Pontifícia Universidade Católica
do Rio de Janeiro

Guilherme Brandão Guglielmo

Path Connectivity of Anosov Metrics on Surfaces

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática, do Departamento de Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Rafael Oswaldo Ruggiero Rodriguez

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Dedicated to my beloved wife, Carla.

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Abstract

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We are interested in the investigation paths of conformal deformations of a metric defined on a compact surface, aiming to study the connectedness of the set of metrics without conjugate points.

It is known that the set of Anosov metrics, in the C^2 topology, is in the interior of the set of metrics without conjugate points. But it is not known if this set is connected or contractible. Hamilton showed, using the Ricci flow, that given any metric on a compact surface of genus greater than 1, there exists a differentiable curve of metrics that starts at the given metric and ends at a metric with negative curvature. However, it is not known whether, when the initial metric has no conjugate points, this property is preserved along the curve.

Our study has two main objectives. The first is to present a family of compact surfaces of genus greater than 1 that, despite having a finite number of simply connected regions that admit positive curvature, do not present focal points, and whose metrics are Anosov. The second goal is to demonstrate that this family contains a subfamily whose metrics can be continuously deformed through Anosov metrics without focal points until reaching a metric of negative curvature.

Keywords

Conformal deformation of metrics; Anosov's metric; Free focal point metric; Geodesic flow; Path connectivity.

Resumo

Guglielmo, Guilherme Brandão; Ruggiero Rodriguez, Rafael Oswaldo. **Conexidade por Arcos de Métricas de Anosov em Superfícies**. Rio de Janeiro, 2025. 77p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Estamos interessados na investigação de caminhos de deformações conformes de uma métrica definida em uma superfície compacta, visando o estudo da conectividade do conjunto de métricas sem pontos conjugados.

Sabe-se que o conjunto das métricas de Anosov, na topologia C^2, encontra-se no interior do conjunto das métricas sem pontos conjugados. Porém não é conhecido se este conjunto é conexo ou contrátil. Hamilton mostrou, usando o fluxo de Ricci, que dada qualquer métrica em uma superfície compacta de gênero maior que 1, existe uma curva diferenciável de métricas que começa na métrica e termina em uma métrica com curvatura negativa. No entanto, não se sabe se, quando a métrica inicial não possui pontos conjugados, esta propriedade é preservada ao longo da curva.

Nosso estudo tem dois objetivos principais. O primeiro é apresentar uma família de superfícies compactas de gênero maior que 1 que, apesar de possuírem um número finito de regiões simplesmente conexas que admitem curvatura positiva, não apresentam pontos focais, e cujas métricas são Anosov. A segunda meta é demonstrar que esta família contém uma subfamília de superfícies cuja a métrica pode ser deformada continuamente em métricas de Anosov sem pontos focais até alcançar uma métrica de curvatura negativa.

Palavras-chave

Deformação Conforme de Métricas; Métrica de Anosov; Métrica sem pontos focais; Fluxo Geodésico; Conexidade por Arco.

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1
Introduction

The theory of Anosov geodesic flows is well known and studied in the context of stable dynamics. Metrics whose geodesic flow is Anosov have relevant geometric, dynamic and ergodic properties.

The interplay between geometry and dynamics has been very important for the development of the theory of Anosov geodesic flows.

Morse, in [21], studied the global geometry of geodesics in the universal cover of compact surfaces of genus greater than one. He showed that in any metric, a minimizing geodesic in the universal cover is shadowed by a hyperbolic geodesic.

In the decades following Morse's paper [21], much work was done on geodesic flow on compact surfaces with negative curvature, as in [17] or [15] papers by Hopf and Hedlund respectively. On compact surfaces with negative curvature, Hopf proved that geodesic flow is ergodic and Hedlund proved that geodesic flow has topological transitivity.

Many geometric results were important for the theory. Hopf [18] in 1948 proved that on surfaces without conjugate points the integral of the curvature over the surface (total curvature) is negative or zero and, if it is zero, the curvature is zero everywhere on the surface. Green [11] in 1952 studied the behavior of manifolds without conjugate points and proved the existence, in these cases, of Jacobi fields that never vanish. These fields were later known as Green's Jacobi fields.

Then in 1967, Anosov [1] (or in the English version in [2]) formally defined the concept of $U-$flows, which later became known as Anosov Flows, and showed that the geodesic flow on a compact Riemannian manifold of negative curvature is Anosov.

Eberlein [10] in 1973 described and characterized geodesic flows in terms of Jacobi fields. He also noted that, although sufficient, the negative curvature condition may not be necessary for the geodesic flow to be Anosov.

Gulliver [12] in 1975 constructs using 'surgeries' compact Riemannian manifolds of genus greater than 1 with non-strictly negative curvature whose geodesic flows are Anosov. Moreover, these manifolds may or may not have focal points depending on the construction. This demonstrates that, unlike the absence of conjugate points, the absence of focal points is not a necessary hypothesis for the geodesic flow to be Anosov. In other words, the hypothesis that every radial Jacobi field on the manifold be increasing from the point where it vanishes is not necessary; what is required is that the Jacobi field does not vanish again.

There are other works on Anosov flows on surfaces with non-strictly negative curvature. More recently, Donnay and Pugh [9] showed the existence of compact surfaces of high genus in $\mathbb{R}^3$ with geodesic Anosov flow and which do not have negative curvature at all points.

Klingenberg [20] in 1974 explored the relation between global geometry and geodesic flow. He proved that the metrics with Anosov geodesic flow are contained in the subset of metrics without conjugate points. This means that Anosov geodesic flow implies that the geodesics are minimizing in the universal covering. Later, Ruggiero [24] in 1991 proved that the set of Anosov metrics is the interior of the C^2-metrics without conjugate points in the C^2 topology. However, not much is known about the topology of the metric space without conjugate points. Questions such as contractibility, connectedness and path connectedness of the set of metric without conjugate points have not yet been answered.

One of the objectives of this thesis is to study path connectivity in the space of metrics without conjugate points.

To study the space of metrics without conjugate points, it is interesting to understand curvature flows. In 1982, Hamilton [13] introduced and defined the Ricci Flow to study the evolution of the metric of a Riemannian manifold in order to prove the Poincaré conjecture, later proved by Perelman. And in 1988, Hamilton [14] proved that given an initial metric on a compact surface of genus greater than 1, there exists a solution to Ricci Flow that converges to a metric with negative curvature.

However, we do not know whether, if the initial solution of the Ricci Flow is a surface without conjugate points, the Ricci flow preserves this property on the surfaces defined by the flow. In fact, ongoing research suggests that conjugate points may emerge as such a surface evolves under the flow. Therefore, it is not clear whether the Ricci Flow can provide a solution to the problem addressed in this thesis.

More recently, in 2014, Jane and Ruggiero [19] proved that for a compact surface (M, g), of genus greater than or equal to 2, with no focal points and a finite number of bubbles with positive curvature, there exists a conformal family of metrics $\{g_\rho\}_{\rho\in[0,\rho_0]}, g_0 = g$ on M and $\overline{\rho} \in (0, \rho_0]$ such that if $\rho \in (0, \overline{\rho})$, then (M, g_ρ) is Anosov. To construct the perturbation, they used a version of Ricci Flow called Yang-Mills Ricci Flow.

We have two objectives in this thesis. The first is to show conditions for a high genus surface to have no focal points and to have Anosov Geodesic Flow even if it has regions where the curvature is positive. Our second objective is to show conditions for the surfaces studied in the first part of the work to admit a continuous deformation of metrics that preserves the properties of having no focal points and of having Anosov geodesic flow, along the path defined by the deformation.

We will state below the definition of a family of surfaces, a family of metrics and the results proved in this thesis. In the first definition, we will present the concept of generalized bubbles that generalizes the concept of bubbles presented in [19].

Definition 1.0.1. Consider $\delta, \epsilon, \Lambda \in \mathbb{R}$ such that $\delta > 0$, $\Lambda > 0$, and $0 < \epsilon < 1$, and consider $k \in \mathbb{N}$. We denote by $\mathcal{M}(\delta, k, \epsilon, \Lambda)$ the family of compact, orientable, smooth Riemannian surfaces of genus greater than 1 such that, if $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$, then:

1. There exist k pairwise disjoint, simply connected, strongly convex open balls in M that are free of focal points, denoted by $B_\delta(p_1), \cdots, B_\delta(p_k)$. We shall refer to them as **generalized bubbles**.
2. Every point where the curvature is non-negative is contained in $\bigcup_{i=1}^{k} B_\delta(p_i)$.
3. For all $i, j = 1, \ldots, k$ with $i \neq j$, the distance between $B_\delta(p_i)$ and $B_\delta(p_j)$, as well as the return time of a geodesic to $B_\delta(p_i)$, is greater than Λ.
4. The curvature in $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$ is smaller than $-\epsilon$.

Definition 1.0.2. Consider $\overline{\Lambda}, \zeta \in \mathbb{R}$ such that $\zeta > 0$ and $\overline{\Lambda} > 0$. Suppose that $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$, where $B_\delta(p_1), \ldots, B_\delta(p_k)$ are the generalized bubbles of (M, g).

Given a family of metrics conformal to g, denoted by g_ρ and parameterized by $\rho \in [0, 1]$ with $g_0 = g$, we define $\mathcal{M}_\rho(M, g, \delta, k, \zeta, \overline{\Lambda})$ as the set of metrics g_{ρ_l} for $\rho_l \in [0, 1]$ such that

1. Every point where the curvature of (M, g_{ρ_l}) is non-negative is contained in $\bigcup_{i=1}^{k} B_\delta(p_i)$.

2. For $i = 1, \ldots, k$, let $\tilde{B_\delta}^j(p_i)$ be a connected component of the lift of $B_\delta(p_i)$ in the universal cover of M, denoted by $\tilde{M}$. The distance between any two such balls with respect to the metric g_{ρ_l} is greater than $\overline{\Lambda}$.

3. The curvature of (M, g_{ρ_l}) in $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$ is smaller than $-\zeta$.

The results proved in this thesis are about the families defined previously.

Theorem 1. *Consider* $0 < \epsilon < 1$ *and* $\delta > 0$*. There exists* $\Lambda := \Lambda(\epsilon, \delta) > 0$ *such that if* $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$ *and*

$$K^+ < \frac{\sqrt{\epsilon}\left(1 - \frac{\epsilon}{2}\right) - \epsilon\,(2\delta + 1)\left(1 - \frac{\epsilon}{2}\right)^2}{2\delta},$$

where K^+ *denotes the maximum curvature attained in* (M, g)*, then* (M, g) *has no focal points and* g *is an Anosov metric.*

Theorem 2. *Consider* $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$*, with* $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$*. Then, there exist* $w \in C^\infty(M)$ *satisfying* $\min_M w = 0$ *such that if* $g_\rho := e^{2\rho w} g$ *with* $\rho \in [0, 1]$ *and* $\mu := \max_{p \in M} w(p)$ *then*

$$g_{\rho_l} \in \mathcal{M}_\rho(M, g, \delta, k, e^{-2\mu}\epsilon, \Lambda)$$

for all $\rho_l \in [0, 1]$.

Moreover, (M, g_1) *has strictly negative curvature, and for* $i = 1, \ldots, k$*, we have*

$$B_\delta(p_i) \subset B^{\rho_l}_{\delta e^{\rho_l \mu}}(p_i),$$

where $B_\delta(p_i)$ *is a generalized bubble of* (M, g) *and a ball in the metric* g*, while* $B^{\rho_l}_{\delta e^{\rho_l \mu}}(p_i)$ *is a ball of radius* $\delta e^{\rho_l \mu}$ *in the metric* g_{ρ_l}.

Theorem 3. *Let* $0 < \epsilon < 1$ *and* $\delta > 0$ *be constants, and define*

$$\Lambda := \frac{1}{\sqrt{\epsilon}} \operatorname{artanh}\left(1 - \frac{\epsilon}{2}\right).$$

Suppose that the Riemannian manifold (M, g) *satisfies*

$$(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda) \quad \textit{and} \quad \epsilon < \frac{-2\pi\,\chi(M)}{\mathrm{vol}(M)}.$$

Then there exists a smooth function $w \in C^\infty(M)$ *such that, if the maximum sectional curvature* K^+ *of* (M, g) *satisfies*

$$K^+ < \frac{\sqrt{\epsilon}}{4\,e^{2\mu}\,\delta}\left[\tanh\!\left(e^{-\mu}\frac{\ln 3}{3}\right) - \sqrt{\epsilon}\,e^{-\mu}\,\tanh^2\!\left(e^{-\mu}\frac{\ln 3}{3}\right)(4\,e^{\mu}\,\delta + 1)\right],$$

where $\mu := \max_M w$*, then the conformal family of metrics*

$$g_\rho := e^{2\rho\, w}\, g, \qquad \rho \in [0, 1],$$

consists entirely of Anosov metrics without focal points. In particular, (M, g_1) *has strictly negative curvature.*

We shall briefly describe the organization of the text.

In the next chapter, Chapter 2, we present some concepts and results necessary for understanding the development of this thesis. These contents are used implicitly or explicitly throughout the text.

We begin Chapter 3 by presenting a type of surface that satisfies the desired conditions, the Gulliver-type surfaces. Later we define the family $\mathcal{M}(\delta, k, \epsilon, \Lambda)$, we show some properties of this family and we prove Theorem 1.

In Chapter 4, we describe the class of deformations we work with. This class is particularly interesting because it is defined in a more explicit way than the Ricci-Yang-Mills Flow presented in [19]. Furthermore, the final metric resulting from this deformation has strictly negative curvature. We show some useful properties of these deformations that, being more explicit, allow us to control the generalized bubbles and their complement. We define the family $\mathcal{M}_\rho(M, g, \delta, k, \zeta, \overline{\Lambda})$ and, we prove some estimates on the deformation that allow us to demonstrate the Theorem 2.

In Chapter 5 we prove the final result of this thesis, Theorem 3. We start with some technical results relating the conditions for not having focal points shown in Chapter 3 and the deformation shown in Chapter 4.

Finally, in Chapter 6, we present our plans for the future. We would like to extend the result of Jane and Ruggiero [19]. More specifically, for a compact surfaces of genus greater than or equal to 2, free of focal points, with a finite number of bubbles with positive curvature (M, g) we would like to prove that there exists a conformal family of metrics $\{g_\rho\}_{\rho\in[0,1]}, g_0 = g$ on M such that g_ρ is an Anosov metric for all $\rho \in (0, 1]$ and g_1 is a metric with strictly negative curvature as long as we have a certain distance between the bubbles. In this case the surface (M, g_ρ) can have focal points if $\rho \neq 0$.

2 Preliminaries

In this chapter, we introduce the fundamental elements that constitute the basis of this study, primarily referencing works such as [6], [23], [10], [8], [25], and [27]. Within the scope of this work, we focus on base spaces consisting of Riemannian surfaces, i.e., Riemannian manifolds of dimension 2. However, throughout this chapter, whenever possible, we present the concepts in a comprehensive manner, regardless of dimension.

2.1 Conjugate points and focal points

2.1.1 Manifolds without conjugate points

Let (M, g) be a Riemannian manifold C^∞ of dimension n and let $(\tilde{M}, \tilde{g})$ be the universal covering of M with $\tilde{g}$ the pullback of g by the covering. We denote by TM the tangent bundle of M and T_1M its unit tangent bundle. The Riemannian metric g induces the Levi-Civita connection ∇, which, in addition to the properties of an affine connection, is also symmetric and compatible with the metric. By fixing a smooth curve in M, we can also represent the connection in terms of the covariant derivative. For example, we say that a smooth curve $\gamma \in M$ is a g**-geodesic** if

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{D\gamma}{dt} = 0.$$

It is important to note that the tangent bundle locally presents a product structure, resulting in the canonical projection $\pi : TM \to M$. This characteristic allows a representation in local charts, making the (local) understanding of the geodesic flow $(\gamma, \dot{\gamma}) \subset TM$ as a differential equation in terms of Christoffel symbols. Such analysis, combined with the Picard-Lindelöf Theorem, ensures the existence and uniqueness of the solution to $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, once a point in TM is fixed. Additionally, it is relevant to detach that geodesics have the property of being locally minimizing. That is, locally, the length of the geodesic segment connecting two points is the distance between these two points. A classical result about geodesics is that for every point $p \in (M, g)$ there exists a radius $r > 0$ such that the geodesic ball $B_{r'}(p)$ is **totally convex** for $0 < r' < r$, that is, any pair of points inside this ball can be connected by a unique minimizing geodesic segment that remains entirely inside the ball. Thus, a geodesic with initial conditions $\theta = (p, v) \in TM$ can be denoted as γ_θ. In general, in the next chapters of the thesis, we always consider geodesics parameterized by arc length.

Let us denote by R the **curvature tensor** of M associated with g and by K the **sectional curvature** of M associated with g. We define,

Definition 2.1.1. Let $\gamma : [0, t_0] \to M$ be a g-geodesic in M and J a vector field along γ. We say that J is a *Jacobi field* if

$$\frac{D^2 J}{dt^2} + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0$$

for $t \in [0, t_0]$.

For notational convenience, when there is no ambiguity, we can write $\frac{D^2 J}{dt^2} = J''$ and $\dot{\gamma} = \gamma'$.

The Jacobi field "quantifies" the deviation of geodesics in an infinitesimal manner, or more precisely, it denotes the rate at which geodesics diverge. This becomes clearer when we view the Jacobi field through a variation of geodesics. That is, let

$$V(t, s) := exp_p tv(s) \text{ with } t \in [0, 1] \text{ and } s \in (-\epsilon, \epsilon)$$

be a parametrized surface with $v(s) \subset T_pM$. Then $\frac{\partial f}{\partial s}(t, 0) = J(t)$ represents the unique Jacobi field satisfying $J(0) = 0$ and $J'(0) = v(0)$. Indeed, Jacobi fields define a vector space of dimension $2n$ that depends only on the initial conditions of J and J'. In particular, $\gamma'(t)$ and $t\gamma'(t)$ are Jacobi fields. Furthermore, if a Jacobi field is perpendicular to γ at a point then it is always perpendicular to γ. Therefore, in general, only Jacobi fields perpendicular to the geodesic are considered.

Definition 2.1.2. Let $\gamma : [0, t_0] \to M$ be a geodesic. We say that $\gamma(0)$ is *conjugate* to $\gamma(t_1)$ along γ, for $t_1 \in (0, t_0]$, if there exists a non-zero Jacobi field J along γ such that $J(0) = J(t_1) = 0$.

In this work, we consider manifolds without conjugate points, that is,

Definition 2.1.3. We say that M has no conjugate points if no geodesic in M has conjugate points.

A sufficient condition for the absence of conjugate points is that the sectional curvature is smaller than or equal to zero at all points. However, this condition is not necessary, as the manifold may contain regions with positive curvature, and it is precisely this type of manifold that we focus on in the study of the following chapters.

Manifolds without conjugate points have interesting properties. Let's look at some of them:

Proposition 2.1.4 ([4], Proposition 7.1.1)**.** *Let (M, g) be a Riemannian manifold without conjugate points. Then,*

1. *A Jacobi field is zero at two points if and only if the Jacobi field is identically zero;*

2. *Let γ be a geodesic, and consider $V \in T_{\gamma(0)}M$ and $W \in T_{\gamma(t)}M$ with $t \neq 0$. Then there exists a unique Jacobi field J along γ such that $J(0) = V$ and $J(t) = W$;*

3. *The derivative of the map exp_p is non-singular for all $p \in M$;*

4. *Every geodesic is **minimizing** in the universal covering of M, that is, let $\overline{\gamma}$ be in the universal covering of M. Then, for all $a, b \in \mathbb{R}$ we have that $L(\overline{\gamma}\big|_{[a,b]}) \leq L(c)$ where $L(\cdot)$ indicates the length of a curve and c represents any piecewise differentiable curve connecting $\overline{\gamma}(a)$ and $\overline{\gamma}(b)$.*

As a consequence, geodesics are unbounded in the universal covering of a manifold without conjugate points.

Furthermore a general property in the theory of manifolds without conjugate points:

Theorem 4 ([6], Chapter 7, Theorem 3.1 and Remark 3.4)**.** *Let (M, g) be a complete manifold without conjugate points. Then $\exp_p : T_p\tilde{M} \to \tilde{M}$ is a diffeomorphism.*

In particular, $\tilde{M}$ is diffeomorphic to $\mathbb{R}^n$.

Let us now consider $e_1(t), \cdots, e_{n-1}(t), e_n(t) = \gamma'(t)$ as a parallel orthonormal basis along γ. Writing the Jacobi fields perpendicular to γ as

$$J(t) = \sum_{i=1}^{n-1} \langle J(t), e_i(t) \rangle e_i(t)$$

and defining the **curvature matrix** by

$$(\mathbb{K}(t))_{i,j} := \langle R(\gamma'(t), e_i(t)), \gamma'(t), e_j(t) \rangle$$

we have the **Jacobi equation in matrix form**

$$\mathbb{J}''(t) + \mathbb{K}(\gamma(t))\mathbb{J}(t) = 0$$

in the basis $e_1, \cdots, e_n$. The solution of the matrix Jacobi equation is of the form

$$(\mathbb{J}(t))_{i,j} = \langle J_i(t), e_j(t) \rangle$$

and we can recover the solutions of the Jacobi equation by

$$J(t) = \mathbb{J}(t)J(0).$$

An additional important property about Jacobi fields on geodesics without conjugate points is attributed to Green [11]. Green demonstrated that along any geodesic in a manifold without conjugate points, there exists a solution to the matrix Jacobi equation that never becomes singular. In other words, along the geodesic, there exist at least n Jacobi fields that never vanish. Such as the called **Green's Jacobi Fields**. These fields play a crucial role throughout this study.

Restricting ourselves to compact manifolds, these Jacobi fields can be presented as follows:

Definition 2.1.5. Let (M, g) be a compact Riemannian manifold without conjugate points. Let γ be a geodesic parametrized by arc length in M. Suppose V is a vector orthogonal to $\gamma'(0)$. The *stable Jacobi field* J_V^s with initial condition $J_V^s(0) = V$ is given by

$$J_V^s(t) = \lim_{T \to +\infty} J_T(t)$$

where J_T is a Jacobi field along γ with boundary conditions $J_T(0) = V$ and $J_T(T) = 0$. The *unstable Jacobi field* J_V^u with initial condition $J_V^u(0) = V$ is given by

$$J_V^u(t) = \lim_{T \to -\infty} J_T(t).$$

The previously mentioned fields are Green's fields. Note that the stable and unstable Jacobi fields may coincide.

Remark 2.1.6. In the case of surfaces, when the stable and unstable Jacobi fields do not coincide along a geodesic, any Jacobi field orthogonal to geodesic can be expressed as a linear combination of these fields. $\diamond$

2.1.2 Riccati equation

Let (M, g) be a compact Riemannian manifold without conjugate points and let γ be a geodesic. Consider a solution to the matrix Jacobi equation $\mathbb{J}(t)$. Define $\mathbb{U}(t) := \mathbb{J}'(t) \cdot \mathbb{J}^{-1}(t)$; then $\mathbb{U}$ satisfies the **matrix Riccati differential equation**

$$\mathbb{U}'(t) + \mathbb{U}^2(t) + \mathbb{K}(\gamma(t)) = 0.$$

The solutions to the Riccati matrix equation may not be defined for all time, since the Jacobi matrix solutions may not be singular at all points. That is, given a Jacobi matrix solution, there may exist t such that the matrix is not invertible at time t.

What we can assume is the existence of entire solutions due to the previously discussed results.

Now suppose that (M, g) is a compact Riemannian surface without conjugate points, and let γ be a geodesic with unit speed in M. A Jacobi field along γ perpendicular to γ' can be written as

$$J(t) = f(t)e(t)$$

where $e(t)$ is a unit parallel vector field along γ perpendicular to γ' and $f(t)$ is a solution to the scalar Jacobi equation

$$f''(t) + K(t)f(t) = 0,$$

where $K(t)$ is the curvature of M. The structure becomes a little simpler because it is one-dimensional. We then define

$$U(t) := \frac{f'(t)}{f(t)}$$

which is a solution to the **Riccati equation**

$$U'(t) + U^2(t) + K(t) = 0.$$

Again, solutions to the Riccati equation may have asymptotes since $f(t)$ may be zero for some $t \in \mathbb{R}$. However, if $J(t)$ is a Green's Jacobi field, the Riccati solution induced by it is entire. In particular, we have the following:

Definition 2.1.7. Consider $\theta = (p, v)$ and $W \in T_1M$ be perpendicular to v. Define the *stable solution of the Riccati equation* U^s_θ as the solution of the Riccati equation induced by J_W^s and the *unstable solution of the Riccati equation* U_θ^u as the Riccati solution induced by J_W^u.

In particular, U_θ^s and U_θ^u are defined for all $t \in \mathbb{R}$.

Given the one-dimensional representation of the Jacobi field

$$J(t) = f(t)e(t),$$

we can allow ourselves to represent f as the Jacobi field itself, that is, to think of J as a scalar solution. This simplifies the notation and becomes clear in context.

2.1.3 Study of the solution of the Riccati Equation in manifolds with constant curvature and some consequences

The study of the Riccati equation becomes significantly simpler when dealing with surfaces of constant sectional curvature. In the case of zero curvature, the Riccati equation

$$U' + U^2 = 0$$

is a simple ODE with general solution

$$U(t) = \frac{1}{t+c}, c \in \mathbb{R}, t \neq -c,$$

where $-c$ is an asymptote in this case and $U(t) = 0$ is a constant solution.

In the case of constant curvature $-K$ with $K \geq 0$, the situation is similar. We have constant solutions

$$U(t) = -\sqrt{K} \text{ e } U(t) = \sqrt{K}.$$

The general solution is of the form

$$U(t) = \sqrt{K}\frac{\tanh(\sqrt{K}t) + \frac{C}{\sqrt{K}}}{1 + \frac{C}{\sqrt{K}}\tanh(\sqrt{K}t)}$$

where C is a constant depending on the initial value $U(0)$. The general solution is somewhat complex, but we can better understand the limit of the solutions by dividing them into regions defined by the constant solutions.

If $-\sqrt{K} < U(0) < \sqrt{K}$, then $-\sqrt{K} < U(t) < \sqrt{K}$, $U(t) \to -\sqrt{K}$ as $t \to -\infty$ and $U(t) \to \sqrt{K}$ as $t \to +\infty$.

On the other hand, for solutions of the Riccati equation with $U(0) > \sqrt{K}$, these correspond to the hyperbolic cotangent, that is,

$$U(t) = \sqrt{K} \coth(\sqrt{K}t - d)$$

where d is a constant that represents both the translation of the solution with respect to the x-axis and the vertical asymptote of the solution multiplied by $\sqrt{K}$. These solutions are not defined for all time and tend to $+\infty$ as $t \to \frac{d}{\sqrt{K}}$ and to $\sqrt{K}$ as $t \to \infty$.

Similarly, solutions of the Riccati equation with $U(0) < \sqrt{K}$ are of the form

$$U(t) = -\sqrt{K} \coth(-\sqrt{K}t - d),$$

which are not entire and tend to $-\infty$ as $t \to \frac{-d}{\sqrt{K}}$ and to $-\sqrt{K}$ as $t \to -\infty$.

Indeed, Eberlein [10], shows that there is a bound for the solution of the Riccati equation outside the neighborhood of its asymptote. In other words,

Proposition 2.1.8 ([10], Proposition 2.7)**.** *Let (M, g) be a Riemannian manifold of dimension n. Consider the Riccati matrix equation*

$$\mathbb{U}'(t) + \mathbb{U}^2(t) + \mathbb{K}(t) = 0.$$

Suppose that $g(\mathbb{K}(t)x, x) > -r$ for some $r > 0$, for all $x \in \mathbb{R}^n$ such that $||x|| = 1$ and all $t \in \mathbb{R}$. Then, for all $a > 0$, there exists $C = C(\mathbb{K}, a) > 0$ such that if a solution of the Riccati equation U has an asymptote at T, then

$$|g(U(t)x, x)| < C$$

for all $x \in \mathbb{R}^n$ and for all $t \in \mathbb{R} \setminus (T - a, T + a)$.

In particular, if (M, g) is a compact surface, for all $a > 0$ there exists $C = C(K, a) > 0$ such that if a solution of the Riccati equation U has an asymptote at T, then

$$||U(t)|| < C$$

for all $t \in \mathbb{R} \setminus (T - a, T + a)$.

By comparing compact surfaces without conjugate points with those of constant negative curvature, we can control the solutions of the Riccati equation and, consequently, the Jacobi fields. To illustrate, consider (M, d) as a compact surface without conjugate points and K as its curvature. Suppose that $K \geq -r^2$, where $r > 0$. Now, assume the existence of a g-geodesic γ_θ, with $\theta \in T_1M$, and a point t_0 where an entire solution of the Riccati equation along γ_θ satisfies $U(t_0) > r$. Consequently, there exists d such that $U(t_0) = r\coth(rt_0 - d)$.

If we define $V(t) := r\coth(rt - d)$ for $t \neq \frac{d}{r}$, we find that $V(t)$ is a solution of the constant Riccati equation

$$V' + V^2 - r^2 = 0.$$

Comparing this with the Riccati equation

$$U' + U^2 + K = 0$$

we have

$$V'(t_0) - U'(t_0) = r^2 + K \geq 0.$$

Thus, $V(t) > U(t)$ for $t > t_0$ and $V(t) < U(t)$ for $t < t_0$. In other words, for $t < t_0$, $U(t)$ would be bounded from below by $V(t)$, and since $V(t)$ has an asymptote, $U(t)$ would not be entire.

The approach presented above is based on one-dimensional arguments of the Riccati equation applied to surfaces. However, by fixing a direction and using the metric as an aid, it is feasible to extend this construction to higher dimensions. This procedure has been described in detail by Eberlein [10].

We can also control the magnitude of the Green's Jacobi fields with respect to time. That is, similar to what was done above, Eberlein [10] proves that

Proposition 2.1.9 ([10], Proposition 2.12)**.** *Let M be a compact manifold without conjugate points. Then every Jacobi field J with $||J(t)|| \leq r$ for all $t \geq 0$ and some $r > 0$ is stable. If $||J(t)|| \leq r$ for all $t \leq 0$, then J is an unstable Jacobi field.*

And,

Proposition 2.1.10 ([10], Proposition 2.11)**.** *Let M be a compact surface without conjugate points. Let $k_0 > 0$ be such that $-k_0^2$ is the lower bound of the sectional curvature. Then the stable and unstable solutions of the Riccati equation are bounded by k_0.*

2.1.4 Focal points

A conjugate point is, in particular, a focal point. In fact, focal points are associated not only with a single geodesic (like conjugate points), but with a submanifold. In essence, conjugate points are a specific type of focal points.

In order to define focal points, we first recall some basic concepts from the theory of immersions. Consider $f : N \to M$ an imersion. Let X, Y be fields in N. Let $\overline{\nabla}$ and ∇ be denote the Levi-Civita derivations of M and N respectively. The field

$$B(X,Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$$

is a field in M normal to N, whose $\overline{X}, \overline{Y}$ are extensions of X and Y respectively.

Definition 2.1.11. The quadratic form II_η defined on T_pN by

$$II_\eta(x) = H_\eta(x,x)$$

with $H_\eta : T_pN \times T_pN \to \mathbb{R}$ given by

$$H_\eta(x,y) = g(B(x,y),\eta),\ x,y \in T_pN$$

is the *second fundamental form* of f on p with respect to the normal vector η.

Consider now the self-adjoint map $S_\eta : T_pN \to T_pN$ given by

$$g(S_\eta(x),y) = H_\eta(x,y).$$

Definition 2.1.12. Let $N \subset M$ be a Riemannian submanifold of M, $q \in M$ is a *focal point* of N if there exists a geodesic $\gamma : [0,l] \to M$, with $\gamma(0) = p \in N$, $\gamma'(0) \in (T_pN)^\perp$, $\gamma(l) = q$, and a nonzero Jacobi field J along γ such that

- $J(0) \in T_pN$;
- $J'(0) + S_{\gamma'(0)}(J(0)) \in (T_pN)^\perp$;
- $J(l) = 0$.

Definition 2.1.13. Let (M,g) be a C^∞ Riemannian manifold. Let $\gamma : [0,a] \to M$ be a geodesic segment in M. Let $B_\epsilon(0) \subset \gamma'(0)^\perp$ be a ball contained in the orthogonal complement of $\gamma'(0)$ with radius $\epsilon > 0$ and center 0. We say that γ is *free of focal points* in $(0,a]$ if there exists $\epsilon > 0$ such that γ has no focal points relative to the submanifold.

$$\Upsilon_\epsilon := \exp_{\gamma(0)}(B_\epsilon(0)).$$

And so we can define manifold without focal points and neighborhood without focal points.

Definition 2.1.14. Let (M, g) be a C^∞ Riemannian manifold and B an open set in (M, g). We say that B is free of focal points if the geodesic segments contained in B are free of focal points.

Definition 2.1.15. Let (M, g) be a C^∞ Riemannian manifold. We say that (M, g) *has no focal points* if every geodesic in (M, g) is free of focal points.

An equivalent way to understand a manifold without focal points is through the following result that can be seen in [23] or in [10]:

Proposition 2.1.16. *Let (M, g) be a C^∞ Riemannian manifold. Then (M, g) has no focal points if and only if for every geodesic γ and Jacobi field $J(t)$ along γ, perpendicular to $\gamma'(t)$ and satisfying $J(0) = 0$, the norm of $J(t)$ is increasing for $t \geq 0$.*

In fact, in this thesis, when we work with manifolds without focal points we will, in general, be using the equivalence of Proposition 2.1.16, that is, radial Jacobi fields are increasing after the point where it is zero.

And, by [10] and [23],

Lemma 2.1.17 ([19], Lemmas 4.6 and 4.7)**.** *Let (M, g) be a compact surface without focal points. Then*

1. *The norms of stable Jacobi fields are non-increasing. The norms of unstable Jacobi fields are non-decreasing.*
2. *A Jacobi field is parallel if and only if it is both unstable and stable at the same time.*
3. *Given $\theta \in T_1M$, we denote by U^s_θ and U^u_θ the stable and unstable solutions of the Riccati equation on γ_θ. The other solutions of the Riccati equation tend to $U^u_\theta(t)$ as $t \to +\infty$ and to $U^s_\theta(t)$ as $t \to -\infty$.*
4. *The solutions U^s_θ and U^u_θ depend continuously on θ.*

2.2 Characterization of surfaces without conjugate points and surfaces without focal points in terms of the solution of the Riccati equation

In this section, we prove some results about surfaces without conjugate points and surfaces without focal points. The equivalences to be proven are well-known, but the proofs are not clearly presented in much of the literature.

These results are used in this thesis. Let us begin with surfaces without conjugate points.

Lemma 2.2.1. *On a compact surface, a geodesic has no conjugate points if and only if there exists a solution of the Riccati equation defined in* $\mathbb{R}$.

Proof.

The first side of the equivalence is a well-known result from the works of Green [11] and Eberlein [10]. A solution to the Riccati equation that never vanishes naturally arises from the Jacobi fields of Green.

On the other hand, if there exists a solution to the Riccati equation defined on $\mathbb{R}$, any other solution, by the existence and uniqueness theorem, would be bounded above or below by it. Thus, if J is a non-zero Jacobi field that vanishes at two points, i.e., if the solution of the Riccati equation associated with J has two vertical asymptotes, the solution would have to tend to either $+\infty$ or $-\infty$ at the asymptotes. This does not happen because J' cannot change sign between any two successive asymptotes.

Or, seen another way, notice that

$$U' = -U^2 - K.$$

Therefore, near the asymptotes U' is negative and U is decreasing. Then it is clear that, as the solution of the Riccati equation defined for all time limits superiorly or inferiorly the solutions of the Riccati equation with an asymptote, these solutions present a single asymptote.

□

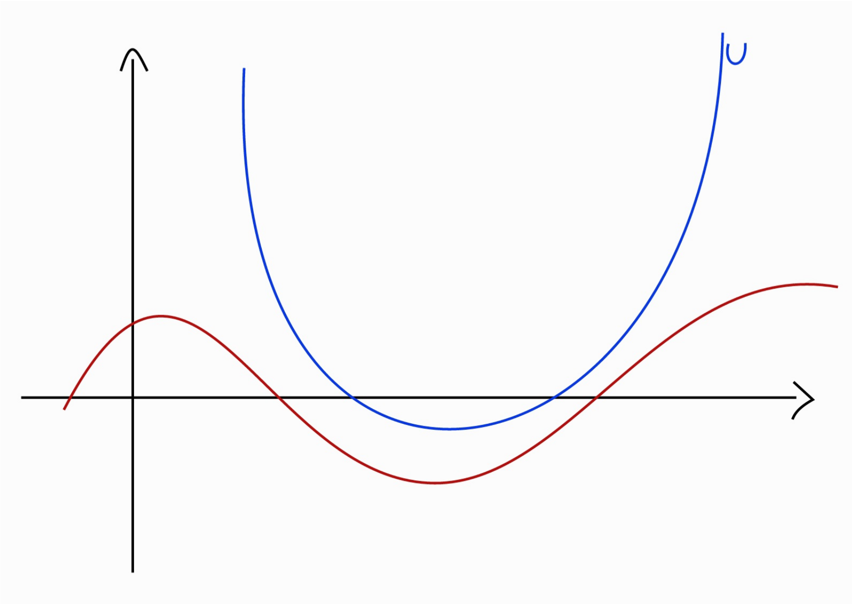

Figure 2.1: The blue curve illustrates a behavior of the Riccati equation solution that is not admissible.

Having established the equivalence between manifolds without conjugate points and the existence of Riccati equation solutions defined for all time, let us now consider an equivalence to having no focal points.

Lemma 2.2.2. *Let (M, g) be a compact surface. (M, g) has no focal points if and only if $U^u(t) \geq 0, U^s(t) \leq 0$ for all t and for all γ.*

Proof.

$\Rightarrow$) Suppose that (M, g) has no focal points. Let γ be a geodesic and J_T be a Jacobi field perpendicular along γ such that $J_T(0)$ has norm 1 and $J_T(T) = 0$. Then, since (M, g) is a surface (with no focal points), we permit the abuse of notation and write $J_T'(t) > 0$ for all $t \geq T$.

But,

$$J^u(t) = \lim_{T \to -\infty} J_T(t)$$

and

$$(J^u)'(t) = \lim_{T \to -\infty} (J_T)'(t).$$

It follows that $(J^u)'(t) \geq 0$, again by an abuse of notation, and $U^u(t) \geq 0$ for all t. Similarly, we have $(U^s)(t) \leq 0$.

$\Leftarrow$) Now suppose that $U^u(t) \geq 0, U^s(t) \leq 0$ for all t and for all γ. The existence of U^u and U^s guarantees the absence of conjugate points. Suppose J is a Jacobi field perpendicular to γ such that $J(T) = 0$. Consider U, the Riccati solution associated with J for $t > T$, that is,

$$U(t) = \frac{J'(t)}{J(t)} \; \forall\, t > T.$$

Clearly, U has a vertical asymptote at $t = T$.

On the other hand, since the surface is compact (and thus the curvature attains a maximum), a brief study of the solutions of the Riccati equation using comparison arguments in the style of Sturm-Liouville, as done in the preliminaries or in [10], guarantees that

$$\lim_{t \to T^+} U(t) = +\infty.$$

Thus, since the unstable solution is defined for all time, by existence and uniqueness of solutions, U^u is a lower bound for U and, consequently, $U(t) > 0$ for all $t > t_0$. This guarantee that $J(t)$ is increasing for $t \geq T_0$.

Therefore, since this holds for any radial Jacobi field, (M, g) has no focal points.

□

Before proceeding, it is important to present a way that we use in this thesis to construct solutions of the Riccati equation that are non-negative and definite for all time from other solutions of the Riccati equation. An analogous argument constructs non-positive solutions of the Riccati equation.

Lemma 2.2.3. *Let (M, g) be a compact surface and let γ be a geodesic in (M, g).*

Suppose there exists a sequence of solutions of the Riccati equation $\{U_n\}_n$ in γ such that for all n, U_n has an asymptote in t_n, $U_n(t) > 0$ for all $t \in (t_n, +\infty)$ and $\lim_{n\to+\infty} t_n = -\infty$. *Then there exists a solution of the Riccati equation U in γ such that $U(t) \geq 0$ for all $t \in (-\infty, +\infty)$.*

Similarly, if there exists a sequence of solutions of the Riccati equation $\{V_n\}_n$ in γ such that for all n, V_n has an asymptote in s_n, $V_n(t) < 0$ for all $t \in (-\infty, s_n)$ and $\lim_{n\to+\infty} s_n = +\infty$. *Then there exists a solution of the Riccati equation V in γ such that $V(t) \leq 0$ for all $t \in (-\infty, +\infty)$.*

Proof.

Consider U_n the solution of the Riccati equation with asymptote at t_n with $U_n(t) > 0$ for $t \in (t_n, +\infty)$. By Proposition 2.1.8, given $a > 0$, there exists a uniform constant C that does not depend on n such that $||U_n(t)|| < C$ for all $t \in [t_n + a, +\infty)$. That is, the sequence $\{U_i\}_{i\geq n}$ is uniformly bounded in $[t_n + a, +\infty)$ and, in particular, in any compact subset of $[t_n + a, +\infty)$.

Moreover, $U_n' = -U_n^2 - K$, i.e.,

$$\|U_n'\| \leq \|U_n^2\| + \|K\|.$$

Thus, there exists D such that if $t \geq t_n + a$ then $\|U_n'(t)\| < D$. Therefore, the sequence $\{U_i\}_{i\geq n}$ is equicontinuous in $[t_n + a, +\infty)$ and, in particular, in any compact subset of $[t_n + a, +\infty)$.

Therefore, by the Arzelà-Ascoli theorem, there exists a subsequence of $\{U_i\}_{i\geq n}$ that converges uniformly in $[t_n + a, +\infty)$. This holds for any t_n, so since $t_n \to -\infty$, $\{U_n\}$ admits a convergent subsequence that converges to a function U defined for all $\mathbb{R}$. And since the limit of solutions of a differential equation is also a solution, U is a solution of the Riccati equation. In particular, U is non-negative. Since the unstable solution is the supremum of the solutions defined for all t, the unstable solution is non-negative.

An analogous argument proves the second statement of Lemma.

□

2.3 Anosov geodesic flow

2.3.1 Sasaki metric

In this subsection, we present a Riemannian structure on the unit bundle T_1M of the Riemannian manifold (M, g) of dimension n. To do this, we first decompose the tangent bundle TTM of TM into vertical and horizontal spaces.

Consider the canonical projection

$$\pi : TM \to M \qquad \theta = (p, v) \mapsto p.$$

Its derivative is a map between tangent spaces

$$d\pi : TTM \to TM$$

$$d_\theta\pi : T_\theta TM \to T_pM.$$

The **vertical subspace** is defined as the kernel of this operation, that is,

$$V(\theta) = \ker(d_\theta \pi) \subset T_\theta TM.$$

For the horizontal space, consider the Levi-Civita connection ∇ associated with g. Then the map

$$k : TTM \to TM$$

is defined as follows: Consider $\xi \in T_\theta TM$ and $z : (-\epsilon, \epsilon) \to TM$ with $z(0) = \theta$ and $z'(0) = \xi$. Then

$$k_\theta(\xi) := \nabla_{\alpha'} Z(0),$$

where $z(t) = (\alpha(t), Z(t))$ and $Z(t) \in T_{\alpha(t)}M$. The **horizontal subspace** is then defined as

$$H = \bigcup_{\theta \in TM} H(\theta).$$

The definition of k_θ does not depend on the chosen curve z.

Now, for every $\theta \in TM$, $V(\theta) \cap H(\theta) = 0$ and the restrictions

$$d_\theta \pi|_{H(\theta)} : H(\theta) \to T_pM, \qquad k_\theta|_{V(\theta)} : V(\theta) \to T_pM$$

are linear isomorphisms. Therefore, we have the natural decomposition

$$T_\theta TM = H(\theta) \oplus V(\theta)$$

into horizontal and vertical spaces and the linear isomorphism

$$j_\theta : T_\theta TM = H(\theta) \oplus V(\theta) \to T_pM \times T_pM$$

$$\xi \mapsto j_\theta(\xi) = (d_\theta \pi(\xi), k_\theta(\xi)) := (\xi_h, \xi_v).$$

The map j_θ helps define the Sasaki metric.

Definition 2.3.1. Given $\theta = (p, v) \in TM$, we define the *Sasaki metric* by

$$<,>_s : T_\theta TM \times T_\theta TM \to \mathbb{R}$$

$$(\xi, \nu) \mapsto < \xi, \nu >_{s,\theta} := g_p(\xi_h, \nu_h) + g_p(\xi_v, \nu_v).$$

The Sasaki metric is a Riemannian metric on TM that depends only on g. Note that the spaces $H(\theta)$ and $V(\theta)$ are orthogonal.

Consider a geodesic γ_θ in M and the geodesic flow $\phi_t(\theta) = (\gamma_\theta(t), \gamma'_\theta(t))$ in TM, we have a characterization of the geodesic field $G : TM \to TTM$ in terms of the identification j_θ, that is, under the usual identification we have

$$G(\theta) = \frac{\partial}{\partial t}\Big|_{t=0} \phi_t(\theta) = \frac{\partial}{\partial t}\Big|_{t=0} (\gamma_\theta(t), \gamma'_\theta(t)),$$

however, since it is a geodesic, its tangent field is parallel along the curve, and thus, under the identification j_θ

$$G(\theta) = (v, 0).$$

The identification j_θ assist in the study of dynamical properties of the geodesic flow, such as the behavior of the flow action on the tangent space. Let $\xi \in T_\theta TM$, z be a curve with $z(0) = \theta$ and $z'(0) = \xi$, also consider $f(s,t) = \pi \circ \phi_t(z(s))$ as the variation of the geodesic $\gamma_\theta(t) = \pi \circ \phi_t(\theta)$ and $J_\xi := \frac{\partial f}{\partial s}(t, 0)$ as the Jacobi field. Then,

$$\begin{aligned} d_\theta\phi_t : T_\theta TM &\longrightarrow T_{\phi_t(\theta)}TM \\ \xi &\longmapsto (J_\xi(t), J'_\xi(t)) \end{aligned}$$

under the identification generated by j_θ.

On the other hand, $T_\theta T_1 M$ is a vector subspace of $T_\theta TM$. The Sasaki metric induces a Riemannian metric on T_1M, and thus T_1M is a Riemannian manifold of dimension $(2n-1)$ with the induced Sasaki metric. Note that $G(\theta)$, $\theta = (x, v) \in T_1M$, defines a 1-dimensional subspace in the tangent space of T_1M, and under the identification generated by j_θ, $T_\theta T_1 M \cong T_xM \times \{w \in T_xM \mid w \perp v\}$.

2.3.2 Green subspace

We previously discussed the existence of stable and unstable Jacobi fields along any geodesic in a manifold without conjugate points. In fact, more than that, given a geodesic and a base point along the curve, each vector perpendicular to the curve at the base point generates a stable Jacobi field and an unstable Jacobi field. That is, if (M, g) is a compact n-dimensional Riemannian manifold without conjugate points and γ_θ is a geodesic in M, the subspace spanned by the stable Jacobi fields along γ_θ has dimension $(n-1)$ and the same is true for the subspace spanned by the unstable Jacobi fields along γ_θ.

We define the *stable subspace* $\mathcal{J}^s_\theta$ as the subspace spanned by the stable Jacobi fields along γ_θ. Similarly, $\mathcal{J}^u_\theta$ represents the *unstable subspace* along γ_θ.

Both have dimension $(n-1)$.

On the other hand, for every $\theta \in T_1TM$ and $\xi \in T_\theta T_1M$, the map $\xi \mapsto J_\xi$ is a linear isomorphism between $G(\theta)^\perp$ and the space of Jacobi fields normal to γ_θ.

Definition 2.3.2. We define the *stable Green subspace* at $\theta \in T_1M$, $G^s(\theta)$, as the image of $\mathcal{J}_\theta^s$ under the linear isomorphism $J_\xi \mapsto \xi$. Similarly, we define $G^u(\theta)$ as the image of $\mathcal{J}_\theta^u$ as the *unstable Green subspace* at θ.

Then along the geodesic flow defined by θ we are considering three important subspaces of $T_\theta T_1M$, $G(\theta), G^s(\theta)$ and $G^u(\theta)$.

2.3.3
Characterization of Anosov flow

Let us start this section with the definition of hyperbolic geodesic flow and Anosov geodesic flow.

Definition 2.3.3. Let (M, g) be a compact manifold and ϕ_t the geodesic flow. A compact subset $X \subset T_1M$ is said to be *hyperbolic* if there exist constants $C, \lambda > 0$ and $d\phi_t$-invariant subspaces $E^s(\theta), E^u(\theta) \subset T_\theta T_1M$ for all $\theta \in X$ such that:

1. $T_\theta T_1M = E^s(\theta) \oplus E^u(\theta) \oplus G(\theta)$.

2. $\|d_\theta\phi_t(\xi^s)\| \leq C \cdot \exp(-\lambda t)$ for all $\xi^s \in E^s(\theta)$ and all $t \geq 0$.

3. $\|d_\theta\phi_t(\xi^u)\| \geq C \cdot \exp(\lambda t)$ for all $\xi^u \in E^u(\theta)$ and all $t \leq 0$.

If $X = T_1M$, we say that the geodesic flow is **Anosov**. In this case, $E^s(\theta) = G^s(\theta)$ and $E^u(\theta) = G^u(\theta)$. Moreover, we say that g is an **Anosov metric**.

Anosov [2] proved that in the case of compact manifolds of negative curvature the geodesic flow is always Anosov. This result is a particular case of what was later proven by Eberlein [10].

Theorem 5 ([10], Theorem 3.2)**.** *Let M be a compact manifold without conjugate points. The following properties are equivalent:*

1. *The geodesic flow on T_1M is Anosov.*

2. *For all $\theta \in T_1M$, $G^s(\theta) \cap G^u(\theta) = \{0\}$.*

3. *For all $\theta \in T_1M$, $T_\theta T_1M = G^s(\theta) \oplus G^u(\theta) \oplus G(\theta)$.*

4. *A Jacobi field has bounded norm for all $t \in \mathbb{R}$ if and only if the field is zero.*

By a convexity argument, property 2 above is immediate in the case of compact manifolds with negative curvature. Indeed, if (M, g) has negative curvature then it has no conjugate points and there are $n-1$ stable Jacobi fields and $n-1$ unstable Jacobi fields. It suffices then to show that for all $\theta \in T_1M$, $G^s(\theta) \cap G^u(\theta) = \{0\}$.

Suppose J is a stable and unstable Jacobi field. Consider $f(t) = ||J(t)||^2$. Then

$$\begin{aligned} f''(t) &= 2g(J''(t), J(t)) + 2g(J'(t), J'(t)) \\ &= -2g(R(\gamma'(t), J(t))\gamma'(t), J(t)) + 2||J'||^2 \\ &= -2K(t).||J(t)||^2 + 2||J'||^2 \geq 0 \end{aligned}$$

since $K(t) < 0$.

Therefore f is convex and bounded, since the stable Jacobi field is bounded for $t \geq 0$ and the unstable Jacobi field is bounded for $t \leq 0$. It follows that f is constant and hence $f'' \equiv 0$. Therefore $J \equiv 0$.

In the statement of theorem, we assume that M has no conjugate points, but this condition is implicitly included in the hypothesis of the geodesic flow being Anosov, since Klingenberg [20] proved the following:

Theorem 6 ([20], Theorem)**.** *Let (M, g) be a compact Riemannian manifold. If the geodesic flow on (M, g) is Anosov, then (M, g) has no conjugate points.*

2.4
The space of metrics on a manifold

Considering M as a compact differentiable manifold of class C^∞, it is interesting to observe, as discussed in [16], that the C^m functions for $m \geq 0$ on M form a topological space. This same principle extends to metrics on a compact differentiable manifold. One way to understand this is through local charts, where the choice of a coordinate basis $\left\{\frac{\partial}{\partial x_i}\right\}_{1\leq i\leq n}$ in an open set U and a metric g results in

$$g_{i,j} := g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) : U \to \mathbb{R},$$

which represents a function. Therefore, the idea of a topological space of functions extends to the topological space of metrics, depending on the degree of differentiability of the metric.

Hyperbolic systems are known for their stability. In the case of Anosov systems, the C^1-structural stability holds, which implies that Anosov metrics form an open set inside of the set of C^2 metrics on M. Furthermore, as shown by Klingenberg [20], Anosov geodesic flows on compact manifolds do not have conjugate points, which means that Anosov metrics lie in the interior of the set of C^2 metrics without conjugate points. Additionally, the set of metrics without conjugate points on M is a closed set in the C^k-topology for $k \geq 2$. Later, Ruggiero [24] established the following result:

Theorem 7 ([24], Theorem A)**.** *Let M be a compact C^∞-manifold. Then the interior of the metrics without conjugate points in the C^2-topology coincides with the set of Anosov metrics.*

Figure 2.4 illustrates the known behavior of the space of metrics without conjugate points.

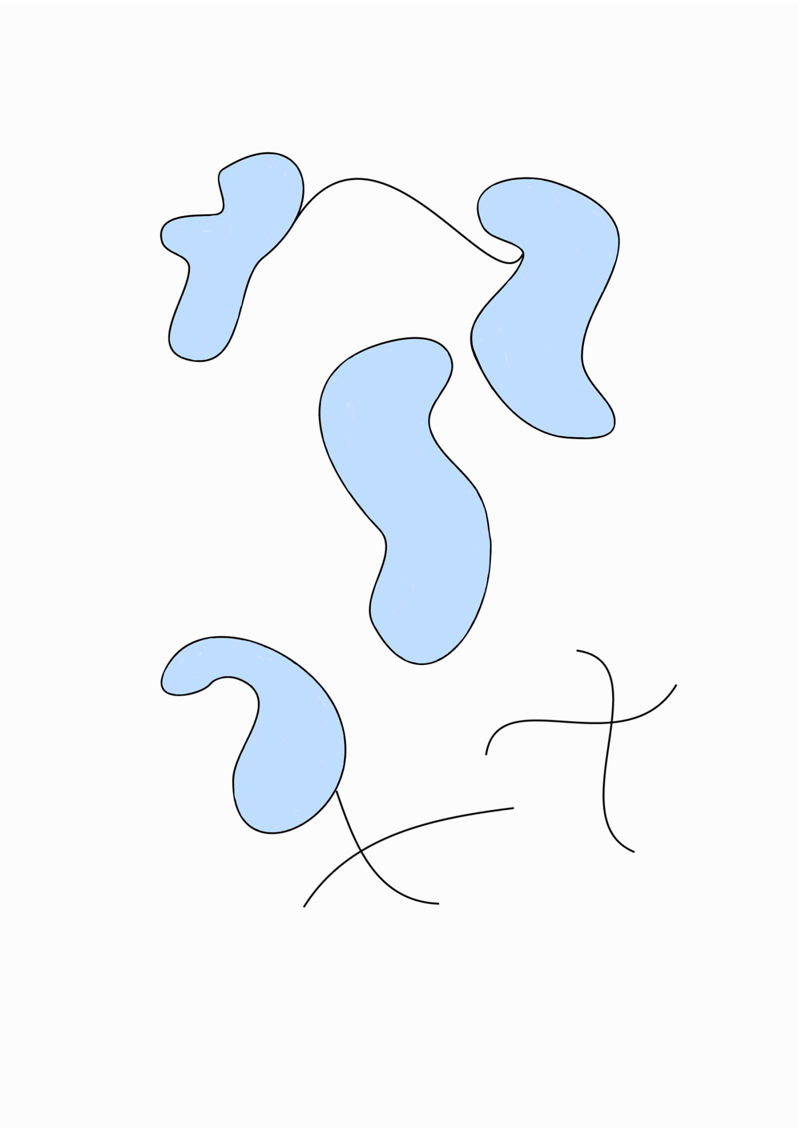

Figure 2.2: The lines represent the limit of metrics without conjugate points and in blue the Anosov metrics in a two-dimensional sketch.

2.4.1 Conformal deformation

Definition 2.4.1. Let M be a C^∞-manifold and g a metric on M. We say that $\overline{g}$ is a *conformal deformation* of g if there exists a function $h : M \to \mathbb{R}$, with $h \in C^\infty$ and $h > 0$, such that $\forall p \in M$

$$\overline{g}_p = h(p)g_p,$$

where the subscript p denotes the restriction of the metric to the point p.

Conformal deformation has the property of preserving angles.

We naturally have a defined path in the space of metrics between two conformal deformations through the function that deforms one into the other. That is, let $w : M \to \mathbb{R}$ be a C^∞ function. Then

$$g_\rho^w = e^{2\rho w}g,$$

where $\rho \in [0, 1]$, defines a path of conformal deformations between g and $e^{2w}g$.

Fix a metric g in M and let $g_w := e^{2w}g$ be conformal to g, with $w \in C^\infty(M)$. The following calculations can be found in [22].

Denote by grad, div, and $\triangle$ the gradient, divergence, and Laplacian with respect to g, respectively. When referring to g_w, the notations will appear with subscript or superscript w.

The gradient of a function f can be viewed from a orthonormal frame. That is, consider $(E_1, \cdots, E_n)$ a orthonormal frame at $p \in M$. Then

$$grad(f)(p) = \sum_{i=1}^{n}(E_i(f))(E_i(p)).$$

On the other hand, it is immediate that $(e^{-w}E_1 := \overline{E_1}, \cdots, e^{-w}E_n := \overline{E_n})$ is a orthonormal frame with respect to g_w. Thus,

$$(grad_w(f))(p) = \sum_{i=1}^{n} e^{-2w}E_i(f)E_i = e^{-2w} \cdot grad(f).$$

An application of the Koszul formula allows us to relate the Levi-Civita connections. Indeed, given vector fields X, Y, and Z on M, we have that

$$\begin{aligned} 2e^{2w}g(\nabla^w_X Y, Z) &= 2g_w(\nabla^w_X Y, Z) \\ &= X(g_w(Y,Z)) + Y(g_w(g_w(Z,X)) - Z(g_w(X,Y)) \\ &+ g_w([X,Y],Z) - g_w([Y,Z],X) + g_w([Z,X],Y) \\ &= 2X(w)e^{2w}g(Y,Z) + 2Y(w)e^{2w}g(Z,X) - 2Z(w)e^{2w}g(X,Y) \\ &+ e^{2w}X(g(Y,Z)) + e^{2w}X(g(Y,Z)) + e^{2w}Y(g(Z,X)) - e^{2w}Z(g(X,Y)) \\ &- e^{2w}g([Y,Z],X) + e^{2w}g([Z,X],Y) + e^{2w}g([X,Y],Z). \end{aligned}$$

Dividing by $2e^{2w}$, we obtain

$$\begin{aligned} g(\nabla^w_X Y, Z) &= X(w)g(Y,Z) + Y(w)g(Z,X) - Z(w)g(X,Y) + g(\nabla_X Y, Z) \\ &= g(\nabla_X Y + X(w)Y + Y(w)X - g(X,Y)grad(w), Z). \end{aligned}$$

Therefore,

$$\nabla^w_X Y = \nabla_X Y + X(h)Y + Y(h)X - g(X,Y)\cdot grad(w).$$

The divergence of a vector field X at a point p is defined as the trace of the linear map $Y(p) \to \nabla_Y X(p)$. Alternatively, using the previously presented frame of reference, the divergence can be expressed as $div(X)(p) = \sum_{i=1}^n E_i(f_i)(p)$, where $X = \sum_{i=1}^n f_i E_i$. Thus,

$$div_w(X) = \sum_{i=1}^n g_w(\nabla^w_{\overline{e_i}} X, \overline{e_i}) = \sum_{i=1}^n g(\nabla^w_{e_i} X, e_i).$$

Therefore,

$$\begin{aligned} div_w(X) &= \sum_{i=1}^n g(\nabla_{E_i} X + E_i(w)X + X(w)E_i - g(E_i, X)grad(w), E_i) \\ &= div(X) + nX(w) \end{aligned}$$

Now, the Laplacian is the divergence of the gradient, that is, $\triangle f = \mathrm{div}(\mathrm{grad}(f))$. Thus

$$\begin{aligned} \triangle_w f &= \mathrm{div}_w(\mathrm{grad}_w(f)) \\ &= \mathrm{div}_w(e^{-2w}\mathrm{grad}(f)) \\ &= \mathrm{div}(e^{-2w}\mathrm{grad}(f)) + ne^{-2w}\mathrm{grad}(f(w)) \\ &= e^{-2w}\triangle f + g(\mathrm{grad}(e^{-2w}), \mathrm{grad}(f)) + ne^{-2w}g(\mathrm{grad}(f), \mathrm{grad}(w)). \end{aligned}$$

Then, since $\mathrm{grad}(e^{-2w}) = -2e^{-2w}\mathrm{grad}(w)$, we have that

$$\triangle_w f = e^{-2w}(\triangle f + (n-2)g(\mathrm{grad}(w), \mathrm{grad}(f))).$$

A useful relation concerns the sectional curvatures of conformally deformed metrics. The formula describing this deformation, as shown, can be found in [7].

Theorem 8. *Let (M, g) be a smooth compact Riemannian surface. Consider $w \in C^\infty(M)$ and $g_w = e^{2w}g$. Then*

$$-\triangle_g w + K_g = K_{g_w} e^{2w},$$

where K_g and K_{g_w} are the sectional curvatures of g and g_w, respectively.

It is important to remember that although curvature may vary with deformation, its integral over the surface remains the same, thanks to the Gauss-Bonnet theorem

$$\int_M K_g \, d\sigma_g = 2\pi\chi(M) \quad \forall g \text{ metric on } M,$$

where σ_g is the volume form of M with respect to g and $\chi(M)$ is the Euler characteristic of M.

By the curvature formula of Theorem 8, it is possible to control the curvature of the metric after the deformation of the regions with negative curvature and the regions with non-negative curvature through the Laplacian of the function that defines the deformation. The following theorem is interesting because it shows that it is possible to obtain a function from its Laplacian. The result is a specific case of Theorem 4.7 in [3]:

Theorem 9 ([3], Theorem 4.7)**.** *Consider (M, g) a compact C^∞ manifold and h a C^∞ function. Then there exists $w \in C^\infty(M)$ such that $h = \Delta_g w$ if and only if $\int h \, d\sigma_g = 0$.*

3
Surfaces without focal points

Motivated by Gulliver's paper [12], our goal is to show reasonable conditions for a compact Riemannian surface where the positive curvature is located in bubbles so as to have no focal points, and to explore the possibility of deforming the original metric into one with negative curvature while maintaining the property of having no focal points along the entire metric path. Furthermore, hyperbolicity is also preserved along the path defined by the deformation.

In this chapter we define a family of surfaces without focal points that in some way extends the surfaces constructed by Gulliver through surgeries. In the family of surfaces that is defined, the regions of non-negative curvature can be more complex.

So, first we present the surfaces constructed by Gulliver, then we present a family of surfaces over which we show conditions for having no focal points and which have Anosov geodesic flow.

3.1
Example: Gulliver-type Surface

Before presenting our reference space, it is important to note that the set of surfaces that satisfy the conditions that will be presented is not empty. On the contrary, it is known that a large number of surfaces satisfy stronger conditions.

In this section, we present an example (or a class) of surfaces that satisfy the hypotheses. The idea is to show, briefly, the construction carried out by Gulliver [12]. This was one of the first examples constructed under such conditions.

In Gulliver's article, reasonable conditions are presented to control the length of geodesics and the non-existence of conjugate points in a given control region. Gulliver also demonstrates, through examples, the existence of manifolds with non-strictly negative curvature that can be Anosov and that may or may not have focal points.

The idea of Gulliver's examples is to make surgery on a higher genus surface of negative curvature by cutting some normal ball and gluing a disk endowed with a new metric with some points of positive curvature.

To begin with, we can observe that by Borel [5], there are compact manifolds of curvature -1. Furthermore, in [12], Gulliver proves the following:

Lemma 3.1.1 ([12], Lemma 4)**.** *For any $h \geq 2$, there is a compact surface M of genus h with constant sectional curvature -1, and a point $p \in M$ such that the exponential map at p is injective within a ball of radius R, where*

$$\cosh R = \frac{1}{2} \csc\left(\frac{\pi}{12h-6}\right).$$

In particular, $R > 1.71$.

These results guarantee that the conditions of the following result establish a non-empty set of Riemannian manifolds. The condition of constant sectional curvature -1 can be relaxed by the effect of conformal deformations. In any case, what Gulliver [12] establishes is the following:

Theorem 10 ([12], Theorem 3)**.** *Suppose that (M, g') is a Riemannian manifold. Consider $p \in M$ a point such that a ball D of radius R centered at p is the injective image of the corresponding ball in the tangent space at p under the exponential map. Suppose that g' has sectional curvature $\leq -c^2$ and has constant sectional curvature $-\beta^2$ on D, where $c \geq 0$ and $\beta > 0$. Then there exists another Riemannian metric g on M, with $g = g'$ except in a compact subset of D, which may be chosen to have any or all of the following four properties:*

1. *The g-sectional curvature is a positive constant K^+ on a neighborhood of p, and $\leq K^+$ everywhere;*
2. *g has no conjugate points;*
3. *The geodesic flow of g is Anosov, provided M is compact and $c > 0$;*
4. *One of the following properties holds:*
   - (a) *g has no focal points;*
   - (b) *Focal points occur along a certain geodesic through p, provided $\beta R > 1.70$.*

Gulliver adds that the same method used to prove the mentioned theorem extends to a disjoint union of balls, provided each ball has constant negative sectional curvature.

One way to see this is to consider n points $p_1, \cdots, p_n$ in M and n disjoint balls $D_1, \cdots, D_n$ in M that satisfy the conditions of Theorem 10. Then, by Theorem 10, there exist metrics $g_1, \cdots, g_n$ on M such that $g_i = g'$ outside a compact set C_i in D_i, and g_i satisfies:

1. The g_i-curvature is a positive constant K^+ in a neighborhood of p_i, and $\leq K^+$ everywhere;

2. g_i has no focal points.

Consider smooth functions $f_i : M \to \mathbb{R}$ such that $f_i|_{C_i} = 1$, $f_i|_{C_j} = 0$ for $j \neq i$, and $f_i|_{M \setminus \bigcup_{j=1}^n D_j} = \frac{1}{n}$.

Then,

$$g = \sum_{j=1}^{n} f_j g_j$$

is a metric on M with constant positive sectional curvature K^+ in neighborhoods of $p_1, \cdots, p_n$ and $\leq K^+$ everywhere. Moreover, $g = g'$ outside $\bigcup_{j=1}^n D_j$.

A construction of this type would increase the number of bubbles with positive curvature. For convenience, let us denote a surface of this type as a **Gulliver-type Surface**.

Figure 3.1 represents a Gulliver-type surface.

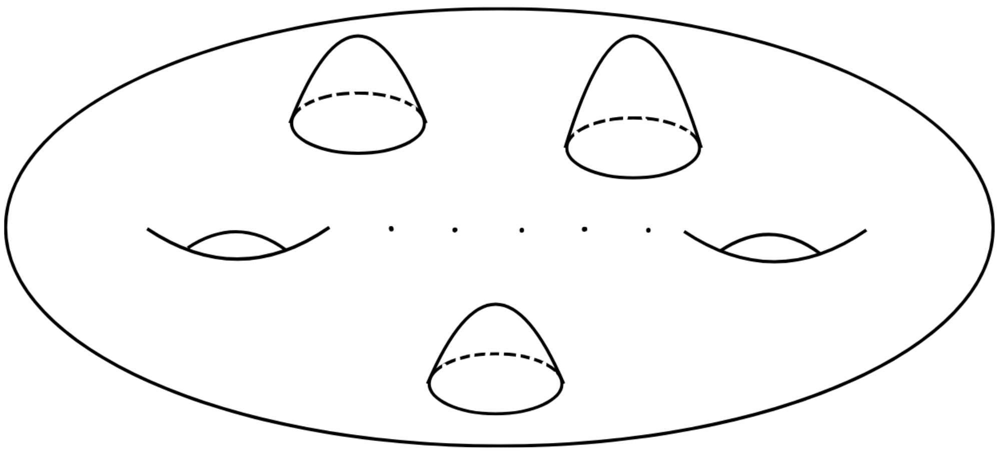

Figure 3.1: Sketch of a Gulliver-type surface.

The construction made by Gulliver [12] is one of the first works where surfaces (Anosov or not) without focal points with curvature that is not strictly negative are shown. In the article, Gulliver demonstrates how to construct these surfaces by means of a kind of gluing or replacing a ball on the original surface with a Euclidean ball that satisfies the desired conditions. It is important to note that this construction does not apply universally to all surfaces with the hypotheses we intend to consider.

3.2 The family of surfaces

In this section we define and present the set of surfaces that we work with. Naturally, it is important to have control over the region where the curvature is positive and consequently where the curvature is negative.

If we study in more detail the behavior of Jacobi fields in metrics without conjugate points, metrics without focal points or Anosov metrics, we see that the convexity of the norm of the Jacobi fields and, consequently, the negative curvature, is important in the behavior of these fields. Thus, for the surface to be free of focal points or for the metric to be Anosov, it is natural to think that a large part of the surface has negative curvature or, better, the integral of the curvature on the surface is negative. In other words, the surface has a genus greater than 1.

Let us recall the definition of the family $\mathcal{M}(\delta, k, \epsilon, \Lambda)$.

Definition (1.0.1)**.** Consider $\delta, \epsilon, \Lambda \in \mathbb{R}$ such that $\delta > 0$, $\Lambda > 0$, and $0 < \epsilon < 1$, and consider $k \in \mathbb{N}$. We denote by $\mathcal{M}(\delta, k, \epsilon, \Lambda)$ the family of compact, orientable, smooth Riemannian surfaces of genus greater than 1 such that, if $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$, then:

1. There exist k pairwise disjoint, simply connected, strongly convex open balls in M that are free of focal points, denoted by $B_\delta(p_1), \cdots, B_\delta(p_k)$. We shall refer to them as **generalized bubbles**.

2. Every point where the curvature is non-negative is contained in $\bigcup_{i=1}^{k} B_\delta(p_i)$.

3. For all $i, j = 1, \ldots, k$ with $i \neq j$, the distance between $B_\delta(p_i)$ and $B_\delta(p_j)$, as well as the return time of a geodesic to $B_\delta(p_i)$, is greater than Λ.

4. The curvature in $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$ is smaller than $-\epsilon$.

Note that for any choice of parameters it is possible to find a surface M in $\mathcal{M}(\delta, k, \epsilon, \Lambda)$. The volume, and consequently Λ, can be as large as we want by simply increasing the genus of the surface. Furthermore, it is possible to introduce regions of positive curvature into surfaces of negative curvature through surgeries or perturbations.

As previously mentioned, we shall refer to $B_\delta(p_1), \cdots, B_\delta(p_k)$ as **generalized bubbles**. A subset of Gulliver-type surfaces satisfy the conditions of the definition. But note that the hypotheses cover more surfaces than just Gulliver-type surfaces. For example, we do not need to assume that the curvature is constant in a compact subset contained in a generalized bubble.

Remark 3.2.1. Because they are strongly convex, simply connected and free of focal points, generalized bubbles cannot contain entire geodesics, that is, every geodesic that intersects a generalized bubble necessarily crosses it. Furthermore, any geodesic segment contained in a generalized bubble has a maximum length of 2δ. In fact, if there were a geodesic segment with a length greater than 2δ, there would be another geodesic segment contained in the generalized bubble connecting its ends with a length smaller than 2δ, which contradicts the absence of focal points. Finally, there cannot be closed geodesics in the generalized bubble, since the region is simply connected and has no focal points. $\diamond$

Remark 3.2.2. The third assumption of the Definition 1.0.1 is equivalent to saying that the distance between the lifts of the generalized bubbles in the universal covering is greater than Λ. More precisely, for $i = 1, \cdots, k$, denote by $\tilde{B_\delta}^j(p_i)$ a connected component of the lift of $B_\delta(p_i)$. Then, the assumption (3) of the Definition 1.0.1 is equivalent to requiring that the distance between the balls $\tilde{B}^j_\delta(p_i)$ is greater than Λ and the return time of the geodesic balls $\tilde{B}^j_\delta(p_i)$ is greater than Λ. $\diamond$

3.3 Neighborhoods without focal points

We show in this section that given a surface, for every point there is an open neighborhood of this point without focal points.

Proposition 3.3.1. *Let (M, g) be a compact surface and K^+ be the maximum of the sectional curvature in M. Consider $p \in M$ and denote by $B(p, r)$ the ball with center p and radius $r > 0$ in (M, g).*

If for all $p_1, p_2 \in M$ we have

$$p_1, p_2 \in B(p, \frac{\pi}{4\sqrt{K^+}})$$

then p_1 and p_2 are not each other's focal points.

Proof.

Fix $p \in M$ and let K be the sectional curvature of (M, g). Let γ be a geodesic such that $\gamma(0) = p$ and let J be the Jacobi field along γ such that $J(0) = 0$ and $J'(0) > 0$ (the construction would be analogous if $J'(0) < 0$).

Denote by S_{K^+} the surface with constant curvature K^+ and let J_{K^+} be a Jacobi field on S_{K^+} such that $J_{K^+}(0) = 0$ and $J'_{K^+}(0) = J'(0)$.

The general solution of a Jacobi equation in S_{K^+} is

$$A\cos(\sqrt{K^+}t) \ + \ B\sin(\sqrt{K^+}t),$$

where A, B are chosen according to the initial conditions.

Then since $J_{K^+}(0) = 0$ we have that, in this case, $A = 0$ and $B > 0$, that is,

$$J_{K^+}(t) = B\sin(\sqrt{K^+}t)$$

and

$$J'_{K^+}(t) = \sqrt{K^+}B\cos(\sqrt{K^+}t).$$

Therefore, J_{K^+} is increasing in $[0, \frac{\pi}{2\sqrt{K^+}})$. Or, the solution of the Riccati equation V associated with J_{K^+} satisfies that $V(t) > 0$ for $t \in (0, \frac{\pi}{2\sqrt{K^+}})$.

Denote by J_δ, with $0 < \delta < \frac{\pi}{2\sqrt{K^+}}$ the solution of the Riccati equation along γ such that $J_\delta(\delta) = J_{K^+}(\delta)$ and $J'_\delta(\delta) = J'_{K^+}(\delta)$. Let U_δ be the solution of the Riccati equation associated with J_δ. Then consider the Riccati equations:

- $U'_\delta(t) + U^2_\delta(t) + K(t) = 0$
- $V'(t) + V^2(t) + K^+ = 0.$

Subtracting the first equation from the second equation when $t = \delta$ we have that

$$U'(\delta) - V'(\delta) = K^+ - K(t) \geq 0.$$

Therefore $U_\delta(t) > V(t)$ for t in a neighborhood of δ with $t > \delta$. And then the local argument extends to the interval where U_δ and V are defined and $K(t) \leq K^+$.

On the other hand, by continuity,

$$\lim_{\delta \to 0} J_\delta = J$$

and then if U is a solution of the Riccati equation associated to J, we have that

$$\lim_{\delta\to 0} U_\delta = U$$

and since $U_\delta \geq V$ we have that $U(t) \geq V(t) > 0$ for all $t \in (0, \frac{\pi}{2\sqrt{K^+}})$.

This proves that if $p_1 \in B(p, \frac{\pi}{2\sqrt{K^+}})$ then p_1 and p are not focal points of each other. Furthermore, if $p_1, p_2 \in B(p, \frac{\pi}{4\sqrt{K^+}}))$ then the distance between p_1 and p_2 is smaller than $\frac{\pi}{2\sqrt{K^+}}$ and therefore p_1 and p_2 are not focal points of each other.

$\square$

We can apply the previous argument in the other direction, that is, by defining a radius we can find a quota for the curvature that maintains neighborhoods without focal points inside balls with the fixed radius. In other words;

Corollary 3.3.2. *Given $\delta > 0$ there exists $K^+ > 0$ such that if the surface (M, g) has sectional curvature smaller than or equal to K^+ each ball of radius δ is free of focal points.*

Proof.

By constructing of Proposition 3.3.1, if $\delta \leq \frac{\pi}{4\sqrt{K^+}}$, for all $p \in M$, we have no focal points in $B(p, \delta)$. On the other hand,

$$\delta \leq \frac{\pi}{4\sqrt{K^+}} \iff K^+ \leq \frac{\pi^2}{16\delta^2}.$$

Therefore it is sufficient to take K^+ such that $K^+ \leq \frac{\pi^2}{16\delta^2}$.

$\square$

3.4
Surfaces free of focal points and with Anosov metric

In this section we present conditions for the family $\mathcal{M}(M, \delta, k, \epsilon, \Lambda)$ to have no focal points. In fact, we use the equivalence presented in 2.2.2 to show such conditions. For clarity, we restate Theorem 1 below and dedicate this section to its proof.

Theorem (1)**.** Consider $0 < \epsilon < 1$ and $\delta > 0$. There exists $\Lambda := \Lambda(\epsilon, \delta) > 0$ such that if $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$ and

$$K^+ < \frac{\sqrt{\epsilon}\left(1 - \frac{\epsilon}{2}\right) - \epsilon\,(2\delta + 1)\left(1 - \frac{\epsilon}{2}\right)^2}{2\delta},$$

where K^+ denotes the maximum curvature attained in (M, g), then (M, g) has no focal points and g is an Anosov metric.

Having fixed a surface (M, g), we denote by K the curvature of the surface and by K^+ the maximum curvature on the surface.

Lemma 3.4.1. *Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$. Let γ be a geodesic and let $\{(a_m, b_m)\}_{m \in I \subset \mathbb{Z}}$ be the collection of parametrized intervals corresponding to the intersection of $\gamma(t)$ with $\bigcup_{i=1}^{k} B_\delta(p_i)$. Let U be a solution of the Riccati solution in γ.*

Suppose that $U(b_m) > 0$. Then $U(t) > 0$ for all $t \in [b_m, a_{m+1}]$. In particular, for all $n > 0, n \in \mathbb{N}$, there exists $\Lambda(n) > 0$ such that if $\Lambda \geq \Lambda(n)$ then $U(a_2) > \sqrt{\epsilon}(1 - \frac{1}{n})$.

Proof.

First, since $U(t) > 0$ and considering the behavior of the norm of Jacobi fields in the region of negative curvature, we have that $U(t) > 0$ for all $t \in [b_m, a_{m+1}]$.

Now, let us study the solution of the Riccati equation outside of $\bigcup_{i=1}^{k} B_\delta(p_i)$.

The idea is to use Sturm-Liouville type arguments and compare the equations

1. $U'(t) + U^2(t) + K(t) = 0$
2. $V'(t) + V^2(t) - \epsilon = 0$

with the same initial conditions at b_m, since we know that $K < -\epsilon$ in $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$.

Recall that we are working on surfaces, and therefore, the equations are essentially one-dimensional.

The solutions of the equation $V'(t) + V^2(t) - \epsilon = 0$ are of the form

$$V(t) = \frac{\sqrt{\epsilon} e^{2\sqrt{\epsilon} t} + \sqrt{\epsilon} C}{e^{2\sqrt{\epsilon} t} - C}$$

where C is a constant. Note that $\sqrt{\epsilon}$ and $-\sqrt{\epsilon}$ are solutions. Additionally, the up solutions $\sqrt{\epsilon}$ behave like $V(t) = \epsilon \coth(\epsilon t - d)$ where d is a constant that translates the solutions. On the other hand, solutions between $\sqrt{\epsilon}$ and $-\sqrt{\epsilon}$ are solutions that tend to $-\sqrt{\epsilon}$ in the past and $\sqrt{\epsilon}$ in the future. Since we are working with a solution that at the starting point b_m we have $U(b_m) > 0$, the two types of solutions presented above are the two possibilities. That is, we are considering $V(t)$ with the same conditions as $U(t)$ at $t = b_m$.

Moreover, since $U(b_m) = V(b_m)$, we have

$$U'(b_m) - V'(b_m) = -K(b_m) - \epsilon > 0$$

and, therefore, $U(t) > V(t)$ for all $t \in [b_m, a_{m+1}]$ (a local argument that extends to the interval due to the curvature assumptions). Since $V(t) > 0$ for $t \in [b_m, a_{m+1}]$, the same holds for $U(t)$.

Now, by the definition of $V(t)$, either $V(t) > \sqrt{\epsilon}$ for all $t > b_m$ or $V(t)$ tends to $\sqrt{\epsilon}$ when $t \to \infty$. Therefore, for all n there is $\Lambda(n)$ such that for any $\Lambda \geq \Lambda(n)$ we have that $V(a_{m+2}) > \sqrt{\epsilon}(1 - \frac{1}{n})$. Then, in particular, if $b_m - a_{m+1} \geq \Lambda(n)$ we have $U(a_{m+1}) > \sqrt{\epsilon}(1 - \frac{1}{n})$.

See the sketch in Figure 3.4.

□

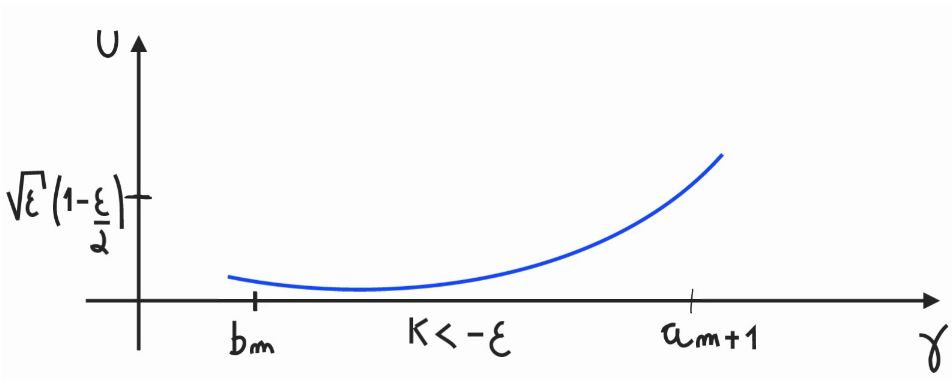

Figure 3.2: Control of the Riccati solution outside the generalized bubble.

Lemma 3.4.2. *Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$. Let γ be a geodesic and let $\{(a_m, b_m)\}_{m \in I \subset \mathbb{Z}}$ be the collection of parametrized intervals corresponding to the intersection of $\gamma(t)$ with $\bigcup_{i=1}^{k} B_\delta(p_i)$.*

Then for each $n > 0$ there is a constant $\Lambda(n) > 0$ such that if $\Lambda \geq \Lambda(n)$ and

$$K^+ < \frac{\sqrt{\epsilon}(1 - \frac{1}{n})}{2\delta} - \epsilon(1 - \frac{1}{n})^2,$$

then for all m there exists a solution of the Riccati equation $U(t)$ in γ such that $U(t) > 0$ for all $t \in [a_m, \infty)$.

In particular, along this geodesic any solution of the Riccati equation that at some point on a generalized bubble is greater than or equal to $\sqrt{\epsilon}(1 - \frac{1}{n})$ is positive *on this generalized bubble.*

Proof.

By Lemma 3.4.1, for each n there exists $\Lambda(n)$ such that if $\Lambda \geq \Lambda(n)$, then any solution of the Riccati equation that is positive when a geodesic leaves a generalized bubble is greater than $\sqrt{\epsilon}(1-\frac{1}{n})$ when the geodesic enters another generalized bubble. Furthermore, by assumptions about the generalized bubble, for all m, $b_m - a_m < 2\delta$.

Fix m and take U such that $U(b_{m-1}) > 0$. Then $U(a_m) > \sqrt{\epsilon}(1-\frac{1}{n})$. Note that if $U(t)$ is greater than $\sqrt{\epsilon}(1-\frac{1}{n})$ while the geodesic is in the generalized bubble, then the solution of the Riccati equation comes out positive for the generalized bubble and there is nothing to be done. So let us assume, without loss, that there exists $c \in (a_m, b_m)$ such that $U(c) = \sqrt{\epsilon}(1-\frac{1}{n})$.

Consider the Riccati equation in the second generalized bubble:

$$U' + U^2 + K = 0 \Longrightarrow U' = -U^2 - K.$$

That is,

$$\begin{aligned}
U(t) &= U(c) + \int_c^t U'(s)\,ds \\
&= U(c) - \int_c^t (U^2(s) + K(s))\,ds \\
&\geq U(c) - \int_c^t (\max_{[c,b_m]} U(s))^2\,ds - \int_c^t (\max_{[c,b_m]} K(s))\,ds \\
&\geq \sqrt{\epsilon}(1-\frac{1}{n}) - \int_c^t \epsilon(1-\frac{1}{n})^2\,ds - \int_c^t K^+\,ds \\
&= \sqrt{\epsilon}(1-\frac{1}{n}) - (\epsilon(1-\frac{1}{n})^2 + K^+)(t-c).
\end{aligned}$$

Thus, if

$$K^+ < \frac{\sqrt{\epsilon}(1-\frac{1}{n})}{2\delta} - \epsilon(1-\frac{1}{n})^2,$$

then $U(t) > 0$ for all $t \in [a_m, b_m]$ and, inductively, $U(t) > 0$ for all $t \in [a_m, \infty)$.

□

Corollary 3.4.3. *With the same assumptions as Lemma 3.4.2 there exists $\Lambda_0 > 0$ such that if $\Lambda \geq \Lambda_0$ and*

$$K^+ < \frac{\sqrt{\epsilon}(1-\frac{\epsilon}{2})}{2\delta} - \epsilon(1-\frac{\epsilon}{2})^2,$$

then for all m there exists a solution of the Riccati equation $U(t)$ in γ such that $U(t) > 0$ for all $t \in [a_m, \infty)$.

In particular, if

$$K^+ < \frac{\sqrt{\epsilon}\left(1-\frac{\epsilon}{2}\right) - \epsilon\,(2\delta+1)\left(1-\frac{\epsilon}{2}\right)^2}{2\delta},$$

then $U(t) > \epsilon(1-\frac{\epsilon}{2})^2$ for all $t \in [a_m, \infty)$.

Proof.

It suffices to apply the Lemma 3.4.2 with n such that $\frac{1}{n} < \frac{\epsilon}{2}$, $\Lambda(n) = \Lambda_0$ and

$$\sqrt{\epsilon}(1-\frac{\epsilon}{2}) - (\epsilon(1-\frac{\epsilon}{2})^2 + K^+)2\delta \geq \epsilon(1-\frac{\epsilon}{2})^2.$$

□

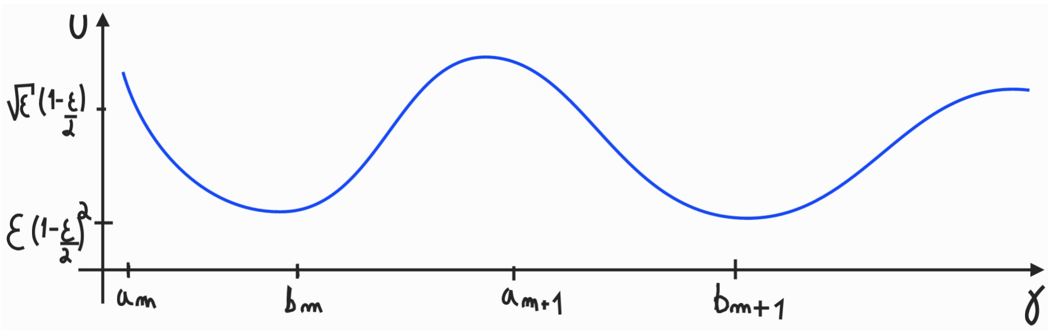

Figure 3.3: Control of Riccati solution inside and outside the generalized bubble.

Now, with the previous results, we can prove the Theorem 1.

Proof. [Theorem 1]

Let Λ be as in Corollary 3.4.3, that is, $\Lambda > 0$ such that if

$$V'(t) + V^2(t) - \epsilon = 0$$

and $V(0) > 0$ then $V(\Lambda) > \sqrt{\epsilon}(1-\frac{\epsilon}{2})$.

Let γ be a geodesic. In the case where γ does not intersect a generalized bubble, i.e., it remains in the region of negative curvature, we know that there exists a solution to the Riccati equation that never vanishes on γ. The same is true if γ is tangent to some generalized bubbles.

Furthermore, by hypothesis, geodesics that have points inside the generalized bubble must have an entry and exit point of this generalized bubble.

Now, suppose that γ intersects generalized bubbles in the sequence of parametrized intervals $\{(a_m, b_m)\}_{m\in I}$. We know that $b_m - a_m < 2\delta$.

Let J be a non-trivial Jacobi field along γ such that $J(b_1) = 0$. Let U be the solution of the Riccati equation associated with J. Then U is positive in (b_1, a_2) and by Corollary 3.4.3, $U(t) > \epsilon(1 - \frac{\epsilon}{2})^2$ for all $t \in (b_1, +\infty)$.

We claim that this is sufficient to ensure a global, non-negative solution of the Riccati equation. Indeed, note that the choice of the radial point is irrelevant (in this context, the radial point of a Jacobi field characterizes the asymptotic of the solution to the associated Riccati equation).

We can always assume that the radial points are either close to a generalized bubble or far enough from the next generalized bubble that the geodesic intersects so that the norm of U is greater than $\sqrt{\epsilon}(1 - \frac{\epsilon}{2})$, as in Lemma 3.4.1.

In other words, for every $t_n \in \mathbb{R}$ and $n \in \mathbb{N}$, there exists a solution U_n of the Riccati equation along γ such that $U_n(t) > \epsilon(1 - \frac{\epsilon}{2})^2$ for all $t \in (t_n, +\infty)$.

Consider U_n the solution of the Riccati equation with asymptote at t_n. Then $U_n(t) > \epsilon(1-\frac{\epsilon}{2})^2$ for $t \in (t_n, +\infty)$ and, by Lemma 2.2.3, there is a solution of the Riccati equation U defined for all $t \in \mathbb{R}$ and non-negative.. Since the unstable solution is the supremum of the solutions defined for all t, the unstable solution is non-negative.

A similar argument along with Lemma 2.2.3 can be made to show the existence of a solution to the stable non-positive Riccati equation. Therefore, as is true for all geodesics, by Lemma 2.2.2, the surface has no focal points.

In particular, as

$$K^+ < \frac{\sqrt{\epsilon}\left(1 - \frac{\epsilon}{2}\right) - \epsilon\,(2\delta + 1)\left(1 - \frac{\epsilon}{2}\right)^2}{2\delta},$$

along any geodesic there is a solution of the Riccati equation defined for all t with lower bound $\epsilon(1 - \frac{\epsilon}{2})^2$ and a solution of the Riccati equation defined for all t with upper bound $-\epsilon(1 - \frac{\epsilon}{2})^2$.The proof is analogous to that of Lemma 2.2.3 but with the modulus of the solutions of the Riccati equations with asymptotes being greater than $\epsilon(1-\frac{\epsilon}{2})^2$.

Let us now prove that g is an Anosov metric.

First, since g is a metric without conjugate points, we have the existence of both stable and unstable Jacobi fields. Therefore, according to [10], it suffices to show that there are no simultaneously stable and unstable Jacobi fields. This, of course, could only happen if the reference geodesic passes through the region of non-negative curvature.

However, by our assumptions, along any geodesic the solution of the unstable Riccati equation is positive and the solution of the stable Riccati equation is negative. This implies that these fields are not generated by the same Jacobi field, i.e., there is no stable and unstable Jacobi field simultaneously. And this proves the Theorem.

□

In the next chapters, we construct a conformal deformation of metrics that, when we restrict the set $\mathcal{M}(\delta, k, \epsilon, \Lambda)$, not only maintains the property of having no focal points along the path, but also defines a segment of Anosov metrics. Furthermore, the final metric of this deformation has strictly negative curvature.

We conclude the chapter with a lemma on the behavior of geodesics in generalized bubbles and later with an observation on the parameter Λ.

Lemma 3.4.4. *Let (M, g) be a surface and $\tilde{\gamma}$ be a geodesic in $(\tilde{M}, \tilde{g})$. Suppose $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$ is free of focal points. Then, for all $i = 1, \cdots, k$, if $\tilde{B_\delta}^j(p_i)$ is a connected component of the lifting of $B_\delta(p_i)$ in the universal covering, the intersection of $\tilde{\gamma}$ and $\tilde{B_\delta}^j(p_i)$ has at most one connected component. In other words, after leaving the lifting of a generalized bubble, $\tilde{\gamma}$ does not return to it.*

Proof.

Indeed, suppose $\tilde{\gamma}$ leaves $\tilde{B_\delta}^j(p_i)$ at $\tilde{\gamma}(a)$ and returns at $\tilde{\gamma}(b)$. Since generalized bubbles are strongly convex, the minimizing geodesic connecting $\tilde{\gamma}(a)$ and $\tilde{\gamma}(b)$ is contained in $\tilde{B_\delta}^j(p_i)$ Moreover, since (M, g) has no conjugate points, $\tilde{\gamma}$ is a closed geodesic. However, this leads to a contradiction, as geodesics in the universal covering of a surface without conjugate points are unbounded.

□

Remark 3.4.5. Although we do not present the value of Λ explicitly, we can estimate its length in terms of ϵ. In fact, note that the solution to the Riccati equation on a compact surface of constant curvature $-\epsilon$ such that it vanishes at 0 is given by

$$\sqrt{\epsilon}\tanh(\sqrt{\epsilon}t).$$

Then, if $\gamma : [0,\Lambda] \to M$ is a geodesic segment in (M,g) outside the generalized bubbles, we know that if U is a solution to the Riccati equation associated to γ and $U(0) \geq 0$, then

$$U(t) \geq \sqrt{\epsilon}\tanh(\sqrt{\epsilon}t)$$

for $t \in [0,\Lambda]$.

Therefore, in order to have $U(\Lambda) \geq \sqrt{\epsilon}(1-\frac{\epsilon}{2})$, it suffices that

$$\sqrt{\epsilon}(1-\frac{\epsilon}{2}) \leq \sqrt{\epsilon}\tanh(\sqrt{\epsilon}\Lambda),$$

that is, it is enough to have

$$\operatorname{artanh}(1-\frac{\epsilon}{2}) \leq \sqrt{\epsilon}\Lambda,$$

and thus, if $\Lambda \geq \frac{1}{\sqrt{\epsilon}}\operatorname{artanh}(1-\frac{\epsilon}{2})$ we have $U(\Lambda) \geq \sqrt{\epsilon}(1-\frac{\epsilon}{2})$.

Hence, we may define Λ in Theorem 1 as

$$\Lambda := \frac{1}{\sqrt{\epsilon}}\operatorname{artanh}(1-\frac{\epsilon}{2}).$$

$\diamond$

4
One parameter conformal deformations of metrics

In this chapter, we present a method of metric deformation and analyze how this deformation influences the estimates on geodesics in the deformed metrics.

The idea of deforming the metric was inspired by the work of Jane and Ruggiero [19] although the deformation model here is quite different. Jane and Ruggiero used the so-called Ricci Yang-Mills Flow a PDE system, to construct a metric perturbation. The deformation we proposed is defined in a well-defined interval $[0,1]$, and the final metric generated by the deformation has strictly negative curvature.

4.1
The family of surfaces generated by a deformation

Let us recall the definition of the family $\mathcal{M}_\rho(M, g, \delta, k, \zeta, \overline{\Lambda})$.

Definition (1.0.2)**.** Consider $\overline{\Lambda}, \zeta \in \mathbb{R}$ such that $\zeta > 0$ and $\overline{\Lambda} > 0$. Suppose that $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$, where $B_\delta(p_1), \dots, B_\delta(p_k)$ are the generalized bubbles of (M, g).

Given a family of metrics conformal to g, denoted by g_ρ and parameterized by $\rho \in [0,1]$ with $g_0 = g$, we define $\mathcal{M}_\rho(M, g, \delta, k, \zeta, \overline{\Lambda})$ as the set of metrics g_{ρ_l} for $\rho_l \in [0,1]$ such that

1. Every point where the curvature of (M, g_{ρ_l}) is non-negative is contained in $\bigcup_{i=1}^k B_\delta(p_i)$.
2. For $i = 1, \dots, k$, let $\tilde{B_\delta}^j(p_i)$ be a connected component of the lift of $B_\delta(p_i)$ in the universal cover of M, denoted by $\tilde{M}$. The distance between any two such balls with respect to the metric g_{ρ_l} is greater than $\overline{\Lambda}$.
3. The curvature of (M, g_{ρ_l}) in $(\bigcup_{i=1}^k B_\delta(p_i))^c$ is smaller than $-\zeta$.

Note that $B_\delta(p_i)$ represents a ball of radius δ centered at p_i only in the original metric, g. As the metric changes, the notion of distance also changes.

Denote by $B_\delta^{\rho_l}(p_i)$ a ball in the metric g_{ρ_l}. For clarity, we restate Theorem 2 below and dedicate this chapter to its proof.

Theorem (2)**.** Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$, with $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$. Then, there exist $w \in C^\infty(M)$ satisfying $\min_M w = 0$ such that if $g_\rho := e^{2\rho w} g$ with $\rho \in [0, 1]$ and $\mu := \max_{p\in M} w(p)$ then

$$g_{\rho_l} \in \mathcal{M}_\rho(M, g, \delta, k, e^{-2\mu}\epsilon, \Lambda)$$

for all $\rho_l \in [0, 1]$.

Moreover, (M, g_1) has strictly negative curvature, and for $i = 1, \ldots, k$, we have

$$B_\delta(p_i) \subset B^{\rho_l}_{\delta e^{\rho_l \mu}}(p_i),$$

where $B_\delta(p_i)$ is a generalized bubble of (M, g) and a ball in the metric g, while $B^{\rho_l}_{\delta e^{\rho_l \mu}}(p_i)$ is a ball of radius $\delta e^{\rho_l \mu}$ in the metric g_{ρ_l}.

As we will see, the assumption $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$ is important for the construction of the deformation. However, in a certain sense, the choice of $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$ may not be changing the choice of the initial ϵ. Let us take as an example the Gulliver-type surfaces. The region of positive curvature is constructed from surfaces with negative curvature, that is, a small strongly convex ball is removed from the surface with the original metric and replaced by a strongly convex neighborhood in a metric that admits non-negative curvature in the neighborhood. In this way, the rest of the surface remains with the same metric.

Consider the following situation, let (N, g) be a surface of genus greater than 1 with constant curvature -1. Therefore,

$$\int_N K = -vol_g(N) = 2\pi\chi(N) \Longrightarrow -\frac{2\pi\chi(N)}{vol_g(N)} = 1.$$

Now let us do some surgery and replace a ball $B_\delta(p)$ with an open B with the same points as $B_\delta(p)$ but with a metric that admits points with non-negative curvature. The paper [12] shows that it is possible to do this smoothly. Over this new metric $(N, \overline{g_\delta})$, outside B the curvature remains constant -1. And, on the other hand, for δ close to 0 we have that $vol_{\overline{g_\delta}}(N) \approx vol_g(N)$. In other words, ϵ could be chosen as close to 1 as one wants.

The same can be done with any metric with negative constant curvature.

4.2 Construction of the deformation

In this section, we construct the metric deformation function. By means of a parameter, this function defines a metric path along which the curvature inside the generalized bubbles decreases.

Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$. Let K be the sectional curvature of (M, g) and K^+ be the maximum sectional curvature. Furthermore, $vol(\cdot)$ represents the volume and dv the volume form on M. Suppose that $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$.

Let us begin by considering two Theorems already presented in the preliminaries:

- (Theorem 8) Let w be a C^∞ function, and define $g_w = e^{2w} g$. Then

$$e^{-2w}(-\triangle_g w + K) = K_w,$$

  where K_w is the sectional curvatures of g_w.
- (Theorem 9) Let h be a C^∞ function. Then there exists $w \in C^\infty(M)$ such that $h = \Delta_g w$ if and only if $\int h\, dv = 0$.

Then,

Proposition 4.2.1. *Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$, with $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$. Then, there exists $w \in C^\infty(M)$ satisfying $\min_M w = 0$ such that the deformation*

$$g_\rho := e^{2\rho w} g, \quad \textit{with } \rho \in [0, 1],$$

reduces the curvature of points with positive curvature as ρ increases, preserves the negative curvature of points that already have negative curvature, and ensures that the negative curvature remains smaller than $-\zeta$ in $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$, where $\zeta = e^{-2\mu}\epsilon$ and μ is the maximum of w in M.

Moreover, when $\rho = 1$ the curvature of g_ρ is strictly negative in M.

Proof.

First, note that if $\int h\, dv = 0$ for $h \in C^\infty$, then there exists $w \in C^\infty(M)$ such that $h = \triangle_g w$. The idea is to construct w this way.

By hypothesis, $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$, then $-\epsilon > \frac{2\pi\chi(M)}{vol(M)}$.

On the other hand, by the Gauss-Bonnet Theorem, we have that

$$\int_M K\, dv = 2\pi\chi(M) \leq -4\pi.$$

Then define $h : M \to \mathbb{R}$ by

$$h := -\Big(K - \frac{2\pi\chi(M)}{vol(M)}\Big).$$

Therefore,

<u>If $K(x) < 0$:</u>

$$K(x) + \rho h(x) = K(x)(1-\rho) + \rho\frac{2\pi\chi(M)}{vol(M)} < 0$$

for all $\rho \in [0,1]$, since $K(x) < 0$ and $\frac{2\pi\chi(M)}{vol(M)} < -\epsilon$.

<u>If $K(x) < -\epsilon$:</u>

$$K(x) + \rho h(x) = K(x)(1-\rho) + \rho\frac{2\pi\chi(M)}{vol(M)} < -\epsilon(1-\rho) - \epsilon\rho = -\epsilon$$

for all $\rho \in [0,1]$, since $K(x) < -\epsilon$ and $\frac{2\pi\chi(M)}{vol(M)} < -\epsilon$.

<u>If $K(x) \geq 0$:</u>

$$K(x) + \rho h(x) = K(x)(1-\rho) + \rho\frac{2\pi\chi(M)}{vol(M)} < K(x)$$

for all $\rho \in [0,1]$, since $K(x) \geq 0$ and $\frac{2\pi\chi(M)}{vol(M)} < -\epsilon$. In particular, when $\rho = 1$

$$K(x) + \rho h(x) = \frac{2\pi\chi(M)}{vol(M)} < 0.$$

Furthermore, it is immediate that $\int_M h\, dv = 0$.

Therefore, knowing that adding a constant to a function does not change its Laplacian, we can take $w \in C^\infty(M)$ such that $\triangle_g w = -h$ and assuming, without loss of generality, that $\min_M w = 0$. This w satisfies the conditions of the statement.

In fact, we know that

$$e^{-2\rho w}(-\rho\triangle_g w + K) = e^{-2\rho w}(-\rho h + K) = K_{\rho w}.$$

Therefore,

1. <u>If $(x \in \bigcup_{i=1}^{k} B_\delta(p_i))^c$ (by hypothesis, $K(x) < -\epsilon$):</u>

$$K_{\rho w}(x) = e^{-2\rho w(x)}(-\rho \triangle_g w(x) + K(x)) < -e^{-2\rho w(x)}\epsilon \leq -e^{-2\mu}\epsilon$$

where $\mu := \max_M w$, since $\rho \in [0,1]$, $K(x) < -\epsilon$ and $w(x) \geq 0$.

2. <u>If $K(x) < 0$:</u>

$$K_{\rho w}(x) = e^{-2\rho w(x)}(-\rho \triangle_g w(x) + K(x)) < 0,$$

since $-\rho \triangle_g w(x) + K(x) < 0$ as previously done and $e^{-2\rho w(x)} > 0$.

3. <u>If $K(x) \geq 0$:</u>

$$K_{\rho w}(x) = e^{-2\rho w(x)}(-\rho \triangle_g w(x) + K(x)) < e^{-2\rho w(x)} K(x) \leq K(x),$$

because $-\rho \triangle_g w(x) + K(x) < K(x)$ and $w(x) \geq 0$. In particular, $K_{\rho w}(x) < 0$ if $\rho = 1$ since $-\triangle_g w(x) + K(x) < 0$.

In other words, the curvature function decreases in the regions of positive curvature as ρ increases and maintains a negative upper bound in $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$. Furthermore, when $\rho = 1$ the curvature is strictly negative.

This proves the theorem.

□

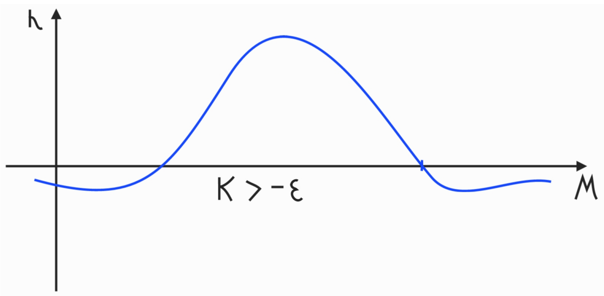

Figure 4.1: Behavior of h.

We use the deformation constructed in Proposition 4.2.1 for the next lemmas of this chapter, and to prove Theorem 2.

4.3
Estimates of length and distance between generalized bubbles.

When we change the metric, we inevitably change the geodesics, Jacobi fields, curvatures, and other important features. However, it is crucial to estimate these changes. We start working on some of these estimates below.

Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$ and w as in Proposition 4.2.1.

The deformation we have constructed makes it possible to estimate the length of the curves.

Lemma 4.3.1. *Let $\sigma : [a, b] \to M$ be a smooth curve. Then the length of σ in the g metric is smaller than (or equal to) the length of σ in the g_ρ metric.*

Proof.
In fact, since $g_\rho = e^{2\rho w}$ with $w \geq 0$, we have

$$g_\rho(X, X) \geq g(X, X)\, \forall x \in TM.$$

Then,

$$L(\sigma) - L_\rho(\sigma) = \int_a^b ||\sigma'(t)||_g \, dt - \int_a^b ||\sigma'(t)||_{g_\rho} \, dt \leq 0.$$

□

Corollary 4.3.2. *The distance between points in M does not decrease as ρ increases.*

Proof.
Let p and q be points in M. Let γ_ρ be the geodesic segment connecting p and q in the metric g_ρ. By Lemma 4.3.1,

$$L(\gamma_\rho) \leq L_\rho(\gamma_\rho).$$

Therefore, since the distance between p and q in the metric g is less than $L(\gamma_\rho)$ the result follows.

□

Remark 4.3.3. Although the region of non-negative curvature is decreasing as a set of points during the conformal deformation, this does not guarantee that the maximum distance between points with non-negative curvature is decreasing. And, in fact, the distance between points with non-negative curvature may be increasing in their respective metrics,as we show in the Corollary 4.3.2. ◇

Let us now verify how the distance between the generalized bubbles behaves as the parameter ρ increases.

Lemma 4.3.4. *Consider $\tilde{B}^{j_1}_{\delta}(p_{i_1}), \tilde{B}^{j_2}_{\delta}(p_{i_2}) \in (\tilde{M}, \tilde{g})$ generalized bubble lifting of (M, g) such that $i_1 \neq i_2$ or $j_1 \neq j_2$. Then, if $\rho_2, \rho_1 \in [0, 1]$ with $\rho_2 > \rho_1$,*

$$\tilde{d}_{\rho_2}(\tilde{B}^{j_1}_{\delta}(p_{i_1}), \tilde{B}^{j_2}_{\delta}(p_{i_2})) \geq \tilde{d}_{\rho_1}(\tilde{B}^{j_1}_{\delta}(p_{i_1}), \tilde{B}^{j_2}_{\delta}(p_{i_2})),$$

where $\tilde{d}_{\rho_2}$ and $\tilde{d}_{\rho_1}$ are the distances functions on $\tilde{g}_{\rho_2}$ and $\tilde{g}_{\rho_1}$ respectively.

Proof.

Given ρ we consider $(\tilde{M}, \tilde{g}_{\rho})$ the covering of M with the metric given by the pullback of g_{ρ}. Then, since $w \geq 0$,

$$\tilde{g}_{\rho_2}(X, X) = e^{2\rho_2(w\circ\pi)}\tilde{g}(X, X) \geq e^{2\rho_1(w\circ\pi)}\tilde{g}(X, X) = \tilde{g}_{\rho_1}(X, X) \ , \ \forall X \in T\tilde{M}.$$

Consider $q_1 \in \tilde{B}^{j_1}_{\delta}(p_{i_1})$ and $q_2 \in \tilde{B}^{j_2}_{\delta}(p_{i_2})$. Let $\tilde{\gamma}_{\rho_2} : [c, d] \to \tilde{M}$ be the minimizing geodesic segment connecting q_1 and q_2. Then

$$\tilde{L}_{\rho_2}(\tilde{\gamma}_{\rho_2}) - \tilde{L}_{\rho_1}(\tilde{\gamma}_{\rho_2}) = \int_c^d (||\tilde{\gamma}'_{\rho_2}(t)||_{\tilde{g}_{\rho_2}} - ||\tilde{\gamma}'_{\rho_2}(t)||_{\tilde{g}_{\rho_1}})dt \geq 0.$$

On the other hand,

$$\tilde{d}_{\rho_1}(q_1, q_2) \leq \tilde{L}_{\rho_1}(\tilde{\gamma}_{\rho_2}) \leq \tilde{L}_{\rho_2}(\tilde{\gamma}_{\rho_2}) = \tilde{d}_{\rho_2}(q_1, q_2).$$

Therefore, as it is true for any pair of points in $\tilde{B}^{j_1}_{\delta}(p_{i_1}), \tilde{B}^{j_2}_{\delta}(p_{i_2})$ the result is true.

□

4.4 The maximum radius of the generalized bubble

It is also important to estimate the length of a generalized bubble after the metric is deformed.

Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$ and w as in Proposition 4.2.1.

Lemma 4.4.1. *$B_{\delta}(q) \subset B^{\rho}_{\delta e^{\rho\mu}}(q)$ where $B^{\rho}_{\delta e^{\rho\mu}}(q)$ represents the ball in the metric g_{ρ}. In particular, for all $\rho \in [0, 1]$, $B_{\delta}(q) \subset B^{\rho}_{\delta e^{\mu}}(q)$.*

Proof.

Fix $\rho \in [0, 1]$. Consider $p \in B_{\delta}(q)$.

Let $\gamma : [a, b] \to M$ be the geodesic segment connecting q and p. In a similar way to what we did in Lemma 4.3.1, we have

$$L_\rho(\gamma) = \int_a^b ||\gamma'(t)||_{g_\rho}\, dt = \int_a^b \sqrt{g_\rho(\gamma'(t), \gamma'(t))}\, dt,$$

such that,

$$L_\rho(\gamma) = \int_a^b e^{\rho w}\sqrt{g(\gamma'(t), \gamma'(t))}\, dt \leq e^{\rho\mu}L(\gamma)$$

Then, since $p \in B_\delta(q)$,

$$L_\rho(\gamma) \leq \delta e^{\rho\mu}.$$

Note that γ may not be minimizing with respect to the metric g_ρ. However, we have

$$d_\rho(p, q) \leq \delta e^{\rho\mu} \quad \text{and thus} \quad B_\delta(q) \subset B^\rho_{\delta e^{\rho\mu}}(q).$$

□

4.5
Proof of Theorem 2

Now we prove Theorem 2.

Proof. [Theorem 2]

By Proposition 4.2.1, for $i = 1, \cdots, k$, there exists a deformation $g_\rho := e^{2\rho w}g$ with $\rho \in [0, 1]$ that maintains the curvature smaller than $-e^{-2\mu}\epsilon$ in $(\bigcup_{i=1}^k B_\delta(p_i))^c$, the non-negative curvature in $\bigcup_{i=1}^k B_\delta(p_i)$ for all $\rho \in [0, 1]$ and defines a metric with strictly negative curvature when $\rho = 1$.

By Lemma 4.4.1, $B_\delta(p_i) \subset B^\rho_{\delta e^\mu}(p_i)$. Note that, for all $\rho \in [0, 1]$, the curvature of g_ρ is smaller than $-e^{-2\mu}\epsilon$ in $(\bigcup_{i=1}^k B_{\delta e^\mu}(p_i))^c$.

Furthermore, the distance between points does not decrease by Corollary 4.3.2 as ρ increase. On the other hand, since $(M, g) \in \mathcal{M}(M, \delta, k, \epsilon, \Lambda)$,the distance between the lifting of the generalized bubbles is bigger than Λ in the original metric. Then, by Lemma 4.3.4,for all for all $\rho \in [0, 1]$, the distance g_ρ between the lifting of the generalized bubbles is bigger than Λ.

Therefore, $(M, g_{\rho_l}) \in \mathcal{M}_\rho(M, B_\delta(p_i), k, e^{-2\mu}\epsilon, \Lambda)$, $B_\delta(pi) \subset B^{\rho_l}_{\delta e^{\rho_l\mu}}(p_i)$ for all $\rho_l \in [0, 1]$ and (M, g_1) has strictly negative curvature.

□

5
Deformation without focal points and Anosov

In this chapter we show conditions so that, in addition to the original surface not having focal points, the surfaces generated by the deformation of the metric do not have focal points and are formed by Anosov metrics.

Our goal in this section is to prove the final result of this thesis, that is, to prove the following theorem:

Theorem (3)**.** Let $0 < \epsilon < 1$ and $\delta > 0$ be constants, and define

$$\Lambda := \frac{1}{\sqrt{\epsilon}} \operatorname{artanh}\left(1 - \tfrac{\epsilon}{2}\right).$$

Suppose that the Riemannian manifold (M, g) satisfies

$$(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda) \quad \text{and} \quad \epsilon < \frac{-2\pi\,\chi(M)}{\operatorname{vol}(M)}.$$

Then there exists a smooth function $w \in C^\infty(M)$ such that, if the maximum sectional curvature K^+ of (M, g) satisfies

$$K^+ < \frac{\sqrt{\epsilon}}{4\,e^{2\mu}\,\delta}\left[\tanh\left(e^{-\mu\,\frac{\ln 3}{3}}\right) - \sqrt{\epsilon}\,e^{-\mu}\,\tanh^2\left(e^{-\mu\,\frac{\ln 3}{3}}\right)(4\,e^{\mu}\,\delta + 1)\right],$$

where $\mu := \max_M w$, then the conformal family of metrics

$$g_\rho := e^{2\rho\,w}\,g, \qquad \rho \in [0, 1],$$

consists entirely of Anosov metrics without focal points. In particular, (M, g_1) has strictly negative curvature.

Remark 5.0.1. It is worth noting that there are surfaces whose metric admits regions of positive curvature and satisfies the hypotheses of Theorem. Indeed, one can consider neighborhoods of metrics on surfaces with nonnegative curvature.

Let us consider the following situation: let $(M, \mathfrak{g})$ be a surface with nonpositive curvature such that $(M, \mathfrak{g}) \in \mathcal{M}(\frac{\delta}{2}, k, 2\epsilon, 2\Lambda)$. Then, if δ is sufficiently small, there exists a small C^∞-neighborhood $V_{\mathfrak{g}}$ of the metric $\mathfrak{g}$ such that for every $g \in V_{\mathfrak{g}}$, we have $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$, since the assumptions of strong convexity and absence of focal points for the generalized bubbles hold for sufficiently small balls.

On the other hand, consider the function

$$h_{\mathfrak{g}} := -\left(K_{\mathfrak{g}} - \frac{2\pi\chi(M)}{\mathrm{vol}_{\mathfrak{g}}(M)}\right).$$

Associated with the function $h_{\mathfrak{g}}$, we define the function $w_{\mathfrak{g}}$ as in Proposition 4.2.1. In fact, since both the curvature and the volume depend continuously on the metric, the function

$$h_g := -\left(K_g - \frac{2\pi\chi(M)}{\mathrm{vol}_g(M)}\right)$$

also depends continuously on g. Therefore, by the definition of w_g as the solution to the inverse Laplacian problem, we have that, in a C^∞-neighborhood of $\mathfrak{g}$, the map

$$F : V_{\mathfrak{g}} \to C^\infty(M), \quad g \mapsto w_g,$$

where $w_g = \Delta_g h_g$ with $\min_M w_g = 0$, is continuous.

Thus, since by hypothesis $K_{\mathfrak{g}} \leq 0$, the surface $(M, \mathfrak{g})$ satisfies the curvature assumptions of Theorem 3, and by continuity, the same holds for metrics in a neighborhood of $(M, \mathfrak{g})$. Consequently, there exist metrics admitting regions of positive curvature that still satisfy the assumptions of Theorem 3. $\diamond$

Consider $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$ such that $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$,

$$\Lambda = \frac{1}{\sqrt{\epsilon}} \operatorname{artanh}\left(1 - \frac{\epsilon}{2}\right)$$

and let w be as in Proposition 4.2.1.We will continue to denote $B_\delta(p)$ a ball in the metric g and $B_\delta(p_1), \cdots, B_\delta(p_k)$ the generalized bubbles of (M, g). Recall that $\tilde{B}^1_\delta(p_1), \cdots, \tilde{B}^1_\delta(p_k), \cdots, \tilde{B}^l_\delta(p_k), \cdots$ represent the lifting of generalized bubbles in $(\tilde{M}, \tilde{g})$. Recall that by the definition of w and by Lemma 4.3.4 the distance between bubbles remains greater than Λ when deforming the metric.

Furthermore, denote by $B^\rho_r(p_i)$ a ball of radius r on the surface (M, g_ρ) and $\tilde{B}^\rho_r(p^j_i)$ a ball on $(\tilde{M}, \tilde{g}_\rho)$ where p^j_i is a lifting of p_i.

To prove the result we need to display and demonstrate some technical results. Let us start with an estimate of the growth of a solution of the Riccati equation outside the generalized bubbles after deforming the initial metric. Remember that by Remark 3.4.5, we can take $\Lambda = \frac{1}{\sqrt{\epsilon}}\operatorname{artanh}(1-\frac{\epsilon}{2})$ in Theorem 1. Furthermore,

Lemma 5.0.2. *Let γ_ρ be a geodesic in (M, g_ρ) such that $\gamma_\rho\big|_{[0,\overline{\Lambda}]} : [0,\overline{\Lambda}] \to \left(\cup_{i=1}^k B_\delta(p_i)\right)^c$, with $\overline{\Lambda} \geq \Lambda = \frac{1}{\sqrt{\epsilon}}\operatorname{artanh}\left(1-\frac{\epsilon}{2}\right)$. Then, if U_ρ is a solution to the Riccati equation along γ_ρ such that $U_\rho(0) \geq 0$, we have*

$$U_\rho(\overline{\Lambda}) > e^{-\mu}\sqrt{\epsilon}\tanh\left(\frac{1}{2}e^{-\mu}\ln(3)\right).$$

Proof.

To prove the result, observe that for every $\rho \in [0,1]$, the curvature K_ρ in

$$\left(\bigcup_{i=1}^{k} B_\delta(p_i)\right)^c$$

satisfies $K_\rho < -\zeta = -e^{-2\mu}\epsilon$. Hence, by comparison with the solution of the Riccati equation of constant curvature $-\zeta$, we obtain

$$U_\rho(\Lambda) > \sqrt{\zeta}\,\tanh\big(\sqrt{\zeta}\,\Lambda\big),$$

and therefore

$$\begin{aligned} U_\rho(\Lambda) &> e^{-\mu}\sqrt{\epsilon}\,\tanh\big(e^{-\mu}\sqrt{\epsilon}\,\Lambda\big) \\ &> e^{-\mu}\sqrt{\epsilon}\,\tanh\Big(e^{-\mu}\sqrt{\epsilon}\,\frac{1}{\epsilon}\,\operatorname{artanh}\big(1-\tfrac{\epsilon}{2}\big)\Big) \\ &> e^{-\mu}\sqrt{\epsilon}\,\tanh\big(e^{-\mu}\,\operatorname{artanh}\big(1-\tfrac{\epsilon}{2}\big)\big). \end{aligned}$$

Since $\epsilon < 1$, we have

$$\operatorname{artanh}\big(1-\tfrac{\epsilon}{2}\big) \geq \operatorname{artanh}\big(\tfrac{1}{2}\big) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \tfrac{1}{2}\ln(3),$$

and consequently

$$U_\rho(\Lambda) > e^{-\mu}\sqrt{\epsilon}\,\tanh\Big(\tfrac{1}{2}\,e^{-\mu}\ln(3)\Big).$$

Therefore

$$U_\rho(\overline{\Lambda}) > e^{-\mu}\sqrt{\epsilon}\,\tanh\Big(\tfrac{1}{2}\,e^{-\mu}\ln(3)\Big)$$

by the behavior of the solution of the Riccati equation at curvature less than $-\zeta$.

∎

Lemma 5.0.3. *Let $\tilde{B}^j_\delta(p_i)$ be a connected component of the lifting of a generalized bubble in $(\tilde{M}, \tilde{g})$. Then if (M, g_ρ) has no focal points and $\tilde{\gamma}_\rho$ is a g_ρ−geodesic with*

$\tilde{\gamma}_\rho(0) \in \tilde{B}^j_\delta(p_i)$ then $\tilde{\gamma}_\rho(t) \notin \tilde{B}^j_\delta(p_i)$ if $t \notin (-2\delta e^\mu, 2\delta e^\mu)$.

In particular, after leaving $\tilde{B}^\rho_{2\delta e^\mu}(p^j_i)$ the geodesic $\tilde{\gamma}_\rho$ does not return to $\tilde{B}^j_\delta(p_i)$.

Proof.

By Lemma 4.4.1, $\tilde{B}^j_\delta(p_i)$ is contained in $\tilde{B}^\rho_{\delta e^\mu}(p^j_i)$, that is, in the ball of metric $\tilde{g}_\rho$ of radius δe^μ and center p^j_i. Therefore, for every $\tilde{q}_1, \tilde{q}_2 \in \tilde{B}^j_\delta(p_i)$ there exists a g_ρ−piecewise geodesic of length smaller than $2\delta e^\mu$ connecting $\tilde{q}_1$ and $\tilde{q}_2$. Therefore, the length of the minimizing geodesic segment connecting $\tilde{q}_1$ and $\tilde{q}_2$ is smaller than $2\delta e^\mu$.

Then, since by hypothesis (M, g_ρ) has no focal points and, therefore, no conjugate points, $\tilde{\gamma}_\rho$ leaves $\tilde{B}^j_\delta(p_i)$ for some $t \in (0, 2\delta\epsilon^\mu)$ and, by Lemma 3.4.4, does not return to $\tilde{B}^j_\delta(p_i)$.

□

Remark 5.0.4. In fact what we are proving in Lemma 5.0.3 is that if (M, g_ρ) has no focal points, for all $r > 0$ if $\tilde{\gamma}_\rho$ is a geodesic in $(\tilde{M}, \tilde{g}_\rho)$ such that $\tilde{\gamma}_\rho(0) \in \tilde{B}^\rho_r(p^j_i)$ we have that $\tilde{\gamma}_\rho(t) \notin \tilde{B}^\rho_r(p^j_i)$ for all $t \in \mathbb{R} \setminus (-2r, 2r)$, or even, after leaving $\tilde{B}^\rho_{2r}(p^j_i)$, $\tilde{\gamma}_\rho$ does not return to $\tilde{B}^\rho_r(p^j_i)$ ⋄

Lemma 5.0.5. *Consider $\zeta_1 = e^{-2\mu}\epsilon$. There exists $r > 0$ such that if*

- *$(M, g_{\overline{\rho}})$ is free of focal points,*
- *for each generalized bubble $B_\delta(p_i)$ and $g_{\overline{\rho}}$−geodesic $\gamma_{\overline{\rho}}$ such that $\gamma_{\overline{\rho}}(0) \in B_\delta(p_i)$, if $\tilde{\gamma}_{\overline{\rho}}$ is the lifting of $\gamma_{\overline{\rho}}$ such that $\tilde{\gamma}_{\overline{\rho}}(0) \in \tilde{B}^j_\delta(p_i)$, then if (a, b) is the connected component of $\tilde{\gamma}_{\overline{\rho}} \cap \tilde{B}^\rho_{2\delta e^\mu}(p^j_i) = (a, b)$ which contains $\tilde{\gamma}_{\overline{\rho}}(0)$, if U is a solution of the Riccati equation in $\gamma_{\overline{\rho}}$ such that $U(c) > e^{-\mu}\sqrt{\epsilon}\tanh(e^{-\mu}\frac{1}{3}\ln(3))$ for some $c \in (a, b)$ then $U(t) > e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{3}\ln(3))$ for all $t \in (a, b)$,*
- *for each $g_{\overline{\rho}}$−geodesic $\gamma_{\overline{\rho}}$ that intersects a generalized bubble $B_\delta(p_i)$ in an interval (a, b) we have that if U is a solution of the Riccati equation such that $U(b) > 0$ then $U(b + \overline{\Lambda}) > e^{-\mu}\sqrt{\epsilon}\tanh(e^{-\mu}\frac{1}{2}\ln(3))$ if $\gamma_{\overline{\rho}}(t) \notin B_\delta(p_i)$ for all $t \in (b, b + \Lambda)$;*

then (M, g_ρ) is free of focal points if $\rho \in [\overline{\rho}, \rho_1]$ with $\rho_1 = \min\{\overline{\rho} + r, 1\}$.

Proof.

In fact, let γ_ρ be a g_ρ−geodesic. The proof will be done in the following stages:

1. Intersection of geodesics and generalized bubbles in the universal covering.

If γ_ρ does not intersect a generalized bubble then γ_ρ has no focal points. Suppose then that there exists s_1 such that $\gamma_\rho(s_1)$ is in a generalized bubble. Let us denote this generalized bubble by $B_\delta(p_i)$. Let $\tilde{\gamma}_\rho$ be the lifting of γ_ρ in $(\tilde{M}, \tilde{g})$ such that $\tilde{\gamma}_\rho(s_1) \in \tilde{B}^j_\delta(p_i)$.

Let $\gamma^{s_1}_{\overline{\rho}}$ be a $g_{\overline{\rho}}$−geodesic with the same initial conditions as γ_ρ in $\gamma_\rho(s_1)$, that is,

$$\gamma^{s_1}_{\overline{\rho}}(s_1) = \gamma_\rho(s_1) \text{ and } \gamma'_{\overline{\rho}}(x) = \frac{\gamma'_\rho(x)}{||\gamma'_\rho(x)||_{\overline{\rho}}}.$$

Let $\tilde{\gamma}^{s_1}_{\overline{\rho}}$ be the lifting of $\gamma^{s_1}_{\overline{\rho}}$ in $(\tilde{M}, \tilde{g_{\overline{\rho}}})$ such that $\tilde{\gamma}_\rho(s_1) = \tilde{\gamma}_{\overline{\rho}}(s_1)$. Since $(M, g_{\overline{\rho}})$ has no focal points, $\tilde{\gamma}^{s_1}_{\overline{\rho}}$ is not contained in any ball and if it leaves $\tilde{B}^{\overline{\rho}}_{2\delta e^\mu}(p^j_i)$ it does not return to $\tilde{B}^{\overline{\rho}}_{\delta e^\mu}(p^j_i)$, by Lemma 5.0.3. In particular, there exists (a, b) such that $s_1 \in (a, b)$ and (a, b) is the unique connected component of $\tilde{\gamma}^{s_1}_{\overline{\rho}} \cap \tilde{B}^{\overline{\rho}}_{2\delta e^\mu}(p^j_i)$ that contains points of $\tilde{B}^j_\delta(p_i)$.

2. Estimate of the solution of the Riccati equation of the metric $g_{\overline{\rho}}$ in the generalized bubble.

   Let $U^{s_1}_\rho(t)$ be a solution of the Riccati equation in γ_ρ such that $U^{s_1}_\rho(s_1) > e^{-\mu}\sqrt{\epsilon}\tanh(e^{-\mu}\frac{1}{2}\ln(3)) > e^{-\mu}\sqrt{\epsilon}\tanh(e^{-\mu}\frac{1}{3}\ln(3))$.

   Consider also $U^{s_1}_{\overline{\rho}}$ solution of the Riccati equation in $\gamma^{s_1}_{\overline{\rho}}$ such that

$$U^{s_1}_{\overline{\rho}}(s_1) = U^{s_1}_\rho(s_1).$$

   By the hypotheses, $U^{s_1}_{\overline{\rho}}(t) > e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{3}\ln(3))$ for all $t \in (a, b)$. On the other hand, since $\tilde{\gamma}^{s_1}$ does not return to $\tilde{B}^{\overline{\rho}}_{\delta e^\mu}(p^j_i)$ after (a, b) and, furthermore, the distance between the generalized bubbles is greater than Λ since the distances increase after deformation, then, by the behavior of the solution of the Riccati equation in the region of curvature less than $-\zeta$, $U^{s_1}_{\overline{\rho}}(t) > e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{3}\ln(3))$ for all $t \in (a, b + \Lambda)$. Fix $\lambda < \frac{\Lambda}{2}$.

3. The intersection between $\tilde{\gamma}_\rho$ and $\partial\tilde{B}^j_\delta(p_i)$.

   By the continuous dependence of geodesics, there exists $r_1 > 0$ such that for every $\overline{\rho} \in [0, 1]$ if $r_1 \geq r' > 0$, $\gamma_{\overline{\rho}}$ is a $g_{\overline{\rho}}$−geodesic and $\gamma_{\overline{\rho}+r'}$ is a

$g_{\overline{\rho}+r}$−geodesic such that

$$\gamma_{\overline{\rho}+r'}(x) = \gamma_{\overline{\rho}}(x) \text{ and } \gamma'_{\overline{\rho}+r'}(x) = \frac{\gamma'_{\overline{\rho}}(x)}{||\gamma'_{\overline{\rho}}(x)||_{\overline{\rho}+r'}}$$

then

$$d_{\overline{\rho}}(\tilde{\gamma}_{\rho_{\overline{\rho}+r}}(t), \tilde{\gamma}_{\overline{\rho}}(t)) \leq d_1(\tilde{\gamma}_{\overline{\rho}+r}(t), \tilde{\gamma}_{\overline{\rho}}(t)) < \frac{\lambda}{3}$$

for all $t \in [x - 4\delta\epsilon^{\mu} - \Lambda, x + 4\delta\epsilon^{\mu} + \Lambda]$. Note that this interval was chosen to ensure that $\tilde{\gamma}_{\overline{\rho}}$ does not return to the neighborhood of $\tilde{B}^j_{\delta}(p_i)$.

Therefore, if $\rho \in [\overline{\rho}, \rho_l]$ with $\rho_l = \min\{\overline{\rho} + r_1, 1\}$ then there exists $\overline{s_2} \in (s_1, b + \lambda)$ such that $\tilde{\gamma}_{\rho}(\overline{s}_2) \in \partial\tilde{B}^{\overline{\rho}}_{2\delta\epsilon^{\mu}+\lambda}(p^j_i)$.

Note that $d(\tilde{\gamma}^{s_1}_{\overline{\rho}}(s_1 + 4\delta\epsilon^{\mu} + 2\lambda), \tilde{B}^j_{\delta}(p_i)) > \lambda$ and $d(\tilde{\gamma}^{s_1}_{\overline{\rho}}(s_1 - 4\delta\epsilon^{\mu} - 2\lambda), \tilde{B}^j_{\delta}(p_i)) > \lambda$. The same is true for each $t \notin [s_1 - 4\delta\epsilon^{\mu} - 2\lambda, s_1 + 4\delta\epsilon^{\mu} + 2\lambda]$.

Then consider s_2 to be the largest $s \in (s_1, \overline{s}_2)$ such that $\tilde{\gamma}_{\rho}(s) \in \partial\tilde{B}^j_{\delta}(p_i)$ and hence $\gamma_{\rho}(s_2) \in \partial B_{\delta}(p_i)$.

4. Estimation of the solution of the Riccati equation of the metric g_{ρ} in the generalized bubble.

   By continuous dependence of geodesics and solutions of the Riccati equation, there exists $r_2 > 0$ such that for every $\overline{\rho} \in [0, 1]$ if $r_2 \geq r' > 0$, $\gamma_{\overline{\rho}}$ is a $g_{\overline{\rho}}$−geodesic and $\gamma_{\overline{\rho}+r'}$ is a $g_{\overline{\rho}+r}$−geodesic such that

$$\gamma_{\overline{\rho}+r'}(x) = \gamma_{\overline{\rho}}(x) \text{ and } \gamma'_{\overline{\rho}+r'}(x) = \frac{\gamma'_{\overline{\rho}}(x)}{||\gamma'_{\overline{\rho}}(x)||_{\overline{\rho}+r'}}$$

and $U_{\overline{\rho}}, U_{\overline{\rho}+r'}$ are solutions of the Riccati solution respectively associated with $\gamma_{\overline{\rho}}$ and $\gamma_{\overline{\rho}+r'}$ with $U_{\overline{\rho}}(x) = U_{\overline{\rho}+r'}(x)$ then

$$|U_{\overline{\rho}}(y) - U_{\overline{\rho}+r'}(y)| < \frac{e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{3}\ln(3))}{4}$$

for $y \in [x - 4\delta\epsilon^{\mu} - \Lambda, x + 4\delta\epsilon^{\mu} + \Lambda]$, that is, greater than the length of any $g_{\overline{\rho}}$-geodesic segment in a generalized bubble.

Then, if $\rho_1 = \min\{\overline{\rho} + r_1, \overline{\rho} + r_2, 1\}$ and $\gamma_{\rho}(s_2)$ is the exit point of γ_{ρ} from $B_{\delta}(p_i)$, we have that $U^{s_1}_{\rho}(s_2) > \frac{3e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{3}\ln(3))}{4}$.

5. Additional Remarks on the Exit from a Neighborhood of a Generalized Bubble

   In the case where γ_ρ does not cross a generalized bubble again, there is nothing more to do since the geodesic will be contained in the region of negative curvature. Suppose then that γ_ρ crosses a generalized bubble $B_\delta(p_l)$.

   Then let $\gamma_{\overline{\rho}}^{s_2}$ be a $g_{\overline{\rho}}$−geodesic with the same initial conditions as γ_ρ in $\gamma_\rho(s_2)$ and let $U_{\overline{\rho}}^{s_2}$ be the solution of the Riccati equation in $\gamma_{\overline{\rho}}^{s_2}$ such that

$$U_{\overline{\rho}}^{s_2}(s_2) = U_\rho^{s_1}(s_2).$$

   Let us denote by $\overline{s}_3 \in [s_1, s_1 + 4\delta\epsilon^\mu + 2\lambda]$ the time such that $\tilde{\gamma}_{\overline{\rho}}^{s_2}(s) \notin \tilde{B}_\delta^j(p_i)$ for all $s \in [\overline{s}_3, +\infty)$. The existence of such a time follows from the fact that $\tilde{\gamma}_\rho(\overline{s}_2) \in \partial\tilde{B}_{2\delta e^\mu+\lambda}^{\overline{\rho}}(p_i^j)$ and that the distance between $\tilde{\gamma}_\rho$ and $\tilde{\gamma}_{\overline{\rho}}^{s_1}$ is smaller than $\frac{\lambda}{3}$ in the neighborhood of the generalized bubble, since $\rho_l \leq \overline{\rho} + r_1$.

   Observe that $\gamma_\rho(s) \notin \bigcup_{i=1}^k B_\delta(p_i)$ for all $s \in [s_2, \overline{s}_3]$, as $\overline{s}_3 \in [s_2, \tilde{s}_2]$.

   Since $K_\rho < -\zeta_1$ in $(\bigcup_{i=1}^k B_\delta(p_i))^c$, it follows that

$$U_\rho^{s_1}(t) > \frac{3e^{-2\mu}\epsilon \tanh^2(e^{-\mu}\frac{1}{3}\ln(3))}{4}, \quad \text{for } t \in [s_2, \overline{s}_2].$$

   We should also note that, since the distance between $\tilde{\gamma}_\rho$ and $\tilde{\gamma}_{\overline{\rho}}^{s_1}$ is smaller than $\frac{\lambda}{3}$ on the interval, it follows that $\tilde{\gamma}_{\overline{\rho}}^{s_2}(s) \notin \tilde{B}_{\delta+\frac{2\lambda}{3}}^j(p_i)$ for all $s \in [\overline{s}_2, +\infty)$.

6. Estimation of the solution of the Riccati equation in the metric g_ρ.

   By the continuous dependence of geodesics and the assumptions regarding the choice of r_1, we know that

$$d_1(\gamma_\rho(t), \tilde{\gamma}_{\overline{\rho}}^{s_2}(t)) < \frac{\lambda}{3}$$

   for all $t \in [s_2 - 4\delta\epsilon^\mu - \Lambda,\ s_2 + 4\delta\epsilon^\mu + \Lambda]$.

   Thus, since $\tilde{\gamma}_\rho(t) \notin \tilde{B}_\delta(p_i^j)$ for all $t \in [s_2, \overline{s}_2]$, and $\tilde{\gamma}_{\overline{\rho}}^{s_2}(s) \notin \tilde{B}_{\delta+\frac{2\lambda}{3}}^j(p_i)$ for all $s \in [\overline{s}_2, +\infty)$, it follows that

$$\tilde{\gamma}_\rho(t) \notin \tilde{B}_\delta(p_i^j)$$

for all $t \in [s_2, s_2 + \Lambda]$. Hence, since distances do not decrease under the deformation of the metrics, we have

$$\gamma_\rho(t) \in \left(\bigcup_{i=1}^{k} B_\delta(p_i)\right)^c$$

for all $t \in [s_2, s_2 + \Lambda]$, and

$$U_\rho^{s_1}(s_2) > \frac{3e^{-2\mu}\epsilon \tanh^2\left(e^{-\mu}\frac{1}{3}\ln(3)\right)}{4}.$$

Then, by Lemma 5.0.2, it follows that $U_\rho^{s_1} > e^{-\mu}\sqrt{\epsilon}\tanh\left(e^{-\mu}\frac{1}{2}\ln(3)\right)$ when γ_ρ enters another generalized bubble. In particular, $U_\rho^{s_1} > 0$ for all $s \in [s_1, s_2+\Lambda]$. Therefore, steps 1 through 6 repeat.

7. (M, g_ρ) is free of focal points.

   Inductively we have $U_\rho^{s_1}(t) > 0$ for $t > s_1$. Note that the choice of s_1 is arbitrary. If we choose a starting point outside a generalized bubble, when the geodesic enters the generalized bubble the lower bound condition still holds.

Then by Lemma 2.2.3, just like in Theorem 1, there is a solution of the non-negative Riccati equation defined for all time and a solution of the non-positive Riccati equation. Furthermore, the module of these solutions is greater than or equal to $e^{-2\mu}\epsilon \tanh^2(e^{-\mu}\frac{1}{3}\ln(3))$.

In particular, by Lemma 2.2.2, γ_ρ has no focal points if $\rho \in [\overline{\rho}, \rho_1]$ with

$$\rho_1 = \min\{\overline{\rho} + r, 1\}$$

with

$$r = \min\{r_1, r_2, r_3\}.$$

Since it holds for all γ_ρ, (M, g_ρ) has no focal points and since $r > 0$ is chosen uniformly the lemma is proved for all $\overline{\rho} \in [0, 1)$.

□

Note that $r > 0$ is chosen uniformly, i.e., it does not depend on the choice of $\overline{\rho} \in [0, 1]$.

Now, with the last lemmas, we can finally prove the results about the non-existence of focal points and, subsequently, the result of the metric path generated by the deformation being Anosov metrics.

Proposition 5.0.6. *Consider* $0 < \epsilon < 1$, $\delta > 0$ *and* $\Lambda := \frac{1}{\sqrt{\epsilon}} \operatorname{artanh}\left(1 - \frac{\epsilon}{2}\right)$. *Suppose that* $(M, g) \in \mathcal{M}(\delta, k, \epsilon, \Lambda)$ *such that* $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$. *Then, there exists a function* $w \in C^\infty(M)$ *with* $\min_M w = 0$ *such that if the maximum curvature* K^+ *of* (M, g) *satisfies*

$$K^+ < \frac{\sqrt{\epsilon}}{4e^{2\mu}\,\delta}\left[\tanh\left(e^{-\mu}\tfrac{1}{3}\ln 3\right) - \sqrt{\epsilon}\,e^{-\mu}\,\tanh^2\left(e^{-\mu}\tfrac{1}{3}\ln 3\right)\,(4e^{\mu}\delta + 1)\right],$$

where $\mu := \max_M w$, *then the conformal family of metrics*

$$g_\rho := e^{2\rho w} g, \quad \rho \in [0, 1],$$

consists entirely of metrics without focal points.

Proof.

Consider the deformation constructed in Proposition 4.2.1.

Then by hypotheses about the deformation, the curvature is smaller than $-\epsilon$ and therefore smaller than $-\zeta_1 = -e^{-2\mu}\epsilon$ in $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$.

Then, as proved in Lemma 5.0.2, if γ is a $g-$geodesic that intersects a generalized bubble in the interval (a, b), then, by hypothesis, every solution of the Riccati equation U such that $U(b) > 0$ satisfies that $U(b + \Lambda) > e^{-\mu}\sqrt{\epsilon}\,\tanh\left(\frac{1}{2}\,e^{-\mu}\ln(3)\right)$.

Let us fix γ a geodesic and $\{(a_i, b_i)\}_{i\in I}$ the parameterized intervals of the intersection of γ with the generalized bubbles.

Using the same arguments from Lemma 3.4.2 and from Theorem 1 we have that, inside the generalized bubble, if there exist $c \in (a_2, b_2)$ and $U(c) = e^{-\mu}\sqrt{\epsilon}\,\tanh\left(\frac{1}{2}\,e^{-\mu}\ln(3)\right)$ then, inside the generalized bubble,

$$
\begin{aligned}
U(t) &= U(c) + \int_c^t U'(s)\, ds \\
&= U(c) - \int_c^t (U^2(s) + K(s))\, ds \\
&\geq U(c) - \int_c^t (\max_{[c,b_2]} U(s))^2\, ds - \int_c^t (\max_{[c,b_2]} K(s))\, ds \\
&\geq e^{-\mu}\sqrt{\epsilon}\,\tanh\Big(\tfrac{1}{2}\, e^{-\mu}\ln(3)\Big) - \int_c^t \Big(e^{-\mu}\sqrt{\epsilon}\,\tanh\Big(\tfrac{1}{2}\, e^{-\mu}\ln(3)\Big)\Big)^2 ds - \int_c^t K^+\, ds \\
&= e^{-\mu}\sqrt{\epsilon}\,\tanh\Big(\tfrac{1}{2}\, e^{-\mu}\ln(3)\Big) - (U(t) > e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{2}\ln(3)) + K^+)(t-c).
\end{aligned}
$$

Then, if

$$K^+ < \frac{\sqrt{\epsilon}}{2\delta}\left[\tanh\Big(e^{-\mu}\tfrac{1}{3}\ln 3\Big) - \sqrt{\epsilon}\, e^{-\mu}\,\tanh^2\Big(e^{-\mu}\tfrac{1}{3}\ln 3\Big)\ (2\delta+1)\right],$$

we have that $U(t) > e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{3}\ln(3))$ in the generalized bubble.

Then, as per hypothesis

$$K^+ < \frac{\sqrt{\epsilon}}{2\delta}\left[\tanh\Big(e^{-\mu}\tfrac{1}{3}\ln 3\Big) - \sqrt{\epsilon}\, e^{-\mu}\,\tanh^2\Big(e^{-\mu}\tfrac{1}{3}\ln 3\Big)\ (2\delta+1)\right],$$

just as in Theorem 1, (M, g) has no focal points. In fact an argument analogous to the proof of Lemma 5.0.5 proves that (M, g_ρ) has no focal points if $\rho \in [0, r]$ since the bubbles are simply connected. But we can use Lemma 5.0.5 directly. If

$$K^+ < \frac{\sqrt{\epsilon}}{4e^{2\mu}\,\delta}\left[\tanh\Big(e^{-\mu}\tfrac{1}{3}\ln 3\Big) - \sqrt{\epsilon}\, e^{-\mu}\,\tanh^2\Big(e^{-\mu}\tfrac{1}{3}\ln 3\Big)\ (4e^{\mu}\delta+1)\right],$$

(M, g_ρ) has no focal points if $\rho \in [0, r]$ since the solutions of the Riccati solution are in the conditions of Lemma 5.0.5 by Corollary 3.4.3, Lemma 5.0.2 and Lemma 5.0.3.

Now, suppose that (M, g_ρ) has no focal points for $\rho \in [0, \overline{\rho}]$. Then, in particular, there are no focal points in the $\bigcup_{i=1}^{k} B_\delta(p_i)$.

On the other hand, if the distance between the generalized bubbles is greater than Λ in the metric g, then since the distance between the generalized bubbles in the universal covering does not decreases as we deform by Lemma 4.3.4, the distance remains greater than Λ in the metric $g_{\overline{\rho}}$. Therefore, as before, if the solution of the Riccati equation comes out positive from the generalized bubble, since the curvature remains smaller than $-\zeta_1$ at $(\bigcup_{i=1}^{k} B_\delta(p_i))^c$, the solution of the Riccati equation comes will be greater than $e^{-\mu}\sqrt{\epsilon}\,\tanh\Big(\frac{1}{2}\, e^{-\mu}\ln(3)\Big)$ when it enters another generalized bubble, by Lemma 5.0.2. So we must be careful with the case where the geodesic returns to the same generalized bubble, or better, case where in the lifting of the geodesic, it returns to the same lifting of the generalized bubble.

But by Lemma 5.0.5, it suffices that if $\gamma_{\overline{\rho}}$ is such that $\gamma_{\overline{\rho}}(0) \in B_\delta(p_i)$, if $\tilde{\gamma}_{\overline{\rho}}$ is the lifting of $\gamma_{\overline{\rho}}$ such that $\tilde{\gamma}_{\overline{\rho}}(0) \in \tilde{B}^j_\delta(p_i)$, then if (a, b) is the connected component of $\tilde{\gamma}_{\overline{\rho}} \cap \tilde{B}^{\rho}_{2\delta\epsilon^\mu}(p^j_i) = (a, b)$ which contains $\tilde{\gamma}_{\overline{\rho}}(0)$, if U is a solution of the Riccati equation in $\gamma_{\overline{\rho}}$ such that $U(c) > e^{-\mu}\sqrt{\epsilon}\tanh(e^{-\mu}\frac{1}{3}\ln(3))$ for some $c \in (a, b)$ then $U(t) > e^{-2\mu}\epsilon\tanh^2(e^{-\mu}\frac{1}{3}\ln(3))$ for all $t \in (a, b)$. That is, it is sufficient to control the solution of the Riccati equation at the moment when the lifted geodesic exits the δe^μ-neighborhood of the generalized bubble in the universal cover.

Furthermore, since the length of the generalized bubbles in the deformed metric is smaller than $2e^{\overline{\rho}\mu}\delta$, $\Lambda = \frac{1}{\sqrt{\epsilon}}\operatorname{artanh}\left(1 - \frac{\epsilon}{2}\right)$ as in Lemma 5.0.2 and the region of positice curvature $g_{\overline{\rho}}$ is contained in the generalized bubbles, denoting $K^+_{\overline{\rho}}$ the maximum curvatuve in $(M, g_{\overline{\rho}})$, we have, by an analogous calculation made previously, that if

$$K^+_{\overline{\rho}} < \frac{\sqrt{\epsilon}}{4e^{2\mu}\,\delta}\left[\tanh\!\left(e^{-\mu}\tfrac{1}{3}\ln 3\right) - \sqrt{\epsilon}\,e^{-\mu}\,\tanh^2\!\left(e^{-\mu}\tfrac{1}{3}\ln 3\right)\,(4e^{\mu}\delta + 1)\right],$$

then (M, g_ρ) has no focal points for $\rho \in [0, \overline{\rho} + r]$ and the argument repeats inductively. That is, (M, g_ρ) is free of focal points and satisfies the conditions of Lemma 5.0.5.

Thus, for (M, g_ρ) to have no focal points for $\rho \in [0, 1]$ it is sufficient that, for all $\rho \in [0, 1]$, we have

$$K^+_{\rho} < \frac{\sqrt{\epsilon}}{4e^{2\mu}\,\delta}\left[\tanh\!\left(e^{-\mu}\tfrac{1}{3}\ln 3\right) - \sqrt{\epsilon}\,e^{-\mu}\,\tanh^2\!\left(e^{-\mu}\tfrac{1}{3}\ln 3\right)\,(4e^{\mu}\delta + 1)\right].$$

Therefore, since $K^+_\rho < K^+$, if

$$K^+ < \frac{\sqrt{\epsilon}}{4e^{2\mu}\,\delta}\left[\tanh\!\left(e^{-\mu}\tfrac{1}{3}\ln 3\right) - \sqrt{\epsilon}\,e^{-\mu}\,\tanh^2\!\left(e^{-\mu}\tfrac{1}{3}\ln 3\right)\,(4e^{\mu}\delta + 1)\right],$$

(M, g_ρ) has no focal points for $\rho \in [0, 1]$.

□

Let us conclude with Theorem 3.

Proof. [**Theorem 3**]

Just consider w from Proposition 5.0.6.

In fact, (M, g_ρ) has no focal points for any $\rho \in [0,1]$ as a consequence of

$$K^+ < \frac{\sqrt{\epsilon}}{4e^{2\mu}\,\delta}\left[\tanh\left(e^{-\mu}\tfrac{1}{3}\ln 3\right) - \sqrt{\epsilon}\, e^{-\mu}\ \tanh^2\left(e^{-\mu}\tfrac{1}{3}\ln 3\right)\ (4e^{\mu}\delta + 1)\right],$$

and of Theorem 3.

On the other hand, (M, g_1) has strictly negative curvature is consequence of Theorem 2.

Let us now prove hyperbolicity. In fact the proof is the same as in Theorem 1, but we will do the demonstration again.

Let's observe that since g_ρ is a metric without conjugate points, we have the existence of stable and unstable Jacobi fields. Therefore, according to [10], it suffices to show that there are no simultaneously stable and unstable nontrivial Jacobi fields. This, of course, could only happen if the reference geodesic passes through the region of non-negative curvature.

However, by our assumptions, along any g_ρ-geodesic the solution of the unstable Riccati equation is positive and the solution of the stable Riccati equation is negative. This implies that these fields are not generated by the same Jacobi field, i.e., there is no stable and unstable Jacobi field simultaneously. And this proves the Theorem.

$\square$

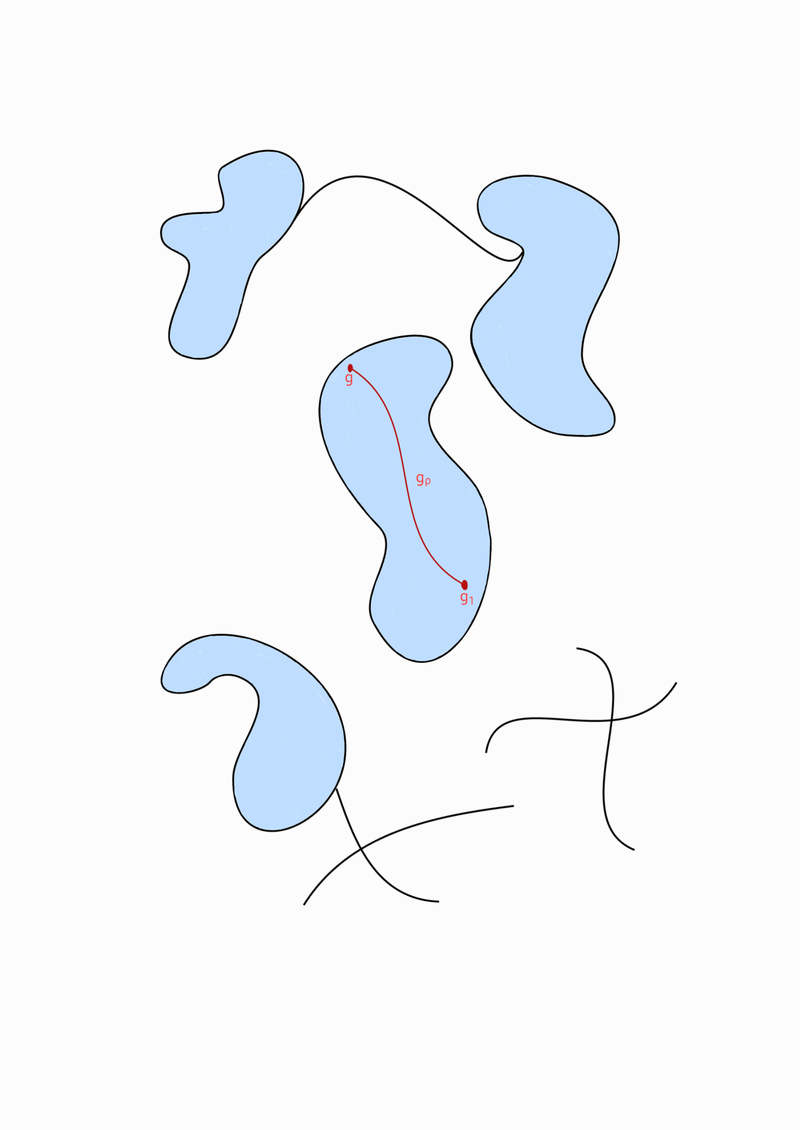

Figure 5.1: Deformation result and (M, g_1) with negative curvature.

6
Further projects

The last chapter is devoted to show some further applications of the ideas developed before and some further projects.

The first application of our previous results concerns the main result in [19]. Let us explain in detail.

Definition 6.0.1 ([19], Definition 4.8)**.** Let (M, g) be a complete C^∞ surface. A **non-degenerate bubble** $B \subset M$ is the closure of a non-empty open, simply connected set $A \subset M$ of positive curvature satisfying the following properties:

1. The boundary of B is a C^1 strictly convex, simple closed curve where the Gaussian curvature of the surface vanishes everywhere.

2. There exists an open neighborhood U of B such that $U \setminus B$ in U is an open set of negative curvature.

Definition 6.0.2. A surface (M, g) is called **non-degenerate** if its bubbles are non-degenerate.

In the paper [19], Jane and Ruggiero proved the following:

Theorem 11 ([19], Theorem 1.2)**.** *Let (M, g) be a compact C^∞ surface of genus greater than one that is non-degenerate. Then there exists a conformal family of metrics $g_\rho, \rho \in [0, \rho_0], g_0 = g$ on M with the following property: let K_ρ be the curvature of g_ρ and P^+_ρ the set of points where $K_\rho \geq 0$. Then*

- *$K_\rho(x) < K_s(x)$ for every $x \in P^+_0 = P^+$ and $\rho > s$.*
- *$P^+_\rho \subset P^+_s$ for every $\rho > s$.*

The conformal family of metrics was found from the Ricci-Yang-Mills Flow. The Ricci-Yang-Mills Flow is a solution for the PDE system

$$\frac{\partial g_\rho}{\partial \rho} = (m_\rho^2 - 2K_\rho) \cdot g_\rho$$

$$\frac{\partial m_\rho}{\partial \rho} = \triangle_\rho m_\rho + 2K_\rho m_\rho - m_\rho^3,$$

where $m_\rho : M \to \mathbb{R}$ is the magnetic potential. A well chosen potential, as in [19], allows us to conclude the existence of a perturbation that satisfies the properties shown in the theorem. The existence of a short term solution is verified in papers such as [26] and [28].

Unfortunately, the function that generates the deformed metrics varies with the deformation parameter, that is, for each value of ρ, we have a specific m_ρ. This, in turn, makes it difficult to determine the feasibility of continuing deformation beyond ρ_0. Since we do not have an explicit formula for the deformation, extending the process is not trivial and is not the focus of the work by Jane and Ruggiero in [19].

In fact, what they proved is this:

Theorem 12 ([19],Theorem 1.1)**.** *Let (M, g) be a compact C^∞ surface of genus greater than one, without focal points and that the closure of points of positive curvature is contained in a finite union of non-degenerate bubbles. Then there exists a conformal family of metrics $g_\rho, \rho \in [0, \rho_0], g_0 = g$ on M and $\overline{\rho} \in (0, \rho_0]$ such that if $\rho \in (0, \overline{\rho})$, then (M, g_ρ) is Anosov.*

The hypothesis that the surface has no focal points is important because it controls the behavior of the unstable solution of the Riccati equation, that is, by the definition of bubbles, if (M, g) has no focal points, the zeros of the unstable solution of the Riccati equation of geodesics that cross bubbles can only occur when the geodesics are leaving the bubble and, therefore, on their boundary.

Note that (M, g) does not need to have Anosov geodesic flow. This is actually the great importance of theorem. It is possible to perturb a metric on the boundary of the set of Anosov metrics in such a way that the family of metrics defined by the perturbation is formed by Anosov metrics. Of course, this is provided that the initial metric is not only without conjugate points, but also without focal points. The property of having no focal points can (and in general will) be lost after perturbing the metric.

Now is a good time to state the problem we want to solve:

Conjecture 1. *Let (M, g) be a compact C^∞ surface of genus greater than one, without focal points and non-degenerate. Then, as long as the bubbles are sufficiently far apart from each orther, there exists a conformal family of metrics g_ρ, $\rho \in [0, 1], g_0 = g$ on M such that g_ρ is an Anosov metric for all $\rho \in (0, 1]$ and g_1 is a metric with strictly negative curvature as long as we have a certain distance between the bubbles.*

As discussed previously, the Jane and Ruggiero perturbation in [19] only exists in an interval that depends on the metric. So the idea would be to replace the initial perturbation by a deformation defined in an interval $[0, 1]$ that maintains the properties:

- $K_\rho(x) < K_s(x)$ for every $x \in P_0^+ = P^+$ and $\rho > s$.
- $P_\rho^+ \subset P_s^+$ for every $\rho > s$.

These two properties make Theorema 12 true.

What we will do is replace the pertubation generated by the Ricci Yang-Mills Flow by the deformation defined by Proposition 4.2.1. In other words, Proposition 4.2.1 allows proving Theorem 12 without using the Ricci Yang-Mills Flow.

Note que if (M, g) is a compact C^∞ surface of genus greater than one and non-degenerate, by the definition of a non-degenerate surface and compactness, there exists $\delta' > 0$ such that for all $0 < \delta < \delta'$, if $B(\delta) := \{x \in M \mid d(P^+, x) < \delta\}$, then there exists $\epsilon = \epsilon(\delta)$ such that $B(\delta)$ is a disjoint union of open sets and $K(x) < -\epsilon$ for all $x \in M \setminus B(\delta)$. We can assume that $\epsilon < \frac{-2\pi\chi(M)}{vol(M)}$.

In essence, compact surfaces C^∞ of genus greater than one and non-degenerate are a special case of the surfaces we have been working with before. But now the region of non-negative curvature is better behaved.

Therefore, by a proof analogous to that of Proposition 4.2.1 we have that:

Proposition 6.0.3. *Let (M, g) be a C^∞ surface, compact, of genus greater than 1, and non-degenerate.*

Then there exists $w \in C^\infty(M)$ with $\min_M w = 0$ such that the deformation $g_\rho := e^{2\rho w} g$ with $\rho \in [0, 1]$ reduces the region of positive curvature and the curvature of points in this region as ρ increases, and keeps the negative curvature smaller than $-\zeta$ for some $\zeta > 0$, for points outside a bubble neighborhood.

Moreover, when $\rho = 1$ the curvature of g_ρ is strictly negative in M.

In proposition, $B(\delta)$ represents the neighborhood of the bubble and $\zeta = e^{-2\mu}\epsilon$ with μ the maximum of w in M. Note that for the deformation defined in proposition we have that

- $K_\rho(x) < K_s(x)$ for every $x \in P_0^+ = P^+$ and $\rho > s$.
- $P_\rho^+ \subset P_s^+$ for every $\rho > s$.

Therefore, the same type of technique presented in this thesis can redemonstrate Theorem 12, but the deformation is better defined and can be extended beyond a perturbation. Moreover, (M, g_1) has strictly negative curvature.

The problem is that Theorem 12 remains valid only for a small perturbation. Our goal is to obtain a result that holds for the entire interval $[0, 1]$, at least in some controlled cases.

The main difficulty in proving Conjecture 1 is in the fact that the deformation can generate metrics with focal points. Consequently, it is necessary to adapt the main arguments of the thesis to deal with the existence of focal points, which is not trivial and requires the development of new arguments and estimates.

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