Pontifícia Universidade Católica
do Rio de Janeiro

**Luca Abrahão Paiva**

# Quantum Features of Harmonic Oscillators: investigating the limits of Quantum Mechanics

**Dissertação de Mestrado**

Dissertation presented to the Programa de Pós–graduação em Física of - PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Física.

Advisor: Prof. Thiago Barbosa dos Santos Guerreiro

Rio de Janeiro
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To my family, for their love and encouragement.

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## Abstract

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Quantum Mechanics is one of the most successful theories of all time. It not only has a great predictive power, but also has completely changed the way physics is understood. However, Quantum Mechanics does not predict its own range of validity, and, in principle, the probabilistic description should be valid in our macroscopic world. But it does not happen. In the core of the explanation of why the description from Quantum Mechanics is substituted by Classical Mechanics lies decoherence. The interaction of the many unseen degrees of freedom from a macroscopic environment with a quantum system makes it extremely hard to measure its quantum properties. In this context, we explore how one can still detect non-classicality of oscillators in a intermediate regime, whether in a mesoscopic scale or a oscillator with a macroscopic number of excitations, via an optomechanical description. In this work, we present the basics of the formalism of optomechanics, both in the unitary dynamics and in an open quantum system approach. We then discuss two different optomechanical systems, highlighting how we can perceive its quantum features. At last, we discuss other possible schemes to identify the quantum nature of harmonic oscillators in situations of increasing macroscopic nature.



## Keywords

Quantum Features; Decoherence; Macroscopic Oscillators; Non-Classicality; Quantum Harmonic Oscillators.

## Resumo

Abrahão, Luca; Guerreiro, Thiago. **Aspectos Quânticos de Osciladores Harmônicos: investigando os limites da Mecânica Quântica**. Rio de Janeiro, 2024. 113p. Dissertação de Mestrado – Departamento de Física, Pontifícia Universidade Católica do Rio de Janeiro.

A Mecânica Quântica é uma das teorias mais bem sucedidas de todos os tempos. Não apenas tem um grande poder preditivo, como também mudou completamente a maneira como entendemos a física. No entanto, a Mecânica Quântica não prevê o próprio alcance, e, em princípio, a descrição probabilística deveria ser válida em nosso mundo macroscópico. Mas isso não acontece. Um ponto central do porquê a descrição quântica é substituída pela Mecânica Clássica, é a descoerência. A interação dos muitos graus de liberdade de um ambiente macroscópico faz com que seja extremamente difícil medirmos as propriadades quânticas de um sistema. Nesse contexto, exploramos como ainda podemos detectar efeitos não-clássicos de osciladores harmônicos em um regime intermediário, através da optomecânica. Neste trabalho apresentamos fundamentos do formalismo da optomecânica, tanto a dinâmica unitária, quanto para sistemas quânticos abertos. Depois, discutimos dois sistemas optomecânicos distintos, ressaltando como podemos investigar a presença de características quânticas. Além disso, discutiremos outras abordagens para identifcar características quânticas de osciladores harmônicos em situações cada vez mais próximas ao regime macroscópico.

## Palavras-chave

Características Quânticas; Descoerência; Osciladores Macroscópicos; Não-Classicalidade; Oscilador Harmônico Quântico.

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*Poets say science takes away from the beauty of the stars — mere globs of gas atoms. I too can see the stars on a desert night, and feel them. But do I see less or more? The vastness of the heavens stretches my imagination — stuck on this carousel my little eye can catch one-million-year-old light. A vast pattern — of which I am a part.... What is the pattern, or the meaning, or the why? It does not do harm to the mystery to know a little about it. For far more marvelous is the truth than any artists of the past imagined it. Why do the poets of the present not speak of it? What men are poets who can speak of Jupiter if he were a man, but if he is an immense spinning sphere of methane and ammonia must be silent?*

**Richard P. Feynman**, *Footnote in "Lectures on Physics".*

# 1
# Introduction

Quantum Mechanics (QM) is one of the greatest theories ever developed. From the beginning its predictive power was impressive. Considered by many as the birth of the Quantum theory, in 1900, Max Planck's idea of quantizing the radiation emitted by a black body [4], was the very first of many rightful predictions. What seemed, at first, only a mathematical tool, turned out to be a complete revolution in our understanding of nature.

In the early years of the twentieth century, QM was centered in most of the discussions within the physical world. Presented in different, and in principle unconnected, formulations, such as the wave-like mechanics from Schröedinger [5] and Heisenberg's matrix theory [6], getting to a consent was far way from happening. Even among the "founding fathers" there were disbelievers. Einstein's work on the photoelectric effect [7], which awarded him as the Nobel laureate in 1921, is one of the stepping stones of QM. Nevertheless, years later he was known as one of the most diligent against the Quantum theory, as exemplified by his famous work together with Podolski and Rosen, the EPR Paradox [8]. This lack of consent on the foundations of QM persists until today and aspects such the collapse of the wave function are still a subject of discussion, for example, in Zeilinger's review [9].

On the practical side, however, the work developed in the mid and late years of the twentieth century buried any possible doubts. In a fundamental perspective, the development of Quantum Electrodynamics (QED) [10] from Feynman, Schwinger, Tomonaga and Dyson (the first three awardad as Nobel laureates in 1965), brought a precision degree that was unmatched at the time. Even now, the precision is astonishing: the prediction of the theory matches the measured value up to $10^{-12}$ [11]. The success of QED motivated the study of more general theories with gauge symmetries. The quantization of the so called Yang-Mills theory [12] ultimately resulted in the Standard Model of Particles (SM), the model of all the fundamental interactions (except gravity) in a weak energy regime [13]. Moreover, the success of gauge theories reshaped the paradigm of fundamental physics.

Furthermore, in a applied context, the invention of the LASER and advances in Quantum Optics enabled a huge amount of practical tests of QM.

For instance, the work of Clauser [14] showed the inherent need of a quantized theory of the electric field. Also, intrinsic quantum effects, such as squeezing of light, are used in extremely precise experiments. For example, in the Laser Interferometer Gravitational-Wave Observatory (LIGO) [15].

Despite all of its successes and impact, there are two issues in the Quantum Theory that remain unresolved: The first regards whether we need for a quantized theory of gravity and, if so, what is this theory; the second regards the limits of QM and how we transition from quantum to classical. As we will see, these seemingly unrelated questions are getting closer and closer.

The discussion of a quantized theory of gravity [16] arose soon after Einstein's theory of General Relativity (GR) was published. The positive history of QFT and the SM earned the formalism of gauge theories a spot as the natural approach to a fundamental theory of Quantum Gravity (QG) This approach, however, could not describe gravity at a fundamental level, due to the non-renormalisability of the theory (an introductory explanation can be found in [17], as well as in textbooks such [18, 19]). Many other theories such as String Theory [20] and Loop Quantum Gravity [21], to name a few, were developed pursuing a way to put together QM and GR. Even though containing a rich mathematical structure, many of the unifying models lack predictions that are experimentally feasible to test within the near future, thus leaving the problem open. In addition, even the existence of a fundamental quantum theory of gravity is not a consent. Dyson, for instance, argues that detecting a single graviton is, in principle, impossible [22]; Penrose on the other hand, argues that gravitation is intrinsically classical and "induces classicality" into, a priori, quantum systems [23].

As in any physical theory, we need experimental data to resolve the debate. The detection of Gravitational Waves (GW) [24] opened another window to gravitational phenomena and many proposals of detecting quantum effects appeared, such as in [25, 26]. On top of that, many tabletop experiments looking for quantum effects of gravity have been proposed (for a general view, look [27] and references therein). The most famous tabletop approach is due a thought experiment proposed by Feynman [28], which basically consists of a gravitational double-slit experiment: if we were able to put a test mass with net spin 1/2 in a state with superposition on the location of the center of mass, via a Stern-Gerlach magnet, and then interact with a source mass, we would see that the interaction would lead to entanglement of the test and the source masses [29]. If the interaction between the particles was only via the gravitational field and assuming that no entanglement can be created from local operators and classical communication [30], this ensures the quantum nature of the gravitational field.

In the example above, and in most proposals of tabletop QG, we must be able to engineer a system that is simultaneously described by QM and with a appreciable gravitational field. This is a remarkably difficult experimental achievement mostly to the aforementioned issue of precisely stating the limits of when a quantum description of the system remains valid, instead of a classical one. In principle, the laws of QM impose no bounds in the size of the system under study, and many experiments of matter-wave interference of increasing number of atoms have been made, either with Bose-Einstein Condensates (BEC) [31] or with molecules [32, 33]. However, these systems are still too light to to present any appreciable gravitational interaction and the bigger the system of interest gets, the higher the coupling with the environment, leading to one of the main forms of imposing classicality: Decoherence.

To better understand the mechanism of decoherence, it is fundamental to state that no quantum system is absolutely closed. For instance, in order to get information about a system we need to make measurements, which are by principle an interaction with an external system (the detector). The idea of decoherence is that, despite every superposition of states being equally valid (whilst dealing with unitary closed systems), not of all these superpositions behave the same under the influence of a external environment. This leads to the *einselection* of the states that we observe and describe via the classical formalism [34]. In this way, as the number of degrees of freedom of the quantum system of interest increases, the channels for decoherence also increase. Since interactions with the environment are the main cause of the decoherence, in order to maintain the properties of the quantum state we need an extremely isolated system. A good candidate for the task are optically levitated nanoparticles [35, 36].

The technique of trapping particles using highly focused lasers is not new [37]. Since Arthur Ashkin's seminal work, in 1970, the improvements in optical tweezers technology has skyrocketed. In the next years, he and his collaborators developed many experiments ranging from trapping atoms [38] to living organisms, as cells and bacteria [39]. In the midst of it, working with Steven Chu and others, they achieved the trapping of a silica particle in water [40]. These experiments were the beginning of the study of optical tweezers to trap dieletric nanoparticles. The work of Ashkin awarded him as a Nobel laureate in 2018, and the use of optical tweezers as a tool for studying physical phenomena is only increasing.

In the 2010's the study trapped nanoparticles in vacuum was propelled by works that emphasized the possibility of unmatched isolation and control of the levitated particles [41, 42]. Combining features of quantum optics [43], control theory [44] and stochastic thermodynamics [45], the study of these systems, often called levitodynamics, has become a important tool to study fundamental physics. In the goal of demonstrating quantum features of mesoscopic objects, levitated optomechanics has achieved stunning results, such driving the particles to its center of mass motion ground state [46, 47]. However, this is the step 0 of actually demonstrating the quantum behavior of these systems and much needs to be done, both increasing our theoretical knowledge and experimental capacities.

Within this picture, this work intends to provide a introduction to the formalism and the physics underlying both the study of levitated optomechanical systems as well as a way to characterize possible quantum behavior from gravitational waves. The common ground of these two seemingly uncorrelated phenomena is a old companion of all physics students: the Harmonic Oscillator (HO).

We begin in chapter 2 reviewing the formalism that we'll need to approach the next parts of the texts. The idea of this is not to provide a complete description of the formalism, rather it is to be taken as a collection of the principles. This chapter ranges from many applications of the formalism of Harmonic Oscillators, starting from the basic features, passing by open quantum system dynamics and QFT, in a direct and straightforward way.

In chapter 3 we apply the formalism to understand the dynamics of a Gravitational Wave interacting with a detector and how to detect quantum effects. As we will see, this does not mean that we will be able to detect a graviton, instead we focus in a description that encompass a macroscopic number of gravitons populating the GW.

In chapter 4 we turn ourselves to the open dynamics of optomechanical systems, regarding fluctuation and dissipation. We analyze the interaction of a few cases of interest, focusing in how the quantum state of a mesoscopic oscillator (either the mechanical nano particle, or a lightfield popullated by a high number of photons) could, in principle, alter the dynamics of the system.

Finally, we end up in chapter 5, concluding the work and setting up the discussion to what comes next.

# 2
# Fundamentals of Quantum Harmonic Oscillators

The study of Quantum Harmonic Oscillators (QHO) is an essential step towards comprehending theoretical physics. Due to it's wide range of applications, from Condensed Matter physics to Quantum Field theory, many problems are accurately described by this formalism.

In this chapter, we will briefly review some of the concepts that will be useful throughout this work as well as define the notations that will be used in the following chapters.

We start from the basics, defining the bosonic operators and its commutation relationship and the proceed to study some important states: Coherent, Squeezed and Thermal. Afterwards, we describe the system of QHO subjected to damping through the input-output formalism [48]. Diving deeper in the discussion of open quantum systems, we follow the seminal work of Caldeira and Legget [49], and introduce the path integral approach for dealing with quantum systems with dissipation[50]. We then establish a connection with Quantum Field theory, enabling us to discuss a effective theory of Gravitational Waves via the Einstein-Hillbert action and connecting it to optomechanics [51].

## 2.1
## Basic Concepts

In this section we rapidly cover the basics of QHO. A more comprehensible and complete discussion can be found in most introductory textbooks in quantum mechanics (for example, [52]). In what follows, we will consider only single mode oscillators, unless otherwise stated.

The Hamiltonian of a Quantum Harmonic oscillator with mass $m$ and frequency $\omega$ is written as

$$\hat{H} = \frac{m\omega^2\hat{x}^2}{2} + \frac{\hat{p}^2}{2m}, \tag{2-1}$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators, respectively.

We define the bosonic creation and annihilation operators $a$ and $a^\dagger$, respectively, which satisfy

$$[a, a^\dagger] = 1. \tag{2-2}$$

Acting on a excited state $|n\rangle$, we have

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{2-3}$$

justifying the names "creation" and "annihilation" operators. Furthermore, we see that

$$a^\dagger a\,|n\rangle = n\,|n\rangle \tag{2-4}$$

so that we define the number operator $a^\dagger a \equiv \hat{n}$.

The relation of these bosonic operators with the position and momentum operators is expressed as

$$\begin{aligned} \hat{x} &= x_{zpm}(a + a^\dagger) \\ \hat{p} &= ip_{zpm}(a^\dagger - a) \end{aligned} \tag{2-5}$$

where $x_{zpm}$ and $p_{zpm}$ are, respectively, the zero point fluctuation of the position and momentum given by

$$\begin{aligned} x_{zpm} &= \sqrt{\frac{\hbar}{2m\omega}} \\ p_{zpm} &= \sqrt{\frac{m\omega\hbar}{2}}. \end{aligned} \tag{2-6}$$

Its also worth defining the dimensionless quadratures, $X$ and $P$, which are the position and momentum operators in units of zero point fluctuations.

In this way, the Hamiltonian, written in terms of creation and annihilation operators is simply

$$\hat{H} = \left(\hat{n} + \frac{1}{2}\mathbb{1}\right)\hbar\omega \tag{2-7}$$

and we can evaluate the energy levels of the oscillator $\hat{H}\,|n\rangle = E_n\,|n\rangle$ where

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{2-8}$$

The equation above highlights the main feature of QHO: the energy levels are equally spaced and there is a minimum possible energy, known as the ground state.

So far, we have described our system in the Fock state basis, which rely in the number of excitations of the oscillator. We now introduce other 3 states that will be useful in what is to come.

### 2.1.1 Thermal States

As the excitations (phonons, in case of a mechanical oscillator and photons in case of a light field) in QHO are bosonic degrees of freedom, if a oscillator is in equilibrium with its environment in a finite temperature $T$, the occupancy number will follow the Bose-Einstein distribution, given by

$$p(n) = \exp\left(-\frac{n\hbar\omega}{k_B T}\right)\left[1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)\right] \tag{2-9}$$

where $k_B$ is the Boltzmann constant.

In this way, a thermal state $\rho_{th}$ is a statistical mixture of Fock states in different occupation numbers, wheighted by the Bose-Einstein distribution.

$$\rho_{th} = \sum_{n=o}^{\infty} p(n) |n\rangle \langle n| \,. \tag{2-10}$$

We can show that the mean number of phonons $\langle n\rangle \equiv \bar{n}$ is

$$\bar{n} = \sum_{n=o}^{\infty} np(n) = \left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right]^{-1} . \tag{2-11}$$

In the classical limit, where $k_B T >> \hbar\omega$, the mean occupation number is

$$\bar{n} \approx \frac{k_B T}{\hbar\omega}, \tag{2-12}$$

precisely the thermal energy, $k_B T$, divided by the energy of each phonon, $\hbar\omega$.

### 2.1.2 Coherent States

Another set of important states are the coherent states [53]. These states are the eigenstates of the annihilation and creation operators, that is

$$a |\alpha\rangle = \alpha |\alpha\rangle \,. \tag{2-13}$$

Coherents states are considered the closest quantum mechanical states to a classical description. This arises from the fact that in the coherent state basis, the uncertainty principle is satisfied with an equality, that is, the minimum allowed value.

Unlike Fock states, Coherent states don't have a definite number of excitations. We can see that by expressing a Coherent state in the number basis

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,, \tag{2-14}$$

showing that the number of excitations in the coherent states follows a Poisson distribution.

We can also describe Coherent states as generated by a unitary displacement operator

$$\mathrm{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a), \tag{2-15}$$

where $\alpha$ is a complex number.

From the Baker-Campbell-Hausdorff [54] theorem we see that

$$e^{A+B} = e^A e^B e^{-[A,B]/2} \tag{2-16}$$

if

$$[A,[A,B]] = [B,[A,B]] = 0$$

and in this way, we can rewrite the displacement operator as

$$\mathrm{D}(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha a}. \tag{2-17}$$

A coherent state $|\alpha\rangle$ is generated by the application of the displacement operator in the vacuum state,

$$|\alpha\rangle = \mathrm{D}(\alpha)\,|0\rangle \tag{2-18}$$

There are many properties of the displacement operator that facilitates calculations. We will not present them here, instead we will state them as needed.

Its also worth noticing that the Coherent states form a overcomplete basis, since

$$|\,\langle\alpha|\beta\rangle\,|^2 = e^{-|\alpha-\beta|^2} \tag{2-19}$$

and the completeness relation is given by

$$\frac{1}{\pi}\int |\alpha\rangle\,\langle\alpha|\, d^2\alpha = 1, \tag{2-20}$$

where the integration is to be taken in the real and complex part of $\alpha$.

### 2.1.3 Squeezed States

The last state that we will explicitly mention are the Squeezed states [55].

These states also satisfy the the uncertainty principle with equality and therefore they are minimum-uncertainty states. However, a Squeezed state may have less noise in one of the quadratures than the coherent state, at the expense of increasing it in its conjugate quadrature. This property is very useful when one needs to make extremely precise measurements, even enabling to surpass the standard quantum limit [56].

We can generate Squeezed states by applying the unitary squeeze operator

$$S(\epsilon) = \exp\left(\frac{1}{2}\epsilon^* a^2 - \frac{1}{2}\epsilon a^{\dagger 2}\right) \tag{2-21}$$

where $\epsilon = re^{2i\phi}$, with $r$ being the squeezing radius and $\phi$ is the squeezing phase.

It is worth noticing that we can create the so called squeezed-coherent states, by first squeezing the vacuum and then applying the displacement operator

$$|\alpha, \epsilon\rangle = D(\alpha)S(\epsilon)\,|0\rangle\,. \tag{2-22}$$

We can better visualize the effect of the squeezing operator when looking at the phase space representation, where a coherent state may be represented as a "uncertainty circle". After applying the squeezing operator, the circle becomes an ellipse squeezed in one direction, originating the name.

Some properties of the squeezing operator will be used to perform calculations. As well as with the displacement operator, we will present them as needed.

## 2.2 Optomechanical Hamiltonians

After a brief introduction of special states that we will encounter in the future, we now proceed to describe two important optomechanical Hamiltonians: the dispersive and the coherent scattering.

Consider an empty Fabry-Perot optical cavity consisting of two completely reflecting mirrors(we will relax this condition in the next sections), one fixed at one end and one movable at the other side. Given a cavity with $L$, the resonance frequency of the cavity is given by

$$\omega_c = \pi \frac{c}{L} n \tag{2-23}$$

where $c$ is the speed of light and $n$ is an integer. The Hamiltonian of the system is

$$\frac{\mathrm{H}}{\hbar} = \omega_c a^\dagger a + \omega_m b^\dagger b - g a^\dagger a (b + b^\dagger) \tag{2-24}$$

where $a(a^\dagger)$ and $b(b^\dagger)$ are bosonic annihilation(creation) operators of the light field and the mechanical oscillator, respectively. The last term of (2-24) is the interaction term, with coupling ($g$) given by

$$g = \frac{\omega_c}{L} \sqrt{\frac{\hbar}{2m\omega_m}}. \tag{2-25}$$

This interaction term can be understood as follows: the light field inside the cavity exerts a displacement in the movable mirror (recall that $b + b^\dagger$ is related to the position quadrature) proportional to the number of photons inside ($a^\dagger a$). As the mirror moves, it induces a small shift in the length $L$, resulting in a change in the energy of the photons inside. Furthermore, the dispersive Hamiltonian can be achieved by properly positioning a nanoparticle inside a cavity [57].

We see that this Hamiltonian is of third order in the bosonic operators, giving rise to nonlinearities, as we will see below. Following [58] and [59], we can write the unitary evolution operator generated by 2-24

$$U(t) = e^{-ib^\dagger b t} e^{-ira^\dagger a} e^{k a^\dagger a \left(\eta(t) b - \eta^*(t) b^\dagger\right)} e^{-i(a^\dagger a)^2 B(t)} \tag{2-26}$$

with the following definitions

$$\begin{aligned} k &= \frac{g}{\omega_m}, \\ r &= \frac{\omega_c}{\omega_m}, \\ \eta(t) &= 1 - e^{-it}, \\ B(t) &= -k^2(t - \sin t) \end{aligned} \tag{2-27}$$

and we also used the scaled time $\omega_m t \to t$.

This unitary operator contains an optically driven displacement term, as well as a Kerr-like term [60], leading to an effective nonlinearity in the dynamics.

Empowered with the unitary evolution, we can evaluate the dynamics of a state evolving via the dispersive Hamiltonian. We leave the application to chapter 3, where we will explore its features in a situation of interest.

We now turn ourselves to the linear optomechanical interaction, sometimes referred to as coherent scattering interaction. We will once again start from the dispersive Hamiltonian in eq. 2-24, but to make it closer to situations of interest, such as feedback cooling [46, 47], we will make some adjustments. The Hamiltonian of interest, now reads

$$\frac{\mathrm{H}}{\hbar} = \tilde{\Delta} a^\dagger a + \omega_m b^\dagger b - g a^\dagger a (b + b^\dagger) + d(a^\dagger + a), \tag{2-28}$$

where we consider a laser pumping the cavity, with frequency $\omega$, generating a detuning $\tilde{\Delta} = \omega_c = \omega$ and a drive term, $d$, which constantly pumps the cavity. The detuning is related to the cooling or heating of the particle [61].

We then move to a reference frame that is centered around the mean number of photons inside the cavity and the mean number of phonons in the mechanical oscillator, displacing the Hamiltonian with the operators $D(\alpha)$ and $D(\beta)$. This is equivalent as transforming the bosonic operators as

$$\begin{aligned} a &\to a + \alpha \\ b &\to b + \beta. \end{aligned} \tag{2-29}$$

By a suitable choice of displacements and disregarding terms higher than second order in the operators, we can simplify the Hamiltonian to

$$\frac{\mathrm{H}}{\hbar} = \Delta a^\dagger a + \omega_m b^\dagger b - g_c (a^\dagger + a)(b + b^\dagger). \tag{2-30}$$

Despite seeming quite arbitrary, the freedom to choose the correct displacement $\alpha$ and $\beta$ are due experimental feasibility, such modulating the laser power or changing the parameters of the cavity [62].

Within linearizing the Hamiltonian, we guarantee that we will be able to solve the equations of motion in the input-output formalism (see below). Despite losing some information regarding the nonlinearities of the dispersive Hamiltonian, the linearized coupling, $g_c$, is also stronger in many cases, comparing to the dispersive coupling. So this approximation is a good one for many of the applications of the area.

## 2.3 Input-output approach

So far, we described the dynamics of an optomechanical system in the unitary regime, where the system is completely closed. However, this is an idealization which typically does not hold for two main reasons. The first is straightforward: no system is perfectly closed. We have photons leaking from the cavity and the mechanical oscillator suffers from dissipation coming from the environment, to name a couple. The second reason, is fundamental: in order to perform measurements on the system, we must interact. In this way, the dynamics becomes intrinsically open and we also need a open quantum system approach.

In this section, we will follow a seminar work from Gardiner and Collet [48] and get to the Quantum Langevin Equations (QLE). Then we will show the optomechanical equations of motion for a system of interest.

We start by considering a picture of a harmonic oscillator with creation(annihilation) operators $a^{\dagger}(a)$, interacting with a bath of harmonic oscillators with creation(annihilation) operators $b^{\dagger}(b)$. The total Hamiltonian reads

$$H_{tot} = H_{sys} + H_{bath} + H_{int} \tag{2-31}$$

where

$$\begin{aligned} H_{sys} &= \hbar\omega_a a^{\dagger} a \\ H_{bath} &= \hbar \int_{-\infty}^{\infty} d\omega b^{\dagger}(\omega) b(\omega), \\ H_{int} &= i\hbar \int_{-\infty}^{\infty} d\omega \kappa(\omega) \left( a^{\dagger} b(\omega) - a b^{\dagger}(\omega) \right), \end{aligned} \tag{2-32}$$

with the usual bosonic commutation relations

$$[b(\omega), b^{\dagger}(\omega')] = \delta(\omega - \omega'). \tag{2-33}$$

This is a rather idealised Hamiltonian, which we consider only linear interactions between the system and the bath. We can understand the interaction term as follows: a phonon of frequency $\omega$ is annihilated in the bath, creating a phonon in the system. Similarly, the bath can also absorb a phonon from the system, dissipating the energy.

We can now write can write the Heisenberg's equation of motion for the operators $a$ and $b$, leading to

$$\dot{a}(t) = -i\omega_a a(t) + \int_{-\infty}^{\infty} d\omega \kappa(\omega) b(\omega) \tag{2-34}$$

$$\dot{b}(\omega, t) = -i\omega b(\omega, t) + \kappa(\omega) a(t). \tag{2-35}$$

Solving (2-35), we find

$$b(\omega, t) = b_0(\omega) e^{-i\omega(t-t_0)} + \kappa(\omega) \int_{t_0}^{t} dt' e^{-i\omega(t-t')} a(t') \tag{2-36}$$

where we defined $b_(\omega, t_0) = b_0(\omega)$ and we consider $t_0 < t$, so that the system is at a initial state at $t = t_0$. Inserting back on eq. 2-34,

$$\begin{aligned} \dot{a}(t) = -i\omega_a a(t) + \int_{-\infty}^{\infty} d\omega \kappa(\omega) b_0(\omega) e^{-i\omega(t-t_0)} \\ + \int_{-\infty}^{\infty} d\omega \int_{t_0}^{t} dt' \kappa^2(\omega) e^{-i\omega(t-t')} a(t') \end{aligned} \tag{2-37}$$

We now perform the first Markov approximation, which consists of assuming a constant coupling between the bath and the system, $\kappa(\omega) = \sqrt{\frac{\gamma_c}{2\pi}}$. We have

$$\begin{aligned} \dot{a}(t) = -i\omega_a a(t) + \sqrt{\frac{\gamma_c}{2\pi}} \int_{-\infty}^{\infty} d\omega b_0(\omega) e^{-i\omega(t-t_0)} \\ + \frac{\gamma_c}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{t_0}^{t} dt' e^{-i\omega(t-t')} a(t'). \end{aligned} \tag{2-38}$$

Performing the integration in the last term, remembering that

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t') \tag{2-39}$$

and

$$\int_{t_0}^{t} dt' f(t') \delta(t-t') = \frac{1}{2} f(t), \tag{2-40}$$

we have

$$\dot{a}(t) = -i\omega_a a(t) + \sqrt{\frac{\gamma_c}{2\pi}} \int_{-\infty}^{\infty} d\omega b_0(\omega) e^{-i\omega(t-t_0)} + \frac{\gamma_c}{2} a(t). \tag{2-41}$$

Finally, we define the input field, which

$$b_{in}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega b_0(\omega) e^{-i\omega(t-t_0)} \tag{2-42}$$

so that

$$\dot{a}(t) = -i\omega_a(t) + \frac{\gamma_c}{2} + \sqrt{\gamma_c} b_{in}(t). \tag{2-43}$$

We could as well have solved (2-35) considering a future time $t_1 > t$ and this would lead to

$$b(\omega, t) = b_1(\omega)e^{-i\omega(t-t_0)} + \kappa(\omega)\int_t^{t_1} dt' e^{-i\omega(t-t')}a(t') \tag{2-44}$$

where $b_1$ is $b(\omega, t_1)$. In this way, we define the output field as

$$b_{out}(t) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega b_1(\omega)e^{-i\omega(t-t_1)}. \tag{2-45}$$

This leads to a time reversed QLE, and we can also see that

$$b_{in}(t) + b_{out}(t) = \sqrt{\gamma_c}a(t). \tag{2-46}$$

Physically, this can be thought of as a conservation law: the dynamics of the system of interest is given by the difference of the input (what goes in) and output (what comes out) fields, considering the dissipation of the process.

We also note that the dynamics of the system of interest depends only on the initial condition of the bath, which can be interpreted as a external force acting upon the system. Furthermore, dissipation appears naturally in the QLE, due to the fact that we are unaware of the dynamics of the many degrees of freedom of the bath, and when we trace them out, its influence on the system of interest comes in the form of dissipation. More on this in the next section.

Taking the conjugate of (2-43), we are able to write down the equations of motion

$$\dot{X}(t) = \omega_a P(t) - \frac{\gamma_c}{2}X(t) + \sqrt{\gamma_c}X_{in}(t) \tag{2-47}$$

$$\dot{P}(t) = -\omega_a X(t) - \frac{\gamma_c}{2}P(t) + \sqrt{\gamma_c}P_{in}(t) \tag{2-48}$$

where $X(t)$ and $P(t)$ are the dimensionless position and momentum quadratures.

A good physical situation that fits perfectly the input-output formalism is the one of a leaky cavity. We consider an optical cavity with one perfectly reflecting mirror and one imperfect mirror. The damping is related to the bandwith of the cavity: the broader the bandwith, higher the dissipation.

Following a analogous formalism, we can also write the QLE for a mechanical oscillator interacting with a discrete bath of harmonic oscillators

[63]. Despite being two different physical situations, the input-output formalism is pretty much the same. We will derive this equation in another fashion within the next section.

Now, we can look at the equations of motion of a more general system than the one described in 2.2. The system consists of a one perfectly reflecting movable mirror, in contact with a environment and one partially reflecting fixed mirror. In this way, there is dissipation both due to the leaking of photons from the cavity and also from the brownian motion from the mechanical mirror. The equations of motion are

$$\dot{X}_c(t) = \omega_a P_c(t) - \frac{\gamma_c}{2} X_c(t) + \sqrt{\gamma_c} X_{in}(t) + \frac{i}{\hbar}[H_{int}, X_c(t)] \tag{2-49}$$

$$\dot{P}_c(t) = -\omega_a X_c(t) - \frac{\gamma_c}{2} P_c(t) + \sqrt{\gamma_c} P_{in}(t) + \frac{i}{\hbar}[H_{int}, P_c(t)] \tag{2-50}$$

$$\dot{X}_m(t) = \omega_m P_m(t) + \frac{i}{\hbar}[H_{int}, X_m(t)] \tag{2-51}$$

$$\dot{P}_m(t) = -\omega_m X_m(t) - \gamma_m P_m(t) + \sqrt{2\gamma_m} f + \frac{i}{\hbar}[H_{int}, P_m(t)] \tag{2-52}$$

where we have the dimensionless quadratures

$$X_c(t) = a(t) + a^\dagger(t),$$
$$X_m(t) = b(t) + b^\dagger(t),$$
$$X_{in}(t) = c_{in}(t) + c_{in}^\dagger \tag{2-53}$$

and analogous for the dimensionless $P_i(t)$ $(i = c, m, in)$ quadratures. Also, $\gamma_m$ is the mechanical dissipation and $f$ is the input function of the mechanical bath.

We see that the major part of the challenges when solving this system of equations comes from the coupling rising from the interaction Hamiltonian.

## 2.4 Influence Functional

In this section we will deal with the problem of a dissipative quantum system via the Feynman-Vernon influence functional formalism [50]. In this approach we will treat the quantum systems via Path Integrals [64] and we will describe the interaction of a system interacting with a bath of harmonic oscillators. We will follow the work of Caldeira and Legget [49], where this formalism was used to derive features of classical Brownian motion from a completely quantum perspective.

The system is described by the total Hamiltonian

$$H_{tot} = H_B + H_S + H_{Int} \tag{2-54}$$

where

$$\begin{aligned} H_S &= \frac{p^2}{2M} + v(x) \\ H_B &= \sum_k \frac{p_k^2}{2m} + \sum_k \frac{1}{2} m\omega_k^2 R_k^2 \\ H_{Int} &= x \sum_k C_k R_k, \end{aligned} \tag{2-55}$$

and $R_k$, $p_k$ and $\omega_k$ are, respectively, the position, the momentum and frequency of the k-th oscillator of the bath. The interaction is assumed to be linear, and the coupling is given by $C_k$.

The density matrix of the system evolves as

$$\rho(t) = U(t)\rho(0)U^\dagger(t) \tag{2-56}$$

with the unitary evolution

$$U(t) = \exp\left(\frac{-iH_{tot}t}{\hbar}\right). \tag{2-57}$$

We now write the density matrix in the position basis, where bold symbols are position vectors $\boldsymbol{R} = (R_1, ..., R_k)$

$$\begin{aligned} \langle x, \boldsymbol{R}| \rho(t) |x', \boldsymbol{R'}\rangle \quad &= \int dx_0 dx_0' d\boldsymbol{R_0} d\mathbf{R_0}' \langle x, \boldsymbol{R}| U(t) |x_0, \boldsymbol{R_0}\rangle \\ &\langle x_0, \boldsymbol{R_0}| \rho(0) |x_0', \boldsymbol{R_0'}\rangle \langle x_0', \boldsymbol{R_0'}| U^\dagger(t) |y, \boldsymbol{Q}\rangle \end{aligned} \tag{2-58}$$

from which we identify the propagators

$$K(x, \boldsymbol{R}, t; x_0, \boldsymbol{R_0}, 0) = \langle x, \boldsymbol{R}| U(t) |x_0, \boldsymbol{R_0}\rangle \tag{2-59}$$

and its conjugate. We can think of this integration as our ignorance of the initial conditions of the system, so that we are summing over all the possible initial states.

From the Path Integral formulation of Quantum Mechanics [64] we can write the propagators as

$$K(x, \boldsymbol{R}, t; x_0, \boldsymbol{R_0}, 0) = \int \mathcal{D}x\mathcal{D}\boldsymbol{R} \exp\left(\frac{i}{\hbar} S_{tot}[x, \boldsymbol{R}]\right) \tag{2-60}$$

where the integration with respect to $\mathcal{D}$ is to be taken over all the trajectories from $x(0), \boldsymbol{R}(0)$ to $x(t), \boldsymbol{R}(t)$ and $S_{tot}[x, \boldsymbol{R}]$ is the total action of the system, given by

$$S_{tot} = \int_0^t dt' L_{tot}(t') \tag{2-61}$$

Therefore the system plus bath is described by

$$\begin{aligned} \langle x, \boldsymbol{R}| \rho(t) |x', \boldsymbol{R}'\rangle \quad &= \int dx_0' dx_0' d\boldsymbol{R_0} d\mathbf{R_0}' K(x, \boldsymbol{R}, t; x_0, \boldsymbol{R_0}, 0) \\ &\langle x_0, \boldsymbol{R_0}| \rho(0) |x_0', \boldsymbol{R_0'}\rangle\, K^*(x', \boldsymbol{R}', t; x_0', \boldsymbol{R_0'}, 0). \end{aligned} \tag{2-62}$$

However, given the many unobservable degrees of freedom of the bath, we wish to write the evolution of our system alone. We do this by tracing out the bath degrees of freedom. Thus we are interested in the dynamics of the reduced density operator $\tilde{\rho}(x, x', t)$, given by

$$\begin{aligned} \tilde{\rho}(x, x', t) = \quad & \int d\boldsymbol{R}\, \langle x, \boldsymbol{R}| \rho(t) |x', \boldsymbol{R}\rangle = \int dx_0 dx_0' d\boldsymbol{R_0} d\mathbf{R_0}' d\mathbf{R} K(x, \boldsymbol{R}, t; x_0, \boldsymbol{R_0}, 0) \\ & \langle x_0, \boldsymbol{R_0}| \rho(0) |x_0', \boldsymbol{R_0'}\rangle\, K^*(x', \boldsymbol{R}, t; x_0', \boldsymbol{R_0'}, 0). \end{aligned} \tag{2-63}$$

Assuming the bath is in a separable state at $t = 0$, we have

$$\rho(0) = \rho_S(0)\rho_B(0) \tag{2-64}$$

and the dynamics of the reduced density operator reads

$$\tilde{\rho}(x, x', t) = \int dx_0 dx_0' \mathcal{J}(x, x', t; x_0, x_0', 0) \rho_S(x_0, x_0', 0) \tag{2-65}$$

where the superpropagator $\mathcal{J}(x, x', t; x_0, x_0', 0)$ is

$$\mathcal{J}(x, x', t; x_0, x_0', 0) = \int \mathcal{D}x\mathcal{D}x' \exp\left(\frac{i}{\hbar} S_S[x]\right) \exp\left(-\frac{i}{\hbar} S_S[x']\right) \mathcal{F}[x, x'] \tag{2-66}$$

and the Influence Functional, $\mathcal{F}[x, y]$, is

$$\begin{aligned} \mathcal{F}[x, x'] = & \int d\boldsymbol{R} d\boldsymbol{R_0} d\boldsymbol{R_0'} \rho_B(\boldsymbol{R_0}, \boldsymbol{R_0'}, 0) \\ & \int \mathcal{D}\boldsymbol{R}\mathcal{D}\boldsymbol{R}' \exp\left\{\frac{i}{\hbar}\left(S_B[\boldsymbol{R}] - S_B[\boldsymbol{R}'] + S_{Int}[x, \boldsymbol{R}] - S_{Int}[x', \boldsymbol{R}']\right)\right\}. \end{aligned} \tag{2-67}$$

The mathematical properties of Influence functional can be found in [50].

We notice here that if there is no interaction among the system and the bath, the Influence Functional reduces to the identity, and the evolution of the system is given by a product of a forward time propagator and a time reversed propagator that are completely uncoupled. This is related to the fact that when one deals with closed systems, the probabilities of transition are written in terms of a modulus squared of wave functions. Furthermore, we see that the Influence Functional couples the forward and backward time evolution, leading, for example, to dissipation.

The actions of the the bath and the interaction between the bath and the system are

$$S_B[\boldsymbol{R}, \dot{\boldsymbol{R}}] = \int_0^t dt' \sum_k \frac{p_k^2}{2m} - \sum_k \frac{1}{2} m\omega_k^2 R_k^2 \tag{2-68}$$

and

$$S_{Int}[x, \boldsymbol{R}] = -\int_0^t dt' x \sum_k C_k R_k. \tag{2-69}$$

Taking the bath at a initial state in equilibrium with a temperature $T$, is density matrix reads

$$\rho_B(\boldsymbol{R_0}, \boldsymbol{R_0'}, 0) = \prod_k \rho_B^{(k)}(R_{0k}, R_{0k}', 0) \tag{2-70}$$

where

$$\rho_B^k(R_{0k}, R_{0k}', 0) = \frac{m\omega_k}{2\pi\hbar \sinh(\hbar\omega_k/k_BT)} \exp\left\{ \frac{m\omega_k}{2\pi\hbar \sinh(\hbar\omega_k/k_BT)} \times\left[ (R_{0k}^2 + R_{0k}'^2)\cosh(\hbar\omega_k/k_BT) - 2R_{0k}R_{0k}' \right] \right\}. \tag{2-71}$$

After inserting the initial state of the bath in (2-67) we are able to analytically solve the Influence Functional [65], due to the many considerations that were made, such as the linearity of the interaction and the separability of the initial state, for instance, we were able to.

In this way, the Influence Functional for the system reads

$$\mathcal{F}[x, x'] = \exp\left\{ -\frac{1}{\hbar} \int_0^{t_f} \int_0^t dt\, dt' [x(t) - x'(t)][\alpha(t-t')x(t') - \alpha^*(t-t')x'(t')] \right\} \tag{2-72}$$

where

$$\alpha(t-t') = \sum_k \frac{C_k^2}{2m\omega_k} \left[ \exp(-i\omega_k(t-t')) + \frac{\exp(i\omega_k(t-t'))}{\exp\left(\frac{\hbar\omega_k}{k_BT}\right) - 1} + \frac{\exp(-i\omega_k(t-t'))}{\exp\left(\frac{\hbar\omega_k}{k_BT}\right) - 1} \right]. \tag{2-73}$$

Before substituting in eq. 2-65, we first split $\alpha(t-t')$ in its real and imaginary part, $\alpha_{fl}$ and $\alpha_{diss}$, respectively. We will later see that the real part is associated with fluctuations and the imaginary to dissipation, reasoning the names. Therefore, the propagator of eq 2-65 is

$$\mathcal{J}(x,x',t;x_0,x_0',0) = \int \mathcal{D}x\mathcal{D}x' \exp\left(\frac{i}{\hbar}(S_S[x]-S_S[x'])\right)$$
$$\times \exp\left\{-\frac{i}{\hbar}\int_0^{t_f}\int_0^{t} dtdt' \,[x(t)-x'(t)]\,\alpha_{diss}(t-t')\,[(x(t)+x(t')]\right\}$$
$$\times \exp\left\{-\frac{1}{\hbar}\int_0^{t_f}\int_0^{t} dtdt' \,[x(t)-x'(t)]\,\alpha_{fl}(t-t')\,[(x(t')-x(t')]\right\} \tag{2-74}$$

where we have

$$\alpha_{fl}(t-t') = \sum_k \frac{C_k^2}{2m\omega_k}\coth\left(\frac{\hbar\omega_k}{k_BT}\right)\cos\omega_k(t-t') \tag{2-75}$$

and

$$\alpha_{diss}(t-t') = -\sum_k \frac{C_k^2}{2m\omega_k}\sin\omega_k(t-t'). \tag{2-76}$$

Now, we can perform the so-called Feynman-trick [65] in the propagator in eq.2-74. This consists of writing the term containing the fluctuation as

$$\exp\left\{-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\,dt'(x(t)-x'(t))\alpha_{fl}(t,t')(x(t')-x'(t'))\right\} =$$
$$\int \mathcal{D}\zeta \exp\left\{-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\,dt'\zeta(t)\alpha_{fl}^{-1}(t,t')\zeta(t') + i\int_0^{t_f} dt\zeta(t)(x(t)-x'(t))\right\}. \tag{2-77}$$

The Feynman trick basically decouples the forward and backward paths, at the expense of introducing a random procces $\zeta$ with probability density functional given by

$$P[\zeta(t)] = \exp\left(-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\,dt'\zeta(t)\alpha_{fl}(t,t')\zeta(t')\right). \tag{2-78}$$

Furthermore, we notice that

$$\langle\zeta(t)\rangle_\zeta = 0, \tag{2-79}$$
$$\langle\zeta(t)\zeta(t')\rangle_\zeta = \hbar\alpha_{fl}(t-t'), \tag{2-80}$$

with $\langle\cdot\rangle_\zeta$ denoting the stochastic average over the distribution $P[\zeta(t)]$.

The explicit form of the superpropagator reads

$$\mathcal{J}(x,x',t;x_0,x_0',0) = \int \mathcal{D}x\mathcal{D}x' \exp\left(\frac{i}{\hbar}(S_S[x]-S_S[x'])\right)$$
$$\times \exp\Big\{-\frac{i}{\hbar}\int_0^{t_f}\int_0^{t} dtdt' \left[x(t)-x'(t)\right]\alpha_{diss}(t-t')\left[(x(t)+x(t')\right]\Big\}$$
$$\times \int \mathcal{D}\zeta \exp\Big\{-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\, dt'\zeta(t)\alpha_{fl}^{-1}(t,t')\zeta(t') + i\int_0^{t_f} dt\zeta(t)(x(t)-x'(t)\Big\} \quad (2\text{-}81)$$

where in the first line we have the dynamics of system of interest, in the second line we have the dissipative term and in the last line we have the stochastic density kernel and the coupling of the stochastic variable with the forward and backward paths.

Up until now, the description of the model has been completely quantum mechanical. In order to get to the equation of classical brownian motion, we need to make a semiclassical approximation. Instead of considering all possible paths, as we do in the full quantum picture, we will extremize the exponentials in (2-81). This is analogous as the classical formalism, but now we also have the contributions of the interaction with the bath (read, the fluctuation and dissipation terms), leading us to the Langevin equations. Considering that there is no coupling between the forward and backward paths, and take the ansatz $x(t) = x'(t)$ [66, 67]. Therefore, the equation of motion derived from the propagator is

$$m\ddot{x}(t) + v'(x) + f_{diss}(t) = \zeta(t), \quad (2\text{-}82)$$

where we have

$$f_{diss}(t) = \int_0^t dt' x(t')\alpha_{diss}(t-t'). \quad (2\text{-}83)$$

We now want to see how this formalism relates to the classical equation of motion that is normally used to described Brownian motion, that is

$$m\ddot{x}(t) + v'(x) + \eta\dot{x}(t) = F(t) \quad (2\text{-}84)$$

where $F(t)$ is a random force with

$$\langle F(t)\rangle = 0 \quad (2\text{-}85)$$
$$\langle F(t)F(t')\rangle = 2\eta k_B T\delta(t-t') \quad (2\text{-}86)$$

where $\eta$ is a dissipation constant. We can see that there is a great resemblance between eq's 2-84 and 2-82. In fact, for the appropriate regime, they are the same. In order to achieve that, we will work with the expressions for $\alpha_{fl}$ and

$\alpha_{diss}$, which as we will see, lead to the terms $f_{diss}(t)$ and $\zeta(t)$ in the equation of motion. Let's begin by looking at the autocorrelator of the stochastic process, $\zeta(t)$, 2-80, explicitly

$$\langle\zeta(t)\zeta(t')\rangle_\zeta = \hbar\sum_k \frac{C_k^2}{2m\omega_k}\coth\left(\frac{\hbar\omega_k}{k_BT}\right)\cos\omega_k(t-t'). \tag{2-87}$$

Taking the high temperature regime, i.e., $k_BT \gg \hbar\omega_k$, we have

$$\hbar\alpha_{fl}(t-t') \approx \frac{k_BT}{m}\sum_k \frac{C_k^2}{\omega_k^2}\cos\omega_k(t-t') + \mathcal{O}(\hbar^2). \tag{2-88}$$

Next, we consider the bath as a continuum of oscillators, with density, $\rho_D(\omega)$, leading to

$$\hbar\alpha_{fl}(t-t') = \frac{k_BT}{m}\int_0^\infty d\omega\rho_D(\omega)\frac{C^2(\omega)}{\omega^2}\cos\omega(t-t') \tag{2-89}$$

and we choose

$$\rho_D(\omega)C^2(\omega) = \begin{cases} \frac{2m\eta\omega^2}{\pi}\,, & \omega < \Omega\,, \\ 0\,, & \omega > \Omega\,, \end{cases} \tag{2-90}$$

introducing a high frequency cutoff. Solving the integral, we are left with

$$\hbar\alpha_{fl}(t-t') = 2\eta k_BT\frac{1}{\pi}\frac{\sin\Omega(t-t')}{(t-t')} \tag{2-91}$$

which in the limit as $\Omega$ goes to infinity,

$$\langle\zeta(t)\zeta(t')\rangle_\zeta = \hbar\alpha_{fl}(t-t') = 2\eta k_BT\delta(t-t'). \tag{2-92}$$

This means that the stochastic term in eq. 2-82 satisfies the same autocorrelation as the usual term in in the Classical Brownian motion, in the regime of high temperatures and in the limit of large $\Omega$. Physically, this means that we are interested in the low frequency regime of the system. One way to understand this is as follows: we do not resolve every single interaction of the environment with the system (for example, every gas molecule hitting a mesoscopic particle), instead we get an averaged out behavior, where many collisions happens in one time interval $\Omega^{-1}$.

We now turn ourselves to the dissipative term, $\alpha_{diss}(t-t')$. Since the distribution of oscillators is already fixed from eq. 2-90, all we need to do is insert it in eq. 2-76, leading to

$$\alpha_{diss}(t-t') = \frac{\eta}{2\pi}\frac{d}{d(t-t')}\int_{-\Omega}^{\Omega} d\omega\cos\omega(t-t') \tag{2-93}$$

which in the limit of $\Omega \to \infty$ gives

$$\alpha_{diss} = \eta\delta'(t-t') \tag{2-94}$$

where the prime is a derivation with respect to $t-t'$. We can now insert it back at eq. 2-76,

$$f_{diss}(t) = \eta \int_0^t dt' x(t')\delta'(t-t'). \tag{2-95}$$

Using the properties of distributions in A, its easy to see that this integral simply gives

$$f_{diss}(t) = \eta\dot{x}(t) \tag{2-96}$$

precisely recovering the term proportional to the velocity in the equation of Classical Brownian motion.

In this way, we showed how to derive the classical equation of Brownian motion from a completely quantum formalism, with use of the Influence Functional approach.

## 2.5 A digression: Quantum Field Theory

In the last section, we discussed a formalism for describing the effective dynamics of a subsystem of interest, tracing out the unobservable part. We did this via the Path Integral formalism which is also used in many formulations of Quantum Field Theory [68]. In this section, we aim at pointing out some similarities between what we have discussed so far and more standard quantum field theory. To do so, we will focus on the dynamics of a scalar field, via the Path Integral formalism and then establish a link with the dynamics of a open system. Afterwards, we briefly describe the quantization method known as second quantization, which highlights the importance of the formalism of Harmonic Oscillator even in a more abstract and fundamental point of view. Throughout this section, unless otherwise stated, we work in natural units, where $\hbar = c = 1$.

### 2.5.1 Path Integrals and Effective Interactions

We begin by stating the Path Integral for a scalar field in 3+1 dimensions, where we will follow [18].

$$Z = \langle 0| e^{-iHt} |0\rangle = \int \mathcal{D}\phi e^{i\int d^4x(\frac{1}{2}(\partial\phi)^2 - V(\phi))} \tag{2-97}$$

where $H$ is the Hamiltonian, the field $\phi(x)$ depends on the four-vector $x = (t, \vec{x})$ and we use a mostly minus metric, as usual in QFT. This functional integral gives us the transition amplitude of going from the ground state, to the ground state, i.e., vacuum to vacuum. Despite many interesting phenonema regarding the vacuum fluctuations present in QFT, we will not cover them in this brief introduction.

A more interesting approach is to actively perturb the vacuum, including in our description source and sink terms, responsible for creation and annihilation of excitations in the field. The Path Integral of interest, than becomes

$$Z = \int \mathcal{D}\phi e^{i\int d^4x(\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))}. \tag{2-98}$$

We notice that the source term that appears in the action, contributes as a force term in the equations of motion for $\phi$.

This functional integral is extremely difficult to solve, except when we have the so called free theory, for $V(\phi) = \frac{1}{2}m^2\phi^2$. In this way we end up with the Klein-Gordon field equation, which is the massive scalar field, usually the first example in any QFT course

$$Z = \int \mathcal{D}\phi e^{i\int d^4x(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + J(x)\phi(x))}. \tag{2-99}$$

By integrating by parts, we can write eq. 2-99 as

$$Z = \int \mathcal{D}\phi e^{i\int d^4x(\phi(\partial^2 - m^2)\phi + J(x)\phi(x))} \tag{2-100}$$

where we can disregard the boundary terms, as we consider all the fields are sufficiently well behaved and vanish at infinity.

The last step to finish the analogy will be to integrate eq. 2-100 over the field, leading to [18]

$$Z(J) = Z_0 e^{iW(J)} \tag{2-101}$$

where $Z_0 = Z(J=0)$ and

$$W(J) = -\frac{1}{2}\int\int d^4xd^4yJ(x)D(x-y)J(y) \tag{2-102}$$

in which $D(x-y)$ is called the propagator and is the solution of

$$-(\partial^2 + m^2)D(x-y) = \delta^{(4)}(x-y). \tag{2-103}$$

In our case, the propagator is of the form

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4)}\frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}. \tag{2-104}$$

Writting in terms of the Fourier transform, we are left with

$$\tilde{W}(J) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}J^*(k)\frac{1}{k^2 - m^2 + i\epsilon}J(k) \tag{2-105}$$

and here we can explicitly see the resemblance with a effective system.

We see that we have a sum over all the modes of the scalar field, due to the integration of $d^4k$. This is actually the same of what happened in the last section, when, in order to find the effect of the environment of the bath in the system of interest (2-89), we summed over all the frequencies of the bath's oscillators. As before this induces a characteristic time in the dynamics of the system, and performing this integration is equivalent to state that the dynamics of the field $\phi$ is way faster than the rest of the system, leading to very short-lived excitations. In field theory jargon, the virtual excitations of the field lead to interaction among the currents $J$.

As an example, consider a current $J = J_1 + J_2$ where $J_i = q_i\delta^{(4)}(x - x_i)$. Calculating

$$W(J_1, J_2) = \int dtE(J_1, J_2) \tag{2-106}$$

where we neglect self interacting terms (which only would lead to additional terms in the action) we end up with

$$E(J_1, J_2) = -q_1q_2\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2 + m^2} = -\frac{q_1q_2}{4\pi r}e^{-mr}. \tag{2-107}$$

In the limit of $m \to 0$ we recover the Coulomb interaction, showing how to achieve it from an effective interaction description.

It's worth noticing that there are many subtleties when quantizing the theory of electromagnetism, mainly to gauge issues. In this example, of the massive scalar field, things are simpler and we can understand the concept in a qualitative way.

The connection with open quantum system dynamics is as follows. Picture a system contained of two subsystems, say A and B, that individually interact with the same environment. The total Hamiltonian of the system is analogous as in the previous section,

$$H_{tot} = H_B + H_S + H_{Int} \tag{2-108}$$

where now we have

$$\begin{aligned} H_S &= \frac{p_x^2}{2M} + v(x) + \frac{p_y^2}{2M} + v(y) \\ H_B &= \sum_k \frac{p_k^2}{2m} + \sum_k \frac{1}{2} m\omega_k^2 R_k^2 \\ H_{Int} &= (x+y)\sum_k C_k R_k, \end{aligned} \tag{2-109}$$

where $x$ and $y$ are the positions and $p_x$ and $p_y$ the momenta regarding the subsystems A and B, respectively. Also, we assumed that the coupling is the same for both systems, to simplify the calculations.

Since the interaction term is linear, it is not hard to show, following exactly the same procedure as in sec 2.4, that the reduced evolution of the system it's given by

$$\tilde{\rho}(x,y,x',y',t) = \int dx_0 dx'_0 dy_0 dy'_0 \mathcal{J}(x,y,x',y',t,;x_0,x'_0,y_0,y'_0,0)\rho_S(x_0,x'_0,y_0,y'_0,0). \tag{2-110}$$

where the superpropagator $\mathcal{J}(x,y,x',y',t,;x_0,x'_0,y_0,y'_0,0)$ is

$$\mathcal{J}(x_i,x'_i,t;x_i(0),x'_i(0),0) = \int \mathcal{D}x\mathcal{D}x' \exp\left(\frac{i}{\hbar}S_S[x_i]\right)\exp\left(-\frac{i}{\hbar}S_S[x'_i]\right)\mathcal{F}[x_i,x'_i]. \tag{2-111}$$

In order to shorten the notation, we have written the dependence in $x, y$ as only $x_i$, and there is a implicit summation. For example, $\exp\left(\frac{i}{\hbar}S_S[x_i]\right)$ is to be understood as $\exp\left(\frac{i}{\hbar}\left(S_S[x]\right)+\left(S_S[y]\right)\right)$. At last, the influence functional for this system is

$$\begin{aligned} \mathcal{F}[x_i,x'_i] = \int & d\boldsymbol{R} d\boldsymbol{R_0} d\boldsymbol{R'_0} \rho_B(\boldsymbol{R_0},\boldsymbol{R'_0},0) \\ & \int \mathcal{D}\boldsymbol{R}\mathcal{D}\boldsymbol{R'} \exp\left\{\frac{i}{\hbar}\left(S_B[\boldsymbol{R}] - S_B[\boldsymbol{R'}] + S_{Int}[x_i,\boldsymbol{R}] - S_{Int}[x'_i,\boldsymbol{R'}]\right)\right\}. \end{aligned} \tag{2-112}$$

At this point, we will begin a more formal discussion. Formal, in this context, is not related to the rigour of the formalism, but rather to the actual form of the equations. Taking the real and imaginary parts of the influence functional we showed that we can write the propagator in terms of two kernels, one involving the fluctuations and the other involving the dissipation, namely

$$\mathcal{J}(x_i, x_i', t; x_i(0), x_i'(0), 0) = \int \mathcal{D}Q\mathcal{D}Q'\exp\left(\frac{i}{\hbar}(S_S[x_i] - S_S[x_i'])\right)$$
$$\times \exp\left\{-\frac{i}{\hbar}\int_0^{t_f}\int_0^t dtdt' \, [Q(t) - Q'(t)] \, \mathcal{A}_{diss}(t-t') \, [(Q(t) + Q(t')]\right\}$$
$$\times \exp\left\{-\frac{1}{\hbar}\int_0^{t_f}\int_0^t dtdt' \, [Q(t) - Q'(t)] \, \mathcal{A}_{fl}(t-t') \, [(Q(t') - Q(t')]\right\} \quad (2\text{-}113)$$

where we have defined $Q(t) = x(t) + y(t)$. Here we explicitly exchanged the previous kernels $\alpha_{fl}(t-t')$ and $\alpha_{diss}(t-t')$ to arbitrary kernels $\mathcal{A}_{fl}(t-t')$ and $\mathcal{A}_{diss}(t-t')$ where we assumed that the time dependence is of the form $(t-t')$. Furthermore, we now that the kernels depends on a continuous distribution of oscillators, analogous to eq.(2-89) and the definition of the density itself depends on physical constraints of the problem (2-90).

We will now turn ourselves to the dissipation term and will argue that its is precisely from there that we will achieve a effective interaction between the two subsystems, mediated by the environment. When summing over all modes dissipative term will be of the form

$$\Phi_{diss} = \int_0^\infty \int_0^{t_f}\int_0^t dtdt'd\omega\mathcal{N}(\omega)\mathcal{A}_{diss}(t-t') \, [Q(t) - Q'(t)] \, [(Q(t) + Q(t')] \quad (2\text{-}114)$$

The argument is as follows: since we have some freedom to choose the density of states, we could, in principle, end up with something proportional to

$$\Phi \propto \int_0^{t_f} dt[Q^2(t) - Q'^2(t')]. \quad (2\text{-}115)$$

As $Q(t) = x(t) + y(t)$, the squared terms would lead to a interaction among the systems, mediated by the environment. Hence, tracing out a subsystem could in principle generate an effective interaction amongst the others.

We have work in progress regarding two particles in a cavity where we wish to derive the effective interaction among them via the Feynman-Vernon formalism, motivated by [69].

### 2.5.2 Canonical Quantization

In this approach, we will see how starting from a classical field equation, namely, once again, the Klein-Gordon field, we can get to a quantum theory. We will assume some familiarity with introductory ideas of classical field theory such as found in [70], for example.

Firstly, we start by the Klein-Gordon Lagrangian (to be precise, this is actually a *Lagrangian Density*, but we will use it as a language abuse) for a real scalar field, $\phi$, that is

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{2-116}$$

From the Euler-Lagrange equations, we see that the equations of motion for the field reads

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0. \tag{2-117}$$

Lastly, we can write the Hamiltonian as

$$H = \int d^3x\mathcal{H} = \int d^3x\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right) \tag{2-118}$$

where $\pi(x)$ is the canonical momentum density conjugate to the field $\phi(x)$ and $\mathcal{H}$ is the Hamiltonian density.

Now, in the same way that is done when quantizing a system with particles, instead of fields, we elevate the field and its canonical conjugate momenta to operators, imposing the following commutation relations

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{x'})] = i\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x'}) \tag{2-119}$$

and

$$[\phi(\boldsymbol{x}), \phi(\boldsymbol{x'})] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{x'})] = 0 \tag{2-120}$$

where we now work in the Schrödinger picture, $\phi(x)$ and $\pi(x)$ does not depend on time. We will relax this condition in the next section, but this serve well for our purposes in here.

We now expand the field into its Fourier transform giving

$$\phi(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{x}\cdot\boldsymbol{k}}\phi(\boldsymbol{k}) \tag{2-121}$$

so that the Klein-Gordon equation turns into

$$\left(\frac{\partial^2}{\partial t^2} + |\boldsymbol{k}|^2 + m^2\right)\phi = 0. \tag{2-122}$$

Defining $\omega_k^2 = |\boldsymbol{k}|^2 + m^2$ we see that the Klein-Gordon field in momenta space follows the same equation as the harmonic oscillator.

Thus, the way of solving is analogous. We will define creation and annihilation operators, such as in section 2.1, but in this time they range through all modes, i.e,

$$\phi(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{1}{2\omega_k}}\left(a_{\boldsymbol{k}}e^{i\boldsymbol{x}\cdot\boldsymbol{k}} + a_{\boldsymbol{k}}^{\dagger}e^{-i\boldsymbol{x}\cdot\boldsymbol{k}}\right) \tag{2-123}$$

$$\pi(\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\left(a_{\boldsymbol{k}}^{\dagger}e^{-i\boldsymbol{x}\cdot\boldsymbol{k}} - a_{\boldsymbol{k}}e^{i\boldsymbol{x}\cdot\boldsymbol{k}}\right), \tag{2-124}$$

where the commutation relation for each of the operators reads

$$[a_{\boldsymbol{k}}, a_{\boldsymbol{k'}}] = (2\pi)^3\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k'}). \tag{2-125}$$

Thus, we can interpret the field as a collection of harmonic oscillators, permeating the whole space.

In the next section, we will generalize this idea for a field with higher degrees of freedom and with more subtleties. The ideas of the formalism, however, remains the same.

## 2.6 From Graviton Physics to Optomechanics

We saw how Quantum Field theory deeply rely in harmonic oscillators and now we take one step further: we aim to describe Quantum Gravity as harmonic oscillators too! Well, not quite. What we will deal, in fact, is closer to a rather new approach called graviton physics [71]. We will not be concerned with fundamentals aspects of Quantum Gravity, for example, what happens at the Planck scale. What we'll do is to consider that gravitons are quantized weak perturbations of spacetime, not its basic constituents. The analogy is precisely what is done with phonons, in condensed matter: phonons are not fundamental constituents of matter and, in fact, they don't even exist in the atomic scales.

Bearing this in mind, we will follow the discussion in [72] and establish the connection between graviton physics and optomechanics. Later on in, chapter

3, we will deepen the discussion analyzing the dynamics of such Hamiltonian.

We will be working within the framework of linearized gravity, employing a flat metric background to enable us to consider weak gravity far from sources. With this choice, the quadratic part of the Einstein-Hilbert action (in vacuum) in the harmonic gauge $\partial_\mu h^{\mu\nu} = 0$ reduces to

$$S_{EH} = \frac{c^4}{32\pi G} \int d^4x \left( \frac{1}{2} \partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} - \frac{1}{4} \partial_\mu h \partial^\mu h \right) , \qquad (2\text{-}126)$$

where, as usual, the field $h^{\mu\nu}$ represents the small perturbations of the otherwise flat metric $\eta_{\mu\nu}$, and $h = \eta^{\mu\nu} h_{\mu\nu}$ is contracted by the flat metric tensor. We neglect higher order terms (that is, gravitational self-interactions) throughout, as their impact is negligible in GWs far from their source.

In the transverse traceless (TT) gauge, we expand the field into Fourier components as

$$h_{ij}^{TT}(t, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}} \epsilon_{ij}^{\lambda}(\boldsymbol{k})\, h_\lambda(t, \boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} , \qquad (2\text{-}127)$$

where the $\epsilon_{ij}^{\lambda}(\boldsymbol{k})$ are the tensors for the two polarization states $\lambda = +, \times$, satisfying the due conditions of orthonormality ($\epsilon_{ij}^{\lambda} \epsilon_{jk}^{\lambda'} = \delta_{ik} \delta^{\lambda\lambda'}$), transverseness ($\epsilon_{ij}^{\lambda} k^j = 0$) and tracelessness ($\mathrm{Tr}\left[\epsilon_{ij}^{\lambda}\right] = 0$). Notice that Greek indices have become Latin indices, as the time components of the field in the TT gauge are null ($h_{0\mu} = 0$).

With the field expressed in this form, we can execute canonical quantization by rewriting the field to operators. We promote the Fourier coefficients to annihilation and creation operators as follows

$$h_\lambda(t, \boldsymbol{k}) \quad \rightarrow \quad \hat{\mathfrak{b}}_{\boldsymbol{k}}^{\lambda} , \qquad (2\text{-}128)$$

$$h_\lambda^*(t, \boldsymbol{k}) \quad \rightarrow \quad \hat{\mathfrak{b}}_{\boldsymbol{k}}^{\lambda\dagger} , \qquad (2\text{-}129)$$

which obey the standard commutation relations (from here on we work in units where $\hbar = 1$),

$$\left[\hat{\mathfrak{b}}_{\boldsymbol{k}}^{\lambda}, \hat{\mathfrak{b}}_{\boldsymbol{k}'}^{\lambda'\dagger}\right] = \delta_{\lambda\lambda'} \delta^{(3)}(\boldsymbol{k}, \boldsymbol{k}') . \qquad (2\text{-}130)$$

The classical field now gets promoted to a quantum field operator and we can write explicitly

$$\hat{h}_{ij}(t, \boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}} \left( \sqrt{\frac{8\pi G}{k}} \epsilon_{ij}^{\lambda}(\boldsymbol{k})\, \hat{\mathfrak{b}}_{\boldsymbol{k}}^{\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \Omega_k t)} + h.c. \right) . \qquad (2\text{-}131)$$

Equation (2-131) concludes the description of the setup we will be using for the gravitational part of our problem; we will mostly consider single-mode metric perturbations in the following, $\sim h_{\mu\nu} e^{ikx}$, corresponding to planar waves of well-defined frequency. This does not limit the scope of our calculations since, under the conditions discussed in the introduction, we will be able to express any potential initial (quantum) state of gravity as a superposition of plane waves.

As far as the detector is concerned, we want to model a GW interferometer. Arvanitaki and Geraci [73] have shown that already a single-mode Fabry-Pérot cavity is sensitive to gravitational waves. This can be achieved either by inserting a nanosphere in the setup [73], or by letting one of the two mirrors be free of moving, as described by Buonanno and Chen in [74], and by Pang and Chen in [75]. Let us narrow our focus to the second case of study. Pang and Chen have demonstrated, by making realistic assumptions (see [75]), that a complete model of a GW interferometer (including power recycling and signal recycling mirrors) can be mapped to a single Fabry-Pérot cavity where one mirror is fixed and the other is free to move. This simplifies the complexity of the interferometer, and we can work with a single cavity of length $L_0$ as the only degree of freedom to describe our model detector. When a GW of strain $h$ passes through such a cavity perpendicularly to its axis, its length changes in the following way:

$$L_0 \quad \to \quad L_0\left(1+\frac{1}{2}h\right) . \tag{2-132}$$

This can be seen as a "gravitomechanical" coupling between the GW and the detector, much like an optomechanical coupling between the electromagnetic field and a mechanical oscillator [76].

Instead of working with the GW coupled to the detector's mirror, one can move to a perspective in which the GW couples directly to the cavity's electromagnetic field (the laser beam). When the cavity is stretched, its resonance frequency changes accordingly as

$$\omega_0 = \frac{n\pi}{L_0} \quad \to \quad \omega = \frac{n\pi}{L_0\left(1+\frac{1}{2}h\right)} , \tag{2-133}$$

which can be expanded as

$$\omega = \omega_0\left(1-\frac{1}{2}h+\mathcal{O}\left(h^2\right)\right) . \tag{2-134}$$

The induced frequency shift can be interpreted as producing an effective coupling between the GW and the electromagnetic field inside the cavity. It turns out, following [77, 78], that for a + polarized GW propagating in the $z$ direction perpendicularly to the cavity axis ($x$ direction), and satisfying $k_x L_0 \ll 1$, such a coupling is represented by an interaction Hamiltonian of the form

$$\hat{H}^{\text{int}}_{\text{GW}} = -\frac{\omega_0}{4}\hat{a}^\dagger\hat{a}\int\frac{d^3\boldsymbol{k}}{\sqrt{(2\pi)^3}}\left(\sqrt{\frac{8\pi G}{k}}\hat{\mathfrak{b}}_{\boldsymbol{k}}+\text{h.c.}\right) . \tag{2-135}$$

In this expression, $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the cavity field, which we take to be in a single mode state for simplicity, while the operators $\hat{\mathfrak{b}}$ are defined in (2-128). Notice that, having fixed the polarization of the GW, the $\lambda$ index has dropped.

Following a procedure which is standard in quantum optics [79], we introduce a quantization volume $V$ to define a dimensionless quantity $\hat{b}_{\boldsymbol{k}} = \hat{\mathfrak{b}}_{\boldsymbol{k}}/\sqrt{V}$, and transform the continuous integral in Eq. (2-135) into its discretized version [77]

$$\hat{H}_{\text{GW}}^{\text{int}} = -\frac{\omega_0}{4}\hat{a}^\dagger\hat{a}\sum_{\boldsymbol{k}}\left(\sqrt{\frac{8\pi G}{Vk}}\hat{b}_{\boldsymbol{k}} + \text{h.c.}\right) . \tag{2-136}$$

Let us now define, respectively, the single graviton strain $f_{\boldsymbol{k}}$, the opto-gravitational coupling constant $g_{\boldsymbol{k}}$, and the dimensionless coupling $q_{\boldsymbol{k}}$, in the following way:

$$f_{\boldsymbol{k}} = \sqrt{\frac{8\pi G}{Vk}}, \quad g_{\boldsymbol{k}} = \frac{\omega_0 f_{\boldsymbol{k}}}{4}, \quad q_{\boldsymbol{k}} = \frac{g_{\boldsymbol{k}}}{\Omega_k} . \tag{2-137}$$

Here, $\boldsymbol{k}$ represents the GW frequency for the mode $\boldsymbol{k}$, where $|\boldsymbol{k}| = \Omega_k$. Since $q_{\boldsymbol{k}}$ is a small number by definition, we will treat it as a perturbative parameter. With these definitions, the Hamiltonian for the complete system, including the GW, the cavity field, and their effective interaction, can be defined as

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{GW}}^{\text{int}} , \tag{2-138}$$

with the free Hamiltonian given by

$$\hat{H}_0 = \omega\hat{a}^\dagger\hat{a} + \sum_{\boldsymbol{k}}\Omega_k\hat{b}_{\boldsymbol{k}}^\dagger\hat{b}_{\boldsymbol{k}} , \tag{2-139}$$

and the interaction Hamiltonian further reduced to

$$\hat{H}_{\text{GW}}^{\text{int}} = -\hat{a}^\dagger\hat{a}\sum_{\boldsymbol{k}} q_{\boldsymbol{k}}\Omega_{\boldsymbol{k}}\left(\hat{b}_{\boldsymbol{k}} + \hat{b}_{\boldsymbol{k}}^\dagger\right) , \tag{2-140}$$

which is precisely the interaction term of the dispersive Hamiltonian, discussed in 2.2. The derivation of the explicit form of the time evolution operator for the interaction term (2-140) is a lengthy but straightforward calculation, using an approach which is standard in quantum optics. As reported in [77, 80], we can express the result for a single mode $\boldsymbol{k}$ (omitting the index for the sake of readability) as

$$\hat{U}(t)\,|\Psi(t)\rangle = e^{q\hat{a}^\dagger\hat{a}\left[\eta(t)\hat{b}-\eta^*(t)\hat{b}^\dagger\right]}e^{iB(t)\left(\hat{a}^\dagger\hat{a}\right)^2}\,|\Psi(t)\rangle , \tag{2-141}$$

where the time evolution is contained in the definition of

$$\eta(t) = 1 - e^{-it} , \tag{2-142}$$

$$\eta^*(t) = 1 - e^{it} , \tag{2-143}$$

$$B(t) = q^2\left(t - \sin(t)\right) , \tag{2-144}$$

and the time-evolving state that appears in (2-141) is defined as $|\Psi(t)\rangle = e^{-i\hat{b}^{\dagger}\hat{b}t}|\Psi\rangle$.

Note that we are neglecting the effects of optical loss and decoherence in the cavity, which is a valid approximation for times smaller than the inverse cavity decay rate, or linewidth. We consider optical coherent states as the detector's probe state, for which the main effect of optical loss is to reduce the state's amplitude [81]. Within this approximation, we will then be interested in how the electric field quadratures are affected by the interaction with different quantum states of the GWs, as predicted by the low energy EFT description of gravity.

# 3
# Quantum Optics of Gravitational Waves

This chapter is based on the article: Luca Abrahão, Francesco Coradeschi, Antonia Micol Frassino, Thiago Guerreiro, Jennifer Rittenhouse West and Enrico Junior Schioppa 2024 Class. Quantum Grav. 41 015029, *Quantum Optics of Gravitational Waves* [72].

In the last section, we have showed how one achieves the dispersive optomechanical interaction, starting from a linearized theory of gravity. In the following chapter, we will focus in the experimental implications of this interaction. Furthermore we discuss how the possible quantum nature of the gravitational wave would induce new behaviour in our detection. We do that by combining tools used in quantum optics with the description of gravitational waves via a effective field theory.

Furthermore, we highlight that the formalism presented consists of many approximations, such as a unitary dynamics and possible quantum fluctuations arising from the complete theory of quantum gravity, i.e, the contributions of higher energies. In order to provide a full description, one would need to consider, for instance, a open quantum system approach, similar from one described in sec.2.4. This formalism is known in the literature as stochastic gravity [82]. In addition we will disregard higher energy effects that could, in principle, be present in the proper quantum gravity theory (here, "proper quantum gravity" is to be understood according [71]).

We therefore look into how a simplified model of gravitational waves could, at least in principle, lead to insights regarding the quantum nature of gravity.

This chapter is disposed as follows. In section 3.1 we apply the unitary dynamics previously derived in 2.6 to different gravitational waves states, such as vacuum, coherent and thermal states. In section 3.2 we see how the gravitational wave affects fluctuations in the electric field and how it could be used to reconstruct (at least partly) the gravitational wave state. In section 3.3 we discuss, rather speculative, the effects of squeezed gravitational waves that could be (optimistically) measurable. Nonetheless, until any signal following the predicted deviations from the classical one shows in the detector, we remain in the guessing department. We do not prove the existence of such non-classical states, but we argue in favor of it in 3.3.1, discussing possible sources. Lastly, in section 3.4 we briefly discuss the results.

## 3.1 GW state reconstruction

When examining how a GW interacts with an optical cavity, the most suitable observable is the electric field operator that characterizes the cavity field's state

$$\hat{\mathcal{E}} = \sqrt{\frac{\omega}{V_c}} \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} \,. \tag{3-1}$$

This is indeed the physical quantity that one measures at an interferometer to produce a detectable signal. Note that here $V_c$ denotes the cavity mode volume.

Now that we have defined the operator, we can proceed to calculate the matrix elements of its time evolution. Our focus will be on the mean value, specifically when $n = 1$. The classical GW signal at an interferometer is sensed as a variation of the phase of the field quadrature:

$$\mathcal{E} \quad \to \quad \mathcal{E} e^{i\phi} \,, \tag{3-2}$$

where $\phi$ changes in time. For example, this variation of the phase produces typical chirp-like signatures observed for binary merger events. Any result we find that produces a departure of $\phi$ from its classical behavior, namely which has the form

$$\mathcal{E} \quad \to \quad \mathcal{E} e^{i(\phi + \delta\phi)} \,, \tag{3-3}$$

is interpreted as an effect on the *signal*. Contrariwise, if we find a modification of the form

$$\mathcal{E} \quad \to \quad \mathcal{E} + \epsilon \,, \tag{3-4}$$

then we are witnessing an effect on the *noise*.

To begin with, we observe that the expression for the time evolution operator (2-141) includes an exponential term in $q^2$ (the one defined as $B(t)$ in (2-144)). However, since the terms in $q$ is dominant, we can disregard this term for now (although we will revisit it in Sec. 3.1.3). Based on this assumption, we can use the time evolution of the electric field operator to obtain the following result

$$\mathcal{E}(t) = \sqrt{\frac{\omega}{V_c}} \left( \frac{\langle \Psi(t) | \hat{\mathcal{D}}[q\eta(t)] \, \hat{a} | \Psi(t) \rangle + h.c.}{\sqrt{2}} \right) , \tag{3-5}$$

where we have defined the operator in parenthesis as

$$\hat{\mathcal{D}}[q\eta(t)] = e^{q\hat{a}^\dagger \hat{a} \left[ \eta(t) \hat{b} - \eta^*(t) \hat{b}^\dagger \right]} \,. \tag{3-6}$$

This operator acts on the gravitational field as a *displacement operator* whose amplitude is proportional to the optical field's intensity.

Let us proceed with the selection of specific states within the Hilbert space, commencing with the detector component. To a very good approximation, the electromagnetic field inside the cavity is a monochromatic wave at frequency $\omega$. From the quantum point of view, we can model it as a single mode coherent state $|\alpha\rangle$ for some complex number $\alpha$. With this hypothesis, we can easily trace out the detector component of the state $|\Psi(t)\rangle$, and we are left with

$$\mathcal{E}(t) = \sqrt{\frac{\omega}{V_c}} \left( \frac{\alpha \langle \Psi_g(t) | \hat{\mathcal{D}}[q\eta(t)] | \Psi_g(t) \rangle + c.c.}{\sqrt{2}} \right) , \tag{3-7}$$

where now the (quantum) GW wavefunction $|\Psi_g\rangle$ enters into play independently.

### 3.1.1 Vacuum state

Now, let us consider the possibility of preparing GW states in specific quantum states, with the simplest one being the vacuum state. The vacuum state for the generic mode $\boldsymbol{k}$ (that is, a state with no gravitons of energy $\boldsymbol{k}$) can be written as:

$$|\Psi_g\rangle = |0_{\boldsymbol{k}}(t)\rangle = e^{-i\hat{b}^\dagger_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}}\Omega_k t} |0_{\boldsymbol{k}}\rangle = |0\rangle \ , \tag{3-8}$$

where we are following the standard harmonic oscillator convention of writing the eigenstates of the number operator $a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}$ as $|n_{\boldsymbol{k}}\rangle$, and $|0\rangle$ is the state with no gravitons at all. Therefore, to obtain the mean field we must evaluate the following expression:

$$\mathcal{E}(t) = \sqrt{\frac{\omega}{V}} \left( \frac{\alpha \prod_{\boldsymbol{k}} \langle 0 | \hat{\mathcal{D}}[q_{\boldsymbol{k}}\eta(\Omega_k t)] | 0 \rangle + c.c.}{\sqrt{2}} \right) . \tag{3-9}$$

The matrix element is easily calculated by considering that the vacuum can be seen as the coherent state with $\alpha = 0$. The displacement operator acting on such a state gives

$$\hat{\mathcal{D}}[q_{\boldsymbol{k}}\eta(\Omega_k t)] |0\rangle = |q_{\boldsymbol{k}}\eta(\Omega_k t)\rangle \ . \tag{3-10}$$

Using the normalization condition for coherent states,[1] we can rewrite the previous expression as

$$\langle 0 | \hat{\mathcal{D}}[q_{\boldsymbol{k}}\eta(\Omega_k t)] | 0 \rangle = \langle 0 | q_{\boldsymbol{k}}\eta(\Omega_k t) \rangle = e^{-\frac{1}{2}q^2_{\boldsymbol{k}}|\eta(\Omega_k t)|^2} , \tag{3-12}$$

and thus define

[1] The normalization condition is that given two coherent states characterized by complex numbers $\alpha$ and $\beta$, one has

$$\langle \alpha | \beta \rangle = e^{-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - \alpha^*\beta - \alpha\beta^*\right)} . \tag{3-11}$$

$$D\prod_{\boldsymbol{k}} \langle 0|\hat{\mathcal{D}}\left[q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle = e^{-\frac{1}{2}\sum_{\boldsymbol{k}} q_{\boldsymbol{k}}^2 |\eta(\Omega_k t)|^2} . \tag{3-13}$$

This quantity (3-13) was calculated in [77] by introducing both an infrared and an ultraviolet cutoff to avoid divergence and by noting that by simple algebra

$$|\eta\left(\Omega_k t\right)|^2 = 2\left[1 - \cos\left(\Omega_k t\right)\right] . \tag{3-14}$$

The final result for the the mean field, in the case of a vacuum state, is

$$\mathcal{E}\left(t\right) = \sqrt{\frac{2\omega}{V_c}} D\,\mathrm{Re}\{\alpha\} . \tag{3-15}$$

We can express this as equation (3-4), which represents the effect of the detector's noise. Essentially, we have determined how the gravitational vacuum affects the interferometer's sensitivity curve. However, it is important to note that this effect is orders of magnitude below any reasonably achievable sensitivity [77, 83]. In fact, it's even lower than the theoretical quantum thermal noise of gravity ( $\sim 10^{-37}\ 1/\sqrt{\mathrm{Hz}}$) across a wide range of frequencies, as demonstrated in Figure 3 of [84], where it is clear that such a limit is at least 15 orders of magnitude below the currently achievable sensitivity curve near $\sim 10^{-25}\ 1/\sqrt{\mathrm{Hz}}$. Ultimately, this effect is impractical to measure.

When analyzed more closely, however, we do find an interesting side result. We performed this calculation after we had neglected the subdominant $q^2$ term in equation (2-141). Had we retained it, we would have arrived to the surprising conclusion that the gravitational vacuum induces squeezing of the cavity field [77]. Once again, after plugging in the right numbers, this effect turns out to be practically unmeasurable. However, in general, we find that, unsurprisingly, to achieve a measurable effect in an experiment where gravity couples to optical observables, one needs to start from gravity modes populated with a large mean number of gravitons, as we will see in the following.

### 3.1.2 Coherent state

The simplest state in which we can collect a large mean number of gravitons together is a coherent state. A calculation similar to the one we performed to arrive at equation (3-15) can be carried out to derive the effect on the cavity's field quadrature of a single mode coherent state of gravity. We write such state as

$$|\Psi_g\rangle = \begin{cases} \left|he^{i\Omega_{GW}t}\right\rangle & \text{if } k = k_{GW}\,, \\ |0\rangle & \text{otherwise}\,. \end{cases} \tag{3-16}$$

Here $h$ is real (we set the phase to zero, for simplicity) and is indeed a large number linked to the population of the mode. Now we must calculate

$$\mathcal{E}(t) = \sqrt{\frac{\omega}{V_c}} \left( \frac{\alpha \left\langle h e^{i\Omega_{GW} t} \middle| \hat{\mathcal{D}}\left[q_{GW}\eta\left(\Omega_{GW} t\right)\right] \middle| h e^{i\Omega_{GW} t} \right\rangle \prod_{\boldsymbol{k}\neq \boldsymbol{k}_{GW}} \langle 0 | \hat{\mathcal{D}}\left[q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right] |0\rangle + c.c.}{\sqrt{2}} \right). \tag{3-17}$$

When evaluating the $\boldsymbol{k} \neq \boldsymbol{k}_{GW}$ product, we would obtain a product of terms in $\sim e^{\frac{1}{2}q_{\boldsymbol{k}}^2}$ which - because of the small value of $q_{\boldsymbol{k}}$ - are all of order 1, much in the same way as we calculated expression (3-13). This means we can neglect all but

$$\mathcal{E}(t) \sim \sqrt{\frac{\omega}{V_c}} \left( \frac{\alpha \left\langle h e^{i\Omega_{GW} t} \middle| \hat{\mathcal{D}}\left[q_{GW}\eta\left(\Omega_{GW} t\right)\right] \middle| h e^{i\Omega_{GW} t} \right\rangle + c.c.}{\sqrt{2}} \right). \tag{3-18}$$

Again, the full calculation is straightforward, arriving at

$$\mathcal{E}(t) \sim e^{i2qh\sin\Omega t}. \tag{3-19}$$

This expression is of the form (3-3), and it tells us that we are measuring a signal oscillating in phase with the GW. We have thus recovered the classical GW signal from a representation of the quantum analog of a classical monochromatic wave: the single-mode coherent state. This gives us a first validation check of our program.

### 3.1.3 GW-induced decoherence

In Sec. 3.1, we initially neglected the $q^2$ terms in the time evolution operator (2-141). This choice was justified as these terms give rise to effects, such as the aforementioned squeezing of the cavity field induced by the gravitational quantum vacuum, that is by far dominated by effects that are linear in $q$. However, it is worth paying some more attention to such $q$-quadratic terms, as it turns out they produce gravity-induced decoherence. Although the effect is weak and difficult to measure, it serves as a secondary validation check by linking our setup to established results.

To see this, let us now repeat the calculations presented in Sec. 3.1 but this time by preparing our system as an electromagnetic (EM) qubit interacting with the gravitational vacuum

$$|\Psi(0)\rangle = \frac{|0\rangle_{\text{EM}} + |N\rangle_{\text{EM}}}{\sqrt{2}} \otimes |0\rangle_{\text{GW}} \tag{3-20}$$

where, once again, $|N\rangle_{\text{EM}}$ ($|N\rangle_{\text{GW}}$) denotes a state with $N$ photons (gravitons). We can now perform the following three steps:

1. Evolve the state using the simplified time evolution operator as expressed

in equation (3-6);

2. write the total density matrix $\rho\left(t\right)$ associated to the time-evolving state;

3. calculate the density matrix $\rho_{EM}\left(t\right)$ associated to the EM subsystem, by tracing out the GW degrees of freedom.

When carrying out the calculations, we arrive at the following denity matrix [85]:

$$\rho_{\mathrm{EM}}\left(t\right) = \begin{pmatrix} \frac{1}{2} & \rho_{01} \\ \rho_{01}^{*} & \frac{1}{2} \end{pmatrix}, \tag{3-21}$$

where

$$\rho_{01} = \langle 0|qN\eta\rangle = e^{-\frac{1}{2}q^2N^2|\eta|^2}. \tag{3-22}$$

The presence of such time dependent off-diagonal terms (via the time dependence of $\eta\left(t\right)$) shows indeed that the $q^2$ component of the time evolution operator is inducing decoherence.

This can be taken farther. By replacing the vacuum $|0\rangle_{\mathrm{GW}}$ in equation (3-20) with a GW single mode coherent state $|\alpha\rangle_{\mathrm{GW}}$, and repeating the same calculations, we obtain an EM density matrix of the same form as (3-21), but now with

$$\rho_{01} = e^{-\frac{1}{2}\left[q^2N^2|\eta|^2+qN(\eta^*\alpha-\eta\alpha^*)\right]}. \tag{3-23}$$

When we extend it to a single mode GWs in a thermal state, we obtain

$$\rho_{01} = e^{-\frac{1}{2}q^2N^2|\eta|^2(1+\overline{n})} \tag{3-24}$$

where $\overline{n}$ is the mean number of gravitons[2]. This result is useful in that it finally allows us to consider an ensemble of modes in thermal states. In such a case, we would need to reintroduce the state index $\boldsymbol{k}$, and calculate

$$\prod_{\boldsymbol{k}} e^{-\frac{1}{2}q_{\boldsymbol{k}}^2N^2|\eta\boldsymbol{k}|^2(1+\overline{n}_{\boldsymbol{k}})} = e^{-\frac{1}{2}N^2\sum_{\boldsymbol{k}}q_{\boldsymbol{k}}^2|\eta\boldsymbol{k}|^2(1+\overline{n}_{\boldsymbol{k}})} \tag{3-26}$$

For large temperatures $T$, one approximates $(1+\overline{n}_{\boldsymbol{k}})$ with $\overline{n}_{\boldsymbol{k}}$, and $T$ simply counts the number of gravitons in each mode as

$$\overline{n}_{\boldsymbol{k}} = \frac{k_BT}{\Omega_{\boldsymbol{k}}}. \tag{3-27}$$

where $k_B$ is Boltzmann's constant. Using the explicit forms of $q_{\boldsymbol{k}}$ and $\eta_{\boldsymbol{k}}$, and averaging over the Bose-Einstein distribution (see [85]), we arrive at an expression of the form

$$\rho_{01} \approx e^{-\Gamma t}, \tag{3-28}$$

[2]To arrive at equation (3-24), one should remember that the density matrix of a thermal state with mean number of gravitons $\overline{n}$, can be related to the continuum of coherent states $|\alpha\rangle$ as

$$\rho = \int \frac{d^2\alpha}{\pi\overline{n}}e^{-\frac{|\alpha|^2}{\overline{n}}}|\alpha\rangle\langle\alpha|. \tag{3-25}$$

where

$$\Gamma \propto k_B T \left(\frac{\Delta E}{E_{\rm pl}}\right)^2 , \tag{3-29}$$

and $\Delta E = N\omega$ is the energy of the state $|N\rangle_{\rm EM}$, in accordance with previous results on gravitational-induced decoherence [86].

## 3.2 GW-induced electric field fluctuations

Before we continue discussing other gravity wave states, it is important to comment on the practicality of measuring deviations from the classical theory with elecrtomagnetic probes. In a previous work [85], we showed that the measurement problem for our quantum gravitational states can be stated in terms of photon-number tomography of the optical mode that interacts with the wave. In particular, we showed that if the GW state is Gaussian, it can be reconstructed from experimentally accessible data that can be measured from non-classical (yet macroscopic) observables. Here, we would like to point out that information on the GW states can also be obtained from field (homodyne) measurements, which is more practical than a photon-number resolving measurement experiments.

Reconstruction of the second moments of a general GW state $|\Psi\rangle$ can be achieved by measuring expectation values of the form $\langle\Psi|\mathcal{D}(nq\eta(t))|\Psi\rangle$, where $n$ is an integer [85]. General reconstruction of the first and second moments can be achieved if we measure these expectation values for $n = 1, 2, 3$. For coherent states, this can be done by measuring the first three moments of the electric field.

The variance of the field is $\Delta\mathcal{E} = \langle\mathcal{E}^2\rangle - \langle\mathcal{E}\rangle^2$ and we compute $\langle\mathcal{E}^2\rangle$ by tracing out the detector component. Noticing that

$$\mathcal{E}^2(t) = \frac{\omega}{2V_c}\left(2a^\dagger a - 1 + a^2\mathcal{D}(-2q\eta(t)) + \left(a^\dagger\right)^2\mathcal{D}^*(-2q\eta(t))\right), \tag{3-30}$$

we find that for general states,

$$\langle\mathcal{E}^2(t)\rangle = \frac{\omega}{2V_c}\left(|\alpha|^2 - 1 + \alpha^2\langle\Psi_g(t)|\hat{\mathcal{D}}(2q\eta(t)|\Psi_g(t)\rangle + c.c.\right) . \tag{3-31}$$

Assuming that the gravitational wave is initially a vacuum state, we have

$$\langle\mathcal{E}^2\rangle = \frac{\omega}{2V_c}\left(\alpha^2\langle 0|\hat{\mathcal{D}}\left[2q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle + \alpha^{*2}\langle 0|\hat{\mathcal{D}}^*\left[2q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle - 1 + |\alpha|^2\right) . \tag{3-32}$$

Analogously,

$$\langle 0|\hat{\mathcal{D}}\left[2q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle = \langle 0|2q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\rangle = e^{-2q_{\boldsymbol{k}}^2|\eta(\Omega_k t)|^2} , \tag{3-33}$$

and we can define the quantity,

$$D_2 \prod_{\boldsymbol{k}} \langle 0|\hat{\mathcal{D}}\left[2q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle = e^{-2\sum_{\boldsymbol{k}} q_{\boldsymbol{k}}^2|\eta(\Omega_k t)|^2}\,. \tag{3-34}$$

Using the previous equations, the mean square value of the electric field interacting with the GW (3-32) becomes

$$\langle \mathcal{E}^2\rangle = \frac{\omega}{2V_c}\left[2D_2\,\mathrm{Re}(\alpha^2) - 1 + |\alpha|^2\right]. \tag{3-35}$$

As was shown in [85], to determine the second order correlation functions of the GW, we need to evaluate terms proportional up to $\langle 0|\hat{\mathcal{D}}\left[3q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle$. In order to achieve that, we need to go up to the third order moment, the skewness ($\Delta s$), defined as

$$\Delta s = \langle \mathcal{E}^3\rangle - \langle \mathcal{E}\rangle\langle \mathcal{E}^2\rangle + 2\langle \mathcal{E}\rangle^3\,. \tag{3-36}$$

Note that all the terms in the above definition have already been computed, except for $\langle \mathcal{E}^3\rangle$. We now turn our attention to this particular term. Notice that

$$\begin{aligned}\mathcal{E}^3(t) \;=\; & \left(\frac{\omega}{2V_c}\right)^{3/2}\Big\{a^3\mathcal{D}[-3q\eta(t)] + (2a^\dagger aa - 3a)\mathcal{D}[-q\eta(t)] \\ & +(2a^\dagger a^\dagger a - 3a^\dagger)\mathcal{D}^*[-q\eta(t)] + (a^\dagger)^3\mathcal{D}^*[-3q\eta(t)]\Big\}\,.\end{aligned} \tag{3-37}$$

For the initial vacuum state we find

$$\langle \mathcal{E}^3(t)\rangle = \left(\frac{\omega}{2V_c}\right)^{3/2}\left[\alpha^3\langle 0|\hat{\mathcal{D}}\left[3q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle + \left(|\alpha|^2\alpha - 3\alpha\right)\langle 0|\hat{\mathcal{D}}\left[q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle + c.c\right], \tag{3-38}$$

and

$$\langle \mathcal{E}^3\rangle = \left(\frac{\omega}{2V_c}\right)^{3/2}\left[2D_3\,\mathrm{Re}(\alpha^3) - 2D\,\mathrm{Re}(\alpha)(3-|\alpha|^2)\right], \tag{3-39}$$

where $D_3$ is defined as

$$D_3 \prod_{\boldsymbol{k}} \langle 0|\hat{\mathcal{D}}\left[3q_{\boldsymbol{k}}\eta\left(\Omega_k t\right)\right]|0\rangle = e^{-\frac{9}{2}\sum_{\boldsymbol{k}} q_{\boldsymbol{k}}^2|\eta(\Omega_k t)|^2}\,. \tag{3-40}$$

With this, we see that the quantities $D, D_2$ and $D_3$ can be obtained from measurements of the first three moments of the electric field, which in turn can be measured via homodyne detection. In possession of these quantities, we can then reconstruct the first and second moments of GW vacuum fluctuations. This calculation can easily be extended to the case of a coherent sate. GW-induced electric field fluctuations can also be calculated for other states following the recipe introduced above, although in general perfect state reconstruction cannot be achieved from measurements of the electric field moments alone. These fluctuations could, however, lead to interesting signatures [87].

## 3.3 Squeezed gravity

So far, we have considered the vacuum, coherent states, and thermal states for the gravity modes. While interesting theoretical insight can be gained by studying these cases, none of them will yield measurable quantum effect under our assumptions. It is still interesting (and, indeed, a required sanity check of our approach) that in the case of a highly populated coherent state, we recover the classical signal even though we started from a completely quantum picture.

The natural next step in our investigation is asking the question: can any quantum states of gravity exist that have no classical analog and have a chance of yielding a detectable effect?

We give a tentative answer to this question by drawing from quantum optics experience and investigating what happens if we *assume* that gravity can live in a *squeezed state* – the analogue of squeezed states of light that are routinely produced in optics laboratories. Before going on, we stress that this assumption implies the existence of some mechanism that does put gravity into such a state, which is as yet unknown for GWs in the LIGO band. To be more precise, there is at least one candidate in the category: an established consensus exists on the hypothesis that inflation might indeed have squeezed gravity at primordial times; however, once again – much like in the cases of vacuum corrections or gravitational decoherence – this effect leads to a weak signal [84]. Nonetheless, calculating the potential signature of a gravitational squeezed vacuum on our model detector is instructive. After all, we cannot a priori exclude mechanisms, other than inflation, that could produce squeezing (see section 3.3.1 for more detailed discussion on this point).

Let us thus prepare gravity in a squeezed state, which we may model as a mode of the form

$$|\Psi_g\rangle = \left|\beta e^{2i\Omega t}\right\rangle = \hat{S}\left(\beta\right)|0\rangle \ , \tag{3-41}$$

where the complex number $\beta$ is the squeezing parameter and $\hat{S}\left(\beta\right)$ is the squeezing operator, as defined in textbooks. The preparation of such squeezed state (as well as more general non-classical GW states) assumes that the low energy quantum EFT of gravity is a valid description and that no additional decoherence mechanisms emerging from a more complete theory of quantum gravity, for instance stochastic gravity [82], are at play either in the state preparation of its propagation throughout spacetime. Within this assumption, the matrix element for the mean electric field (3-5) contains terms of the form

$$\alpha \left\langle 0\right|\hat{S}^{\dagger}\left(\beta\right)\hat{\mathcal{D}}\left[q\eta\left(t\right)\right]\hat{S}\left(\beta\right)\left|0\right\rangle \ , \tag{3-42}$$

and after some calculations, which were slightly more involved but nonetheless straightforward, we arrive at our result:

$$\mathcal{E}\left(t\right) \sim 2\alpha\left[1 - 8q^2 e^{2|\beta|}\sin^4\left(\frac{\Omega t}{2}\right)\right] . \tag{3-43}$$

This term is of the form of equation (3-4) and it thus tells us that we found an effect on the *noise*: a squeezed gravitational vacuum would manifest itself at an interferometer as an additional, oscillating term, in the noise spectrum of the instrument [77, 83, 88]. The most interesting part is that the amplitude of such an oscillating term contains an exponential factor. Such a factor could behave as an enhancement term to the noise, depending on the magnitude of the squeezing parameter $\beta$. The magnitude of $\beta$, in turn, depends on the details of the source dynamics, which at this moment we cannot foresee. Nonetheless, if mechanisms exist in nature that would produce such an exponentially enhanced effect, we cannot exclude that future, more sensitive detectors could actually see it. In the subsequent section, we shall delve deeper into this topic.

Something even more interesting happens when we prepare gravity in a *squeezed-coherent* state. For a single mode, this would mean

$$|\Psi_g\rangle = \hat{S}\left(\beta\right)\hat{\mathcal{D}}\left(he^{i\Omega_{GW}t}\right)|0\rangle \ , \tag{3-44}$$

where we have used the same notation as in eq. (3-16). Such a state would represent a squeezed quantum gravitational wave mode propagating from the source to the detector. The electromagnetic analog would be a squeezed laser beam, which we are able to generate and propagate in a laboratory.

After calculating the electric field matrix element as usual, for the first time we find an effect of the type described by equation (3-3) which deviates from the classical behavior. Specifically, after rewriting $\beta = re^{i\xi}$, we get

$$\delta\phi = 2hq\left[\sin\left(\Omega t\right)\cosh\left(r\right) + \sin\left(2\xi - \Omega t\right)\sinh\left(r\right) - \sin\left(2\xi\right)\sinh\left(r\right)\right] \tag{3-45}$$

Thus, an exponentially enhanced or suppressed effect, but this time on the *signal*. Once again, the magnitude of the effect depends on the dynamics at the source, which at this point remains unknown. Nonetheless, equation (3-45) clearly shows how a purely quantum effect (squeezing) involving a state with a macroscopically high number of gravitons $|h|$, would produce a signal which can be exponentially enhanced – even to order one – and can thus be detectable with current or near future technology [85].

### 3.3.1 Are squeezed gravitational waves produced in nature?

In the framework in which we are working, where GR is seen as a classical limit of an intermediate-energy effective quantum field theory of (self)-interacting gravitons, squeezed gravitational waves definitely exist *theoretically*, that is, they are allowed states in the Hilbert space of the theory.

The question however remains open on whether there are any realistic astrophysical sources that could produce GWs with a sizable (i.e. potentially measurable) amount of squeezing. While the aforementioned hypothesis on the squeezing of the relic gravitational background induced by inflation seems to be widely accepted, it is predicted to be too small to be observed at gravitational interferometers. Drawing on our experience on quantum optics, we can outline two basic conditions that we can expect to be met in order to have measurable squeezing in a physical process:

1. The process should involve states characterized by a high – macroscopic – occupancy number;
2. The resulting state should be capable of propagating (ideally) undisturbed from source to detector.

Both conditions are naturally met by the processes that produce the gravitational waves that we are able to observe. First of all, there is no doubt that mergers of black holes (or really any other sources of high-intensity GWs) involve macroscopic number of gravitons (assuming, of course, that gravitons do exist). Furthermore, gravity is naturally weak-interacting at low energy, which means a GW basically stops interacting as soon as it leaves its source, traveling the distance to the detector nearly undisturbed. Note that this contrasts sharply with the behaviour of squeezed sources in electromagnetism: in quantum optics laboratories, squeezed beams of light are commonly produced by using intense laser beams, which are prone to losing coherence because of the high probability of interaction with any medium present in the laboratory. Transporting the quantum state of a laser beam (e.g. a squeezed coherent state) over long distances from source to detector is thus challenging, and can only be achieved with great effort in a laboratory. Gravitational waves can be expected to be free of this problem.

However, even if merger events (or other sources of strong GWs) do satisfy the minimum requirements to be candidate producers of squeezed gravitational waves, it is a challenge to understand if they *actually* produce such states. Answering this question appears to involve the theoretical treatment of quantum effects at strongly nonlinear GR regimes, which is beyond our current capacities.

However, while we are not (yet) capable of providing a formal argument in favour of the production of squeezed GWs at mergers and similar events (but for some discussion on the topic, see [85]), we would still like to argue, more heuristically, that this possibility does deserve further investigation. Let us think once again of the analogy with quantum optics. Squeezed states of light are produced in the lab by making intense laser beams interact with anisotropic crystals. In fact, the mechanism that turns a coherent laser beam into a squeezed coherent laser beam involves the interaction of intense light with a highly non-linear optical medium. As electromagnetism is a linear theory at the classical level (nonlinear effects are only manifest in the quantum regime), producing squeezed light is in a sense an "exotic" process, in that it needs well-controlled laboratory conditions and is not found spontaneously in nature. Compare this with the case of gravity. Contrary to Maxwell's equations, Einstein's equations are already nonlinear at the classical level, and nonlinear effects affect phenomena taking place in strong regimes [89, 90]. This means that squeezing of macroscopic gravitational waves might well be a natural effect, provided the source is strong enough – which is definitely true in the case of mergers. Strong nonlinear astronomical sources, perhaps those already known to emit GWs, seem therefore reasonable candidates to investigate. A recent step in this direction has been proposed in [91] where nonlinear effects present in black hole's ringdown [89, 90, 92–95] have been considered.

## 3.4 Discussion

In this chapter, we have shown explicitly how to make use of quantum optics in order to derive phenomenological results in quantum gravity in the weak gravity regime. We focused on the treatment of the problem from the point of view of the equations of motion and applied the result to a model GW interferometer interacting with a few possible quantum states of gravity. We have examined various quantum states ranging from the basic vacuum state to the coherent state, and ultimately concluded with an evaluation of squeezed states. Among the ones we evaluated, *squeezed-coherent* gravitational waves have proven to be the most promising candidates for providing potentially detectable quantum aspects of gravity. The findings of Sec. 3.3.1, however basic, together with the results reported in Sec. 3.3, show how a squeezed coherent GW could produce an effect on the *signal* of a GW interferometer, and that such an effect has the potential of being of order 1. This indicates to us that further research on the topic - especially regarding the existence of possible sources - has promise and is worth pursuing.

# 4
# Quantum Induced Stochasticity

This chapter is based on the article: Pedro Paraguassú, Luca Abrahão, Thiago Guerreiro, ArXiv preprint 2401.16511v1 *Quantum Induced Stochastic Optomechanical Dynamics* [96].

In this chapter we will, once again, look for quantum signatures in a, in principle, non quantum situation. Given the increasing capability of optomechanical systems in controlling and measuring the nanoparticle's dynamics, we are reaching a point where possible quantum features within the system could start affecting the experimental data.

Motivated by the work of Parikh, Wilzcek and Zahariade [26], regarding the effect on the detector after we trace out the gravitational wave, we investigate the analogous effect in an optomechanical system.

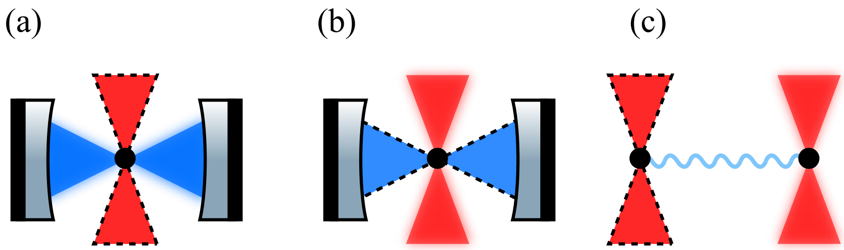


Figure 4.1: Different quantum-classical optomechanical interactions. (a) Quantum light influencing a classical particle. (b) Quantum particle influencing classical light. (c) Quantum particle influencing a classical particle.

We study two different systems, showed in fig 4.1: the first consists of a trapped nanoparticle inside an optical cavity, in which we consider tracing out both the cavity field and the particle; in the second, we consider two individually trapped particles interacting via Coulomb.

This procedure allows to seek for how different quantum states of the traced out system impacts the dynamics of the observed system, since the extra noise term depends on the quantum state.

To quantify this effects, we use the Influence Functional formalism stated in the beginning of sec. 2.4 and plug in the actual values of the parameters currently achieved or pursued by the levitated optomechanics community.

In 4.1, we develop the whole formalism needed to quantitative describe each of the cases. We start describing the Hamiltonian of the system in a general case. Then we proceed to describe the dynamics via a Coherent Path Integral [97] approach. Afterwards we evaluate the Influence Functional for some quantum states of interest, such as the squeezed. The final step is to generalize the formalism, that was up to this point only a mono mode description, to a more general form. The subsequent sections 4.2, 4.3 are applications of the formalism considering that we first trace out the quantum states of the cavity and observe the semiclassical dynamics of the particle and the effect of one nanoparticle into another, respectively. For completeness, the effects of possible quantum states of the mechanical particle in the semiclassical dynamics of the lightfield in the cavity is made in the appendix B. Finally, we briefly discuss the results.

## 4.1 Path Integral formulation of linear optomechanical system

Path integrals fundamentally deal with fluctuations, classical or quantum. We are interested in deriving a classical stochastic equation where fluctuations arise from interactions with a quantum system. Therefore, double path integrals for density matrices and the Feynman-Vernon method are the natural tools. In order to bridge the gap between the Feynman-Vernon method and quantum optics and optomechanics, we have adapted the theory by combining coherent state and configuration space path integrals in a single expression for the density matrix. We find it convenient to express the dynamics of the semiclassical system in terms of variables in configuration space (position and velocity) while treating the quantum system in terms of the coherent state basis.

### 4.1.1 Hamiltonian

We consider a linear optomechanical system consisting of a levitated nanoparticle of mass $m$ and frequency $\omega_m$ coupled to an optical cavity mode with frequency $\omega$. The total Hamiltonian of the system reads

$$H = H_c + H_m + H_I \tag{4-1}$$

where the free cavity and mechanical Hamiltonians are

$$H_c/\hbar = \omega\, a^\dagger a \tag{4-2}$$

$$H_m/\hbar = \omega_m b^\dagger b \tag{4-3}$$

and the interaction reads

$$H_I/\hbar = gq(t)X(t) \tag{4-4}$$

Here, $q(t)$ and $X(t)$ denote the dimensionless mechanical position and cavity mode amplitude quadratures, respectively. In terms of creation and annihilation operators these quantities are $q = b + b^\dagger$ and $X = a + a^\dagger$. To recover the dimensionful position we multiply by the zero point fluctuation,

$$\mathbf{q} \equiv q_0 q = q_0(b + b^\dagger) \tag{4-5}$$

where $q_0 = \sqrt{\hbar/2m\omega_m}$.

Following [26,66,71,98], we are interested in studying how state-dependent quantum fluctuations of one oscillator affects the semiclassical dynamics of the other via the interaction $H_I$. To this end, we will construct a path integral expression for the system's density matrix and employ the theory of Feynman-Vernon influence functionals. In doing this, we introduce coherent state path integrals, a convenient resource for when the Hamiltonian is expressed in terms of bosonic creation and annihilation operators [97].

### 4.1.2 Density matrix

We start with the reduced mechanical density matrix in the position basis,

$$\rho_b(q_t, q_t', t) = \frac{1}{\pi}\int d^2\beta \, \langle q, \beta| \, U\rho_0 U^\dagger \, |q', \beta\rangle \tag{4-6}$$

where $|q', \beta\rangle \equiv |q'\rangle \otimes |\beta\rangle$ is an element of the position-coherent state product bases with a similar definition for its dual, $\beta$ is the coherent state amplitude variable, $U$ is the unitary evolution of the system generated by (4-1) and $\rho_0$ is the initial optomechanical joint density matrix, assumed to be a product state $\rho_0 = \rho_{0,a} \otimes \rho_{0,b}$, which is justified provided each subsystem is prepared independently and the interaction Hamiltonian (4-4) is switched on over a time scale much shorter than the inverse characteristic frequencies $\omega_m^{-1}, \omega^{-1}$. We will denote $q_t \equiv q(t)$ the *forward* position variable, while $q_t' \equiv q'(t)$ will be referred to as the *backward* position variable. Inserting complete bases $\frac{1}{\pi}\int dq_0 d^2\beta_0 |q_0, \beta_0\rangle\langle q_0, \beta_0|$ of joint position-coherent state states we can rewrite the reduced density matrix in terms of coherent state propagators,

$$\begin{aligned} \rho_b(q_t, q_t', t) &= \frac{1}{\pi^3}\int d^2\beta dq_0 dq_0' d^2\beta_0 d^2\beta_0' \\ &\times K(q_t, \beta; q_0, \beta_0) \, \langle q_0, \beta_0|\rho_0|q_0', \beta_0'\rangle \, K^*(q_t', \beta; q_0', \beta_0') \end{aligned} \tag{4-7}$$

where,

$$K(q_t,\beta;q_0,\beta_0) = \langle q_t,\beta|\, U\, |q_0,\beta_0\rangle \quad (4\text{-}8)$$
$$K^*(q_t',\beta;q_0',\beta_0') = \langle q_0',\beta_0'|\, U^\dagger\, |q_t',\beta\rangle \quad (4\text{-}9)$$

These propagators can in turn be written as the path integral [97];

$$K(q_t,\beta|q_0,\beta_0) = \int_{q_0}^{q_t}\int_{\beta_0}^{\beta}\mathcal{D}q\mathcal{D}\alpha \times e^{\int_0^{t_f} dt\left[\frac{1}{2}(\alpha\dot{\alpha}^*-\alpha^*\dot{\alpha})-\frac{i}{\hbar}(H_c(\alpha,\alpha^*)+H_I(q,\alpha,\alpha^*)-L_{\mathbf{q}})\right]}, \quad (4\text{-}10)$$

where $H_c(\alpha,\alpha^*)$ and $H_I(q,\alpha,\alpha^*)$ denote the optical and interaction Hamiltonians in Eqs. (4-2) and (4-4) with the substitutions $a\to\alpha, a^\dagger\to\alpha^*$, and $L_{\mathbf{q}}$ is the Lagrangian of the mechanical oscillator,

$$L_{\mathbf{q}} = \frac{1}{2}m\left(\dot{\mathbf{q}}^2-\omega_m^2\mathbf{q}^2\right) \quad (4\text{-}11)$$

The conjugate propagator $K^*(q_t',\beta;q_0',\beta_0')$ is obtained from Eq. (4-10) by complex conjugation.

For a separable initial joint state $\langle q_0,\beta_0|\rho_0|q_0',\beta_0'\rangle = \rho_{0,a}(\beta_0,\beta_0')\rho_{0,b}(q_0,q_0')$ and we may write the reduced mechanical density matrix as

$$\rho_b(q_t,q_t',t) = \int dq_0dq_0'\,\rho_{0,a}(q_0,q_0')\mathcal{J}(q_t,q_t'|q_0,q_0'), \quad (4\text{-}12)$$

where we define the *double path integral propagator*,

$$\mathcal{J}(q_t,q_t'|q_0,q_0') = \int_{q_0,q_0'}^{q_t,q_t'}\mathcal{D}q\mathcal{D}q' e^{\frac{i}{\hbar}\int_0^{t_f}dt\left(L_{\mathbf{q}}-L_{\mathbf{q}'}\right)}\mathcal{F}[q,q'], \quad (4\text{-}13)$$

and $\mathcal{F}[q,q']$ denotes the Feynman-Vernon *influence functional* [65],

$$\begin{aligned}\mathcal{F}[q,q'] &= \frac{1}{\pi^3}\int d^2\beta d^2\beta_0 d^2\beta_0'\,\rho_{0,a}(\beta_0,\beta_0')\\ &\times\int_{\beta_0}^{\beta}\int_{\beta_0'}^{\beta'}\mathcal{D}\alpha\mathcal{D}\alpha'\, e^{\int_0^{t_f}dt\left[\frac{1}{2}(\alpha\dot{\alpha}^*+\alpha'\dot{\alpha}'^*-\alpha^*\dot{\alpha}-\alpha'^*\dot{\alpha}')\right]}\\ &\times e^{\int_0^{t_f}dt\left[-\frac{i}{\hbar}(H_c(\alpha,\alpha^*)-H_c(\alpha',\alpha'^*)+H_I(q,\alpha)-H_I(q',\alpha'))\right]}.\end{aligned} \quad (4\text{-}14)$$

Because we are dealing with the density matrix, i.e. probabilities rather than amplitudes, we have a double path integral. Furthermore, the path integral treats the mechanical oscillator in the position bases while the cavity mode is dealt with in the coherent state bases. Moreover, we can see that both the Lagrangian and Hamiltonian make an appearance in the expression of the stochastic propagator, reminiscent of Routhian mechanics [70].

Since the interaction Hamiltonian is symmetric with respect to the cavity and mechanical modes, the same derivation can be carried over with the mechanical system in the coherent state bases, which will be convenient later on, when we want to study the effective dynamics of the optical mode influenced by the mechanical oscillator.

The influence functional can be evaluated using the canonical formalism. Define the time-ordered evolution operator,

$$U_q = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^{t_f} dt\, H_q[q(t)]\right), \tag{4-15}$$

and the reduced propagators,

$$K_q(\beta;\beta_0) = \langle\beta| U_q |\beta_0\rangle \tag{4-16}$$

$$K_q^\dagger(\beta;\beta_0') = \langle\beta_0| U_q^\dagger |\beta\rangle \tag{4-17}$$

where

$$H_q[q(t)] = H_c + H_I \tag{4-18}$$

is a restricted cavity-interaction Hamiltonian. For an initial pure state $\rho_{0,a} = |\psi\rangle\langle\psi|$ the influence functional becomes

$$\begin{aligned}
\mathcal{F}[q,q'] &= \frac{1}{\pi^3}\int d^2\beta d^2\beta_0 d^2\beta_0' \rho_{0,a}(\beta_0,\beta_0') \\
&\times K_q(\beta;\beta_0)K_q^\dagger(\beta;\beta_0') \\
&= \frac{1}{\pi}\int d^2\beta\, \langle\beta| U_q |\psi\rangle \langle\psi| U_{q'}^\dagger |\beta\rangle \\
&= \langle\psi| U_{q'}^\dagger U_q |\psi\rangle .
\end{aligned} \tag{4-19}$$

Note that this expression can be generalized to mixed states by using the trace formula for the expectation value. We can move to the interaction picture via the transformation $U_q^\dagger \to U_{q'}^{I\dagger} = U_{q'}^\dagger e^{-\frac{i}{\hbar}H_c t}$, where without loss we choose $t_0 = 0$. The influence functional is then given by

$$\mathcal{F}[q,q'] = \langle\psi| U_{q'}^{I\dagger} U_q^I |\psi\rangle\ , \tag{4-20}$$

where we define the interaction picture evolution

$$U_q^I = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^{t_f} dt\, H_I[q(t)]\right) \tag{4-21}$$

When the commutator of the interaction Hamiltonian is a c-number the time ordered symbol $\mathcal{T}$ can be exchanged by a commutator term [99],

$$U_q^I = e^{-\frac{i}{\hbar}\int_0^\tau dt\, H_I} e^{-\frac{1}{2\hbar^2}\int_0^{t_f}\int_0^t dt\, dt' [H_I(t), H_I(t')]} \tag{4-22}$$

with

$$H_I/\hbar = g\, q(t)\left(a^\dagger(t) + a(t)\right), \tag{4-23}$$

and

$$a^\dagger(t) = a^\dagger e^{i\omega t}, \;\; a(t) = a e^{-i\omega t}. \tag{4-24}$$

After evaluation of the influence functional all the dependence on the optical degrees of freedom is removed, leaving only path integrals in the forward and backward position variables $q$ and $q'$.

### 4.1.3 Influence functional

As observed in [26], by manipulating Eq. (4-20) via the Baker–Campbell–Hausdorff formula, we can further simplify the influence functional to

$$\mathcal{F}[q, q'] = e^{i\Phi_0[q,q']} F_\psi[q, q'] \tag{4-25}$$

where the *influence phase* $\Phi_0[q, q']$ splits into two parts,

$$\Phi_0[q, q'] = i\Phi_0^{\text{fl}} + i\Phi_0^{\text{diss}} \tag{4-26}$$

one corresponding to fluctuations,

$$i\Phi_0^{\text{fl}} = -\frac{g^2}{2}\int_0^{t_f}\int_0^{t_f} dt\, dt' \left(q(t) - q'(t)\right)\left(q(t') - q'(t')\right)\cos\left(\omega(t - t')\right) \tag{4-27}$$

and another to dissipation

$$i\Phi_0^{\text{diss}} = ig^2\int_0^{t_f}\int_0^{t} dt\, dt' \left(q(t) - q'(t)\right)\left(q(t') + q'(t')\right)\sin\left(\omega(t - t')\right) \tag{4-28}$$

Further, Eq. (4-25) has the state-dependent factor

$$F_\psi[q, q'] = \langle\psi|\, e^{-a^\dagger W^*} e^{aW} \,|\psi\rangle\,, \tag{4-29}$$

where $W$ is

$$W = -ig\int_0^{t_f} dt\, \left(q(t) - q'(t)\right)e^{-i\omega t}. \tag{4-30}$$

Note that $F_\psi[q, q']$ is the quantum optical characteristic function evaluated at $W$ [62]. The influence phase $\Phi_0[q, q']$ encodes the effects of vacuum quantum fluctuations of the optical mode upon the mechanical oscillator and is present for all quantum states of the optical field. In general, the vacuum influence will be modified by the state-dependent term $F_\psi[q, q']$. We next discuss each of these terms in detail, starting from a calculation of the effects associated to the vacuum influence phase. We then generalize the results to squeezed-coherent and thermal states. We note that the Gaussian character of these states significantly simplifies the problem at hand, but in principle nothing prevents one from extending the path integral method to include the effects associated to non-Gaussian states [100].

### 4.1.4 Fluctuation and Dissipation

The fluctuation contribution to the influence phase can be written as

$$i\Phi_0^{\mathrm{fl}} = -\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\,dt' J(t)\mathcal{A}(t,t')J(t') \tag{4-31}$$

where we define

$$J(t) = q(t) - q'(t) \tag{4-32}$$

and the two-time noise kernel

$$\mathcal{A}(t,t') = g^2\cos(\omega(t-t')) \tag{4-33}$$

We now perform the so-called Feynman trick [65], which consists in the observation that Eq. (4-31) can be written in terms of a path integral over an auxiliary variable $\zeta(t)$,

$$e^{-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\,dt' J(t)\mathcal{A}(t,t')J(t')} = \int \mathcal{D}\zeta e^{-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\,dt'\zeta(t)\mathcal{A}^{-1}(t,t')\zeta(t') + i\int_0^{t_f} dt\zeta(t)J(t)} \tag{4-34}$$

where $\int_0^{t_f} ds\mathcal{A}(t,s)\mathcal{A}^{-1}(s,t') = \delta(t-t')$. Effectively, the Feynman trick decouples the forward and backward variables $q$ and $q'$ at the expense of introducing the variable $\zeta$, which is interpreted as a random process with the probability density functional of the stochastic variable,

$$P[\zeta(t)] = \exp\left(-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dt\,dt'\zeta(t)\mathcal{A}^{-1}(t,t')\zeta(t')\right). \tag{4-35}$$

This random process can be thought of as the noise induced by quantum fluctuations of the optical mode on the mechanical oscillator.

Denoting the stochastic average over $P[\zeta(t)]$ by $\langle \ldots \rangle_\zeta$, Eq. (4-34) can be written as,

$$\langle e^{\frac{i}{\hbar}\int_0^{t_f} dt\zeta(t)J(t)}\rangle_\zeta \ . \tag{4-36}$$

The mean value and time correlation function of this noise are,

$$\langle\zeta(t)\rangle_\zeta = 0, \tag{4-37}$$
$$\langle\zeta(t)\zeta(t')\rangle_\zeta = \mathcal{A}(t,t'), \tag{4-38}$$

Since the stochastic variable is Gaussian, the first and second moments are sufficient to completely describe the process. Note that $\mathcal{A}(t,t') \propto g^2$, which implies $\zeta(t) \propto g$. It will be convenient to write

$$\zeta(t) = g\xi(t) \ . \tag{4-39}$$

where we define the *dimensionless random variable* $\xi(t)$.

We now turn to the dissipation contribution given in Eq. (4-28). This term is non-local in time [101, 102] and is given by the integral of a product of two distinct functions, namely $J(t) = (q(t) - q'(t))$ and $(q(t) + q'(t))$. Unlike $i\Phi_0^{\text{fl}}$, the Feynman trick cannot be performed and it cannot be associated to fluctuations. Consequently, the equations of motion for the forward and backward variables $q$ and $q'$ are coupled [101, 102]. This coupling is associated to the breaking of time-reversal symmetry [103], but as we will see, its effects are subleading when compared to the fluctuations.

### 4.1.5 Equation of motion

For the ground state $|\psi\rangle = |0\rangle$, the total influence phase yields the propagator,

$$\begin{aligned}
\mathcal{J}(q_t, q'_t|q_0, q'_0) = & \int_{q_0,q'_0}^{q_t,q'_t} \mathcal{D}q\mathcal{D}q' \int \mathcal{D}\zeta \ \exp\left(-\frac{1}{2}\int_0^{t_f}\int_0^{t_f} dtdt' \ \zeta(t)\mathcal{A}^{-1}(t,t')\zeta(t')\right) \\
& \times \exp\left(\frac{i}{\hbar}\int_0^{t_f} dt \ (L_{\mathbf{q}} - L_{\mathbf{q}'})\right) \\
& \times \exp\left(\frac{i}{\hbar}\int_0^{t_f} dt \ \left(\frac{\hbar\zeta(t)}{q_0}\right)(\mathbf{q}(t) - \mathbf{q}'(t))\right) \\
& \times \exp\left(\frac{i}{\hbar}\left(\frac{\hbar g^2}{q_0^2}\right)\int_0^{t_f}\int_0^{t} dt\,dt' \ (\mathbf{q}(t) - \mathbf{q}'(t))(\mathbf{q}(t') + \mathbf{q}'(t'))\sin\left(\omega(t-t')\right)\right)
\end{aligned} \tag{4-40}$$

The first line shows the stochastic density kernel. The second line contains the dynamics of the mechanical oscillator. In the third line there is the coupling between the stochastic variable $\zeta(t)$ and the forward and backward position variables. Note here that writing the arguments in the exponential in terms of the dimensionful oscillator position forces the appearance of the combination $\hbar\zeta/q_0 = \hbar g\xi/q_0$, which has dimension of force. Finally, the last term consists in the non-local dissipation coupling the $\mathbf{q}$ and $\mathbf{q}'$ variables.

Extremising the argument in the exponentials of (4-40) with respect to $\mathbf{q}(t)$ and $\mathbf{q}'(t)$ leads to effective Langevin-like equations for the forward and backward paths. In general these are coupled stochastic differential equations, with the coupled terms arising from the cross-terms in the dissipation phase $\Phi_0^{\text{diss}}$. As customary, we will neglect the coupling between forward and backward paths and take the ansatz $\mathbf{q}(t) = \mathbf{q}'(t)$ [26, 67]. As consequence, the dynamics of the forward and backward paths obey the same equation of motion. This corresponds to expressing the action in terms of symmetric $(q(t) + q'(t))/2$ and antisymmetric $(q(t) - q'(t))$ paths and expanding to leading order terms in the anti-symmetric path, as was also done in [26].

We find the Langevin-like equation of motion

$$m\ddot{\mathbf{q}}(t) + m\omega_m^2\mathbf{q} = \mathbf{f}_Q(t) + \mathbf{f}_{\text{diss}}(t) \tag{4-41}$$

where we define the *quantum fluctuation force*,

$$\mathbf{f}_Q(t) \equiv \frac{\hbar\zeta(t)}{q_0} = \left(\frac{\hbar g}{q_0}\right)\xi(t) \tag{4-42}$$

with correlation function

$$\langle \mathbf{f}_Q(t)\mathbf{f}_Q(t')\rangle = \left(\frac{\hbar g}{q_0}\right)^2 \cos(\omega(t-t')), \tag{4-43}$$

where the expected value is taken over the distribution of $\mathbf{f}_Q(t)$, which is analogous to Eq. (4-76) but rescaled with $\hbar g/q_0$, and the dissipative force,

$$\mathbf{f}_{\text{diss}}(t) = 2\frac{\hbar g^2}{q_0^2}\int_0^t dt'\mathbf{q}(t')\sin(\omega(t-t')) \tag{4-44}$$

An analogous equation is obtained for $\mathbf{q}'(t)$. In the semiclassical regime, Newton's second law is modified by an additional fluctuation term originating from quantum fluctuations and a dissipation force with memory.

It is interesting to observe the scaling of the stochastic quantum force $\mathbf{f}_Q(t)$ with Planck's constant. Substituting the zero point motion of the oscillator we find $\mathbf{f}_Q(t) \propto \sqrt{\hbar}$, while $\langle \mathbf{f}_Q(t)\mathbf{f}_Q(t')\rangle \propto \hbar$, making the quantum origin of the fluctuations explicit. Note that $\hbar$ drops out of the dissipation force $\mathbf{f}_{\text{diss}}$.

For more general quantum states, additional noise contributions and deterministic forces might arise. In that case, the stochastic variable $\zeta$ will in general be substituted by a sum of uncorrelated stochastic forces $\zeta(t) \to \sum_i \zeta_i$ satisfying the independence condition $\langle \zeta_i \zeta_j \rangle = g^2 \delta_{ij} A_i$. We now turn to the calculation of the noise for quantum states of interest to optomechanical experiments, notably squeezed-coherent and squeezed-thermal states.

### 4.1.6 Squeezed-coherent states

As in 2.1.3 we define a squeezed coherent state as,

$$|\Psi\rangle = S(z)\,|\alpha\rangle \tag{4-45}$$

where $\alpha = |\alpha| e^{i\theta}$ is the coherent state amplitude and the squeezing operator reads,

$$S(z) = e^{\frac{1}{2} z^* a^2 + \frac{1}{2} z a^{\dagger 2}} \tag{4-46}$$

with $z = re^{i\phi}$. We refer to $r$ as the squeezing parameter and $\phi$ as the squeezing phase. Using the identities,

$$\begin{aligned} S^\dagger(r,\phi) a S(r,\phi) &= a \cosh r - a^\dagger e^{2i\phi} \sinh r, \\ S^\dagger(r,\phi) a^\dagger S(r,\phi) &= a^\dagger \cosh r - a e^{-2i\phi} \sinh r. \end{aligned} \tag{4-47}$$

the influence functional can be brought to the form,

$$\mathcal{F}_{\rm sq}[q,q'] = e^{i\Phi_0[q,q'] + i\Phi_{0,\rm sq}[q,q']} F_\alpha[q,q'], \tag{4-48}$$

where,

$$F_\alpha[q,q'] = \langle \alpha |\, e^{-a^\dagger f(W)} e^{a f(W)^*}\, |\alpha\rangle \tag{4-49}$$

and we define,

$$f(W) = W^* \cosh r + W e^{2i\phi} \sinh r, \tag{4-50}$$

In addition to the vacuum phase, squeezed-coherent states acquire a squeezing phase which can be split into a stationary and a non-stationary transient contribution,

$$i\Phi_{0,\rm sq}[q,q'] = i\Phi_{0,\rm sq}^{\rm st} + i\Phi_{0,\rm sq}^{\rm n-stat} \tag{4-51}$$

with,

$$i\Phi_{0,\rm sq}^{\rm st} = -\frac{g^2}{2}(\cosh 2r - 1) \int_0^\tau \int_0^\tau dt dt' \; J(t) \cos\left(\omega(t-t')\right) J(t')$$

(4-52)

and

$$i\Phi_{0,\mathrm{sq}}^{\mathrm{n-st}} = -\frac{g^2}{2}(\sinh 2r) \int_0^{t_f} \int_0^{t_f} dtdt' \; J(t) \cos\left(\omega(t+t') - 2\phi\right) J(t'). \tag{4-53}$$

The stationary contribution of the influence phase due to squeezing adds to the vacuum phase $i\Phi_0^{\mathrm{fl}}$, yielding an enhanced phase proportional to $\cosh(2r)$. At the same time, we find a non-stationary contribution carrying the information on the squeezing phase $\phi$. No additional contribution to the dissipation appears due to squeezing.

We can proceed to perform the Feynman trick for the stationary and non-stationary phase contributions. Introducing auxiliary stochastic variables for each phase in the path integral we arrive at

$$\langle \zeta^{\mathrm{st}}(t)\zeta^{\mathrm{st}}(t')\rangle = g^2 \cosh(2r)\cos(\omega(t-t')) \tag{4-54}$$

for the stationary and

$$\langle \zeta^{\mathrm{n-stat}}(t)\zeta^{\mathrm{n-stat}}(t')\rangle = g^2 \sinh(2r)\cos(\omega(t+t') - 2\phi) \tag{4-55}$$

for the non-stationary terms. Note both stochastic forces are exponentially enhanced in the squeezing parameter, $\zeta^{\mathrm{st}} \propto g\sqrt{\cosh(2r)}$ and $\zeta^{\mathrm{n-stat}} \propto g\sqrt{\sinh(2r)}$. This implies an enhancement in the quantum force due to squeezing,

$$\mathbf{f}_{\mathcal{Q}}^{\mathrm{st}}(t) = \sqrt{\cosh(2r)} \left(\frac{\hbar g}{q_0}\right) \xi^{\mathrm{st}}(t) \tag{4-56}$$

$$\mathbf{f}_{\mathcal{Q}}^{\mathrm{n-st}}(t) = \sqrt{\sinh(2r)} \left(\frac{\hbar g}{q_0}\right) \xi^{\mathrm{n-st}}(t) \tag{4-57}$$

where $\xi^{\mathrm{st}}(t)$ and $\xi^{\mathrm{n-st}}(t)$ denote the stationary and non-stationary dimensionless random force variables with correlation functions

$$\langle \xi^{\mathrm{st}}(t)\xi^{\mathrm{st}}(t')\rangle = \cos(\omega(t-t')) \tag{4-58}$$

$$\langle \xi^{\mathrm{n-st}}(t)\xi^{\mathrm{n-st}}(t')\rangle = \cos(\omega(t+t') - 2\phi) \tag{4-59}$$

Besides the stationary and non-stationary stochastic forces, the squeezed-coherent state also yields a deterministic force arising from the $F_\alpha[q,q']$ factor in Eq. (4-48). We have,

$$F_\alpha[q,q'] = e^{-\alpha^* f(W)} e^{\alpha f(W)^*} = e^{i\Phi_{\alpha,\mathrm{sq}}}, \tag{4-60}$$

where $i\Phi_{\alpha,\mathrm{sq}}$ denotes the additional influence phase containing the effects of both the coherent and squeezed nature of the state $|\Psi\rangle$. Direct calculation shows that,

$$i\Phi_{\alpha,\mathrm{sq}}[q,q'] = \frac{i}{\hbar}\int_0^\tau dt\,(\mathbf{q}(t) - \mathbf{q}'(t))\,\mathbf{f}_\Psi(t) \tag{4-61}$$

where,

$$\mathbf{f}_\Psi(t) = -2|\alpha|\left(\frac{\hbar g}{q_0}\right)(\sin(\omega t - \theta - 2\phi)\sinh r + \cos(\omega t + \theta)\cosh r)\,. \tag{4-62}$$

Note that $\mathbf{f}_\Psi(t)$ represents a deterministic force enhanced by the coherent state amplitude $|\alpha|$ and by the exponential squeezing factors $\sinh r$ and $\cosh r$. This deterministic force is also quantum mechanical in origin and $\mathbf{f}_\Psi(t) \propto \sqrt{\hbar}$. It is also interesting to note that this force performs work on the mechanical system. In stochastic thermodynamics this work behaves as a random variable and is described by $W[\mathbf{q}(t)] = \int \mathbf{f}_\Psi(t)\cdot\dot{\mathbf{q}}(t)dt$ [104–106]. A statistical approach can be employed to characterize this work function [107–109].

### 4.1.7 Squeezed-thermal states

We define squeezed-thermal states as

$$\rho_{\mathrm{sq,th}} = S(z)\rho_{\mathrm{th}}S^\dagger(z) \tag{4-63}$$

where

$$\rho_{\mathrm{th}} = \left(1 - e^{-\beta\hbar\omega}\right)\sum_n e^{\beta\hbar\omega n}\,|n\rangle\,\langle n|\,, \tag{4-64}$$

and $\beta$ is the inverse temperature associated to the quantum oscillator. The influence functional assumes the same form as in Eq. (4-48),

$$\mathcal{F}_{\mathrm{sq,th}}[q,q'] = e^{i\Phi_0 + i\Phi_{0,\mathrm{sq}}}F_{\mathrm{sq,th}}[q,q'] \tag{4-65}$$

where,

$$F_{\mathrm{sq,th}}[q,q'] = \left(1 - e^{-\beta\hbar\omega}\right)\sum_n e^{\beta\hbar\omega n}\,\langle n|\,e^{-a^\dagger f(W)}e^{af(W)^*}\,|n\rangle \tag{4-66}$$

This sum can be solved analytically, as showed in [26], and the total influence functional (4-65) is expressed in terms of a phase $e^{i\Phi_{\mathrm{sq,th}}}$. The final result again contains stationary and non-stationary terms,

$$i\Phi_{\mathrm{sq,th}}[q,q'] = i\Phi^{\mathrm{st}}_{\mathrm{sq,th}} + i\Phi^{\mathrm{n-stat}}_{\mathrm{sq,th}} \tag{4-67}$$

with

$$i\Phi^{\mathrm{st}}_{\mathrm{sq,th}} = -\frac{g^2}{2}\cosh(2r)\coth\left(\frac{\beta\hbar\omega}{2}\right)\int_0^{t_f}\int_0^{t_f} dtdt'\; J(t)\cos\left(\omega(t-t')\right)J(t') \tag{4-68}$$

$$i\Phi^{\mathrm{n-st}}_{\mathrm{sq,th}} = -\frac{g^2}{2}\sinh(2r)\coth\left(\frac{\beta\hbar\omega}{2}\right)\int_0^{t_f}\int_0^{t_f} dtdt'\; J(t)\cos\left(\omega(t+t')-2\phi\right)J(t'). \tag{4-69}$$

Once again, we apply the Feynman trick to obtain the stationary and non-stationary stochastic forces, now appearing with enhancement factors due to to the squeezing and thermal nature of the state,

$$\mathbf{f}^{\mathrm{st}}_{Q}(t) = \left[\cosh(2r)\coth\left(\frac{\beta\hbar\omega}{2}\right)\right]^{1/2}\left(\frac{\hbar g}{q_0}\right)\xi^{\mathrm{st}}(t) \tag{4-70}$$

$$\mathbf{f}^{\mathrm{n-st}}_{Q}(t) = \left[\sinh(2r)\coth\left(\frac{\beta\hbar\omega}{2}\right)\right]^{1/2}\left(\frac{\hbar g}{q_0}\right)\xi^{\mathrm{n-st}}(t) \tag{4-71}$$

where $\xi^{\mathrm{st}}(t)$ and $\xi^{\mathrm{n-st}}(t)$ also satisfy (4-58) and (4-59). No additional contribution to the dissipation and deterministic force appears. This is expected, since the mean value of the field amplitude quadrature is zero for squeezed-thermal states.

### 4.1.8
### Sum over modes

So far, we have only dealt with the noise contribution arising from a single mode of the optical field. In many applications, we are interested in considering multimode systems. The Feynman-Vernon influence functional then acquires a contribution from each mode [49]. Consider for example a multimode cavity. In a discrete $N$-mode approximation, the Hamiltonian is schematically written as

$$H_c/\hbar = \sum_k^N \omega_k a_k^\dagger a_k \tag{4-72}$$

where $k = \omega/c$, and the quantum state of the cavity reads,

$$|\Psi\rangle = \bigotimes_k^N |\psi_{\omega_k}\rangle \tag{4-73}$$

The Feynman-Vernon influence functional becomes,

$$\mathcal{F}[q,q'] = \prod_k^N \mathcal{F}_{\omega_k}[q,q']. \tag{4-74}$$

In the continuum limit, we must take into account the cavity density of states $\mathcal{N}(\omega)$ [49]. The total influence phase is then,

$$\Phi[q,q'] = \sum_k^N \Phi_{\omega_k}[q,q'] \xrightarrow{N\to\infty} \int_0^\infty d\omega \mathcal{N}(\omega)\Phi_\omega[q,q'] \tag{4-75}$$

For an optical cavity, the density of modes is [110],

$$\mathcal{N}(\omega) = \frac{1}{\pi}\frac{\gamma}{(\omega-\omega_c)^2+\gamma^2}, \tag{4-76}$$

where $\omega_c$ is the cavity central frequency and $\gamma$ the cavity damping rate.

Note that the interaction strength of the mechanical oscillator with a given cavity mode $k$ is frequency dependent $g = g(\omega)$, which must be taken into account in the integration in the r.h.s. of Eq. (4-75). The dissipation force (4-44) becomes

$$\mathbf{f}_{\text{diss}} = \frac{2\hbar}{q_0^2}\int_0^\infty d\omega \mathcal{N}(\omega) g^2(\omega) \int_0^t dt' \mathbf{q}(t') \sin(\omega(t-t')) \tag{4-77}$$

In the case of squeezed-coherent states, the deterministic force reads,

$$\mathbf{f}_\Psi(t) = -2|\alpha|\frac{\hbar}{q_0}\int_0^\infty d\omega \mathcal{N}(\omega) g(\omega)\left(\cos(\omega t+\theta)\cosh r + \sin(\omega t - \theta - 2\phi)\sinh r\right) \tag{4-78}$$

Lastly, the stationary and non-stationary noise correlators are,

$$\langle \mathbf{f}_\mathcal{Q}^{\text{st}}(t)\mathbf{f}_\mathcal{Q}^{\text{st}}(t')\rangle = \left(\frac{\hbar}{q_0}\right)^2 \int_0^\infty d\omega \mathcal{N}(\omega) g^2(\omega)\cos(\omega(t-t')) \tag{4-79}$$

$$\langle \mathbf{f}_\mathcal{Q}^{\text{n-st}}(t)\mathbf{f}_\mathcal{Q}^{\text{n-st}}(t')\rangle = \left(\frac{\hbar}{q_0}\right)^2 \int_0^\infty d\omega \mathcal{N}(\omega) g^2(\omega)\cos(\omega(t+t')-2\phi) \tag{4-80}$$

where, in the case of squeezed and thermal states the appropriate enhancement factors must be included. Note that this sum over modes approach can be complemented by an open quantum system model of a cavity interacting with free electromagnetic modes within the formalism of path integrals [111] in an analogous fashion to the quantum Langevin equations [48].

## 4.2 Semiclassical particle as probe of quantum light

We are now in the position to apply the Feynman-Vernon theory to a semiclassical levitated nanoparticle in an optical cavity interacting with a quantum light reservoir via coherent scattering.

### 4.2.1 Optomechanical parameters

The coherent scattering coupling rate is given by [1]

$$\hbar g = \alpha \mathcal{E}_0 \mathcal{E}_c k q_0 \sin(k\mathbf{r}_0) \sin\theta \tag{4-81}$$

where $\alpha = 3\epsilon_0 V \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$ is the polarizability of a dielectric particle of volume $V$, refractive index $n$ and relative permittivity $\epsilon_r \approx n^2$, $\mathcal{E}_0$ is the tweezer field, $\mathbf{r}_0$ is the mean particle position within the cavity, $\theta$ is the angle between the tweezer polarization and the cavity axis and $\mathcal{E}_c$ is the cavity electric field strength, given by

$$\mathcal{E}_c = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V_c}} \tag{4-82}$$

where $V_c$ is the cavity mode volume. We will assume the particle is placed at a cavity node and the tweezer polarization is orthogonal to the cavity axis such that $\sin(k\mathbf{r}_0) = \sin\theta = 1$; placing the particle at a node only mildly affects the cavity finesse [1], and hence we will neglect any additional losses introduced by the particle. We note also that the coherent scattering interaction strength can be tuned and switched on and off by placing the particle at different positions within the cavity or by controlling the polarization of the tweezer field [112].

For a confocal cavity of length $L$ the mode volume in Eq. (4-82) is given by $V_c = \mathrm{w}_c^2 \pi L/4$, where the cavity waist is $\mathrm{w}_c = \sqrt{cL/\omega}$. In terms of the mode frequency $\omega$ the optomechanical coupling rate then reads

$$g \equiv g(\omega) = a\omega^2 \tag{4-83}$$

where

$$a = \frac{\alpha \mathcal{E}_0}{\sqrt{\pi \epsilon_0 c^3 L^2 m \omega_m}} \tag{4-84}$$

Note that $g$ is independent of $\hbar$ as it drops out of Eq. (4-81).

We will find it convenient to define the optomechanical coupling rate at the cavity central frequency,

$$g_c \equiv g(\omega_c) \tag{4-85}$$

as well as the dimensionless frequency ratios,

$$\varepsilon \equiv g_c/\omega_m \tag{4-86}$$

$$\nu \equiv \gamma/\omega_c \tag{4-87}$$

and the characteristic force associated to the central optomechanical coupling,

$$\mathbf{f}_0 \equiv \hbar g_c/q_0 \tag{4-88}$$

Table 4.1 shows a list of the parameters we will assume for the cavity optomechanical system, similar to the ones described in [1]. Note the dimensionless parameters $\varepsilon, \nu \ll 1$.

### 4.2.2 Vacuum fluctuations and dissipation

We can calculate the stochastic force $\mathbf{f}_\mathcal{Q}(t)$ associated to the vacuum fluctuations in the optical cavity using Eq. (4-79). In this case we only have the stationary process with noise kernel,

$$\langle \mathbf{f}_\mathcal{Q}(t)\mathbf{f}_\mathcal{Q}(t')\rangle = \frac{\gamma}{\pi}\left(\frac{\hbar a}{q_0}\right)^2 \int_0^\infty d\omega \frac{\omega^4 \cos(\omega(t-t'))}{(\omega-\omega_c)^2+\gamma^2} \tag{4-89}$$

where we omit the stationary 'st' superscript for simplicity. The frequency integral (4-89) can be solved in terms of distributions – we refer to Appendix A for the details. We find that the vacuum state introduces three independent stationary stochastic force components,

$$\mathbf{f}_\mathcal{Q}(t) = \mathbf{f}_0 \sum_i \xi_i(t) \tag{4-90}$$

with the correlation functions

$$\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij} A_i(\tau) \tag{4-91}$$

where, $\tau \equiv t - t'$ and

$$A_1(\tau) = e^{-\gamma|\tau|}\Big((1-6\nu^2+\nu^4)\cos(\omega_c\tau) - 4(\nu-\nu^3)\sin(\omega_c|\tau|)\Big) \tag{4-92}$$

$$A_2(\tau) = \Big(3\nu-\nu^3\Big)\frac{\delta(\tau)}{\omega_c} \tag{4-93}$$

$$A_3(\tau) = \nu\frac{\delta''(\tau)}{\omega_c^3} \tag{4-94}$$

The correlator in Eq. (4-92) resembles active noise with a relative phase shift between the sine and cosine components [113,114]. Colored noise can be generally expressed as an expansion in terms of derivatives of the delta distribution [115], hence the second and third terms represent additional colored noise. Moreover, the second derivative delta noise has appeared previously in the context of stochastic gravity [116].

The leading contribution to the stochastic force correlator comes from the zeroth order term in $\nu$. We approximate the stochastic force correlator as

$$\langle \mathbf{f}_{\mathcal{Q}}(t)\mathbf{f}_{\mathcal{Q}}(t')\rangle \approx \mathbf{f}_0^2\, e^{-\gamma|\tau|}\cos(\omega_c\tau) \tag{4-95}$$

Observe the characteristic scale of the stochastic force is given by $\mathbf{f}_0$.

We now turn to the dissipation. Substituting Eqs. (4-76) and (4-83) into (4-77) we arrive at

$$\mathbf{f}_{\text{diss}} = \frac{2}{\pi}\frac{\hbar\gamma}{q_0}a^2\int_0^t dt' q(t')\int_0^\infty d\omega\frac{\omega^4\sin(\omega\tau)}{(\omega-\omega_c)^2+\gamma^2} \tag{4-96}$$

Once again, the frequency integral can be evaluated in terms of distributions, see Appendix A for details. We find the dissipation force,

$$\mathbf{f}_{\text{diss}} = \mathbf{f}_0\left[12\nu\left(\frac{g_c}{\omega_c}\right)\frac{\dot q(t)}{\omega_c} + \varepsilon u(t)\right] \tag{4-97}$$

where

$$\begin{aligned} u(t) = 2\int_0^t d(\omega_m t')e^{-\gamma|\tau|}\Big((1-6\nu^2+\nu^4)\sin(\omega_c\tau) \\ +4(\nu-\nu^3)\text{sgn}(\tau)\cos(\omega_c\tau)\Big)\, q(t'). \end{aligned} \tag{4-98}$$

We see the total dissipation force $\mathbf{f}_{\text{diss}}$ consists of a standard velocity-dependent term plus a modification of the mechanical spring constant with a memory kernel.

The dissipation is small when compared to the fluctuation force. From Table 4.1, $g_c \approx 0.1\gamma$, while $\omega_m \approx \gamma$. Moreover, $\dot{q}(t) \approx \omega_m q(t)$. Hence, the first term in Eq. (4-97) is $\mathcal{O}(\nu^3)$, while the second term is at most of order $\mathcal{O}(\varepsilon)$ in units of $\mathbf{f}_0$. We will henceforth neglect dissipation effects arising from the influence functional.

### 4.2.3 Particle dynamics

We arrive at the equation of motion for the mechanical oscillator,

$$m\ddot{\mathbf{q}} + \Gamma_m\dot{\mathbf{q}} + m\omega_m^2\mathbf{q} = \mathbf{f}_{\mathcal{Q}}(t) + \mathbf{f}_{\text{diss}} + \eta(t) \tag{4-99}$$

where we have added phenomenologically a dissipation with damping coefficient $\Gamma_m$ and the thermal white noise $\eta(t)$ with

$$\langle\eta(t)\eta(t')\rangle = 2\Gamma_m k_B T_{\text{bath}}\delta(\tau) \tag{4-100}$$

where $T_{\text{bath}}$ is the temperature of the particle's environment. This term accounts for the Brownian motion of the particle, as derived in sec.2.4. Note the stochastic quantum force $\mathbf{f}_{\mathcal{Q}}(t)$ might contain stationary and non-stationary components and is accompanied by a state-dependent correlation function. For example, for the vacuum state, the correlation function of $\mathbf{f}_{\mathcal{Q}}(t)$ is given in Eq. (4-95).

Considering only the zeroth order terms in the dimensionless parameters $\nu$ and $\varepsilon$, the equation of motion simplifies to

$$m\ddot{\mathbf{q}} + \Gamma_m\dot{\mathbf{q}} + m\omega_m^2\mathbf{q} \approx \mathbf{f}_{\mathcal{Q}}(t) + \eta(t) \tag{4-101}$$

We have the formal solution

$$\mathbf{q}(t) = \frac{e^{-\gamma_m t}}{m\Omega_m}\int_0^t ds e^{\gamma_m s}\sin\left(\Omega_m(s-t)\right)\left(\mathbf{f}_{\mathcal{Q}}(s) + \eta(s)\right) \tag{4-102}$$

where,

$$\gamma_m \equiv \Gamma_m/2m \tag{4-103}$$

$$\Omega_m \equiv \sqrt{\omega_m^2 - \gamma_m^2} \tag{4-104}$$

and, for simplicity, we assume initial conditions $\mathbf{q}(0) = \dot{\mathbf{q}}(0) = 0$. For a particle with a radius of 70 nm at pressures around $10^{-9}$ mbar, $\gamma_m \approx 5 \times 10^{-6}$ Hz so we have $\Omega_m \approx \omega_m$ [44].

Let $\sigma_{\mathbf{q}} \equiv \sqrt{\langle \mathbf{q}^2(t) \rangle}$ be the particle's position root-mean-square (rms). We are interested in calculating $\sigma_{\mathbf{q}}$ while the particle interacts with a quantum state populating the cavity and an external thermal reservoir. The position rms $\sigma_{\mathbf{q}}^2$ then has two independent contributions, one from the quantum force $\mathbf{f}_{\mathcal{Q}}(t)$ and the other from the uncorrelated thermal noise $\eta(t)$. We define the *excess* quantum-induced fluctuations as

$$\Delta\sigma_{\mathbf{q}}^2 = \sigma_{\mathbf{q}}^2 - \sigma_0^2 \tag{4-105}$$

where $\sigma_0^2$ denotes the non-quantum contribution to $\sigma_{\mathbf{q}}^2$ arising from the thermal fluctuations $\eta(t)$ and

$$\Delta\sigma_{\mathbf{q}}^2 = \frac{e^{-2\gamma_m t}}{m^2\Omega_m^2}\int_0^t\int_0^t ds ds' e^{\gamma_m(s+s')}\sin\left(\Omega_m(s-t)\right)\sin\left(\Omega_m(s'-t)\right)\langle \mathbf{f}_{\mathcal{Q}}(s)\mathbf{f}_{\mathcal{Q}}(s')\rangle \tag{4-106}$$

The solution to this integral is exact but cumbersome. In essence, $\Delta\sigma_{\mathbf{q}}$ reaches a steady state with characteristic value is given by

$$\Delta\sigma_{\mathbf{q}} \approx \frac{\mathbf{f}_0}{m\sqrt{\Omega_m^3\omega_c}} \tag{4-107}$$

As we will see in Sec. 4.2.6, this quantum contribution to the particle's position rms is negligible for current experiments with levitated nanoparticles.

### 4.2.4
### Squeezed-coherent states

We now consider squeezed-coherent states. Due to their displacement in phase space, these states introduce a deterministic force on the particle given by,

$$\mathbf{f}_\psi = -\frac{2|\alpha|}{\pi}\frac{\hbar\gamma}{q_0}a\int_0^\infty d\omega\frac{\omega^2}{(\omega-\omega_c)^2+\gamma^2}\left(\cos\left(\omega t+\theta\right)\cosh r+\sin\left(\omega t-\theta-2\phi\right)\sinh r\right). \tag{4-108}$$

Again, this integral can be solved following the steps in Appendix A.

For simplicity, consider a real coherent state amplitude ($\theta = 0$). To leading order in $\mathcal{O}(\nu^0)$ the deterministic force reads

$$\mathbf{f}_\psi = \mathbf{f}_0|\alpha|\; e^{-\gamma t}\left[2\cosh r\cos(\omega_c t)+\sinh r\left(\sin 2\phi\sin(\omega_c t)-2\cos 2\phi\cos(\omega_c t)\right)\right]. \tag{4-109}$$

This is a fast oscillating force, which time averages to zero. Note, however, that it exhibits an exponential enhancement due to squeezing and, depending on the squeezing phase, this force can be exponentially enhanced or suppressed. The same enhancement/suppression effects as a function of squeezing phase have been shown to appear in the context of a quantized gravitational wave in a squeezed-coherent state interacting with an optical cavity [25].

The stationary noise associated to squeezing assumes the same form as the vacuum, but is enhanced exponentially in the squeezing parameter,

$$\mathbf{f}_{\mathcal{Q}}^{\text{st}}(t) = \sqrt{\cosh(2r)}\ \mathbf{f}_0\ \sum_i \xi_i(t) \tag{4-110}$$

where the dimensionless stochastic variables $\xi_i(t)$ satisfy Eqs. (4-91)-(4-94). The noise correlation function to zeroth order in $\nu$ is approximated by

$$\langle \mathbf{f}_{\mathcal{Q}}^{\text{st}}(t)\mathbf{f}_{\mathcal{Q}}^{\text{st}}(t')\rangle \approx \cosh(2r)\ \mathbf{f}_0^2\ e^{-\gamma|\tau|}\cos(\omega_c\tau) \tag{4-111}$$

Squeezed-coherent states also exhibit the non-stationary noise defined in Eq. (4-80). To zeroth order in $\nu$,

$$\langle \mathbf{f}_{\mathcal{Q}}^{\text{n-st}}(t)\mathbf{f}_{\mathcal{Q}}^{\text{n-st}}(t')\rangle \approx \sinh(2r)\ \mathbf{f}_0^2\ e^{-\gamma(t+t')}$$
$$\times\left(\frac{\cos(2\phi)}{2}\cos(\omega_c(t+t')) + \sin(2\phi)\sin(\omega_c(t+t'))\right) \tag{4-112}$$

This non-stationary noise is also exponentially enhanced, despite decaying with a characteristic time given by $\gamma^{-1}$. Note, however, that $\mathbf{f}_{\mathcal{Q}}^{\text{n-st}}(t)$ depends on the squeezing angle $\phi$, and is maximized for $\phi = \pi/2$. In that case, for short times $\mathbf{f}_{\mathcal{Q}}^{\text{n-st}}(t) \approx \mathbf{f}_{\mathcal{Q}}^{\text{st}}(t)$ and the effects of the non-stationary and stationary noises becomes comparable.

Finally, no additional contribution to the dissipation arises from squeezing. We conclude that for long times, the squeezed-coherent state contributes to the particle's rms as

$$\Delta\sigma_{\mathbf{q}} \approx \sqrt{\cosh(2r)}\frac{\mathbf{f}_0}{m\sqrt{\Omega_m^3\omega_c}} \tag{4-113}$$

### 4.2.5 Squeezed-thermal states

Squeezed-thermal states also present stationary and non-stationary noise defined by,

$$\langle \mathbf{f}_Q^{\mathrm{st}}(t)\mathbf{f}_Q^{\mathrm{st}}(t')\rangle = \cosh(2r)\frac{\gamma}{\pi}\left(\frac{\hbar a}{q_0}\right)^2\int_0^\infty d\omega \frac{\omega^4\coth\left(\frac{\beta\hbar\omega}{2}\right)\cos\left(\omega\tau\right)}{(\omega-\omega_c)^2+\gamma^2}, \tag{4-114}$$

$$\langle \mathbf{f}_Q^{\mathrm{n-st}}(t)\mathbf{f}_Q^{\mathrm{n-st}}(t')\rangle = \sinh(2r)\frac{\gamma}{\pi}\left(\frac{\hbar a}{q_0}\right)^2\int_0^\infty d\omega \frac{\omega^4\coth\left(\frac{\beta\hbar\omega}{2}\right)\cos\left(\omega(t+t')-2\phi\right)}{(\omega-\omega_c)^2+\gamma^2}. \tag{4-115}$$

In the high temperature limit $\coth(\beta\hbar\omega/2)\to 2/(\beta\hbar\omega)$ and the integrals can be solved analytically, following similar steps given in the Appendix A. We find for the stationary noise,

$$\mathbf{f}_Q^{\mathrm{st}}(t) = \left[\cosh(2r)\left(\frac{k_BT}{\hbar\omega_c}\right)\right]^{1/2}\mathbf{f}_0\sum_i \xi_i(t) \tag{4-116}$$

where $\xi_{i=1,2}$ are independent random variables with correlators,

$$A_1(\tau) = 2e^{-\gamma|\tau|}\left((1-3\nu^2)\cos(\omega_c\tau)-(\nu-\nu^3)\sin(\omega_c|\tau|)\right) \tag{4-117}$$

$$A_2(\tau) = 8\nu\frac{\delta(\tau)}{\omega_c} \tag{4-118}$$

Once again, considering only the zeroth order terms in $\nu$ we can approximate the stationary noise correlator as

$$\langle \mathbf{f}_Q^{\mathrm{st}}(t)\mathbf{f}_Q^{\mathrm{st}}(t')\rangle \approx 2\cosh(2r)\left(\frac{k_BT}{\hbar\omega_c}\right)\mathbf{f}_0^2\ e^{-\gamma|\tau|}\cos(\omega_c\tau) \tag{4-119}$$

We see that besides the exponential squeezing enhancement the thermal state further increases the noise in proportion to its temperature.

Moving on to the non-stationary noise, we find to zeroth order in $\nu$,

$$\langle \mathbf{f}_Q^{\mathrm{n-st}}(t)\mathbf{f}_Q^{\mathrm{n-st}}(t')\rangle \approx \sinh(2r)\left(\frac{k_BT}{\hbar\omega_c}\right)\mathbf{f}_0^2\ e^{-\gamma(t+t')}$$
$$\times\left[2\cos(2\phi)\cos(\omega_c(t+t'))-\sin(2\phi)\sin(\omega_c(t+t'))\right] \tag{4-120}$$

For the squeezed-thermal state the particle's position rms is modified to

$$\Delta\sigma_{\mathbf{q}} \approx \left[2\cosh(2r)\left(\frac{k_B T}{\hbar\omega_c}\right)\right]^{1/2} \frac{\mathbf{f}_0}{m\sqrt{\Omega_m^3\omega_c}} \tag{4-121}$$

presenting an enhancement both due to squeezing and temperature of the optical reservoir.

In the absence of squeezing we can show that the fluctuation-dissipation relation holds [117]. Define

$$T(t,t') = \frac{2}{\pi}\frac{\gamma}{\hbar\omega_c^4}\,\mathbf{f}_0^2\int_0^\infty d\omega\frac{\omega^3\cos(\omega\tau)}{(\omega-\omega_c)^2+\gamma^2} \tag{4-122}$$

and observe that Eq. (4-77) can be written as,

$$\begin{aligned}\mathbf{f}_{\text{diss}} &= \int_0^t \dot{T}(t,t')\mathbf{q}(t')dt' \\ &= -\int_0^t T(t,t')\dot{\mathbf{q}}(t')dt' \end{aligned} \tag{4-123}$$

where we have integrated by parts and assumed vanishing boundary terms. Moreover, from Eqs. (4-114) and (4-122) we have

$$\langle \mathbf{f}_{\mathcal{Q}}^{\text{st}}(t)\mathbf{f}_{\mathcal{Q}}^{\text{st}}(t')\rangle = \frac{1}{\beta}T(t,t') \tag{4-124}$$

Hence, for a high occupation thermal state the dissipation and fluctuation kernels are related as usual.

### 4.2.6
### Quantitative estimations

We can now estimate how large is the quantum contribution to the particle's position rms due to interaction with an optical reservoir in the vacuum, squeezed-coherent and squeezed-thermal states given the optomechanical parameters in Table 4.1.

The characteristic value of the quantum stochastic force is

$$\mathbf{f}_0 \approx 10^{-18}\ \text{N} \tag{4-125}$$

which is about 100 times stronger than the gravitational force between two Planck masses at a 1 mm separation distance [118]. For the vacuum, the parameters in Table 4.1 give

$$(\Delta\sigma_{\mathbf{q}})_{\text{vac}} \approx 2\times 10^{-6} q_0\ , \tag{4-126}$$

well below the zero point fluctuations of the mechanical ground state and hence negligible.

The exponential enhancement provided by squeezing can be used to improve (4-126). For example, for a squeezing parameter of $r = 14$, corresponding to $\approx 60$ dB of quadrature squeezing, we find

$$(\Delta\sigma_{\mathbf{q}})_{\mathrm{sq}} \approx 2q_0 \tag{4-127}$$

This is arguably an enormous amount of squeezing, to be compared to state-of-the-art squeezing sources operating producing states around 10 dB [119].

We can relax the amount of squeezing if we populate the cavity with a squeezed-thermal state. For instance, a state with $k_B T/\hbar\omega_c = 10^5$ and 12 dB squeezing produces $(\Delta\sigma_{\mathbf{q}})_{\mathrm{sq,th}} \approx 2q_0$.

| Parameter | Symbol | Units | Value |
|---|---|---|---|
| Cavity length | $L$ | cm | 3.0 |
| Cavity central frequency | $\omega_c$ | PHz | 1.22 |
| Cavity linewidth | $\gamma$ | kHz | $2\pi \times 193$ |
| Particle mass | $m$ | fg | 2.8 |
| Mechanical frequency | $\omega_m$ | kHz | $2\pi \times 190$ |
| Zero point fluctuation | $q_0$ | m | $3.6 \times 10^{-12}$ |
| Damping rate at | $\gamma_m$ | Hz | $5 \times 10^{-6}$ |
| Coupling rate at $\omega_c$ | $g_c$ | kHz | $2\pi \times 18$ |
| Coupling-to-mech. freq. ratio | $\varepsilon$ | - | 0.1 |
| Cavity linewidth-to-freq. ratio | $\nu$ | - | $10^{-9}$ |

Table 4.1: Coherent scattering optomechanical parameters. Values adapted from [1].

## 4.3 Semiclassical particle as probe of quantum particle

We turn to two linearly coupled mechanical oscillators. Different coupling mechanisms between levitated nanoparticles have been recently demonstrated, such as the Coulomb interaction [2, 120], optical binding [2, 121] and cavity-mediated interactions [122].

From now on modes $a$ and $b$ in the Hamiltonian (4-1) are interpreted as two identical harmonically trapped nanoparticles with frequencies $\omega_a, \omega_b$. We consider $\omega_a \gtrsim \omega_b$ and trace out mode $a$, assumed to be the quantum system. For simplicity, we will neglect external decoherence acting on the quantum system, although we note that these effects can also be taken into account using

| Parameter | Symbol | Units | Value |
|---|---|---|---|
| Charge | $Q_{a,b}$ | $e$ | 250 |
| Interparticle distance | $d$ | $\mu$m | 2.0 |
| Bare frequency $a$ | $\omega_a$ | kHz | $2\pi \times 190$ |
| Bare frequency $b$ | $\omega_b$ | kHz | $2\pi \times 180$ |
| Modified frequency $a$ | $\Omega_a$ | kHz | $2\pi \times 147$ |
| Modified frequency $b$ | $\Omega_b$ | kHz | $2\pi \times 134$ |
| Zero point fluctuation $a$ | $q_{0,a}$ | m | $4.1 \times 10^{-12}$ |
| Zero point fluctuation $b$ | $q_{0,b}$ | m | $4.3 \times 10^{-12}$ |
| Coulomb coupling rate | $g_e$ | kHz | $2\pi \times 51$ |
| Coupling-to-freq. ratio | $g_e/\Omega_{a,b}$ | - | $\approx 0.34$ |

Table 4.2: Coulomb interaction parameters. Values adapted from [2,3].

the path integral formalism [111]. We also neglect the mechanical damping $\gamma_m$ and as in Sec. B, our conclusions will be valid for times $t \ll \gamma_m^{-1}$.

As an example of coupling mechanism we consider the Coulomb interaction between charged particles. For small displacements with respect to the trap center the Coulomb potential is approximated by [3],

$$V_e = -\frac{Q_a Q_b}{8\pi\epsilon_0 d^3}(\mathbf{q}_a - \mathbf{q}_b)^2 \tag{4-128}$$

where $Q_{a,b}$ denotes the charges of the particles and $d$ the interparticle separation. This leads to a change in the bare mechanical frequencies $\omega_{a,b}$ given by

$$\Omega_{a,b} = \sqrt{\omega_{a,b}^2 - \frac{Q_a Q_b}{4\pi\epsilon_0 m d^3}} \tag{4-129}$$

and a linear coupling with rate

$$g_e = -\frac{Q_a Q_b}{4\pi\epsilon_0 \hbar}\left(\frac{q_{0,a} q_{0,b}}{d^3}\right) \tag{4-130}$$

where $q_{0,a}, q_{0,b}$ are the zero point fluctuations of each oscillator now defined in terms of $\Omega_a$ and $\Omega_b$, respectively. The sign of the interaction depends on the charges of the particles. Oppositely charged particles have $g_e > 0$, while like charged particles have $g_e < 0$. Note that the interaction can also be switched on and off by controlling the charge of the nanoparticles [123–125]. This can be exploited to prepare the particle in an initial separable state and subsequently turn the interaction on over a time scale much shorter than $\omega_a^{-1}, \omega_b^{-1}$. Throughout this section, we will consider the parameters in Table 4.2 with values adapted from [2,3].

### 4.3.1 Quantitative estimations

Following Secs. 4.2 we can write the excess rms of the semiclassical particle linearly coupled to the quantum particle in the ground state,

$$\Delta\sigma^2_{\mathbf{q}_b} = \frac{1}{(\kappa^2-1)^2}\left(\frac{\mathbf{f}_0}{m\Omega_b^2}\right)^2 h(t) \tag{4-131}$$

where $h(t)$ is given in Eq. (B-21) and we have redefined $\kappa = \Omega_a/\Omega_b$. Here the characteristic force $\mathbf{f}_0$ defined in Eq. (4-88) is evaluated at the Coulomb rate $g_e$. For the parameters in Table 4.2, $\mathbf{f}_0 \approx 7\times 10^{-18}$ N. Figure 4.2(a) shows the position rms, given by Eq. (4-105), of the semiclassical particle as a function of time, initially in a thermal state with mean occupation number $\bar{n}_b = 10$ and standard deviation $\sigma_0 = \sqrt{2\bar{n}_b+1}\times q_{0,b} \approx 4.6\times q_{0,b}$. As a consequence of the interaction, the standard deviation cyclically oscillates between $\sigma_0$ and $\approx 9\times q_{0,b}$, representing heating and recooling of the particle motion. Note the position rms can become smaller than $\sigma_0$, as shown in the inset in Figure 4.2(b).

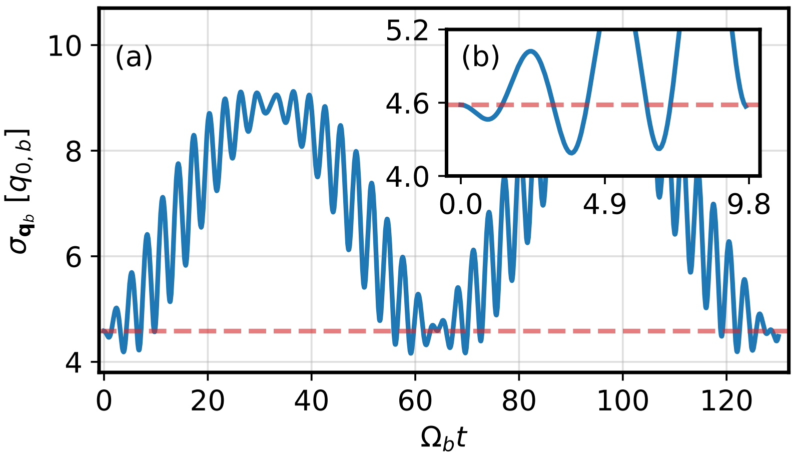


Figure 4.2: (a) Position uncertainty of a thermal semiclassical particle with $\bar{n}_b = 10$ phonons influenced by a quantum particle in the ground state via Coulomb interaction (blue curve), in comparison to the thermal state position standard deviation $\sigma_0 \approx 4.6q_{0,b}$ (red dashed line). (b) Inset: repeated cooling and heating of the semiclassical particle.

A squeezed mechanical state will yield an exponentially enhanced effect, with the excess stationary quantum noise given by,

$$(\Delta\sigma^{\mathrm{st}}_{\mathbf{q}_b})^2 = \cosh{(2r)}\Delta\sigma^2_{\mathbf{q}_b} \tag{4-132}$$

and a non-stationary contribution

$$(\Delta\sigma^{\text{n-st}}_{\mathbf{q}_b})^2 = \frac{\sinh(2r)}{(\kappa^2-1)^2}\left(\frac{\mathbf{f}_0}{m\Omega_b^2}\right)^2 h_\phi(t) \tag{4-133}$$

where $h_\phi(t)$ carries the information on the squeezing phase and is defined as

$$\begin{aligned} h_\phi(t) = \cos(2(\phi-\Omega_a t)) + \cos(\Omega_b t)(\cos(2\phi)\cos(\Omega_b t) \\ -2\cos(2\phi-\Omega_a t)) + 2\kappa\sin(\Omega_b t)(\sin(2\phi)\cos(\Omega_b t) \\ -\sin(2\phi-\Omega_a t)) - \kappa^2\cos(2\phi)\sin^2(\Omega_b t). \end{aligned} \tag{4-134}$$

Figure 4.3 shows the individual stationary (a) and non-stationary (b) contributions and the total (c) position rms for a semiclassical particle in a thermal state with occupation number $\bar{n}_b = 10$ in contact with a squeezed quantum particle with squeezing parameter $r = 3$ ($\approx 30$ dB) [126], for $\phi = 0$, and for different angles in the supplemental material. The effect of different angles can be calculated from Eq. (4-134) and Eq. (4-133). The individual stationary and non-stationary contributions can become negative, but observe the total position rms remains positive due to the thermal bath contribution. Moreover, the total position rms $\sigma_{\mathbf{q}_b}$ depends strongly on the squeezing phase, as can be seen from the traces in Figure 4.3(c); see Supplementary video for a complete sweep of the squeezing phase from $\phi = 0$ to $\phi = 2\pi$.

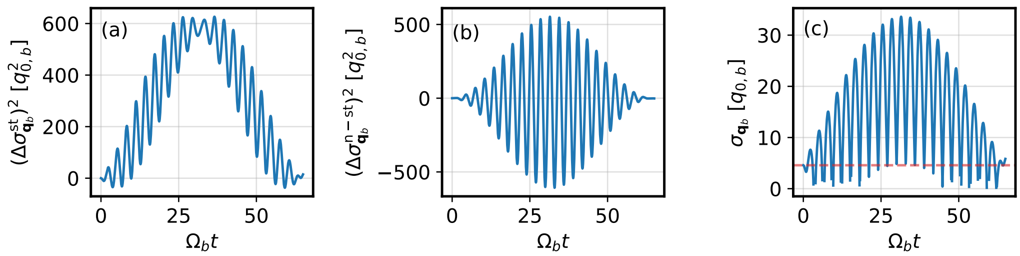


Figure 4.3: Effect of a squeezed quantum particle with $r = 3$ ($\approx 30$ dB) in contact with a semiclassical particle in an initially thermal state with occupation number $\bar{n}_b = 10$: (a) stationary contribution, (b) non-stationary contribution for a squeezing phase $\phi = 0$ (solid blue curve), (c) total position rms for $\phi = 0$ (solid blue curve) compared to the initial uncertainty $\sigma_0 = \sqrt{2\bar{n}_b + 1} \times q_{0,b} \approx 4.6 \times q_{0,b}$ (red dashed line).

Figure 4.4 shows the maximum value of $\sigma_{\mathbf{q}_b}$ as a function of squeezing and the Coulomb coupling strength, changed by varying the particles' charge. Squeezing is measured in decibels (dB) by $S = 10\log(2\text{Var}(q_b))$, with $\text{Var}(q_b) = e^{2r}/2$. We see that for a quantum particle with 30 dB squeezing [126] and moderate values of the coupling strength, $g_e/\Omega_b \approx 0.2$, significant enhancement of the position rms can be achieved when compared to the initial semiclassical

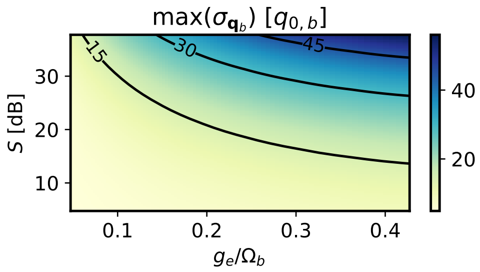


Figure 4.4: Dependence of the maximum value of the position rms $\sigma_{\mathbf{q}_b}$ with the interaction coupling strength $g_e/\Omega_b$ and quantum particle squeezing $S$ (dB). The position rms is given in units of zero point fluctuations $q_{0,b}$, and should be compared to the initial semiclassical uncertainty of $\sigma_0 \approx 4.6 \times q_{0,b}$. The coupling strength is changed by varying the electric charge of the particles from 100 to 260 elementary charges.

rms value $\sigma_0 \approx 4.6 \times q_{0,b}$. In terms of thermal occupation number, this represents an increase from an initial value of $\bar{n}_b = 10$ to $\bar{n}'_b = 220$, while for more modest 10 dB of squeezing, the increase of the occupation number will be from $\bar{n}_b = 10$ to $\bar{n}'_b = 71$. A possible way of measuring this effect of quantum-induced noise would be to employ a generalization of the Kalman filter as used in [44] to achieve zero point fluctuation-level position uncertainty in the presence of colored noise [127]. For comparison, Figure 4.5 displays the maximum position rms $\sigma_{\mathbf{q}_b}$ compared to $\sigma_0$ as a function of the number of phonons of the semiclassical oscillator and squeezing of the quantum particle, at a fixed optomechanical coupling of $g_e/\Omega_b = 0.2$; as expected, the higher the number of phonons (the initial temperature of the semiclassical particle), the harder it becomes to observe the rms oscillations, unless more squeezing is added to the quantum oscillator. For a single mode squeezed thermal state the excess position rms acquires an additional enhancement factor,

$$(\Delta\sigma^{\text{st}}_{\mathbf{q}_b})^2_{\text{th}} = \coth\left(\frac{\beta_a \Omega_a \hbar}{2}\right)(\Delta\sigma^{\text{st}}_{\mathbf{q}_b})^2, \tag{4-135}$$

which becomes significant when $\beta_a^{-1} \gg \hbar\Omega_a/2$. Considering a levitated nanoparticle cooled to a number of phonons of $n_b = 0.5$ ($T_a = 45\,\text{mK}$, $\beta_a = 1.6 \times 10^{27}$ J/K) [44] we have $\coth(\beta_a \hbar \Omega_a/2) \sim 2.12$, further enhancing the fluctuations in the semiclassical particle.

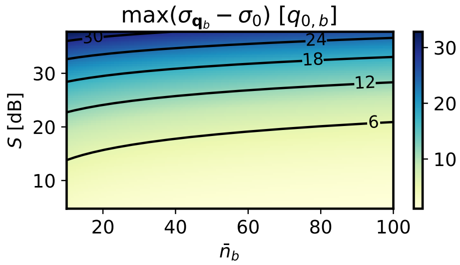


Figure 4.5: Dependence of the maximum value of the position rms $\sigma_{\mathbf{q}_b}$ (in units of zero point fluctuations $q_{0,b}$) with the initial number of phonons $n_b$ in the semiclassical oscillator versus quantum particle squeezing $S$ (dB), compared to the initial semiclassical uncertainty of $\sigma_0 \approx 4.6 \times q_{0,b}$. The coupling strength is assumed to be $g_e/\Omega_b = 0.2$.

### 4.3.2 Gravitational field of a delocalized particle

A this point, we cannot resist exploiting the analogy between the Coulomb and Newtonian potentials to draw some comments on the stochastic gravitational field of a delocalized quantum particle.

We consider two identical particles of mass $m$ at a center-of-mass separation $d$. To leading order in the particles' displacements, the Newtonian gravitational potential yields an interaction Hamiltonian of the form [128] $H_N \approx (Gm^2/d^3)(\mathbf{q}_a - \mathbf{q}_b)^2$, which translates into an effective frequency shift of

$$\Omega_{a,b} \approx \sqrt{\omega_{a,b}^2 + 2Gm/d^3} \tag{4-136}$$

and a Newtonian coupling rate

$$g_N \approx -\frac{2G}{\hbar}\frac{m^2 q_{0,a} q_{0,b}}{d^3} \tag{4-137}$$

According to Eqs. (4-56) and (4-57), for large squeezing parameters, we can recast both the stationary and non-stationary stochastic forces according to

$$\mathbf{f}_{\mathcal{Q}}(t) \approx e^r \frac{\hbar g_N}{q_{0,b}} \xi(t) = \frac{\hbar \Gamma_{\text{ent}}}{q_{0,b}} \xi(t) \tag{4-138}$$

where $\xi(t)$ satisfies either the stationary or non-stationary correlators in Eqs. (4-58) and (4-59) and we define the *entanglement rate*

$$\Gamma_{\text{ent}} \equiv 2\left(\frac{G}{\hbar}\right)\left(\frac{m^2 \Delta\mathbf{q}_a \Delta\mathbf{q}_b}{d^3}\right) \tag{4-139}$$

where $\Delta\mathbf{q}_a = e^r q_{0,a}$ and $\Delta\mathbf{q}_b = q_{0,b}$ are the wavefunction uncertainties in the position basis for a squeezed particle and a particle in the ground state.

To arrive at the stochastic force, we have considered a stationary phase approximation in the stochastic propagator in Eq. (4-40). If we do not make the semiclassical approximation, we can see that the gravitational interaction generates entanglement between the oscillators [128–131] at a rate given by $\Gamma_{\text{ent}}$ – indeed Eq. (4-139) is the short-time approximation of the entanglement rate between two continuously delocalized oscillators interacting via gravity as derived in [131]. Arguably, the rate $\Gamma_{\text{ent}}$ is extremely small given current experiments, but the result (4-138) is conceptually interesting as it makes the connection between entanglement and the quantum-induced stochastic force of a subsystem manifest. Naturally, the same conclusions apply to the much stronger case of Coulomb interactions.

## 4.4 Discussion

In this chapter, we have applied the formalism of double path integrals and Feynman-Vernon influence functionals to linear optomechanical systems. We have analysed the effective stochastic dynamics induced by the interactions between a semiclassical and a quantum system in different states, notably the ground state, squeezed-coherent and squeezed-thermal states. Colored noise and dissipation with memory arising from quantum fluctuations are ubiquitous in linear optomechanical quantum-classical interactions. Microscopically, these fluctuations can be understood as a 'semiclassical' manifestation of the entanglement generated by the interaction between the two subsystems. We have studied these effects in the context of levitated nanoparticles, both in cavity and free space multi-particle scenarios. Notably, the quantum-classical stochastic dynamics induced by the Coulomb interaction between two levitated particles is potentially measurable in near future experiments with delocalized quantum states, where squeezing of the mechanical wavefunction yields an exponential enhancement of the quantum-induced stochastic forces. The analogy between the Coulomb and Newtonian potentials has been used to comment on the connection between the effective stochastic dynamics and gravitational-induced entanglement.

Throughout our study, we find strong similarities between a linear optomechanical system and the effective field theory of a quantized gravitational wave mode interacting with a GW detector, despite both systems being governed by seemingly different Hamiltonians, one originating from the Einstein-Hilbert action [26] while the other from the interaction of dielectrics in electromagnetic fields [57, 132]. This reinforces the analogies between optomechanics and the quantum theory of GWs [133] and opens the possibility of investigating novel ways of probing the quantum nature of GWs through proof-of-principle laboratory experiments.

Our findings also open the way to novel investigations at the interface between stochastic thermodynamics and fundamental tests of quantum theory. For instance, the formalism of path integrals allows for the calculation of the work distribution [108] and probability of rare violations of the second law of thermodynamics [107] springing from state dependent quantum-induced fluctuations in levitated optomechanical systems. Moreover, the formalism can be extended to investigate the effective stochastic dynamics induced by multipartite entangled quantum states in contact with a classical probe. In principle, such investigation could lead to new forms of witnessing entanglement and non-classicality in the mesoscopic scale.

# 5
# Outlook

In this work, we have investigated how the quantum features of harmonic oscillators of macroscopic nature affects the dynamics of a system of interest. After developing the basics of the formalism needed throughout the text, we apply it in two different contexts. The first, was to analyze the dynamics of a gravitational wave interacting with a detector, via the dispersive optomechanical interaction. Assuming that intrinsically quantum states of GW could be generated in astrophysical events, due to the nonlinearity of Einstein's field equations, we estimated its effect and how it would induce a change in the observable signal. We also worked in a full optomechanical frame, discussing how different quantum states of the unobserved subsystems (the light field or a different nanoparticle) would affect the dynamics of the observable particle. We employed the Feynman-Vernon formalism and achieved the Langevin equations for the nanoparticle dynamics. Furthermore, the theoretical approach discussed in this work could be deepened and extended for other scopes. We briefly discuss some possibilities.

## Theoretical Prospects in Levitated Optomechanics

Following what was discussed in this work, we aim at showing intrinsic quantum behavior in levitated optomechanical systems. To this end, the usual approach in the literature is to put the particle in its ground state of the center of mass motion, and then be able to expand its wave function, such as in [134, 135]. After that, when the uncertainty of the position is of the order of the size of the nanoparticle, one would make a double slit-like experiment, expliciting the quantum nature of the particle by observing the interference fringes that would show up.

One of the main obstacles in this program is due to the decoherence caused by laser, called recoil heating [136]. This is a fundamental source of noise, arising from the fact that in order to measure the system, one needs to interact with it, so that the unobserved scattered photons carry the information away. At the same time, we need a high detection efficiency to perform the feedback cooling needed to bringing it to the ground state. So we need a way to maximize the information that we extract from the particle, while minimally disturbs it. One of the ways that we believe we could achieve that, is by the use of structured light [137].

It was shown that when measuring the particle with a gaussian beam, mostly of the information regarding the particle's position is contained in the light that is scattered backwards by the particle [138]. Motivated by recent advances in using structured light to tweeze particles [139], one could use different light modes to manipulate the information patterns scattered by the particle, increasing it in one direction or another, for example. Moreover, understanding the behavior of the information patterns for an arbitrary beams opens the possibility to other features, such as beams that don't extract any information from the particle, minimizing the recoil heating, for instance.

Another interesting possibility is to slightly change the paradigm. The previous approach relied in a *extreme* capacity of isolating, controlling and measuring the system. Yet achievable, this yields a fragile quantum coherence, susceptible to many decoherence mechanisms, given the complexity of the setups and the time needed to perform the double slit-like experiment. Here, we propose investigating the quantum nature of these systems via the scope of quantum thermodynamics and quantum information [140].

Since the nanoparticle is in a intermediate regime, with many degrees of freedom, but also close to its motional ground state, quantum thermodynamical effects may play a important role in the dynamics. Moreover, the formalism of quantum information is the best suited for treating features such as entanglement and decoherence. In this way, this seems a good path to understand intrinsic quantum behavior such as in [141–143].

Moreover, in the quantum information/thermodynamical approach, one could maybe find some fundamental inequality that assures nonclassicality of a system via, for instance, correlations in the position. These kind of inequality would be analogous to a Bell-like inequality, and, in principle, could be more resilient to decoherence. Of course that even though one expects to relax the experimental constraints, the feasibility of this kind of experiment is still very reliable in sophisticated experimental techniques.

### Quantization of Nonlinear Gravitational Waves

This is by far the boldest assumption of the outlook. In chapter 3 we assumed no knowledge of the source of the GW's, and argued that intrinsic quantum states could be, in principle, generated due to nonlinearities of the Einstein's equations. Given recent works outlining the importance of nonlinear effects on the detection of gravitational waves coming from the ringdown of black holes [144], these arguments are reinforced.

Nonetheless, to quantize a theory of nonlinear gravitational waves is not simple. Due to issues regarding gauge invariance and ghosts [145], we cannot simply add higher order terms in the action 2-126. We need to look for second order propagating gauge invariant perturbations.

This could perhaps be better described in the formalism of Newman-Penrose [146], and seminal works regarding gauge invariant perturbations of spacetime use this formalism [147]. Perturbations of second order that are gauge invariant were also proposed [148].

In this way, we believe that combining perks of the Newman-Penrose formalism and QFT in curved space times, one could, in principle find a (gauge, coordinate invariant) quantized theory of GWs that explicitly exhibits nonlinearities. Moreover, if we achieve that, we can look for intrinsic nonlinear quantum phenomena in gravitational waves, that are usual in quantum optics [149] and its effects on observational data, in analogy to the optomechanical program of detecting the quantum nature of mesoscopic oscillators.

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# A
# Fluctuation and Dissipation in a cavity

Summation over modes in a cavity leads us to the fluctuation and dissipation integrals in Eqs. (4-79) and (4-80). Taking into account the density of modes in Eq. (4-76) results in the two main integrals,

$$I_1 = \int_0^\infty d\omega \frac{\omega^4 \cos(\omega(t-t'))}{(\omega-\omega_c)^2+\gamma^2} \tag{A-1}$$

$$I_2 = \int_0^\infty d\omega \frac{\omega^4 \sin(\omega(t-t'))}{(\omega-\omega_c)^2+\gamma^2} \tag{A-2}$$

for the fluctuation and dissipation terms, respectively. These can be rewritten as

$$I_1 = \frac{d^4}{d\tau'^4} J_1, \tag{A-3}$$

$$I_2 = \frac{d^4}{d\tau'^4} J_2. \tag{A-4}$$

where,

$$J_1 = \int_0^\infty d\omega \frac{\cos(\omega\tau')}{(\omega-\omega_c)^2+\gamma^2} \tag{A-5}$$

$$J_2 = \int_0^\infty d\omega \frac{\sin(\omega\tau')}{(\omega-\omega_c)^2+\gamma^2} \tag{A-6}$$

and $\tau' = t - t'$. Both $I_1$ and $I_2$ can be evaluated in terms of distributions. We now proceed to calculate each integral, starting with $J_{1,2}$ and then $I_{1,2}$.

## A.1
## Evaluation of $J_{1,2}$

Eq. (A-4) can be written in terms of a Fourier transform,

$$\int_0^\infty d\omega \frac{\sin\omega\tau}{\left(\omega-\omega_c\right)^2+\gamma^2} = \operatorname{Im}\int_{-\infty}^\infty d\omega \frac{H(\omega)e^{i\omega\tau}}{\left(\omega-\omega_c\right)^2+\gamma^2}, \tag{A-7}$$

We can use the convolution theorem to solve (A-7). We use the following convention for the convolution of two functions,

$$\frac{1}{2\pi}(f*g)(\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty d\alpha f(\alpha)g(\tau-\alpha). \tag{A-8}$$

For the Heaviside, we have the Fourier transform,

$$f(\tau) = \int_{-\infty}^{\infty} d\omega H(\omega) e^{i\omega\tau} = \pi\delta(\tau) + \mathcal{P}\left(\frac{i}{\tau}\right), \tag{A-9}$$

where $\mathcal{P}$ denotes the Cauchy principal value. Using [18],

$$\mathcal{P}\left(\frac{1}{\tau}\right) = \frac{1}{\tau + i\epsilon} + \pi i\delta(\tau), \tag{A-10}$$

we have

$$f(\tau) = \int_{-\infty}^{\infty} d\omega H(\omega) e^{i\omega\tau} = \frac{i}{\tau + i\epsilon}. \tag{A-11}$$

For the Lorentzian the Fourier transform reads,

$$g(\tau) = \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{(\omega - \omega_c)^2 + \gamma^2} = \frac{\pi}{\gamma} e^{i\omega_c\tau} e^{-\gamma|\tau|}. \tag{A-12}$$

Using the convolution theorem,

$$\frac{1}{2\pi}(f * g)(\tau) = \frac{i}{2\gamma} \int_{-\infty}^{\infty} d\alpha \frac{1}{\tau - \alpha + i\epsilon} e^{i\omega_c\alpha} e^{-\gamma|\alpha|}. \tag{A-13}$$

We can again use the convolution theorem to solve (A-13). We have,

$$\int_{-\infty}^{\infty} d\beta e^{i\omega_c\beta} e^{-\gamma|\beta|} = \frac{2\gamma}{\gamma^2 + \omega_c^2}, \tag{A-14}$$

$$\int_{-\infty}^{\infty} d\beta \frac{1}{\tau - \beta \pm i\epsilon} e^{i\omega_c\beta} = -2\pi i e^{i\omega_c\tau}, \tag{A-15}$$

where $\omega_c > 0$. The convolution of Eq. (A-14) and (A-15) reads,

$$\int_{-\infty}^{\infty} d\beta \frac{e^{i\omega_c\beta} e^{-\gamma|\beta|}}{\tau - \beta \pm i\epsilon} = -2i\gamma e^{i\omega_c\tau} \int_{-\infty}^{\infty} d\alpha \frac{e^{-i\tau\alpha}}{\gamma^2 + \alpha^2}. \tag{A-16}$$

Finally, using

$$\int_{-\infty}^{\infty} d\alpha \frac{e^{-i\tau\alpha}}{\gamma^2 + \alpha^2} = \frac{\pi e^{-\gamma|\tau|}}{\gamma}, \tag{A-17}$$

we find,

$$\frac{i}{2\gamma} \int_{-\infty}^{\infty} d\alpha \frac{e^{i\omega_c\alpha} e^{-\gamma|\alpha|}}{\tau - \alpha \pm i\epsilon} = \frac{\pi}{\gamma} e^{i\omega_c\tau} e^{-\gamma|\tau|}. \tag{A-18}$$

Putting it all together and taking the imaginary part of the result we arrive at

$$\int_{0}^{\infty} d\omega \frac{\sin\omega\tau}{(\omega - \omega_c)^2 + \gamma^2} = \frac{\pi}{\gamma} e^{-\gamma|\tau|} \sin(\omega_c\tau). \tag{A-19}$$

The integral $J_1$ can be written as a Fourier transform,

$$J_1 = \frac{1}{2} \mathrm{Re} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega\tau}}{(\omega - \omega_c)^2 + \gamma^2} \tag{A-20}$$

Again using (A-12) we have,

$$J_1 = \frac{\pi}{2\gamma} e^{-\gamma|\tau|} \cos\omega_c\tau. \tag{A-21}$$

## A.2 Distributional derivatives

We now need to take derivatives of $J_{1,2}$. Note these are discontinuous and exist only in the sense of distributions [150]. Define the action of a distribution $T$ on a test function $\phi$ as the inner product

$$\langle T, \phi \rangle = \int_{-\infty}^{\infty} dx T(x)\phi(x) \tag{A-22}$$

and the derivative of a distribution as

$$\langle T', \phi \rangle = -\langle T, \phi' \rangle \tag{A-23}$$

To evaluate the derivatives of $J_{1,2}$ we note that discontinuities spring from the the $e^{-\gamma|\tau|}$ terms, for which the first derivative is

$$\frac{d}{d\tau} e^{-\gamma|\tau|} = -\gamma \,\mathrm{sign}(\tau) \mathrm{e}^{-\gamma|\tau|}. \tag{A-24}$$

We can evaluate higher derivatives by considering their effects on a test function $\phi(\tau)$. We have,

$$\begin{aligned} \left\langle \frac{d}{d\tau}\left(-\gamma\,\mathrm{sign}(\tau)\mathrm{e}^{-\gamma|\tau|}\right), \phi(\tau)\right\rangle &= -\left\langle \left(-\gamma\,\mathrm{sign}(\tau)\mathrm{e}^{-\gamma|\tau|}\right), \phi'(\tau)\right\rangle \\ &= \gamma \int_0^\infty e^{-\gamma\tau}\phi'(\tau)d\tau - \gamma\int_{-\infty}^0 e^{\gamma\tau}\phi'(\tau)d\tau \\ &= -2\gamma\phi(0) - \gamma\int_{-\infty}^{\infty} \frac{d}{d\tau}\left(e^{-\gamma|\tau|}\right)\phi(\tau)d\tau \\ &= -2\gamma\langle\delta(\tau), \phi(\tau)\rangle + \gamma^2\langle e^{-\gamma|\tau|}, \phi(\tau)\rangle. \end{aligned} \tag{A-25}$$

We conclude,

$$\frac{d^2}{d\tau^2}\left(e^{-\gamma|\tau|}\right) = -2\gamma\delta(\tau) + \gamma^2 e^{-\gamma|\tau|}. \tag{A-26}$$

Generalizing to higher orders we find,

$$\frac{d^3}{d\tau^3}\left(e^{-\gamma|\tau|}\right) = -2\gamma\delta'(\tau) - \gamma^3\mathrm{sign}(\tau)\mathrm{e}^{-\gamma|\tau|}, \tag{A-27}$$

$$\frac{d^4}{d\tau^4}\left(e^{-\gamma|\tau|}\right) = -2\gamma\delta''(\tau) - 2\gamma^3\delta(\tau) + \gamma^4 e^{-\gamma|\tau|}. \tag{A-28}$$

Through repeated applications of the above derivative rules we can compute Eqs. (A-3) and (A-4). We have,

$$\begin{aligned} \frac{d^4}{d\tau^4}\left(e^{-\gamma|\tau|}\sin(\omega_c\tau)\right) = \omega_c^4 \Bigg[&-2(\nu^3 - 6\nu)\frac{\delta(\tau)}{\omega_c}\sin(\omega_c\tau) - 8\nu\frac{\delta'(\tau)}{\omega_c^2}\cos(\omega_c\tau) - 2\nu\frac{\delta''(\tau)}{\omega_c^3}\sin(\omega_c\tau) \\ &+ e^{-\gamma|\tau'|}\left((1 - 6\nu^2 + \nu^4)\sin(\omega_c\tau) + 4(\nu - \nu^3)\mathrm{sgn}(\tau)\cos(\omega_c\tau)\right)\Bigg] \end{aligned} \tag{A-29}$$

$$\frac{d^4}{d\tau^4}\left(e^{-\gamma|\tau'|}\cos(\omega_c\tau')\right) = \omega_c^4\left[-2(\nu^3-6\nu)\frac{\delta(\tau)}{\omega_c}\cos(\omega_c\tau) - 8\nu\frac{\delta'(\tau)}{\omega_c^2}\sin(\omega_c\tau) - 2\nu\frac{\delta''(\tau)}{\omega_c^3}\cos(\omega_c\tau)\right.$$
$$\left. + e^{-\gamma|\tau|}\left((1-6\nu^2+\nu^4)\cos(\omega_c\tau) - 4(\nu-\nu^3)\mathrm{sgn}(\tau)\sin(\omega_c|\tau|)\right)\right] \tag{A-30}$$

where as in the main text we define $\nu = \gamma/\omega_c$. We can further simplify this result by introducing certain distribution identities.

## A.3 Distribution identities

Consider the action of the $\delta''(\tau)\cos(\omega_c\tau)$ distribution on a test function,

$$\begin{aligned}
\langle\delta''(\tau)\cos(\omega_c\tau), \phi(\tau)\rangle &= \langle\delta''(\tau), \cos(\omega_c\tau)\phi(\tau)\rangle = \langle\delta(\tau), (\cos(\omega_c\tau)\phi''(\tau))\rangle \\
&= \langle\delta(\tau), (-2\omega_c\sin(\tau\omega_c)\phi'(\tau) + \cos(\tau\omega_c)\phi''(\tau) + -\omega_c^2\phi(\tau)\cos(\tau\omega_c)\big)\rangle \\
&= \phi''(0) - \omega_c^2\phi(0) = \langle\delta(\tau'') - \omega_c^2\delta(\tau), \phi(\tau)\rangle
\end{aligned} \tag{A-31}$$

Therefore,

$$\delta''(\tau)\cos(\omega_c\tau) = \delta''(\tau) - \omega_c^2\delta(\tau). \tag{A-32}$$

Following similar steps, we also have the following identities, used throughout the main text,

$$\delta(\tau)\cos(\omega_c\tau) = \delta(\tau), \tag{A-33}$$
$$\delta'(\tau)\sin(\omega_c\tau) = -\omega_c\delta(\tau), \tag{A-34}$$
$$\delta'(\tau)\cos(\omega_c\tau) = -\delta'(\tau), \tag{A-35}$$
$$\delta''(\tau)\sin(\omega_c\tau) = 2\omega_c\delta'(\tau). \tag{A-36}$$

# B
# Semiclassical Light as probe of quantum particle

In this appendix we will analyze the results when we trace off a particle in a quantum state, looking at the resulting semiclassical equations of the cavity field. Since the formalism and the conclusions are much alike the ones in 4.2, we chose to leave this part in the appendix. With the appropriate modifications, we can use the methods developed so far to describe a semiclassical optical field interacting with a quantum mechanical oscillator.

## B.1
## Optical equations of motion

The derivation of the optical semiclassical equations of motion is very analogous to that of the mechanical case and with the proper substitution of constants we can fast-forward to the optical version of the results in Sec. 4.1.5.

For simplicity, we will consider a single cavity mode. The Hamiltonian describing our system is the same as in Eq. (4-1). Up to a constant shift, the free optical Hamiltonian in Eq. (4-2) can be written as

$$H_c/\hbar = \frac{\omega}{4}\left(X(t)^2 + Y(t)^2\right) \tag{B-1}$$

where the optical field quadratures are defined as

$$X(t) = ae^{-i\omega t} + a^\dagger e^{i\omega t} \tag{B-2}$$

$$Y(t) = i\left(a^\dagger e^{i\omega t} - ae^{-i\omega t}\right) \tag{B-3}$$

and $\dot{X} = \omega Y$. We define the electromagnetic position and momentum quadratures

$$Q \equiv \sqrt{\frac{\hbar}{2m_0\omega}}X \quad , \quad P = m_0\dot{Q} \tag{B-4}$$

where $m_0$ is a constant with dimension of mass introduced to establish the analogy between an optical mode and a harmonic oscillator [151]. In terms of

$Q, P$ the Hamiltonian (B-1) becomes

$$H_c = \frac{1}{2}\left(m_0\omega^2 Q^2 + \frac{P^2}{m_0}\right) \tag{B-5}$$

with the corresponding Lagrangian,

$$\begin{aligned} L_c &= \frac{m_0}{2}\left(\dot{Q}^2 - \omega^2 Q^2\right) \\ &= \frac{\hbar}{4\omega}\left(\dot{X}^2 - \omega^2 X^2\right) \end{aligned} \tag{B-6}$$

Note that $m_0$ drops out of $L_c$.

Repeating the steps up to Sec. 4.1.5 we obtain the electromagnetic version of the stochastic propagator in Eq. (4-40) after tracing out the mechanical degree of freedom. For the mechanical oscillator initially in the ground state we have

$$\begin{aligned} \mathcal{J}(X_t, X_t'|X_0, X_0') = &\int_{X_0,X_0'}^{X_t,X_t'} \mathcal{D}X\mathcal{D}X' \int \mathcal{D}\zeta\, P[\zeta(t)] \exp\left(\frac{i}{\hbar}\int_0^{t_f} dt\,(L_c - L_c')\right) \\ &\times \exp\left(\frac{i}{\hbar}\int_0^{t_f} dt\hbar\zeta(t)(X(t) - X'(t))\right. \\ &\left.+\frac{i}{\hbar}\left(\hbar g^2\right)\int_0^{t_f}\int_0^{t} dt\,dt'\,(X(t) - X'(t))\,(X(t') + X'(t'))\sin\left(\omega_m(t-t')\right)\right) \end{aligned} \tag{B-7}$$

where the probability density $P[\zeta(t)]$ is defined in (4-76) and $L_c'$ is the Lagrangian in Eq. (B-6) in terms of the backward quadrature $X \to X'$. Note that we can express $\zeta(t)$ in terms of the dimensionless random variable $\xi(t)$ by using Eq. (4-39).

From (B-7) we can calculate the semiclassical Eqs. of motion for the cavity field quadrature. As in the mechanical case we neglect the coupling between the forward and backward variables $X$ and $X'$. We have,

$$\ddot{X} + \gamma\dot{X} + \omega^2 X = \mathbf{F}_Q(t) + \mathbf{F}_{\text{diss}}(t) \tag{B-8}$$

where $\mathbf{F}_Q(t)$ is the stochastic "force" arising from the quantum fluctuations of the mechanical oscillator,

$$\mathbf{F}_Q(t) = 2\omega g\xi(t) \tag{B-9}$$

with

$$\langle \xi(t)\xi(t')\rangle = \cos(\omega_m \tau) \tag{B-10}$$

and $\mathbf{F}_{\rm diss}(t)$ is the dissipation term given by

$$\mathbf{F}_{\rm diss}(t) = 2\omega g^2 \int_0^t dt' X(t') \sin(\omega_m(t-t')). \tag{B-11}$$

Note that we have introduced a phenomenological dissipative term $\gamma\dot{X}$ due to the coupling of the cavity mode with the external electromagnetic field [62]. This comes from the input-output formalism, discussed in sec.2.3.

To compare the order-of-magnitude of the terms in Eq. (B-8) we rescale time according to $t \to \omega_m t$. The equation of motion becomes,

$$\ddot{X} + \left(\frac{\gamma}{\omega_m}\right)\dot{X} + \left(\frac{\omega}{\omega_m}\right)^2 X = 2\varepsilon\left(\frac{\omega}{\omega_m}\right)\xi(t) + 2\varepsilon^2\left(\frac{\omega}{\omega_m}\right)\int_0^t dt' X(t')\sin(t-t') \tag{B-12}$$

where here derivatives and integration are taken with respect to rescaled time. Again we see the fluctuation is of order $\varepsilon$ while the dissipation arising from the influence functional is of order $\varepsilon^2$, which from now on will be neglected.

In the mechanical case we have summed over the cavity modes weighted by the Lorentzian density of states of width $\gamma$. Similarly, the mechanical mode also couples to external environmental degrees of freedom and a sum over modes procedure is in order. However, the broadening of the mechanical mode given by the mechanical damping rate $\gamma_m$ is comparatively much smaller than that of the cavity. We have $\gamma_m/\omega_m \approx 10^{-11} \ll \nu$. We will therefore consider the mechanical oscillator as a single mode system. This approximation is valid for times much smaller than the inverse mechanical damping rate, $t \ll \gamma_m^{-1}$.

We arrive at the final form of the semiclassical optical equations for the field quadratures,

$$\dot{X} = \omega Y \tag{B-13}$$

$$\dot{Y} \approx -\omega X - \gamma Y + 2g\xi(t) \tag{B-14}$$

where $\xi(t)$ satisfies (B-10).

## B.2 Fluctuations

Let us estimate the order of magnitude of the fluctuations imprinted on the cavity by a quantum mechanical oscillator in the ground state. For simplicity we neglect the cavity dissipation. In that case the optical equations are formally solved by

$$X(t) = X_0 \cos(\omega t) + Y_0 \sin(\omega t) + 2g \int_0^t ds \sin(\omega(t-s))\xi(s), \tag{B-15}$$

$$Y(t) = -X_0 \sin(\omega t) + Y_0 \cos(\omega t) + 2g \int_0^t ds \cos(\omega(t-s))\xi(s). \tag{B-16}$$

where the field quadratures have initial conditions $X(0) = X_0, Y(0) = Y_0$. The field quadrature rms reads

$$\sigma_X^2 = \sigma_0^2 + \Delta\sigma_X^2 \tag{B-17}$$

where

$$\sigma_0^2 = \sigma_{X_0}^2 \cos^2(\omega t) + \sigma_{Y_0}^2 \sin^2(\omega t) + 2\mathrm{Cov}(X_0, Y_0) \sin(\omega t) \cos(\omega t) \tag{B-18}$$

and the excess quantum-induced fluctuations are

$$\Delta\sigma_X^2 = 4g^2 \int_0^t \int_0^t ds ds' \sin(\omega(t-s)) \sin(\omega(t-s'))\langle \xi(s)\xi(s')\rangle \tag{B-19}$$

Note that with the appropriate changes of constants $\Delta\sigma_X^2$ assumes the same form as Eq. (4-106) with zero damping rate $\gamma_m = 0$.

## B.3 Quantitative Estimations

Since we are interested in the changes in the quadrature rms due to interaction with the mechanical oscillator we need only look into $\Delta\sigma_X^2$. We have,

$$\Delta\sigma_X^2 = \frac{4g^2\omega^2}{(\omega^2 - \omega_m^2)^2} h(t) \tag{B-20}$$

where

$$h(t) = 1 + \cos^2(\omega t) - 2\cos(\omega t)\cos(\omega_m t) - 2\kappa \sin(\omega t)\sin(\omega_m t) + \kappa^2 \sin^2(\omega_m t) \tag{B-21}$$

and $\kappa = \omega_m/\omega$. Similar expressions for $\sigma_Y^2$ can be obtained. We see that to leading order in $(\omega_m/\omega)$, the optical quadrature rms oscillates with a characteristic amplitude given by $\Delta\sigma_X^2 \approx (g/\omega)^2$, which for the optomechanical parameters in Table 4.1 is $g/\omega \approx 10^{-10}$. This number is to be compared with the standard deviation of the optical coherent state $\sigma_{X_0} = 1/2$. Again, ground state fluctuations are too small to have any practical effects in levitated optomechanics.

For the case of squeezed states, the result in Eq. (B-20) will acquire an exponential enhancement and a non-stationary contribution dependent on the squeezing phase as in the mechanical case. However, to elevate the amplitude of the oscillations in (B-20) to a level comparable to the standard deviation of a coherent state will require a 23 e-fold enhancement factor, corresponding to an impractical amount of squeezing. It is interesting to observe that a room-temperature levitated nanoparticle yields an appreciable effect, $\Delta\sigma_X \propto (k_B T_{\text{bath}}/\hbar\omega_m)(g/\omega) \approx 10^{-3}$ for $T_{\text{bath}} \approx 293$ K.

As a final observation, note that the rms in Eq. (B-20) has a factor inversely proportional to the square difference of the oscillators' frequencies. For a cavity interacting with a mechanical oscillator the higher optical frequency dominates, leading to a suppression of $\Delta\sigma_X$. If two oscillators with similar frequencies interact via the linear optomechanical Hamiltonian we can expect a resonantly enhanced effect. This motivates our final application of two linearly coupled levitated nanoparticles.