30 and to the left cell in Figure 15.31. The attachment to the cell in Figure 15.30 is through the square with ancestry $\varepsilon_{46} = (\diamond \bullet \bullet \bullet \diamond \bullet \diamond \bullet \bullet \bullet \bullet)$. The attachment to the cell in Figure 15.31 is through the square with ancestry $\varepsilon_{47} = (\diamond \bullet \bullet \bullet \diamond \diamond \bullet \bullet \bullet \bullet \diamond \circ)$.

Step 6: The sixth part of the CW complex consists of attaching two 3-cells to Figure 15.29.

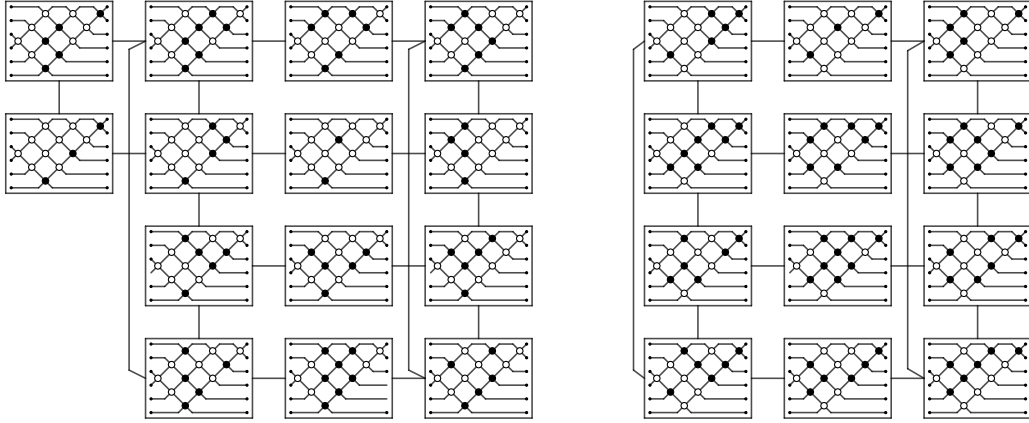


Figure 15.33: Sixth part of the CW complex of dimension 3 with ancestries $\varepsilon_{48} = (\circ \blacklozenge \blacklozenge \circ \blacklozenge \circ \blacklozenge \bullet \circ \blacklozenge \circ)$ and $\varepsilon_{49} = (\circ \blacklozenge \blacklozenge \bullet \blacklozenge \circ \blacklozenge \circ \circ \blacklozenge \bullet)$.

The cell on the left of Figure 15.33 attaches to Figure 15.29 through the hexagon with ancestry $\varepsilon_{50} = (\circ \blacklozenge \blacklozenge \circ \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \circ)$. Furthermore, this cell is attached to the left cell in Figure 15.30 and to the right cell Figure 15.31. The attachment to the cell in Figure 15.30 is through the square with ancestry $\varepsilon_{51} = (\circ \circ \blacklozenge \bullet \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \bullet)$. The attachment to the cell in Figure 15.31 is through the hexagon with ancestry $\varepsilon_{52} = (\circ \blacklozenge \circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \circ \circ)$.

The cell on the right of Figure 15.33 attaches to Figure 15.29 through the hexagon with ancestry $\varepsilon_{53} = (\circ \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \circ \circ \circ \blacklozenge \bullet)$. Furthermore, this cell is attached to the right cell in Figure 15.30 and to the left cell in Figure 15.31. The attachment to the cell in Figure 15.30 is through the square with ancestry $\varepsilon_{54} = (\circ \circ \blacklozenge \bullet \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \bullet)$. The attachment to the cell in Figure 15.31 is through the hexagon with ancestry $\varepsilon_{55} = (\circ \blacklozenge \circ \bullet \blacklozenge \bullet \bullet \circ \blacklozenge \blacklozenge \circ \bullet)$.

To help with understanding, there are two 4-cells that share a common 3-cell. Moreover, eight 3-cells are attached, encircling the 4-cells.

Upon completing all attachments, we have a contractible CW complex. Therefore, BL_σ has a total of 16 connected components of this type, all contractible. Summing up, BL_σ has a total of 96 connected components, all contractible.

Type 15.3. For $\sigma = [463521] = a_2a_1a_3a_4a_3a_2a_1a_5a_4a_3a_2a_1 \in S_6$ it follows that

$$\begin{aligned} \acute{\sigma} = \frac{1}{4\sqrt{2}} & (1 - \hat{a}_1 - \hat{a}_2\hat{a}_3 + \hat{a}_1\hat{a}_2\hat{a}_3 - \hat{a}_4 + \hat{a}_1\hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 \\ & - \hat{a}_1\hat{a}_5 - \hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_4\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If the diamonds are in r_4 and the remaining rows have equal signs, we have the CW complex in Figure 15.34. This results in 16 copies. Therefore, BL_σ has a total of 16 connected components of this type, all contractible.

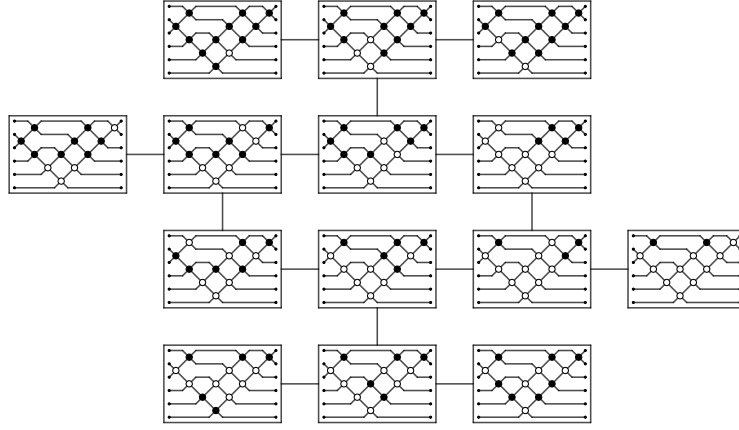


Figure 15.34: CW complex of dimension 2 with ancestry $\varepsilon_1 = (\blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet)$.

The remaining ancestries of dimension 1 and 2 appear in higher-dimensional CW complexes.

If the diamonds are in r_1 with signs $(\bullet \circ \circ)$, we have a component that has seven 3-cells. Now, Let us describe the step by step construction of the component.

Step 1: Start with three 3-cells attached in the first part of the CW complex, as shown in Figure 15.35. The cells fill one convex solid with twelve faces and two prisms, with ancestries $\varepsilon_2 = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \blacklozenge \bullet)$, $\varepsilon_3 = (\circ \circ \blacklozenge \blacklozenge \bullet \circ \blacklozenge \circ \blacklozenge \blacklozenge \bullet \blacklozenge)$ and $\varepsilon_4 = (\circ \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \circ \blacklozenge \blacklozenge \circ \circ)$, respectively.

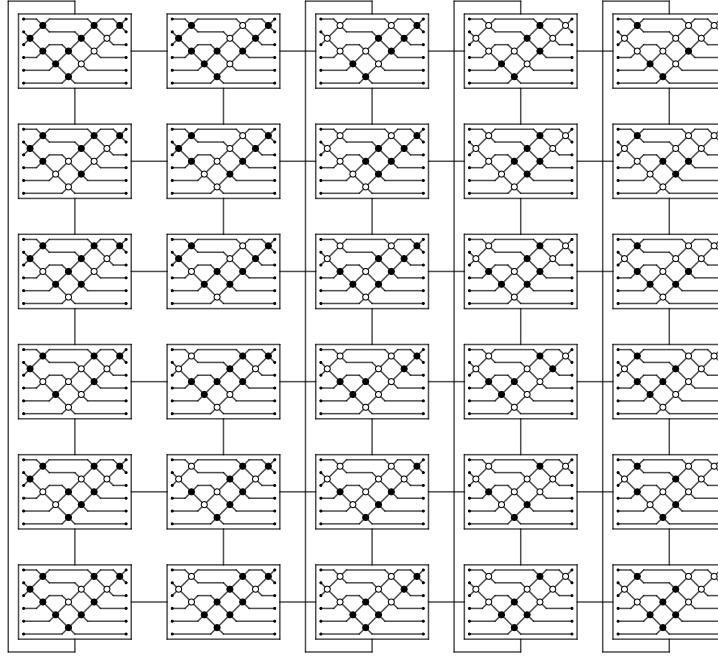
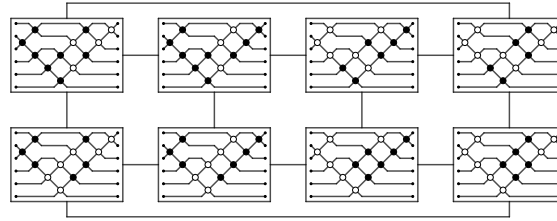


Figure 15.35: First part of the CW complex of dimension 3.

Step 2: The second step consists of attaching a cube to the second and third squares in the first line of squares. The attachment is through the squares in Figure 15.36 with ancestries $\varepsilon_6 = (\blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \circ \bullet \blacklozenge \bullet \bullet \bullet)$ and $\varepsilon_7 = (\circ \circ \circ \blacklozenge \circ \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge)$.

Figure 15.36: Second part of the CW complex of dimension 3, with ancestries $\varepsilon_5 = (\blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge)$.

Step 3: The third part consists of attaching a cell that fills the prism in Figure 15.37. The attachment is through the three central squares in Figure 15.37 to the three last squares in the fourth line of squares in Figure 15.35, with ancestries $\varepsilon_9 = (\circ \circ \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet)$, $\varepsilon_{10} = (\circ \circ \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge)$ and $\varepsilon_{11} = (\circ \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \bullet \circ)$.

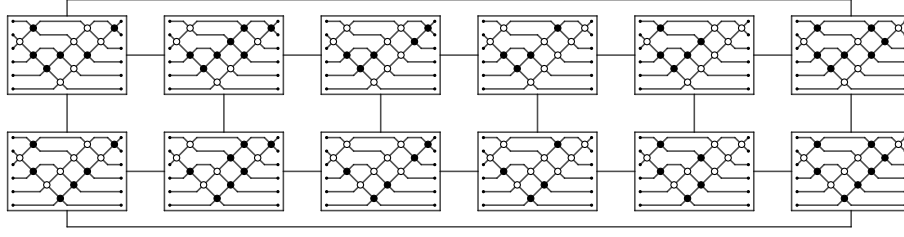


Figure 15.37: Third part of the CW complex of dimension 3, with ancestries $\varepsilon_8 = (\circ \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \circ \diamond \circ \diamond \diamond)$.

Step 4: The fourth part involves attaching a cell that fills the cube in Figure 15.38 to the two central squares in the second line of squares in Figure 15.35, with ancestries $\varepsilon_{12} = (\blacklozenge \bullet \blacklozenge \circ \diamond \diamond \circ \bullet \bullet \bullet \bullet)$ and $\varepsilon_{13} = (\circ \circ \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \bullet \circ \diamond)$. Furthermore, the upper square in Figure 15.38 glues to the square at the bottom of Figure 15.36, with ancestry $\varepsilon_{15} = (\blacklozenge \bullet \bullet \circ \circ \diamond \blacklozenge \bullet \bullet \circ \diamond)$. Note that there are three 2-cells attached to the cube.

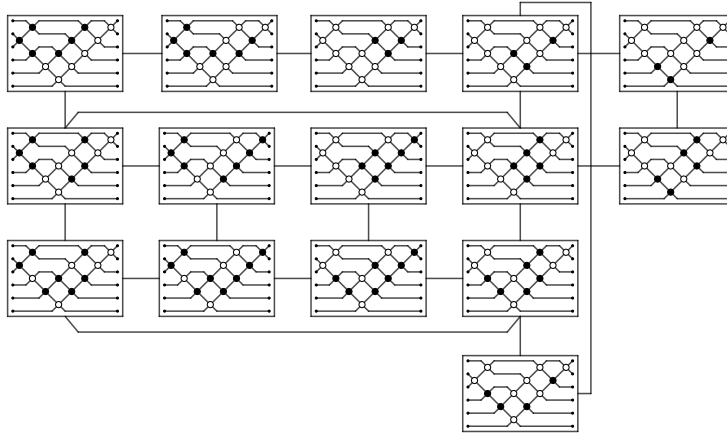


Figure 15.38: Fourth part of the CW complex of dimension 3, with ancestries $\varepsilon_{14} = (\blacklozenge \bullet \blacklozenge \circ \diamond \diamond \blacklozenge \bullet \bullet \circ \circ \diamond)$.

Step 5: The fifth part consists to attach a cell that fills the prism in Figure 15.39. The attachment is made by connecting the three last squares in the prism to the three last squares in the last line of squares in Figure 15.39, with ancestries $\varepsilon_{17} = (\circ \circ \blacklozenge \circ \diamond \blacklozenge \bullet \bullet \bullet \bullet \diamond \bullet)$, $\varepsilon_{18} = (\circ \circ \blacklozenge \circ \diamond \circ \blacklozenge \bullet \bullet \circ \circ \diamond)$, and $\varepsilon_{19} = (\circ \blacklozenge \blacklozenge \circ \diamond \bullet \blacklozenge \bullet \bullet \circ \circ \circ)$. The upper hexagon in the prism attaches to the hexagon at the bottom of Figure 15.37, with ancestry $\varepsilon_{20} = (\circ \blacklozenge \bullet \circ \circ \blacklozenge \bullet \bullet \circ \circ \diamond \diamond)$. Note that there is a 2-cell attached to the prism.

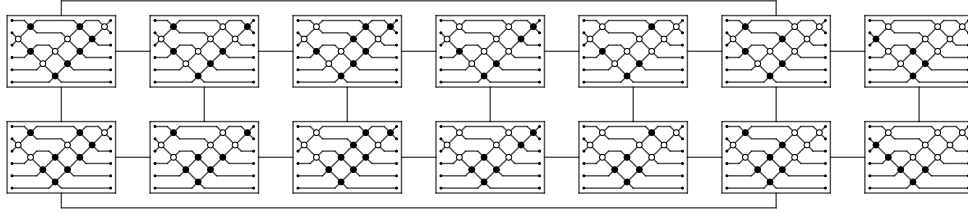


Figure 15.39: Fifth part of the CW complex of dimension 3, with ancestries $\varepsilon_{16} = (\circ \blacklozenge \blacklozenge \circ \diamond \blacklozenge \bullet \bullet \circ \diamond \diamond)$.

Step 6: In this step, we attach seven 2-cells in Figure 15.40 to Figure 15.35, Attachment occurs through four 0-cells and two 1-cells.

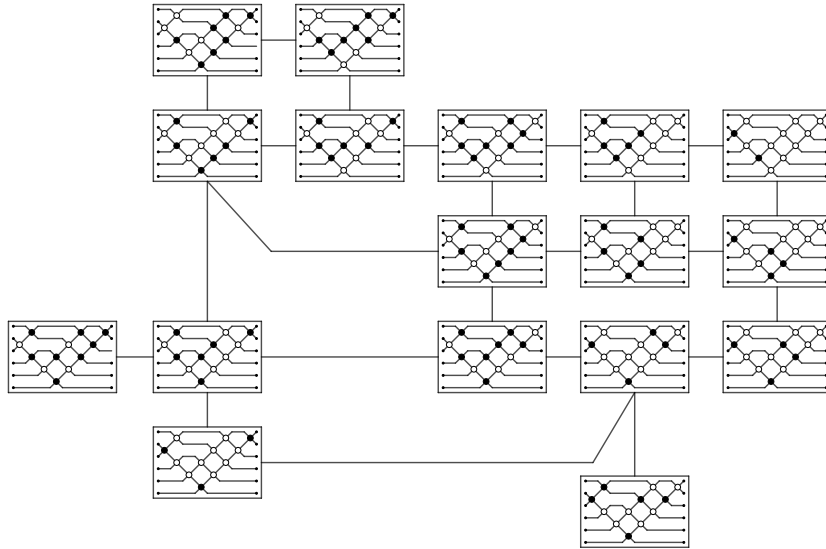


Figure 15.40: Sixth part of the CW complex.

Step 7: To finish the attachment of this CW complex, attach three 0-cells and two 1-cells from Figure 15.41 to Figure 15.35.

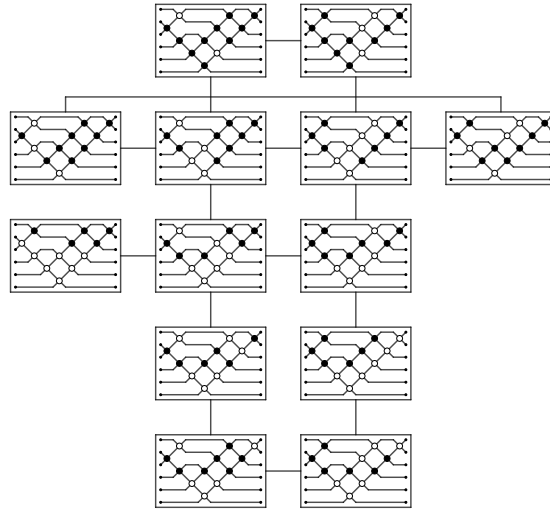


Figure 15.41: Seventh part of the CW complex with four 2-cells.

Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 32 connected components of this type, all contractible.

The remaining ancestries of dimensions 1 and 2 appear in higher-dimensional CW complexes.

For dimension 3, there are eleven possible positions for the diamonds, and there are two components with dimension 3: one contains four 3-cells, and the other contains seven, the latter corresponds to the previous case. We proceed with a step by step construction for the component with four 3-cells. The procedure involves attaching one cell at a time.

Step 1: Begin with the 3-cell with ancestry $\varepsilon_{21} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge)$ fills the “parallelepiped” in Figure 15.42.

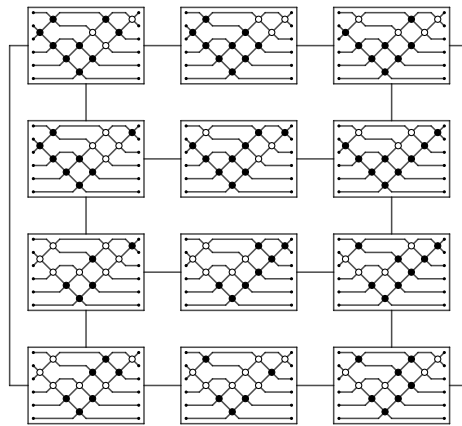


Figure 15.42: First part of the CW complex of dimension 3.

Step 2: The second part consists in attach the “parallelepiped” in Figure 15.43, with ancestry $\varepsilon_{22} = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \bullet)$, to the previous one. The attachments occurs through the hexagon at the bottom of Figure 15.43 to the one in the center of Figure 15.42 with ancestry $\varepsilon_{23} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \circ \blacklozenge \bullet \bullet \circ \blacklozenge \bullet)$.

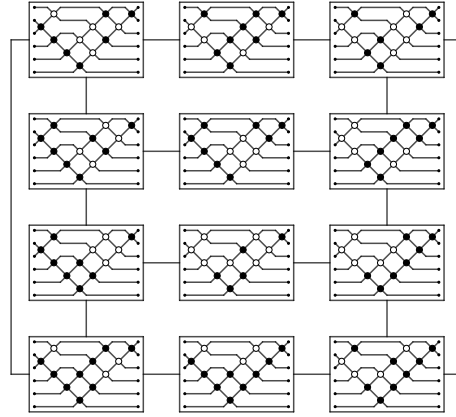


Figure 15.43: Second part of the CW complex of dimension 3.

Step 3: For the third part, the prism in Figure 15.44 will be attached to the upper hexagon in the previous figure.

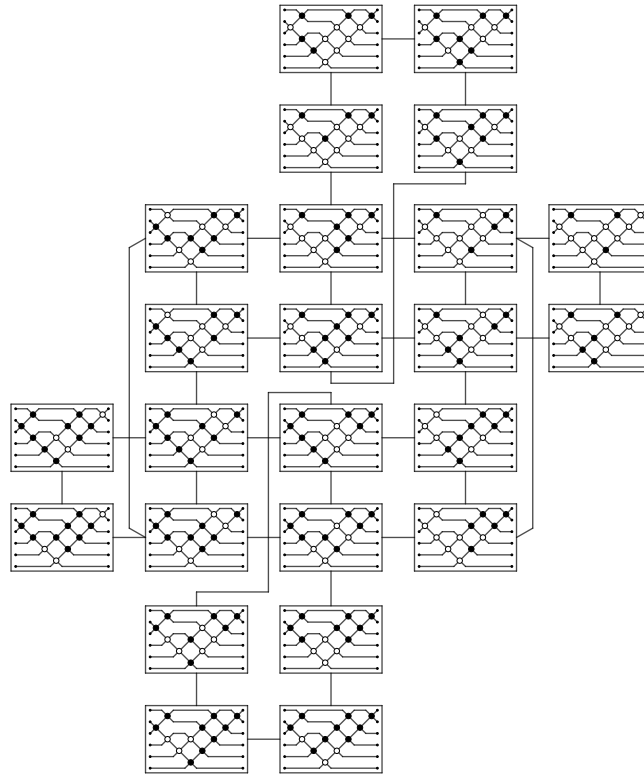


Figure 15.44: Third part of the CW complex of dimension 3.

Attachment occurs through the hexagon in the center of Figure 15.44 with ancestry $\varepsilon_{25} = (\blacklozenge \blacklozenge \bullet \bullet \circ \bullet \blacklozenge \bullet \bullet \circ \bullet \bullet)$. Note that four 2-cells are attached like “wings” to the 3-cell, with ancestry $\varepsilon_{24} = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \circ \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet)$ that fills the prism.

Step 4: The last 3-cell fills the prism in Figure 15.45 and will be attached to the same 2-cell as the third part. Note that there are two 2-cells attached to the prism.

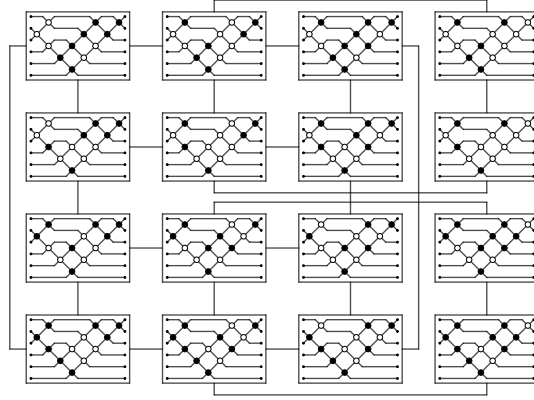


Figure 15.45: Fourth part of the CW complex of dimension 3, with ancestry $\varepsilon_{26} = (\blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \circ \bullet \bullet \blacklozenge \bullet \bullet)$.

Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 16 contractible connected components of this type.

The remaining ancestries of dimension 3 appear in a 4-dimensional CW complex.

For dimension 4, there are three 4-cells that appear alongside three 3-cells in a CW complex. Alternatively, in dimension 1, if the diamonds are located in r_3 with signs $(\bullet \bullet \circ)$, we obtain the 4-dimensional component. All 4-cells are homotopically equivalent to $\mathbb{D}^4$, as we have seen before. This results in 16 copies. The structure is complex and requires a step by step construction.

Step 1: The first part consists of a 4-cell with ancestry of dimension 4 $\varepsilon_{27} = (\blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet)$, and cells of lower dimension attached.

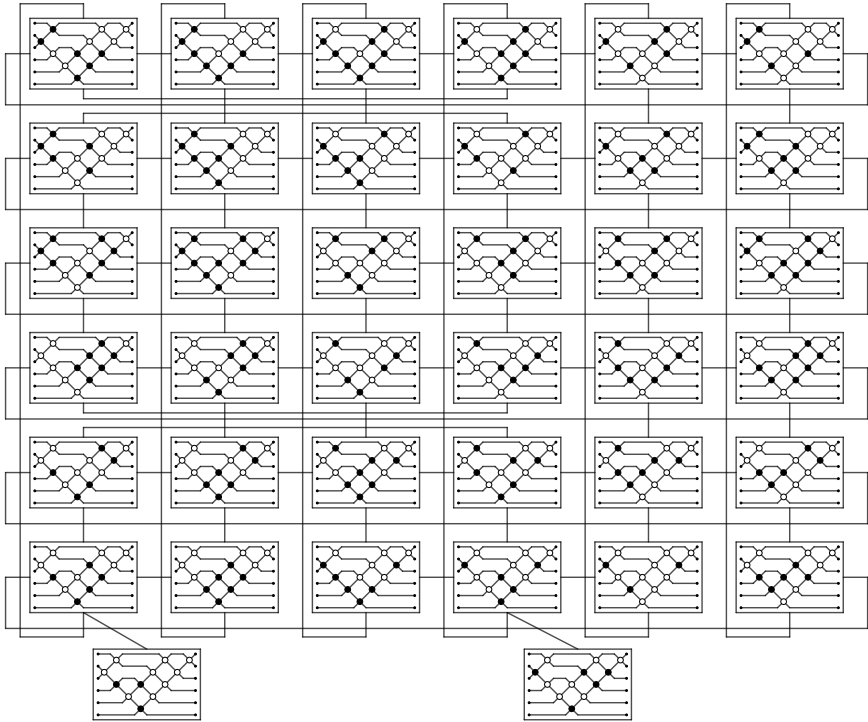


Figure 15.46: First part of the CW complex.

Step 2: Next, attach the second 4-cell with ancestry of dimension 4
 $\varepsilon_{28} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \diamond \bullet \diamond \bullet \diamond \diamond)$.

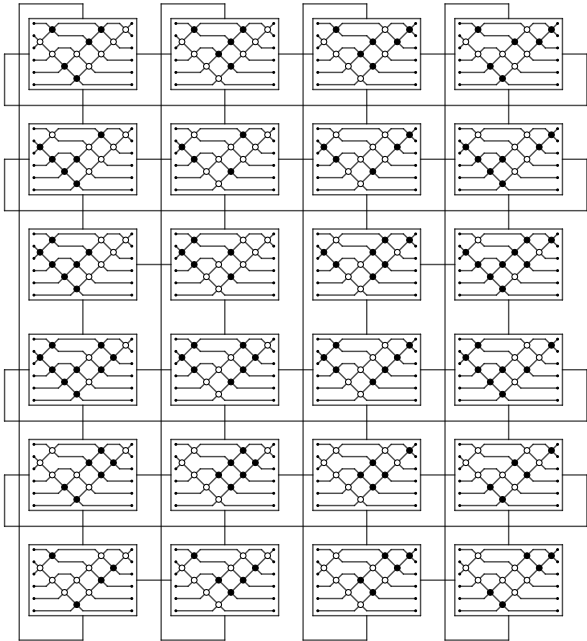


Figure 15.47: Second part of the CW complex.

Attachment occurs through the first vertical prism in Figure 15.47 to the 3-cell in Figure 15.46 with ancestry $\varepsilon_{29} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \circ \circ \blacklozenge \circ)$.

Step 3: The third part consists of attaching the last 4-cell with ancestry $\varepsilon_{30} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \circ \bullet \circ \blacklozenge \blacklozenge)$.

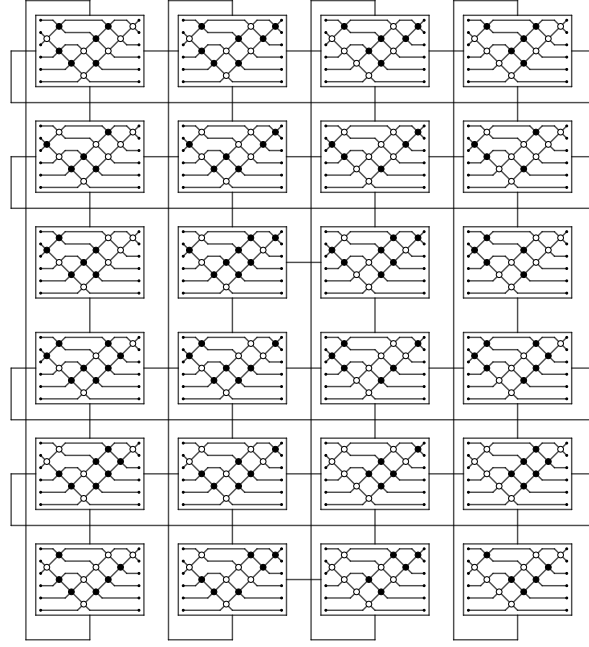


Figure 15.48: Third part of the CW complex.

Attachment occurs through the last vertical “parallelepiped” in Figure 15.48 to the 3-cell in the vertical center of Figure 15.47, with ancestry $\varepsilon_{31} = (\blacklozenge \blacklozenge \bullet \circ \circ \blacklozenge \blacklozenge \circ \bullet \circ \blacklozenge \blacklozenge)$. The vertical prism in the center of Figure 15.48 attaches to the last vertical prism in Figure 15.46, with ancestry $\varepsilon_{32} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \bullet)$.

Step 4: The fourth part is a 3-cell that fills the “parallelepiped” with ancestry $\varepsilon_{33} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \circ \bullet \circ \blacklozenge \blacklozenge)$ in Figure 15.49. The attachment is through the hexagon on the left side of Figure 15.49 to the hexagon with ancestry $\varepsilon_{34} = (\blacklozenge \blacklozenge \bullet \circ \circ \circ \blacklozenge \circ \bullet \bullet \blacklozenge \bullet)$ in Figure 15.48.

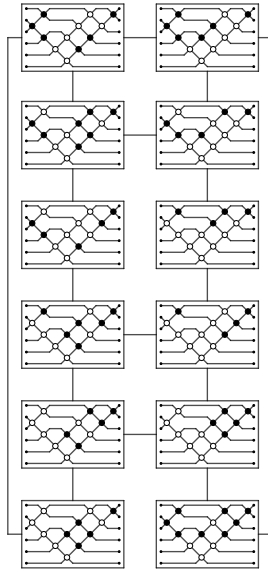


Figure 15.49: Fourth part of the CW complex.

Step 5: The fifth part is a 3-cell that fills the prism in Figure 15.50. This cell is attached to all cells of dimension 4. The hexagon with ancestry $\varepsilon_{36} = (\circ \circ \blacklozenge \blacklozenge \bullet \circ \bullet \circ \blacklozenge \blacklozenge \bullet \circ)$ attaches to Figure 15.46. The square with ancestry $\varepsilon_{37} = (\circ \circ \circ \blacklozenge \circ \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge)$ attaches to Figure 15.47. The square with ancestry $\varepsilon_{38} = (\circ \circ \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge)$ attaches to Figure 15.48.

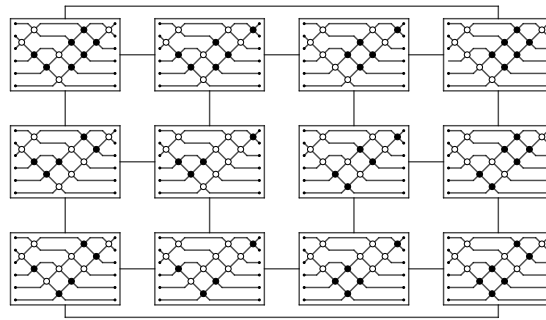


Figure 15.50: Fifth part of the CW complex with ancestries of dimension 3 $\varepsilon_{35} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \circ \bullet \circ \blacklozenge \blacklozenge)$.

Step 6: The last part is a 3-cell that fills the prism with ancestry $\varepsilon_{39} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \circ \bullet \circ \blacklozenge \blacklozenge)$ in Figure 15.51.

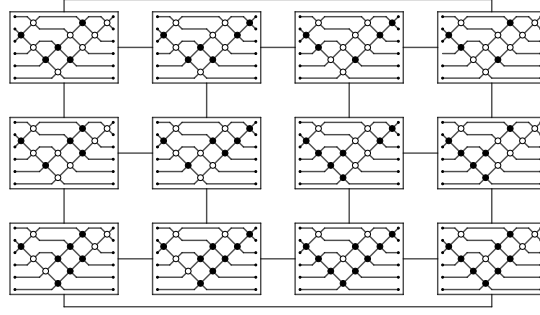


Figure 15.51: Sixth part of the CW complex.

This cell attaches to all 4-cells. The hexagon with ancestry $\varepsilon_{40} = (\bullet \circ \blacklozenge \blacklozenge \bullet \circ \bullet \circ \blacklozenge \blacklozenge \bullet \circ \bullet)$ attaches to Figure 15.46. The square with ancestry $\varepsilon_{41} = (\bullet \circ \blacklozenge \circ \blacklozenge \circ \blacklozenge \bullet \circ \bullet \circ \blacklozenge)$ attaches to Figure 15.48. The square with ancestry $\varepsilon_{42} = (\bullet \circ \bullet \bullet \bullet \bullet \circ \blacklozenge \bullet \circ \blacklozenge \bullet \circ \blacklozenge)$ attaches to Figure 15.47.

The CW complex consists of three 4-cells connected through 3-cells and sharing one common 2-cell, with three additional 3-cells attached.

Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 16 connected components of this type, all contractible. Summing up, BL_σ has a total of 112 connected components, all of them contractible.

The permutations

$$\sigma = a_1 a_3 a_2 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1, \quad \sigma = a_2 a_1 a_3 a_2 a_1 a_4 a_3 a_5 a_4 a_3 a_2 a_1 \in S_6$$

have a CW complex structure similar to the one described.

Type 15.4. For $\sigma = [465231] = a_2 a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1 \in S_6$ it follows that

$$\begin{aligned} \sigma = \frac{1}{4\sqrt{2}} & (-\hat{a}_2 + \hat{a}_1 \hat{a}_2 + \hat{a}_3 - \hat{a}_1 \hat{a}_3 - \hat{a}_4 + \hat{a}_1 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_5 \\ & - \hat{a}_1 \hat{a}_5 - \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 + \hat{a}_2 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 - \hat{a}_3 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If the diamonds are in row r_4 and the remaining rows have equal signs, we have a CW complex with one 3-cell and some 2-cells attached. This will be constructed through three steps.

Step 1: First, we have a 3-cell that fills the cube with ancestry $\varepsilon_1 = (\blacklozenge \circ \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \circ \blacklozenge \bullet \blacklozenge \bullet)$ in Figure 15.52, with 2-cells attached.

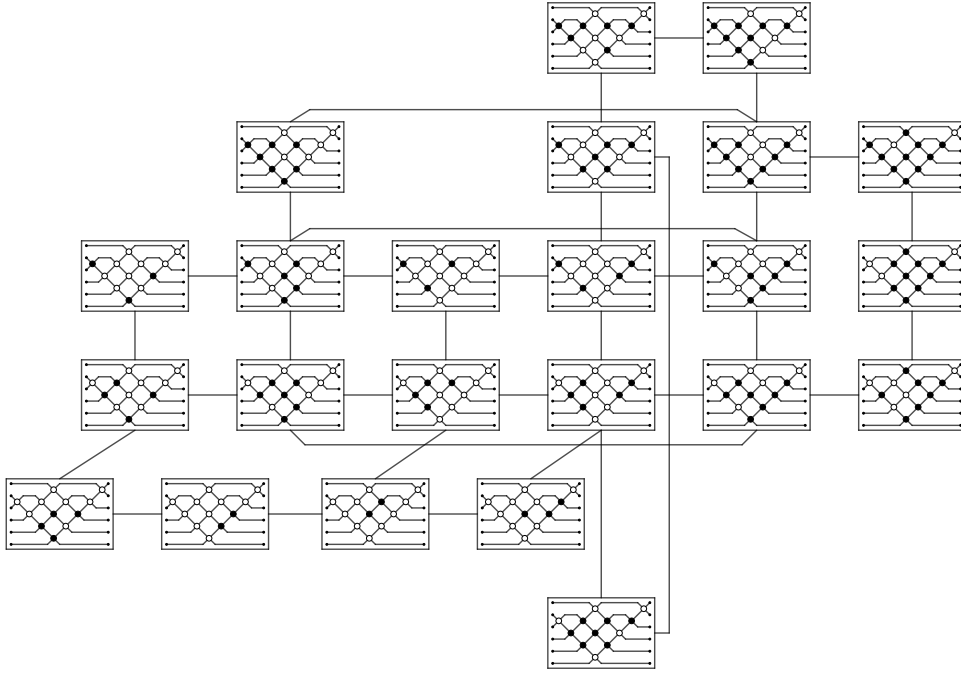


Figure 15.52: First part of the CW complex.

Step 2: Attach nine 2-cells to Figure 15.52. Attachment occurs through five 0-cells and four 1-cells.

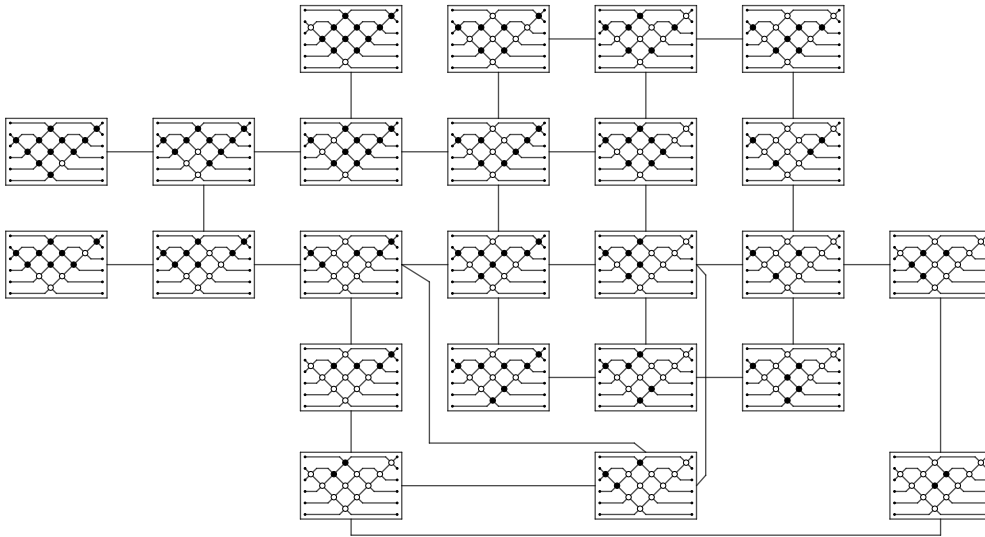


Figure 15.53: Second part of the CW complex.

Step 3: To finish, attach eight 2-cells to Figure 15.52. Attachment occurs through six 0-cells and five 1-cells.

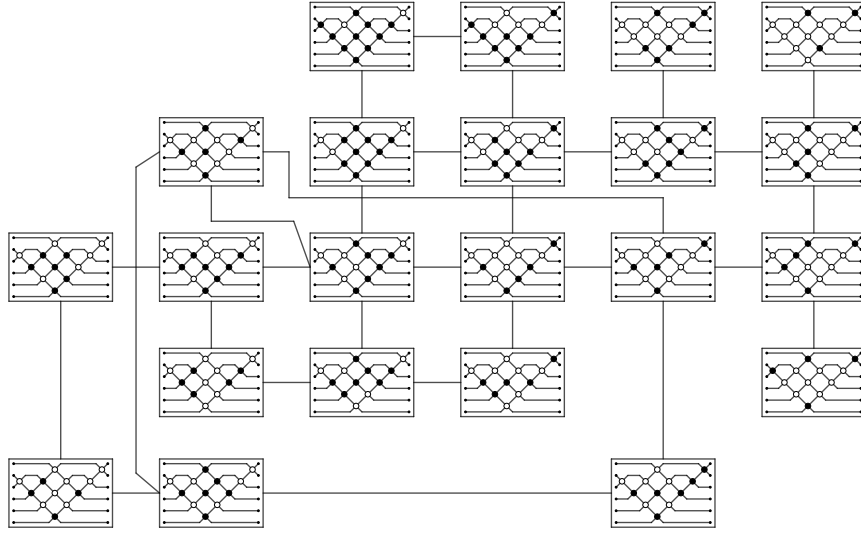
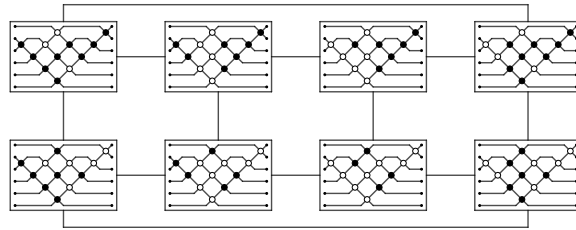


Figure 15.54: Third part of the CW complex.

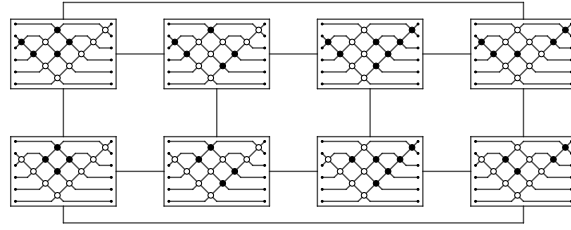
Upon completing the attachments, we obtain a contractible CW complex. Therefore, BL_σ has a total of 16 connected components of this type, all contractible.

If the diamonds are in row r_1 and the remaining rows have equal signs, we have a CW complex with ten 3-cells. This CW complex will be constructed step by step, attaching one 3-cell at time.

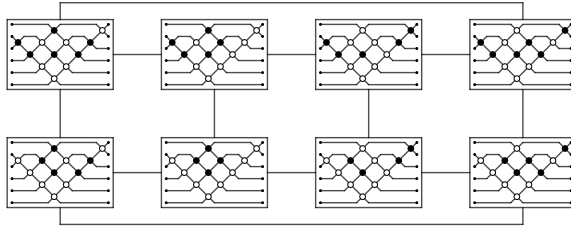
Step 1: First, we have a 3-cell that fills the cube in Figure 15.55.

Figure 15.55: 3-Cell with ancestry $\varepsilon_2 = (\blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge)$.

Step 2: Attach a 3-cell that fills the cube in Figure 15.56. The attachment to Figure 15.55 occurs through the 2-cell with ancestry $\varepsilon_4 = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge)$.

Figure 15.56: 3-Cell with ancestry $\varepsilon_3 = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \circ \circ \blacklozenge \circ \blacklozenge)$.

Step 3: Attach a 3-cell that fills the cube in Figure 15.57. The attachment to Figure 15.56 occurs through the 2-cell with ancestry $\varepsilon_6 = (\blacklozenge \bullet \blacklozenge \blacklozenge \circ \bullet \bullet \bullet \circ \circ \circ \blacklozenge)$.

Figure 15.57: 3-Cell with ancestry $\varepsilon_5 = (\blacklozenge \bullet \blacklozenge \blacklozenge \circ \bullet \blacklozenge \circ \circ \circ \blacklozenge \blacklozenge)$.

Step 4: Attach a 3-cell with ancestry $\varepsilon_7 = (\blacklozenge \blacklozenge \blacklozenge \circ \circ \blacklozenge \blacklozenge \circ \circ \circ \blacklozenge \circ)$ that fills the “parallelepiped” in Figure 15.58. The attachment to Figure 15.56 occurs through the 2-cell with ancestry $\varepsilon_8 = (\blacklozenge \bullet \blacklozenge \bullet \circ \bullet \bullet \bullet \circ \circ \blacklozenge \circ \circ)$.

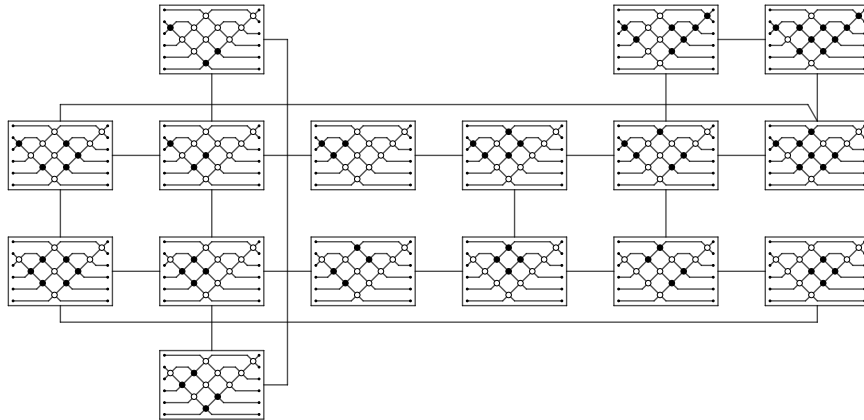


Figure 15.58: Fourth part of the CW complex.

Step 5: Attach a 3-cell with ancestry $\varepsilon_9 = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \circ \bullet \bullet \blacklozenge \blacklozenge \bullet)$ that fills the convex solid in Figure 15.59, with four 0-cells and four 1-cells

attached. The attachment to Figure 15.56 occurs through the 2-cell with ancestry $\varepsilon_{10} = (\blacklozenge \bullet \diamond \circ \circ \blacklozenge \circ \circ \circ \diamond \bullet \bullet)$. This cell is also attached to the one in Figure 15.55 through the 2-cell with ancestry $\varepsilon_{11} = (\blacklozenge \bullet \diamond \circ \blacklozenge \bullet \bullet \bullet \diamond \bullet \bullet \bullet)$.

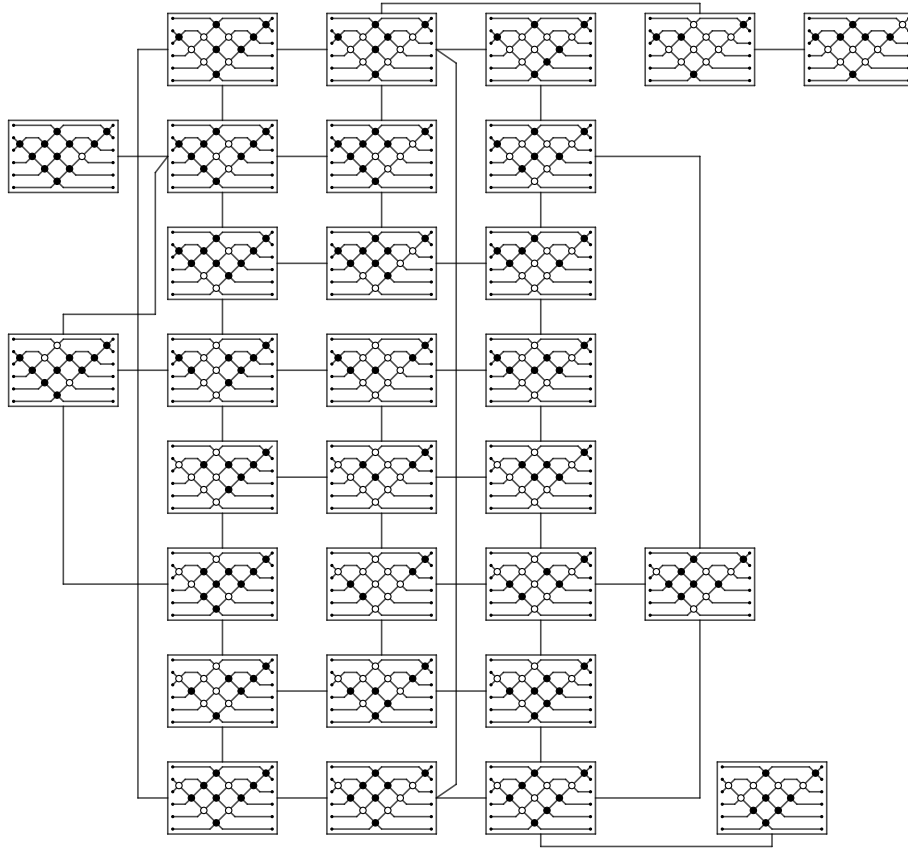


Figure 15.59: Fifth part of the CW complex.

Step 6: Attach a 3-cell that fills the prism in Figure 15.59. The attachment to Figure 15.56 occurs through the 2-cell with ancestry $\varepsilon_{13} = (\circ \circ \bullet \blacklozenge \circ \blacklozenge \bullet \circ \circ \diamond \circ \diamond)$.

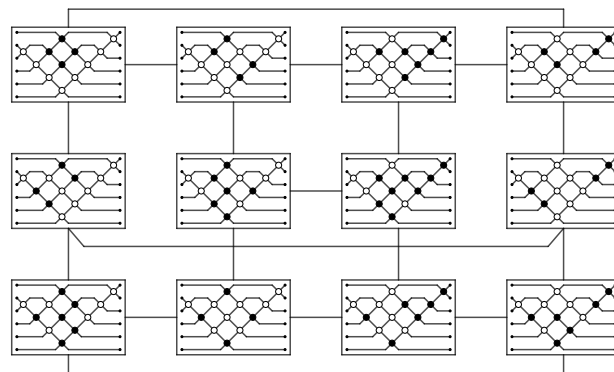


Figure 15.60: 3-Cell with ancestry $\varepsilon_{12} = (\diamond \blacklozenge \circ \blacklozenge \bullet \circ \bullet \circ \diamond \diamond \circ \diamond)$.

This cell is also attached to the one in Figure 15.59 through the 2-cell with ancestry $\varepsilon_{14} = (\circ \blacklozenge \circ \circ \blacklozenge \circ \circ \circ \blacklozenge \blacklozenge \bullet)$. And to the one in Figure 15.55 through the 2-cell with ancestry $\varepsilon_{15} = (\circ \circ \bullet \blacklozenge \blacklozenge \bullet \circ \bullet \blacklozenge \bullet \circ \blacklozenge)$.

Step 7: Attach a 3-cell that fills the cube in Figure 15.61. The attachment to Figure 15.60 occurs through the 2-cell with ancestry $\varepsilon_{17} = (\circ \blacklozenge \circ \blacklozenge \bullet \blacklozenge \bullet \circ \circ \circ \blacklozenge)$. This cell is also attached to the one in Figure 15.57 through the 2-cell with ancestry $\varepsilon_{18} = (\circ \circ \bullet \blacklozenge \circ \bullet \blacklozenge \circ \circ \circ \blacklozenge \blacklozenge)$.

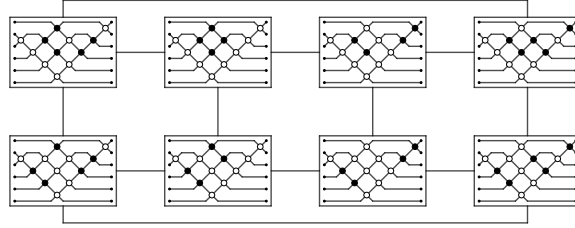


Figure 15.61: 3-Cell with ancestry $\varepsilon_{16} = (\circ \blacklozenge \circ \blacklozenge \bullet \blacklozenge \bullet \circ \circ \circ \blacklozenge \blacklozenge)$.

Step 8: Attach a 3-cell that fills the cube in Figure 15.62. The attachment to Figure 15.60 occurs through the 2-cell with ancestry $\varepsilon_{20} = (\circ \bullet \circ \blacklozenge \blacklozenge \circ \bullet \circ \blacklozenge \circ \circ \blacklozenge)$. This cell is also attached to the one in Figure 15.59 through the 2-cell with ancestry $\varepsilon_{21} = (\circ \bullet \circ \circ \blacklozenge \circ \blacklozenge \circ \blacklozenge \bullet \blacklozenge \bullet)$.

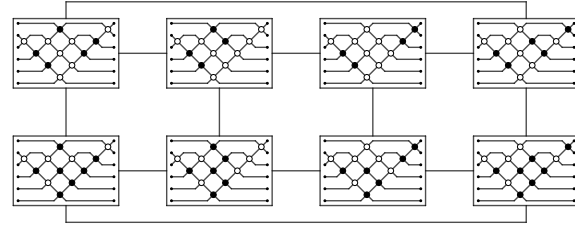


Figure 15.62: 3-Cell with ancestry $\varepsilon_{19} = (\circ \bullet \circ \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge \circ \blacklozenge \blacklozenge)$.

Step 9: Attach a 3-cell that fills the “parallelepiped” in Figure 15.63, with a 2-cell attached. The attachment to Figure 15.60 occurs through the 2-cell with ancestry $\varepsilon_{23} = (\circ \blacklozenge \circ \bullet \blacklozenge \circ \bullet \circ \blacklozenge \blacklozenge \circ \circ)$.

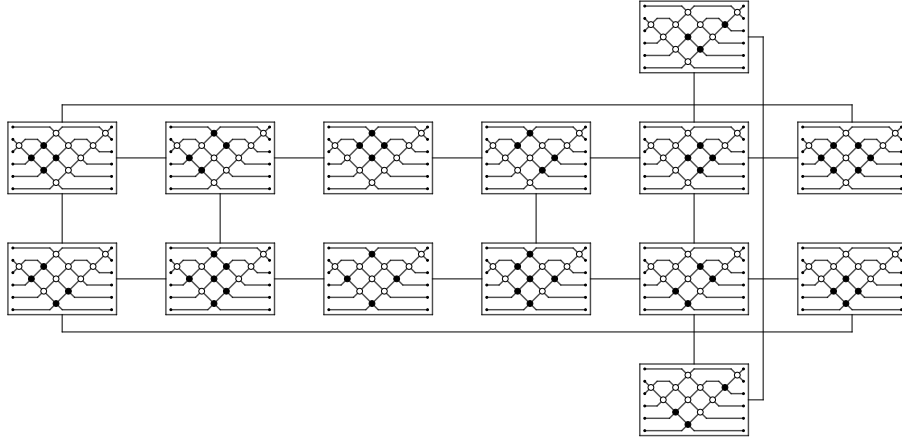


Figure 15.63: Ninth part of the CW complex with ancestry of dimension 3:
 $\varepsilon_{22} = (\circ \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \circ \diamond \diamond \circ \circ)$.

Step 10: Attach a 3-cell that fills the “parallelepiped” with ancestry $\varepsilon_{24} = (\circ \blacklozenge \blacklozenge \circ \bullet \blacklozenge \diamond \circ \circ \diamond \diamond \circ)$ in Figure 15.64, with two 2-cells attached. The attachment to Figure 15.63 and Figure 15.58 occurs through the 2-cell with ancestry $\varepsilon_{25} = (\circ \blacklozenge \blacklozenge \circ \bullet \bullet \diamond \circ \circ \diamond \circ \circ)$.

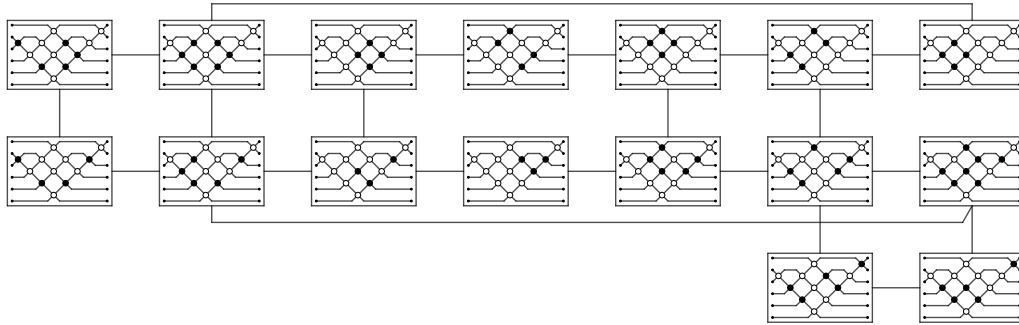


Figure 15.64: Tenth part of the CW complex.

Upon completing all attachments, we obtain a contractible CW complex. Therefore, BL_σ has a total of 32 connected components of this type, all contractible.

The remaining ancestries of dimensions 1, 2 and 3 appear in higher-dimensional CW complexes.

For dimension 4, there are two possible positions for the diamonds, which always appear together. The construction will proceed step by step.

Step 1: First, we have a 4-cell with ten 3-cells. This cell is homotopically equivalent to a $\mathbb{D}^4$.

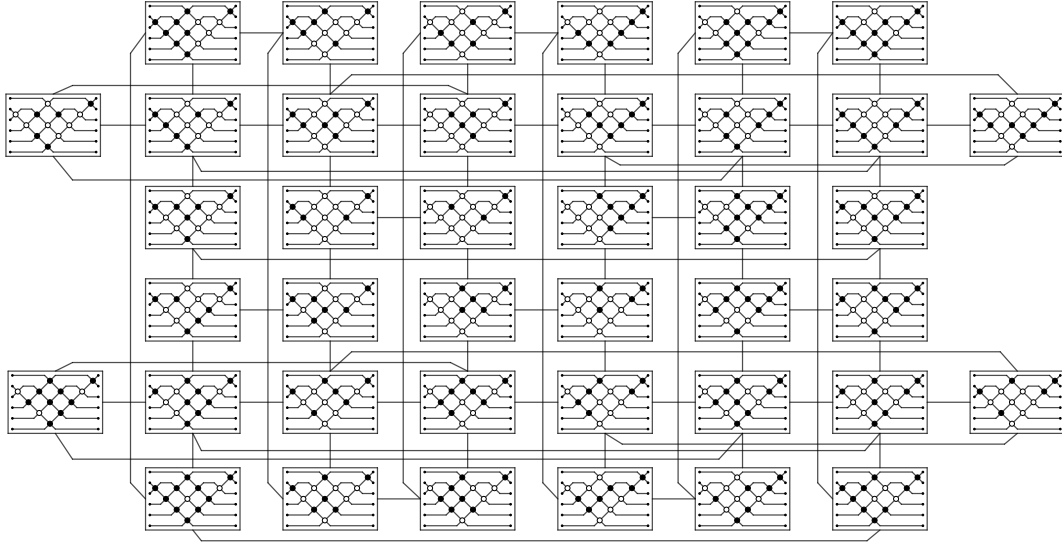


Figure 15.65: 4-Cell with ancestry $\varepsilon_{26} = (\blacklozenge \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \bullet)$.

Step 2: Next, attach the second 4-cell, which is another $\mathbb{D}^4$, with three 2-cells attached. This cell is attached to the one in Figure 15.65 through the 3-cell with ancestry $\varepsilon_{28} = (\blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet)$.

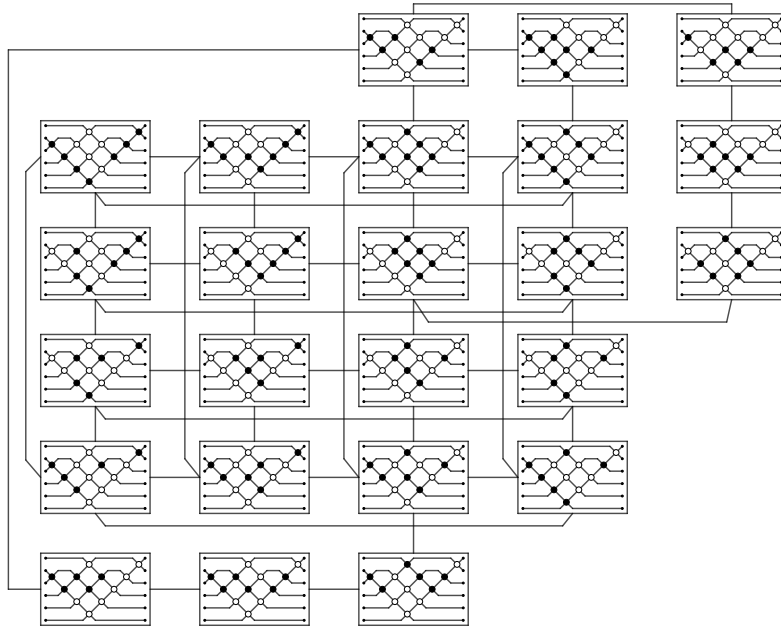


Figure 15.66: Second part of the CW complex with ancestry of dimension 4: $\varepsilon_{27} = (\blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge)$.

Step 3: Now we attach the 3-cells. The first fills the prism in Figure 15.67. This cell is attached to the one in Figure 15.65 through the 2-cell with

ancestry $\varepsilon_{30} = (\bullet \blacklozenge \circ \circ \blacklozenge \circ \bullet \bullet \blacklozenge \circ \circ \bullet)$, and to the one in Figure 15.66 through the 2-cell with ancestry $\varepsilon_{31} = (\bullet \bullet \circ \blacklozenge \blacklozenge \circ \circ \bullet \blacklozenge \circ \bullet \blacklozenge)$.

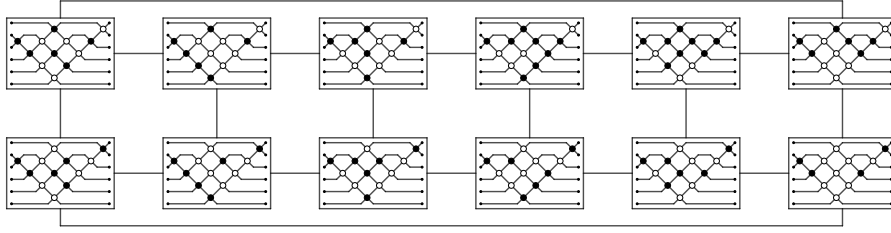


Figure 15.67: 3-Cell with ancestry $\varepsilon_{29} = (\bullet \blacklozenge \circ \blacklozenge \blacklozenge \circ \circ \bullet \blacklozenge \circ \bullet \blacklozenge)$.

Step 4: The second 3-cell fills the prism in Figure 15.68. This cell is attached to the one in Figure 15.65 through the 2-cell with ancestry $\varepsilon_{33} = (\circ \blacklozenge \circ \circ \blacklozenge \bullet \bullet \circ \blacklozenge \circ \bullet \bullet)$, and to the one in Figure 15.66 through the 2-cell with ancestry $\varepsilon_{34} = (\circ \circ \bullet \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \circ \bullet \blacklozenge)$.

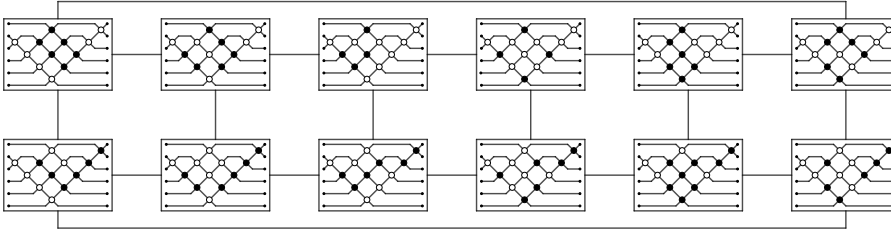


Figure 15.68: 3-Cell with ancestry $\varepsilon_{32} = (\circ \blacklozenge \circ \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \circ \bullet \blacklozenge)$.

Step 5: The third 3-cell fills the cube in Figure 15.69.

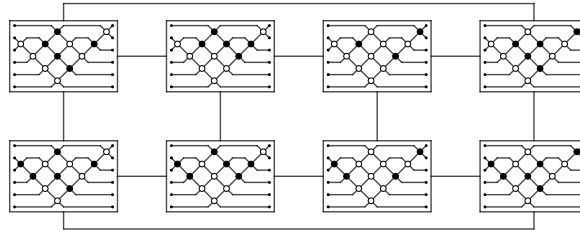


Figure 15.69: 3-Cell with ancestry $\varepsilon_{35} = (\blacklozenge \bullet \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \circ \bullet \blacklozenge)$.

This cell is attached to the one in Figure 15.65 through the 2-cell with ancestry $\varepsilon_{36} = (\blacklozenge \bullet \blacklozenge \circ \circ \blacklozenge \bullet \bullet \circ \blacklozenge \circ \bullet)$. To the one in Figure 15.66 through the

2-cell with ancestry $\varepsilon_{37} = (\blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge)$. And to the one in Figure 15.67 through the 2-cell with ancestry $\varepsilon_{38} = (\bullet \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge)$.

Step 6: The fourth 3-cell fills the cube in Figure 15.70. This cell is attached to the one in Figure 15.65 through the 2-cell with ancestry $\varepsilon_{40} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet)$. To the one in Figure 15.66 through the 2-cell with ancestry $\varepsilon_{41} = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge)$. And to the one in Figure 15.68 through the 2-cell with ancestry $\varepsilon_{42} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge)$.

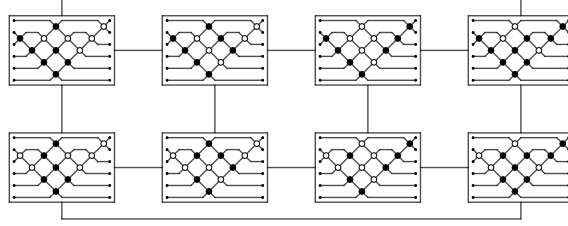


Figure 15.70: 3-Cell with ancestry $\varepsilon_{39} = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge)$.

Step 7: The fifth 3-cell fills the cube in Figure 15.71. This cell is attached to the one in Figure 15.65 through the 2-cell with ancestry $\varepsilon_{44} = (\bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet)$, and to the one in Figure 15.66 through the 2-cell with ancestry $\varepsilon_{45} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge)$.

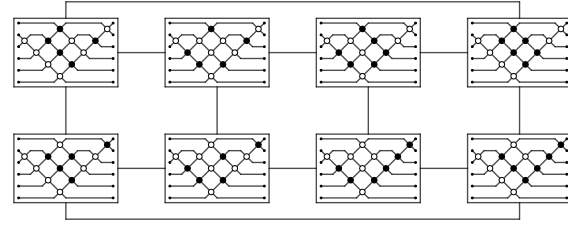


Figure 15.71: 3-Cell with ancestry $\varepsilon_{43} = (\bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge)$.

Step 8: The sixth 3-cell fills the cube in Figure 15.72.

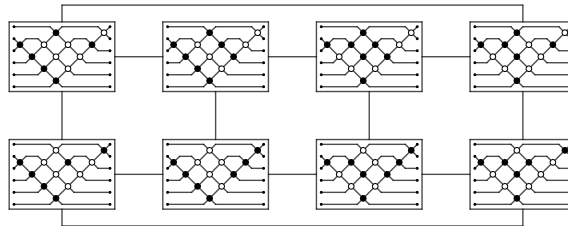


Figure 15.72: 3-Cell with ancestry $\varepsilon_{46} = (\bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge)$.

This cell is attached to the one in Figure 15.65 through the 2-cell with ancestry $\varepsilon_{47} = (\bullet \blacklozenge \circ \circ \bullet \blacklozenge \bullet \circ \circ \blacklozenge \bullet)$, and to the one in Figure 15.66 through the 2-cell with ancestry $\varepsilon_{48} = (\bullet \circ \bullet \blacklozenge \circ \bullet \blacklozenge \bullet \circ \bullet \blacklozenge \blacklozenge)$.

Note that this CW complex comprises two $\mathbb{D}^4$ attached through a cube. Additionally, these $\mathbb{D}^4$ have some 3-cells attached to them, but these cells do not alter the homotopy type.

Upon completing all attachments, we obtain a contractible CW complex. Therefore, BL_σ has a total of 16 connected components of this type, all contractible. Summing up, BL_σ has a total of 96 connected components, all of them contractible.

Type 15.5. For $\sigma = [465312] = a_2a_3a_2a_1a_4a_3a_2a_1a_5a_4a_3a_2 \in S_6$ it follows that

$$\begin{aligned} \acute{\sigma} = \frac{1}{4\sqrt{2}} & (-\hat{a}_1 - \hat{a}_2 - \hat{a}_1\hat{a}_3 - \hat{a}_2\hat{a}_3 + \hat{a}_1\hat{a}_4 - \hat{a}_2\hat{a}_4 + \hat{a}_1\hat{a}_3\hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4 \\ & - \hat{a}_1\hat{a}_5 - \hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_4\hat{a}_5 - \hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If the diamonds are in row r_4 and the remaining rows have equal signs, we have a CW complex with one 3-cell and cells of lower dimensions attached. This will be constructed through three steps.

Step 1: First, we have a 3-cell that fills the convex solid, with two vertexes and two edges attached.

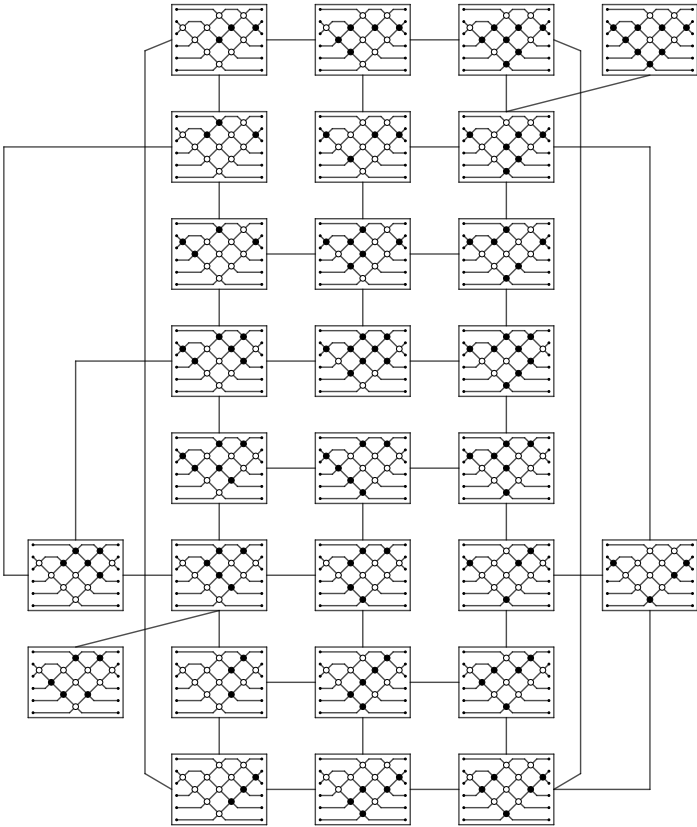


Figure 15.73: Cell of dimension 3 with ancestry $\varepsilon_1 = (\blacklozenge \blacklozenge \circ \bullet \blacklozenge \circ \circ \bullet \blacklozenge \blacklozenge \blacklozenge)$.

Step 2: Attach the Figure 15.74 to the previous one. Attachment occurs through four 1-cells and five 0-cells.

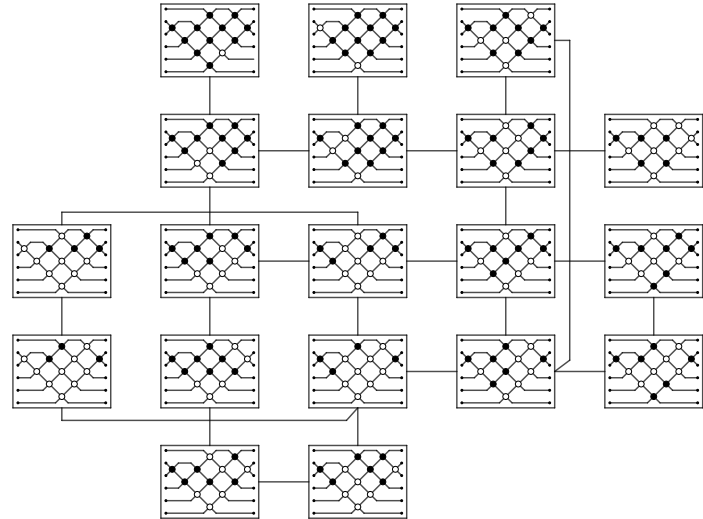


Figure 15.74: Second part of the CW complex.

Step 2: For the second part, attach a 3-cell that fills the “parallelepiped” in Figure 15.77. The attachment to Figure 15.76 occurs through the 2-cell with ancestry $\varepsilon_4 = (\bullet \bullet \blacklozenge \blacklozenge \bullet \circ \circ \blacklozenge \bullet \circ \circ \blacklozenge)$.

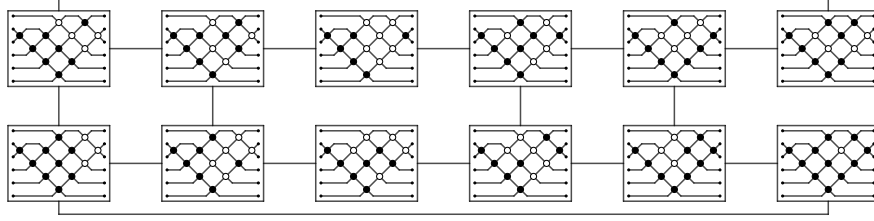


Figure 15.77: 3-Cell with ancestry of dimension 3: $\varepsilon_3 = (\bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge)$.

Step 3: For the third part attach a 3-cell with with ancestry $\varepsilon_5 = (\bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge)$ that fills the “parallelepiped” in Figure 15.78, with a 2-cell attached. The attachment to Figure 15.76 occurs through the 2-cell with ancestry $\varepsilon_6 = (\bullet \blacklozenge \bullet \circ \circ \blacklozenge \blacklozenge \bullet \bullet \circ \bullet \blacklozenge)$, and to Figure 15.77 through the 2-cell with ancestry $\varepsilon_7 = (\bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge)$.

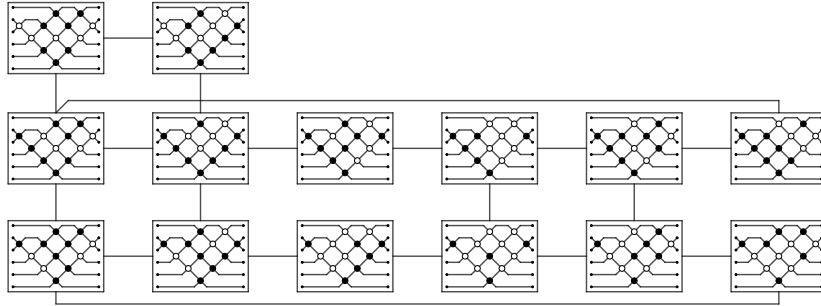


Figure 15.78: Third part of the CW complex.

Step 4: For the fourth part, attach to Figure 15.78 a 3-cell with ancestry $\varepsilon_8 = (\bullet \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet)$ that fills the “parallelepiped” in Figure 15.79, with a 2-cell attached. Attachment occurs through the cell of dimension 2 with ancestry $\varepsilon_9 = (\bullet \blacklozenge \blacklozenge \bullet \bullet \circ \bullet \circ \blacklozenge \bullet \bullet \blacklozenge)$.

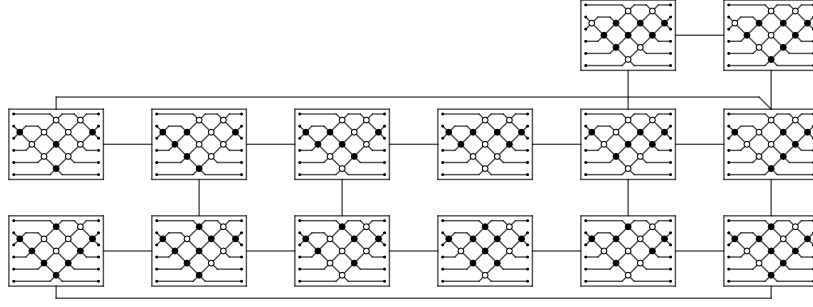
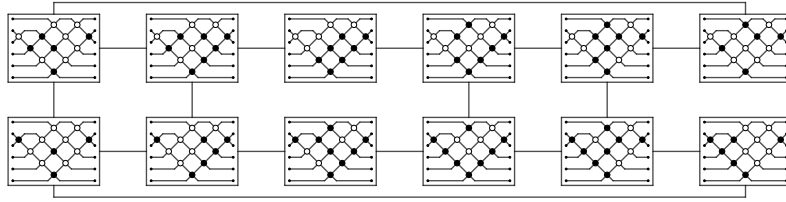
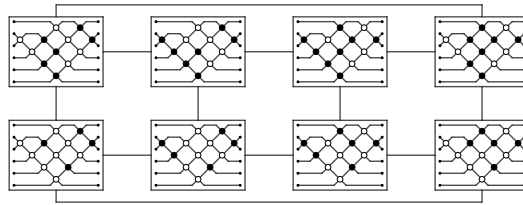


Figure 15.79: Fourth part of the CW complex.

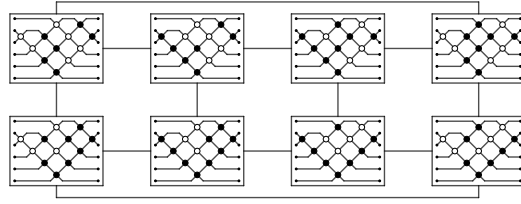
Step 5: For the fifth part, attach a 3-cell that fills the “parallelepiped” in Figure 15.80. The attachment to Figure 15.76 occurs through the 2-cell with ancestry $\varepsilon_{11} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet)$, and to Figure 15.79 and to Figure 15.78 through the 2-cell with ancestry $\varepsilon_{12} = (\bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet)$.

Figure 15.80: 3-Cell with ancestry $\varepsilon_{10} = (\blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet)$.

Step 6: For the sixth part, attach a 3-cell that fills the cube in Figure 15.81. Attachment to Figure 15.76 occurs through the 2-cell with ancestry $\varepsilon_{14} = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet)$.

Figure 15.81: 3-Cell with ancestry $\varepsilon_{13} = (\blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet)$.

Step 7: For the seventh part, attach a 3-cell that fills the cube in Figure 15.82. Attachment to Figure 15.76 and Figure 15.81 occurs through the 2-cell with ancestry $\varepsilon_{16} = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet)$, and to Figure 15.80 through the 2-cell with ancestry $\varepsilon_{17} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet)$.

Figure 15.82: 3-Cell with ancestry $\varepsilon_{15} = (\blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \circ \blacklozenge \bullet)$.

Step 8: For the eighth part, attach a 3-cell with ancestry $\varepsilon_{18} = (\blacklozenge \bullet \circ \blacklozenge \bullet \blacklozenge \circ \bullet \blacklozenge \bullet \blacklozenge)$ that fills the cube in Figure 15.83, with five 2-cells attached. The attachment to Figure 15.76 occurs through the 2-cell with ancestry $\varepsilon_{19} = (\blacklozenge \bullet \blacklozenge \circ \bullet \bullet \blacklozenge \circ \bullet \circ \bullet \blacklozenge)$, and to Figure 15.81 through the 2-cell with ancestry $\varepsilon_{20} = (\blacklozenge \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \circ \bullet)$.

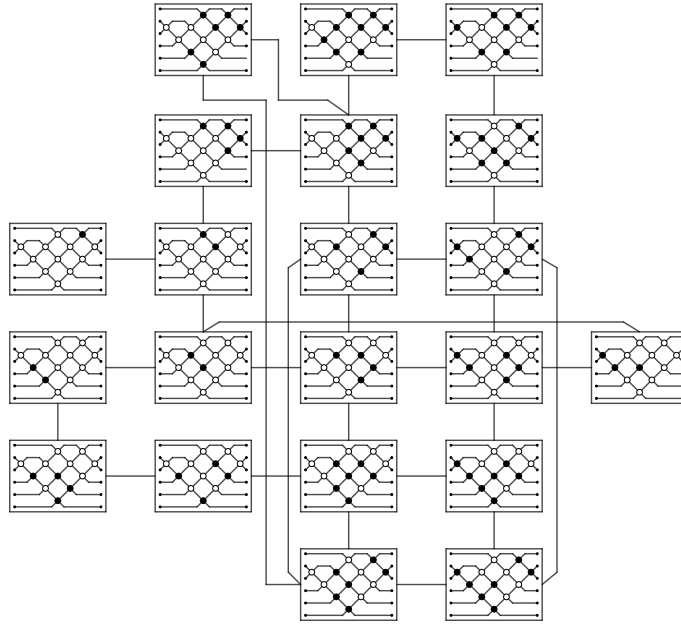


Figure 15.83: Eighth part of the CW complex.

Step 9: For the ninth part attach a 3-cell that fills the prism in Figure 15.84, with a 2-cell attached. The attachment to Figure 15.77 and 15.82 occurs through the 2-cell with ancestry $\varepsilon_{22} = (\bullet \bullet \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \circ \blacklozenge \bullet)$. And to Figure 15.85 and Figure 15.81 through the 2-cell with ancestry $\varepsilon_{23} = (\bullet \bullet \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \circ \bullet)$. And to Figure 15.79 through the 2-cell with ancestry $\varepsilon_{24} = (\bullet \blacklozenge \circ \bullet \bullet \bullet \bullet \bullet \bullet \circ \blacklozenge \bullet)$.

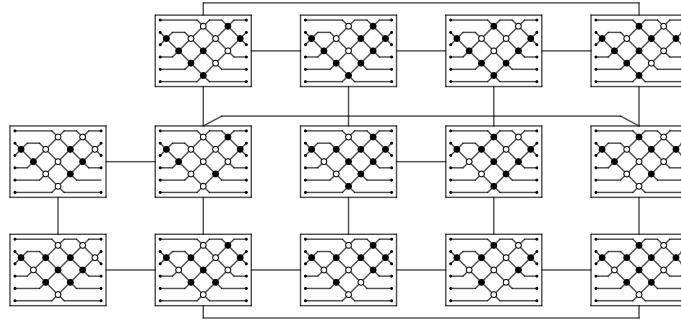


Figure 15.84: Ninth part of the CW complex with ancestry of dimension 3:
 $\varepsilon_{21} = (\bullet \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet)$.

Step 10: For the tenth part, attach a 3-cell that fills the prism in Figure 15.85. Attachment to Figure 15.77 occurs through the 2-cell with ancestry $\varepsilon_{26} = (\bullet \bullet \blacklozenge \bullet \bullet \circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge)$. This cell is also attached to Figure 15.83 through the 2-cell with ancestry $\varepsilon_{27} = (\bullet \bullet \circ \circ \blacklozenge \bullet \bullet \circ \bullet \blacklozenge \bullet \blacklozenge)$, and to Figure 15.81 through the 2-cell with ancestry $\varepsilon_{28} = (\bullet \bullet \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \circ \bullet \bullet)$. To Figure 15.77 Attachment occurs through the 2-cell with ancestry $\varepsilon_{29} = (\bullet \bullet \blacklozenge \bullet \bullet \circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge)$, and to Figure 15.79 through the 2-cell with ancestry $\varepsilon_{30} = (\bullet \bullet \blacklozenge \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \circ \bullet \bullet)$.

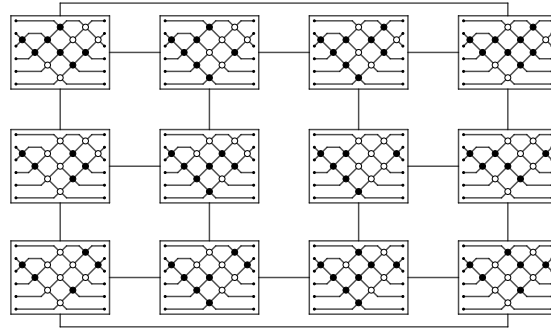


Figure 15.85: 3-Cell with ancestry $\varepsilon_{25} = (\bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge)$.

Upon completing all attachments, we obtain a contractible CW complex. Therefore, BL_σ has a total of 32 connected components of this type, all contractible.

The remaining ancestries of dimensions 1, 2 and 3 appear in a 4-dimensional CW-complex.

For dimension 4, we have two possible positions for the diamonds, which appear together. The CW complex will be constructed step by step.

Step 1: The first 4-cell with ancestry $\varepsilon_{31} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \blacklozenge)$ has ten 3-cells and is homotopically equivalent to a $\mathbb{D}^4$. Figure 15.86 shows the cell, with one 2-cell attached.

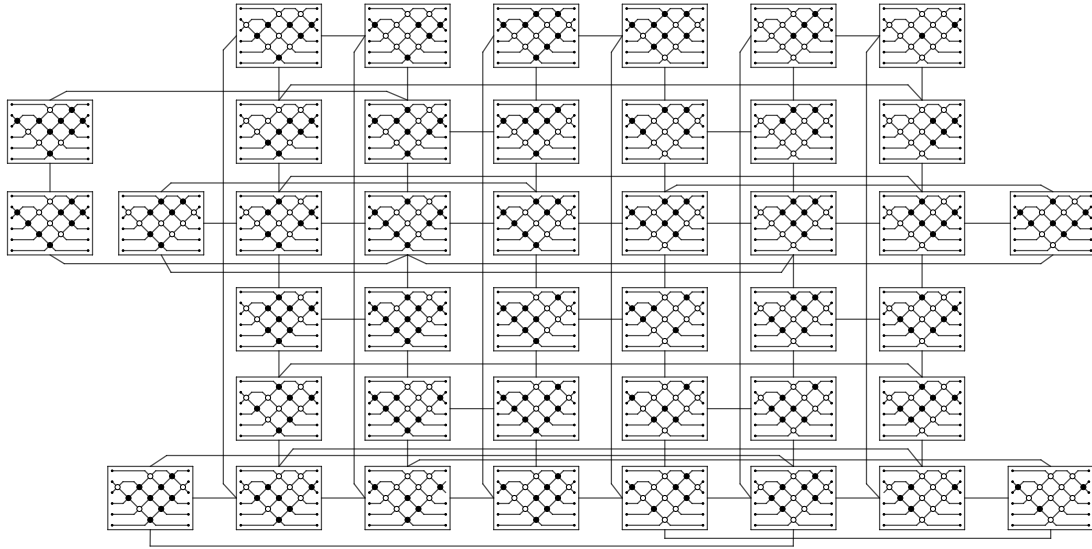


Figure 15.86: First part of the CW complex.

Step 2: The second 4-cell with ancestry $\varepsilon_{32} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \blacklozenge \blacklozenge)$ has sixteen 3-cells and is also homotopically equivalent to a $\mathbb{D}^4$. Its construction requires a step-by-step approach. First, six 3-cells are vertically attached in Figure 15.87.

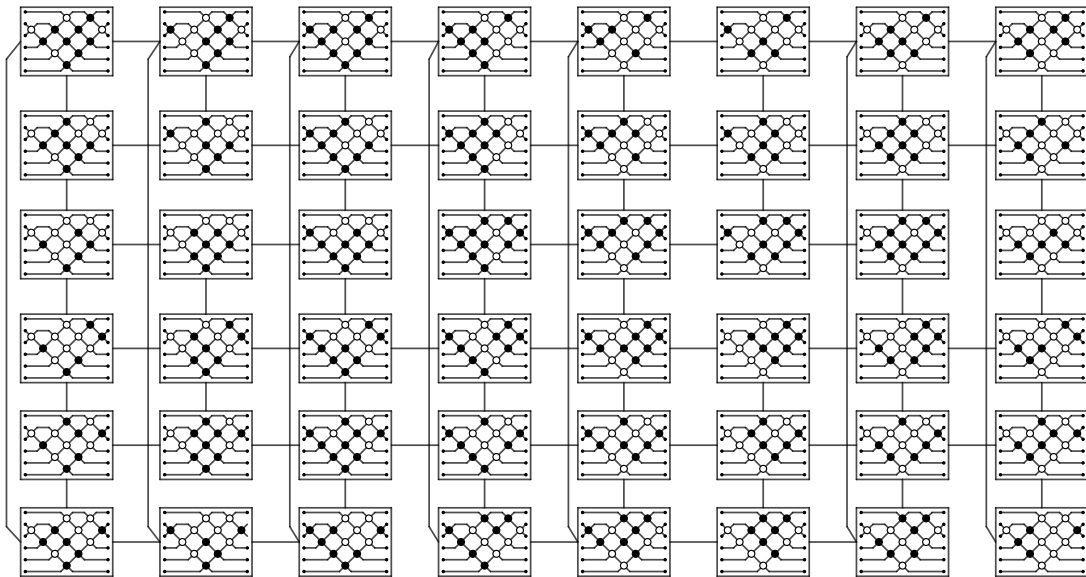


Figure 15.87: Second part of the CW complex.

Now, we attach the remaining ten 3-cells.

Step 3: Attach the two cubes and one prism in Figure 15.88 to Figure 15.87. The attachment occurs through the seven 2-cells with ancestries $\varepsilon_{33} = (\circ \bullet \bullet \blacklozenge \bullet \bullet \diamond \circ \diamond \bullet \circ)$, $\varepsilon_{34} = (\circ \bullet \bullet \blacklozenge \bullet \bullet \diamond \circ \diamond \bullet \circ)$, $\varepsilon_{35} = (\blacklozenge \circ \diamond \bullet \bullet \circ \diamond \bullet \bullet \circ \circ)$, $\varepsilon_{36} = (\bullet \bullet \blacklozenge \circ \diamond \bullet \bullet \circ \bullet \circ \circ)$, $\varepsilon_{37} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \diamond \circ \circ)$, $\varepsilon_{38} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \diamond \circ \diamond \bullet \bullet \circ)$, and $\varepsilon_{39} = (\bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \diamond \circ \diamond)$. Note that we have one 2-cell attached as a wing.

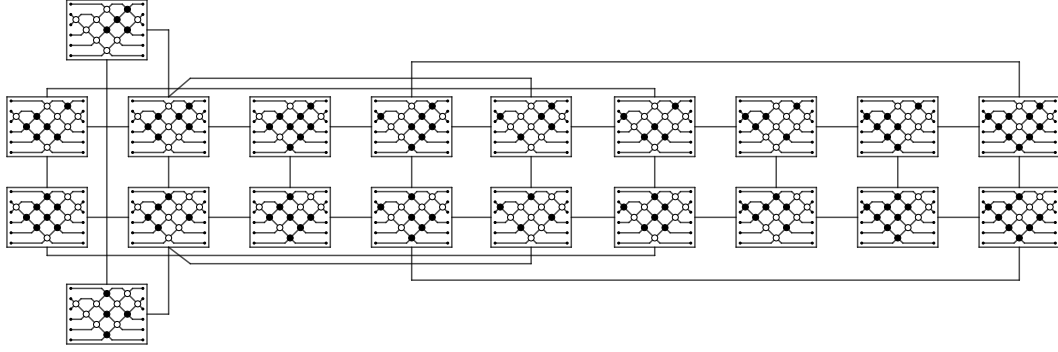


Figure 15.88: Third part of the CW complex.

Step 4: Attach the convex solid in Figure 15.89 to Figure 15.87. The attachment occurs through twelve 2-cells that fill four hexagons and eight squares. Note that there are four 1-cells with four 0-cells attached to the convex solid. Furthermore, attach Figure 15.89 to Figure 15.88 through the 2-cells with ancestries $\varepsilon_{40} = (\diamond \circ \diamond \bullet \bullet \blacklozenge \bullet \bullet \circ \bullet \circ \diamond \circ)$, $\varepsilon_{41} = (\blacklozenge \circ \diamond \bullet \blacklozenge \bullet \bullet \circ \circ \circ \diamond \bullet \circ)$, and $\varepsilon_{42} = (\bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \circ \bullet \diamond \diamond \circ)$.

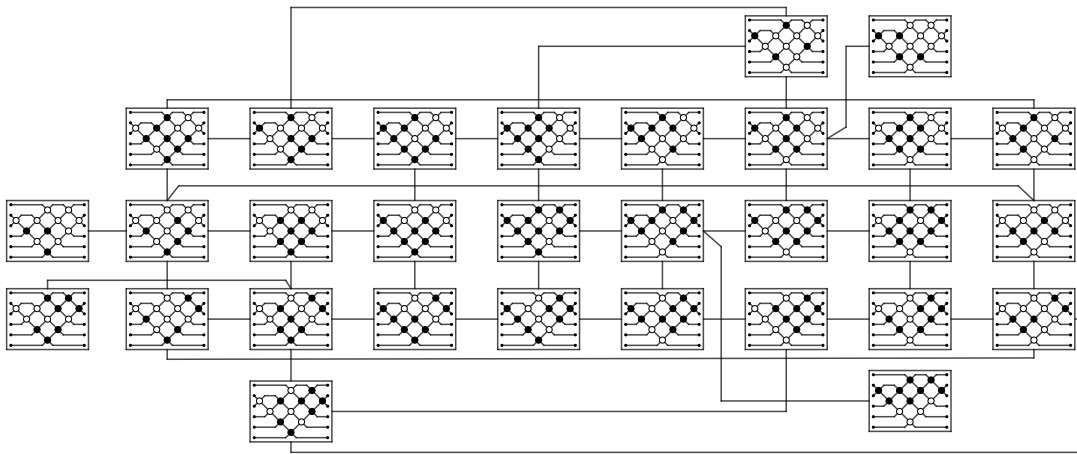


Figure 15.89: Fourth part of the CW complex.

Step 5: Similar to Step 3, attach the two cubes and one prism in Figure 15.90 to Figure 15.87. The attachment occurs through the seven 2-cells with

ancestries $\varepsilon_{43} = (\bullet \bullet \circ \blacklozenge \blacklozenge \circ \circ \blacklozenge \bullet \blacklozenge \bullet \bullet)$, $\varepsilon_{44} = (\circ \blacklozenge \circ \blacklozenge \bullet \blacklozenge \circ \blacklozenge \circ \bullet \bullet \bullet)$, $\varepsilon_{45} = (\circ \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \circ \bullet \bullet)$, $\varepsilon_{46} = (\circ \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \circ \bullet \bullet)$, $\varepsilon_{47} = (\circ \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet)$, $\varepsilon_{48} = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet)$, and $\varepsilon_{49} = (\bullet \bullet \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet)$. Note that we have one 2-cell attached as a wing.

Furthermore, Figure 15.90 attaches to Figure 15.89 through three 2-cells with ancestries $\varepsilon_{50} = (\blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet)$, $\varepsilon_{51} = (\circ \blacklozenge \circ \bullet \bullet \bullet \bullet \circ \bullet \blacklozenge \bullet)$, and $\varepsilon_{52} = (\blacklozenge \bullet \bullet \circ \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet)$.

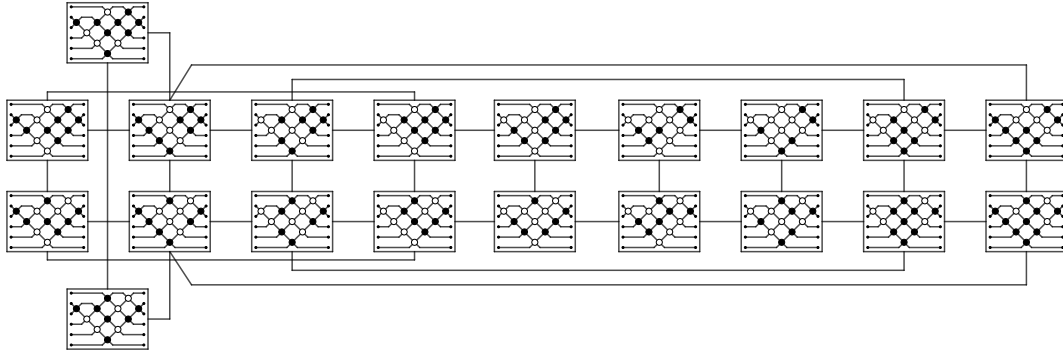


Figure 15.90: Fifth part of the CW complex.

Step 6: Attach the convex solid in Figure 15.91 to Figure 15.87. The attachment occurs through twelve 2-cells that fill four hexagons and eight squares. Furthermore, attach Figure 15.91 to Figure 15.90 through the 2-cells with ancestries $\varepsilon_{53} = (\blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet)$, $\varepsilon_{54} = (\circ \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \circ \bullet \blacklozenge \bullet)$, and $\varepsilon_{55} = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet)$. This 3-cell also attached to Figure 15.88 through the three 2-cells with ancestries $\varepsilon_{56} = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet)$, $\varepsilon_{57} = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$, and $\varepsilon_{58} = (\blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$.

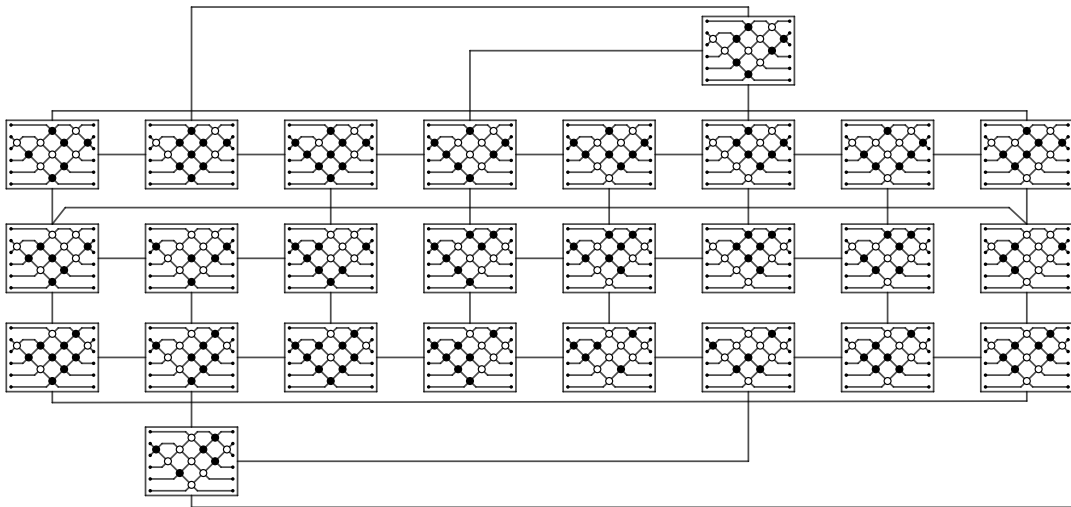


Figure 15.91: Sixth part of the CW complex.

After completing all the attachments, the resulting 4-cell is homotopically equivalent to a $\mathbb{D}^4$. The attachment to Figure 15.86 occurs through three 3-cell with ancestry $\varepsilon_{59} = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \blacklozenge)$.

Note that this CW complex comprises two $\mathbb{D}^4$ attached through a 3-cell. Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 16 connected components of this type, all contractible. In summary, BL_σ has a total of 96 connected components, all contractible.

Type 15.6. For $\sigma = [546231] = a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_5 a_4 a_3 a_2 a_1 \in S_6$ it follows that

$$\begin{aligned} \sigma = \frac{1}{4\sqrt{2}} & (-\hat{a}_1 - \hat{a}_2 + \hat{a}_3 - \hat{a}_1 \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_1 \hat{a}_2 \hat{a}_4 - \hat{a}_1 \hat{a}_3 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4 + \hat{a}_1 \hat{a}_5 \\ & - \hat{a}_2 \hat{a}_5 - \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_3 \hat{a}_5 + \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If r_1 has signs $(\bullet \bullet \bullet)$, and the remaining rows have equal signs, we have a CW complex that will be constructed through three steps.

Step 1: First, we have a 3-cell with four 2-cells attached. The 3-cell with ancestry $\varepsilon_1 = (\blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge)$ fills the convex solid completely.

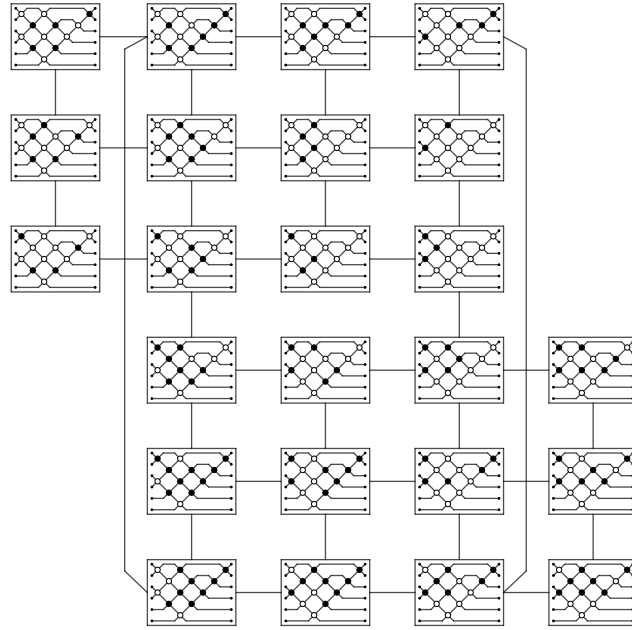
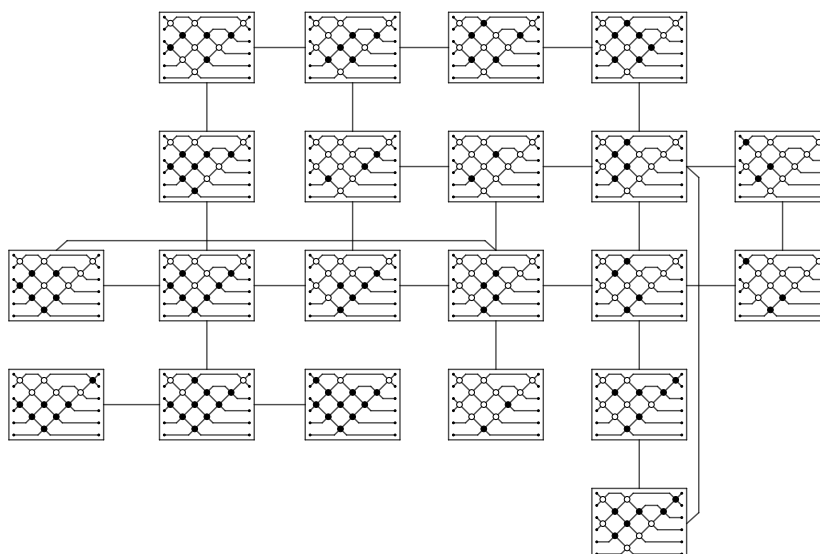
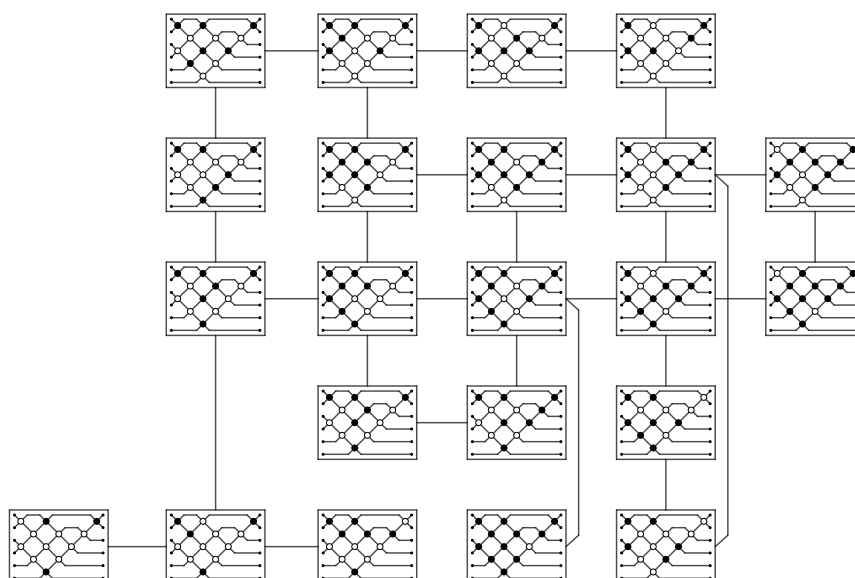


Figure 15.92: First part of the CW complex.

Step 2: Now, attach Figure 15.93 to Figure 15.92 through three 1-cells four 0-cells.



Step 3: To conclude, attach Figure 15.94 to the Figure 15.92 through three 1-cells and four 0-cells.



The component consists of a 3-cell with some lower-dimensional cells attached. Upon completing all attachments, we obtain a contractible CW complex. Therefore, BL_σ has a total of 16 connected components, all contractible.

If the diamonds are in r_4 and the remaining rows have equal signs, we have a CW complex with ten 3-cells. The component will be constructed one 3-cell per step.

Step 1: First we have a 3-cell with ancestry $\varepsilon_2 = (\blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \circ \circ \circ \blacklozenge \blacklozenge \blacklozenge)$ that fills the convex solid in Figure 15.95 with some cells of lower dimension attached.

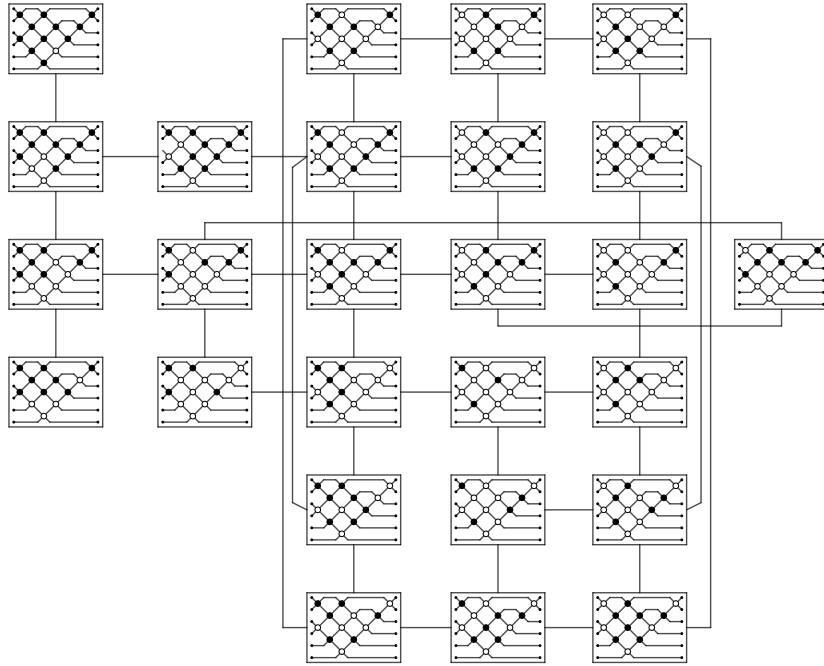


Figure 15.95: First part of the CW complex.

Step 2: Attach a 3-cell that fills the prism in Figure 15.96. Attachment occurs through the 2-cell in Figure 15.95 with ancestry $\varepsilon_3 = (\blacklozenge \circ \circ \blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \bullet \circ)$.

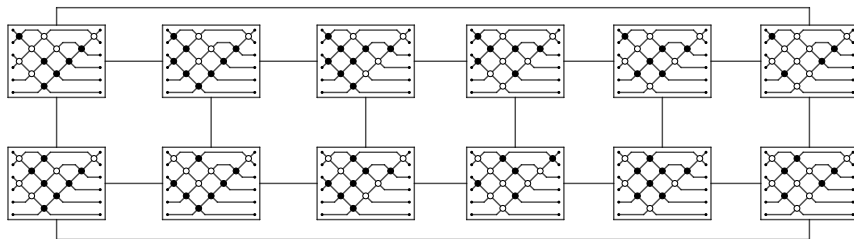


Figure 15.96: 3-Cell with ancestry $\varepsilon_4 = (\blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \circ \circ \bullet \blacklozenge \blacklozenge \bullet \bullet \circ)$.

Step 3: Attach a 3-cell that fills the “parallelepiped” in Figure 15.97, with a 2-cell attached. Attachment occurs through the 2-cell in Figure 15.96

with ancestry $\varepsilon_5 = (\circ \blacklozenge \bullet \circ \blacklozenge \circ \circ \bullet \blacklozenge \blacklozenge \bullet \circ)$.

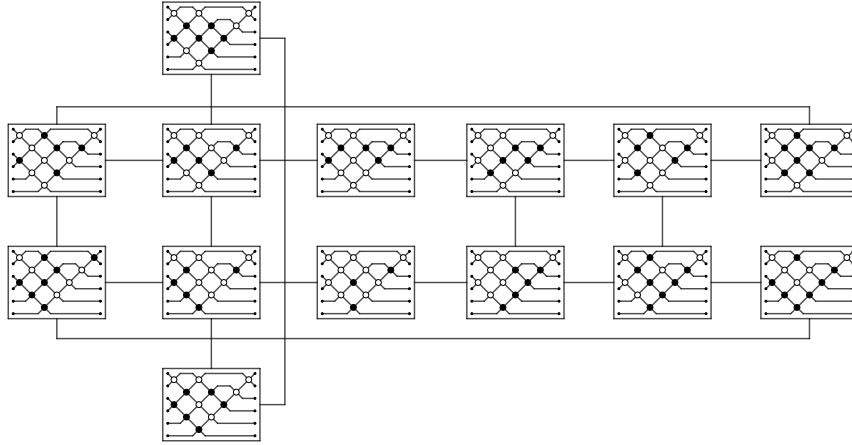


Figure 15.97: Third part of the CW complex with ancestry of dimension 3:

$\varepsilon_6 = (\circ \blacklozenge \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \blacklozenge \bullet \circ)$.

Step 4: Attach a 3-cell that fills the “parallelepiped” in Figure 15.98. Attachment occurs through the 2-cell in Figure 15.96 with ancestry $\varepsilon_7 = (\circ \blacklozenge \blacklozenge \circ \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ)$.

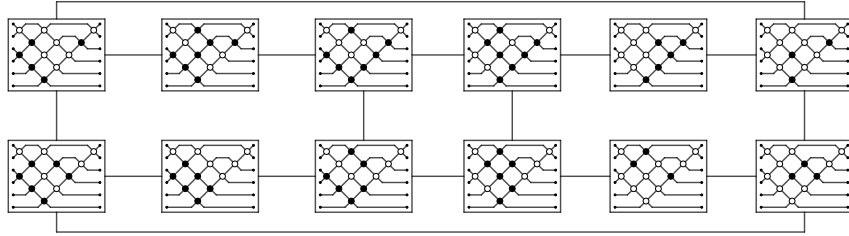


Figure 15.98: 3-Cell with ancestry $\varepsilon_8 = (\circ \blacklozenge \blacklozenge \circ \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ)$.

Step 5: Attach a 3-cell with ancestry $\varepsilon_9 = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \circ \blacklozenge \circ)$ that fills the cube in Figure 15.99, with two 2-cells attached. Attachment occurs through the 2-cell in Figure 15.98 with ancestry $\varepsilon_{10} = (\circ \blacklozenge \circ \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \circ)$. This cell also attaches to Figure 15.96 through the 2-cell with ancestry $\varepsilon_{11} = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \circ \bullet \bullet \bullet \bullet \circ)$.

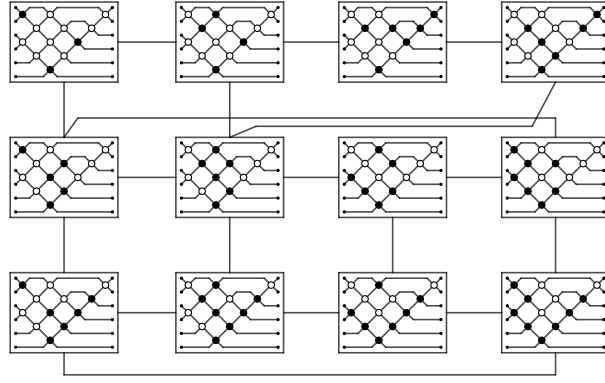


Figure 15.99: Fifth part of the CW complex.

Step 6: Attach a 3-cell with ancestry $\varepsilon_{12} = (\circ \blacklozenge \circ \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \blacklozenge)$ that fills the cube in Figure 15.100, with one 2-cell attached. Attachment occurs through the 2-cell in Figure 15.99 and to Figure 15.98 with ancestry $\varepsilon_{13} = (\circ \blacklozenge \circ \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \circ)$.

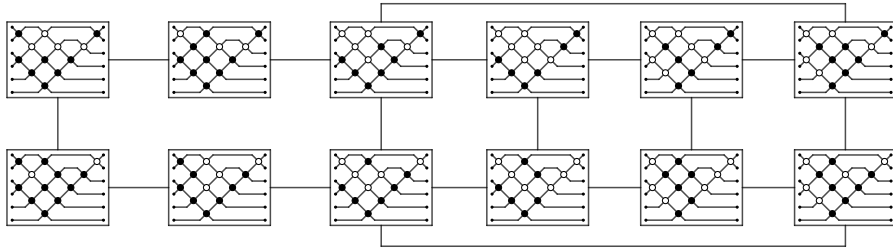
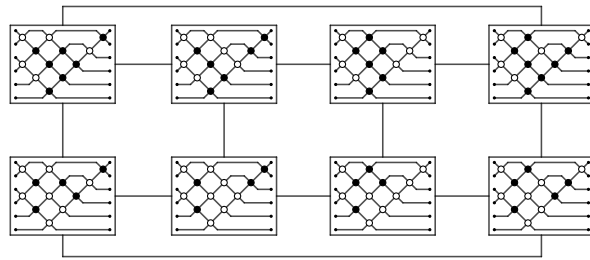


Figure 15.100: Sixth part of the CW complex.

Step 7: Attach a 3-cell that fills the cube in Figure 15.101. Attachment occurs through the 2-cell in Figure 15.100 with ancestry $\varepsilon_{14} = (\circ \circ \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \bullet \circ \blacklozenge \blacklozenge)$.

Figure 15.101: 3-Cell with ancestry $\varepsilon_{15} = (\circ \circ \bullet \blacklozenge \blacklozenge \circ \bullet \blacklozenge \circ \blacklozenge \blacklozenge)$.

Step 8: Attach a 3-cell that fills the prism in Figure 15.102. Attachment occurs through the 2-cell in Figure 15.101 with ancestry $\varepsilon_{16} = (\circ \circ \bullet \blacklozenge \circ \circ \circ \blacklozenge \bullet \bullet \blacklozenge)$. In Figure 15.100 with ancestry $\varepsilon_{17} = (\circ \blacklozenge \circ \blacklozenge \bullet \blacklozenge \circ \bullet \bullet \bullet \bullet \blacklozenge)$. In Figure 15.95 with ancestry $\varepsilon_{18} = (\circ \circ \bullet \bullet \bullet \blacklozenge \bullet \bullet \circ \bullet \blacklozenge \bullet \blacklozenge)$. In Figure 15.96 with ancestry $\varepsilon_{19} = (\circ \blacklozenge \circ \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \circ)$.

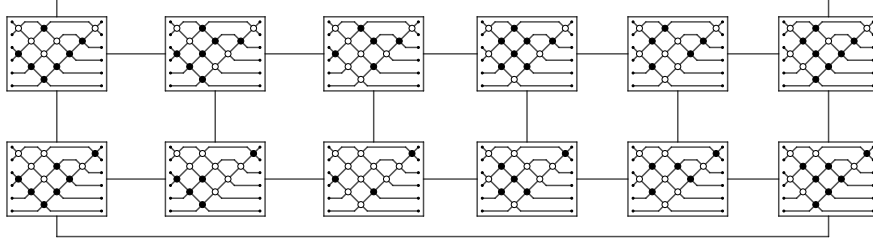


Figure 15.102: 3-Cell with ancestry $\varepsilon_{20} = (\circ \blacklozenge \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \blacklozenge)$.

Step 9: Attach a 3-cell that fills the cube in Figure 15.103. Attachment occurs through the 2-cell in Figure 15.101 with ancestry $\varepsilon_{21} = (\circ \circ \bullet \bullet \blacklozenge \circ \blacklozenge \circ \blacklozenge \bullet \bullet)$.

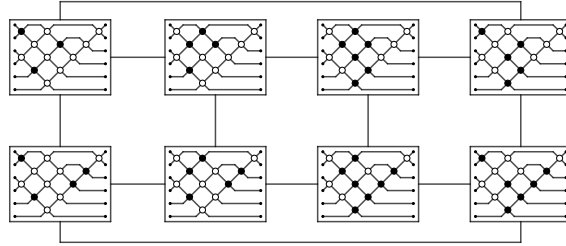
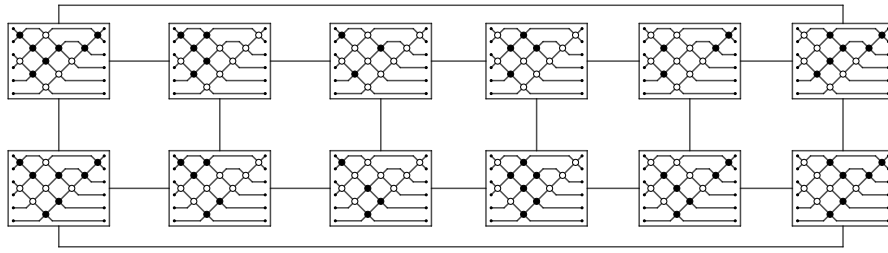


Figure 15.103: Cell of dimension 3 with ancestry $\varepsilon_{22} = (\blacklozenge \circ \circ \blacklozenge \blacklozenge \bullet \blacklozenge \circ \blacklozenge \bullet \blacklozenge \circ)$.

Attachment occurs through the 2-cell in Figure 15.95 with ancestry $\varepsilon_{23} = (\blacklozenge \circ \circ \blacklozenge \bullet \bullet \blacklozenge \circ \circ \circ \blacklozenge \bullet)$. In Figure 15.96 with ancestry $\varepsilon_{24} = (\blacklozenge \circ \circ \blacklozenge \blacklozenge \circ \circ \circ \blacklozenge \bullet \bullet \circ)$. In Figure 15.99 with ancestry $\varepsilon_{25} = (\blacklozenge \circ \circ \blacklozenge \circ \bullet \bullet \bullet \bullet \circ \blacklozenge \circ)$

Step 10: Attach a 3-cell that fills the prism in Figure 15.104. Attachment occurs through the 2-cell in Figure 15.103 with ancestry $\varepsilon_{26} = (\blacklozenge \circ \circ \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \bullet \bullet \bullet)$. In Figure 15.95 with ancestry $\varepsilon_{27} = (\blacklozenge \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \circ \circ \circ \bullet \blacklozenge)$. In Figure 15.101 with ancestry $\varepsilon_{28} = (\circ \circ \bullet \blacklozenge \blacklozenge \circ \bullet \circ \blacklozenge \circ \circ \blacklozenge)$.

Figure 15.104: 3-Cell with ancestry $\varepsilon_{29} = (\blacklozenge \circ \bullet \bullet \blacklozenge \bullet \circ \diamond \circ \diamond \circ \diamond \circ)$.

Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 32 connected components of this type, all contractible.

The remaining ancestries of dimensions 1, 2, and 3 appear in a 4-dimensional CW-complex.

For dimension 4, we have two possible positions for the diamonds, which appear together. Alternatively, in dimension 1, if the diamonds are located in r_2 with signs $(\bullet \circ \circ)$, we obtain this component. The CW complex will be constructed step by step.

Step 1: The first 4-cell with ancestry $\varepsilon_{30} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \circ \diamond \circ \diamond \circ \diamond \circ \diamond)$ has ten 3-cells and is homotopically equivalent to $\mathbb{D}^4$. Figure 15.105 shows the cell with two edges and two vertices attached.

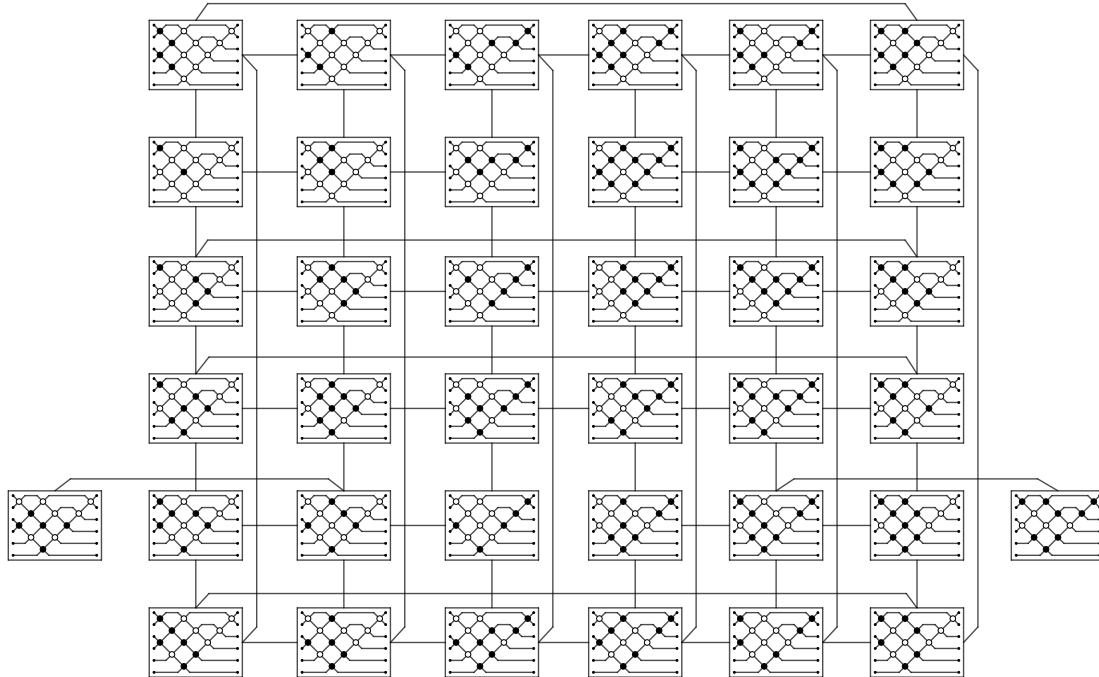


Figure 15.105: First part of the CW complex.

Step 2: The second 4-cell similarly to the previous one, also has ten 3-cells and is homotopically equivalent to a $\mathbb{D}^4$. Attachment occurs via the last 3-cell horizontally in Figure 15.105 and the first one in Figure 15.106, with ancestry $\varepsilon_{31} = (\blacklozenge \blacklozenge \blacklozenge \circ \circ \bullet \blacklozenge \bullet \bullet \blacklozenge \circ \blacklozenge)$.

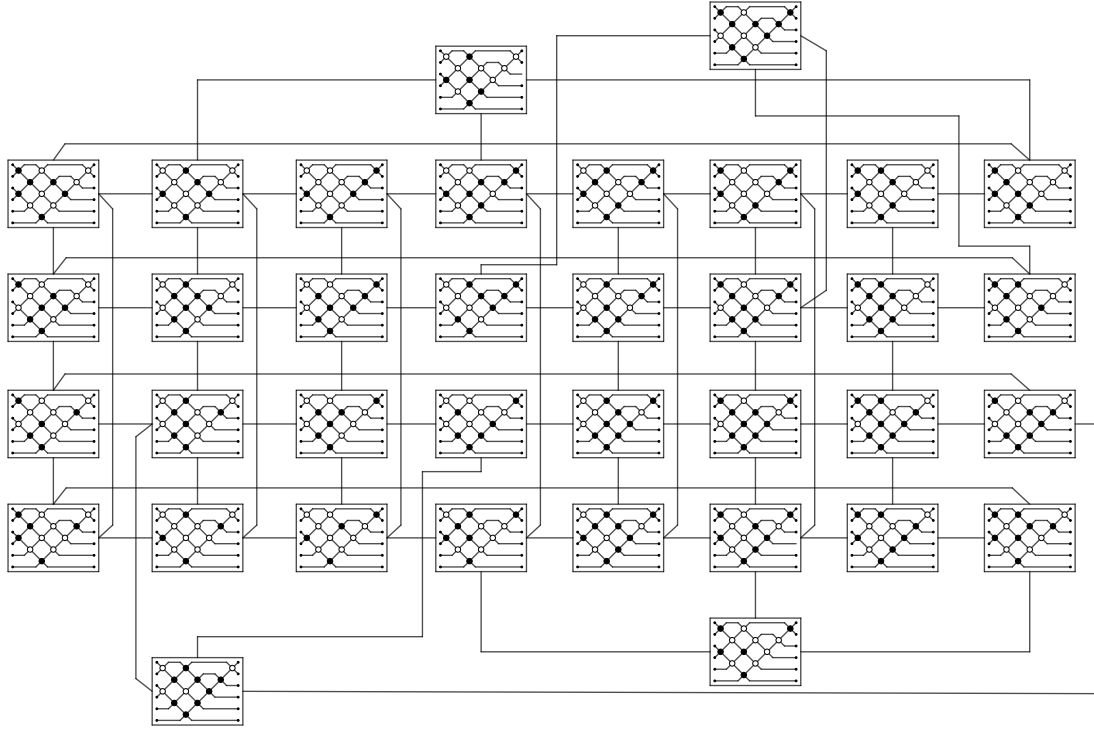


Figure 15.106: 4-Cell with ancestry $\varepsilon_{32} = (\blacklozenge \blacklozenge \blacklozenge \circ \circ \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$.

Step 3: The Figure 15.107 shows the first 3-cell with one 2-cell attached. The cell is attached to Figure 15.105 through the 2-cell with ancestry $\varepsilon_{33} = (\blacklozenge \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \circ \circ \blacklozenge \bullet \bullet)$, and to Figure 15.106 through $\varepsilon_{34} = (\blacklozenge \bullet \bullet \blacklozenge \circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet)$.

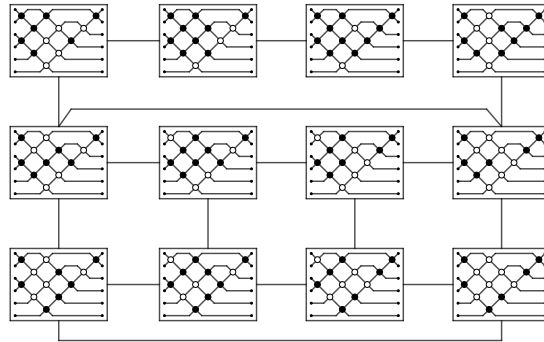


Figure 15.107: Third part of the CW complex with ancestry of dimension 3: $\varepsilon_{35} = (\blacklozenge \bullet \bullet \circ \blacklozenge \blacklozenge \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \bullet)$.

Step 4: The Figure 15.108 shows the second 3-cell with ancestry $\varepsilon_{36} = (\blacklozenge \circ \circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \blacklozenge \circ)$ with one 2-cell attached. The cell is attached to Figure 15.105 through the 2-cell with ancestry $\varepsilon_{37} = (\blacklozenge \circ \circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \circ \circ)$, and to Figure 15.106 through $\varepsilon_{38} = (\blacklozenge \circ \circ \circ \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ \bullet \blacklozenge \circ)$.

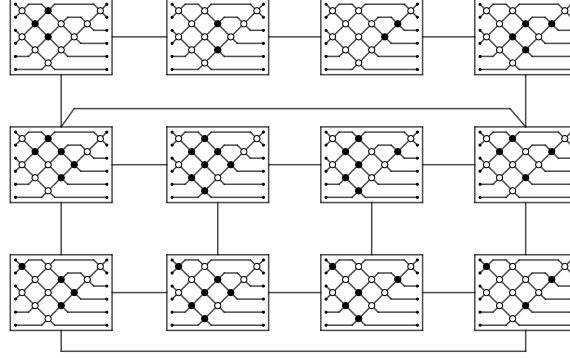


Figure 15.108: Fourth part of the CW complex.

Step 5: The Figure 15.109 shows the third 3-cell with cells of lower dimension attached. The cell is attached to Figure 15.105 through the 2-cell with ancestry $\varepsilon_{39} = (\bullet \bullet \circ \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \circ \circ \blacklozenge)$, and to Figure 15.106 through $\varepsilon_{40} = (\bullet \bullet \circ \blacklozenge \circ \circ \blacklozenge \bullet \bullet \circ \blacklozenge \blacklozenge)$.

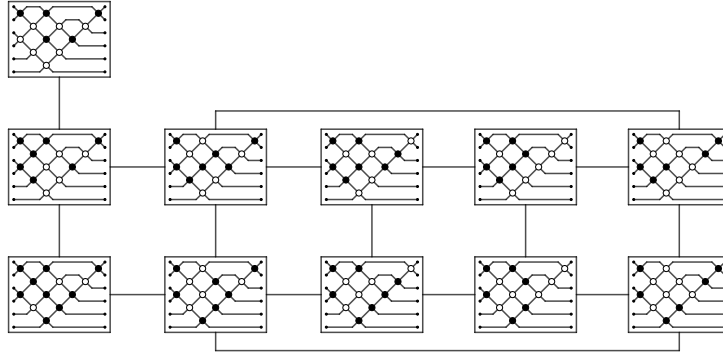


Figure 15.109: Fifth part of the CW complex with ancestry of dimension 3: $\varepsilon_{41} = (\bullet \bullet \circ \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \circ \blacklozenge \blacklozenge)$.

Step 6: To finish, Figure 15.110 shows the fourth 3-cell with ancestry $\varepsilon_{42} = (\circ \circ \bullet \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \blacklozenge)$, with cells of lower dimension attached. The cell is attached to Figure 15.105 through the 2-cell with ancestry $\varepsilon_{43} = (\circ \circ \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet \circ \blacklozenge)$, and to Figure 15.106 through $\varepsilon_{44} = (\circ \circ \bullet \blacklozenge \circ \circ \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge)$.

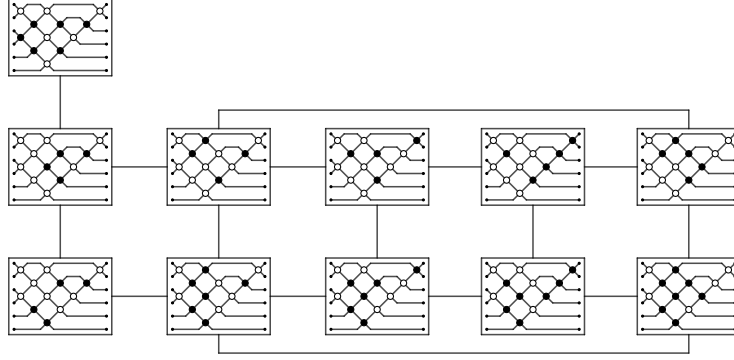


Figure 15.110: Sixth part of the CW complex.

This CW complex comprises two $\mathbb{D}^4$ connected through a single 3-cell, with four 3-cells attached to the 4-cells. Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 16 connected components of this type, all contractible. Summing up, BL_σ has a total of 96 connected components, all of them contractible.

Type 15.7. For $\sigma = [546312] = a_1 a_3 a_2 a_1 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 \in S_6$ it follows that

$$\begin{aligned} \sigma = \frac{1}{4\sqrt{2}} & (-\hat{a}_1 - \hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_3 - \hat{a}_1\hat{a}_2\hat{a}_3 - \hat{a}_4 - \hat{a}_2\hat{a}_4 - \hat{a}_3\hat{a}_4 - \hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 \\ & + \hat{a}_2\hat{a}_5 - \hat{a}_3\hat{a}_5 + \hat{a}_2\hat{a}_3\hat{a}_5 - \hat{a}_1\hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_4\hat{a}_5 - \hat{a}_1\hat{a}_3\hat{a}_4\hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If the diamonds are in row r_4 and the remaining rows have equal signs, we have a CW complex with one 3-cell and cells of lower dimension attached. This will be constructed through three steps.

Step 1: First, we have a 3-cell in Figure 15.111 with cells of lower dimension attached. The 3-cell with ancestry $\varepsilon_1 = (\blacklozenge \circ \blacklozenge \bullet \blacklozenge \bullet \bullet \circ \diamond \circ \diamond \circ \diamond \circ)$ fills the prism completely.

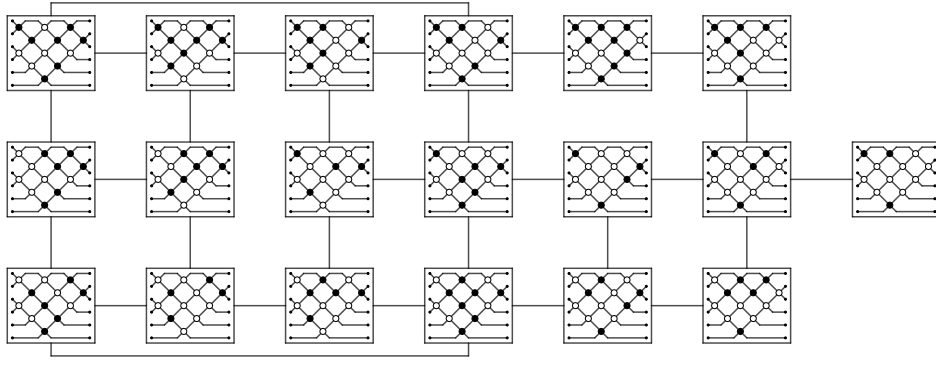


Figure 15.111: First part of the CW complex.

Step 2: Now, attach ten 2-cells to Figure 15.111, with cells of lower dimension as shown in Figure 15.112. Attachment occurs through six 0-cells and five 1-cells.

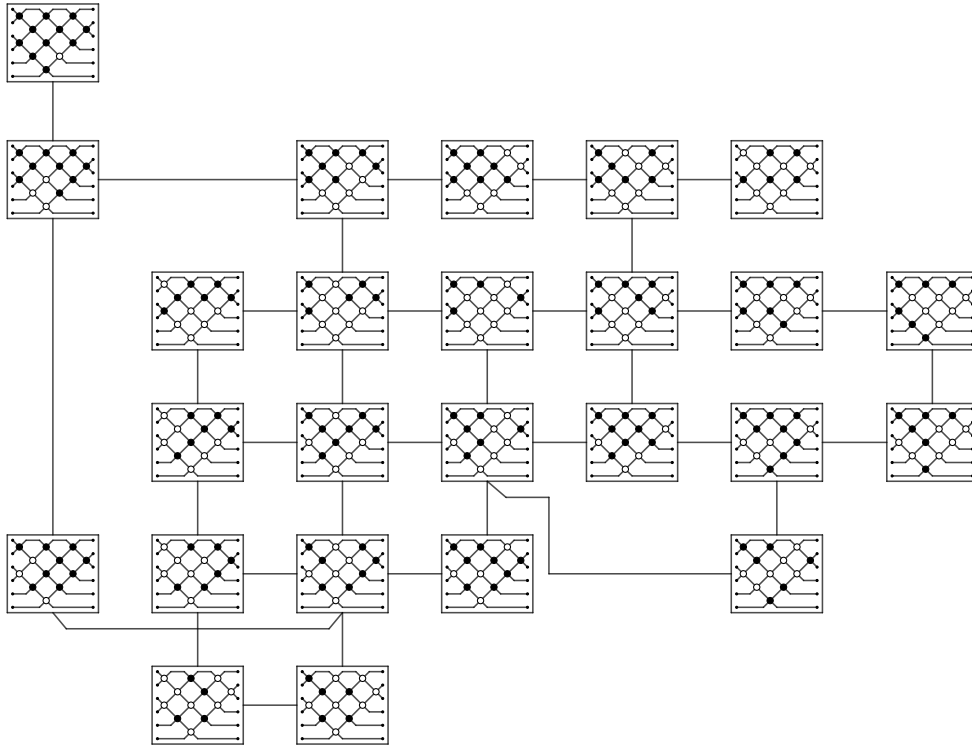


Figure 15.112: Second part of the CW complex with ten 2-cells.

Step 3: To finish, attach ten 2-cells to Figure 15.111 with cells of lower dimension as shown in Figure 15.113. Attachment occurs through six 0-cells and five 1-cells.

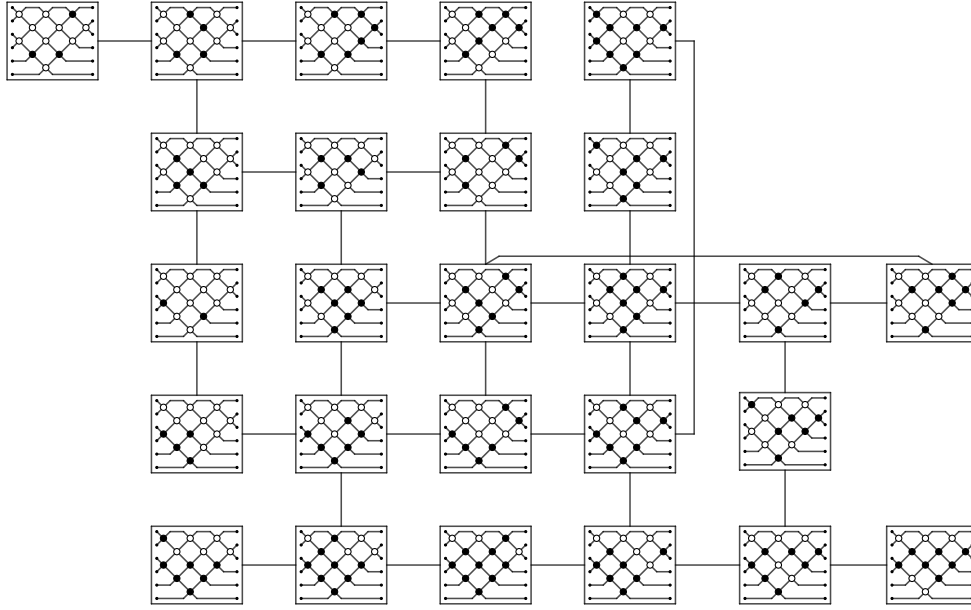
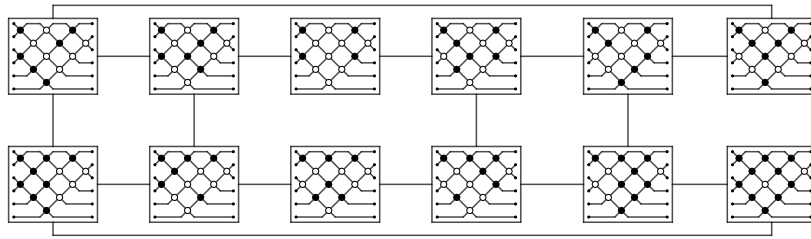


Figure 15.113: Third part of the CW complex with ten 2-cells.

If the diamonds are in row r_2 with the signs $(\bullet \bullet \circ)$, and the remaining rows have equal signs, we have a CW complex with ten 3-cells and lower-dimensional cells attached. This will be constructed one 3-cell at a time.

Step 1: Start with the 3-cell that fills the “parallelepiped” in Figure 15.114.

Figure 15.114: 3-Cell with ancestry $\varepsilon_2 = (\bullet \blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \circ)$.

Step 2: Now, attach the 3-cell that fills the prism in Figure 15.115 with cells of lower dimension attached. The attachment to Figure 15.115 occurs through the 2-cell with ancestry $\varepsilon_3 = (\bullet \blacklozenge \circ \bullet \blacklozenge \circ \circ \bullet \blacklozenge \blacklozenge \circ)$.

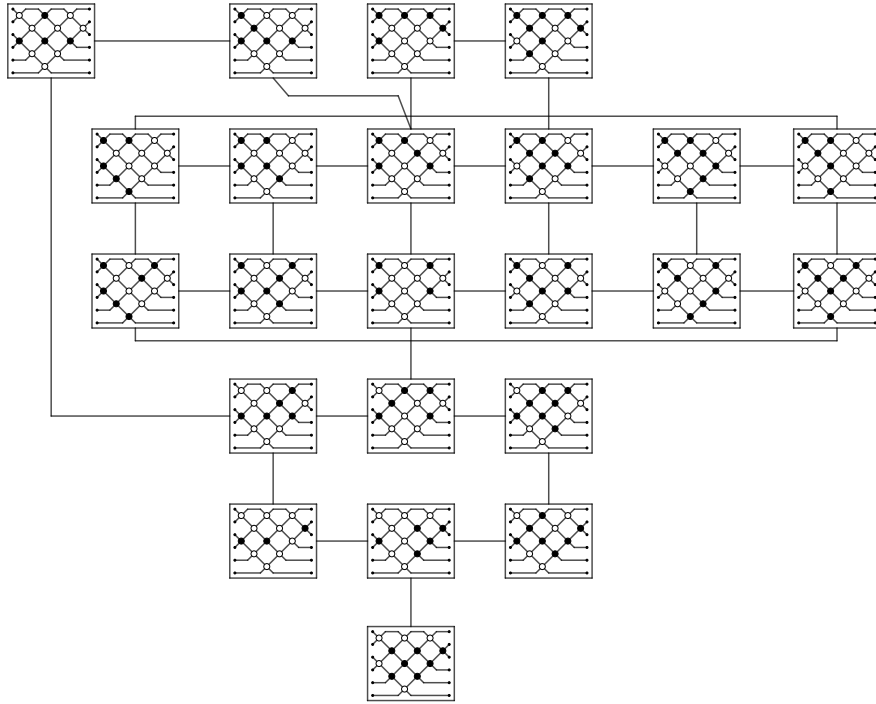


Figure 15.115: Second part of the CW complex with ancestry of dimension 3:
 $\varepsilon_4 = (\bullet \blacklozenge \circ \blacklozenge \bullet \circ \circ \diamond \bullet \diamond \diamond \circ)$.

Step 3: Attach the 3-cell that fills the prism in Figure 15.116. The attachment to Figures 15.114 and 15.115 occurs through the 2-cell with ancestry $\varepsilon_5 = (\bullet \blacklozenge \circ \bullet \blacklozenge \circ \circ \circ \bullet \diamond \diamond \circ)$.

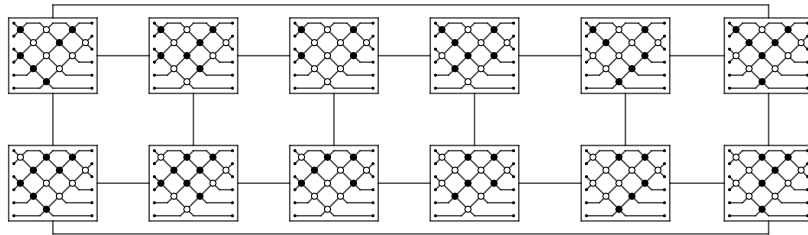
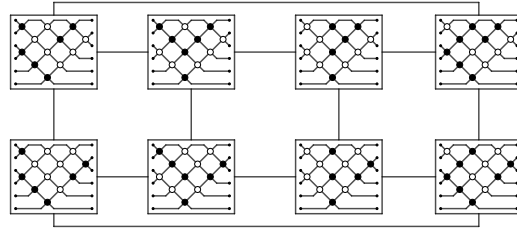


Figure 15.116: 3-Cell with ancestry $\varepsilon_6 = (\blacklozenge \blacklozenge \circ \diamond \blacklozenge \circ \bullet \bullet \bullet \diamond \diamond \circ)$.

Step 4: Attach the 3-cell that fills the cube in Figure 15.117. The attachment to Figure 15.116 occurs through the 2-cell with ancestry $\varepsilon_7 = (\blacklozenge \blacklozenge \circ \diamond \bullet \diamond \bullet \bullet \bullet \circ \circ \circ)$.

Figure 15.117: 3-Cell with ancestry $\varepsilon_8 = (\blacklozenge \blacklozenge \circ \diamond \bullet \diamond \blacklozenge \bullet \bullet \diamond \circ \diamond)$.

Step 5: Attach the 3-cell with ancestry $\varepsilon_9 = (\blacklozenge \blacklozenge \circ \diamond \bullet \diamond \bullet \bullet \diamond \circ \diamond)$ that fills the cube in Figure 15.118 with two 2-cells attached. The attachment to Figure 15.117 occurs through the 2-cell with ancestry $\varepsilon_{10} = (\blacklozenge \bullet \circ \diamond \bullet \circ \bullet \bullet \bullet \circ \circ \diamond)$, and to Figure 15.116 through $\varepsilon_{11} = (\blacklozenge \bullet \circ \diamond \blacklozenge \circ \bullet \bullet \bullet \diamond \circ \circ)$.

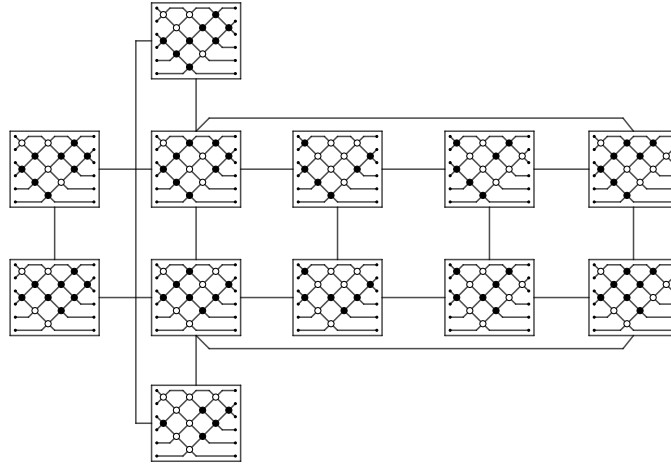
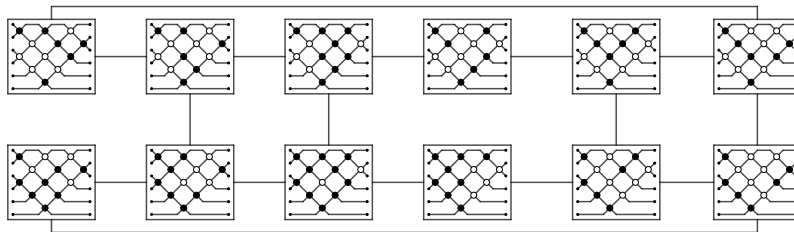


Figure 15.118: Fifth part of the CW complex.

Step 6: Attach the 3-cell that fills the “parallelepiped” in Figure 15.119. The attachment to Figure 15.117 occurs through the 2-cell with ancestry $\varepsilon_{13} = (\bullet \blacklozenge \circ \circ \bullet \diamond \blacklozenge \bullet \bullet \circ \circ \diamond)$, and to Figure 15.114 through $\varepsilon_{13} = (\blacklozenge \bullet \bullet \bullet \bullet \diamond \bullet \bullet \diamond \bullet \circ \diamond \circ)$.

Figure 15.119: 3-Cell with ancestry $\varepsilon_{14} = (\bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \diamond \bullet \bullet \circ \diamond \circ)$.

Step 7: Attach the 3-cell with ancestry $\varepsilon_{15} = (\blacklozenge \blacklozenge \blacklozenge \bullet \bullet \circ \diamond \diamond \bullet \bullet \diamond \bullet)$ that fills the convex solid in Figure 15.120, with cells of lower dimension attached. The attachment to Figure 15.119 occurs through the 2-cell with ancestry $\varepsilon_{16} = (\bullet \blacklozenge \blacklozenge \bullet \bullet \circ \diamond \circ \bullet \bullet \diamond \bullet)$, and to Figure 15.117 through $\varepsilon_{17} = (\blacklozenge \blacklozenge \circ \diamond \bullet \diamond \circ \circ \circ \bullet \bullet)$.

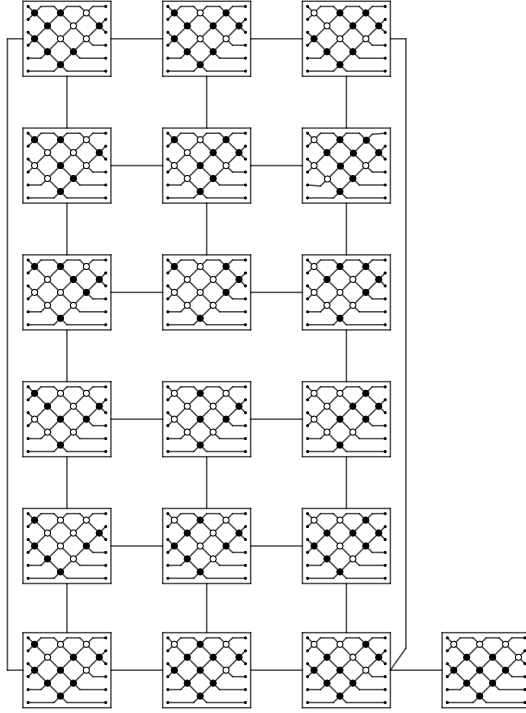


Figure 15.120: Seventh part of the CW complex.

Step 8: Attach the 3-cell that fills the “parallelepiped” in Figure 15.121. The attachment to Figure 15.120 occurs through the 2-cell with ancestry $\varepsilon_{18} = (\blacklozenge \circ \blacklozenge \circ \bullet \bullet \diamond \diamond \bullet \bullet \bullet)$, and to Figure 15.117 through $\varepsilon_{19} = (\blacklozenge \circ \bullet \diamond \circ \bullet \blacklozenge \bullet \bullet \circ \circ \diamond)$.

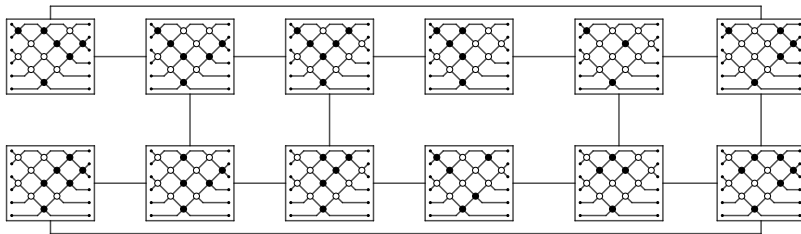


Figure 15.121: 3-Cell with ancestry $\varepsilon_{20} = (\diamond \circ \blacklozenge \diamond \circ \bullet \diamond \diamond \bullet \circ \circ \diamond)$.

Step 9: Attach the 3-cell that fills the “parallelepiped” in Figure 15.122. The attachment to Figure 15.121 occurs through the 2-cell with ancestry $\varepsilon_{21} =$

$(\bullet \circ \blacklozenge \bullet \circ \bullet \circ \blacklozenge \bullet \bullet \bullet \circ)$, and to Figure 15.115 through $\varepsilon_{22} = (\bullet \circ \bullet \circ \blacklozenge \bullet \circ \bullet \circ \bullet \circ \bullet \circ)$. Moreover, to Figure 15.120 through $\varepsilon_{23} = (\bullet \circ \bullet \circ \blacklozenge \bullet \circ \bullet \circ \bullet \bullet \bullet \circ)$, and to Figure 15.119 through $\varepsilon_{24} = (\bullet \circ \blacklozenge \bullet \circ \bullet \bullet \bullet \bullet \bullet \circ \bullet \circ)$.

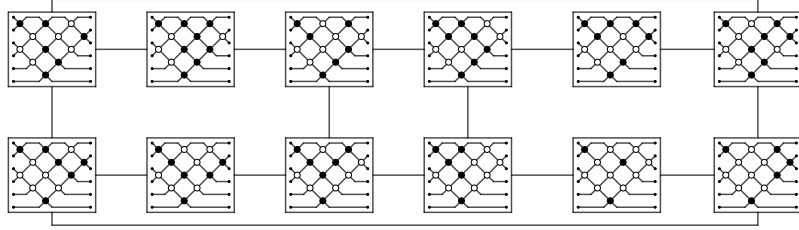


Figure 15.122: 3-Cell with ancestry $\varepsilon_{25} = (\bullet \circ \blacklozenge \blacklozenge \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ)$.

Step 10: Attach the 3-cell that fills the prism in Figure 15.123, with lower-dimensional cells attached. The attachment to Figure 15.122 occurs through the cell of dimension 2 with ancestry $\varepsilon_{26} = (\bullet \circ \blacklozenge \blacklozenge \bullet \circ \bullet \circ \bullet \bullet \bullet \circ)$, and to Figure 15.115 through $\varepsilon_{27} = (\bullet \circ \bullet \circ \blacklozenge \bullet \bullet \bullet \circ \bullet \circ \bullet \circ)$. Moreover, to Figure 15.114 through $\varepsilon_{28} = (\bullet \circ \blacklozenge \bullet \circ \bullet \bullet \bullet \circ \bullet \circ \bullet \circ)$.

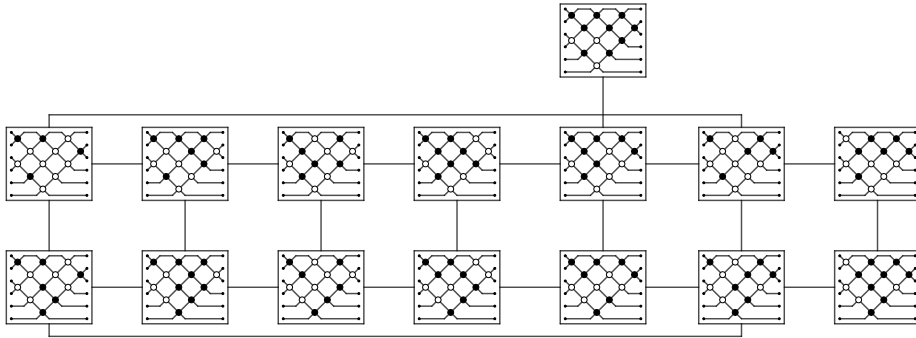


Figure 15.123: Tenth part of the CW complex with ancestry of dimension 3: $\varepsilon_{29} = (\bullet \circ \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \circ \bullet \circ \bullet \circ \bullet \circ)$.

Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 32 connected components of this type, all contractible.

The remaining ancestries of dimensions 1, 2, and 3 appear in a 4-dimensional CW-complex.

For dimension 4, there are two possible positions for the diamonds, which appear together with some 3-cells attached in a CW complex that will be described step by step.

Step 1: First we have a 4-cell that fills the $\mathbb{D}^4$ in Figure 15.124, with some cells of lower dimension attached.

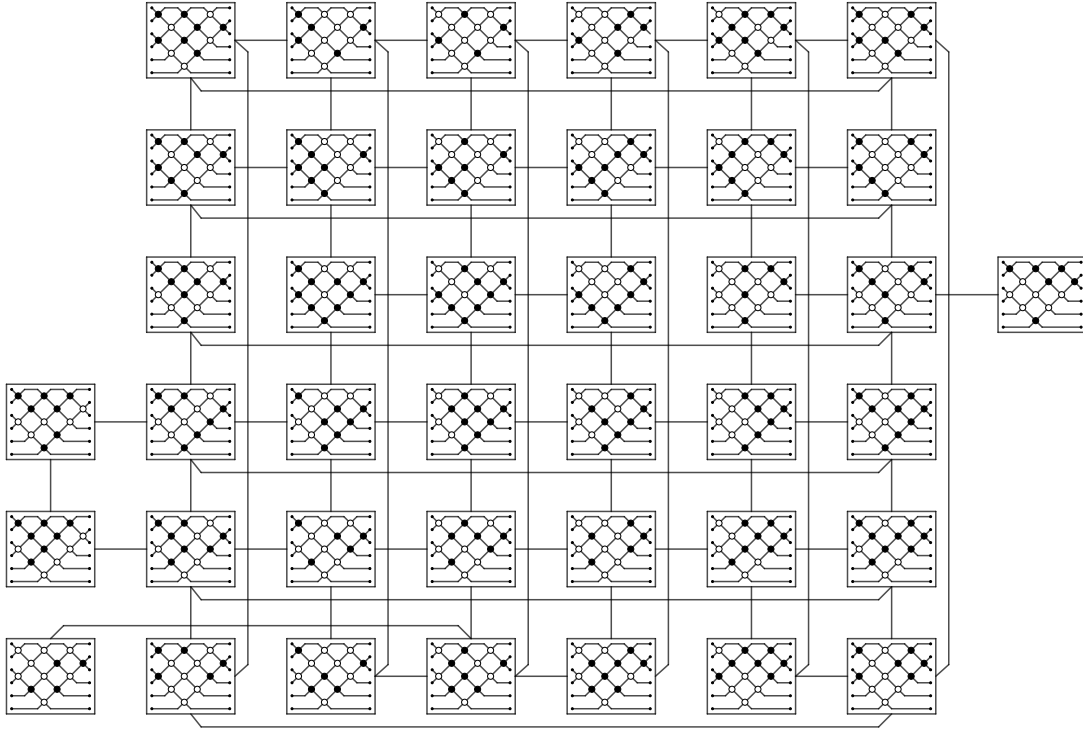


Figure 15.124: 4-Cell with ancestry $\varepsilon_{30} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \bullet \diamond \diamond \bullet \diamond \diamond \bullet)$.

Step 2: Now attach another 4-cell that fills the $\mathbb{D}^4$ in Figure 15.125. The attachment to Figure 15.124 occurs through the 3-cell with ancestry $\varepsilon_{31} = (\bullet \blacklozenge \circ \blacklozenge \blacklozenge \circ \bullet \blacklozenge \bullet \diamond \diamond \bullet)$. This cell has some cells of lower dimension attached.

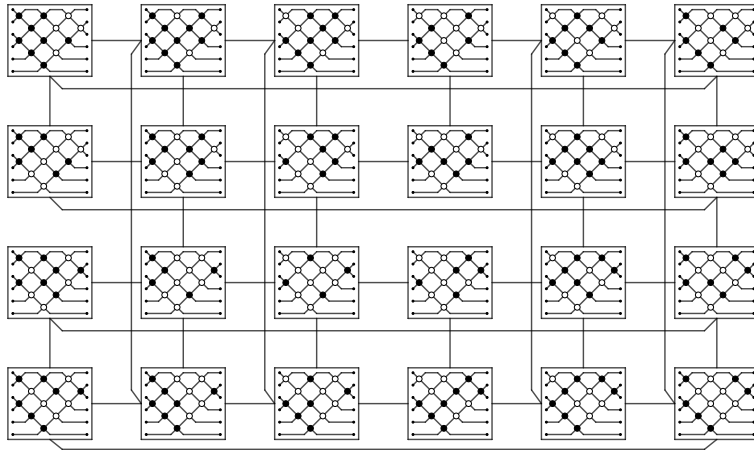


Figure 15.125: 4-Cell with ancestry $\varepsilon_{32} = (\blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \diamond \bullet \diamond \bullet \diamond)$.

Now we attach six 3-cells:

Step 3: Attach a 3-cell that fills the “parallelepiped” in Figure 15.126. This cell is attached to Figure 15.124 through the 2-cell with ancestry $\varepsilon_{33} = (\circ \bullet \circ \blacklozenge \bullet \blacklozenge \circ \diamond \bullet \circ \diamond \bullet)$, and to Figure 15.125 through the 2-cell with ancestry $\varepsilon_{34} = (\circ \bullet \blacklozenge \bullet \bullet \circ \bullet \diamond \bullet \circ \bullet \diamond)$.

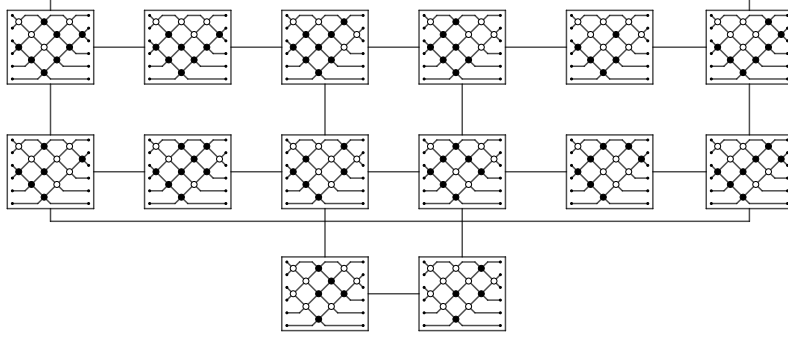


Figure 15.126: Third part of the CW complex with ancestry of dimension 3:
 $\varepsilon_{35} = (\circ \bullet \blacklozenge \bullet \bullet \blacklozenge \circ \diamond \bullet \bullet \diamond \diamond)$.

Step 4: Attach a 3-cell that fills the “parallelepiped” in Figure 15.127. This cell is attached to Figure 15.124 through the 2-cell with ancestry $\varepsilon_{36} = (\bullet \bullet \circ \blacklozenge \circ \blacklozenge \bullet \diamond \bullet \circ \diamond \bullet)$ and to Figure 15.125 through the 2-cell with ancestry $\varepsilon_{37} = (\bullet \bullet \blacklozenge \bullet \circ \circ \circ \diamond \circ \bullet \bullet \diamond)$.

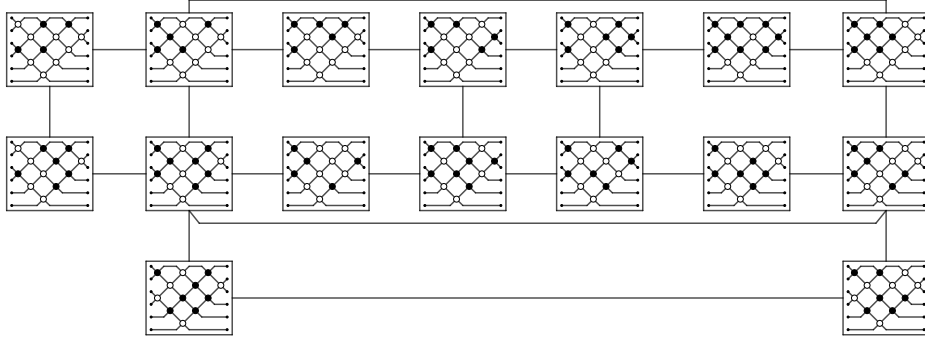
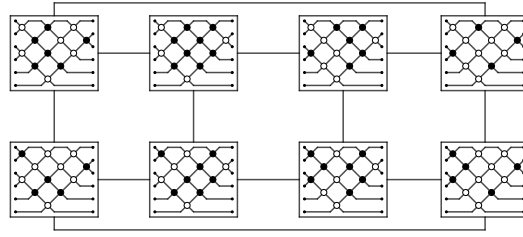


Figure 15.127: Fourth part of the CW complex with ancestry of dimension 3:
 $\varepsilon_{38} = (\bullet \bullet \blacklozenge \bullet \circ \blacklozenge \bullet \diamond \bullet \circ \circ \diamond \diamond)$.

Step 5: Attach a 3-cell that fills the cube in Figure 15.128. This cell is attached to Figure 15.124 through the 2-cell with ancestry $\varepsilon_{39} = (\blacklozenge \blacklozenge \bullet \diamond \circ \diamond \bullet \bullet \circ \bullet \bullet \circ)$ and to Figure 15.125 through the 2-cell with ancestry $\varepsilon_{40} = (\blacklozenge \bullet \bullet \diamond \circ \circ \blacklozenge \bullet \circ \bullet \bullet \diamond)$.

Figure 15.128: 3-Cell with ancestry $\varepsilon_{41} = (\blacklozenge \blacklozenge \bullet \blacklozenge \circ \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge)$.

Step 6: Attach a 3-cell with ancestry $\varepsilon_{42} = (\blacklozenge \blacklozenge \circ \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge)$ that fills the cube in Figure 15.129. This cell is attached to Figure 15.124 through the 2-cell with ancestry $\varepsilon_{43} = (\blacklozenge \blacklozenge \circ \blacklozenge \bullet \blacklozenge \circ \bullet \bullet \circ \circ \bullet)$ and to Figure 15.125 through the 2-cell with ancestry $\varepsilon_{44} = (\blacklozenge \bullet \circ \blacklozenge \bullet \circ \blacklozenge \bullet \circ \bullet \bullet \blacklozenge)$.

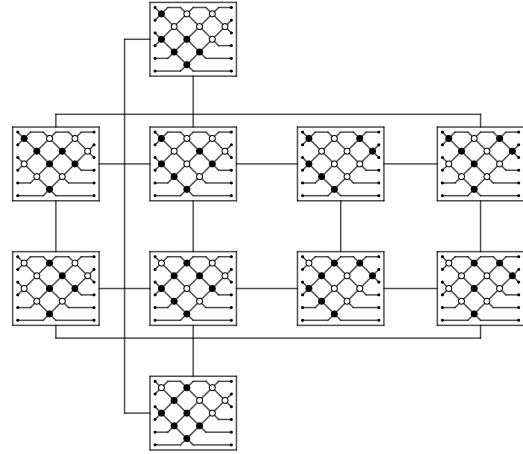
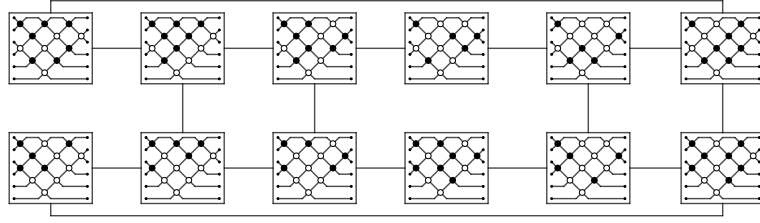


Figure 15.129: Sixth part of the CW complex.

Step 7: Attach a 3-cell that fills the “parallelepiped” in Figure 15.130. This cell is attached to Figure 15.124 through the 2-cell with ancestry $\varepsilon_{45} = (\bullet \blacklozenge \blacklozenge \circ \circ \circ \blacklozenge \circ \bullet \bullet \blacklozenge \bullet)$ and to Figure 15.127 through the 2-cell with ancestry $\varepsilon_{46} = (\bullet \bullet \blacklozenge \circ \circ \blacklozenge \bullet \bullet \circ \circ \blacklozenge \blacklozenge)$. Moreover, to Figure 15.128 through the cell of dimension 2 with ancestry $\varepsilon_{47} = (\bullet \blacklozenge \bullet \circ \circ \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge)$.

Figure 15.130: 3-Cell with ancestry $\varepsilon_{48} = (\bullet \blacklozenge \blacklozenge \circ \circ \blacklozenge \blacklozenge \bullet \circ \circ \blacklozenge \blacklozenge)$.

Step 8: Attach a 3-cell with ancestry $\varepsilon_{49} = (\circ \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \blacklozenge)$ that fills the “parallelepiped” in Figure 15.131. This cell is attached to Figure 15.124 through the 2-cell with ancestry $\varepsilon_{50} = (\circ \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \bullet \bullet \circ \blacklozenge \bullet)$ and to Figure 15.126 through the 2-cell with ancestry $\varepsilon_{51} = (\circ \bullet \blacklozenge \bullet \bullet \blacklozenge \circ \circ \bullet \bullet \blacklozenge \blacklozenge)$. Moreover, to Figure 15.129 through the 2-cell with ancestry $\varepsilon_{52} = (\circ \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \blacklozenge)$.

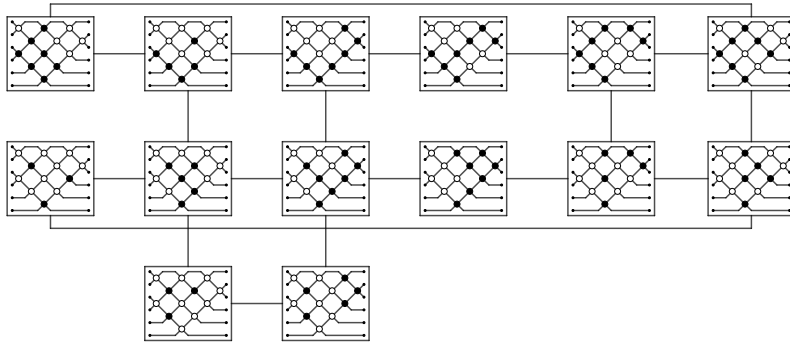


Figure 15.131: Eighth part of the CW complex.

This CW complex comprises two $\mathbb{D}^4$ connected through a single 3-cell, with six 3-cells attached to the 4-cells. Upon completing all attachments, we have a contractible component. Therefore, BL_σ has 16 connected components of this type, all contractible. Summing up, BL_σ has a total of 96 connected components, all of them contractible.

Type 15.8. For $\sigma = [634521] = a_1 a_2 a_3 a_4 a_3 a_2 a_1 a_5 a_4 a_3 a_2 a_1 \in S_6$ it follows that

$$\begin{aligned} \acute{\sigma} = \frac{1}{4\sqrt{2}} & (1 - \hat{a}_2 - \hat{a}_3 - \hat{a}_2 \hat{a}_3 - \hat{a}_4 + \hat{a}_2 \hat{a}_4 - \hat{a}_3 \hat{a}_4 - \hat{a}_2 \hat{a}_3 \hat{a}_4 - \hat{a}_1 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_5 \\ & + \hat{a}_1 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_5 - \hat{a}_1 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_2 \hat{a}_4 \hat{a}_5 - \hat{a}_1 \hat{a}_3 \hat{a}_4 \hat{a}_5 + \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \hat{a}_5). \end{aligned}$$

There exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible.

For dimension 1, there are seven possible positions for the diamonds. If the diamonds are in r_4 and the remaining rows have equal signs, we have the CW complex in Figure 15.132. This results in 32 copies. Therefore, BL_σ has 32 connected components of this type, all contractible.

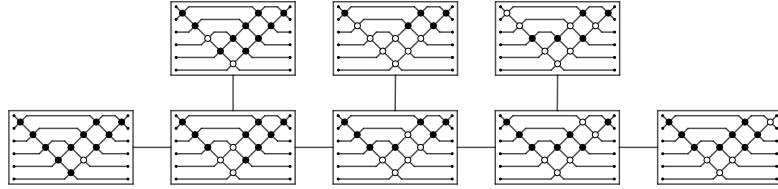


Figure 15.132: CW complex of dimension 1.

The remaining possible positions for the diamonds appear in cells of higher dimensions.

For dimension 2, there are 15 possible positions for the diamonds. If the diamonds are in the first two inversions of r_1 with signs $(\bullet \circ \circ)$, and in r_4 , we obtain the CW complex in Figure 15.133. This results in 32 copies. Therefore, BL_σ have 32 connected components of this type, all contractible.

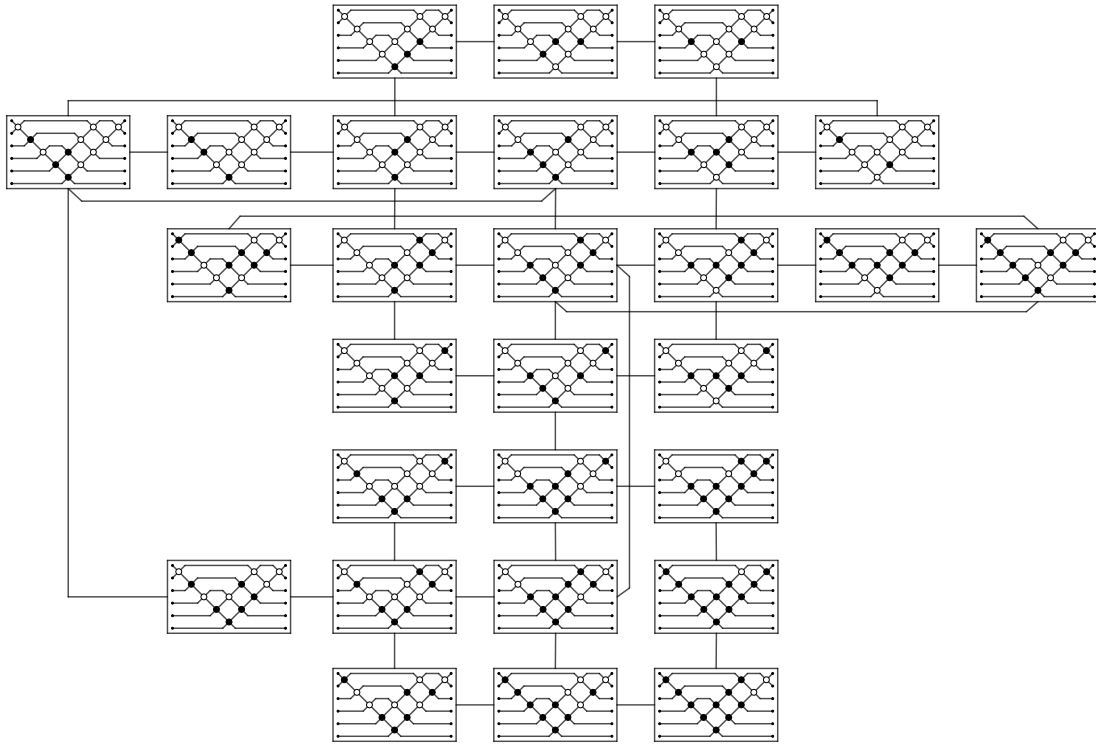


Figure 15.133: CW complex of dimension 2.

The remaining possible positions for the diamonds appear in the cells of higher dimensions.

For dimension 3, there are ten possible positions for the diamonds. If r_2 has signs $(\circ \circ \bullet)$ and the remaining rows have equal signs, we obtain a CW complex with ten 3-cells, each corresponding to one of the possible diamond positions. The structure of this CW complex can be complex to visualize, so to confirm its contractibility, we examine some cells separately and observe where they attach to generate the CW complex.

Step 1: First, six 3-cells are attached as illustrated in Figure 15.134. These cells comprise four cubes and two prisms.

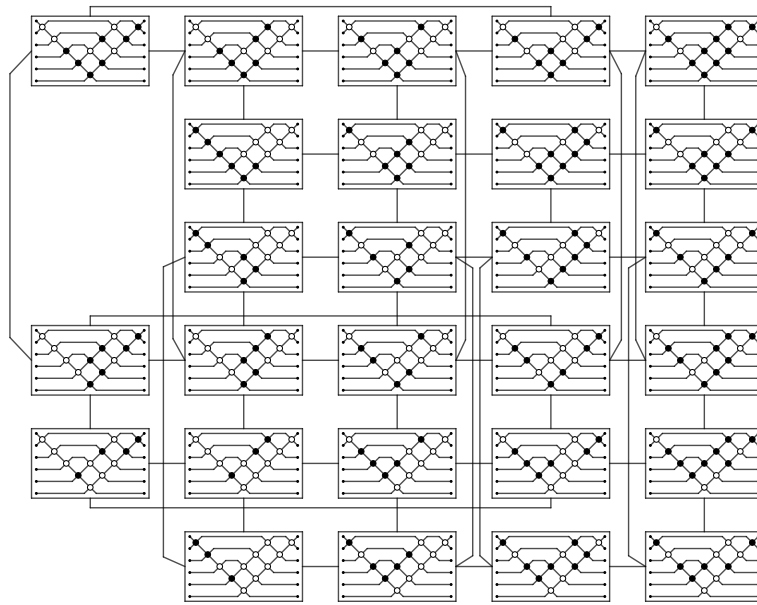


Figure 15.134: First part of the CW complex with six 3-cells.

Step 2: Attach a 3-cell that is a convex solid with twelve faces, consisting of eight squares and four hexagons. The hexagon in the center of Figure 15.135 attaches to the hexagon that is a common face of the two prisms in Figure 15.134, with ancestry $\varepsilon_2 = (\blacklozenge \circ \bullet \circ \circ \blacklozenge \circ \bullet \bullet \circ \diamond \diamond)$. The square with ancestry $\varepsilon_3 = (\circ \bullet \blacklozenge \circ \diamond \circ \blacklozenge \bullet \bullet \circ \circ \diamond)$, and the square with ancestry $\varepsilon_4 = (\blacklozenge \blacklozenge \circ \circ \bullet \diamond \diamond \bullet \bullet \circ \circ \circ)$ attach to the corresponding squares in Figure 15.134.

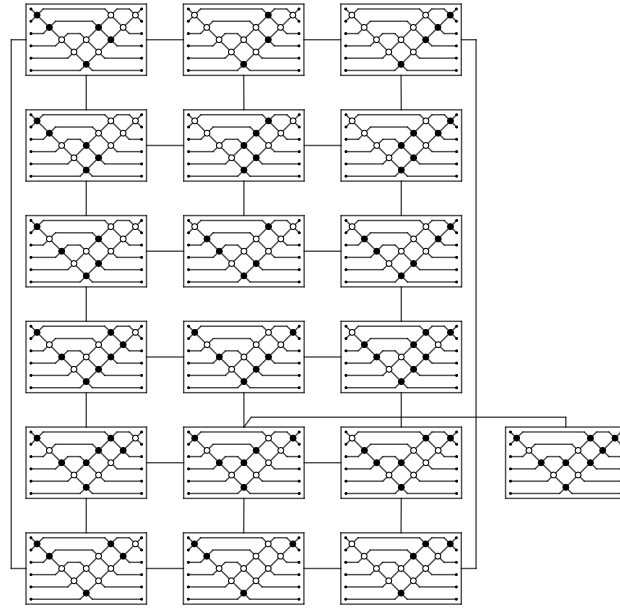


Figure 15.135: Second part of the CW complex with ancestry of dimension 3:
 $\varepsilon_1 = (\blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \circ \bullet \bullet \blacklozenge \blacklozenge \blacklozenge)$.

Step 3: Attach another 3-cell similar to the previous one. The square with ancestry $\varepsilon_6 = (\circ \blacklozenge \blacklozenge \circ \blacklozenge \blacklozenge \bullet \bullet \bullet \circ \circ \circ)$, and the one with ancestry $\varepsilon_7 = (\circ \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \circ \circ \circ \circ \blacklozenge)$ in Figure 15.136 attach to the corresponding squares in Figure 15.134.

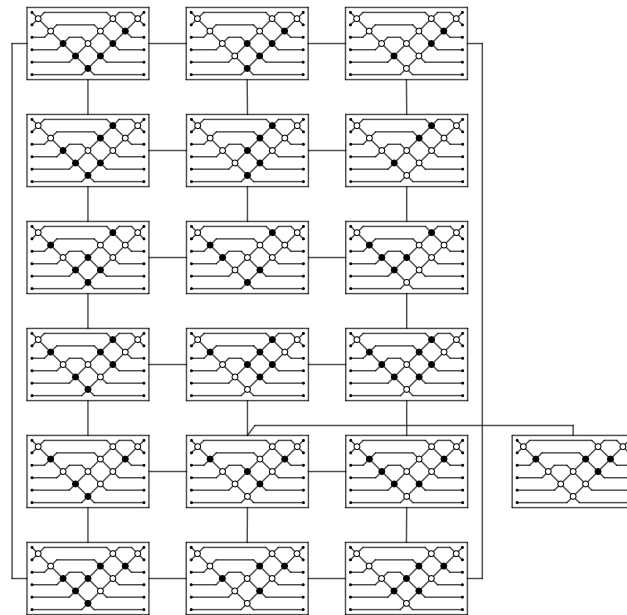


Figure 15.136: Third part of the CW complex with ancestry of dimension 3:
 $\varepsilon_5 = (\circ \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \circ \circ \blacklozenge \blacklozenge \blacklozenge \circ)$.

Step 4: Attach one more 3-cell, which fills the prism completely. The hexagon at the top of Figure 15.137 attaches to the hexagon in the middle of the previous cell, with ancestry $\varepsilon_8 = (\circ \blacklozenge \blacklozenge \circ \diamond \diamond \bullet \bullet \circ \circ \circ)$. The square on the left side of Figure 15.137, with ancestry $\varepsilon_9 = (\blacklozenge \circ \blacklozenge \circ \diamond \bullet \diamond \bullet \bullet \circ \circ \circ)$, attaches to the corresponding square in Figure 15.134. The square on the left side of Figure 15.137, with ancestry $\varepsilon_{10} = (\blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \diamond \circ \diamond \circ \circ \circ)$, attaches to the corresponding square in Figure 15.134.

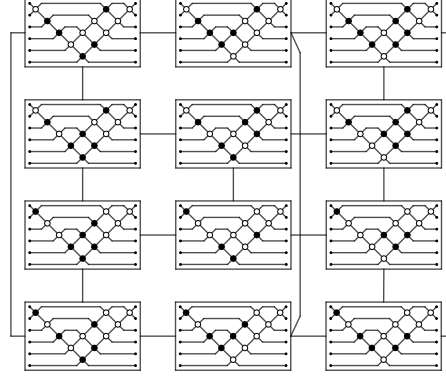


Figure 15.137: 3-Cell with ancestry $\varepsilon_{11} = (\blacklozenge \circ \blacklozenge \blacklozenge \bullet \bullet \diamond \circ \diamond \diamond \circ \circ)$.

Step 5: Attach one last 3-cell, which is a prism. The hexagon at the top of Figure 15.138 attaches to the hexagon in the middle of Figure 15.137, with ancestry $\varepsilon_{13} = (\circ \bullet \blacklozenge \blacklozenge \bullet \circ \circ \circ \diamond \diamond \bullet \bullet)$. The square with ancestry $\varepsilon_{14} = (\circ \bullet \blacklozenge \circ \diamond \circ \blacklozenge \bullet \bullet \circ \circ \diamond)$ on the right side of Figure 15.138 attaches to Figure 15.134. The square in Figure 15.138 with ancestry $\varepsilon_{15} = (\circ \bullet \bullet \blacklozenge \bullet \circ \blacklozenge \circ \diamond \circ \circ \diamond)$ attaches to Figure 15.134.

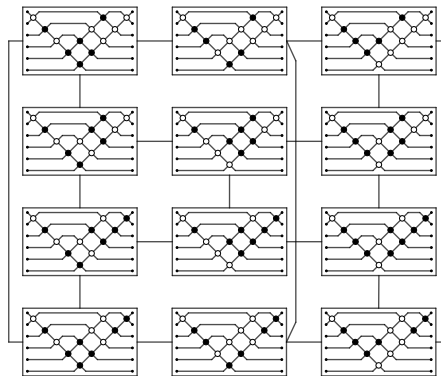


Figure 15.138: 3-Cell with ancestry $\varepsilon_{12} = (\circ \bullet \blacklozenge \blacklozenge \bullet \circ \blacklozenge \circ \diamond \diamond \circ \diamond)$.

Upon completing all the attaching, we have a component that is con-

tractible. Therefore, BL_σ has 32 connected components of this type, all contractible.

The remaining 3 dimensional ancestries appear in a 4-dimensional CW-complex.

For dimension 4, there is only one possible arrangement for the diamonds. The structure of this CW complex is large, containing many cells; we will first examine the 4-cell and then attach several 2-cells, which do not change the homotopy type of the component.

Step 1: First, we have twenty 3-cells attached that form two solid tori as shown in Figure 15.139. This construction results in $\mathbb{D}^4$, as previously demonstrated.

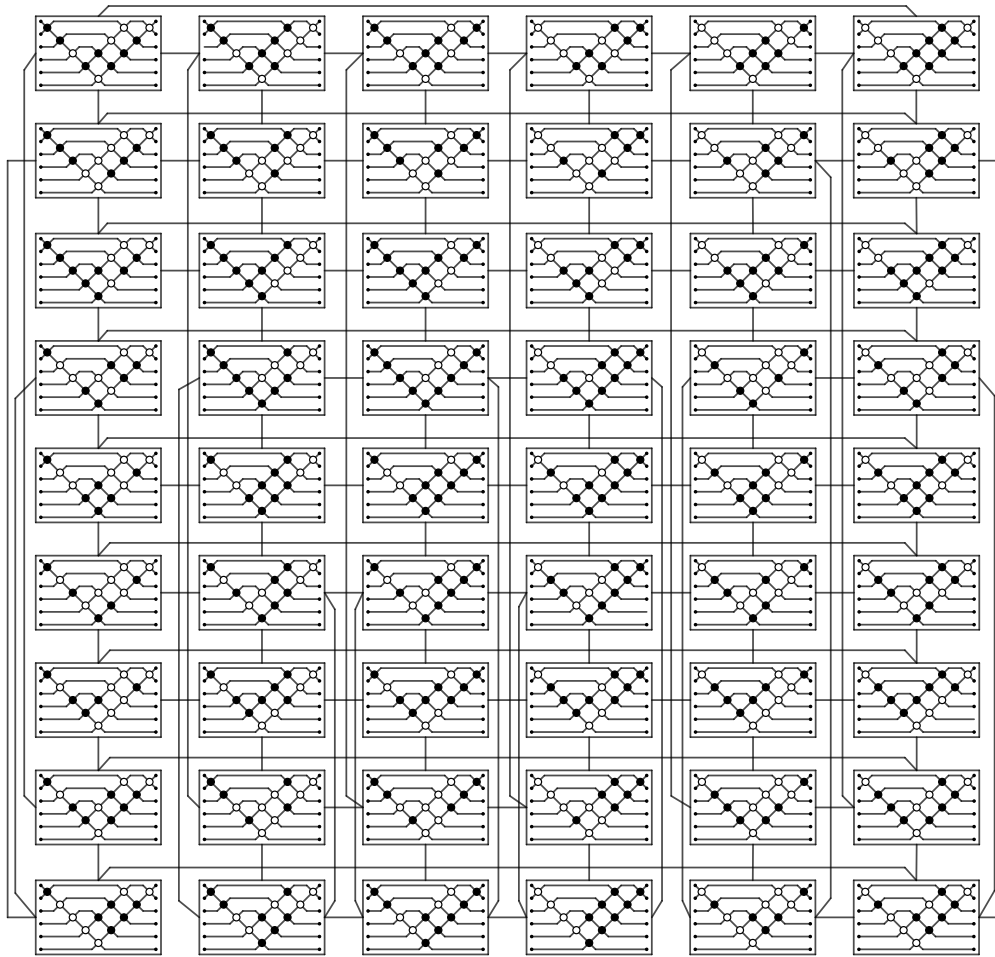


Figure 15.139: 4-cell with ancestry $\varepsilon = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \blacklozenge \blacklozenge)$.

Step 2: Figure 15.140 represents two additional parts that we attach to Figure 15.139, each attachment occurs through five 1-cells.

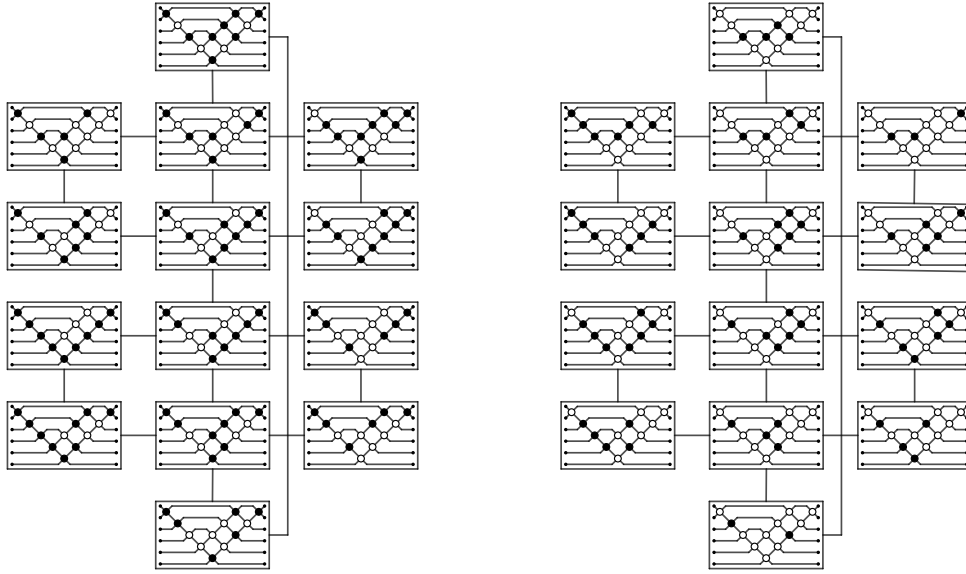


Figure 15.140: Additional two parts with five 2-cells each.

Upon completing all attachments, we have a contractible connected component. Therefore, BL_σ has 16 connected components of this type, all contractible. In summary, BL_σ has a total of 144 connected components, all contractible.

The permutation $\sigma = a_1 a_2 a_3 a_2 a_1 a_4 a_3 a_5 a_4 a_3 a_2 a_1 \in S_6$ has a CW complex structure similar to the one described.

16

The Homotopy Type of BL_σ for $\text{inv}(\sigma) = 12$ - Case 9

For $\sigma = [563412] = a_2a_1a_3a_2a_4a_3a_2a_1a_5a_4a_3a_2 \in S_6$, it follows that

$$\sigma = \frac{1}{2}(-\hat{a}_1 - \hat{a}_2\hat{a}_3\hat{a}_4 - \hat{a}_5 + \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5) \in \tilde{B}_6^+.$$

In the first section, we explore the orbits of the elements $z \in \sigma \text{Quat}_6$, as well as the count of cells of each dimension present in the component. The following sections investigate each component.

16.1

The Orbits

The set σQuat_6 consists of nine orbits each of size 4 or 8:

$$\begin{aligned} \mathcal{O}_\sigma &= \left\{ \frac{\pm\hat{a}_1 \pm \hat{a}_2\hat{a}_3\hat{a}_4 \pm \hat{a}_5 \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1\sigma} &= \left\{ \frac{1 \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 \pm \hat{a}_1\hat{a}_5 \pm \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_2\sigma} &= \left\{ \frac{\pm\hat{a}_1\hat{a}_2 \pm \hat{a}_3\hat{a}_4 \pm \hat{a}_2\hat{a}_5 \pm \hat{a}_1\hat{a}_3\hat{a}_4\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1\hat{a}_2\sigma} &= \left\{ \frac{\pm\hat{a}_2 \pm \hat{a}_1\hat{a}_3\hat{a}_4 \pm \hat{a}_1\hat{a}_2\hat{a}_5 \pm \hat{a}_3\hat{a}_4\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_3\sigma} &= \left\{ \frac{\pm\hat{a}_1\hat{a}_3 \pm \hat{a}_2\hat{a}_4 \pm \hat{a}_3\hat{a}_5 \pm \hat{a}_1\hat{a}_2\hat{a}_4\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_1\hat{a}_3\sigma} &= \left\{ \frac{\pm\hat{a}_3 \pm \hat{a}_1\hat{a}_2\hat{a}_4 \pm \hat{a}_1\hat{a}_3\hat{a}_5 \pm \hat{a}_2\hat{a}_4\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_2\hat{a}_3\sigma} &= \left\{ \frac{\pm\hat{a}_1\hat{a}_2\hat{a}_3 \pm \hat{a}_4 \pm \hat{a}_1\hat{a}_4\hat{a}_5 \pm \hat{a}_2\hat{a}_3\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{\hat{a}_4\sigma} &= \left\{ \frac{\pm\hat{a}_2\hat{a}_3 \pm \hat{a}_1\hat{a}_4 \pm \hat{a}_4\hat{a}_5 \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_5}{2} \right\}, \\ \mathcal{O}_{-\hat{a}_1\sigma} &= \left\{ \frac{-1 \pm \hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4 \pm \hat{a}_1\hat{a}_5 \pm \hat{a}_2\hat{a}_3\hat{a}_4\hat{a}_5}{2} \right\}. \end{aligned}$$

In the expressions within the Clifford algebra notation, the signs must be such that there is an odd number of equal signs.

The elements $z \in \sigma \text{Quat}_6$ have $\Re(z) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. Using the Formula 4-3 of the number of ancestries of dimension 0 for a given $z \in \sigma \text{Quat}_6$, it

follows that $N(z) \in \{48, 64, 80\}$. The number of ancestries per dimension can be determined using the Formulas 4-1 and 4-2 (see Section 4.3), and this can be cross-verified using Maple.

1. For $z \in \mathcal{O}_\sigma$, $\mathfrak{R}(z) = 0$ and $N(z) = 64$ and $N_{thin}(z) = 4$. Consequently, BL_σ has 40 connected components of this type. The CW complex BLC_z are described in Section 16.4. Thus, for each $z \in \mathcal{O}_\sigma$, the sets BL_z have four thin components and one thick.

The component has sixty 0-cells, one hundred and twelve 1-cells, sixty-eight 2-cells, sixteen 3-cells, and one 4-cell. Moreover, the Euler characteristic of this component is equal to 1.

2. If $\mathfrak{R}(z) = \frac{1}{2}$, then $N(z) = 80$ and $N_{thin}(z) = 0$. Therefore, BL_σ has 4 connected components of this type. The CW complex BLC_z is described in Section 16.9. Then, for each $z \in \mathcal{O}_{\hat{a}_1\sigma}$, the sets BL_z have one connected component.

The component has eighty 0-cells, one hundred and sixty-eight 1-cells, one hundred and twenty-eight 2-cells, forty-eight 3-cells, ten 4-cells, and one 5-cell. Additionally, the Euler characteristic of this component is 1.

3. For $z \in \mathcal{O}_{\hat{a}_2\sigma}$, $\mathfrak{R}(z) = 0$ and $N(z) = 64$ and $N_{thin}(z) = 0$. Consequently, BL_σ has 8 connected components of this type. The CW complex BLC_z are described in Section 16.5. Thus, for each $z \in \mathcal{O}_{\hat{a}_2\sigma}$, the sets BL_z have one connected component.

The component has sixty-four 0-cells, one hundred and twelve 1-cells, sixty 2-cells, twelve 3-cells, and one 4-cell. In this case, the Euler characteristic is also equal to 1.

4. For $z \in \mathcal{O}_{\hat{a}_1\hat{a}_2\sigma}$, $\mathfrak{R}(z) = 0$ and $N(z) = 64$ and $N_{thin}(z) = 0$. Consequently, BL_σ has 8 connected components of this type. The CW complex BLC_z is described in Section 16.6. Thus, for each $z \in \mathcal{O}_{\hat{a}_1\hat{a}_2\sigma}$, the sets BL_z have one connected component.

The component has sixty-four 0-cells, one hundred and twelve 1-cells, sixty 2-cells, twelve 3-cells, and one 4-cell. Furthermore, this component has an Euler characteristic of 1.

5. For $z \in \mathcal{O}_{\hat{a}_3\sigma}$, $\mathfrak{R}(z) = 0$ and $N(z) = 64$ and $N_{thin}(z) = 0$. Consequently, BL_σ has 8 connected components of this type. The CW complex BLC_z are described in Section 16.3. Thus, for each $z \in \mathcal{O}_{\hat{a}_3\sigma}$, the sets BL_z have one connected component.

The component has sixty-four 0-cells, one hundred and twelve 1-cells, fifty-two 2-cells, and four 3-cells. Moreover, the Euler characteristic of this component is 0.

6. For $z \in \mathcal{O}_{\hat{a}_1\hat{a}_3\hat{\sigma}}$, $\Re(z) = 0$ and $N(z) = 64$ and $N_{thin}(z) = 0$. Consequently, BL_σ has 8 connected components of this type. The CW complex BLC_z is also the one described in Section 16.3. Thus, for each $z \in \mathcal{O}_{\hat{a}_1\hat{a}_3\hat{\sigma}}$, the sets BL_z have one connected component.

The component has sixty-four 0-cells, one hundred and twelve 1-cells, fifty-two 2-cells, and four 3-cells. In addition, the Euler characteristic of this component equals 0.

7. For $z \in \mathcal{O}_{\hat{a}_2\hat{a}_3\hat{\sigma}}$, $\Re(z) = 0$ and $N(z) = 64$ and $N_{thin}(z) = 0$. Consequently, BL_σ has 8 connected components of this type. The CW complex BLC_z are described in Section 16.8. Thus, for each $z \in \mathcal{O}_{\hat{a}_2\hat{a}_3\hat{\sigma}}$, the sets BL_z have one connected component.

The component has sixty-four 0-cells, one hundred and twelve 1-cells, sixty 2-cells, twelve 3-cells and one 4-cell. Moreover, the Euler characteristic of this component is equal to 1.

8. For $z \in \mathcal{O}_{\hat{a}_4\hat{\sigma}}$, $\Re(z) = 0$ and $N(z) = 64$ and $N_{thin}(z) = 0$. Consequently, BL_σ has 8 connected components of this type. The CW complex BLC_z are described in Section 16.7. Thus, for each $z \in \mathcal{O}_{\hat{a}_4\hat{\sigma}}$, the sets BL_z have one connected component.

The component has sixty-four 0-cells, one hundred and twelve 1-cells, sixty 2-cells, twelve 3-cells and one 4-cell. Furthermore, this component has an Euler characteristic of 1.

9. If $\Re(z) = -\frac{1}{2}$, then $N(z) = 48$ and $N_{thin}(z) = 0$. Therefore, BL_σ has 8 connected components of this type. The CW complex BLC_z are described in Section 16.2. Then, for each $z \in \mathcal{O}_{-\hat{a}_1\hat{\sigma}}$, the sets BL_z have two connected components.

The component has twenty-four 0-cells, twenty-eight 1-cells and four 2-cells. Furthermore, the Euler characteristic of this component is 0.

16.2

The Known Component

In the Introduction, we already presented a connected component of BL_σ that is homotopically equivalent to $\mathbb{S}^1$ and thus non-contractible. In this section, we will discuss this component in greater detail.

If r_4 has opposite signs and the remaining rows have equal signs, we obtain the component shown in Figure 16.1, which is the one in Figure 1.1 (as described in [2]) and corresponds to the CW complex depicted in Figure 16.2. Therefore, BL_σ has 8 connected components of this type.

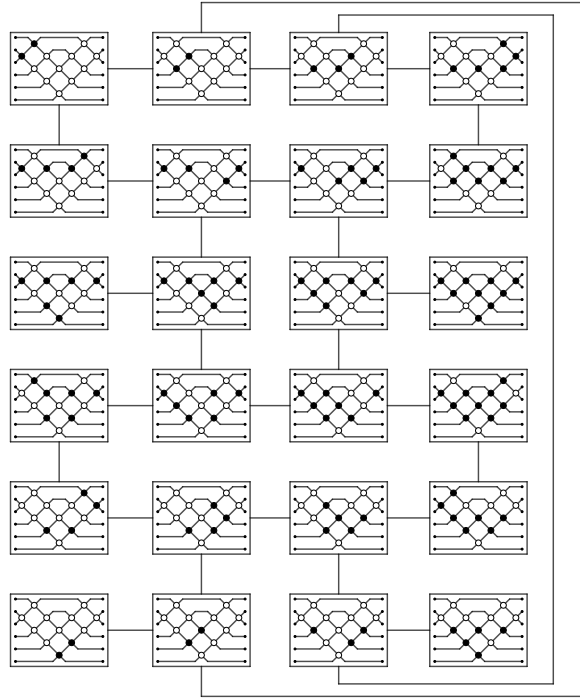


Figure 16.1: Connected component homotopically equivalent to $\mathbb{S}^1$.

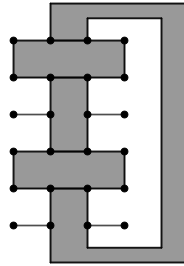


Figure 16.2: CW complex homotopically equivalent to $\mathbb{S}^1$.

Furthermore, using equation 6-1, we generate the paths that correspond to the edges yielding $\mathbb{S}^1$, by concatenating these paths, we obtain the circle. The paths are given by:

$$\Gamma_i : [-1, 1] \rightarrow \text{Lo}_6^1, \quad i \in \{1, \dots, 10\},$$

$$\begin{aligned} \Gamma_1(t) &= \begin{pmatrix} 1 & & & & & \\ -t+1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 2+t & 4+2t & 2+t & 1 & & \\ 1 & 4 & 3 & 2 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \Gamma_2(t) = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 2 & -2t & 1 & & & \\ 3 & 3-3t & 3 & 1 & & \\ 1 & 3-t & 3 & 2 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \\ \Gamma_3(t) &= \begin{pmatrix} 1 & & & & & \\ 1+t & 1 & & & & \\ -2t & -2 & 1 & & & \\ 2-t & 0 & 2-t & 1 & & \\ 1 & 2 & 1-2t & 2 & 1 & \\ 0 & 1 & -t & 1 & 1 & 1 \end{pmatrix}, \quad \Gamma_4(t) = \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ -3-t & -3-t & 1 & & & \\ -t & -1-t & 1 & 1 & & \\ 1 & 2 & -1 & 1-t & 1 & \\ 0 & 1 & -1 & -t & 1 & 1 \end{pmatrix}, \\ \Gamma_5(t) &= \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ -3+t & -3+t & 1 & & & \\ -1 & -2 & 1 & 1 & & \\ 1 & 3+t & -2-t & -1-t & 1 & \\ 0 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}, \quad \Gamma_6(t) = \begin{pmatrix} 1 & & & & & \\ -t+1 & 1 & & & & \\ -2 & -2 & 1 & & & \\ -2-t & -4-2t & 2+t & 1 & & \\ 1 & 4 & -3 & -2 & 1 & \\ 0 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}, \\ \Gamma_7(t) &= \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ -2 & 2t & 1 & & & \\ -3 & -3+3t & 3 & 1 & & \\ 1 & 3-t & -3 & -2 & 1 & \\ 0 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}, \quad \Gamma_8(t) = \begin{pmatrix} 1 & & & & & \\ t+1 & 1 & & & & \\ 2t & 2 & 1 & & & \\ -2+t & 0 & 2-t & 1 & & \\ 1 & 2 & -1+2t & -2 & 1 & \\ 0 & 1 & 0 & -1 & 1 & 1 \end{pmatrix}, \\ \Gamma_9(t) &= \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 3+t & 3+t & 1 & & & \\ t & t+1 & 1 & 1 & & \\ 1 & 2 & 1 & -1+t & 1 & \\ 0 & 1 & 1 & t & 1 & 1 \end{pmatrix}, \quad \Gamma_{10}(t) = \begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 3-t & 3-t & 1 & & & \\ 1 & 2 & 1 & 1 & & \\ 1 & 3+t & 2+t & t+1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

To obtain these paths, we consider the product of λ_i that generates the matrices in the stratum of dimension 0 as in equation 6-1, for instance

$$\Gamma_1(t) = \lambda_2\left(\frac{-1}{t}\right)\lambda_1(-t)\lambda_3(t)\lambda_2\left(\frac{1}{t}\right)\lambda_4(1)\lambda_3(1)\lambda_2(1)\lambda_1(1)\lambda_5(1)\lambda_4(1)\lambda_3(1)\lambda_2(1).$$

Notice that, even though there are fractions in the right hand side, the matrix Γ_1 has polynomial entries.

These paths illustrate how the matrices in the strata of codimension 0 connect via the matrices in the strata of codimension 1. For instance, applying $t = 1$ to Γ_1 we have a matrix in the stratum of codimension 0 represented by the second diagram from the left at the top of Figure 16.1. Applying $t = -1$ produces a matrix corresponding to the adjacent stratum of codimension 0 on the left. Finally, for $t = 0$, we obtain a matrix in the stratum of codimension 1, the edge connecting these diagrams.

Note that applying $t = -1$ in Γ_1 and $t = 1$ in Γ_{10} results in the same matrix, indicating that the concatenation of these paths forms a closed curve homotopically equivalent to $\mathbb{S}^1$.

16.3

The New non-Contractible Component

Another connected component homotopically equivalent to $\mathbb{S}^1$, and thus non-contractible, was found with CW complex of dimension 3. This component consists of four 3-cells that are attached together, generating a solid torus. Additionally, some 2-cells are attached like wings, which do not alter the homotopy type of the component.

Let us go through the step by step construction of this component, adding the 3-cells one by one until we attach the last one with the first to generate the solid torus. Note that in some cells, we have vertices connected to only one edge. In some of these cases, we connect them with an edge in another solid, thus generating the mentioned wings.

Step 1: First, we have a 3-cell that fills the cube in Figure 16.3.

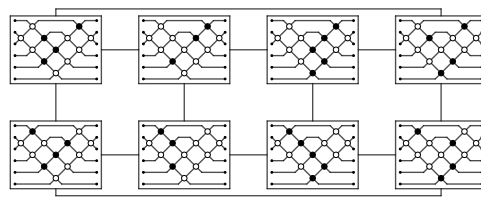


Figure 16.3: Cube with ancestry $\varepsilon_1 = (\circ \blacklozenge \circ \blacklozenge \circ \blacklozenge \circ \blacklozenge \circ \circ \circ \circ)$.

Some lower-dimensional cells are attached to the cube, resulting in the structure shown in Figure 16.4.

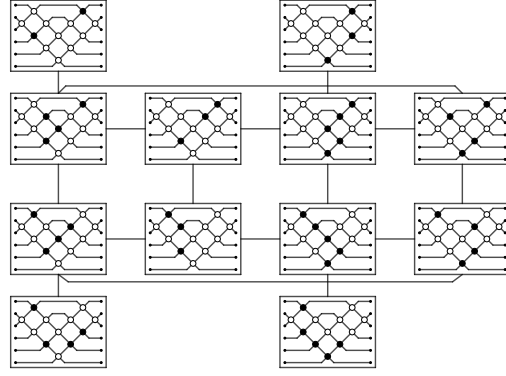
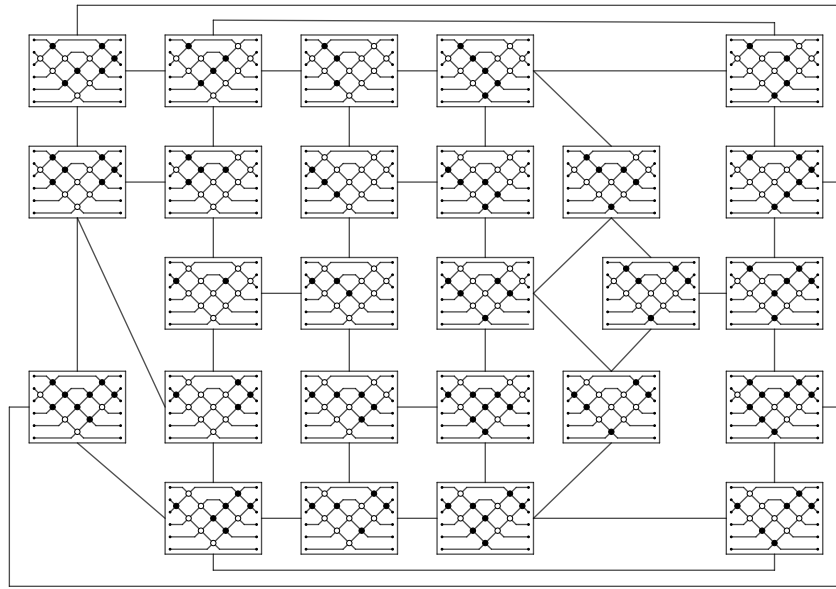


Figure 16.4: First part of the CW complex.

Step 2: Attach a 3-cell that fills the convex solid with 18 faces in Figure 16.5. Attachment occurs through the square face with ancestry $\varepsilon_2 = (\circ \bullet \circ \blacklozenge \blacklozenge \circ \circ \circ \circ \circ \circ \circ \circ)$.

Figure 16.5: Convex solid with ancestry $\varepsilon_3 = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge)$.

Some lower-dimensional cells are attached to the convex solid, resulting in the structure shown in Figure 16.6.

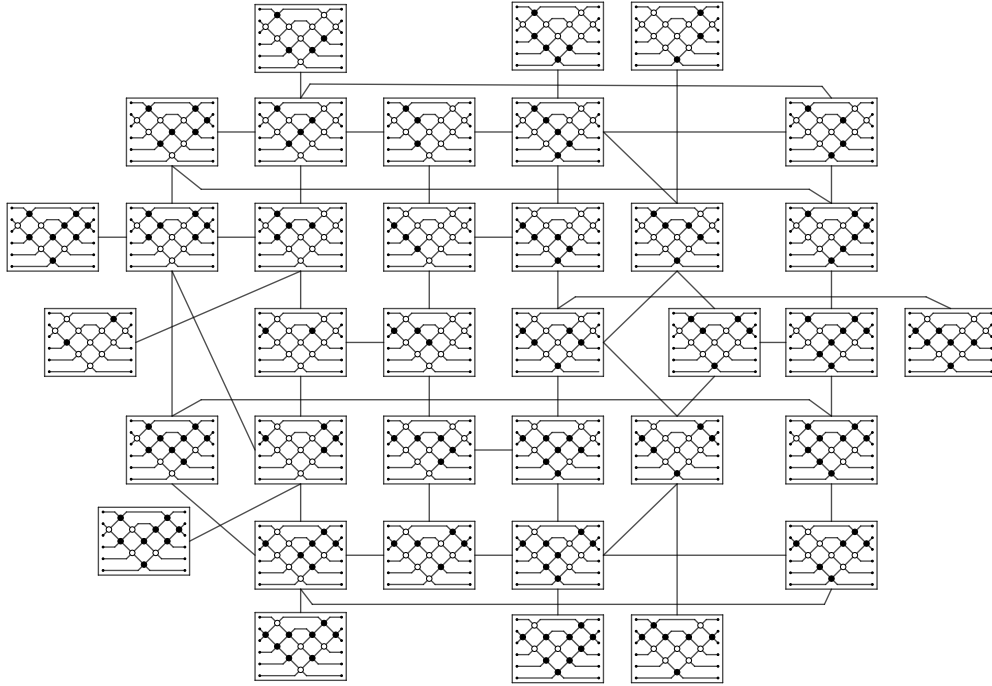


Figure 16.6: Second part of the CW complex.

Following this attachment, two 2-cells appear as wings in the component when we attach the previous two.

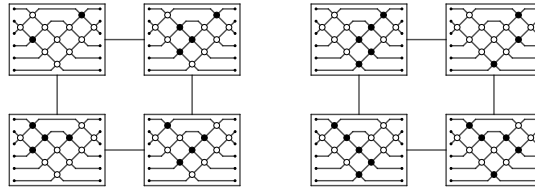


Figure 16.7: 2-Cells with ancestries $\varepsilon_4 = (\circ \blacklozenge \blacklozenge \bullet \circ \diamond \bullet \diamond \circ \circ \circ \circ)$ and $\varepsilon_5 = (\circ \blacklozenge \circ \bullet \circ \blacklozenge \circ \diamond \bullet \bullet \diamond \circ)$.

Step 3: Attach another 3-cell that fills a cube, as shown in Figure 16.8. Attachment occurs through the square face with ancestry $\varepsilon_6 = (\bullet \circ \circ \blacklozenge \blacklozenge \bullet \diamond \bullet \bullet \diamond \circ \bullet)$.

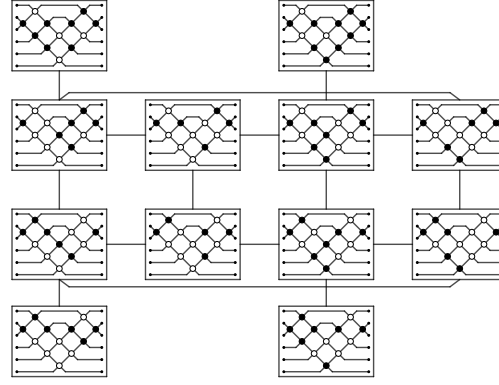


Figure 16.8: Third part of the CW complex with ancestry of dimension 3:
 $\varepsilon_7 = (\bullet \blacklozenge \circ \blacklozenge \blacklozenge \circ \diamond \diamond \bullet \diamond \circ \bullet)$.

Following this attachment, similar to the previous case, some 2-cells appear as wings in the component.

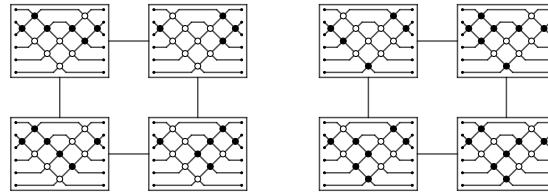


Figure 16.9: 2-Cells with ancestries $\varepsilon_8 = (\bullet \blacklozenge \circ \bullet \circ \blacklozenge \circ \diamond \circ \bullet \diamond \bullet)$ and
 $\varepsilon_9 = (\bullet \blacklozenge \blacklozenge \bullet \circ \diamond \bullet \diamond \bullet \circ \circ \bullet)$.

Step 4: Attach a 3-cell that fills another convex solid with 18 faces, similar to the previous one. Attachment occurs through the square face with ancestry $\varepsilon_{10} = (\bullet \bullet \circ \blacklozenge \blacklozenge \circ \diamond \circ \bullet \diamond \circ \bullet)$.

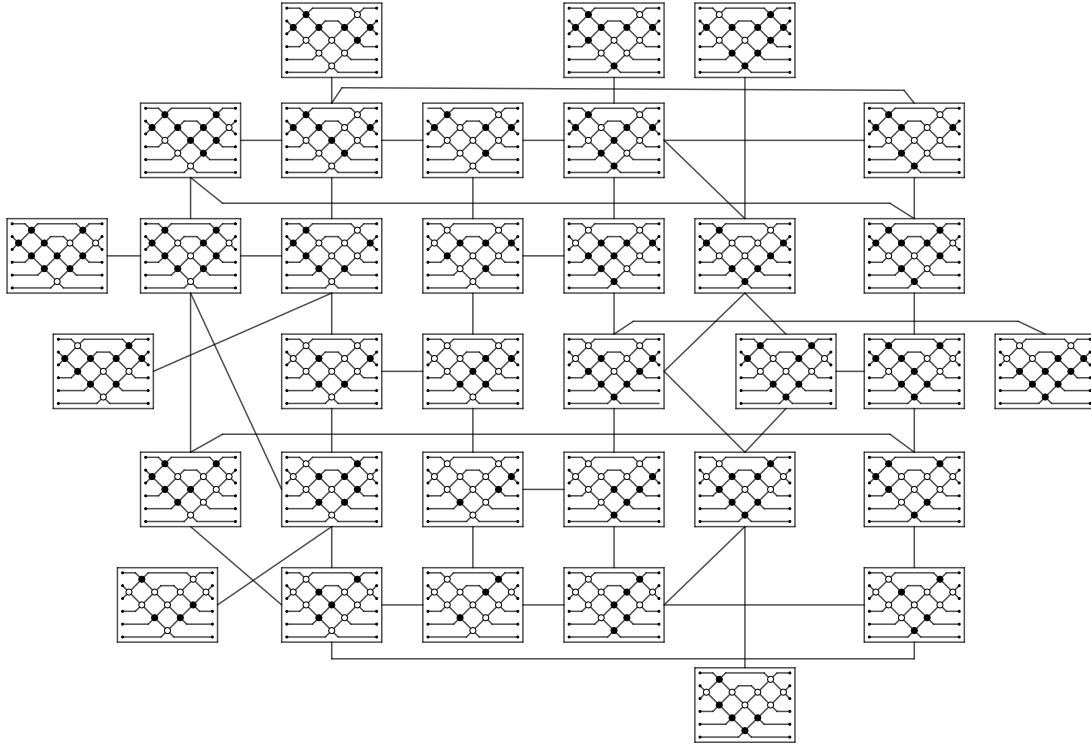
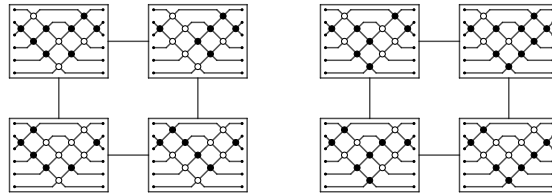


Figure 16.10: Fourth part of the CW complex with ancestry of dimension 3:

$$\varepsilon_{11} = (\blacklozenge \bullet \blacklozenge \circ \blacklozenge \bullet \circ \bullet \circ \diamond \diamond \diamond).$$

Following this attachment, we have some 2-cells that also appear as wings in the component.

Figure 16.11: 2-Cells with ancestries $\varepsilon_{12} = (\bullet \blacklozenge \blacklozenge \circ \bullet \diamond \circ \diamond \circ \bullet \circ \bullet)$ and

$$\varepsilon_{13} = (\circ \bullet \blacklozenge \circ \circ \bullet \blacklozenge \blacklozenge \bullet \circ \diamond \bullet \circ).$$

Step 5: To complete the attachment, the cell in Figure 16.11 is attached to the cell in Figure 16.4, resulting in the formation of the solid torus. The attachment is realized on the square face with ancestry $\varepsilon_{14} = (\circ \circ \circ \blacklozenge \blacklozenge \bullet \diamond \bullet \circ \diamond \circ \circ)$.

After this last attachment, we have some 2-cells that appear as wings in the component.

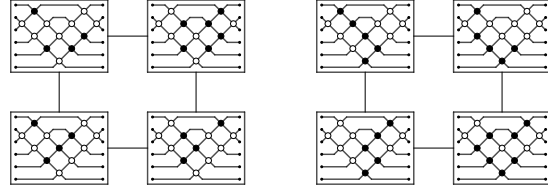


Figure 16.12: 2-Cells with ancestries $\varepsilon_{15} = (\circ \blacklozenge \circ \circ \bullet \blacklozenge \bullet \diamond \circ \circ \diamond \circ)$ and $\varepsilon_{16} = (\circ \blacklozenge \blacklozenge \circ \bullet \diamond \circ \diamond \bullet \bullet \circ \circ)$.

Upon completing all the attachments, we have a component that is homotopically equivalent to $\mathbb{S}^1$. Therefore, BL_σ has 16 components of this type.

For easier visualization, Figure 16.13 first displays the CW complex without the 1-cells and 2-cells attached. It then shows the same CW complex with these cells added, with the red cells representing those not shown in the initial diagram. In this representation, cells of dimension greater than 1 are not filled for clarity.

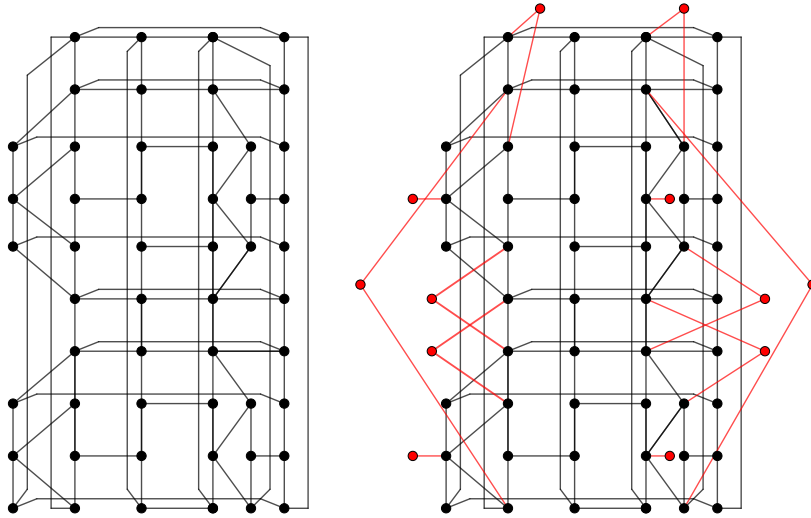


Figure 16.13: Non-contractible CW complex.

Similarly to the previous case, we can also present the paths that correspond to the edges yielding $\mathbb{S}^1$, by concatenating these paths we obtain the circle. The paths are given by:

$$\Gamma_i : [-1, 1] \rightarrow \text{Lo}_6^1, \quad i \in \{1, \dots, 10\},$$

$$\begin{aligned} \Gamma_1(t) &= \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 2 & 1 & & & \\ 1 & t+1 & 1 & 1 & & \\ -1 & 1-t & 1 & 0 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \Gamma_2(t) = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ -1-t & 1-t & 1 & & & \\ 1 & 2 & 1 & 1 & & \\ -1 & t+1 & 2+t & t+1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \\ \Gamma_3(t) &= \begin{pmatrix} 1 & & & & & \\ t+1 & 1 & & & & \\ -2 & 0 & 1 & & & \\ -t & 2 & 2+t & 1 & & \\ -1 & 2 & 3 & 2 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \Gamma_4(t) = \begin{pmatrix} 1 & & & & & \\ t-1 & 1 & & & & \\ -2 & 0 & 1 & & & \\ -t-2 & 2 & 2-t & 1 & & \\ -1 & 2 & 1-2t & 2 & 1 & \\ 0 & 1 & -t & 1 & 1 & 1 \end{pmatrix}, \\ \Gamma_5(t) &= \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ -1+t & -1-t & 1 & & & \\ t-2 & 1-t & 1 & 1 & & \\ -1 & 2 & -1 & 1-t & 1 & \\ 0 & 1 & -1 & -t & 1 & 1 \end{pmatrix}, \quad \Gamma_6(t) = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & -2 & 1 & & & \\ -1 & -1-t & 1 & 1 & & \\ -1 & 1-t & -1 & 0 & 1 & \\ 0 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}, \\ \Gamma_7(t) &= \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ t+1 & -1+t & 1 & & & \\ -1 & -2 & 1 & 1 & & \\ -1 & t+1 & -2-t & -1-t & 1 & \\ 0 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}, \quad \Gamma_8(t) = \begin{pmatrix} 1 & & & & & \\ 1+t & 1 & & & & \\ 2 & 0 & 1 & & & \\ t & -2 & 2+t & 1 & & \\ -1 & 2 & -3 & -2 & 1 & \\ 0 & 1 & -1 & -1 & 1 & 1 \end{pmatrix}, \\ \Gamma_9(t) &= \begin{pmatrix} 1 & & & & & \\ 1-t & 1 & & & & \\ 2 & 0 & 1 & & & \\ 2+t & -2 & 2-t & 1 & & \\ -1 & 2 & -1+2t & -2 & 1 & \\ 0 & 1 & t & -1 & 1 & 1 \end{pmatrix}, \quad \Gamma_{10}(t) = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 1-t & t+1 & 1 & & & \\ 2-t & -1+t & 1 & 1 & & \\ -1 & 2 & 1 & -1+t & 1 & \\ 0 & 1 & 1 & t & 1 & 1 \end{pmatrix}. \end{aligned}$$

Note that applying $t = -1$ in Γ_1 and $t = 1$ in Γ_{10} results in the same matrix, indicating that the concatenation of these paths forms a closed curve homotopically equivalent to $\mathbb{S}^1$.

From now on, all the six found components are contractible. In the next sections, we examine these components in detail, where five have dimension 4 and one has dimension 5.

16.4

The First Contractible Component of Dimension 4

For dimension 4, there are six possible positions for the diamonds, each resulting in one additional component, all of which are contractible.

If r_4 has equal signs and the remaining rows have diamonds, the CW complex will be described below and consists of one 4-cell with four 3-cells attached.

Step 1: First, we have a 4-cell with twelve 3-cells that is a $\mathbb{D}^4$ and fills the CW complex in Figure 16.14.

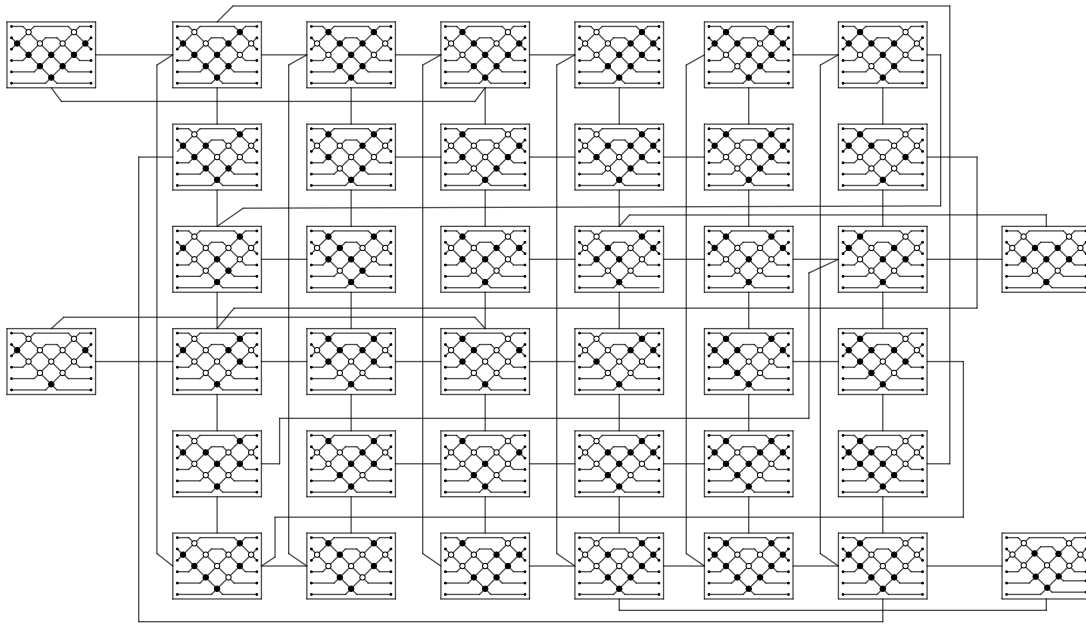


Figure 16.14: 4-cell with ancestry $\varepsilon_1 = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \diamond \diamond \bullet \bullet \diamond \diamond)$.

Step 2: Attach a 3-cell that fills the “parallelepiped” in Figure 16.15. Attachment occurs through the 2-cell with ancestry $\varepsilon_2 = (\bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \diamond \bullet \bullet \diamond \bullet)$.

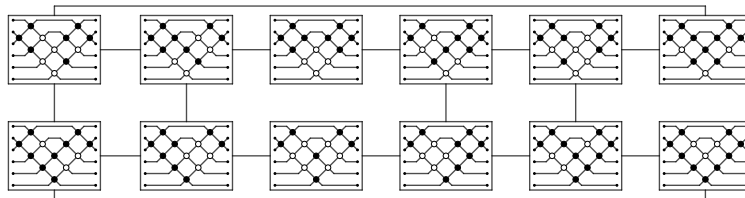


Figure 16.15: 3-cell with ancestry $\varepsilon_3 = (\bullet \bullet \blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \diamond \bullet \bullet \diamond \bullet)$.

Step 3: Attach a 3-cell that fills another “parallelepiped” in Figure 16.16 to Figure 16.14. Attachment occurs through the 2-cell with ancestry

$$\varepsilon_4 = (\circ \bullet \blacklozenge \blacklozenge \circ \bullet \diamond \bullet \bullet \bullet \diamond \circ).$$

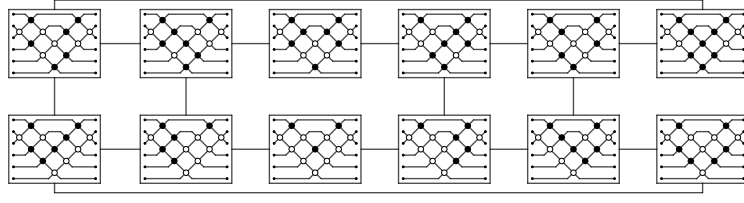


Figure 16.16: 3-cell with ancestry $\varepsilon_5 = (\circ \bullet \blacklozenge \blacklozenge \circ \diamond \bullet \circ \diamond \diamond \circ)$.

Step 4: Attach a 3-cell that fills the prism in Figure 16.17. The attachment to Figure 16.14 occurs through the 2-cell with ancestry $\varepsilon_6 = (\blacklozenge \blacklozenge \circ \bullet \bullet \bullet \diamond \bullet \bullet \circ \diamond)$. This cell is also attached to Figure 16.15 through the 2-cell with ancestry $\varepsilon_7 = (\bullet \bullet \circ \blacklozenge \blacklozenge \circ \diamond \bullet \circ \diamond \bullet \bullet)$, and to Figure 16.16 through the 2-cell with ancestry $\varepsilon_8 = (\circ \bullet \bullet \blacklozenge \blacklozenge \circ \diamond \bullet \circ \diamond \circ \circ)$.

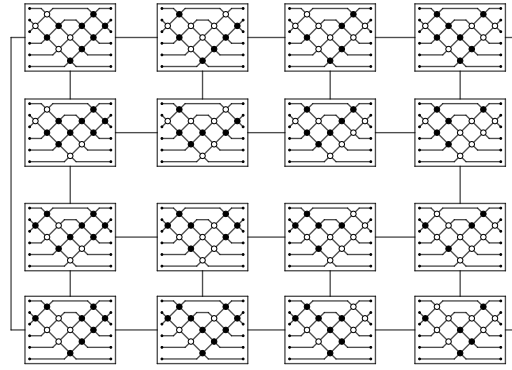


Figure 16.17: 3-cell with ancestry $\varepsilon_9 = (\blacklozenge \blacklozenge \circ \bullet \blacklozenge \circ \bullet \diamond \circ \diamond \circ \diamond)$.

Step 5: Attach a 3-cell that fills another prism in Figure 16.18.

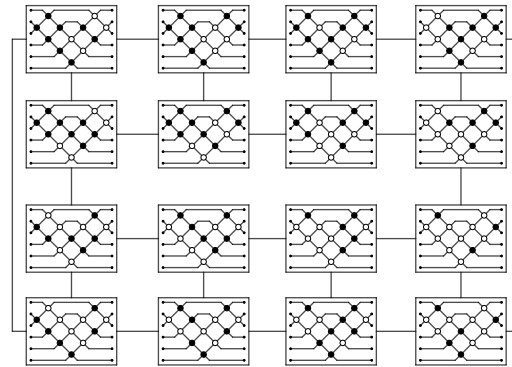


Figure 16.18: 3-cell with ancestry $\varepsilon_{10} = (\blacklozenge \blacklozenge \bullet \bullet \blacklozenge \circ \bullet \diamond \bullet \diamond \bullet \diamond)$.

The attachment to Figure 16.14 occurs through the 2-cell with ancestry $\varepsilon_{11} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \bullet \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge)$. This cell is also attached to Figure 16.15 through the 2-cell with ancestry $\varepsilon_{12} = (\bullet \bullet \bullet \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \circ \bullet)$, and to Figure 16.16 through the 2-cell with ancestry $\varepsilon_{13} = (\circ \bullet \circ \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \bullet \circ)$.

Upon completing all the attachments, we have a contractible connected component. Therefore, BL_σ has 8 connected components, all contractible.

16.5

The Second Contractible Component of Dimension 4

If r_1 has equal signs and the remaining rows have diamonds, the CW complex will be described below.

Step 1: First, we have a 4-cell in Figure 16.19, which is homotopically equivalent to a $\mathbb{D}^4$. Two 3-cells are attached to it.

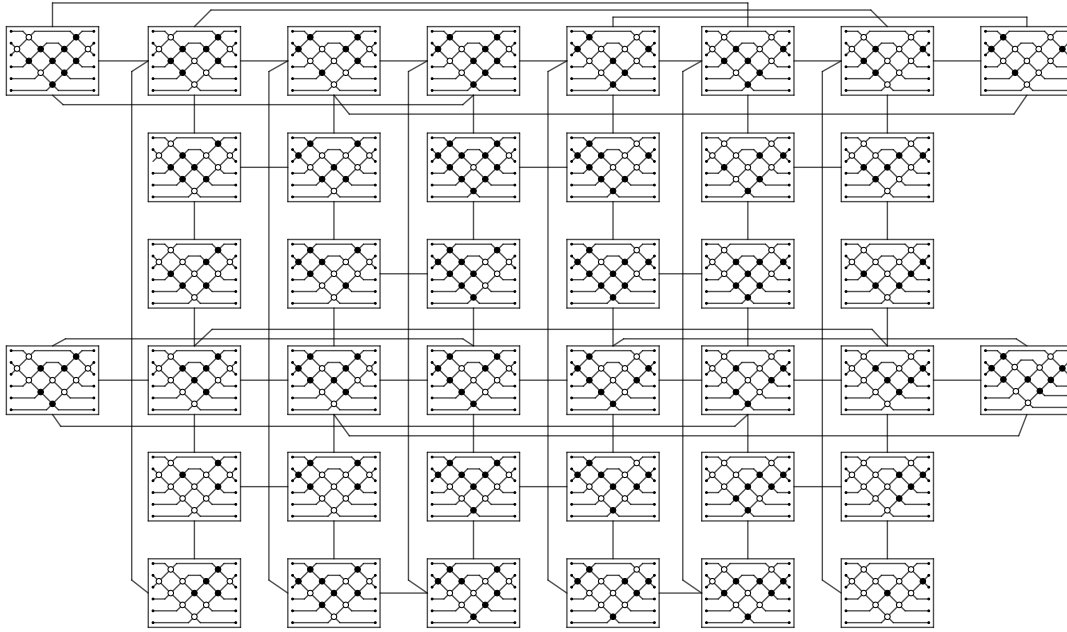


Figure 16.19: 4-cell with ancestry $\varepsilon_1 = (\blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$.

Step 2: Attach a 3-cell with ancestry $\varepsilon_2 = (\circ \blacklozenge \blacklozenge \blacklozenge \bullet \circ \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge)$ that fills the prism in Figure 16.20, with some cells of lower dimension attached. Attachment occurs through the 2-cell with ancestry $\varepsilon_3 = (\circ \circ \blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \bullet \blacklozenge \circ)$.

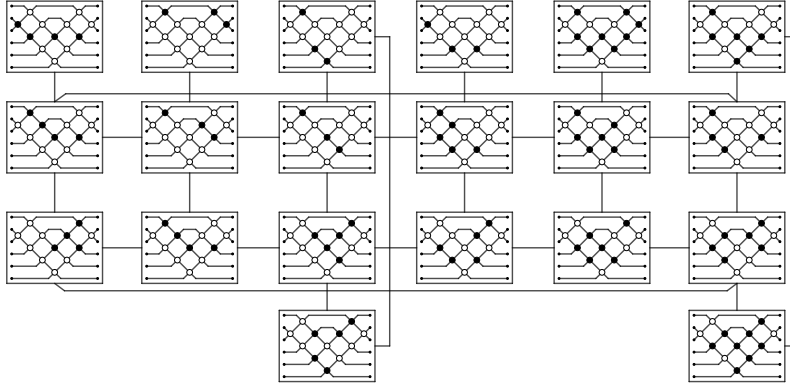


Figure 16.20: Second part of the CW complex.

Step 3: Attach a 3-cell with ancestry $\varepsilon_4 = (\bullet \blacklozenge \blacklozenge \blacklozenge \bullet \circ \diamond \diamond \bullet \bullet \diamond \bullet)$ that fills another prism in Figure 16.21 to Figure 16.19, with some cells of lower dimension attached. Attachment occurs through the 2-cell with ancestry $\varepsilon_5 = (\bullet \bullet \blacklozenge \blacklozenge \bullet \circ \diamond \circ \bullet \bullet \diamond \bullet)$.

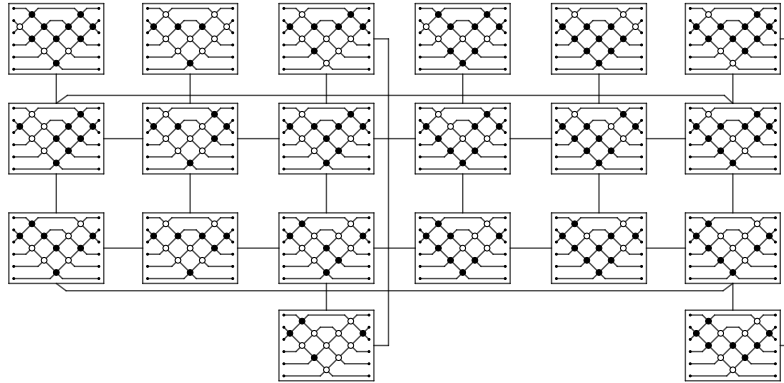


Figure 16.21: Third part of the CW complex.

Upon completing all the attachments, we have a contractible connected component. Therefore, BL_σ has 8 connected components, all of them contractible.

16.6

The Third Contractible Component of Dimension 4

If r_1 has opposite signs and the remaining rows have equal signs, the CW complex will be described below.

Step 1: First, we have a 4-cell that fills the CW complex in Figure 16.22, which is homotopically equivalent to a $\mathbb{D}^4$. Two 3-cells are attached to it.

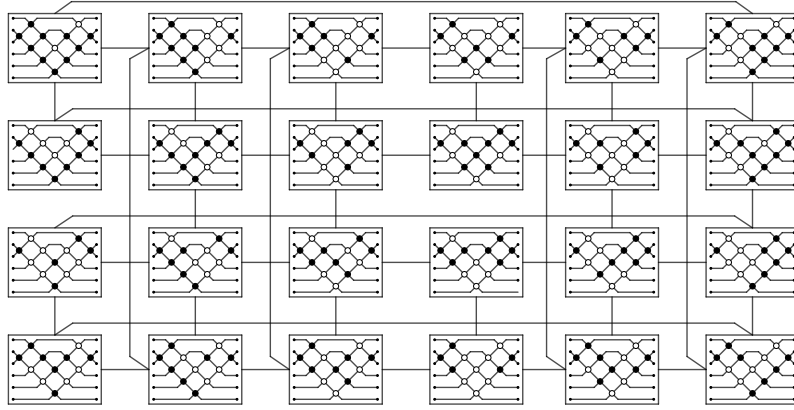


Figure 16.22: 4-cell with ancestry $\varepsilon_1 = (\bullet \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet)$.

Step 2: Attach a 3-cell that fills the “parallelepiped” in Figure 16.23, with some cells of lower dimension attached. The attachment to Figure 16.22 occurs through the 2-cell with ancestry $\varepsilon_2 = (\bullet \bullet \blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \circ \bullet \blacklozenge \bullet)$.

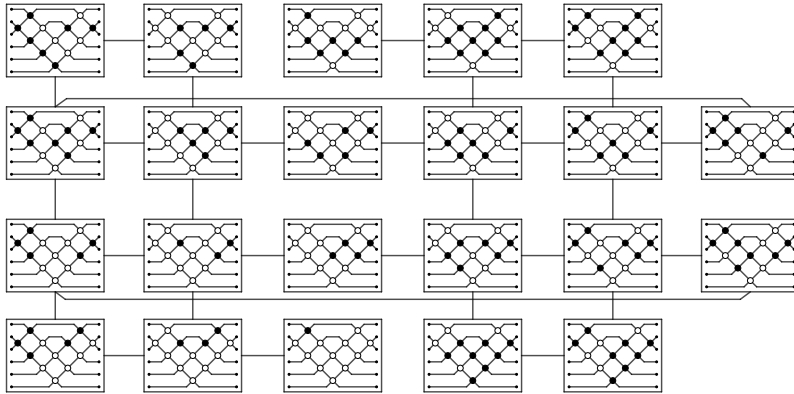


Figure 16.23: Second part of the CW complex with ancestry of dimension 3: $\varepsilon_3 = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge)$.

Step 3: Attach a 3-cell that fills another “parallelepiped” in Figure 16.24 to Figure 16.22, with some cells of lower dimension attached. Attachment occurs through the 2-cell with ancestry $\varepsilon_4 = (\bullet \circ \blacklozenge \blacklozenge \bullet \circ \blacklozenge \bullet \bullet \circ \blacklozenge \bullet)$.

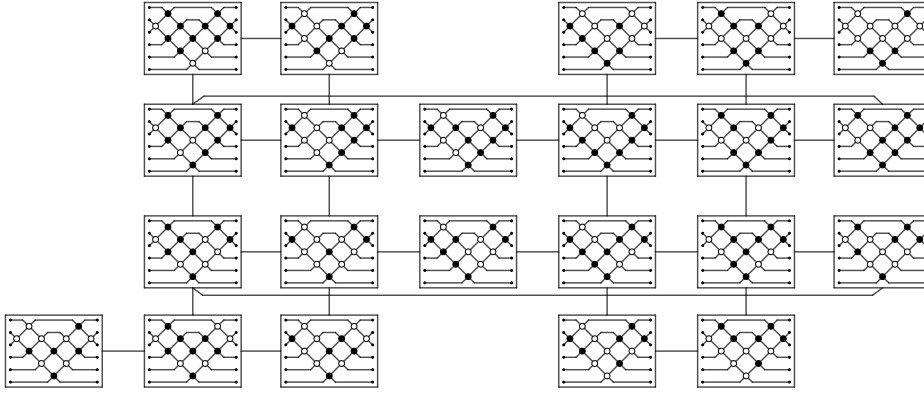


Figure 16.24: Third part of the CW complex with ancestry of dimension 3:
 $\varepsilon_5 = (\blacklozenge \circ \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \bullet)$.

Two more 3-cells are attached in the CW complex; however, they are attached to the previously 3-cells, not to the 4-cell.

Step 4: Attach a 3-cell that fills the “parallelepiped” in Figure 16.25 to the one in Figure 16.24, with some cells of lower dimension attached. Attachment occurs through the 2-cell with ancestry $\varepsilon_6 = (\bullet \circ \blacklozenge \blacklozenge \bullet \circ \blacklozenge \bullet \bullet \circ \blacklozenge \bullet)$.

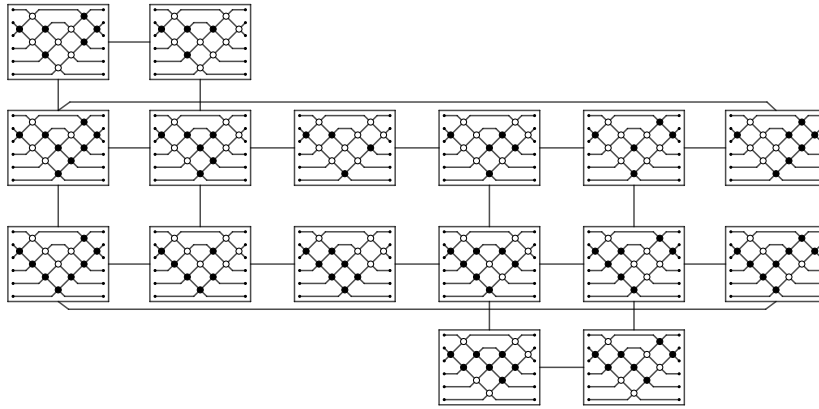


Figure 16.25: Fourth part of the CW complex with ancestry of dimension 3:
 $\varepsilon_7 = (\bullet \circ \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \blacklozenge)$.

Step 5: Attach a 3-cell with ancestry $\varepsilon_8 = (\bullet \bullet \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \circ \bullet \blacklozenge \blacklozenge)$ that fills the “parallelepiped” in Figure 16.26 to the one in Figure 16.23, with some cells of lower dimension attached. The attachment occurs through the 2-cell with ancestry $\varepsilon_9 = (\bullet \bullet \blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \circ \bullet \blacklozenge \bullet)$.

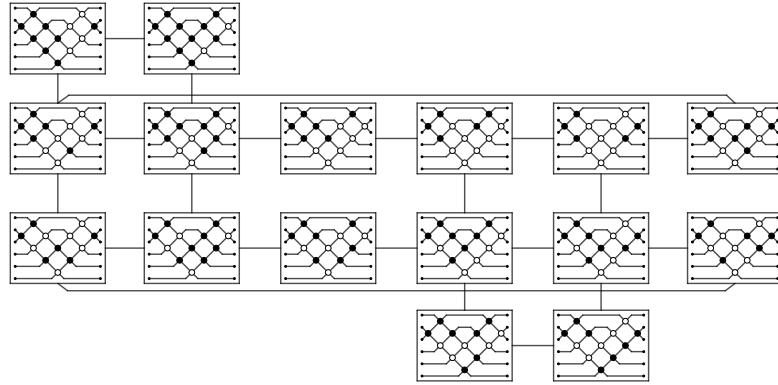


Figure 16.26: Fifth part of the CW complex.

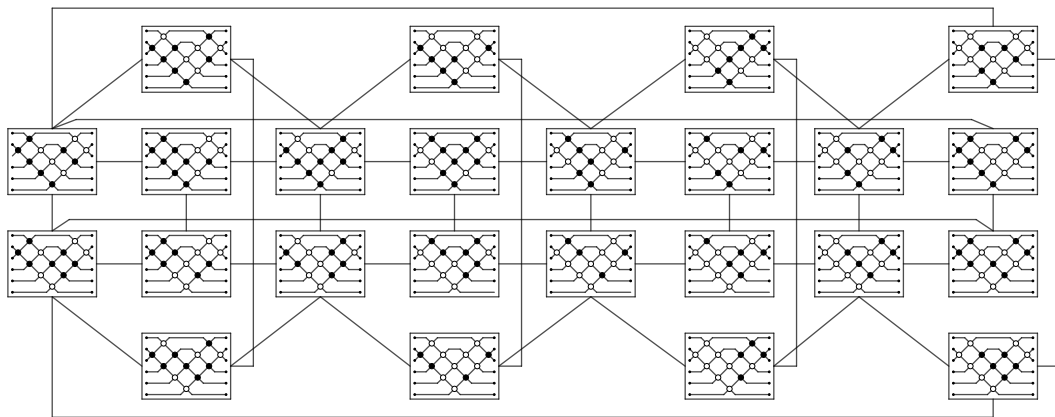
Upon completing all the attachments, we have a contractible connected component. Therefore, BL_σ has 8 connected components, all contractible.

16.7

The Fourth Contractible Component of Dimension 4

For dimension 4, if the only rows without diamonds are r_3 and r_4 , we obtain a CW complex with ten 3-cells. Let us proceed with a step by step construction.

Step 1: First, we have a 4-cell that fills Figure 16.27. This cell is a little bit confusing. As we have already seen, it also contains two tori that form a $\mathbb{S}^3$, to which a $\mathbb{D}^4$ is attached. Its composition consists horizontally of four cubes and vertically of two octagonal prisms and two filled spheres.

Figure 16.27: 4-cell with ancestry $\varepsilon_1 = (\diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond)$.

Step 2: Attach a 3-cell that fills the convex solid in Figure 16.28, with two 2-cells attached. The attachment to Figure 16.27 occurs through the 2-cell

with $\varepsilon_2 = (\blacklozenge \blacklozenge \bullet \bullet \circ \circ \circ \circ \blacklozenge \circ \bullet \bullet \blacklozenge)$.

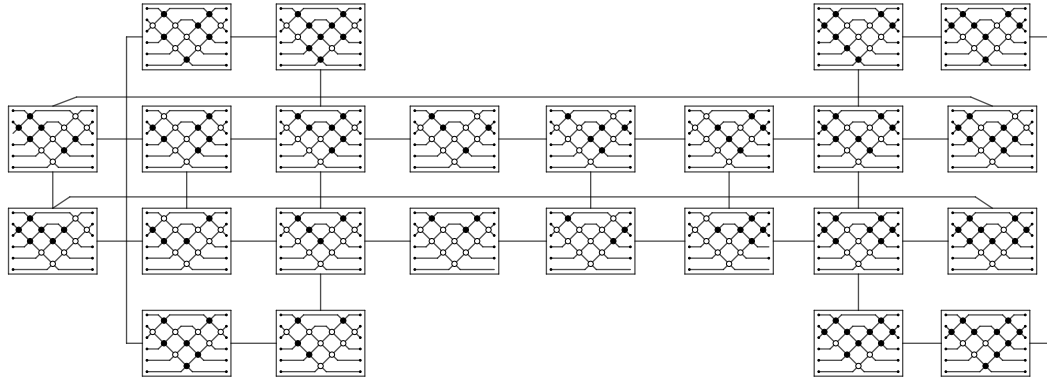


Figure 16.28: Second part of the CW complex with ancestry of dimension 3:
 $\varepsilon_3 = (\blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge \bullet \bullet \blacklozenge \circ \circ \blacklozenge \blacklozenge)$.

Step 3: Attach in Figure 16.27 a 3-cell that fills the convex solid, similar to the previous one, in Figure 16.29, with two 2-cells attached. Attachment occurs through the 2-cell with $\varepsilon_4 = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \circ \bullet \blacklozenge \bullet \bullet \blacklozenge)$.

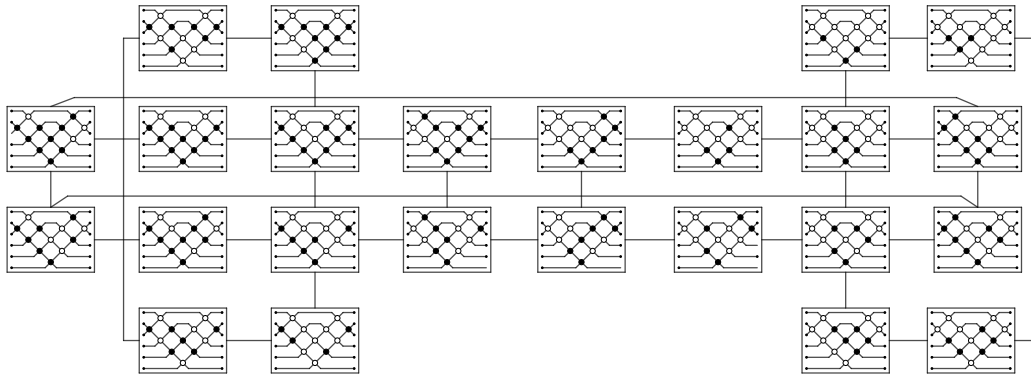
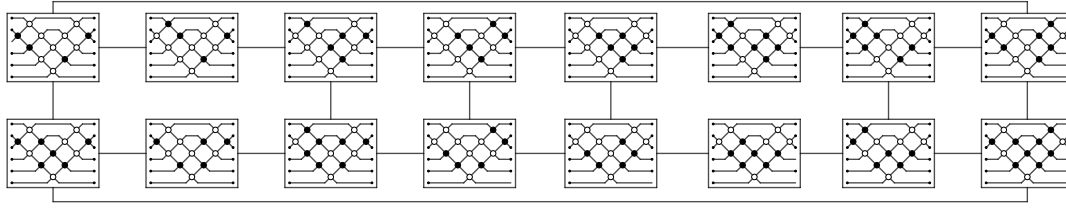
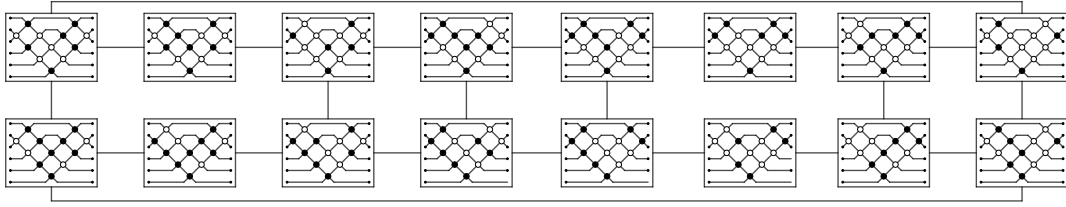


Figure 16.29: Third part of the CW complex with ancestry of dimension 3:
 $\varepsilon_5 = (\blacklozenge \blacklozenge \bullet \bullet \circ \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$.

Step 4: Attach in Figure 16.27 a 3-cell that fills the convex solid, in Figure 16.30. Attachment occurs through the 2-cell with ancestry $\varepsilon_6 = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \circ \circ \blacklozenge \circ \bullet \bullet \blacklozenge)$.

Figure 16.30: 3-cell with ancestry $\varepsilon_7 = (\blacklozenge \blacklozenge \blacklozenge \bullet \circ \diamond \circ \diamond \circ \bullet \bullet \diamond)$.

Step 5: Attach in Figure 16.27 a 3-cell that fills the convex solid, similar to the previous one, in Figure 16.31. Attachment occurs through the 2-cell with ancestry $\varepsilon_8 = (\blacklozenge \blacklozenge \bullet \bullet \circ \bullet \circ \bullet \diamond \bullet \circ \bullet \diamond)$.

Figure 16.31: 3-cell with ancestry $\varepsilon_9 = (\blacklozenge \blacklozenge \blacklozenge \bullet \bullet \diamond \bullet \diamond \bullet \bullet \bullet \diamond)$.

Upon completing all the attachments, we have a contractible connected component. Therefore, BL_σ has 8 connected components, all contractible.

16.8

The Fifth Contractible Component of Dimension 4

In dimension 1, if r_3 has diamonds in any possible position and the remaining rows have equal signs, we obtain a CW complex of dimension 4. The construction will be completed in 3 steps, that follows below.

Step 1: First, we attach ten 3-cells, as illustrated in Figure 16.32, together with additional cells of lower dimensions. For clarity, this construction is presented in two figures. The figure has been rotated for easier visualization.

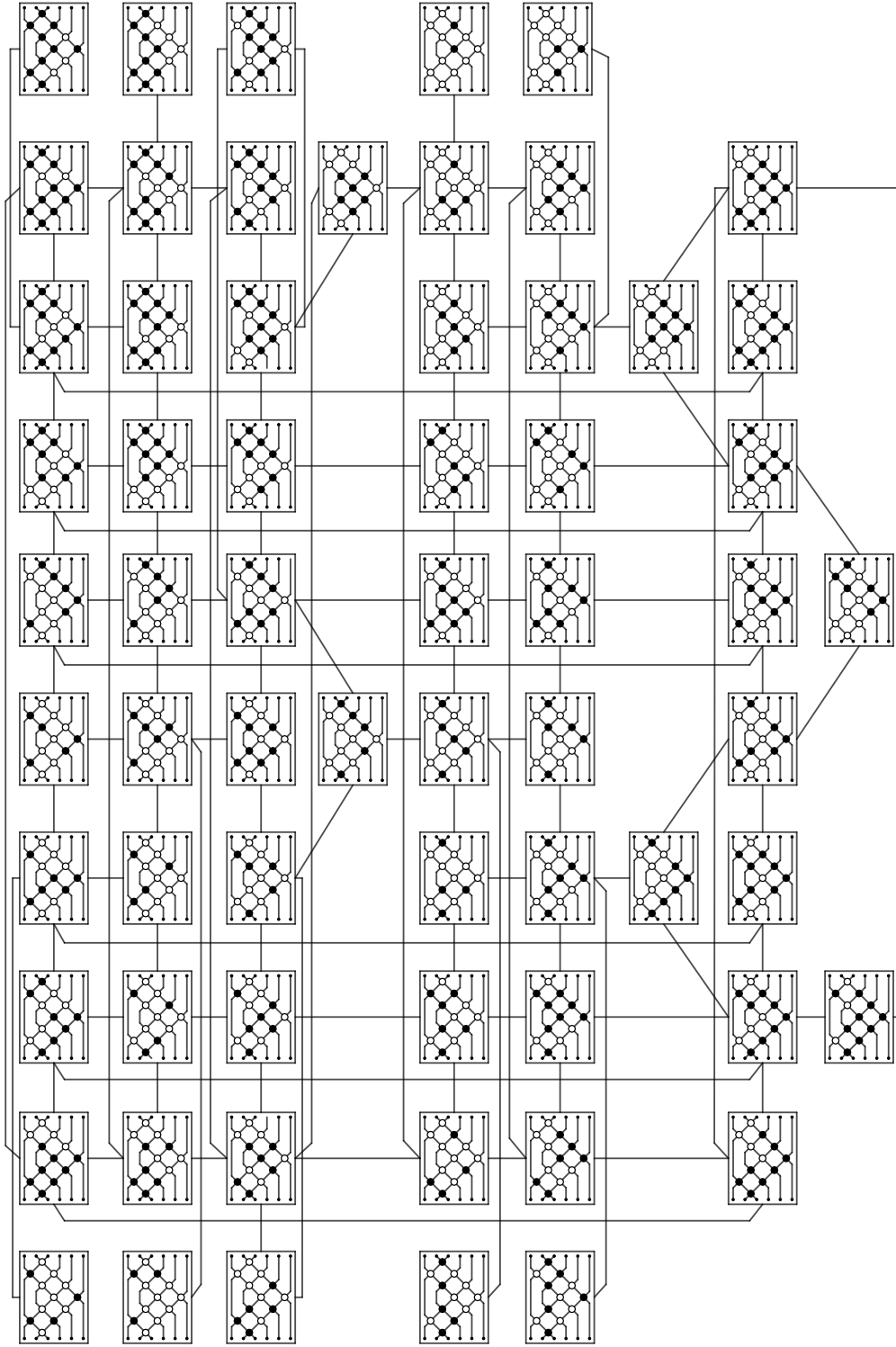
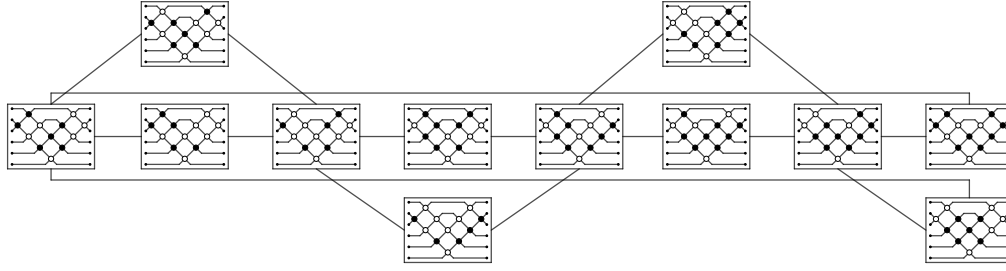
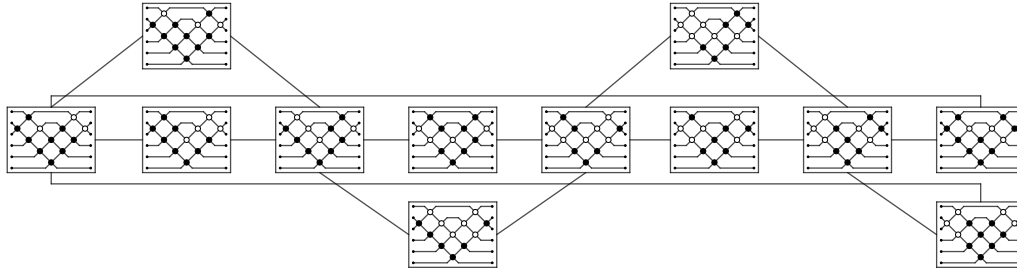


Figure 16.32: First part of the CW complex.

Step 2: Now, attach a 3-cell that fills the S^2 in Figure 16.33. Attachment occurs through the 2-cells with ancestry $\varepsilon_1 = (\blacklozenge \blacklozenge \circ \circ \bullet \bullet \bullet \diamond \circ \bullet \circ \diamond)$, $\varepsilon_2 = (\blacklozenge \circ \circ \diamond \bullet \circ \blacklozenge \bullet \circ \bullet \circ \diamond)$ and $\varepsilon_3 = (\blacklozenge \bullet \circ \circ \bullet \bullet \bullet \circ \circ \bullet \circ \diamond)$.

Figure 16.33: 3-cell with ancestry $\varepsilon_4 = (\blacklozenge \blacklozenge \circ \blacklozenge \bullet \circ \diamond \diamond \circ \bullet \circ \diamond)$.

Step 3: To finish, attach another 3-cell that fills the $\mathbb{S}^2$ in Figure 16.34. Attachment occurs through 2-cells with ancestry $\varepsilon_5 = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \circ \circ \diamond \bullet \bullet \bullet \diamond)$, $\varepsilon_6 = (\diamond \circ \bullet \diamond \bullet \circ \blacklozenge \bullet \bullet \bullet \bullet \diamond)$ and $\varepsilon_7 = (\blacklozenge \bullet \bullet \bullet \bullet \circ \circ \circ \bullet \bullet \bullet \diamond)$.

Figure 16.34: 3-cell with ancestry $\varepsilon_8 = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \circ \diamond \diamond \bullet \bullet \bullet \diamond)$.

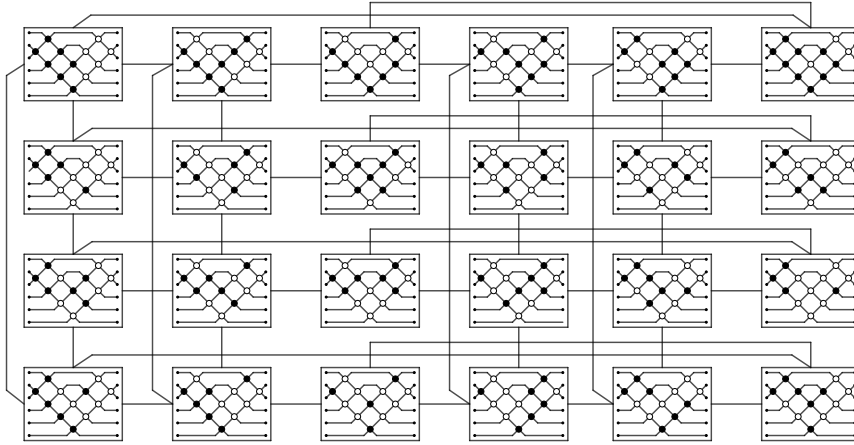
Upon completing all the attachments, we have a contractible connected component. Therefore, BL_σ has 8 connected components, all contractible.

16.9

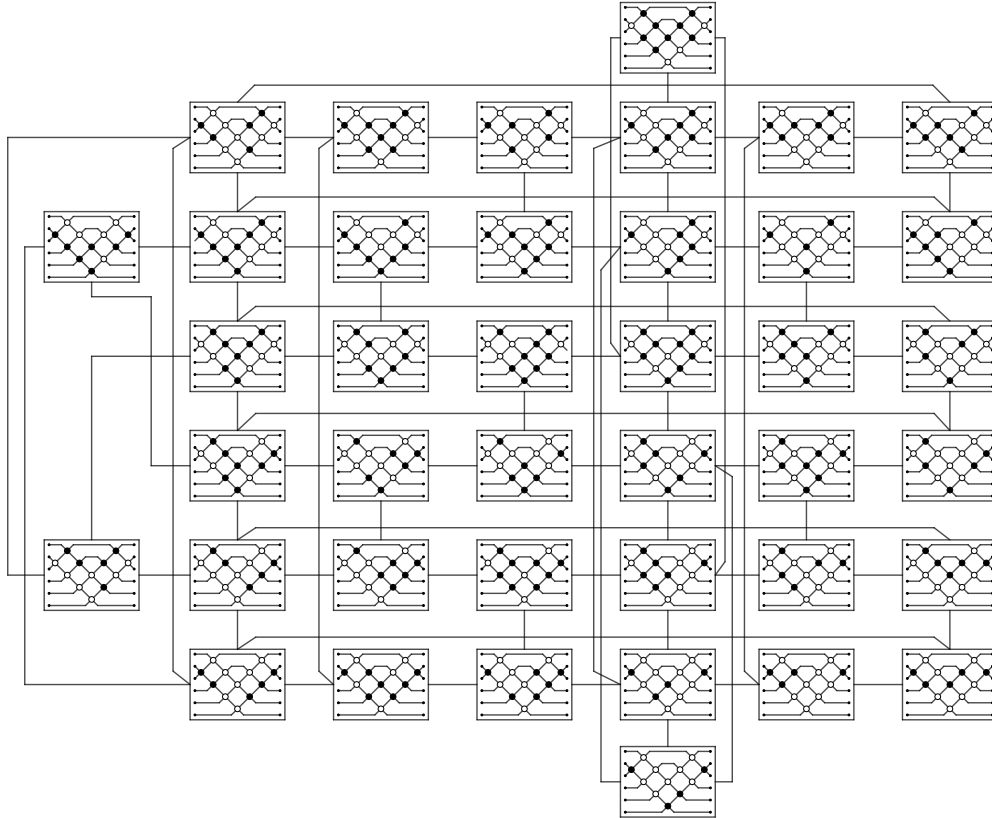
The Component of Dimension 5

For dimension 5, there is only one possible position for the diamonds. The CW complex has an intricate structure. We construct it step by step, attaching ten 4-cells; see the construction in the following.

Step 1: First, we have a 4-cell similar to the one in Figure 16.22.

Figure 16.35: 4-cell with ancestry $\varepsilon_1 = (\bullet \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge)$.

Step 2: Attach the 4-cell below to Figure 16.35, this cell is similar to the one described in Figure 16.19. Attachment occurs via the 3-cell with ancestry $\varepsilon_2 = (\bullet \circ \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ)$. This cell fills the “parallelepiped” which is the second 3-cell vertically, from left to right in Figure 16.35, and the first horizontally in Figure 16.36.

Figure 16.36: 4-cell with ancestry $\varepsilon_3 = (\blacklozenge \circ \blacklozenge \blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \blacklozenge)$.

Step 3: Attach the 4-cell below to Figure 16.36, this cell is similar to the one described in Figure 16.32. Attachment occurs via the 3-cell with ancestry $\varepsilon_4 = (\diamond \circ \diamond \circ \diamond \bullet \bullet \bullet \bullet \diamond \diamond \diamond)$. This cell fills the convex solid, which is the second vertical 3-cell, from left to right in Figure 16.36, and the last vertically in Figure 16.37.

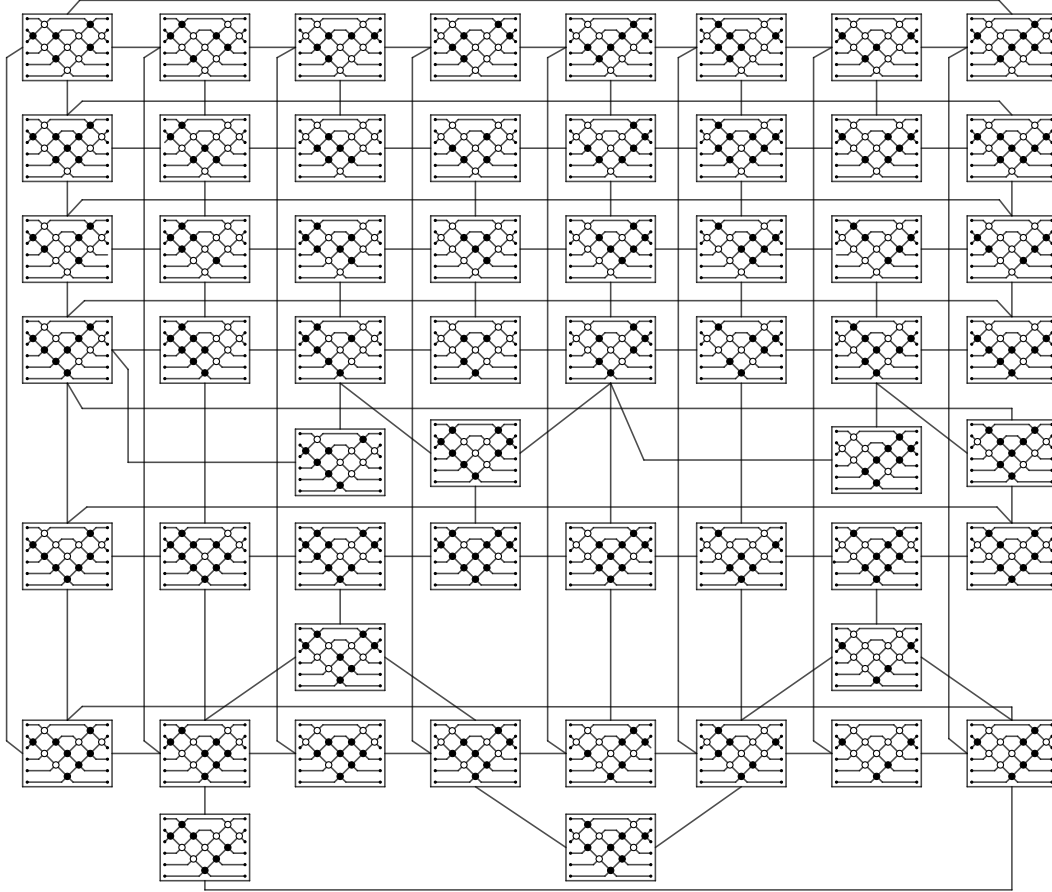
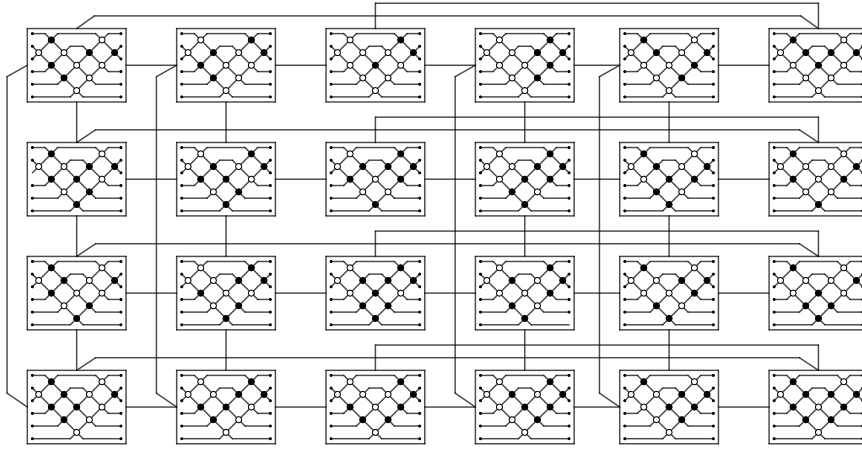
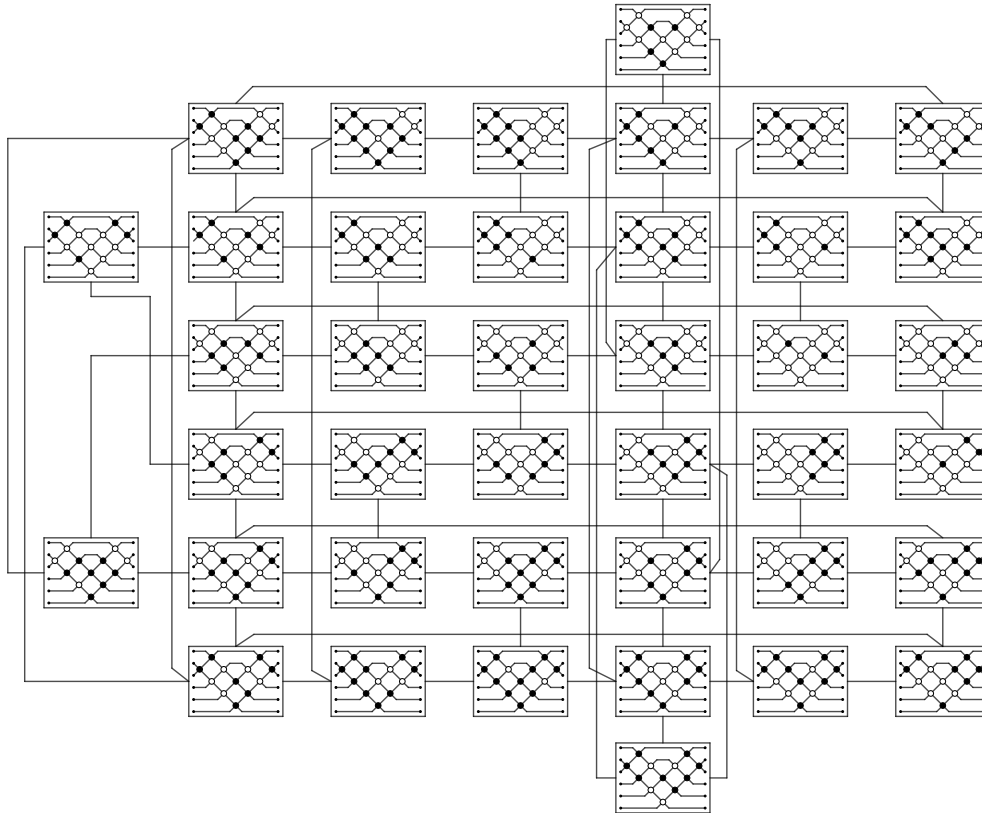


Figure 16.37: 4-cell with ancestry $\varepsilon_5 = (\diamond \diamond \diamond \bullet \bullet \bullet \bullet \circ \diamond \bullet \diamond \diamond \diamond)$.

Step 4: Now, attach the 4-cell in Figure 16.38, which is similar to the one in Figure 16.35, through the 3-cell with ancestry $\varepsilon_6 = (\circ \diamond \diamond \bullet \bullet \bullet \bullet \circ \diamond \circ \diamond \bullet)$. The cell fills the prism at the bottom of Figure 16.38 and the third vertically in Figure 16.37.

Figure 16.38: 4-cell with ancestry $\varepsilon_7 = (\diamond \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \diamond \diamond \diamond \diamond \diamond)$.

Step 5: Attach the 4-cell below to Figure 16.38, this cell is similar to the one described in Figure 16.36. Attachment occurs via the 3-cell with ancestry $\varepsilon_8 = (\diamond \diamond \blacklozenge \blacklozenge \blacklozenge \diamond \blacklozenge \blacklozenge \diamond \diamond \blacklozenge)$, that fills the “parallelepiped” which is the second 3-cell vertically, from left to right in Figure 16.38, and the fourth horizontally in Figure 16.39.

Figure 16.39: 4-cell with ancestry $\varepsilon_9 = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \blacklozenge \diamond \diamond \diamond \diamond \diamond)$.

Step 6: Attach the 4-cell below to Figure 16.39, this cell is similar to the one described in Figure 16.37. Attachment occurs via the 3-cell with ancestry $\varepsilon_{10} = (\blacklozenge \blacklozenge \circ \blacklozenge \circ \bullet \circ \bullet \blacklozenge \blacklozenge)$, which fills the convex solid, which is the fourth 3-cell vertically, from left to right in Figure 16.39, and the last vertically in Figure 16.40. The cell also attaches to Figure 16.35 through $\varepsilon_{11} = (\bullet \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \bullet \blacklozenge \circ)$, which is the prism at the bottom of Figure 16.35.

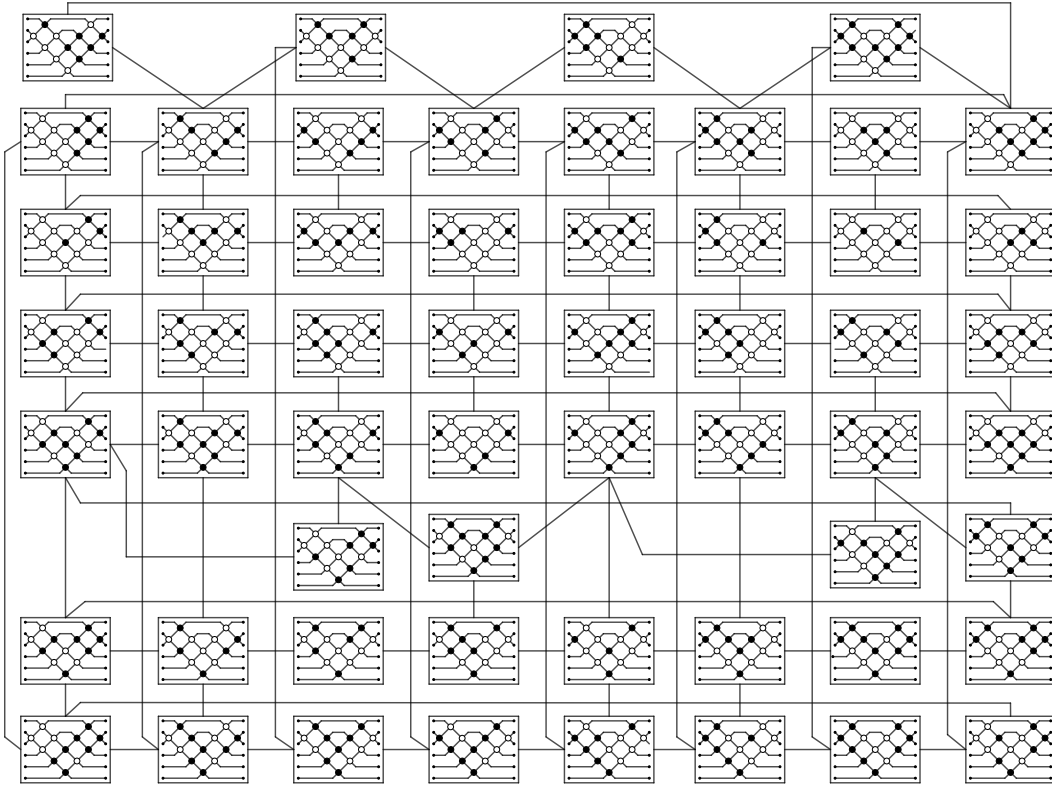


Figure 16.40: 4-cell with ancestry $\varepsilon_{12} = (\blacklozenge \blacklozenge \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \bullet \blacklozenge \blacklozenge)$.

Following these steps, after attaching these six 4-cells, we have a homotopically equivalent structure to $\mathbb{D}^3 \times \mathbb{S}^1$, a 4-dimensional solid torus.

Next, we perform a similar construction with the other four 4-cells.

Step 7: First, we have the 4-cell below to Figure 16.37, this cell is similar to the one described in Figure 16.14.

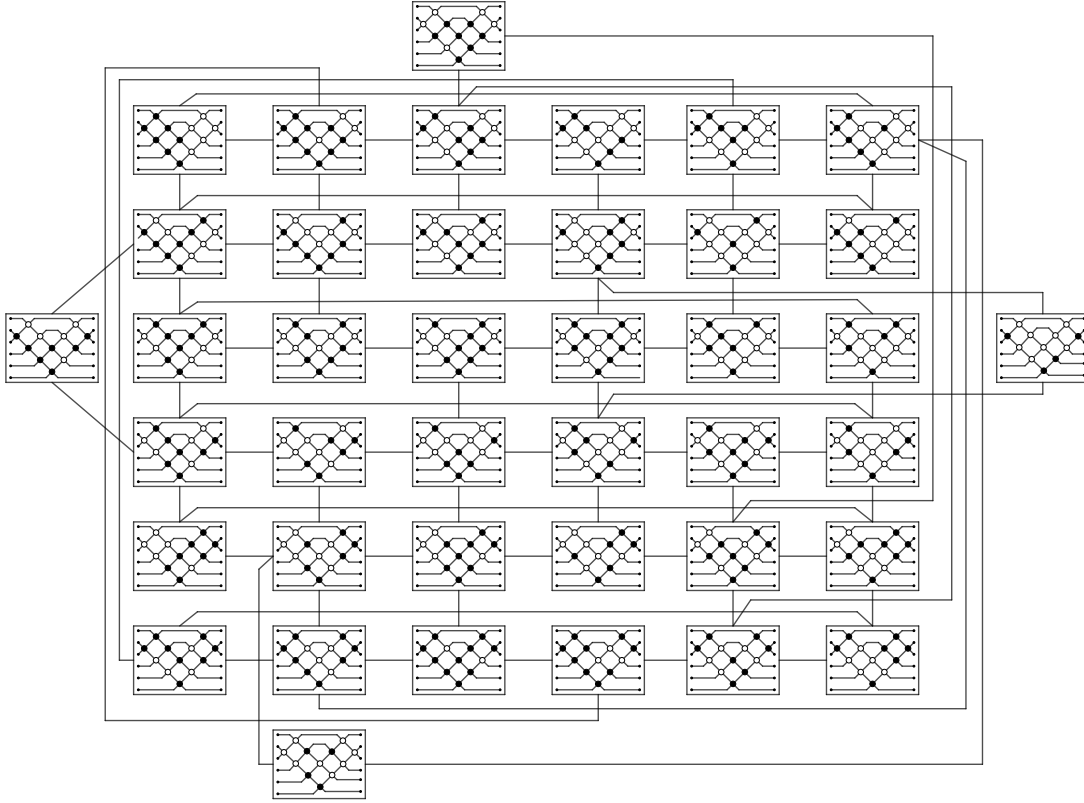


Figure 16.41: 4-cell with ancestry $\varepsilon_{13} = (\blacklozenge \blacklozenge \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \circ \blacklozenge \blacklozenge)$.

Step 8: Attach the 4-cell below, this cell is similar to the one described in Figure 16.27. Attachment occurs via the 3-cell with ancestry $\varepsilon_{14} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \circ \blacklozenge \blacklozenge)$. This cell fills the $\mathbb{S}^2$ which is the last 3-cell horizontally, from left to right in Figure 16.42.

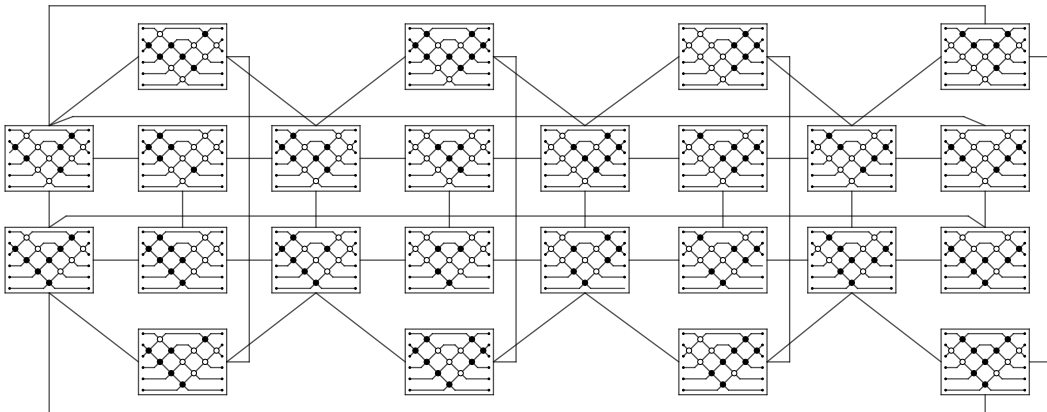


Figure 16.42: 4-cell with ancestry $\varepsilon_{15} = (\blacklozenge \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \circ \blacklozenge \blacklozenge)$.

Step 9: Attach the 4-cell below to Figure 16.43, this cell is similar to the one described in Figure 16.41. Attachment occurs via the 3-cell with ancestry

$\varepsilon_{16} = (\blacklozenge \blacklozenge \bullet \bullet \circ \circ \diamond \diamond \circ \bullet \diamond \diamond)$. This cell fills the $\mathbb{S}^2$, which is a horizontal 3-cell in Figure 16.42.

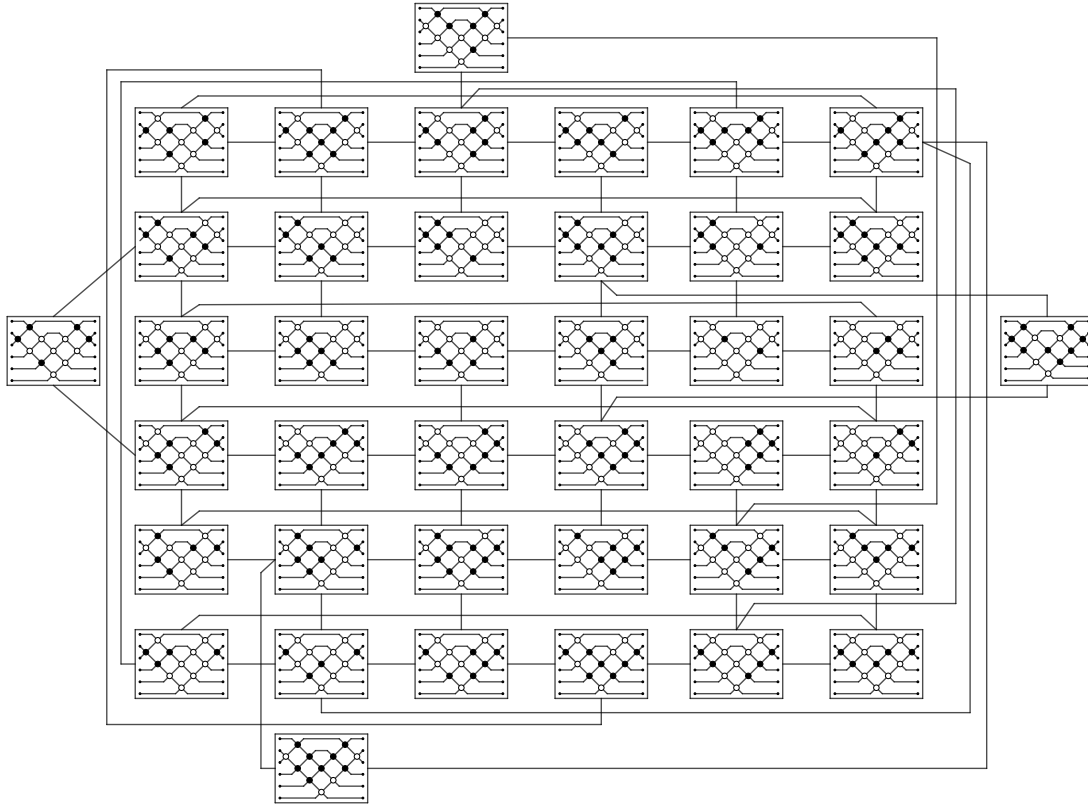


Figure 16.43: 4-cell with ancestry $\varepsilon_{17} = (\blacklozenge \blacklozenge \bullet \bullet \circ \circ \diamond \diamond \circ \bullet \diamond \diamond)$.

Step 10: Attach the 4-cell below to Figure 16.43, this cell is similar to the one described in Figure 16.42. Attachment occurs via the 3-cell with ancestry $\varepsilon_{18} = (\blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \diamond \diamond \circ \circ \bullet \diamond)$. The cell fills the $\mathbb{S}^2$, which is a horizontal 3-cell in Figure 16.44. This cell also attaches to Figure 16.41 through $\varepsilon_{19} = (\blacklozenge \blacklozenge \circ \blacklozenge \circ \circ \diamond \diamond \circ \bullet \bullet \diamond)$, which is a horizontal 3-cell in Figure 16.44 that fills another $\mathbb{S}^2$.

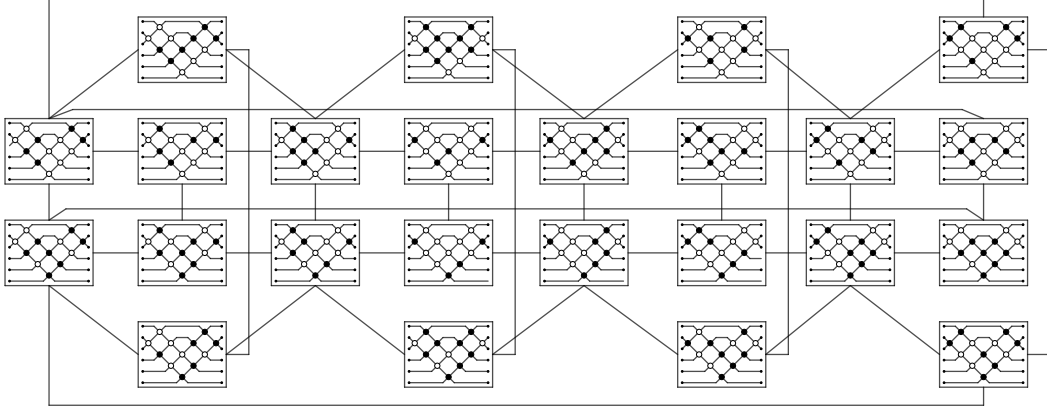


Figure 16.44: 4-cell with ancestry $\varepsilon_{20} = (\blacklozenge \blacklozenge \circ \blacklozenge \bullet \blacklozenge \diamond \diamond \circ \diamond \bullet \diamond)$.

After attaching these four 4-cells, we have another homotopically equivalent structure to $\mathbb{D}^3 \times \mathbb{S}^1$, a 4-dimensional solid torus.

In addition to the attachments described above, the ten 4-cells also connect to several others. In the following, we present the list of attachments:

1. Figure 16.35 attaches to:

- (a) Figure 16.37 through $\varepsilon_{21} = (\bullet \blacklozenge \blacklozenge \bullet \blacklozenge \bullet \circ \diamond \bullet \bullet \diamond \diamond \circ)$;
- (b) Figure 16.39 through $\varepsilon_{22} = (\bullet \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \diamond \circ \bullet \diamond \diamond \circ)$;
- (c) Figure 16.41 through $\varepsilon_{23} = (\bullet \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \diamond \diamond \bullet \circ \diamond \circ)$;
- (d) Figure 16.42 through $\varepsilon_{24} = (\bullet \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \diamond \diamond \bullet \diamond \circ \circ)$;
- (e) Figure 16.43 through $\varepsilon_{25} = (\bullet \blacklozenge \blacklozenge \blacklozenge \circ \circ \diamond \diamond \circ \bullet \diamond \circ)$;
- (f) Figure 16.44 through $\varepsilon_{26} = (\bullet \blacklozenge \circ \blacklozenge \blacklozenge \bullet \diamond \diamond \circ \diamond \bullet \circ)$.

2. Figure 16.36 attaches to:

- (a) Figure 16.38 through $\varepsilon_{27} = (\circ \bullet \blacklozenge \blacklozenge \blacklozenge \bullet \diamond \circ \circ \diamond \diamond \bullet)$;
- (b) Figure 16.40 through $\varepsilon_{28} = (\blacklozenge \circ \blacklozenge \bullet \blacklozenge \circ \circ \bullet \bullet \diamond \diamond \diamond)$;
- (c) Figure 16.41 through $\varepsilon_{29} = (\blacklozenge \circ \blacklozenge \bullet \bullet \diamond \diamond \bullet \bullet \circ \diamond \circ)$
and $\varepsilon_{30} = (\circ \bullet \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \circ \diamond \diamond)$;
- (d) Figure 16.42 through $\varepsilon_{31} = (\blacklozenge \circ \bullet \diamond \blacklozenge \bullet \blacklozenge \bullet \bullet \diamond \circ \diamond)$;
- (e) Figure 16.43 through $\varepsilon_{32} = (\blacklozenge \circ \blacklozenge \blacklozenge \circ \diamond \diamond \circ \circ \circ \diamond \bullet)$
and $\varepsilon_{33} = (\bullet \circ \blacklozenge \blacklozenge \circ \blacklozenge \bullet \circ \bullet \diamond \diamond)$;
- (f) Figure 16.44 through $\varepsilon_{34} = (\blacklozenge \circ \circ \diamond \blacklozenge \bullet \blacklozenge \bullet \circ \diamond \bullet \diamond)$.

3. Figure 16.37 attaches to:

- (a) Figure 16.39 through $\varepsilon_{35} = (\blacklozenge \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \circ \bullet \blacklozenge \blacklozenge \blacklozenge)$;
- (b) Figure 16.41 through $\varepsilon_{36} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \circ \blacklozenge \bullet \circ \blacklozenge \blacklozenge)$
and $\varepsilon_{37} = (\blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$;
- (c) Figure 16.42 through $\varepsilon_{38} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \circ \bullet \blacklozenge \blacklozenge)$
and $\varepsilon_{39} = (\blacklozenge \blacklozenge \bullet \bullet \bullet \bullet \bullet \circ \blacklozenge \bullet \blacklozenge \blacklozenge \blacklozenge)$;
- (d) Figure 16.43 through $\varepsilon_{40} = (\blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \blacklozenge)$
and $\varepsilon_{41} = (\blacklozenge \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge \blacklozenge)$;
- (e) Figure 16.44 through $\varepsilon_{42} = (\blacklozenge \blacklozenge \circ \blacklozenge \circ \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge)$
and $\varepsilon_{43} = (\blacklozenge \blacklozenge \circ \circ \blacklozenge \circ \bullet \blacklozenge \circ \blacklozenge \bullet \blacklozenge)$.

4. Figure 16.38 attaches to:

- (a) Figure 16.40 through $\varepsilon_{44} = (\circ \blacklozenge \bullet \blacklozenge \bullet \bullet \circ \bullet \blacklozenge \circ \blacklozenge \blacklozenge \bullet)$;
- (b) Figure 16.41 through $\varepsilon_{45} = (\circ \blacklozenge \blacklozenge \blacklozenge \circ \circ \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet)$;
- (c) Figure 16.42 through $\varepsilon_{46} = (\circ \blacklozenge \circ \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet)$;
- (d) Figure 16.43 through $\varepsilon_{47} = (\circ \blacklozenge \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ \circ \blacklozenge \bullet)$;
- (e) Figure 16.44 through $\varepsilon_{48} = (\circ \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \blacklozenge \blacklozenge \circ \blacklozenge \circ \bullet)$.

5. Figure 16.39 attaches to:

- (a) Figure 16.41 through $\varepsilon_{49} = (\blacklozenge \bullet \blacklozenge \bullet \bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge \bullet)$
and $\varepsilon_{50} = (\bullet \bullet \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \circ \bullet \circ \blacklozenge \blacklozenge)$;
- (b) Figure 16.42 through $\varepsilon_{51} = (\blacklozenge \bullet \bullet \blacklozenge \blacklozenge \circ \blacklozenge \bullet \bullet \blacklozenge \circ \blacklozenge)$;
- (c) Figure 16.43 through $\varepsilon_{52} = (\blacklozenge \bullet \bullet \blacklozenge \circ \blacklozenge \blacklozenge \circ \circ \bullet \blacklozenge \circ)$
and $\varepsilon_{53} = (\circ \circ \blacklozenge \blacklozenge \bullet \bullet \blacklozenge \bullet \circ \bullet \blacklozenge \blacklozenge)$;
- (d) Figure 16.44 through $\varepsilon_{54} = (\blacklozenge \bullet \circ \blacklozenge \blacklozenge \circ \blacklozenge \circ \circ \blacklozenge \bullet \blacklozenge)$.

6. Figure 16.40 attaches to:

- (a) Figure 16.41 through $\varepsilon_{55} = (\blacklozenge \blacklozenge \circ \circ \circ \blacklozenge \bullet \blacklozenge \bullet \bullet \circ \blacklozenge)$
and $\varepsilon_{56} = (\blacklozenge \blacklozenge \blacklozenge \circ \bullet \bullet \blacklozenge \bullet \blacklozenge \bullet \circ \blacklozenge \blacklozenge)$;
- (b) Figure 16.42 through $\varepsilon_{57} = (\blacklozenge \blacklozenge \bullet \circ \blacklozenge \circ \bullet \blacklozenge \bullet \blacklozenge \circ \blacklozenge)$
and $\varepsilon_{58} = (\blacklozenge \blacklozenge \bullet \blacklozenge \circ \circ \blacklozenge \blacklozenge \circ \bullet \blacklozenge \blacklozenge)$;
- (c) Figure 16.43 through $\varepsilon_{59} = (\blacklozenge \blacklozenge \bullet \circ \circ \blacklozenge \bullet \blacklozenge \blacklozenge \bullet \bullet \circ \blacklozenge)$
and $\varepsilon_{60} = (\blacklozenge \blacklozenge \blacklozenge \circ \circ \blacklozenge \blacklozenge \circ \blacklozenge \circ \bullet \blacklozenge \blacklozenge)$;
- (d) Figure 16.44 through $\varepsilon_{61} = (\blacklozenge \blacklozenge \circ \bullet \blacklozenge \bullet \bullet \circ \blacklozenge \circ \blacklozenge \circ \blacklozenge)$
and $\varepsilon_{62} = (\blacklozenge \blacklozenge \circ \blacklozenge \circ \circ \blacklozenge \blacklozenge \bullet \bullet \bullet \blacklozenge)$.

The structure of the attachment of the two 4-dimensional tori is highly complex and challenging to describe precisely with our currently tools. Therefore, we can only conclude that BL_σ has 4 connected components of this type, with Euler characteristic equal to 1.

Furthermore, there exist $2^5 = 32$ thin ancestries. Consequently, BL_σ has 32 thin connected components, all contractible. Summing up, BL_σ has a total of 100 connected components, distributed as follows:

- (i) 72 are contractible;
- (ii) 4 are inconclusive (for the moment), with Euler characteristic 1;
- (iii) 24 are homotopically equivalent to a $\mathbb{S}^1$.

After completing the analysis of all the components of BL_σ for each $\sigma \in \text{S}_6$ with $\text{inv}(\sigma) \leq 12$, we arrive at our main result:

Theorem 3. *Consider $\sigma \in \text{S}_6$ and $\text{BL}_\sigma \subset \text{Lo}_6^1$.*

1. *For $\text{inv}(\sigma) \leq 11$, every component of every set BL_σ is contractible;*
2. *For $\text{inv}(\sigma) = 12$, except for $\sigma = [563412]$, every component of every set BL_σ is contractible;*
3. *For $\sigma = [563412]$, the set BL_z has*
 - (a) *8 values of $z \in \sigma \text{Quat}_6$ where there are five contractible connected components: 4 thin and 1 thick;*
 - (b) *32 values of $z \in \sigma \text{Quat}_6$ where there are a single contractible connected component;*
 - (c) *4 values of $z \in \sigma \text{Quat}_6$ where there are two connected components homotopically equivalent to $\mathbb{S}^1$;*
 - (d) *16 values of $z \in \sigma \text{Quat}_6$ where there are a single connected component homotopically equivalent to $\mathbb{S}^1$;*
 - (e) *4 values of $z \in \sigma \text{Quat}_6$ where there are a single inconclusive connected component, with Euler characteristic equal to 1.*

17

Some Information About BL_σ for $\text{inv}(\sigma) \geq 13$

For permutations $\sigma \in S_6$ with $\text{inv}(\sigma) \geq 13$, the difficulty increases significantly. While we are not able to determine the homotopy type of the components, we do have information about the orbits and the Euler characteristics of their components.

The maximum dimension of the preancestries for σ with $\text{inv}(\sigma) = 13$ or 14 is 5, and for $\text{inv}(\sigma) = 15$ it reaches 6. This significantly complicates the visualization of the components and, more importantly, makes interpreting these drawings increasingly challenging and uncertain.

17.1

Some Information About BL_σ for $\text{inv}(\sigma) = 13$

There are 14 permutations $\sigma \in S_6$ with $\text{inv}(\sigma) = 13$. Let us present some important information regarding these permutations.

1. There are 6 permutations with 2 orbits, each containing 32 elements. For these permutations, the Euler characteristic of the components is 1, indicating that they are possibly contractible.
2. There are 8 permutations with five orbits: three containing 16 elements and two with 8 elements.
 - (a) Four permutations have four orbits whose components have Euler characteristics equal to 1, suggesting potential contractibility, while one orbit has a component with an Euler characteristic equal to 0, suggesting a nontrivial homotopy type. Although we have drawn the CW complexes for these components, which suggest they are homotopically equivalent to S^1 , the complexity of these drawings makes them difficult to present in full detail at this time.
 - (b) Four permutations exhibit three orbits with components that have Euler characteristics equal to 1, indicating potential contractibility. Two orbits, however, show components with Euler characteristics equal to 2. For these, we have drawn the CW complexes and found that, in one orbit, there are two copies of a contractible CW complex

consisting of 56 0-cells, 96 1-cells, 46 2-cells, and 5 3-cells. In the other orbit, there are two distinct components: one with 16 0-cells, 16 1-cells, and 1 2-cell, which is contractible, and another with 128 0-cells, 240 1-cells, 175 2-cells, 52 3-cells, and 6 4-cells. This latter component has an Euler characteristic of 1, suggesting a potentially trivial homotopy type.

17.2

Some Information About BL_σ for $\text{inv}(\sigma) = 14$

There are 5 permutations $\sigma \in S_6$ with $\text{inv}(\sigma) = 14$.

1. Four of these permutations have three orbits: two with 16 elements and one with 32 elements. The Euler characteristic of the components for these permutations is 1, suggesting that they are potentially contractible.
2. There is one permutation with nine orbits: six of these have components with an Euler characteristic of 1, indicating a potentially trivial homotopy type. Two orbits have components with an Euler characteristic of 0, and, based on the distribution of the ancestries, we hypothesize that these components are homotopically equivalent to $\mathbb{S}^1$. Specifically, these components consist of 256 0-cells, 576 1-cells, 416 2-cells, 100 3-cells, and 4 4-cells. Unfortunately, representing these components graphically exceeds the capabilities of our current tools. The remaining orbit has an Euler characteristic of 2, for these components, we have drawn the CW complexes and observed that they disconnect, resulting in two copies of a contractible CW complex. The component has 112 0-cells, 216 1-cells 128 2-cells, 24 3-cells and 1 4-cell. The complexity of these drawings makes them difficult to present in full detail at this time.

17.3

Some Information About BL_σ for $\text{inv}(\sigma) = 15$

There is only one permutation $\eta \in S_6$ with $\text{inv}(\sigma) = 15$, known as the top permutation. Its five orbits are separated into two with 8 elements and three with 16 elements. Furthermore, by Proposition 6.1.1, $BL_{z, \text{thick}}$ is nonempty and connected.

From Chapter 15 in [1] we already know that there exists a noncontractible component of BL_η , with Euler characteristic equal to 2 and 480 0-cells, 1120 1-cells, 864 2-cells, 228 3-cells and 6 4-cells. The homotopy type of this component remains unknown; additional techniques will be necessary to solve this question.

The remaining four orbits have components with Euler characteristic equal to 1, suggesting that they are potentially contractible. One of these orbits, the one with a positive real part, includes one 6-dimensional cell.

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