Pontifícia Universidade Católica
do Rio de Janeiro

Raul Steven Rodriguez Chavez

Statistical behavior of skew products: Schwarzian derivative and arc-sine laws

Dissertação de Mestrado

Dissertation presented to the Programa de Pós–graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Matemática.

Advisor : Prof. Lorenzo Justiniano Díaz Casado
Co-advisor: Prof. Pablo Gutierrez Barrientos

Rio de Janeiro
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Abstract

Rodriguez Chavez, Raul Steven; Díaz, Lorenzo (Advisor); Gutierrez, Pablo (Co-Advisor). **Statistical behavior of skew products: Schwarzian derivative and arc-sine laws**. Rio de Janeiro, 2024. 89p. Dissertação de mestrado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

We consider skew products over Bernoulli shifts, whose fibred dynamics is given by diffeomorphisms of the interval. We study the predictable and/or historical behavior, referring to convergence and/or non-convergence, of the Birkhoff average, respectively. We employ the Schwarzian derivative of the fiber maps and the arc-sine law to identify conditions under which these skew products exhibit these types of behavior. We identify distinct types of behavior according to the Schwarzian derivative. When the Schwarzian derivative is negative, the skew product has intermingled basins. Conversely, when the Schwarzian derivative is positive, the skew product has a physical measure. Finally, when the Schwarzian derivative is zero, the skew product has historical behavior. In the latter scenario, we establish a connection between historical behavior and the arc-sine law that allows us to obtain results in other settings independent of the sign of the Schwarzian derivative.

Keywords

Intermingled basins; Physical measures; Historical behavior; Schwarzian derivative; Arc-sine law.

Resumo

Rodriguez Chavez, Raul Steven; Díaz, Lorenzo; Gutierrez, Pablo. **Comportamento estatístico de produtos tortos: derivada Schwarziana e leis do arco-seno**. Rio de Janeiro, 2024. 89p. Dissertação de Mestrado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Consideramos produtos tortos sobre “shifts” de Bernoulli, cuja dinâmica fibrada é dada por difeomorfismos do intervalo. Estudamos o comportamento previsível e/ou histórico destes sistemas, referindo-nos à convergência e/ou não convergência, da média de Birkhoff, respectivamente. Utilizamos a derivada Schwarziana das fibras e a lei do arco-seno para identificar condições nas quais esses produtos tortos apresentam esses tipos de comportamento. Identificamos distintos tipos de comportamento em relação à derivada Schwarziana. Quando a derivada Schwarziana é negativa, o produto torto tem bacias entrelaçadas. Por outro lado, quando a derivada Schwarziana é positiva, o produto torto possui uma medida física. Finalmente, quando a derivada Schwarziana é nula, o produto torto tem comportamento histórico. No último cenário, estabelecemos uma conexão entre o comportamento histórico e a lei do arco-seno que nos permite obter resultados em outras configurações independentes do sinal da derivada Schwarziana.

Palavras-chave

Bacias entrelaçadas; Medidas físicas; Comportamento histórico; Derivada Schwarziana; Lei do arco-seno.;

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1
Introduction

For a continuous function $f\colon M \to M$ on a compact metric space, given a point $x \in M$, one can analyze the behavior of the *forward orbit* of the point x defined by

$$\mathcal{O}^+(x) \stackrel{\text{def}}{=} \left\{x, f(x), f^2(x), \dots, f^n(x), \dots\right\}.$$

When the function is invertible, we can consider the *backward orbits* defined similarly. In what follows, we focus on the case where invertibility is not necessary. The orbit has a *predictable behavior* if, for every continuous function $\varphi\colon M \to \mathbb{R}$, the *time average*

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) \tag{1-1}$$

converges. On the other hand, an orbit has *historical behavior* if, there is a continuous function $\varphi\colon M \to \mathbb{R}$, the time average does not converge. This terminology was introduced by Ruelle [19] and develop by Takens [23]. We are interested in studying the abundance of predictable and historical behaviors. We begin by considering the historical case. Historical behavior is abundant if there exists a set $\mathcal{B}$ of positive measure with respect to some reference measure, such that for each point in $\mathcal{B}$, its orbit exhibits historical behavior.

A paradigmatic dynamical configuration that leads to historical behavior is the so-called *Bowen Eye* in [22], see Figure 1.1. This configuration, is a 2-dimensional vector field with two saddle singularities A and B, connected by a heteroclinic cycle (i.e., a branch of the stable manifold of A coincides with a branch of the unstable manifold of B, and vice versa).

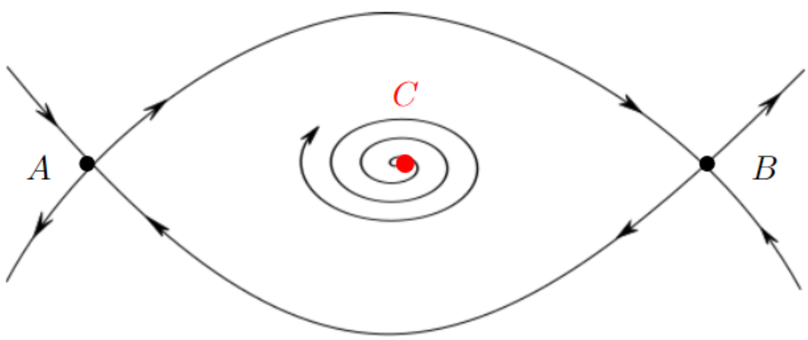

Figure 1.1: Bowen eye.

The union of these invariant manifolds and the singularities A and B (we call this set Γ) bounds a disk Δ. Inside this disk, the vector field has a repelling singularity C. In this way, backward orbits of point in $\mathrm{int}(\Delta)$ converge to C while forward orbits accumulate on Γ. However, forward orbits of points in $\mathrm{int}(\Delta)$ oscillate between the singularities A and B. It was noted by Bowen that if $x \in \mathrm{int}(\Delta) \setminus \{C\}$ and if φ is a continuous function on Δ with $\varphi(A) \neq \varphi(B)$, then the time averages in (1-1) along the forward orbit of x does not converge as $n \to \infty$. A consequence of this oscillating behavior is that for Lebesgue almost every (as follows a.e.) point, the system has historical behavior. The construction of the Bowen eye and the results in [22] had very important impact in the study of predictable behavior of dynamical systems. Extensive research has been conducted to explore the existence of systems with historical behavior from different perspectives. Here we analyze the predictable and historical for skew products. We will now proceed to explain.

We study *skew product* maps $F\colon \Sigma^+ \times I \to \Sigma^+ \times I$ of the form

$$F(\xi, y) \stackrel{\text{def}}{=} (\sigma(\xi), f_\xi(y)). \tag{1-2}$$

Here, $\sigma\colon \Sigma^+ \to \Sigma^+$ is the one-sided Bernoulli shift on $(\Sigma^+, \mathbb{P}) = (\mathcal{A}^{\mathbb{N}}, \rho^{\mathbb{N}})$, where $\mathcal{A}$ is an alphabet with finite symbols. Moreover, the fiber maps $f_\xi\colon I \to I$ are measurable functions on the interval $I = [0,1]$ endowed with the Lebesgue measure λ. We also consider the *reference measure* in $\Sigma^+ \times I$ given by the product probability measure $\mathbb{P} \times \lambda$.

Given any point $(\xi, y) \in \Sigma^+ \times I$, its *forward orbit* under F has the form

$$F^n(\xi, y) \stackrel{\text{def}}{=} (\sigma^n(\xi), f_\xi^n(y)),$$

where

$$f_\xi^n(y) \stackrel{\text{def}}{=} f_{\sigma^{n-1}(\xi)} \circ \cdots \circ f_{\sigma(\xi)} \circ f_\xi(y) \quad \text{for } n \geq 1.$$

Orbits may be studied from both topological and statistical point of views. From a topological perspective, one defines the *ω-limit* of a point (ξ, y) by

$$\omega(\xi, y) \stackrel{\text{def}}{=} \left\{(\bar{\xi}, \bar{y}) \in \Sigma^+ \times I \colon \exists\, n_j \to +\infty \text{ with } F^{n_j}(\xi, y) \to (\bar{\xi}, \bar{y})\right\}.$$

The points in $\omega(\xi, y)$ are called *ω-limit points.* From a statistical perspective, one can associate with the iterates of point (ξ, y) its sequence of *empirical probability measures* defined by

$$\mu_n(\xi, y) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j(\xi, y)}, \quad n \geq 1,$$

where δ_z is the Dirac probability measure supported on the point z. The orbit of a point (ξ, y) may have two types of statistical behaviors concerning the convergence of the sequence of measures $\mu_n(\xi, y)$ in the weak* topology:

- *Predictable behavior*: the sequence $\mu_n(\xi, y)$ converges, that is, there is a probability measure μ on $\Sigma^+ \times I$ such that $\lim_{n\to\infty} \mu_n(\xi, y) = \mu$.
- *Historical behavior*: the sequence $\mu_n(\xi, y)$ does not converge.

The convergence or non-convergence of the *empirical measures* can also be analyzed in an equivalent way from the perspective of *time average*, as presented before.

The *basin of attraction* of an F-invariant measure μ (i.e., $\mu \circ F^{-1} = \mu$) is defined by

$$\mathcal{B}(\mu) \stackrel{\text{def}}{=} \left\{(\xi, y) \in \Sigma^+ \times I : \lim_{n\to\infty} \mu_n(\xi, y) = \mu\right\}.$$

Note that $\mathcal{B}(\mu)$ is an F-invariant set. Moreover, the basins of attraction of two different measures are disjoint. A particularly interesting case with predictable behavior occurs when there is a limit measure whose basin of attraction has positive reference measure. Such measures are called *physical*. Note that a system may have several physical measures. In this case, given two physical measures μ and ν, their basins $\mathcal{B}(\mu)$ and $\mathcal{B}(\nu)$ may be *intermingled*, meaning that for every open set $S \subset \Sigma^+ \times I$, the following holds:

$$(\mathbb{P} \times \lambda)(\mathcal{B}(\mu) \cap S) > 0 \quad \text{and} \quad (\mathbb{P} \times \lambda)(\mathcal{B}(\nu) \cap S) > 0.$$

There are examples of skew products that have more than two physical measures with intermingled basins (see [3, 16]). Also, there are systems having several physical measures. In recent years, numerous works have explored the existence of physical measures in different settings. Observe that when the system has historical behavior almost everywhere, then there are no physical measure. An example of this case is the Bowen eye mentioned before.

Concerning these behaviors, there are the following problems, intentionally formulated imprecisely:

- ***Palis Conjecture* [18]:** Typical dynamical systems have finitely many ergodic physical measures. Moreover, the union of their basins has full Lebesgue measure.
- ***Last Takens Problem* [23]:** Are there persistent classes of dynamical systems that have a set of points with positive Lebesgue measure, where the orbits of these points exhibit historical behavior?

In what follows, we focus on a type of skew products introduced by Kan [13]. For such class, Kan [13] showed that some of them have two physical measures that additionally have intermingled basins. The maps considered by Kan satisfy the following condition

(H1) The fiber maps $f_\xi \colon I \to I$, $\xi \in \Sigma^+$ are order-preserving and satisfy $f_\xi(0) = 0$ and $f_\xi(1) = 1$.

Under this condition, we have the invariant sets

$$A_0 \stackrel{\text{def}}{=} \Sigma^+ \times \{0\} \quad \text{and} \quad A_1 \stackrel{\text{def}}{=} \Sigma^+ \times \{1\},$$

as well as the measures $\mu_0 \stackrel{\text{def}}{=} \mathbb{P}\times\delta_0$ and $\mu_1 \stackrel{\text{def}}{=} \mathbb{P}\times\delta_1$, both of which are supported on these sets. We are interested in analyzing the interaction between these two measures.

To study this interaction, a first step is to determine the "attracting" or "repelling" nature of the sets A_0 and A_1 using their *transverse Lyapunov exponents*. The transverse *Lyapunov exponent* along the set A_i is defined by

$$\mathrm{L}(i) \stackrel{\text{def}}{=} \int \log(f'_\xi(i))\, d\mathbb{P}, \quad i = 0, 1. \tag{1-3}$$

When the Lyapunov exponent $\mathrm{L}(i)$ is negative, the basin $\mathcal{B}(\mu_i)$ has positive probability measure, i.e., μ_i is physical. When $\mathrm{L}(i)$ is positive, the basin $\mathcal{B}(\mu_i)$ has zero probability measure. The case $\mathrm{L}(i) = 0$ is more tricky, and we will also be considering in this dissertation.

When both probability measures are physical, one aims to determine when their basins are intermingled. To address this type of question and to induce some interaction between the basins, some additional properties are required. In the family of examples introduced by Kan, two conditions are imposed to force this interaction:

– Existence of elevators: there are two fixed points $\xi^-, \xi^+ \in \Sigma^+$ such that $f_{\xi^-}(y) < y$ and $f_{\xi^+}(y) > y$ (see Figure 1.2).

– Negative transverse Lyapunov exponents of μ_0 and μ_1.

The existence of elevators is a natural and simple hypothesis that forces the interaction between μ_0 and μ_1. Let us note some parallelism with the Bowen eye. The sets A_0 and A_1 act as the singularities A and B in Figure 1.1 and the fibers take the role of the heteroclinic connection, see Figure 1.2.

In the context above, Kan [13] introduced the first examples of skew products on $\mathbb{S}^1 \times I$, where the measures μ_0 and μ_1 on $\mathbb{S}^1 \times I$ are physical. He also proved that their basins are intermingled.

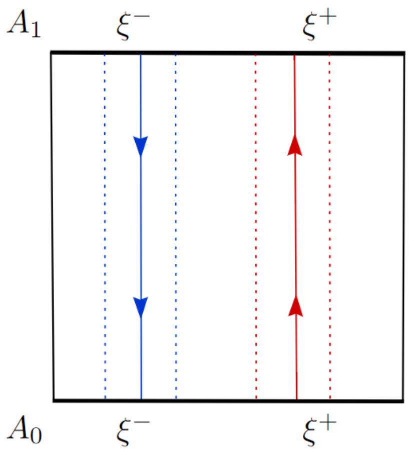

Figure 1.2: Kan's elevators.

So far, the hypotheses on the fiber maps are relatively mild. By adding additional structure to these maps, one can obtain more information about the basins of the measures μ_0 and μ_1. To do this, Bonifant and Milnor [4] consider the *Schwarzian derivative* (see Section (2.3)) of these fiber maps. They proved that under the assumption of negative Schwarzian derivative of fiber maps almost everywhere, when the measures μ_0 and μ_1 are physical, there is a measurable function whose graph splits $\mathbb{S}^1 \times I$ into the basins $\mathcal{B}(\mu_0)$ and $\mathcal{B}(\mu_1)$. They also prove under the existence of elevators that the basins of μ_0 and μ_1 are intermingled. Note that in this case, the fiber maps f_ξ are required to be C^3-diffeomorphisms (otherwise, it is not possible to define the Schwarzian derivative).

In [4], it is also noted the occurrence of two different type of behaviors depending on the sign of the Schwarzian derivative. First, if the Schwarzian derivative of the fiber maps is positive almost everywhere and the probability measures μ_0 and μ_1 have basins of attraction with zero measure, there is a physical measure on $\mathbb{S}^1 \times I$ whose basin has full measure. The second behavior corresponds to the case of a zero Schwarzian derivative of fiber maps almost everywhere and both transverse Lyapunov exponents $\mathrm{L}(i) = 0$, $i = 0, 1$. In this scenario, they replaced $\mathbb{S}^1$ with an infinite product space $\mathcal{A}^{\mathbb{N}}$, where $\mathcal{A}$ is any probability space. They claimed that, in this particular scenario, the skew product F has historical behavior almost everywhere. The proof of this claim involves the use of certain probability laws, which we will analyze later as a tool to obtain historical behavior. Note that in the last two situations, interactions between the basins of attraction of the probability measures μ_0 and μ_1 cannot exist.

This dissertation has two main goals. The first one is to analyze the results of Bonifant and Milnor [4] for skew products. The second goal is to obtain conditions implying that skew products have historical behavior almost everywhere, independently of the Schwarzian derivative. We focus on the class of skew products called one-step (or locally constant). A skew product F as in (1-2) is one-step if that the fiber maps f_ξ only depend on the zero-coordinate of $\xi = (\xi_0, \xi_1, \xi_2, \dots) \in \Sigma^+$, that is, $f_\xi \stackrel{\text{def}}{=} f_{\xi_0}$.

Concerning the first goal, we study different behaviors depending on the sign of the Schwarzian derivative (denoted by Sf_ξ). The following results are obtained:

– Negative derivative (Theorem 3.6): If $Sf_\xi(y) < 0$ for $(\mathbb{P} \times \lambda)$-a.e. point, and both transverse Lyapunov exponents are negative, then the two basins of attraction are intermingled, and the union of the basins has full probability measure.

– Positive derivative (Theorem 4.2): If $Sf_\xi(y) > 0$ for $(\mathbb{P} \times \lambda)$-a.e. point, and both transverse Lyapunov exponents are positive, then there is a physical probability measure whose basin has full probability measure, and therefore there is a unique physical probability measure.

– Zero derivative (Corollary 7.1.1): If $Sf_\xi(y) = 0$ for $(\mathbb{P} \times \lambda)$-a.e. point, and both transverse Lyapunov exponents are zero, then F has historical behavior for $(\mathbb{P} \times \lambda)$-a.e. point.

Concerning the second goal, we consider one-step skew products whose fiber maps satisfy (H1) and the following topological (H2) and statistical (H0) and (H3) conditions:

(H0) For every $y \in I$, the sequence $\{f_\xi^j(y)\}_{j\geq 0}$ of random variables has a trivial tail σ-algebra (see Definition A.10). That is, for every $A \in \mathcal{T}(\{f_\omega^j(y)\}_{j\geq 0})$, it holds that $\mathbb{P}(A) \in \{0, 1\}$.

(H2) For every $y \in (0, 1)$, there exist α, $\beta \in \Sigma^+$ and $k, j \in \mathbb{N}$ such that

$$f_\alpha^k(y) < y < f_\beta^j(y).$$

(H3) There are $y^* \in (0, 1)$ and a non-negative, increasing, non-constant function $\varphi\colon I \to \mathbb{R}$ such that for every $\gamma \in (m, M)$, where $M = \max \varphi$ and $m = \min \varphi$,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\varphi(f_\xi^j(y^*))) = 1, \\ \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\varphi(f_\xi^j(y^*))) = 1. \end{aligned} \tag{1-4}$$

for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$. Here $\mathbb{1}_A$ denotes the indicator map of a set A (i.e., $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise).

Note that condition (H0) can be obtained when the sequence of random variables $\{f^j_\xi(y)\}$ is conjugated to a random walk in $\mathbb{R}$ (see Remark 5.2). On the other hand, condition (H2) is a weak version of the elevators mentioned above. Moreover, condition (H3) is satisfied if the skew product follows the arc-sine law, which we discuss later. We obtain the following result.

Theorem 1.1 *Let F be a one-step skew product as in* (1-2) *whose fiber maps satisfy conditions* (H0)–(H3) *and φ the non-negative, increasing, non-constant map in* (H3). *Then, for every $y \in (0,1)$, it holds*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j_\xi(y)) = \max\varphi \quad \textit{and} \quad \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j_\xi(y)) = \min\varphi,$$

for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$. *In particular, F has historical behavior for* $(\mathbb{P}\times\lambda)$-a.e. *point.*

Lévy [14] and Erdös and Kac [10] introduced the Arc-sine Law. Bonifant and Milnor in [4] realized a connection between the Arc-sine Law and historical behavior. They assert that when the Schwarzian derivative and both Lyapunov exponents are zero, the Arc-sine law is satisfied, and the skew product has historical behavior almost everywhere. However, the proof is incomplete. We analyze this connection and introduce weak versions of the Arc-sine Law, below. We establish a relation between these versions and the statistical condition (H3) (see Proposition 6.6) showing that this property implies that F has historical behavior almost everywhere. A different type of relation between probability laws and historical behavior is established in Crovisier et al. [8], using Brownian motion.

We introduce the following weak versions of the arc-sine law. Let $\psi\colon I \to \mathbb{R}$ be a non-negative, increasing, and non-constant continuous function. Consider the function $\Psi\colon \Sigma^+ \times I \to \mathbb{R}$ defined as $\Psi(\xi, y) \stackrel{\text{def}}{=} \psi(y)$ and define $m \stackrel{\text{def}}{=} \min\psi$ and $M \stackrel{\text{def}}{=} \max\psi$. We introduce the following properties:

- The pair (F, Ψ) satisfies the *skew product arc-sine law* if, for every $\alpha \in (0,1)$ and every $\gamma \in (m, M)$, it simultaneously holds

$$\lim_{n\to\infty} (\mathbb{P}\times\lambda)\left(\left\{(\xi,y)\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\Psi(F^j(\xi,y))) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha},$$
$$\lim_{n\to\infty} (\mathbb{P}\times\lambda)\left(\left\{(\xi,y)\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\Psi(F^j(\xi,y))) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha}.$$

– The pair (F, ψ) satisfies the *fiber weak arc-sine law* if, there is $y^* \in J$ such that, for every $\gamma \in (m, M)$ and every $\alpha \in (0,1)$ it simultaneously holds

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\psi(f_\xi^j(y^*))) < \alpha\right\}\right) < 1, \\
&\limsup_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\psi(f_\xi^j(y^*))) < \alpha\right\}\right) < 1.
\end{aligned} \tag{1-5}$$

We prove that the skew product arc-sine law implies the fiber weak arc-sine law (Proposition 6.5). We obtain the following result, which establishes that any arc-sine law implies historical behavior.

Corollary 1.1.1 *Let F be a one-step skew product as in* (1-2) *whose fiber maps satisfy conditions* (H0)–(H2). *Let $\psi\colon I \to I$ be a non-negative, increasing, non-constant function such that (F, ψ) satisfies the fiber weak arc-sine law. Then, F has historical behavior for* $(\mathbb{P} \times \lambda)$-a.e. *point.*

Finally, building on the ideas of the previous results, we analyze measurable interval functions $f\colon I \to I$ fixing the points 0 and 1. Observe that when the alphabet $\mathcal{A}$ is a singleton, skew products can be interpreted as this measurable interval function. We establish the fiber weak arc-sine law in this context, below.

Let f be a measurable function fixing the points 0 and 1 and $\psi\colon I \to I$ a non-negative, monotone increasing continuous function. Let $m \stackrel{\text{def}}{=} \min \psi$ and $M \stackrel{\text{def}}{=} \max \psi$. The pair (f, ψ) satisfies the *weak arc-sine law* if, for every $\gamma \in (m, M)$ and every $\alpha \in (0, 1)$, it simultaneously holds

$$\begin{aligned}
&\limsup_{n\to\infty} \lambda\left(\left\{y \in I \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\psi(f^j(y))) < \alpha\right\}\right) < 1, \\
&\limsup_{n\to\infty} \lambda\left(\left\{y \in I \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\psi(f^j(y))) < \alpha\right\}\right) < 1.
\end{aligned} \tag{1-6}$$

Recall that a σ-finite measure ν is ergodic if, for every f-invariant set A, we have either $\nu(A) = 0$ or $\nu(I \setminus A) = 0$. We obtain the following result.

Theorem 1.2 *Let f be a measurable function fixing the points* 0 *and* 1 *and admitting a σ-finite ergodic probability measure ν equivalent to λ. Let $\psi\colon I \to \mathbb{R}$ be a non-constant, increasing, non-negative continuous function such that (f, ψ) satisfies the weak arc-sine law. Then, f has historical behavior for* λ-a.e. *point.*

The *Manneville-Pomeau* functions, see Figure 1.3 are a first application of the preceding result. As it established by Thaler in [24, 25, 26], these functions admit a σ-finite ergodic probability measure ν equivalent to λ and satisfy the weak arc-sine law. Consequently, we can apply Theorem 1.2 obtaining that these Manneville-Pomeau functions have historical behavior almost everywhere (Corollary 8.11.1). In a related development, Coates and Luzzatto [6] established the same historical behavior for interval functions f that are similar to the Manneville-Pomeau functions, but may have zero or infinite derivatives at the point of discontinuity.

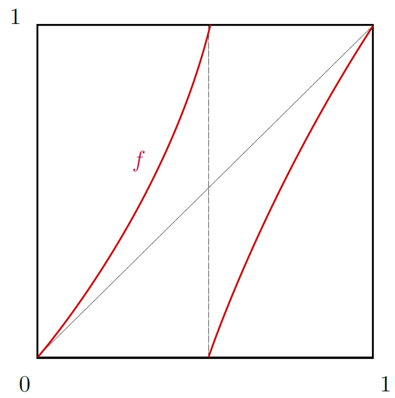

Figure 1.3: Manneville-Pomeu

Organization of the dissertation. This dissertation has two main parts. In the first part (Chapters 3–4), it is dedicated to the predictable behavior. In the second part (Chapters 5–8), it is dedicated to the historical behavior. The Schwarzian derivative plays an essential role in Chapters 3, 4, and 7. The organization is as follows.

In Chapter 2, we review basic notions and introduce notations. In Chapter 3, we analyze the behavior of the skew product when the fiber maps have negative Schwarzian derivatives, specifically examining cases with intermingled basins. In Chapter 4, we explore the existence of a physical probability measure with a basin of attraction of full probability measure when the fiber maps have a positive Schwarzian derivative. In Chapter 5, we study some general properties of functions, specifically the upper and lower limits of the Birkhoff average. We then state Theorem 5.12, stating conditions under which a skew product has historical behavior. In Chapter 6, we introduce a derived definition of the arc-sine law and compare the fiber and skew product arc-sine laws. Additionally, we prove Corollaries 6.8.1 and 6.8.2, connecting historical behavior with the arc-sine law. In Chapter 7, we analyze the behavior of skew products when the fiber maps have a zero Schwarzian derivative. In this case, we prove that the skew product satisfies the arc-sine law and, therefore, has historical behavior. Finally, in Chapter 8, we provide an example of skew products that satisfy the fiber weak arc-sine law independently of the Schwarzian derivative. Moreover, we introduce the arc-sine laws for interval measurable functions. Additionally, we state Theorem 8.7, addressing historical behavior for measurable functions on intervals with two indifferent fixed points. We also introduce the generalized Manneville-Pomeu functions and prove that they exhibit historical behavior.

2
Preliminaries and notations

We now establish some basic definitions and notations

2.1
Skew products systems

Let $(\Sigma^+, \mathbb{P}) = (\mathcal{A}^{\mathbb{N}}, \rho^{\mathbb{N}})$ be a probability product space, where $\mathcal{A}$ is a finite alphabet. Elements in the set Σ^+ are denoted by

$$\xi = (\xi_0, \xi_1, \xi_2, \dots) = (\xi_j)_{j \geq 0},$$

where $\xi_j \in \mathcal{A}$ for every $j \in \mathbb{N}$. Let $\sigma \colon \Sigma^+ \to \Sigma^+$ be the left shift, defined by $\sigma(\xi) = \xi'$, where $\xi'_j = \xi_{j+1}$ for all $j \geq 0$. The shift map is a measure-preserving and ergodic function with respect to $\mathbb{P}$. The space Σ^+ has a product topology generated by cylinders such as C_ξ for Σ^+, defined as follows

$$C_\xi = \left\{\xi' \in \Sigma^+ \colon \xi'_i = \xi_i, \quad \text{for all } i = 0, \dots, n-1\right\}.$$

Let $I = [0,1]$ endowed with the Lebesgue measure λ. Consider a finite family $\mathfrak{F} = \{f_i \colon i \in \mathcal{A}\}$ of measurable function $f_i \colon I \to I$ fixing the points 0 and 1. Now, for every $\xi \in \Sigma^+$, we define $f_\xi \colon I \to I$ by $f_\xi(y) \stackrel{\text{def}}{=} f_{\xi_0}(y)$. Associated with $\mathfrak{F}$ we consider the *one-step (or locally constant) skew product* map $F = F_{\mathfrak{F}}$ given by

$$F \colon \Sigma^+ \times I \to \Sigma^+ \times I, \qquad F(\xi, y) \stackrel{\text{def}}{=} (\sigma(\xi), f_\xi(y)). \tag{2-1}$$

Given a point $(\xi, y) \in \Sigma^+ \times I$ its *orbit* under F is defined by

$$F^j(\xi, y) \stackrel{\text{def}}{=} (\sigma^j(\xi), f^j_\xi(y)), \quad j \geq 0,$$

where

$$f^j_\xi(y) \stackrel{\text{def}}{=} f_{\xi_{j-1}} \circ \dots \circ f_{\xi_0}(y).$$

Definition 2.1 *Given $(\xi, y) \in \Sigma^+ \times I$ its w-limit is defined by*

$$\omega(\xi, y) = \left\{(\bar{\xi}, \bar{y}) \in \Sigma^+ \times I \colon \exists\, n_j \to +\infty \text{ such that } F^{n_j}(\xi, y) \to (\bar{\xi}, \bar{y})\right\}.$$

Definition 2.2 *We say that a compact subset $A \subset \Sigma^+ \times I$ is an* attractor *of F if the set $\{(\xi, y)\colon \omega(\xi, y) = A\}$ has positive measure. Define the* basin of attraction *of every subset $A \subset \Sigma^+ \times I$ by the following set*

$$\mathcal{B}(A) \stackrel{\text{def}}{=} \Big\{(\xi, y) \in \Sigma^+ \times I \,:\, \omega(\xi, y) \subset A\Big\}.$$

Consider the boundary subsets of $\Sigma^+ \times I$ given by

$$A_0 \stackrel{\text{def}}{=} \Sigma^+ \times \{0\} \quad \text{and} \quad A_1 \stackrel{\text{def}}{=} \Sigma^+ \times \{1\} \tag{2-2}$$

and denote by $\mathcal{B}_0$ and $\mathcal{B}_1$ their basins, respectively.

2.2 Lyapunov exponents

Consider the skew product F defined as in (2-1), with f_ξ being C^1-diffeomorphisms. The transverse *Lyapunov exponent* of a point $\xi \in \Sigma^+$ along the boundary A_i is defined by

$$\mathrm{L}_i(\xi) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log \left| D_y\, f_\xi^n(i) \right|, \tag{2-3}$$

whenever this limit exists. By the chain rule, the equation (2-3) can be written as follows:

$$\mathrm{L}_i(\xi) = \lim_{n\to\infty} \frac{1}{n} \log \Big(f'_{\xi_0}(i) f'_{\xi_1}(i) \dots f'_{\xi_{n-1}}(i) \Big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log(f'_{\xi_j}(i)).$$

Note that $\sigma(\xi)$ is a measure-preserving and ergodic function with respect to $\mathbb{P}$. Then, by the Birkhoff ergodic Theorem A.17, for $\mathbb{P}$-almost every $\xi \in \Sigma^+$, the transverse Lyapunov exponent can also be written as:

$$\mathrm{L}(i) \stackrel{\text{def}}{=} \int \log(f'_\xi(i))\, d\mathbb{P}. \tag{2-4}$$

Lemma 2.3 *Let $F\colon \Sigma^+ \times I \to \Sigma^+ \times I$ be a skew product as defined in* (2-1)*. We have that:*

(i) *If* $\mathrm{L}(i) < 0$*, then the basin $\mathcal{B}_i$ has positive measure,*

(ii) *If* $\mathrm{L}(i) > 0$*, then the basin $\mathcal{B}_i$ has zero measure.*

Proof We first prove the statement for $L(i) < 0$ for $i = 0, 1$. We prove the case $i = 0$, as the case $i = 1$ is analogous and hence omitted. Suppose that $\mathrm{L}(0) < 0$. Since f_ξ is a C^1-diffeomorphism with $f_\xi(0) = 0$, we can consider the Taylor expansion of f_ξ at the point 0, obtaining:

$$f_\xi(y) = f'_\xi(0) y + o(y^2).$$

We choose $K > 0$ such that $f_\xi(y)$ is uniformly bounded, i.e.,

$$f_\xi(y) \leq y(f'_\xi(0) + Ky), \quad \text{for every } \ (\xi, y) \in \Sigma^+ \times I.$$

Given any $\eta > 0$ we have that

$$f_\xi(y) \leq y(f'_\xi(0) + \eta) \quad \text{for every} \quad y < \frac{\eta}{K}. \tag{2-5}$$

Since L(0) < 0, we can choose $\eta > 0$ small enough so that

$$\int \log(f'_\xi(0) + \eta)\, d\mathbb{P} < 0. \tag{2-6}$$

We denote by $a(\xi) = \log(f'_\xi(0) + \eta)$. Given $(\xi, y) \in \Sigma^+ \times I$ write $(\xi_j, y_j) = (\sigma^j(\xi), f_{\xi_j}(y))$. Consider the Birkhoff sums and averages

$$B_n(\xi) \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} a(\xi_j) \quad \text{and} \quad A_n(\xi) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=0}^{n-1} a(\xi_j). \tag{2-7}$$

By the Birkhoff ergodic theorem the averages $A_n(\xi)$ are convergent for $\mathbb{P}$-almost every point. Since $\sigma(\xi)$ is ergodic with respect to $\mathbb{P}$, we have that

$$\lim_{n\to\infty} A_n(\xi) = \int a(\xi)\, d\mathbb{P} < 0, \quad \text{for } \mathbb{P}\text{-almost every } \xi \in \Sigma^+.$$

In particular, it follows that the sum $B_n(\xi)$ converges to negative infinity according to $n \to \infty$. Therefore, the maximum

$$B_{\max}(\xi) = \max_{n\geq 0} B_n(\xi) \tag{2-8}$$

is definite and finite for $\mathbb{P}$-almost every ξ. Moreover, $B_{\max}$ is measurable. Now, suppose that

$$y \leq \frac{\eta}{K}\, e^{-B_{\max}(\xi_0)} \tag{2-9}$$

by induction, we have that

$$y_n \leq \frac{\eta}{K}\, e^{B_n(\xi_0) - B_{\max}(\xi_0)} \leq \frac{\eta}{K}$$

for every $n \in \mathbb{N}$. Since the sum $B_n(\xi)$ converges to $-\infty$, it follows that y_n converges to zero. Therefore, (ξ_0, y) belongs to the basin of attraction $\mathcal{B}_0$.

Since the right-hand side of the inequality (2-9) is a measurable function of ξ_0, defined and strictly positive at $\mathbb{P}$-almost every points, it follows that its integral is strictly positive. The integral is a lower bound for the area of $\mathcal{B}_0$. Thus, $\mathcal{B}_0$ has positive measure. Similarly, it can be shown that $\mathcal{B}_1$ has positive measure.

Now we prove the statement for $L(i) > 0$ for $i = 0, 1$. We also prove only the case $i = 0$, and the case $i = 1$ is omitted. We argue by contradiction. Suppose that $\mathcal{B}_0$ has positive reference measures. Consider the Taylor expansion of f_ξ at the point 0, obtaining:

$$f_\xi(y) = f'_\xi(0)y + o(y^2).$$

Since f_ξ preserves order for every $\xi \in \Sigma^+$, $f'_\xi(0)$ is strictly positive. We choose $K \geq 0$ such that $f_\xi(y)$ is uniformly bounded, i.e.,

$$f_\xi(y) \geq y(f'_\xi(0) - Ky). \tag{2-10}$$

Given $\eta > 0$, we have that

$$f_\xi(y) \geq y(f'_\xi(0) - \eta) \quad \text{for every } y < \frac{\eta}{K}.$$

Since $L(0) > 0$, we can choose $0 < \eta \leq f'_\xi(0)$ so that

$$\int \log(f'_\xi(0) - \eta)\, d\mathbb{P} > 0.$$

We denote $a(\xi) \stackrel{\text{def}}{=} \log(f'_\xi(0) - \eta)$. Then, for a set of points (ξ, y) of positive measure, we could find orbits $(\xi_j, y_j) = (\sigma^j(\xi), f_{\xi_j}(y))$. Since σ is ergodic with respect to $\mathbb{P}$, we have that the Birkhoff average in (2-7)

$$\lim_{n\to\infty} A_n(\xi) = \int a(\xi) d\mathbb{P} > 0, \quad \text{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+.$$

Therefore, the Birkhoff sums $B_n(\xi)$ as in (2-7), converge to positive infinity as $n \to \infty$. We can define

$$B_{\min}(\xi) \stackrel{\text{def}}{=} \min_{n\geq 0} B_n(\xi),$$

that is finite for $\mathbb{P}$-a.e. point and is a measurable function. Suppose that

$$\frac{\eta}{K} e^{-B_{\min}(\xi)} \leq y_n.$$

By induction, we have that

$$\frac{\eta}{K} e^{B_n(\xi) - B_{\min}(\xi)} \leq y_n, \quad \text{for every } n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$, we find that $y_n \to \infty$. This contradicts the fact that $y_n < \frac{\eta}{K}$ in (2-10). Therefore, $\mathcal{B}_0$ has measure zero, thus concluding the proof of the lemma. ■

2.3 Schwarzian Derivative

Our goal is to analyze the variation in behavior when the Schwarzian derivative of the fiber map f_ξ has different signs. This relationship was given by Bonifant and Milnor in [4]. The Schwarzian derivative of every C^3-diffeomorphism f is defined by

$$Sf(y) \stackrel{\text{def}}{=} \frac{f'''(y)}{f'(y)} - \frac{3}{2}\left(\frac{f''(y)}{f'(y)}\right)^2.$$

The following lemma establishes the relations between the Schwarzian derivative and the transverse Lyapunov exponents. From now on, whenever we refer to the Schwarzian derivative, the function f_ξ is assumed to be a C^3-diffeomorphism.

Lemma 2.4 *Let $F\colon \Sigma^+ \times I \to \Sigma^+ \times I$ be a skew product as defined in* (2-1)*. We have that:*

(i) *If $Sf_\xi(y) < 0$ for $(\mathbb{P} \times \lambda)$-almost every point (ξ, y), then $\mathrm{L}(0) + \mathrm{L}(1) < 0$.*

(ii) *If $Sf_\xi(y) > 0$ for $(\mathbb{P} \times \lambda)$-almost every point (ξ, y), then $\mathrm{L}(0) + \mathrm{L}(1) > 0$.*

(iii) *If $Sf_\xi(y) = 0$ for $(\mathbb{P} \times \lambda)$-almost every point (ξ, y), then $\mathrm{L}(0) + \mathrm{L}(1) = 0$.*

Proof We prove only item (i), the other items are similar and the proof is hence omitted. Since $Sf_\xi(y) < 0$ for $(\mathbb{P} \times \lambda)$-almost every point (ξ, y), by Lemma B.6 we have that $f'_\xi(0)f'_\xi(1) < 1$. Therefore,

$$\log(f'_\xi(0)f'_\xi(1)) = \log(f'_\xi(0)) + \log(f'_\xi(1)) < 0.$$

Integrating this inequality, we obtain

$$\int_\xi \log(f'_\xi(0))\, d\mathbb{P} + \int_\xi \log(f'_\xi(1))\, d\mathbb{P} < 0.$$

Hence, we have $\mathrm{L}(0) + \mathrm{L}(1) < 0$, proving item (i). ■

Remark 2.5 *Therefore, by Lemma 2.3, either $\mathrm{L}(1) < 0$ or $\mathrm{L}(0) < 0$ it follows that the basin $\mathcal{B}_0$ or $\mathcal{B}_1$ has positive measure. If both exponents may be negative simultaneously, both basins have positive measures.*

3
Negative Schwarzian derivative: intermingled basins

In this chapter, we analyze the intermingled basins of skew products (see (2-1)) of the form:

$$F\colon \Sigma^+ \times I \to \Sigma^+ \times I, \quad F(\xi, y) \stackrel{\text{def}}{=} (\sigma(\xi), f_\xi(y)), \tag{3-1}$$

where $f_\xi\colon I \to I$ are order-preserving C^3-diffeomorphisms that satisfy

$$f_\xi(0) = 0 \quad \text{and} \quad f_\xi(1) = 1.$$

We analyze under which conditions the skew products have *intermingled basins*. The intermingled property was introduced by Alexander et al. [1] as follows:

Definition 3.1 *Let C and D be attractor sets in $\Sigma^+ \times I$ with basins of attraction $\mathcal{B}(C)$ and $\mathcal{B}(D)$, respectively. We say that these basins are* intermingled *if, for every open set $S \subset \Sigma^+ \times I$, we have*

$$(\mathbb{P} \times \lambda)(S \cap \mathcal{B}(C)) > 0 \quad \textit{and} \quad (\mathbb{P} \times \lambda)(S \cap \mathcal{B}(D)) > 0.$$

where $\mathbb{P} \times \lambda$ is the reference measure in $\Sigma^+ \times I$.

The chapter is organized as follows. First, in Section 3.1, we analyze the general properties of skew products whose fiber maps have a negative Schwarzian derivative. In Section 3.2, we investigate under which conditions the skew products exhibit the property of intermingled basins.

3.1
Negative Schwarzian derivative

In this section, we study the skew product F as in (3-1) whose fiber maps have a negative Schwarzian derivative. Let $A_0 \stackrel{\text{def}}{=} \Sigma^+ \times \{0\}$ and $A_1 \stackrel{\text{def}}{=} \Sigma^+ \times \{1\}$ and $\mathcal{B}_0$ and $\mathcal{B}_1$ be the basins of attraction, respectively. The following theorem claims that if the Schwarzian derivative of the fiber maps of the skew product F is almost surely negative, then there is a measurable function whose graph splits $\Sigma^+ \times I$ into these basins.

Theorem 3.2 *Let $F\colon \Sigma^+ \times I \to \Sigma^+ \times I$ be a skew product as in* (3-1)*, $\mathcal{B}_0$ and $\mathcal{B}_1$ be the basins of attraction, and* $\mathrm{L}(0)$ *and* $\mathrm{L}(1)$ *be the Lyapunov exponents, respectively as in* (2-4)*. Suppose that:*

- *$Sf_\xi(y) < 0$ for $(\mathbb{P} \times \lambda)$-a.e. $(\xi, y) \in \Sigma^+ \times I$, and*
- $\mathrm{L}(0) < 0$ *and* $\mathrm{L}(1) < 0$.

Then, there is a measurable function $\gamma\colon \Sigma^+ \to I$ such that for $\mathbb{P}$-a.e. *point, it holds*

$$(\xi, y) \in \mathcal{B}_0 \quad \textit{if} \quad y < \gamma(\xi) \quad \textit{and} \quad (\xi, y) \in \mathcal{B}_1 \quad \textit{if} \quad y > \gamma(\xi).$$

In particular, the union $\mathcal{B}_0 \cup \mathcal{B}_1$ has full $(\mathbb{P} \times \lambda)$ *measure.*

As by Lemma 2.3, both basins have positive measure. Then, the previous theorem justifies calling the sets A_0 and A_1 attractors.

Proof of Theorem 3.2. Consider the following numbers for each $\xi \in \Sigma^+$:

$$\gamma_0(\xi) = \sup \left\{ y \in I\colon F^j(\xi, y) \to A_0 \right\},$$
$$\gamma_1(\xi) = \inf \left\{ y \in I\colon F^j(\xi, y) \to A_1 \right\}.$$

Note that these graphs are invariant, $\gamma_i(\sigma(\xi)) = f_\xi(\gamma_i(\xi))$, $i = 0, 1$.

Since the functions f_ξ preserve the orientation, by definition, we have that

$$0 < \gamma_0(\xi) \leq \gamma_1(\xi) < 1.$$

In particular, if $\gamma_0(\xi) < y < \gamma_1(\xi)$, then $F^j(\xi, y)$ does not converge to any of the boundaries as $j \to \infty$. Thus

$$(\mathbb{P} \times \lambda)(\mathcal{B}_0) = \int \gamma_0(\xi)\, d\mathbb{P}.$$

Since, by hypothesis, $\mathrm{L}(0) < 0$ by Lemma 2.3 we have that $\mathcal{B}_0$ has positive measure, it follows that

$$\mathbb{P}(\Xi_0) > 0, \qquad \Xi_0 \stackrel{\text{def}}{=} \{\xi \in \Sigma^+ \colon \gamma_0(\xi) > 0\}.$$

The invariance $\gamma_0(\sigma(\xi)) = f_\xi(\gamma_0(\xi))$ implies that the set Ξ_0 is σ-invariant. As σ is ergodic with respect to $\mathbb{P}$, one gets that Ξ_0 has a full measure.

Similarly, we define the set

$$\Xi_1 \stackrel{\text{def}}{=} \{\xi \in \Sigma^+ \colon \gamma_1(\xi) < 1\}$$

and prove that $\mathbb{P}(\Xi_1) = 1$.

Lemma 3.3 $\gamma_0(\xi) = \gamma_1(\xi)$ *for* $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.

Proof Suppose, by contradiction, that there is a subset of $\Upsilon \subset \Sigma^+$ of positive measure such that $\gamma_0(\xi) < \gamma_1(\xi)$ for every $\xi \in \Upsilon$. Define the function

$$r(\xi) \stackrel{\text{def}}{=} \rho(0, \gamma_0(\xi), \gamma_1(\xi), 1),$$

where ρ denotes the cross ratio of four points, see Definition B.4. Since $Sf_\xi < 0$ for $(\mathbb{P} \times \lambda)$-a.e. $(\xi, y) \in \Sigma^+ \times I$, by Lemma B.5, the map f_ξ "increases the cross-ratio" ρ and therefore for every $\xi \in \Upsilon$ it holds

$$\begin{aligned} r(\xi) &= \rho(0, \gamma_0(\xi), \gamma_1(\xi), 1) \\ &< \rho(f_\xi(0), f_\xi(\gamma_0(\xi)), f_\xi(\gamma_1(\xi)), f_\xi(1)) \\ &= \rho(0, \gamma_0(\sigma(\xi)), \gamma_1(\sigma(\xi)), 1) = r(\sigma(\xi)). \end{aligned}$$

Integrating over Σ^+, we obtain

$$\int \frac{1}{r(\sigma(\xi))}\, d\mathbb{P} < \int \frac{1}{r(\xi)}\, d\mathbb{P}.$$

This contradicts the fact that the measure $\mathbb{P}$ is σ-invariant. This contradiction implies that $\gamma_0(\xi) = \gamma_1(\xi)$ in $\mathbb{P}$-a.e., proving the lemma. ■

We define $\mathbb{P}$-a.e. the map $\gamma(\xi)$ by the common value $\gamma_0(\xi) = \gamma_1(\xi)$. By construction, the measurable function $\gamma\colon \Sigma^+ \to \mathbb{R}$ has a graph splitting $\Sigma^+ \times I$ into the two basins, as in the theorem. ■

Remark 3.4 *Theorem 3.2 also holds in a more general case: Let* $(X, \mathbb{P})$ *be a standard probability space, and consider the skew product*

$$F\colon X \times I \to X \times I, \quad F(\xi, y) \stackrel{\text{def}}{=} (E(\xi), f_\xi(y)),$$

where $E\colon X \to X$ *is a measure-preserving ergodic map with respect to* $\mathbb{P}$*, and* $f_\xi\colon I \to I$ *are order preserving* C^3 *diffeomorphisms fixing the points 0 and 1.*

3.2
Intermingled basins

Following Kan [13], we study under which conditions skew products have *intermingled* basins. We consider the following skew product

$$F_k\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I, \qquad F_k(\xi, y) \stackrel{\text{def}}{=} (E(\xi), f_\xi(y)), \quad k \geq 2, \tag{3-2}$$

where $E(\xi) = k\xi$ and $f_\xi\colon I \to I$ are order preserving C^3-diffeomorphisms fixing the points 0 and 1. Observe that, by Remark 3.4, this skew product

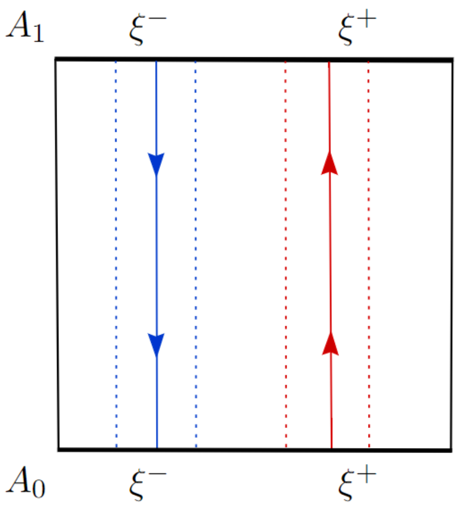

Figure 3.1: Dinamic in (i) and (ii).

satisfies Theorem 3.2. We also have that when considering the alphabet $\mathcal{A} = \{0, \dots, k-1\}$, the system (Σ^+, σ) is ergodically equivalent to the system $(\mathbb{S}^1, E)$. That is, there is a homeomorphism

$$h\colon \mathbb{S}^1 \to \Sigma^+ \quad \text{such that} \quad \sigma \circ h = h \circ E.$$

Then, the ergodic properties of F_k are extended to the skew products F as in (3-1). With a slight abuse of notation, we let

$$A_0 = \mathbb{S}^1 \times \{0\} \quad \text{and} \quad A_1 = \mathbb{S}^1 \times \{1\} \tag{3-3}$$

and analyze the coexistence of the attractors A_0 and A_1 on the set $\mathbb{S}^1 \times I$. Remark 3.4 allows considering their basins by $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively.

We assume the following conditions (see Figure 3.1). There are two fixed points ξ^-, ξ^+ of E such that:

(i) $f_\xi(y) < y$ for every $0 < y < 1$ and every ξ in a neighborhood N^- of ξ^-,

(ii) $f_\xi(y) > y$ for every $0 < y < 1$ and every ξ in a neighborhood of N^+ of ξ^+.

Remark 3.5 *With hypotheses (i) and (ii), the segment* $\{\xi^-\} \times [0, 1)$ *is contained* $\mathcal{B}_0$ *and the segment* $\{\xi^+\} \times (0, 1]$ *is contained* $\mathcal{B}_1$*.*

Theorem 3.6 *Let* $F_k\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ *be a skew product as in* (3-2)*, and* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *be the basins of attraction as in* (3-3)*. Suppose that* F_k *satisfies conditions* (i)–(ii)*,* $Sf_\xi(y) < 0$ *for* $(\mathbb{P} \times \lambda)$-a.e. *point, and both basins* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *have positive measure. Then the two basins of attraction are intermingled.*

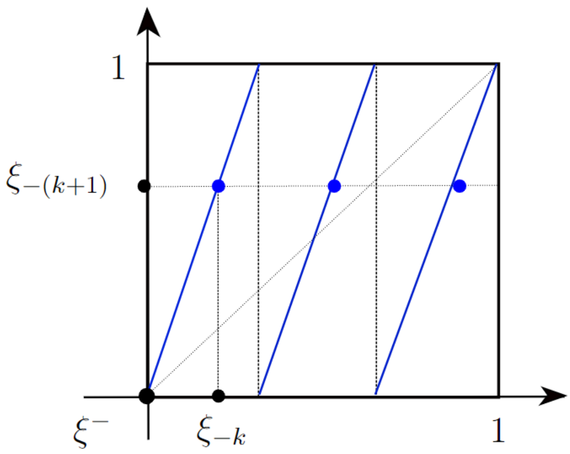

Figure 3.2: Choice of k-preiterate.

Proof Define the measures in the cylinder $\mathbb{S}^1 \times I$ as

$$\mu_i(S) \stackrel{\text{def}}{=} (\mathbb{P} \times \lambda)\,(\mathcal{B}_i \cap S) \quad \text{for every measurable set } S,\ i = 0, 1.$$

To prove that the basins are intermingled, it suffices to show that $\operatorname{supp}(\mu_i)$ covers the entire space $\mathbb{S}^1 \times I$. We will prove the assertion for μ_0; the proof for μ_1 is similar, and hence omitted.

Lemma 3.7 *The set A_1 is contained in* $\operatorname{supp}(\mu_o)$.

Proof Choose any point $(\xi, y_0) \in \operatorname{supp}(\mu_0)$ with $0 < y_0 < 1$ and observe that such points exists since $\mathcal{B}_0$ has positive measure. Now, we inductively define the pre-orbit

$$F^{-j}(\xi, y) \stackrel{\text{def}}{=} (\xi_{-j}, y_{-j}), \text{ where } j \geq 0,$$

where each $\xi_{-(j+1)}$ is the pre-orbit of ξ_{-j} closest to the fixed point ξ^- (See Figure 3.2). This implies that,

$$\lim_{j\to\infty} \xi_{-j} = \xi^-.$$

In particular, ξ_j^- belongs to the neighborhood N^- of ξ^- for sufficiently large j. Thus, by the hypothesis (i), we have that

$$y_0 < y_{-1} < \ldots < y_{-j} < y_{-(j+1)} < \ldots < 1.$$

As a consequence,

$$\lim_{j\to\infty} F^{-j}(\xi_0, y_0) = (\xi^-, 1).$$

Since $\operatorname{supp}(\mu_0)$ is closed and F-invariant, it follows that $(\xi^-, 1) \in \operatorname{supp}(\mu_0)$. Moreover, as the pre-orbit of $(\xi^-, 1)$ is dense in A_1, it follows that A_1 is

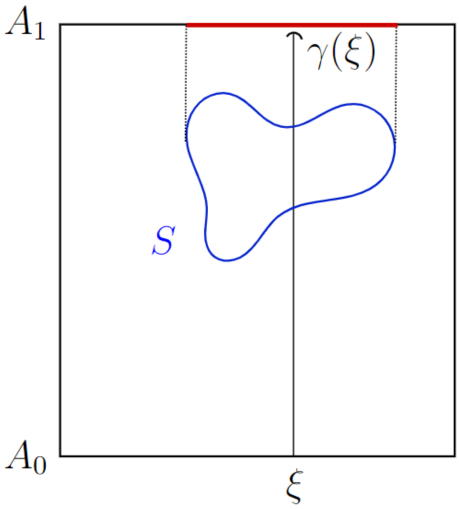

Figure 3.3: Projection of S.

contained in $\operatorname{supp}(\mu_0)$, proving the lemma. ■

By Remark 3.4, there exists a measurable function $\gamma\colon \mathbb{S}^1 \to \mathbb{R}$ whose graph splits $\mathbb{S}^1 \times I$ into the basins $\mathcal{B}_0$ and $\mathcal{B}_1$. Then, for every $S \subset \operatorname{supp}(\mu_0)$, there exists a segment of the form $\{\xi\} \times [0, \gamma(\xi))$ in $\mathcal{B}_0$ that intersects S with $(\mathbb{P} \times \lambda)$-positive measure (see Figure 3.3). By Lemma 3.7, we have that $\gamma(\xi)$ is contained in A_1 for every ξ. According to Remark 3.5 (see Figure 3.3), $\operatorname{supp}(\mu_0)$ covers the entire cylinder. Consequently, for every open subset S of $\Sigma^+ \times I$, we have

$$(\mathbb{P} \times \lambda)(S \cap \mathcal{B}_1) > 0 \quad \text{and} \quad (\mathbb{P} \times \lambda)(S \cap \mathcal{B}_2) > 0.$$

This concludes the proof of the theorem. ■

The following example was given by Kan [13], which presents a C^3-diffeomorphims in the cylinder with two intermingled basins.

Example 1 *For every $\varepsilon > 0$, the skew product $F_\epsilon\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ defined by*

$$F_{k,\epsilon}(\xi, y) = (k\xi, y + ay\varepsilon \cos(2\pi\xi)(1-y)) \quad \textit{when} \quad k \geq 3.$$

Consider the following conditions:

(a) *If $|a| < 1$, then f_ξ has two fixed points $f_\xi(0) = 0$ and $f_\xi(1) = 1$;*

(b) *If $a \neq 0$ we have that $Sf_\xi(y) < 0$ for $(\mathbb{P} \times \lambda)$-a.e. point $(\xi, y) \in \Sigma^+ \times I$; and*

(c) *If we take $\xi^+ = 0$, and choose ξ^- a fixed point in $(\frac{1}{3}, \frac{2}{3})$. For example,*

$$\xi^- = \begin{cases} \frac{1}{2} & \textit{if} \quad k \textit{ is odd,} \\ \frac{k}{2k-2} & \textit{if} \quad k \geq 4 \textit{ is even,} \end{cases}$$

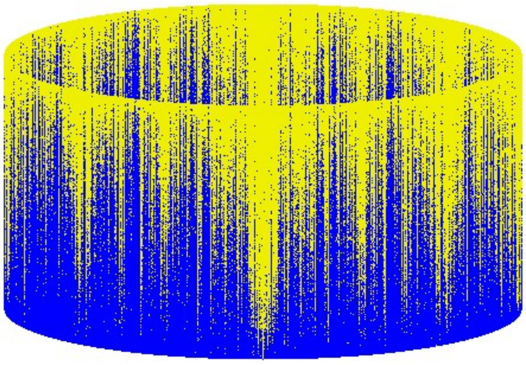

Figure 3.4: Example of intermingled basins

then the application F satisfies hypothesis (i) *and* (ii).

It follows from conditions (a), (b), *and* (c) *by Theorem 3.6 that F has its two basins of attraction intermingled, as can be seen in Figure 3.4.*

4
Positive Schwarzian derivative: physical measures

We continue to study skew products

$$F\colon \Sigma^+ \times I \to \Sigma^+ \times I, \quad F(\xi,y) = (\sigma(\xi), f_\xi(y)),$$

as in (3-1). Let ν be a measure on $\Sigma^+ \times I$, define the basins of attraction of this measure by

$$\mathcal{B}(\nu) \stackrel{\text{def}}{=} \left\{ (\xi,y) \in \Sigma^+ \times I\colon \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j(\xi,y))} = \nu \quad \text{in the weak}^* \text{ topology} \right\}.$$

We analyze under which conditions they skew products F have a physical measure. This concept was introduced as follows:

Definition 4.1 *An F-invariant measure ν is a* physical measure *in $\Sigma^+ \times I$ if the basin $\mathcal{B}(\nu)$ has $(\mathbb{P}\times\lambda)$-positive measure.*

The following theorem claims that if the Schwarzian derivatives of the fiber maps of the skew product F are almost surely positive, then the skew product has a physical measure. This theorem is the main result of this chapter.

Theorem 4.2 *Let $F\colon \Sigma^+ \times I \to \Sigma^+ \times I$ be a skew product as in* (2-1). *Suppose that $Sf_\xi(y) > 0$ for $(\mathbb{P}\times\lambda)$-a.e. $(\xi,y) \in \Sigma^+ \times I$ and* $\mathrm{L}(0) > 0$ *and* $\mathrm{L}(1) > 0$. *Then, F has a physical measure whose basin has full measure.*

Observe that as both Lyapunov exponents are positive, by Lemma 2.4, the basins $\mathcal{B}_0$ and $\mathcal{B}_1$ have zero measure. Consequently, the previous theorem justifies calling the sets $A_0 \stackrel{\text{def}}{=} \Sigma^+ \times \{0\}$ and $A_1 \stackrel{\text{def}}{=} \Sigma^+ \times \{1\}$ as repelling.

Proof of Theorem 4.2. To prove this theorem, we find an appropriate natural extension of F having a physical measure. We denote this extension by $\widetilde{F}$. Thereafter, we extend the properties of this new skew product to the initial skew product F.

Recalling that $(\Sigma^+, \mathbb{P}) \stackrel{\text{def}}{=} (\mathcal{A}^{\mathbb{N}}, \rho^{\mathbb{N}})$ where $\mathcal{A}$ is a finite set. We begin by considering the space of bi-sequences $(\Sigma, \widetilde{\mathbb{P}}) \stackrel{\text{def}}{=} (\mathcal{A}^{\mathbb{Z}}, \rho^{\mathbb{Z}})$ such that there is a projection

$$\pi\colon \Sigma \to \Sigma^+, \quad \pi(\varpi) = \xi.$$

Now, we define the skew product $\widetilde{F}\colon \Sigma \times I \to \Sigma \times I$ as follows:

$$\widetilde{F}(\varpi, y) \stackrel{\text{def}}{=} (\sigma(\varpi), f_{\varpi}(y)), \tag{4-1}$$

where $f_{\varpi}\colon I \to I$ is a C^3-diffeomorphism with $f_{\varpi}(0) = 0$ and $f_{\varpi}(1) = 1$ such that $f_{\varpi}(y) = f_{\xi}(y)$, where $\pi(\varpi) = \xi$. Note that also there exists a projection

$$\Pi\colon \Sigma \times I \to \Sigma^+ \times I \quad \text{defined by} \quad \Pi(\varpi, y) \stackrel{\text{def}}{=} (\pi(\varpi), y) = (\xi, y), \tag{4-2}$$

such that

$$F \circ \Pi = \Pi \circ \widetilde{F}.$$

Remark 4.3 *Since $Sf_{\xi}(y) > 0$ for $(\mathbb{P} \times \lambda)$-a.e. point, it follows that the extension $\widetilde{F}$ also satisfies $Sf_{\varpi}(y) > 0$ for $(\widetilde{\mathbb{P}} \times \lambda)$-*a.e. *point.*

Proposition 4.4 *Let $\widetilde{F}\colon \Sigma\times I \to \Sigma\times I$ be a skew product as in* (4-1). *Suppose that $Sf_{\varpi}(y) > 0$ for $(\widetilde{\mathbb{P}} \times \lambda)$-*a.e. *$(\varpi, y) \in \Sigma \times I$ and* $\mathrm{L}(0) > 0$ *and* $\mathrm{L}(1) > 0$. *Then, there is a measurable function $\gamma\colon \Sigma \to I$ such that:*

- $\gamma(\sigma(\varpi)) = f_{\varpi}(\gamma(\varpi))$ *for* $\widetilde{\mathbb{P}}$-a.e. $\varpi \in \Sigma$,
- *the graph of γ is $\widetilde{\mathbb{P}}$-a.e $\widetilde{F}$-invariant set, i.e., we have*

$$\widetilde{F}(\varpi, \gamma(\varpi)) = (\sigma(\varpi), \gamma(\sigma(\varpi))) \quad \textit{for } \widetilde{\mathbb{P}}\text{-a.e. } \varpi \in \Sigma.$$

Proof As $Sf_{\varpi}(y) > 0$ for $(\widetilde{\mathbb{P}} \times \lambda)$-a.e. point, by Proposition B.1, we have $Sf_{\varpi}^{-1}(y) < 0$ for $(\widetilde{\mathbb{P}} \times \lambda)$-a.e. $(\varpi, y) \in \Sigma \times I$. Moreover, since $\mathrm{L}(0) > 0$ and $\mathrm{L}(1) > 0$ for $\widetilde{F}$, we have $\mathrm{L}(0) < 0$ and $\mathrm{L}(1) < 0$ for $\widetilde{F}^{-1}$. Now, according to Observation 3.4 of Theorem 3.2, applied to the map $\widetilde{F}^{-1}$, there exists a function $\gamma\colon \Sigma \to \mathbb{R}$ such that $\gamma(\sigma(\varpi)) = f\varpi(\gamma(\varpi))$ and

$$\begin{aligned}\widetilde{F}(\varpi, \gamma(\varpi)) &= (\sigma(\varpi), f_{\varpi}(\gamma(\varpi))) \\ &= (\sigma(\varpi), \gamma(\sigma(\varpi))),\end{aligned}$$

for $\widetilde{\mathbb{P}}$-a.e. $\varpi \in \Sigma$. Therefore the graph of γ is $\widetilde{\mathbb{P}}$-a.e. $\widetilde{F}$-invariant set. ■

As by Proposition 4.4, $\gamma(\varpi) \in (0, 1)$ is well-defined for $\widetilde{\mathbb{P}}$-a.e. $\varpi \in \Sigma$. We define the following function:

$$r(\gamma(\varpi), y) \stackrel{\text{def}}{=} |\log \rho(0, \gamma(\varpi), y, 1)| \geq 0, \tag{4-3}$$

where ρ is the cross ratio (Definition B.4). Note that if $\gamma(\varpi) = y$, then $r(\gamma(\varpi), y) = 0$. This function can be viewed as the distance between $\gamma(\varpi)$ and y in the fiber (see Figure 4.1).

Given a point $(\varpi, y) \in \Sigma \times I$, its orbit under $\widetilde{F}$ is defined by:

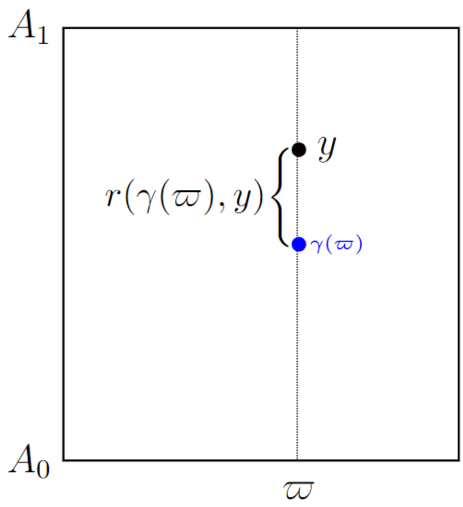

Figure 4.1: Fiber distance

$$\widetilde{F}^j(\varpi, y) \overset{\text{def}}{=} (\sigma^j(\varpi), f^j_\varpi(y)), \quad j \geq 0, \tag{4-4}$$

where

$$f^j_\varpi(y) \overset{\text{def}}{=} f_{\varpi_{j-1}} \circ \cdots \circ f_{\varpi_0}(y).$$

The following proposition claims that for $(\widetilde{\mathbb{P}} \times \lambda)$-a.e. point, the orbit in (4-4) converges appropriately to the graph of γ.

Proposition 4.5 *Let $\widetilde{F}\colon \Sigma \times I \to \Sigma \times I$ be a skew product as in* (4-1)*. Suppose that $Sf_\varpi(y) > 0$ for $(\widetilde{\mathbb{P}} \times \lambda)$-a.e. $(\varpi, y) \in \Sigma \times I$ and $\mathrm{L}(0) > 0$ and $\mathrm{L}(1) > 0$. Then for $\widetilde{\mathbb{P}}$-a.e. $(\varpi, y) \in \Sigma \times I$ it holds*

$$\lim_{n\to\infty} r(\gamma(\sigma^n(\varpi)), f^n_\varpi(y)) = 0.$$

Proof As $Sf_\varpi(y) > 0$ for $(\widetilde{\mathbb{P}} \times \lambda)$-a.e. point, by Lemma B.5, we have that f_ξ decreases the cross ratios for each $\xi \in \Sigma^+$. Then, by (4-3), we have

$$r(f_\varpi(\gamma(\varpi)), f_\varpi(y)) < r(\gamma(\varpi), y) \quad \text{for } \widetilde{\mathbb{P}}\text{-a.e. } (\varpi, y) \in \Sigma \times I. \tag{4-5}$$

By Proposition 4.4, we have $f_\varpi(\gamma(\varpi)) = \gamma(\sigma(\varpi))$, then in (4-5), we can write

$$r(\gamma(\sigma(\varpi)), f_\varpi(y)) < r(\gamma(\varpi), y) \quad \text{for } \widetilde{\mathbb{P}}\text{-a.e. } (\varpi, y) \in \Sigma \times I. \tag{4-6}$$

Given $r_0 > 0$, define the set

$$N(r_0) \overset{\text{def}}{=} \{(\varpi, y) \in \Sigma \times (0,1) : r(\gamma(\varpi), y) < r_0\}.$$

By (4-6) we have $\widetilde{F}(N(r_0)) \subset N(r_0)$. Now, given $0 < r_0 < r_1$, consider the set

$$N(r_1) \setminus N(r_0) = \{(\varpi, y) \in \Sigma \times (0,1) : r_0 < r(\gamma(\varpi), y) < r_1\}$$

and let

$$s(\varpi) \stackrel{\text{def}}{=} \sup\left\{\frac{r(\gamma(\sigma(\varpi)), f_\varpi(y))}{r(\gamma(\varpi), y)} : (\varpi, y) \in N(r_1) \setminus N(r_0)\right\} \tag{4-7}$$

Lemma 4.6 *Let $0 < r_0 < r_1$ it holds $\widetilde{F}^n(N(r_1)) \subset N(r_0)$ for every $n \geq 1$.*

Proof Since $Sf_\varpi(y) > 0$ for $(\mathbb{P} \times \lambda)$-a.e. $(\varpi, y) \in \Sigma \times I$, by (4-6), it follows that $s(\varpi) < 1$ for $\widetilde{\mathbb{P}}$-a.e. $\varpi \in \Sigma$. Thus, $\log(s(\varpi)) < 0$ for $\widetilde{\mathbb{P}}$-a.e. $\varpi \in \Sigma$. As σ is ergodic with respect to $\widetilde{\mathbb{P}}$, and by Birkhoff's Theorem A.17, it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log(s(\sigma^j(\varpi))) = \int \log(s(\varpi))\, d\widetilde{\mathbb{P}} < 0. \tag{4-8}$$

Then,

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\prod_{j=0}^{n-1} s(\sigma^j(\varpi))\right) < 0, \quad \text{for } \widetilde{\mathbb{P}}\text{-a.e. } (\varpi, y) \in \Sigma. \tag{4-9}$$

Claim 4.7 *For every $0 < r_0 < r_1$, there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, it holds*

$$\prod_{j=0}^{n-1} s(\sigma^j(\varpi)) < \frac{r_0}{r_1}.$$

Proof Note that by (4-8), $\prod_{j=0}^{n-1} s(\sigma^j(\varpi))$ is convergent. Now, suppose, by contradiction, there is n_0 such that for every $n \geq n_0$ we have that

$$\prod_{j=0}^{n-1} s(\sigma^j(\varpi)) \to a, \quad a > 0. \tag{4-10}$$

This implies that

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\prod_{j=0}^{n-1} s(\sigma^j(\varpi))\right) = 0,$$

this contradicts (4-9). Therefore, there is n_0 such that for every $n \geq n_0$ it holds

$$\prod_{j=0}^{n-1} s(\sigma^j(\varpi)) \to 0,$$

in particular, there is n_0 such that for every $n \geq n_0$ it holds

$$\prod_{j=0}^{n-1} s(\sigma^j(\varpi)) < \frac{r_0}{r_1}, \tag{4-11}$$

ending the proof of claim. ∎

By definition, we have that $r(\gamma(\sigma^n(\varpi)), f_\varpi^n(y)) > 0$ for every $n \geq 0$. Thus, using (4-7), for every $(\varpi, y) \in N(r_1) \setminus N(r_0)$, we obtain

$$\sup\left\{\frac{r(\gamma(\sigma(\varpi)), f_\varpi(y))}{r(\gamma(\varpi), y)} \times \cdots \times \frac{r(\gamma(\sigma^n(\varpi)), f_\varpi^n(y))}{r(\gamma(\sigma^{n-1}(\varpi), f_\varpi^{n-1}(y))}\right\} \leq \prod_{j=0}^{n-1} s(\sigma^j(\varpi))$$

Now, by the telescopic property, we have

$$\frac{r(\gamma(\sigma(\varpi)), f_\varpi(y))}{r(\gamma(\varpi), y)} \times \cdots \times \frac{r(\gamma(\sigma^n(\varpi)), f^n_\varpi(y))}{r(\gamma(\sigma^{n-1}(\varpi), f^{n-1}_\varpi(y))} = \frac{r(\gamma(\sigma^n(\varpi)), f^n_\varpi(y))}{r(\gamma(\varpi), y)}. \tag{4-12}$$

Thus, by Claim 4-11 and (4-12), we obtain

$$\sup\left\{\frac{r(\gamma(\sigma^n(\varpi)), f^n_\varpi(y))}{r(\gamma(\varpi), y)} : (\varpi, y) \in N(r_1) \setminus N(r_0)\right\} < \frac{r_0}{r_1}.$$

This shows that iterating $\widetilde{F}^n(N(r_1)) \subset N(r_0)$ for $n \geq 1$, concluding the proof of the lemma. ■

As $0 < r_0 < r_1$ can be arbitrary, by Lemma 4.6 we have that

$$\lim_{n\to\infty} r(\gamma(\sigma^n(\varpi)), f^n_\varpi(y)) = 0,$$

ending the proof of proposition. ■

Now, given the graph function $\widetilde{\gamma}\colon \Sigma \to \Sigma \times I$ defined by

$$\widetilde{\gamma}(\varpi) \stackrel{\text{def}}{=} (\varpi, \gamma(\varpi)),$$

consider the measure $\widetilde{\nu}$ on $\Sigma \times I$ to be the push-forward of $\widetilde{\mathbb{P}}$ under the map $\widetilde{\gamma}$. That is, for every measurable set $A \subset \Sigma \times I$, it holds

$$\widetilde{\nu}(A) \stackrel{\text{def}}{=} (\widetilde{\gamma}_* \widetilde{\mathbb{P}})(A) = \widetilde{\mathbb{P}}(\widetilde{F}^{-1}(A)).$$

Lemma 4.8 *The measure $\widetilde{\nu}$ is a physical measure for skew product $\widetilde{F}$.*

Proof We first prove that the measure $\widetilde{\nu}$ is $\widetilde{F}$-invariant. Recall that, by Proposition 4.4, $\widetilde{\gamma}$ is $\widetilde{F}$-invariant. Then for every measurable set $A \subset \Sigma \times I$ we have

$$\begin{aligned}\widetilde{\nu}(\widetilde{F}^{-1}(A)) &= (\widetilde{\gamma}_* \widetilde{\mathbb{P}})(\widetilde{F}^{-1}(A)) = \widetilde{\mathbb{P}}(\widetilde{\gamma}^{-1}(\widetilde{F}^{-1}(A))) \\ &= \widetilde{\mathbb{P}}(\widetilde{\gamma}^{-1}(A)) = (\widetilde{\gamma}_* \mathbb{P})(A) = \widetilde{\nu}(A),\end{aligned}$$

concluding that the measure is $\widetilde{F}$-invariant.

Now, by Proposition 4.5, all orbits of the skew product $\widetilde{F}$ converge to the graph of γ. That is, given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \left|\widetilde{F}^n(\varpi, y) - (\sigma^n(\varpi), \gamma(\sigma^n(\varpi)))\right| = \lim_{n\to\infty} r(\gamma(\sigma^n(\varpi)), f^n_\xi(y)) < \varepsilon,$$

for $\widetilde{\mathbb{P}} \times \lambda$-a.e $(\varpi, y) \in \Sigma \times I$. Given any continuous test function $\widetilde{\varphi}\colon \Sigma \times I \to \mathbb{R}$, we have

$$
\begin{aligned}
\left| \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \widetilde{\varphi}\left(\widetilde{F}^j(\varpi, y)\right) - \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \widetilde{\varphi}(\widetilde{\gamma}(\sigma^j(\varpi))) \right| & \\
= \left| \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left(\widetilde{\varphi}\left(\widetilde{F}^j(\varpi, y)\right) - \widetilde{\varphi}\left(\sigma^j(\varpi), \gamma(\sigma^j(\varpi))\right)\right) \right| & \\
\leq \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left| \widetilde{\varphi}\left(\widetilde{F}^j(\varpi, y)\right) - \widetilde{\varphi}\left(\sigma^j(\varpi), \gamma(\sigma^j(\varpi))\right) \right| & \\
< \varepsilon, &
\end{aligned}
\tag{4-13}
$$

for $\widetilde{\mathbb{P}} \times \lambda$-a.e $(\varpi, y) \in \Sigma \times I$. Since $\widetilde{\varphi} \circ \widetilde{\gamma} \colon \Sigma \to \mathbb{R}$ is a continuous function, and σ is ergodic with respect to $\widetilde{\mathbb{P}}$, by the Birkhoff Ergodic Theorem (Theorem A.17), it follows that

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \widetilde{\varphi}(\widetilde{\gamma}(\sigma^j(\varpi))) = \int \widetilde{\varphi} \circ \widetilde{\gamma}(\varpi)\, d\widetilde{\mathbb{P}} = \int \widetilde{\varphi}(\varpi, y)\, d\widetilde{\nu}. \tag{4-14}
$$

Hence, by (4-13) and (4-14), we conclude that

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(\widetilde{F}^j(\varpi, y)\right) = \int \widetilde{\varphi}(\varpi, y)\, d\widetilde{\nu} \quad \text{for } \widetilde{\mathbb{P}} \times \lambda\text{-a.e } (\varpi, y) \in \Sigma \times I.
$$

This completes the proof of the lemma. ■

Now, we extend the existence of this physical measure to the skew product F in Lemma 4.8. Consider the projection $\pi \colon \Sigma \times I \to \Sigma^+ \times I$ in (4-2), and define the measure $\nu \stackrel{\text{def}}{=} \pi_* \widetilde{\nu}$ on $\Sigma^+ \times I$.

Lemma 4.9 *The measure ν is a physical measure for the skew product F.*

Proof We first prove that the measure ν is F-invariant. Recall that, by Lemma 4.8, the measure $\widetilde{\nu}$ is $\widetilde{F}$-invariant. Then, for every measurable subset $A \subset \xi \times I$, we have

$$
\begin{aligned}
\nu(F^{-1}(A)) = (\Pi_* \widetilde{\nu})(F^{-1}(A)) = \widetilde{\nu}(\Pi^{-1}(F^{-1}(A))) = \widetilde{\mathbb{P}}(\widetilde{\gamma}^{-1}(\widetilde{F}^{-1}(\Pi^{-1}(A))) \\
= \mathbb{P}(\widetilde{\gamma}^{-1}(\Pi^{-1}(A))) = \widetilde{\nu}(\Pi^{-1}(A)) = (\pi_* \widetilde{\nu})(A) = \nu(A),
\end{aligned}
$$

proving that ν is F-invariant.

Now, consider a continuous function $\varphi \colon \Sigma^+ \times I \to \mathbb{R}$. Then, we have

$$
\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(F^j(\xi, y)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(F^j(\pi(\varpi, y))\right).
$$

By projection defined in (4-2), we have $F^n \circ \pi = \pi \circ \widetilde{F}^n$ for every $n \geq 0$. In the previous equality, we can write

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(F^j(\xi, y)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(\pi(\widetilde{F}^j(\varpi, y))\right). \tag{4-15}$$

As $\varphi \circ \pi \colon \Sigma \times I \to \mathbb{R}$ is also a continuous function, and by Lemma 4.8, $\widetilde{\nu}$ is a physical measure, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(\pi(F^j(\varpi, y))\right) = \int \varphi \circ \pi(\varpi, y)\, d\widetilde{\nu},$$

for $(\widetilde{\mathbb{P}} \times \lambda)$-a.e. $(\varpi, y) \in \Sigma \times I$. Thus, by the definition of ν, we have

$$\int \varphi \circ \pi(\varpi, y)\, d\widetilde{\nu} = \int \varphi(\xi, y)\, d(\pi_*\widetilde{\nu}) = \int \varphi(\xi, y)\, d\nu. \tag{4-16}$$

Therefore, by (4-15) and (4-16), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(F^j(\xi, y)\right) = \int \varphi(\xi, y)\, d\nu,$$

for $(\mathbb{P} \times \lambda)$-a.e. $(\xi, y) \in \Sigma^+ \times I$, concluding the proof of the lemma. ■

The measure ν is the required physical measure for F, concluding the proof of the theorem. ■

5
Skew products with historical behavior

In this chapter, we analyze the non-statistical behavior of one-step skew products of the form,

$$F\colon \Sigma^+ \times I \to \Sigma^+ \times I, \quad F(\xi, y) \stackrel{\text{def}}{=} (\sigma(\xi), f_\xi(y)), \tag{5-1}$$

where $f_\xi\colon I \to I$ are measurable functions defined as $f_\xi(y) \stackrel{\text{def}}{=} f_{\xi_0}(y)$ on the interval I endowed with the Lebesgue measure λ. As in Chapter 2, $\sigma\colon \Sigma^+ \to \Sigma^+$ is the lateral shift on the Bernoulli probability space $(\Sigma^+, \mathbb{P}) = (\mathcal{A}^{\mathbb{N}}, \rho^N)$, where $\mathcal{A}$ is at most countable alphabet. We analyze under which conditions a one-step skew product has *historical* behavior almost everywhere.

Definition 5.1 *Let $F\colon \Sigma^+ \times I \to \Sigma^+ \times I$ be a skew product as in* (5-1)*. We say that F has* historical behavior almost everywhere *if, for* $(\mathbb{P} \times \lambda)$-a.e. (ξ, y)*, there is a continuous function $\Phi\colon \Sigma^+ \times I \to \mathbb{R}$ such that the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi\left(F^j(\xi, y)\right)$$

does not exist.

The chapter is organized as follows. First, in Section 5.1, we define the functions upper and lower limit of the Birkhoff average, stating their main properties in our setting. In Section 5.2, we prove the main result of this chapter (Theorem 5.12) dealing with the historical behavior of the skew product F.

5.1
Constant Lyapunov functions

In this section, we analyze general properties of the skew product defined as in (5-1). Given a continuous function $\varphi\colon I \to \mathbb{R}$, we define the functions:

$$\begin{aligned} U_\varphi(\xi, y) &\stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\xi^j(y)), \\ L_\varphi(\xi, y) &\stackrel{\text{def}}{=} \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\xi^j(y)). \end{aligned} \tag{5-2}$$

We analyze the behavior of these functions when the skew product F satisfies the following condition:

(H0) For every $y \in I$, the sequence $\{f_\xi^j(y)\}_{j\geq 0}$ of random variables has a trivial tail σ-algebra (see Definition A.10). That is, for every $A \in \mathcal{T}(\{f_\xi^j(y)\}_{j\geq 0})$, it holds that $\mathbb{P}(A) \in \{0,1\}$.

Remark 5.2 *Condition* (H0) *is obtained when the sequence* $\{f_\xi^j(y)\}_{j\geq 0}$ *is conjugated to a random walk in* $\mathbb{R}$*. That is, if there exist homeomorphisms* $h\colon (0,1) \to \mathbb{R}$ *and* $\hat{f}_\xi\colon \mathbb{R} \to \mathbb{R}$ *such that* $\hat{f}_\xi \circ h = h \circ f_\xi$ *satisfying that the random variables* $X_0 = t$ *and* $X_i = \hat{f}_\xi^i(t) - \hat{f}_\xi^{i-1}(t)$ *are independent and identically distributed (hereafter, i.i.d.),* $i \geq 1$.

Proposition 5.3 *Let* F *be a one-step skew product as in* (5-1) *satisfying* (H0). *Then there are functions* $\ell, h\colon I \to I$ *such that for every* $y \in I$ *it holds that*

$$U_\varphi(\xi, y) = \ell(y) \quad \textit{and} \quad L_\varphi(\xi, y) = h(y) \quad \textit{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+.$$

Proof Given a constant $b \in \mathbb{R}$, define the set

$$A(b, y) \stackrel{\text{def}}{=} \{\xi \in \Sigma^+\colon U_\varphi(\xi, y) < b\}. \tag{5-3}$$

For every $y \in I$, we have that the tail σ-algebra $\mathcal{T}(\{f_\xi^j(y)\}_{j\geq 0})$ is trivial. By Lemma A.11, $A(b, y)$ belongs to the tail algebra $\mathcal{T}(\{f_\xi^j(y)\}_{j\geq 0})$. Hence, the probability of $A(b, y)$ is either zero or one.

Now, let

$$\bar{b}(y) = \inf\{b\colon \mathbb{P}(A(b, y)) = 1\}.$$

Claim 5.4 $U_\varphi(\xi, y) = \bar{b}(y)$ *for* $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.

Proof For simplicity, we write $\bar{b} = \bar{b}(y)$, $A(b) = A(b, y)$ and observe that $\mathbb{P}(A(\bar{b})) \in \{0,1\}$. If $\mathbb{P}(A(\bar{b})) = 0$, then $U_\varphi(\xi, y) \geq \bar{b}$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$. Moreover, by the definition of $\bar{b}$, we have that $U_\varphi(\xi, y) < \bar{b} + \frac{1}{n}$ for every $n \geq 1$ and $\mathbb{P}$-a.e. $\xi \in \Sigma^+$. Hence, by taking $n \to \infty$, we find that $U_\varphi(\xi, y) = \bar{b}$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$ proving the claim in this case.

To conclude the proof, we need to analyze also the case when $\mathbb{P}(A(\bar{b})) = 1$. Suppose, by contradiction, that there exists a set $B \subset A(\bar{b})$ with $\mathbb{P}(B) > 0$ such that $U_\varphi(\xi, y) \neq \bar{b}$ for every $\xi \in B$. By the definition of $\bar{b}$, we have that $\mathbb{P}(A(\bar{b} - \frac{1}{n})) = 0$. Consider the sets

$$B_n \stackrel{\text{def}}{=} \bigcup_{n\geq 0} \left(A\left(\bar{b} - \frac{1}{n}\right) \cap B\right),$$

and note that $\mathbb{P}(B_n) = 0$ and therefore $\lim_{n\to\infty} \mathbb{P}(B_n) = 0$. However, the sequence $B_1 \subset B_2 \subset \dots$ is a monotonically increasing sequence such that

$$\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(A(\bar{b}) \cap B) = \mathbb{P}(B) > 0,$$

which leads to a contradiction. Therefore, $U_\varphi(\xi, y) = \bar{b}$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$, proving the claim. ■

Claim 5.4 implies that we can define the map $\ell \colon I \to I$, $\ell(y) = \bar{b}(y)$. By construction, it holds, $U_\varphi(\xi, y) = \ell(y)$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.

Considering the set

$$C(b, y) \stackrel{\text{def}}{=} \{\xi \in \Sigma^+ \colon L_\varphi(\xi, y) > b\}$$

and arguing similarly to the previous claims, we can prove that $L_\varphi(\xi, y) = h(y)$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$, ending the proof of proposition. ■

Proposition 5.5 *Let F be a one-step skew product as in* (5-1) *satisfying* (H0)*. Consider the interval function ℓ and h defined in Proposition 5.3. Then, for every $y \in I$ it holds that*

$$\ell(y) = \ell(f_\xi^k(y)) \quad \textit{and} \quad h(y) = h(f_\xi^k(y)) \quad \textit{for } \mathbb{P}\text{-a.e. } \xi \textit{ and for all } k \geq 0.$$

Proof We only prove the proposition for the function ℓ, the proof for h is similar and hence omitted. By Proposition 5.3, for every $y \in I$ there exists a set Ω_y with $\mathbb{P}(\Omega_y) = 1$, such that $U_\varphi(\xi, y) = \ell(y)$ for every $\xi \in \Omega_y$. Consider the set $\mathcal{A}^n$ of all words of size n. Now, define the set

$$A_n \stackrel{\text{def}}{=} \bigcap_{\xi \in \Sigma^+} \sigma^{-n}\left(\Omega_{f_\xi^n(y)}\right) = \bigcap_{\xi_0, \dots, \xi_{n-1} \in \mathcal{A}^n} \sigma^{-n}\left(\Omega_{f_{\xi_{n-1}} \circ \dots \circ f_{\xi_0}(y)}\right). \tag{5-4}$$

Note that the set $\mathcal{A}^n$ is countable. (as $\mathcal{A}$ is an alphabet at the most countable.). Since $\mathbb{P}(\Omega_{f_\xi^n(y)}) = 1$ for all $n \geq 0$, and $\mathbb{P}$ is σ-invariant, then

$$\mathbb{P}(\sigma^{-n}(\Omega_{f_\xi^n(y)})) = 1 \quad \text{for all } n \geq 0.$$

Now, as the intersection in (5-4) is a countable, by σ-additivity, we have $\mathbb{P}(A_n) = 1$ for every $n \geq 0$. Define the set

$$\Lambda \stackrel{\text{def}}{=} \bigcap_{n \geq 0} A_n.$$

As Λ is a countable intersection of sets with probability one, it follows that $\mathbb{P}(\Lambda) = 1$. This implies that for every word ω of size k such that the cylinder C_ω of ω satisfies $\Lambda \cap C_\xi \neq \emptyset$. Now, choosing any $\xi \in \Lambda \cap C_\omega$, it holds

$$\begin{aligned}
\ell(f_\xi^k(y)) &= U_\varphi(\sigma^k(\xi), f_\xi^k(y)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j_{\sigma^k(\xi)}(f_\xi^k(y)) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=k}^{n-1+k} \varphi(f_\xi^j(y)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \left(\sum_{j=0}^{n-1} \varphi(f_\xi^j(y)) + \sum_{j=0}^{k-1} \varphi(f_\xi^{n+j}(y)) - \sum_{j=0}^{k} \varphi(f_\xi^j(y)) \right) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\xi^j(y)) = U_\varphi(\xi, y) = \ell(y).
\end{aligned}$$

Note that $f_\xi^k(y) = f_{\omega_k} \circ \cdots \circ f_{\omega_1}(y)$. Now, since ω is an arbitrary word, we conclude that ℓ is constant along the random orbit of y for every $\xi \in \Lambda$ and hence for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$. ■

Remark 5.6 *In Propositions 5.3 and 5.5 one can consider any measurable space X instead of I as the fiber space. This substitution is possible because only the measurability of the fiber maps is used.*

Remark 5.7 *The countability assumption for $\mathcal{A}$ is not necessary for Proposition 5.3. This result holds even if $\mathcal{A}$ is an infinite alphabet or any probability space. However, in Proposition 5.5, the countability assumption of $\mathcal{A}$ is crucial for the existence of the set A_n in* (5-4).

5.2 Historical behavior

In this section, we will analyze conditions implying that F has historical behavior. In what follows, we denote by $\mathbb{1}_A$ the indicator map in a set A (i.e., $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise). We consider the following threee conditions:

(H1) For every $\xi \in \Sigma^+$, the measurable functions $f_\xi \colon I \to I$ are order preserving and satisfy

$$f_\xi(0) = 0 \quad \text{and} \quad f_\xi(1) = 1;$$

(H2) For every $y \in (0,1)$, there exist α, $\beta \in \Sigma^+$ and $k, j \in \mathbb{N}$ such that

$$f_\alpha^k(y) < y < f_\beta^j(y);$$

(H3) There is $y^* \in (0,1)$ and a non-negative monotone increasing function $\varphi\colon I \to \mathbb{R}$ such that for every $\gamma \in (m, M)$ where $M = \max \varphi$ and $m = \min \varphi$, it holds

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\varphi(f_\xi^j(y^*))) = 1, \\ \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\varphi(f_\xi^j(y^*))) = 1, \end{aligned} \tag{5-5}$$

for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.

Remark 5.8 *Condition* (H1) *was assumed in the previous chapters. Note that, if for every* $y \in (0,1)$ *the sets*

$$\Big\{\xi \in \Sigma^+ \colon f_\xi(y) < y\Big\} \quad \textit{and} \quad \Big\{\xi \in \Sigma^+ \colon f_\xi(y) > y\Big\}$$

have both positive measure, then condition (H2) *holds. Finally, condition* (H3) *holds when the skew product satisfies the arc-sine law, see Proposition 6.6 below.*

Remark 5.9 *Conditions* (H1) *and* (H2) *are topological conditions. On the other hand, condition* (H3) *is a statistical condition.*

Proposition 5.10 *Let* F *be a one-step skew product as in* (5-1) *whose fiber maps satisfy conditions* (H0)–(H3)*. Then there exist constants* $\bar{\ell}, \bar{h} \in \mathbb{R}$ *such that*

$$\ell(y) = \bar{\ell} \quad \textit{and} \quad h(y) = \bar{h}, \quad \textit{for every } y \in (0,1).$$

Proof We prove the proposition only for the function ℓ, the proof for h is similar and hence omitted. By (H2), given any $y \in (0,1)$ there are $\alpha, \beta \in \Sigma^+$ and $j, k \in \mathbb{N}$ such that

$$f_\alpha^j(y) < y < f_\beta^k(y) \quad \text{for every } y \in (0,1).$$

By Proposition 5.5 we have that $\ell(f_\alpha^j(y)) = \ell(y) = \ell(f_\beta^k(y))$.

Claim 5.11 *The function* ℓ *is monotone increasing.*

Proof Consider $y_1, y_2 \in (0,1)$ with $y_1 < y_2$. Take $\xi \in \Sigma^+$, since f_ξ preserves the orientation, we have that $f_\xi^n(y_1) < f_\xi^n(y_2)$ for every $n \geq 0$. As the map φ in (H3) also preserves the orientation, we have that $\varphi(f_\xi^n(y_1)) < \varphi(f_\xi^n(y_2))$ for every $n \geq 0$. Therefore, the average satisfies

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\xi^j(y_1)) < \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\xi^j(y_2)).$$

Taking upper limits we have

$$U_\varphi(\xi, y_1) = \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_\xi^j(y_1)) \le \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_\xi^j(y_2)) = U_\varphi(\xi, y_2).$$

By Proposition 5.3, we have $U_\varphi(\xi, y_1) = \ell(y_1)$ and $U_\varphi(\xi, y_2) = \ell(y_2)$, then $\ell(y_1) \le \ell(y_2)$. This proves the claim. ■

Since ℓ is increasing, the claim implies that for any given $y \in (0,1)$, it is constant on the interval $(f_\alpha^j(y), f_\beta^k(y))$. Therefore, for every $y \in (0,1)$, ℓ is locally constant on $(0,1)$. As ℓ is monotone increasing and $(0,1)$ is connected, there exists a constant $\bar\ell \in \mathbb{R}$ such that $\ell(y) = \bar\ell$ for every $y \in (0,1)$. ■

Corollary 5.11.1 *Let F be a one-step skew product as in* (5-1) *whose fiber maps satisfy conditions* (H0)–(H3)*. Then, for every $y \in (0,1)$, it holds*

$$U_\varphi(\xi, y) = \bar\ell \quad \textit{and} \quad L_\varphi(\xi, y) = \bar h, \quad \textit{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+.$$

Proof By Proposition 5.3, we have that $U_\varphi(\xi, y) = \ell(y)$ and $L_\varphi(\xi, y) = h(y)$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$. Now, by Proposition 5.10, $\ell(y) = \bar\ell$ and $h(y) = \bar h$ for every $y \in (0,1)$, proving the corollary. ■

Theorem 5.12 *Let F be a one-step skew product as in* (5-1) *whose fiber maps satisfy conditions* (H0)–(H3) *and φ the non-negative increasing map in* (H3)*. Then, for every $y \in (0,1)$, it holds*

$$U_\varphi(\xi, y) = \max \varphi \quad \textit{and} \quad L_\varphi(\xi, y) = \min \varphi, \quad \textit{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+.$$

In particular, F has historical behavior for $(\mathbb{P}\times\lambda)$-a.e. *point.*

Proof We first prove the statement for the function U_φ. Recall that $m = \min\varphi$ and $M = \max\varphi$. Consider a sequence $\{\gamma_k\}_{k\ge1} \subset (m, M)$ with $\gamma_k \to M$ as $k\to\infty$. By (H3), there exists $y^* \in (0,1)$ such that for every $k \ge 0$ there is a set $\Omega_k \subset \Sigma^+$ with $\mathbb{P}(\Omega_k) = 1$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma_k, M]}(\varphi(f_\xi^j(y^*))) = 1, \quad \text{for every } \xi \in \Omega_k. \tag{5-6}$$

Define the set

$$\Omega^+ \stackrel{\text{def}}{=} \bigcap_{k\ge1} \Omega_k.$$

As Ω^+ is a countable intersection of probability one sets, it holds $\mathbb{P}(\Omega^+) = 1$.

Next lemma corresponds to the first assertion in the theorem.

Lemma 5.13 *For every $y \in (0,1)$ it holds $U_\varphi(\xi, y) = M$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.*

Proof By the definitions of Ω^+ and of Ω_k in (5-6), we have that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma_k,M]}(\varphi(f_\xi^j(y^*))) = 1, \quad \text{for every } \xi \in \Omega^+. \tag{5-7}$$

Since φ is non-negative, for every $k \geq 0$ we have that

$$\mathbb{1}_{[\gamma_k,M]}(\varphi(f_\xi^j(y^*)))\cdot\gamma_k \leq \varphi(f_\xi^j(y^*) \quad \text{for every } \xi \in \Omega^+.$$

Applying the upper limit obtain

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma_k,M]}(\varphi(f_\xi^j(y^*)))\cdot\gamma_k \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_\xi^j(y^*)) = U_\varphi(\xi,y^*).$$

By (5-7) it follows that $\gamma_k \leq U_\varphi(\xi,y^*)$ for every $\xi\in\Omega^+$.

On the other hand, by Corollary 5.11.1, for every $y \in (0,1)$ there is a set $\Omega_y \subset \Sigma^+$ with $\mathbb{P}(\Omega_y) = 1$ such that

$$U_\varphi(\xi,y) = \bar{\ell} \quad \text{for every} \quad \xi\in\Omega_y.$$

Consider $\Omega_y^+ = \Omega_y \cap \Omega^+$. Since Ω_y^+ is an intersection of two sets with probability one, its follows that $\mathbb{P}(\Omega_y^+) = 1$. Moreover, $\gamma_k \leq U_\varphi(\xi,y^*) = \bar{\ell}$ for every $\xi \in \Omega_{y^*}^+$ $\gamma_k \leq \bar{\ell}$. Noting that, by definition, $m \leq \bar{\ell} \leq M$, we have that

$$M = \lim_{k\to\infty} \gamma_k \leq \bar{\ell} \leq M,$$

proving the lemma. ∎

The proof of the statement for the function L_φ is a variation of the proof of Lemma 5.13.

Lemma 5.14 *For every $y\in(0,1)$ it holds $L_\varphi(\xi,y) = m$ for* $\mathbb{P}$-a.e. $\xi\in\Sigma^+$.

Proof Consider $\{\eta_k\} \subset (m,M)$ with $\eta_k \to m$ as $k\to\infty$, arguing as in (5-7), we get a set $\Omega^- \subset \Sigma^+$ with $\mathbb{P}(\Omega^-) = 1$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\eta_k]}(\varphi(f_\xi^j(y^*))) = 1, \quad \text{for every } \xi \in \Omega^-. \tag{5-8}$$

Moreover, since $1 = \mathbb{1}_{[m,\eta_k]} + \mathbb{1}_{(\eta_k,M]}$, by (5-8), we have

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{(\eta_k,M]}(\varphi(f_\omega^j(y^*))) = 0 \quad \text{for every } \omega \in \Omega^-. \tag{5-9}$$

Now, as φ is non-negative, for every $k \geq 0$ we have

$$\varphi(f_\xi^j(y^*)) \leq \mathbb{1}_{[m,\eta_k]}(\varphi(f_\xi^j(y^*)))\cdot\eta_k + \mathbb{1}_{(\eta_k,M]}(\varphi(f_\xi^j(y^*)))\cdot M \quad \text{for every } \xi\in\Omega^-.$$

Applying the lower limit obtain

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_\xi^j(y^*)) \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\eta_k]}(\varphi(f_\xi^j(y^*))) \cdot \eta_k + \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{(\eta_k,M]}(\varphi(f_\xi^j(y^*))) \cdot M \quad (5\text{-}10)$$

Thus, in (5-10) by (5-9) we obtain

$$H_\varphi(\xi, y^*) = \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f_\xi^j(y^*)) \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\eta_k]}(\varphi(f_\xi^j(y^*))) \cdot \eta_k \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\eta_k]}(\varphi(f_\xi^j(y^*))) \cdot \eta_k$$

and by (5-8) we have

$$H_\varphi(\xi, y^*) \leq \eta_k, \quad \text{for every } \xi \in \Omega^-.$$

On the other hand, by Corollary 5.11.1, for every $y \in (0,1)$ there is a set $\Omega_y \subset \Sigma^+$ with $\mathbb{P}(\Omega_y) = 1$ such that

$$H_\varphi(\xi, y) = \bar{h} \quad \text{for every } \xi \in \Omega_y.$$

Consider $\Omega_y^- = \Omega_y \cap \Omega^-$. Since Ω_y^- is the intersection of two sets, each with probability one, it follows that $\mathbb{P}(\Omega_y^-) = 1$. Noting that, by definition, $m \leq \bar{h} \leq M$, we can establish $m \leq \bar{h} \leq \eta_k$.

Taking the limit,

$$m \leq \bar{h} \leq \lim_{k\to\infty} \eta_k = m.$$

Therefore, we conclude that for every $y \in (0,1)$

$$L_\varphi(\xi, y) = \bar{h} = m \quad \text{for every } \xi \in \Omega_{y^*}^-.$$

Proving the lemma. ■

Using Lemmas 5.13 and 5.14, and defining $\Phi(\xi, y) \stackrel{\text{def}}{=} \varphi(y)$, it holds for $(\mathbb{P} \times \lambda)$-a.e. point that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \Phi(F^j(\xi, y)) = m \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \Phi(F^j(\xi, y)) = M.$$

This ends the proof of the theorem. ■

Remark 5.15 *The assumption that the alphabet $\mathcal{A}$ is countable plays a crucial role in the proof of Theorem 5.12, given that Corollary 5.11.1 relies on this assumption. However, when $\mathcal{A}$ is a singleton, we can treat F as a measurable map $f\colon I \to I$. In such a case, conditions* (H1)–(H3) *may not be necessary. The discrete analog of Theorem 5.12 is provided by Theorem 8.7.*

5.2.1 Historical behavior under an ergodic assumption

Now, we will provide alternative conditions to (H0)–(H3), as given above. Consider the following condition:

(H0b) The measure $\mathbb{P} \times \lambda$ is ergodic with respect to F. In other words, for every F-invariant set B, it holds that $(\mathbb{P} \times \lambda)(B) \in \{0, 1\}$.

Remark 5.16 *Condition* (H0) *is more rigid, while* (H0b) *provides more flexibility by only requiring the triviality of the σ-algebra for F-invariant sets. Moreover, it is noteworthy that the measure $\mathbb{P} \times \lambda$ in* (H0b) *is not required to be F-invariant.*

Should the skew product F satisfy (H0b), we can obtain a weak conclusion similar to Corollary 5.11.1, as we will prove in the following proposition.

Proposition 5.17 *Let F be a one-step skew product as in* (5-1) *satisfying* (H0b)*. Then, there are constants $\bar{\ell}, \bar{h} \in \mathbb{R}$ such that*

$$U_\varphi(\xi, y) = \bar{\ell} \quad \textit{and} \quad L_\varphi(\xi, y) = \bar{h} \quad \textit{for } (\mathbb{P} \times \lambda)\textit{-a.e. } (\xi, y) \in \Sigma^+ \times I.$$

Proof Given a constant $\ell \in \mathbb{R}$ define the set

$$A(\ell) \stackrel{\text{def}}{=} \{(\xi, y) \in \Sigma^+ \colon U_\varphi(\xi, y) < \ell\}.$$

Since $A(\ell)$ is an F-invariant set, by (H0b), we have $(\mathbb{P} \times \lambda)(A(\ell)) \in \{0, 1\}$. Let

$$\bar{\ell} \stackrel{\text{def}}{=} \inf\{\ell \colon \mathbb{P}(A(\ell)) = 1\}.$$

Claim 5.18 $U_\varphi(\xi, y) = \bar{\ell}$ *for* $(\mathbb{P} \times \lambda)$-a.e. $(\xi, y) \in \Sigma^+ \times I$.

Proof When $\mathbb{P}(A(\bar{\ell})) = 0$, we have that $U_\varphi(\xi, y) \geq \bar{\ell}$ for $(\mathbb{P} \times \lambda)$-a.e. point. Then, by definition of $\bar{\ell}$, we have that $U_\varphi(\xi, y) < \bar{\ell} + \frac{1}{n}$ for every $n \geq 1$ and $(\mathbb{P} \times \lambda)$-a.e. point. Hence, by taking $n \to \infty$, we find that $U_\varphi(\xi, y) = \bar{\ell}$ for $(\mathbb{P} \times \lambda)$-a.e. point, proving the claim in this case.

To conclude the proof, we need to analyze also the case when $\mathbb{P}(A(\bar{\ell})) = 1$. Suppose, by contradiction, that there is a set $B \subset \Omega \times \Sigma^+$ with $(\mathbb{P}\times\lambda)(B) > 0$ such that $U_\varphi(\xi, y) \neq \bar{\ell}$ for every $(\xi, y) \in B$. By definition of $\bar{\ell}$, we have that $(\mathbb{P}\times\lambda)(A(\bar{\ell} - \frac{1}{n})) = 0$ for all $n \geq 1$. Consider the monotonically increasing sequence of sets

$$B_n \stackrel{\text{def}}{=} A\Big(\bar{\ell} - \frac{1}{n}\Big) \cap B, \quad \text{for } n \geq 1,$$

and note that $(\mathbb{P}\times\lambda)(B_n) = 0$ and therefore $\lim_{n\to\infty}(\mathbb{P}\times\lambda)(B_n) = 0$. However, by the monotonicity and since $(\mathbb{P}\times\lambda)(A(\bar{\ell})) = 1$,

$$\lim_{n\to\infty}(\mathbb{P}\times\lambda)(B_n) = (\mathbb{P}\times\lambda)(A(\bar{\ell})\cap B) = (\mathbb{P}\times\lambda)(B) > 0,$$

which leads to a contradiction. Therefore, $U_\varphi(\xi, y) = \bar{\ell}$ for $(\mathbb{P}\times\lambda)$-a.e. point, proving the claim. ■

The above claim provides $\bar{\ell}$ as in the statement of the proposition. Considering the sets

$$\bar{A}(h) = \{\omega \in \Sigma^+ \colon L_\varphi(\xi, y) > h\}$$

and arguing similarly, we also get that $L_\varphi(\xi, y) = \bar{h}$ for $(\mathbb{P}\times\lambda)$-a.e. point, where

$$\bar{h} = \sup\{h : (\mathbb{P}\times\lambda)(\bar{A}(h)) = 1\} \leq \max\varphi,$$

concludes the proof. ■

In Theorem 5.12, the change from condition (H0) to (H0b) requires a corresponding adjustment of condition (H3). We consider the adjusted condition as follows:

(H3b) There exists $J \subset (0,1)$ with $\lambda(J) > 0$ such that for every $y^* \in J$, there exists a non-negative, monotonically increasing function $\varphi \colon I \to \mathbb{R}$ satisfying, for every $\gamma \in (m, M)$, where $M = \max\varphi$ and $m = \min\varphi$,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\varphi(f_\xi^j(y^*))) = 1, \\ \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\varphi(f_\xi^j(y^*))) = 1, \end{aligned} \tag{5-11}$$

for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.

With this adjusted condition, we can establish, in a manner analogous to Theorem 5.12, the following theorem.

Theorem 5.19 *Let F be a one-step skew product as in* (5-1) *whose fiber maps satisfy conditions* (H0b) *and* (H3b) *and φ the non-negative increasing map* *in* (H3b)*. Then, for* λ-a.e. $y \in (0,1)$*, it holds*

$$U_\varphi(\xi, y) = \max \varphi \quad \textit{and} \quad L_\varphi(\xi, y) = \min \varphi, \quad \textit{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+.$$

In particular, F has historical behavior for $(\mathbb{P} \times \lambda)$-a.e. *point.*

Proof The proof follows the proof of Theorem 5.12. Consider a sequence $\{\gamma_k\}_{k\geq 1} \subset (m, M)$ with $\gamma_k \to M$ as $k \to \infty$. By (H3b), there is a set $J \subset (0,1)$, with $\lambda(J) > 0$ such that for every $y^* \in J$ and every $k \geq 0$ there is a set $\Omega_k(y^*) \subset \Sigma^+$ with $\mathbb{P}(\Omega_k(y^*)) = 1$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\gamma_k, M]}(\varphi(f_\xi^j(y^*))) = 1, \quad \text{for every } \xi \in \Omega_k(y^*). \tag{5-12}$$

Define the set

$$\Omega^+_{y^*} \stackrel{\text{def}}{=} \bigcap_{k\geq 1} \Omega_k(y^*).$$

As $\Omega^+_{y^*}$ is a countable intersection of probability one sets, it holds $\mathbb{P}(\Omega^+_{y^*}) = 1$.

Next lemma corresponds to the first assertion in the theorem.

Lemma 5.20 *For* λ-a.e. $y \in (0,1)$ *it holds* $U_\varphi(\xi, y) = M$ *for* $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.

Proof By the definitions of $\Omega^+_{y^*}$ and of $\Omega_k(y^*)$ in (5-12), we have that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\gamma_k, M]}(\varphi(f_\xi^j(y^*))) = 1, \quad \text{for every } \xi \in \Omega^+_{y^*}. \tag{5-13}$$

Since φ is non-negative, for every $k \geq 0$ we have that

$$\mathbb{1}_{[\gamma_k, M]}(\varphi(f_\xi^j(y^*))) \cdot \gamma_k \leq \varphi(f_\xi^j(y^*)) \quad \text{for every } \xi \in \Omega^+_{y^*}.$$

Applying the upper limit, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[\gamma_k, M]}(\varphi(f_\xi^j(y^*))) \cdot \gamma_k \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\xi^j(y^*)) = U_\varphi(\xi, y^*).$$

By (5-13), it follows that

$$\gamma_k \leq U_\varphi(\xi, y^*) \quad \text{for every} \quad \xi \in \Omega^+_{y^*}. \tag{5-14}$$

Moreover, define the set

$$\bar{J} \stackrel{\text{def}}{=} \left\{(\xi, y) \in \Sigma^+ \times I \colon y \in J \quad \text{and} \quad \xi \in \Omega^+_y\right\}. \tag{5-15}$$

Using Fubini's Theorem, we find

$$(\mathbb{P} \times \lambda)(\bar{J}) = \int_J \mathbb{P}(\Omega^+_y)\, d\lambda,$$

since $\lambda(J) > 0$ and $\mathbb{P}(\Omega_y^+) = 1$ for every $y \in J$, it follows that $(\mathbb{P} \times \lambda)(\bar{J}) > 0$.

On the other hand, by Proposition 5.17, there is a set $\bar{\Omega} \subset \Sigma^+ \times I$ with $(\mathbb{P} \times \lambda)(\bar{\Omega}) = 1$ such that

$$U_\varphi(\xi, y) = \bar{\ell} \quad \text{for every} \quad (\xi, y) \in \bar{\Omega}.$$

Consider $\bar{\Omega}^+ = \bar{\Omega} \cap \bar{J}$, and $(\mathbb{P} \times \lambda)(\bar{\Omega}^+) > 0$. Taking $(\xi^*, y^*) \in \bar{\Omega}^+$, we have that $U_\varphi(\xi^*, y^*) = \bar{\ell}$. Since $(\xi^*, y^*) \in \bar{J}$, then $y^* \in J$ and $\xi^* \in \Omega_{y^*}^+$. Therefore, by (5-14), we have that

$$\gamma_k \leq U_\varphi(\xi^*, y^*) = \bar{\ell} \quad \text{for every } \xi^* \in \Omega_{y^*}^+.$$

Noting that, by definition, $m \leq \bar{\ell} \leq M$, we have that

$$M = \lim_{k \to \infty} \gamma_k \leq \bar{\ell} \leq M,$$

proving the lemma. ■

The proof of the statement that for λ-a.e. $y \in (0,1)$ it holds $L_\varphi(\xi, y) = m$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$ is a variation of the proof of Lemma 5.20 using the ideas of Lemma 5.14, and hence, it is omitted, concluding the proof of the theorem. ■

Remark 5.21 *Theorem 5.19 holds when $\mathcal{A}$ is an infinite alphabet. This is a difference compared to Theorem 5.12 because in this theorem, the assumption that $\mathcal{A}$ is countable plays a crucial role (see Remark 5.15).*

6
Arc-sine Laws

We continue to study one-step skew products

$$F\colon \Sigma^+ \times I \to \Sigma^+ \times I, \qquad F(\xi, y) = (\sigma(\xi), f_\xi(y))$$

as in (5-1). Given any $y \in I$, we study the statistical behavior of the sequence of random variables $\{f_\xi^n(y)\}_{n\geq 0}$. Our goal is to analyze the historical behavior of F when this sequence satisfies the arc-sine law (or similar distributions). The Arc-sine Law, introduced by Lévy [14] and later formalized by Erdős and Kac [10], reads as follows:

Theorem 6.1 (Arc-sine Law) *Let $\{\psi_n\}_{n\geq 0}$ be i.i.d. random variables having mean zero and finite variance. Consider*

$$S_n \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} \psi_j \quad \text{and} \quad N_n \stackrel{\text{def}}{=} \#\{j \in \{0, \dots, n-1\} \colon S_j > 0\}$$

Then,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{n} N_n < \alpha\right) = \frac{2}{\pi} \arcsin\left(\sqrt{\alpha}\right), \qquad \alpha \in (0,1).$$

The chapter is organized as follows. In Section 6.1, we give two auxiliary definitions derived from the Arc-sine law: the fiber maps and the skew product arc-sine laws. In Section 6.2, in combination with the main result of the previous chapter (Theorem 5.12), we prove that if the skew product satisfies the fiber arc-sine law, then F has historical behavior, see Theorem 6.8.1.

6.1
Fiber and skew product arc-sine Laws

In this section, we introduce two types of arc-sine laws, for the fiber maps and to skew products functions. Firstly, we present it for the fiber maps, and additionally we introduce a similar weak property. Recall that λ represents the Lebesgue measure in I.

Definition 6.2 (Fiber arc-sine laws) *Let F be a skew product as in* (5-1) *and $\psi\colon I \to \mathbb{R}$ a non-negative monotone increasing continuous function. Let $m = \min\psi$ and $M = \max\psi$. The pair (F, ψ) satisfies*

– *the* fiber arc-sine law *if, for every $y \in (0,1)$, every $\gamma \in (m,M)$, and every $\alpha \in (0,1)$ it simultaneously holds*

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\psi(f_\xi^j(y))) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha}, \quad (6\text{-}1)$$

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\psi(f_\xi^j(y))) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha}; \quad (6\text{-}2)$$

– *the* fiber weak arc-sine law *if, there is $y^* \in J$ such that, for every $\gamma \in (m,M)$ and every $\alpha \in (0,1)$ it simultaneously holds*

$$\begin{aligned} &\limsup_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\psi(f_\xi^j(y^*))) < \alpha\right\}\right) < 1, \\ &\limsup_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\psi(f_\xi^j(y^*))) < \alpha\right\}\right) < 1. \end{aligned} \quad (6\text{-}3)$$

– *the* fiber λ-weak arc-sine law *if, there is $J \subset (0,1)$, with $\lambda(J) > 0$ such that for every $y^* \in J$, every $\gamma \in (m,M)$ and every $\alpha \in (0,1)$, equations in* (6-3) *hold.*

Remark 6.3 *A pair (F,ψ) satisfying the fiber arc-sine law also satisfies the fiber λ-weak arc-sine law. However, the converse does not hold in general. In [17], Nakamura et al. proved that there is a distribution that satisfies the fiber λ-weak arc-sine law but generally does not satisfy the fiber arc-sine law. We refer to the fiber (λ-)weak arc-sine law because it encompasses any distribution that satisfies* (6-3)*, with the main example being the fiber arc-sine law.*

Now, we present the arc-sine law for a skew product F as in (5-1) with respect to the reference measure $\mathbb{P} \times \lambda$ of the product space.

Definition 6.4 (Skew product arc-sine law) *Let F be a skew product as in* (5-1)*, and let $\psi \colon I \to \mathbb{R}$ be a non-negative, monotone increasing, continuous function. Consider the function $\Psi \colon \Sigma^+ \times I \to \mathbb{R}$ defined by $\Psi(\xi,y) \stackrel{\text{def}}{=} \psi(y)$. The pair (F,Ψ) satisfies the* arc-sine law *if, for every $\alpha \in (0,1)$ and every $\gamma \in (m,M)$, it simultaneously holds*

$$\lim_{n\to\infty} (\mathbb{P}\times\lambda)\left(\left\{(\xi,y) \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\Psi(F^j(\xi,y))) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha},$$

$$\lim_{n\to\infty} (\mathbb{P}\times\lambda)\left(\left\{(\xi,y) \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\Psi(F^j(\xi,y))) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha}.$$

Now, we establish the relation between the fiber arc-sine Law (Definition 6.2) and the arc-sine Law for a skew product (Definition 6.4).

Proposition 6.5 *Let F be a skew product as in* (5-1) *and let $\psi\colon I \to \mathbb{R}$ be a non-negative, monotone increasing, continuous function.*

- *If (F,ψ) satisfies the fiber arc-sine law, then (F,Ψ) satisfies the skew product arc-sine law;*
- *If (F,Ψ) satisfies the skew product arc-sine law, then (F,ψ) satisfies the fiber λ-weak arc-sine law. In particular the (F,ψ) satisfies the fiber weak arc-sine law.*

Proof To prove the first statement, given $\gamma \in (m,M)$ let

$$I_0(\gamma) \stackrel{\text{def}}{=} [m,\gamma] \quad \text{and} \quad I_1(\gamma) \stackrel{\text{def}}{=} [\gamma, M].$$

For every $n \in \mathbb{N}$, and every $\alpha,\gamma \in (0,1)$ define the set

$$A_{i,n} \stackrel{\text{def}}{=} \left\{ (\xi,y) \in \Sigma^+ \times I \colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(\Psi(F^j(\xi,y))) \leq \alpha \right\} \quad \text{for } i = 0,1.$$

where $\Psi(\xi,y) \stackrel{\text{def}}{=} \psi(y)$ for every $(\xi,y) \in \Sigma^+ \times I$. For each $y \in I$, we denote by $A_{i,n}(y)$ the projection on Σ^+ of $A_{i,n} \cap (\Sigma^+ \times \{y\})$. In particular,

$$A_{i,n} = \left\{ (\xi,y) \in \Sigma^+ \times I : y \in I \text{ and } \xi \in A_{i,n}(y) \right\}.$$

Hence, since F satisfies the fiber arc-sine law, by applying Fubini's and Lebesgue's dominated convergence theorems, we find that

$$\begin{aligned} \lim_{n\to\infty} (\mathbb{P}\times\lambda)(A_{i,n}) &= \lim_{n\to\infty} \int \mathbb{P}(A_{i,n}(y))\, d\lambda \\ &= \int \lim_{n\to\infty} \mathbb{P}(A_{i,n}(y))\, d\lambda = \frac{2}{\pi} \arcsin\sqrt{\alpha}. \end{aligned}$$

Now, we prove the second implication. Assume that F satisfies the skew product arc-sine law for a function $\Psi\colon \Sigma^+ \times I \to \mathbb{R}$. As before, we define $A_{i,n}$ and $A_{i,n}(y)$. Notice that these sets depend on $\alpha \in (0,1)$ and $\gamma \in (m,M)$ where $m = \min\varphi$ and $M = \max\varphi$. We write $A_{i,n,\alpha,\gamma}$ and $A_{i,n,\alpha,\gamma}(y)$ to emphasize this dependence here. Suppose, by contradiction, that for λ-a.e. $y \in (0,1)$ it holds that $\limsup_{n\to\infty} \mathbb{P}(A_{i,n,\alpha,\gamma}(y)) = 1$ for some $i \in \{0,1\}$, $\alpha \in (0,1)$ and $\gamma \in (m,M)$. Then, by applying Fubini's and Lebesgue's dominated convergence theorems again,

$$\lim_{n\to\infty}(\mathbb{P}\times\lambda)(A_{i,n,\alpha,\gamma}) = \int \limsup_{n\to\infty}\mathbb{P}(A_{i,n,\alpha,\gamma}(y))\,d\lambda = 1. \tag{6-4}$$

However, since F satisfies the skew product arc-sine law,

$$\lim_{n\to\infty}(\mathbb{P}\times\lambda)(A_{i,n,\alpha,\gamma}) = \frac{2}{\pi}\arcsin\sqrt{\alpha} < 1,$$

which contradicts (6-4). Therefore, there exists $J\subset(0,1)$ with $\lambda(J)>0$ such that, for every $y^*\in J$, it holds

$$\limsup_{n\to\infty}\mathbb{P}(A_{i,n,\alpha,\gamma}(y^*)) < 1 \quad \text{for all } i=0,1,\ \alpha\in(0,1), \text{ and } \gamma\in(m,M).$$

This proves the second statement and, therefore, the proposition. ■

6.2
Arc-sine Laws and historical behavior: a characterization

We establish a connection between historical behavior and the arc-sine Law. Recall that in our settings, a skew product F has historical behavior if it satisfies conditions (H1)–(H3). As mentioned in Remark 5.9, (H3) is a statistical condition. We now present the connection between condition (H3) and the fiber weak arc-sine law in the following proposition:

Proposition 6.6 *Let F be a skew product as in* (5-1) *and let $\psi\colon I\to\mathbb{R}$ be a non-negative, monotone increasing, continuous function.*

- *If F satisfies* (H0) *and (F,ψ) satisfies the fiber weak arc-sine law, then ψ satisfies condition* (H3)*.*
- *If F satisfies* (H0b) *and (F,ψ) satisfies the fiber λ-weak arc-sine law, then ψ satisfies condition* (H3b)*.*

Proof Given $\gamma\in[m,M]$, let $I_0(\gamma)\stackrel{\text{def}}{=}[m,\gamma]$ and $I_1(\gamma)\stackrel{\text{def}}{=}(\gamma,M)$. Since (F,ψ) satisfies the fiber (λ-)weak arc-sine law, there exists $J\subset(0,1)$ such that for every $y^*\in J$, it holds that

$$\limsup_{n\to\infty}\mathbb{P}\left(\left\{\xi\in\Sigma^+\colon \frac{1}{n}\sum_{j=0}^{n-1}\mathbb{1}_{I_i(\gamma)}(\psi(f_\xi^j(y^*)))\geq\alpha\right\}\right) > 0, \quad i=0,1.$$

Define the set

$$B_i(\alpha,y^*)\stackrel{\text{def}}{=}\left\{\xi\in\Sigma^+\colon \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\mathbb{1}_{I_i(\gamma)}(\psi(f_\xi^j(y^*)))\geq\alpha\right\} \quad i=0,1. \tag{6-5}$$

Now, by Lemma A.7, we have

$$\mathbb{P}\left(B_i(\alpha,y^*)\right)\geq\limsup_{n\to\infty}\mathbb{P}\Big(\Big\{\xi\in\Sigma^+\colon \frac{1}{n}\sum_{j=0}^{n-1}\mathbb{1}_{I_i(\gamma)}(\psi(f_\xi^j(y^*)))\geq\alpha\Big\}\Big) > 0. \tag{6-6}$$

Claim 6.7 *If* (H0) *holds, then* $\mathbb{P}(B_i(\alpha, y^*)) = 1$ *for every* $y^* \in J$.

Proof By Lemma A.11, the set $B_i(\alpha, y^*)$ belongs to the tail algebra $\mathcal{T}(\{f_\xi^j(y^*)\}_{j\geq 0})$. By (H0), we have $\mathbb{P}(B_i(\alpha)) \in \{0,1\}$, $i = 0,1$. Thus, since $\mathbb{P}(B_i(\alpha)) > 0$, we conclude $\mathbb{P}(B_i(\alpha)) = 1$, $i = 0,1$. ■

Claim 6.8 *If* (H0b) *and fiber* λ*-weak arc-sine law holds, then*

$$\mathbb{P}(B_i(\alpha, y^*)) = 1 \quad for \quad \lambda - \text{a.e } y^* \in J$$

Proof Note that, in this case $\lambda(J) > 0$. Fix $\alpha \in (0,1)$ and define the set

$$B_i(\alpha) \stackrel{\text{def}}{=} \Big\{(\xi, y^*) \in \Sigma^+ \times I \colon y^* \in J \quad \text{and} \quad \xi \in B_i(\alpha, y^*)\Big\}$$

Note that this set is F-invariant. By (H0b), we have $(\mathbb{P} \times \lambda)(B_i(\alpha)) \in \{0,1\}$. Applying Fubini's Theorem A.5, it holds that

$$(\mathbb{P} \times \lambda)(B_i(\alpha)) = \int_J \mathbb{P}(B_i(\alpha, y))\, d\lambda,$$

since, by (6-6), $\mathbb{P}(B_i(\alpha, y^*)) > 0$ for every $y^* \in J$. Then $(\mathbb{P} \times \lambda)(B_i(\alpha)) > 0$, and therefore $(\mathbb{P} \times \lambda)(B_i(\alpha)) = 1$. Hence, again using Fubini's Theorem, we have

$$\mathbb{P}(B_i(\alpha, y^*)) = 1 \quad \text{for } \lambda\text{-a.e. } y^* \in J,$$

concluding the proof of the claim. ■

Note that Claims 6.7 and 6.8 provide a set $\bar{J} \subset J$ such that $\mathbb{P}(B_i(\alpha, y^*)) = 1$ for every $y^* \in \bar{J}$. In the case of the fiber λ-weak arc-sine law, by Claim 6.8, $\lambda(\bar{J}) > 0$. Now, as $\alpha \to 1$, we obtain that for every $y^* \in \bar{J}$, it holds

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(\psi(f_\xi^j(y^*))) = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+, \quad i = 0,1.$$

Therefore, conditions (H3) and (H3b) hold in the respective cases. ■

By applying Proposition 6.6 to Theorem 5.12, we obtain the following corollary.

Corollary 6.8.1 *Let F be a skew product as in* (5-1) *whose fiber maps satisfy conditions* (H0)–(H2)*. Let $\varphi\colon I \to I$ be a non-negative, increasing, non-constant function. Suppose that (F, φ) satisfies the fiber weak arc-sine law. Then, F has historical behavior for* $(\mathbb{P} \times \lambda)$-a.e. *point.*

Proof As (F, φ) satisfies the fiber weak arc-sine law, by Proposition 6.6, the fiber maps of F satisfy condition (H3). Then, by Theorem 5.12, the skew product F has historical behavior for $(\mathbb{P} \times \lambda)$-a.e. point. ■

Corollary 6.8.2 *Let F be a skew product as in* (5-1) *whose fiber maps satisfy conditions* (H0b)*. Let $\varphi\colon I \to I$ be a non-negative, increasing, non-constant function. Suppose that (F, φ) satisfies the fiber λ-weak arc-sine law. Then, F has historical behavior for* $(\mathbb{P} \times \lambda)$-a.e. *point.*

Proof As (F, φ) satisfies the fiber weak arc-sine law, by Proposition 6.6, the fiber maps of F satisfy condition (H3b). Then, by Theorem 5.19, the skew product F has historical behavior for $(\mathbb{P} \times \lambda)$-a.e. point. ■

Remark 6.9 *In light of Proposition* 6.5 *under the corresponding assumptions of Corollaries* 6.8.1 *and* 6.8.2*, any definition of the arc-sine law in our setting, such as:*

– *Fiber arc-sine law,*

– *Fiber (λ-)weak arc-sine law, and*

– *Skew product arc-sine law.*

implies historical behavior.

7
Zero Schwarzian derivative: historical behavior

In this chapter, we analyze the relations between zero Schwarzian derivative and historical behavior for one-step skew products F as in (5-1). We will see that if the Schwarzian derivative of f_ξ are almost surely zero, then the one-step skew product F satisfies the fiber arc-sine law. Then, by the results in Chapters 5 and 6, the one-step skew product F has historical behavior. These facts were stated by Bonifant and Milnor in [4, Theorem 6.2]. However, they only provide a rough sketch of the proof with several incomplete steps. Here we present a version of this result with a complete proof. Relative to the proof and the results in [4], these are the main differences:

- In [4, Hypothesis 6.1] the authors considers any probability space, without restrictions. Compare with Remark 5.7 where only at most countable spaces are considered. Nowadays, the statement in [4] in this full generality is controversial.
- In [4, page 13], it is claimed, without a proof, that the functions U_φ and L_φ in (5-2) are independent of ξ. This claim is not obvious. In Proposition 5.3 we prove this assertion.
- In [4, page 13], it is claimed, without a proof, that $U_\varphi(y) = U_\varphi(f_\xi(y))$ and $L_\varphi(y) = L_\varphi(f_\xi(y))$. These statements are proven in Propositions 5.3 and 5.5.
- In [4, page 13], it is stated, without a proof, that the functions $y \mapsto U_\varphi(y)$ and $y \mapsto L_\varphi(y)$ are constant. This statement is proved in Proposition 5.10.

7.1
Zero Schwarzian derivative

We consider a one-step skew product F as in (5-1), where $f_\xi \colon I \to I$ are C^3-diffeomorphism. Recall that, F has historical behavior if satisfies contitions (H0)–(H3) (see Theorem 5.12). In what follows, we denote by id the identity map $\mathrm{id} \colon I \to I$. Throughout this chapter, we consider the following conditions on the skew product :

(BM1) $Sf_\xi(y) = 0$ for $(\mathbb{P} \times \lambda)$-a.e. $(\xi, y) \in \Sigma^+ \times I$, (BM2) the Lyapunov exponents (see (2-4)) satisfy $\mathrm{L}(0) = \mathrm{L}(1) = 0$, and

(BM3) $f_\xi \neq \mathrm{id}$ for $\mathbb{P}$-a.e. $\xi \in \Sigma^+$.

As will be seen in Section 7.2, Conditions (H1) and (BM1)–(BM3) are equivalent to the following representation of the fiber maps of F:

$$f_\xi(y) = \frac{a(\xi)y}{1 + (a(\xi) - 1)y} \quad \text{for } y \in I \text{ and } \xi = (\xi_i)_{i \geq 0} \in \Sigma^+ = \mathcal{A}^{\mathbb{N}}, \tag{7-1}$$

where $a\colon \Sigma^+ \to (0, +\infty)$ with $a(\xi) \stackrel{\text{def}}{=} a(\xi_0)$, and it satisfies

$$\int \log a(\xi)\, d\mathbb{P} = 0 \quad \text{and} \qquad a(\xi_0) \neq 1 \text{ for all } \xi_0 \in \mathcal{A}.$$

We will also asume the following asumption:

(BM4) $\int (\log a(\xi))^2\, d\mathbb{P} < +\infty$.

The following theorem shows that if the Schwarzian derivative of the fiber maps of the skew product F are almost surely null (i.e., (BM1) holds), then the skew product has historical behavior.

Theorem 7.1 *Let F be a skew product as in* (5-1) *satisfying* (H1) *and* (BM1)–(BM4)*. Then F satisfies conditions* (H0), (H2)*, and the pair* (F, id) *satisfies the fiber arc-sine law. In particular, the skew product F satisfies* (H3) *with the function* id.

The proof of Theorem 7.1 is completed in Section 7.2.

In our setting, the result by Bonifant and Milnor reads as follows:

Corollary 7.1.1 *Let F be a skew product as in* (5-1) *satisfying* (H1) *and* (BM1)–(BM4)*. Then for every $y \in (0, 1)$,*

$$U_{\mathrm{id}}(\xi, y) = 1 \quad \textit{and} \quad L_{\mathrm{id}}(\xi, y) = 0, \quad \textit{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+.$$

In particular, the skew product F has historical behavior for $(\mathbb{P} \times \lambda)$-a.e. *point.*

Proof By Theorem 7.1, the map F satisfies conditions (H0)–(H3), where $\mathrm{id}\colon I \to I$ serves as the increasing function in (H3). By Theorem 5.12, it follows that for every $y \in (0, 1)$, it holds

$$U_{\mathrm{id}}(\xi, y) = \max \mathrm{id} = 1 \quad \text{and} \quad L_{\mathrm{id}}(\xi, y) = \min \mathrm{id} = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \xi \in \Sigma^+.$$

Consequently, F has historical behavior, proving the corollary. ■

Remark 7.2 *The assumption that the alphabet $\mathcal{A}$ is at most countable plays a crucial role in our proof of Theorem 5.12 and hence in the proof of Corollary 7.1.1.*

7.2 Proof of Theorem 7.1

We first observe that the zero Schwarzian derivative implies that the functions f_ξ are fractional linear, as given in (7-1). Condition (BM1) and Proposition B.1 together imply that f_ξ is a fractional linear map for every $\xi \in \Sigma^+$. Considering condition (H1) which ensures that the fiber maps fix the points 0 and 1, these fiber maps take the form:

$$f_\xi(y) = \frac{a(\xi)y}{1 + (a(\xi) - 1)y},$$

where $a\colon \Sigma^+ \to (0, +\infty)$ and $a(\xi) \stackrel{\text{def}}{=} a(\xi_0)$. Condition (BM3) further ensures that $a(\xi) \neq 1$ for every $\xi \in \Sigma^+$. Now, note that

$$f'_\xi(y) = \frac{a(\xi)}{((a(\xi) - 1)\, y + 1)^2},$$

we find that

$$L(0) = \int \log(f'_\xi(0))\, d\mathbb{P} = \int \log(a(\xi))\, d\mathbb{P}.$$

By (BM2), it follows that $\int \log(a(\xi))\, d\mathbb{P} = 0$.

We split the proof into three parts (Propositions 7.3, 7.5, and 7.12).

Proposition 7.3 *Let F be a skew product as in* (5-1) *satisfying* (H1) *and* (BM1)–(BM4)*. For every $y \in (0, 1)$ the sets*

$$\{\xi \in \Sigma^+ \colon f_\xi(y) < y\} \quad \textit{and} \quad \{\xi \in \Sigma^+ \colon f_\xi(y) > y\}$$

have both positive probability $\mathbb{P}$. In particular, the map F satisfies (H2).

Proof Consider now the sets

$$\Omega_1 \stackrel{\text{def}}{=} \{\xi \in \Sigma^+ \colon a(\xi) < 1\} \quad \text{and} \quad \Omega_2 \stackrel{\text{def}}{=} \{\xi \in \Sigma^+ \colon a(\xi) > 1\}.$$

Note that as $a(\xi) \neq 1$ for every ξ, we have

$$\mathbb{P}(\Omega_1) + \mathbb{P}(\Omega_2) = 1.$$

Claim 7.4 $\mathbb{P}(\Omega_1) > 0$ *and* $\mathbb{P}(\Omega_2) > 0$.

Proof We argue by contradiction, suppose that $\mathbb{P}(\Omega_1) = 1$. Then,

$$L(0) = \int \log(f'_\xi(0))\, d\mathbb{P} = \int_{\Omega_1} \log(a(\xi))\, d\mathbb{P} = \int_{\Omega_1} \log(a(\xi))\, d\mathbb{P} < 0.$$

Contradicting that $L(0) = 0$. A similar contradiction arises assuming that $\mathbb{P}(\Omega_2) = 1$. This proves the claim. ∎

We are now prove that $f_\xi(y) < y$ for every $\xi \in \Omega_1$. For that consider any $\xi \in \Omega_1$ and note than $a(\xi) < 1$. Note that if $y \in (0,1)$ then $y^2 < y$. Thus,

$$a(\xi)y - y = (a(\xi) - 1)y < (a(\xi) - 1)y^2.$$

This inequality implies that

$$a(\xi)y < (a(\xi) - 1)y^2 + y = \big((a(\xi) - 1)y + 1\big)y$$

and therefore

$$\frac{a(\xi)y}{(a(\xi) - 1)y + 1} < y.$$

Therefore, by equation (7-1), we get $f_\xi(y) < y$ for every $\xi \in \Omega_1$, proving the assertion. To prove that $f_\xi(y) > y$, for every $\xi \in \Omega_2$, we argue similarly.

Claim 7.4 now implies that for every $y \in (0,1)$ the sets

$$\{\xi \in \Omega_1 \colon f_\xi(y) < y\} \qquad \text{and} \qquad \{\xi \in \Omega_2 \colon f_\xi(y) > y\},$$

have both positive probability, proving the first part of proposition.

Finally, by Remark 5.8 the skew produc F satisfies condition (H2). ∎

Proposition 7.5 *Let F be a skew product as in* (5-1) *satisfying* (H1) *and* (BM1)–(BM4)*. Then the pair* (F, id) *satisfies the fiber arc-sine law. In particular, F satisfies* (H3).

Proof To prove the proposition, we find appropriate functions associated with $\{f_\xi\}$ to which the Arc-sine Law (Theorem 6.1) can be applied. That is equations (6-1) and (6-2) holds. We denote this new family by $\left\{\hat{f}^n_\xi\right\}_{n\geq 0}$. Thereafter we extend the properties of these new random variables to initial the initial family $\left\{f^n_\xi\right\}_{n\geq 0}$.

The auxiliary family $\hat{f}_\xi$. We start by consider an auxiliary change of variables. For that define the increasing continuous bijection

$$h\colon (0,1) \to \mathbb{R}, \quad h(y) \stackrel{\text{def}}{=} \log\left(\frac{y}{1-y}\right),$$

and the new fiber maps given by

$$\hat{f}_\xi \colon \mathbb{R} \to \mathbb{R}, \quad \hat{f}_\xi(t) \stackrel{\text{def}}{=} h \circ f_\xi \circ h^{-1}(t). \tag{7-2}$$

For the next lemma, recall the functions $a\colon \Sigma^+ \to (0,\infty)$ in (7-1).

Claim 7.6 *For every $t \in \mathbb{R}$, $\hat{f}_\xi(t) = t + \log(a(\xi))$.*

Proof Fix $\xi \in \Sigma^+$. Given any $y \in (0,1)$ and consider $k_\xi(y) = h \circ f_\xi(y)$. Recalling (7-1) it follows

$$\begin{aligned} k_\xi(y) &= \log\left(\frac{\frac{a(\xi)y}{1+(a(\xi)-1)y}}{1-\left(\frac{a(\xi)y}{(a(\xi)-1)y+1}\right)}\right) = \log\left(\frac{a(\xi)y}{1+(a(\xi)-1)y - a(\xi)y}\right) \\ &= \log\left(\frac{a(\xi)y}{1-y}\right) = \log\left(\frac{y}{1-y}\right) + \log(a(\xi)) \\ &= h(y) + \log(a(\xi)). \end{aligned} \tag{7-3}$$

Write now $y = h^{-1}(t) = \frac{e^t}{1+e^t}$. Hence, by (7-3)

$$\hat{f}_\xi(t) = h \circ f_\xi \circ h^{-1}(t) = k_\xi \circ h^{-1}(t) = k_\xi(h^{-1}(t)) = t + \log(a(\xi)),$$

proving the claim. ■

The family $\hat{f}_\xi$ satisfies the Arc-sine Law. The orbit of a point $t \in \mathbb{R}$ under $\hat{f}_\xi$, $\xi \in \Sigma^+$, is defined by

$$\hat{f}^j_\xi(t) = \hat{f}_{\xi_{j-1}} \circ \cdots \circ \hat{f}_{\xi_0}(t) = t + \log(a(\xi_0)) + \cdots + \log(a(\xi_{j-1})), \quad j \geq 1. \tag{7-4}$$

Now, define the following auxiliary random variables for every $t \in \mathbb{R}$,

$$X_{j,t}(\xi) \stackrel{\text{def}}{=} \hat{f}^{j+1}_\xi(t) - \hat{f}^j_\xi(t) = \log(a(\xi_j)) = \log(a(\sigma^j(\xi))), \tag{7-5}$$

$$Y_{j,t}(\xi) \stackrel{\text{def}}{=} \hat{f}^j_\xi(t) - \hat{f}^{j+1}_\xi(t) = -\log(a(\xi_j)) = -\log(a(\sigma^j(\xi))), \tag{7-6}$$

where the last two equalities in (7-5) and (7-6) are result from the orbit of t under $\hat{f}_\xi$ in (7-4). As these random variables do not depend on t we simply write $X_j(\xi) \stackrel{\text{def}}{=} X_{j,t}(\xi)$ and $Y_j(\xi) \stackrel{\text{def}}{=} Y_{j,t}(\xi)$.

Remark 7.7 *Since $\mathbb{P} = \rho^{\mathbb{N}}$ is the Bernoulli measure, the sequences $\{X_j\}_{j\geq 0}$ and $\{Y_j\}_{j\geq 0}$ of random variables are i.i.d.. Now we construc the following random variables*

Claim 7.8 *The sequences of random variables $\{X_j\}_{j\geq 0}$ and $\{Y_j\}_{j\geq 0}$ verifies the hypotheses of the Arc-sine Law (Theorem 6.1).*

Proof We prove the lemma only for the random variables $\{X_j\}_{j\geq 0}$; the proof for $\{Y_j\}_{j\geq 0}$ is similar and hence omitted. Note that, by (7-5), the mean of any X_j is

$$\int X_j(\xi)\, d\mathbb{P} = \int \log(a(\sigma^j(\xi)))\, d\mathbb{P}. \tag{7-7}$$

Since, by hypothesis (BM2),

$$\mathrm{L}(0) = \mathrm{L}(1) = \int \log a(\xi)\, d\mathbb{P} = 0.$$

As $\mathbb{P}$ is σ-invariant, it follows that X_j has mean zero for every $j \geq 0$.

Now consider the variance

$$\mathrm{Var}(X_j) = \int (X_j(\xi))^2\, d\mathbb{P} - \left(\int X_j(\xi)\, d\mathbb{P}\right)^2.$$

Thus, since X_j has zero mean, the measure $\mathbb{P}$ is σ-invariant and (BM4), it holds

$$\mathrm{Var}(X_j) = \int (\log a(\sigma^j(\xi))^2\, d\mathbb{P} = \int (\log a(\xi))^2\, d\mathbb{P} < +\infty.$$

Hence, it verifies the hypotheses of Theorem 6.1, concluding the proof. ■

To see that the pair (F, id) satisfies the fiber arc-sine law, we need to check that equations (6-1) and (6-2) hold for $\{f^j_\xi(t)\}_{j\geq 0}$. For this it is enough to see that these equations hold for $\{\hat{f}^j_\xi(t)\}_{j\geq 0}$.

Lemma 7.9 *Consider any $t \in \mathbb{R}$, then $\{\hat{f}^j_\xi(t)\}_{j\geq 0}$ satisfies* (6-1) *and* (6-2).

Proof We first prove that $\{\hat{f}^j_\xi(t)\}_{j\geq 0}$ satisfies (6-1). Recall that, by Lemma 7.8 we can apply the arc-sine law to the sequences of i.i.d. random variables $\{X_j\}_{j\geq 0}$ and $\{Y_j\}_{j\geq 0}$.

Claim 7.10 *Equation* (6-1) *holds for $\{\hat{f}^j_\xi(t)\}_{j\geq 0}$ for every $t \in \mathbb{R}$.*

Proof For each $n > 0$, consider the sum

$$S_n(\xi) \stackrel{\text{def}}{=} \sum_{j=0}^{n-1} X_j(\xi) = \hat{f}^n_\xi(t) - t \tag{7-8}$$

and define

$$N_n(\xi) \stackrel{\text{def}}{=} \#\left\{j \in \{1, \dots, n\} \colon S_j(\xi) > 0\right\},$$

where $\#B$ denotes the cardinality of the set B. By (7-8) we have

$$\begin{aligned} N_n(\xi) &= \#\left\{j \in \{1, \dots, n\} \colon S_j(\xi) > 0\right\} \\ &= \#\left\{j \in \{1, \dots, n\} \colon \hat{f}^j_\xi(t) - t > 0\right\} \\ &= \sum_{j=0}^{n-1} \mathbb{1}_{[t,\infty]}(\hat{f}^j_\xi(t)). \end{aligned} \tag{7-9}$$

Applying the arc-sin law (Theorem 6.1) to the sequence $\{X_j\}$, we have the following for every $\alpha \in (0,1)$

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n} N_{n,t}(\xi) < \alpha\right\}\right) = \frac{2}{\pi} \arcsin\sqrt{\alpha}. \tag{7-10}$$

Using (7-9), the previous equality reads as follows

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[t,\infty]}(\hat{f}_\xi^j(t)) < \alpha\right\}\right) = \frac{2}{\pi} \arcsin\sqrt{\alpha}, \tag{7-11}$$

completing the proof of the claim. ■

To prove that $\{\hat{f}_\xi^j(t)\}_{j\geq 0}$ satisfies (6-2), we consider a variation of the proof of Claim 7.10. For each $n \geq 0$, consider the sum

$$\bar{S}_n(\xi) = \sum_{j=0}^{n-1} Y_j(\xi) = t - \hat{f}_\xi^n(t) \tag{7-12}$$

and define

$$\bar{N}_n(\xi) \stackrel{\text{def}}{=} \#\left\{j \in \{1,\ldots,n\} \colon \bar{S}_j(\xi) > 0\right\}.$$

By (7-12) we have

$$\begin{aligned}\bar{N}_n(\xi) &= \#\left\{j \in \{1,\ldots,n\} \colon \bar{S}_j(\xi) > 0\right\}\\ &= \#\left\{j \in \{1,\ldots,n\} \colon t - \hat{f}_\xi^n(t) > 0\right\}\\ &= \sum_{j=0}^{n-1} \mathbb{1}_{[-\infty,t]}(\hat{f}_\xi^j(t)).\end{aligned} \tag{7-13}$$

Applying the arc-sin law (Theorem 6.1) to the sequence $\{Y_j\}_{j\geq 0}$, we get the following for every $\alpha \in (0,1)$

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n} \bar{N}_n(\xi) < \alpha\right\}\right) = \frac{2}{\pi} \arcsin\sqrt{\alpha}. \tag{7-14}$$

By (7-13), the previous equality reads as follows

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ \colon \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{[0,y]}(\hat{f}_\xi^j(y)) < \alpha\right\}\right) = \frac{2}{\pi} \arcsin\sqrt{\alpha}, \quad \alpha \in (0,1),$$

ending the proof that sequence $\{\hat{f}_\xi^j\}_{j\geq 0}$ satisfies equation (6-2) and therefore of the lemma. ■

Translating properties to the initial family

Lemma 7.11 *Suppose that $\{\hat{f}_\xi^j(t)\}_{j\geq 0}$ satisfies equations (6-1) and (6-2) for every $t \in \mathbb{R}$. Then $\{f_\xi^j(y)\}_{j\geq 0}$ satisfies equations (6-1) and (6-2) for every $y \in (0,1)$.*

Proof Fix $\xi \in \Sigma^+$. Given any $t \in \mathbb{R}$, by (7-2) we have that for $j \geq 0$,

$$\hat{f}_\xi^j(t) = h \circ f_\xi^j \circ h^{-1}(t). \tag{7-15}$$

Write $y = h^{-1}(t)$ and recall that h is a bijection. Thus, by (7-15) we obtain

$$\begin{aligned}
\mathbb{1}_{[t,+\infty)}(\hat{f}_\xi^j(t)) &= \mathbb{1}_{[t,+\infty)}\left(h \circ f_\xi^j \circ h^{-1}(t)\right) \\
&= \begin{cases} 1 & \text{if} \quad h \circ f_\xi^j \circ h^{-1}(t) \geq t, \\ 0 & \text{if} \quad h \circ f_\xi^j \circ h^{-1}(t) < t, \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad f_\xi^j \circ h^{-1}(t) \geq h^{-1}(t), \\ 0 & \text{if} \quad f_\xi^j \circ h^{-1}(t) < h^{-1}(t), \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad f_\xi^j(y) \geq y, \\ 0 & \text{if} \quad f_\xi^j(y) < y, \end{cases} = \mathbb{1}_{[y,1]}(f_\xi^j(y)).
\end{aligned} \tag{7-16}$$

Since $\{\hat{f}_\xi^j(t)\}_{j\geq 0}$ satisfies equations (6-1) and (6-2), by (7-16) we have that for every $y \in (0,1)$ and every $\alpha \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ : \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[y,1]}(f_\xi^j(y)) < \alpha\right\}\right) = \frac{2}{\pi} \arcsin\sqrt{\alpha},$$
$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma^+ : \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[0,y]}(f_\xi^j(y)) < \alpha\right\}\right) = \frac{2}{\pi} \arcsin\sqrt{\alpha}.$$

This concludes the proof of the lemma. ∎

Lemmas 7.9 and 7.11 imply that (F, id) satisfies the fiber arc-sine law. Then, by Remark 6.3, the pair (F, id) satisfies the fiber weak arc-sine law. Therefore, according to Proposition 6.6, condition (H3) holds. This concludes the proof of Proposition 7.5. ∎

Proposition 7.12 *Let F be a skew product as in* (5-1) *satisfying* (H1) *and* (BM1)–(BM4)*. Then, F satisfies* (H0)*.*

Proof Define the random variable $X_{-1}(\xi) = t$ for every $\xi \in \Sigma^+$ with $t \in \mathbb{R}$. Consider the sums $\{S_0\}_{j\geq -1}$ defined by

$$S_n = \sum_{j=-1}^{n-1} X_j, \quad \text{for every } n \geq 0.$$

Since X_{-1} is independent of itself and hence independent of any other random variable, we have that $\{X_j\}_{j\geq -1}$ is i.i.d. (see Remark 7.7). By the Hewitt-Savage Zero-One Law (see Theorem A.12), the tail σ-algebra

$\mathcal{T}(\{S_j\}_{j\geq 0})$ is trivial. Now, observe that

$$S_0 = t \quad \text{and} \quad S_n = \hat{f}^n_\xi(t), \quad n \geq 1.$$

Hence, the tail σ-algebra $\mathcal{T}(\{\hat{f}^j_\xi(t)\}_{j\geq 0})$ is trivial. Recall that $\hat{f}_\xi(t) = h \circ f_\xi \circ h^{-1}(t)$, where h is an increasing continuous function given in (7-2). Then, the σ-algebra generated by $f_\xi(y)$ for every $y \in I$, is equal to the σ-algebra generated by $\hat{f}_\xi(t)$, for every $t \in \mathbb{R}$. Therefore, the tail σ-algebra $\mathcal{T}(\{\hat{f}^j_\xi(t)\}_{j\geq 0})$ is also trivial, concluding the proof of the proposition. ■

Together, Propositions 7.3, 7.5 and 7.12 conclude the proof of Theorem 7.1.

7.2.1
Zero Schwarzian derivative and ergodic assumption

Now, we study the ergodic properties of the one-step skew product $\hat{F}\colon \Sigma^+ \times \mathbb{R} \to \Sigma^+ \times \mathbb{R}$ defined by

$$\hat{F}(\xi, t) \stackrel{\text{def}}{=} (\sigma(\xi), \hat{f}_\xi(t)), \tag{7-17}$$

where $\hat{f}_\xi(t) = t + \log(a(\xi_0))$ as in (7-2). Note that the measure $\mathbb{P} \times m$ is F-invariant, where m is the Lebesgue measure in $\mathbb{R}$. We are interested in knowing if the product measure $\mathbb{P} \times m$ is ergodic. For this purpose, we consider the following condition:

(E1) The smallest closed subgroup of $(\mathbb{R}, +)$ containing $\log(a(\xi_0))$ for every $\xi_0 \in \mathcal{A}$ is $\mathbb{R}$.

The following theorem can be found in [12, Corollary 2] and [11, Corollary 3], see also [5].

Theorem 7.13 *Let $\hat{F}$ be a skew product as in* (7-17) *satisfying* (E1)*. Then, $\hat{F}$ is ergodic with respect to the measure $\mathbb{P} \times m$.*

In the following proposition, we obtain a weak result related to Theorem 7.1.

Proposition 7.14 *Let F be a skew product as in* (5-1) *satisfying* (H1) *and* (BM1)–(BM4)*. Suppose that F satisfies* (E1)*. Then F satisfies conditions* (H0b) *and the pair* (F, id) *satisfies the fiber arc-sine law. In particular, the skew product F satisfies* (H3b) *with the function* id.

Proof By Theorem 7.13, we conclude that the measure $\mathbb{P} \times m$ is ergodic with respect to $\hat{F}$. Since $\hat{f}_\xi(t) = h \circ f_\xi \circ h^{-1}(t)$, where h is an increasing continuous function defined in (7-2), there is a homeomorphism $H \colon \Sigma^+ \times (0,1) \to \Sigma^+ \times \mathbb{R}$ defined by $H = \mathrm{id} \times h$ such that $H \circ F = \hat{F} \circ H$. As $\hat{F}$ and F are conjugated, then $\nu = (H^{-1})_*(\mathbb{P} \times m)$ is σ-finite ergodic F-invariant measure. Since $H^{-1} = \mathrm{id} \times h^{-1}$ and h^{-1} smooth, we get that ν is equivalent to $\mathbb{P} \times \lambda$. This implies $\mathbb{P} \times \lambda$ satisfies (H0b).

On the other hand, using Proposition 7.5, we establish that (F, id) satisfies the fiber arc-sine law. Then, according to Proposition 6.6, it follows that F satisfies (H3b), proving the proposition. ■

8
Historical behavior in other settings

In this chapter, we analyze the existence of historical behavior and the arc-sine law independently of the Schwarzian derivative in a different settings. First, we consider skew products, which are studied through their core skew product. Second, we study full branch almost expanding interval functions. In Section 8.1, we provide some families of skew products where the fiber (λ-)weak arc-sine law is satisfied. In Section 8.2, following the ideas of Theorem 5.12 and Corollary 6.8.1, we prove Theorem 8.7, dealing with historical behavior for interval functions. In Section 8.3, we provide a class of interval functions (so-called *Thaler functions*) that have historical behavior almost everywhere. This class generalizes the *Manneville-Pomeu functions*.

8.1
Skew products

In this section, we study skew products that satisfy the fiber weak arc-sine law and also have historical behavior. Following Nakamura et al. [17], consider the probability space

$$(\Sigma_2^+, \mathbb{P}) = \left(\{0,1\}^{\mathbb{N}}, \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)^{\mathbb{N}}\right)$$

and let $I = [0,1]$ endowed with the Lebesgue measure λ. Consider the one-step skew product

$$F\colon \Sigma_2^+ \times I \to \Sigma_2^+ \times I, \qquad F(\omega, y) \stackrel{\text{def}}{=} (\sigma(\omega), f_\omega(y)) \tag{8-1}$$

where the fiber maps $f_\xi = f_{\xi_0}$, $\xi_0 \in \{0,1\}$, such that there are $c \in (0, \frac{1}{2}]$, $t > 1$, and g_0 and g_1 are order-preserving measurable functions,

$$g_0\colon \left[c, 1 - \frac{c}{t}\right] \to \left[\frac{c}{t}, 1 - c\right] \quad \text{and} \quad g_1\colon \left[\frac{c}{t}, 1 - c\right] \to [c, 1 - c]\,,$$

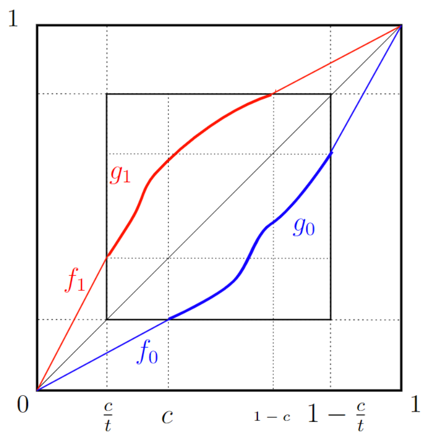

Figure 8.1: Fiber maps

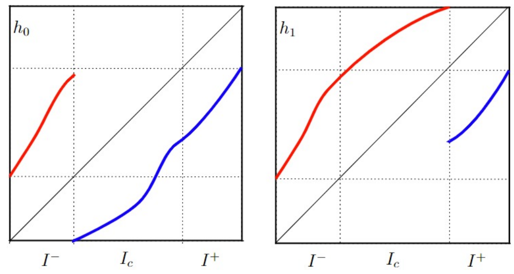

Figure 8.2: Fiber core maps.

satisfying that

$$f_0(y) \stackrel{\text{def}}{=} \begin{cases} \dfrac{y}{t} & \text{if} \quad y \in [0,c), \\ g_0(y) & \text{if} \quad y \in \left[c, 1-\frac{c}{t}\right), \\ ty-(t-1) & \text{if} \quad y \in \left[1-\frac{c}{t}, 1\right], \end{cases}$$

$$f_1(y) \stackrel{\text{def}}{=} \begin{cases} ty & \text{if} \quad y \in \left[0, \frac{c}{t}\right), \\ g_1(y) & \text{if} \quad y \in \left[\frac{c}{t}, 1-c\right), \\ \dfrac{y+(t-1)}{t} & \text{if} \quad y \in [1-c, 1]. \end{cases}$$

See Figure 8.1. Note that these skew products satisfy (H1) and

$$f_0(y) < y \quad \text{and} \quad f_1(y) > y \quad \text{for every } y \in (0,1).$$

This implies that the condition (H2) is also satisfied.

Now, we introduce some notation and definitions. Let $J \subset I$ be the compact interval given by $J \stackrel{\text{def}}{=} I^- \cup I_c \cup I^+$, where

$$I^- \stackrel{\text{def}}{=} \left[\frac{c}{t}, c\right), \qquad I_c \stackrel{\text{def}}{=} [c, 1-c), \quad \text{and} \quad I^+ \stackrel{\text{def}}{=} \left[1-c, 1-\frac{c}{t}\right). \tag{8-2}$$

Associated with F, we define the *core one-step* skew product

$$F_{\text{core}} \colon \Sigma_2^+ \times J \to \Sigma_2^+ \times J, \qquad F_{\text{core}}(\xi, y) \stackrel{\text{def}}{=} (\sigma(\xi), h_\xi(y)), \tag{8-3}$$

where the *fiber core maps* $h_\xi = h_{\xi_0}$, $\xi_0 \in \{0,1\}$, are the form

$$\begin{aligned} h_0(y) &\stackrel{\text{def}}{=} \begin{cases} g_1(y) & \text{if} \quad y \in I^-, \\ g_0(y) & \text{if} \quad y \in I_c \cup I^+, \end{cases} \\ h_1(y) &\stackrel{\text{def}}{=} \begin{cases} g_1(y) & \text{if} \quad y \in I^- \cup I_c, \\ g_0(y) & \text{if} \quad y \in I^+. \end{cases} \end{aligned} \tag{8-4}$$

See Figure 8.2. Recall that a non-zero measure ν on J is called F_{core}-*stationary* if $\mathbb{P} \times \nu$ is F_{core}-invariant (see Definition A.18 and Proposition A.19). Also, ν is said to be *discrete*, its support of ν is finite.

We consider that F satisfies the following condition:

(N1) There is a discrete F_{core}-stationary measure ν such that $\nu(I^-)\nu(I^+) > 0$.

The following theorem, borrowed from Nakamura et al. [17], claims that under that assumption (N1), one-step skew products F as in (8-1) satisfy the fiber weak arc-sine law.

Theorem 8.1 *Let F be a skew product as in* (8-1) *satisfying* (N1) *and ν the measure in* (N1)*. Then, for every $y \in \operatorname{supp}(\nu)$, every $\gamma \in (0,1)$, and every $\alpha \in (0,1)$, for $i = 0,1$, it holds that*

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(f_\xi^j(y)) < \alpha\right\}\right) = \int_0^\alpha \frac{1}{\pi\sqrt{y(1-y)}} \cdot \frac{b}{b^2 y + (1-y)}\, d\lambda,$$

where $I_0(\gamma) \overset{\mathrm{def}}{=} [0,\gamma]$, $I_1(\gamma) \overset{\mathrm{def}}{=} [\gamma, 1]$, $b \overset{\mathrm{def}}{=} \frac{1-\beta}{\beta}$, and $\beta \overset{\mathrm{def}}{=} \frac{\nu(I^-)}{\nu(I^-)+\nu(I^+)}$.

Remark 8.2 *In the case where $\nu(I^-) = \nu(I^+)$, we have $\beta = 1/2$ and $b = 1$. Consequently, for $i = 0,1$, it follows that*

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left\{\xi \in \Sigma_2^+\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(f_\xi^j(y)) < \alpha\right\} &= \int_0^\alpha \frac{1}{\pi\sqrt{y(1-y)}}\, d\lambda \\ &= \frac{2}{\pi}\arcsin\sqrt{\alpha}.\end{aligned}$$

Therefore, Theorem 8.1 *is referred to as the generalized arc-sine law.*

Recall that a skew product that satisfies conditions (H0)–(H2) and the fiber weak arc-sine law has historical behavior almost everywhere (see Corollary 6.8.1). Then, a direct consequence of Theorem 8.1 is that skew products as in (8-1) have historical behavior almost everywhere.

Corollary 8.2.1 *Let F be a skew product as in* (8-1) *satisfying* (H0) *and* (N1)*. Then, F has historical behavior for* $(\mathbb{P} \times \lambda)$-a.e. *point.*

Proof Note that, by construction, F satisfies conditions (H0)–(H2). Let ν be the measure in the assumption (N1). Now, by Theorem 8.1, we have that for every $y \in \operatorname{supp}(\nu)$ it holds

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\xi \in \Sigma_2^+\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(f_\xi^j(y)) < \alpha\right\}\right) < 1, \quad i = 0,1.$$

Then, (F, id) satisfies the fiber weak arc-sine law (Definition 6.2). Since (H0) also hold, by Corollary 6.8.1, we have that F has historical behavior for $(\mathbb{P} \times \lambda)$-a.e. point, ending the proof of proposition. ■

Notice that, in general, the maps h_0 and h_1 generating the core skew-product F_{core} are not continuous, even if both fiber maps f_0 and f_1 of F are continuous (see Figure 8.2). Thus, the existence of an F_{core}-stationary measure might be a non-trivial problem. Nevertheless, we will introduce a class of skew-products where we can easily verify (N1).

Let $t > 1$. Consider the following notations, $I \stackrel{\text{def}}{=} I_0 \cup I^- \cup I^+ \cup I_1$, where

$$I_0 \stackrel{\text{def}}{=} \left[0, \frac{1}{2t}\right), \quad I^- \stackrel{\text{def}}{=} \left[\frac{1}{2t}, \frac{1}{2}\right), \quad I^+ \stackrel{\text{def}}{=} \left[\frac{1}{2}, 1 - \frac{1}{2t}\right), \quad \text{and} \quad I_1 \stackrel{\text{def}}{=} \left[1 - \frac{1}{2t}, 1\right],$$

and $J = I^- \cup I^+$. Note that this notation corresponds to (8-2) when $c = 1/2$. Let $h : J \to J$ be a mesurable function such that

(i) $h|_{I^\pm}$ is order preserving,

(ii) $h(I^-) \subset I^+$ and $h(I^+) \subset I^-$,

(iii) h has a periodic point in J.

Define the one-step skew product

$$F_h \colon \Sigma_2^+ \times I \to \Sigma_2^+ \times I, \qquad F_h(\xi, y) \stackrel{\text{def}}{=} (\sigma(\xi), f_\xi(y)), \tag{8-5}$$

whose fiber maps $f_\xi = f_{\xi_0}$, $\xi_0 \in \{0, 1\}$, are the form:

$$f_0(y) \stackrel{\text{def}}{=} \begin{cases} \dfrac{y}{t} & \text{if } \ y \in I_0 \cup I^-, \\ h(y) & \text{if } \ y \in I^+, \\ ty - (t-1) & \text{if } \ y \in I_1, \end{cases}$$

$$f_1(y) \stackrel{\text{def}}{=} \begin{cases} ty & \text{if } \ y \in I_0, \\ h(y) & \text{if } \ y \in I^-, \\ \dfrac{y + (t-1)}{t} & \text{if } \ y \in I^+ \cup I_1. \end{cases}$$

Figure 8.4 shows an example of piecewise linear fiber maps with a single breakpoint of a skew-product F_h. This map has associated core skew product $H_{\text{core}} = \sigma \times h$, where h is depicted in Figure 8.4. Observe that $f_1 = f_0^{-1}$ and thus, any point in the interval I_+ is a periodic point. Consequently, F_h

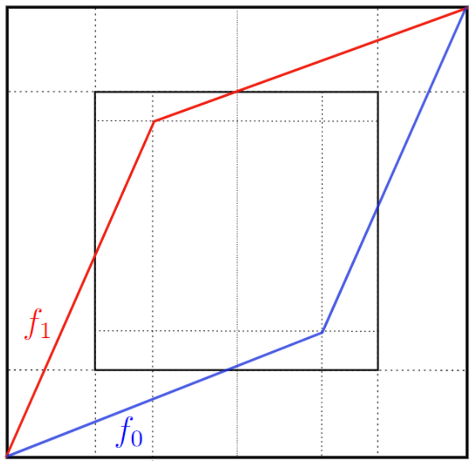

Figure 8.3: Skew product Hata-Yano.

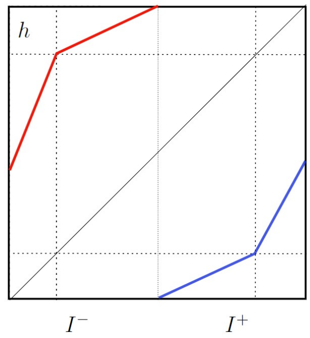

Figure 8.4: Core maps Hata-Yano.

satisfies (N1) on a set of positive measure. The following corollary claims that a similar behaviour also holds under an extra assumption in the periodic point.

Proposition 8.3 *Let $h : J \to J$ be a function satisfying conditions* (i)–(iii)*. Then, F_h satisfies* (N1)*. Moreover, if in addition the periodic point in* (iii) *is attracting, then F_h satisfies the fiber λ-weak arc-sine law.*

Proof Note that by taking $c = 1/2$, the skew-product (8-5) is of the form (8-1) with $g_0 = h|_{I^+}$ and $g_1 = h|_{I^-}$. Consider the core skew product F_{core} as in (8-3) associated with F. By (8-4), we have $h_0 = h_1 = h$. Hence, $H_{\text{core}} = \sigma \times h$. By (iii), h has a periodic point p with period $\pi(p)$. Then, we define the measure

$$\nu \stackrel{\text{def}}{=} \frac{1}{\pi(p)} \sum_{j=0}^{\pi(p)-1} \delta_{h^j(p)}.$$

By (ii), the orbit of p meets I^- and I^+, and thus $\nu(I^+)\nu(I^-) > 0$. Since the point p is periodic, the measure ν is h-invariant. Thus, the product measure $\mathbb{P} \times \nu$ is F_{core}-invariant. Hence, by Proposition A.19, ν is a discrete F_{core}-stationary measure. Therefore, F satisfies condition (N1).

On the other hand, if p is an attracting point of h, Theorem 8.1 holds for any point in the basin of attraction of p. Indeed, observe that if y is close enough to p, $f_\xi^j(y)$ is close to $f_\xi^j(p)$ for every $j \geq 0$ and $\xi \in \Sigma^+$. This follows because while the points are on J, we use h and thus its iterates are closest. When the iterates visit I_0 or I_1, then since in these regions $f_1 = f_0^{-1}$, the distance between the points is either contracted or cannot be expanded much. Therefore, the points close to p follow the same itinerary and thus also satisfy Theorem 8.1, concluding the proof of the proposition. ■

The difficulty arises in constructing examples of skew products as in (8-1), that satisfy either (H0) or (H0b).

8.2
Arc-sin Law in interval functions

We extend the ideas in Chapters 5 and 6 to obtain conditions for interval functions to have historical behavior almost everywhere. Following Remark 5.15, when the alphabet $\mathcal{A}$ is a singleton, the one-step skew products F as in (5-1) can be interpreted as interval measurable functions $f\colon I \to I$ on $I = [0,1]$ satisfying

$$f(0) = 0 \quad \text{and} \quad f(1) = 1. \tag{8-6}$$

Our goal is to establish conditions under which these functions have historical behavior almost everywhere.

Now, we present the arc-sine laws for interval measurable functions as follows (see also the comparison with Definitions 6.2 and 6.4 in Remark 8.5):

Definition 8.4 *Let f be a measurable function as in* (8-6) *and $\psi\colon I \to I$ a non-negative, monotone increasing continuous function. Let $m \overset{\text{def}}{=} \min\psi$ and $M \overset{\text{def}}{=} \max\psi$. The pair (f,ψ) satisfies*

– *the* arc-sine law *if, for every $\gamma \in (m,M)$ and every $\alpha \in (0,1)$, it simultaneously holds*

$$\begin{aligned} \lim_{n\to\infty} \lambda\left(\left\{y \in I\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\psi(f^j(y))) < \alpha\right\}\right) &= \frac{2}{\pi}\arcsin\sqrt{\alpha}, \\ \lim_{n\to\infty} \lambda\left(\left\{y \in I\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\psi(f^j(y))) < \alpha\right\}\right) &= \frac{2}{\pi}\arcsin\sqrt{\alpha}. \end{aligned} \tag{8-7}$$

– *the* weak arc-sine law *if, for every $\gamma \in (m,M)$ and every $\alpha \in (0,1)$, it simultaneously holds*

$$\begin{aligned} \limsup_{n\to\infty} \lambda\left(\left\{y \in I\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,M]}(\psi(f^j(y))) < \alpha\right\}\right) &< 1, \\ \limsup_{n\to\infty} \lambda\left(\left\{y \in I\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[m,\gamma]}(\psi(f^j(y))) < \alpha\right\}\right) &< 1. \end{aligned} \tag{8-8}$$

Remark 8.5 *Definition* 8.4 *is stated with respect to the reference measure in the probability space, similarly to Definition* 6.4*. However, since the alphabet $\mathcal{A}$ has only one element, Definition* 6.2 *of fiber arc-sine laws becomes meaningless, as the same function is always used.*

Remark 8.6 *A pair (f,ψ) satisfying the arc-sine law also satisfies the weak arc-sine law. However, the converse does not hold in general. In [27], Thaler and Zweimüller exhibit a distribution that satisfies the weak arc-sine law but does not satisfy the arc-sine law.*

The following theorem relates the arc-sine laws and the historical behavior.

Theorem 8.7 *Let f be a measurable function as in* (8-6) *admitting a σ-finite ergodic measure ν equivalent to λ. Let $\psi\colon I \to \mathbb{R}$ be a non-constant, increasing, non-negative continuous function such that (f,ψ) satisfies the weak arc-sine law. Then, for λ-a.e. $y \in I$, it holds*

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(y)) = m \quad \textit{and} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi\left(f^j(y)\right) = M.$$

In particular, f has historical behavior for λ-a.e. *point.*

Proof The proof follows using the arguments from the proof of Theorem **??** and Proposition 6.6. We first use the weak arc-sine law to derive the upper bound for the mean visitation of the neighborhoods of 0 and 1. Thereafter, we use this result to conclude that f has historical behavior almost everywhere.

Given any $\gamma \in (m,M)$, let $I_0(\gamma) \stackrel{\text{def}}{=} (m,\gamma)$ and $I_1(\gamma) \stackrel{\text{def}}{=} (\gamma, M)$. Given, $\alpha \in (0,1)$ define the set

$$B_i(\alpha) \stackrel{\text{def}}{=} \left\{ y \in I\colon \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(\psi(f^j(y))) > \alpha \right\}, \quad i = 0,1.$$

Note that, $B_i(\alpha)$ is and f-invariant with respect to f. As (f,ψ) satisfies the weak arc-sine law, it holds

$$\limsup_{n\to\infty} \lambda\left(\left\{ y \in I\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(\psi(f^j(y))) > \alpha \right\} \right) > 0 \quad i = 0,1$$

Now, by Lemma A.7,

$$\lambda\left(B_i(\alpha)\right) \geq \limsup_{n\to\infty} \lambda\left(\left\{ y \in I\colon \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(\psi(f^j(y))) > \alpha \right\} \right) > 0.$$

Since ν is equivalent to λ, we have that $\nu(B_i(\alpha)) > 0$. As ν is ergodic and $B_i(\alpha)$ is f-invariant, it follows that $\nu(I \backslash B_i(\alpha)) = 0$. Then, again from the equivalence of λ and ν, we have that $\lambda(I \setminus B_i(\alpha)) = 0$. This implies that $\lambda(B_i(\alpha)) = 1$. Taking $\alpha \to 1$, we obtain that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_i(\gamma)}(\psi(f^j(y))) = 1, \quad \text{for } \lambda\text{-a.e. } y \in I,\ i = 0,1. \tag{8-9}$$

Now, the following lemma implies that f exhibits historical behavior almost everywhere.

Lemma 8.8 *Let f be a measurable interval function under the assumption of Theorem 8.7. Suppose that* (8-9) *is satisfied. Then*

(i) $\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi\left(f^j(y)\right) = M$ *for λ-a.e. $y \in I$, and*

(ii) $\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi\left(f^j(y)\right) = m$ *for λ-a.e. $y \in I$.*

Proof To prove the first statement, as ψ is non-negative, for every $\gamma \in (0,1)$ we have that

$$\mathbb{1}_{I_1(\gamma)}(\psi(f^j(y))) \cdot \gamma \le \psi(f^j(y)) \quad \text{for every } y \in I,\, j \ge 0.$$

Applying the upper limit we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_1(\gamma)}(\psi(f^j(y))) \cdot \gamma \le \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(y)).$$

By (8-9) it holds that

$$\gamma \le \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(y)) \quad \text{for } \lambda\text{-a.e. } y \in I.$$

Noting that

$$m \le \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(y)) \le M.$$

Since γ is arbitrary, taking $\gamma \to M$ it holds that

$$M = \lim_{\gamma\to M} \gamma \le \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(y)) \le M,$$

proving the first statement. Now, we prove the second statement. The proof is a variation of the proof for the first statement. First, observe that since $1 = \mathbb{1}_{I_0(\gamma)} + \mathbb{1}_{(\gamma,M]}$, by (8-9), we have

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(\gamma,M]}(\psi(f^j(y))) = 0 \quad \text{for Leb-a.e. } x \in I. \tag{8-10}$$

Consider the following inequality:

$$\psi(f^j(y)) \le \mathbb{1}_{I_0(\gamma)}(\psi(f^j(y))) \cdot \gamma + \mathbb{1}_{(\gamma,M]}(\psi(f^j(y))) \cdot M \quad \text{for every } y \in I.$$

Applying the lower limit, we obtain

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(y)) \le \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_0(\gamma)}(\psi(f^j(y))) \cdot \gamma \\ + \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{(\gamma,M]}(\psi(f^j(y))) \cdot M,\end{aligned} \tag{8-11}$$

for every $y \in I$. Applying (8-10) to (8-11), we obtain

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(y)) \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_0(\gamma)}(\psi(f^j(y))) \cdot \gamma,$$

for λ-a.e. $y \in I$. Since

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_0(\gamma)}(\psi(f^j(y))) \cdot \gamma \le \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_0(\gamma)}(\psi(f^j(y))) \cdot \gamma,$$

we have that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(y)) \le \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{I_0(\gamma)}(\psi(f^j(y))) \cdot \gamma.$$

Now, by (8-9), it holds that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(y)) \le \gamma \quad \text{for } \lambda\text{-a.e. } y \in I.$$

Noting that

$$m \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(y)) \le M.$$

Since γ is arbitrary, taking $\gamma \to m$ it holds that

$$m \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(y)) \le \lim_{\gamma\to m} \gamma = m,$$

Therefore, it follows

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(y)) = m, \quad \text{for } \lambda\text{-a.e. } y \in I,$$

proving the second statement and therefore the lemma. ■

Lemma 8.8 implies that f has historical behavior for λ-a.e. point, proving the theorem. ■

Remark 8.9 *Equation* (8-9) *serves as the counterpart of condition* (H3) *and is also derived using the weak arc-sine law for interval functions. Furthermore, the conclusion of Theorem* 8.7 *holds if we replace as assumption* (8-9) *with the weak arc-sine law.*

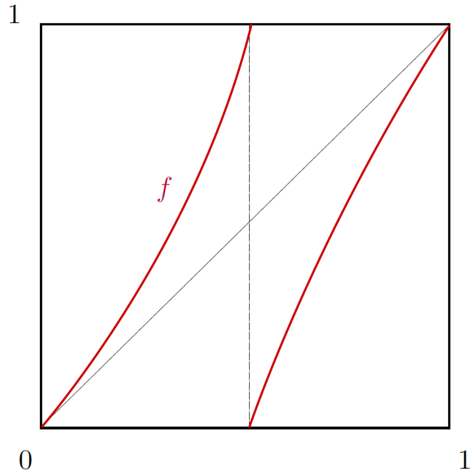

Figure 8.5: Manneville-Pomeu functions.

8.3
Historical behavior in generalized Manneville-Pomeu functions

We analyze interval functions similar to the *Manneville-Pomeau* functions. Namely, we consider piecewise expanding interval functions $f\colon I \to I$ that fix points at the boundaries 0 and 1. Manneville-Pomeau functions are defined as follows (see Figure 8.5) for every $p \geq 1$:

$$f(y) = \begin{cases} y + 2^p y^{p+1} & \text{if} \quad y \in [0, \frac{1}{2}), \\ y - 2^p (1-y)^{p+1} & \text{if} \quad y \in [\frac{1}{2}, 1]. \end{cases}$$

Our goal is to analyze whether these functions satisfy the arc-sine law and possess a σ-finite ergodic measure. When these conditions hold, using Theorem 8.7, we can conclude that they have historical behavior almost everywhere. In fact, Manneville-Pomeau functions belong to a more general family introduced by Thaler in [26] as follows. Consider a measurable function $f\colon I \to I$ as in (8-6), satisfying the following conditions: there exist $c \in (0,1)$ and $p > 1$ such that

(T1) f is *full branch*: the restrictions

$$f^-\colon (0,c) \to (0,1) \quad \text{and} \quad f^+\colon (c,1) \to (0,1)$$

are increasing, onto, and C^2, admit C^2-extensions to the closed intervals $[0,c]$ and $[c,1]$,

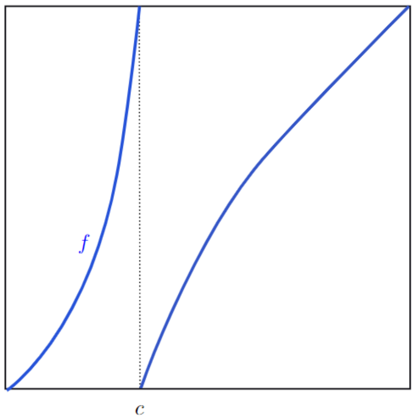

Figure 8.6: Thaler functions.

(T2) f is *almost expanding*: $f'(y) > 1$ for every $y \in (0,1)$, $f'(0) = f'(1) = 1$, and f is convex and concave in neighborhoods of 0 and 1, respectively,

(T3) $f(y) - y = h(y)\, y^{p+1}$, for $y \in (0,c)$ where

$$\lim_{y\to 0} \frac{h(ky)}{h(y)} = 1 \quad \text{for every } k \geq 0, \text{ and}$$

(T4) There is $a \in (0,\infty)$ such that

$$\lim_{y\to 0} \frac{(1-y) - f(1-y)}{a^p(f(y) - y)} = 1.$$

Remark 8.10 *Conditions* $(T1)$ *and* $(T2)$ *are the initial requirements for obtaining measures that are absolutely continuous with respect to the Lebesgue measure. On the other hand,* $(T3)$ *states that the function varies regularly at zero with an index of* $p+1$*. Similarly,* $(T4)$ *implies that* f *varies regularly at one, where the minimum variation is* a^p*.*

In what follows, we refer to functions satisfying (T1)–(T4) as *Thaler functions*, see Figure 8.6. The following theorem has two parts. The first, borrowed from Thaler [24, 25], asserts that every Thaler map has a σ-finite ergodic measure equivalent to λ. The second part, borrowed from [26], claims that (f, id) satisfies the weak arc-sine law.

Theorem 8.11 (Thaler) *Let* f *be a Thaler function. Then, it holds*

(i) *f admits a σ-finite ergodic measure equivalent to λ.*

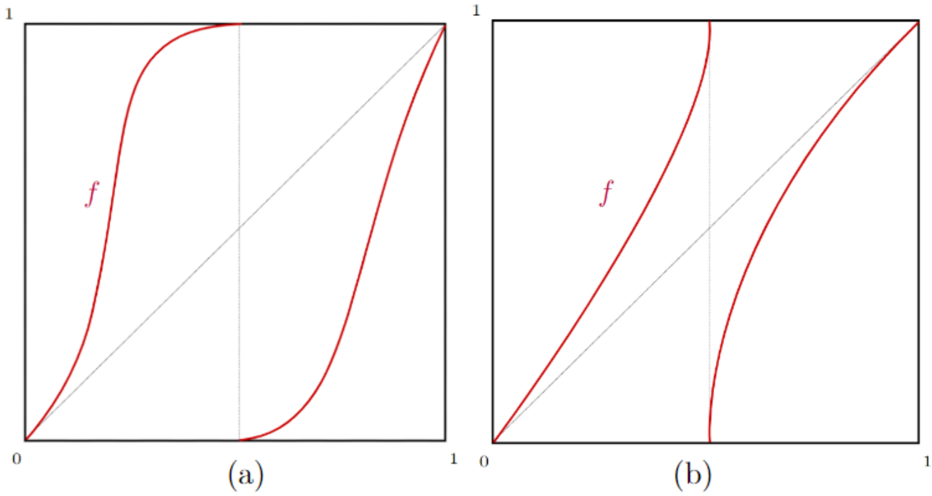

Figure 8.7: (a) Zero derivative; (b) Infinite derivative

(ii) *For every* $\alpha \in (0,1)$*, it simultaneously holds*

$$\lim_{n\to\infty} \lambda\left(\left\{y \in I : \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[0,\gamma]}(f^j(y)) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha},$$
$$\lim_{n\to\infty} \lambda\left(\left\{y \in I : \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{1}_{[\gamma,1]}(f^j(y)) < \alpha\right\}\right) = \frac{2}{\pi}\arcsin\sqrt{\alpha}.$$

A direct consequence of Theorem 8.11 is that every Thaler function have historical behavior.

Corollary 8.11.1 *Every Thaler function has historical behavior for* λ-a.e. *point. In particular, Manneville-Pomeau functions have historical behavior for* λ-a.e. *point.*

Proof By Theorem 8.11 it follows that (f, id) satisfies the arc-sine law and there is a σ-finite ergodic measure. By Remark 8.6, (f, id) also satisfies the weak arc-sine law. Hence, by Theorem 8.7, the function f has historical behavior. ■

Remark 8.12 *Coates et al. [7] consider interval functions f that are similar to the Manneville-Pomeau functions, but may have zero or infinite derivatives at the point of discontinuity, see Figure 8.7. They proved that these functions admit a σ-finite ergodic measure equivalent to λ. Subsequently, Coates and Luzzatto [6] demonstrated that these functions have historical behavior almost everywhere. In this context, they prove* (8-9) *to get historical behavior, see Remark 8.9.*

A
Probability and ergodic theory

In this chapter, we introduce concepts of probability measures and ergodic theory.

Definition A.1 *A* measurable space *is a triplet* $(\Omega, \mathcal{F}, \mu)$ *where*

- $\mathcal{F}$ *is a* σ*-algebra, i.e., a collection of subsets of* Ω *which contains the empty set and is closed under complements and countable unions. The elements of* $\mathcal{F}$ *are called* measurable.
- $\mu\colon \mathcal{F} \to [0, +\infty)$ *is a* σ*-additive function called* measure.

If $\mu(\Omega) = 1$, then we say that μ is a *probability measure* on Ω.

Definition A.2 *Let* $(\Omega, \mathcal{F})$ *and* $(\Sigma, \mathcal{C})$ *be measurable spaces.*

- *The function* $\psi\colon \Omega \to \Sigma$ *is said to be* measurable *if* $\psi^{-1}(C) \subset \mathcal{F}$ *for every* $C \subset \mathcal{C}$.
- *A function* $\psi\colon \Omega \to \mathbb{R}$ *that is measurable is a* random variable *of* $(\Omega, \mathcal{F}, \mu)$.

Lemma A.3 (Fatou Lemma) *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space and* $E_1, E_2, \ldots$ *be measurable sets in* Ω*. Then*

$$\mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n) \leq \limsup_{n\to\infty} \mu(E_n) \leq \mu(\liminf_{n\to\infty} E_n) \tag{A-1}$$

Theorem A.4 (Monotone convergence) *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measurable space and* $\{\psi_n\}_{n\geq 0}$ *be a sequence of non-negative measurable functions. Suppose that* $\psi_1 \leq \psi_2 \leq \ldots$ *and* $\psi_n \to \psi$ *pointwise, then:*

$$\lim_{n\to\infty} \int \psi_n \, d\mu = \int \psi \, d\mu.$$

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_1, \mu_2)$ be two measurable spaces.

Theorem A.5 (Fubini) *Let* $(\Omega_1, \mathcal{F}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{F}_1, \mu_2)$ *be two measurable spaces. Let* $(\Omega, \mathcal{F}, \mu) = (\Omega_1 \times \Omega_2, \sigma(\mathcal{F}_1 \times \mathcal{F}_1), \mu_1 \times \mu_2)$ *and* $\psi\colon \Omega \to \mathbb{R}$ *be a measurable function with* $\psi \geq 0$ *and* $\int |\psi| \, d\mu < \infty$*. Then*

$$\int \psi \, d\mu = \int \int \psi \, d\mu_1 \, d\mu_2 = \int \int \psi \, d\mu_2 \, d\mu_1.$$

Now we present some properties of the random variables (see [20, 21] for more information).

Definition A.6 *Let $\psi\colon \Omega \to \mathbb{R}$ be a random variable of $(\Omega, \mathcal{F}, \mu)$. The expectation and variance of ψ with respect to μ are defined, respectively, by*

$$\mathbb{E}(\psi) \stackrel{\text{def}}{=} \int \psi \, d\mu \quad \textit{and} \quad \operatorname{Var}(\psi) \stackrel{\text{def}}{=} \mathbb{E}(\psi^2) - \mathbb{E}(\psi)^2. \tag{A-2}$$

Lemma A.7 *Let $\{\psi_n\}_{n\geq 0}$ be a sequence of random variables. For every constant, $c > 0$ it holds that*

$$\mu\left(\left\{\xi \in \Omega\colon \limsup_{n\to\infty} \psi_n(\xi) > c\right\}\right) \geq \limsup_{n\to\infty} \mu\left(\{\xi \in \Omega\colon \psi_n(\xi) > c\}\right)$$

Proof Denote by

$$A = \{\xi \in \Omega\colon \limsup_{n\to\infty} \psi_n(\xi) > c\} \quad \text{and} \quad B = \limsup_{n\to\infty} \{\xi \in \Omega\colon \psi_n(\xi) > c\},$$

we first prove the inclusion $A \supset B$. Given any $\xi \in B$, there are infinitely many $n \geq 1$ such that $\psi_n(\xi) > c$. Since there are infinitely many n, we can choose n_0 such that for every $n \geq n_0$, it holds that $\psi_n(\xi) > c$. Therefore, $\xi \in A$ and $B \subset A$. Now consider the set

$$B_n = \{\xi \in \Omega\colon \psi_n(\xi) > c\} \quad \text{such that} \quad \limsup_{n\to\infty} B_n = B$$

Note that by Lemma A.3 we have that

$$\limsup_{n\to\infty} \mu(B_n) \leq \mu(B) \leq \mu(A),$$

ending the proof of lemma. ∎

Definition A.8 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\psi\colon \Omega \to \mathbb{R}$ a random variable. Then the* distribution function *$D\colon \mathbb{R} \to [0, 1]$ of ψ is defined by $D(a) \stackrel{\text{def}}{=} \mu(\psi \leq a)$ with the following properties:*

- *For every $a \leq b$ we have that $D(a) \leq D(b)$,*
- *If $a_n \to a$ then $D(a_n) \to D(a)$*
- *$\lim_{a\to-\infty} D(a) = 0$ and $\lim_{a\to\infty} D(a) = 1$.*

Definition A.9 *Let $\psi, \phi\colon \Omega \to \mathbb{R}$ be random variables of $(\Omega, \mathcal{F}, \mu)$. We say that they are* independent *if and only if, for every $a, b \in \mathbb{R}$, it holds*

$$\mu(\{\psi < a\} \cap \{\phi < b\}) = \mu(\{\psi < a\}) \cdot \mu(\{\phi < b\}). \tag{A-3}$$

Moreover, we say that ψ and ϕ are identically distributed *if they have the same distribution function. When ψ and ϕ are independent and identically distributed, we denote them as i.i.d.*

Definition A.10 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and consider a sequence $\{\psi_n\}_{n\geq 0}$ of random variables. Let $\mathcal{F}_m^\infty = \sigma(\psi_m, \psi_{m+1}, \dots)$ be the σ-algebra generated by $\psi_m, \psi_{m+1}, \dots,$ and defined the* tail algebra *by*

$$\mathcal{T}(\{\psi_n\}_{n\geq 0}) \stackrel{\text{def}}{=} \bigcap_{m=1}^{\infty} \mathcal{F}_m^\infty. \tag{A-4}$$

Lemma A.11 *Let $\{\psi_n\}_{n\geq 0}$ be a sequence of random variables and $\varphi \colon \mathbb{R} \to \mathbb{R}$ measurable function. For every constant $b > 0$, the sets*

$$A(b) = \{\xi \in \Omega \colon \limsup_{j\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(\psi_j(\xi)) < b\}, \ \textit{and}$$
$$B(b) = \{\xi \in \Omega \colon \limsup_{j\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(\psi_j(\xi)) > b\}$$

belong to the tail algebra.

Proof Consider $n > m > 0$, we have that

$$\frac{1}{n}\sum_{j=0}^{n-1} \varphi(\psi_j(\xi)) = \frac{1}{n}\sum_{j=m}^{n-1} \varphi(\psi_j(\xi)) + \frac{1}{n}\sum_{j=0}^{m-1} \varphi(\psi_j(\xi)).$$

Applying the limit to the inequality, we obtain that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(\psi_j(\xi)) = \limsup_{n\to\infty} \frac{1}{n}\sum_{j=m}^{n-1} \varphi(\psi_j(\xi)). \tag{A-5}$$

Therefore, the sets $A(b)$ and $B(b)$ are elements of the tail algebra $\mathcal{T}(\{\psi_n\}_{n\geq 0})$ defined in (A.10). ■

The following theorem, borrowed from [15, §26, Theorem B], demonstrates that if we have a random walk, the tail σ-algebra is trivial. Further insights can be found in [2].

Theorem A.12 (Hewitt-Savage Zero-One Law) *Let $\{\psi_n\}_{n\geq 0}$ be a sequence of random variables, and let $S_0, S_1, S_2, \dots$ be the sums*

$$S_n = \psi_0 + \cdots + \psi_n, \quad n \geq 0.$$

If ψ_i are i.i.d., then the tail σ-algebra $\mathcal{T}(\{S_n\}_{n\geq 0})$ is trivial.

Now we present some concepts of Ergodic Theory (see [28] for more information).

Definition A.13 *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space, and* $T\colon \Omega \to \Omega$ *be a measurable function. We say that* μ *is a* T-invariant measure *on* Ω *or that* T preserves μ *if* $\mu(T^{-1}(A)) = \mu(A)$ *for every measurable set* $A \in \mathcal{F}$.

Definition A.14 *The space* $(\Omega, \mathcal{F}, \mu, T)$ *is called a* measure-preserving dynamical system *if* μ *is* T*-invariant. Let* $\phi\colon \Omega \to \mathbb{R}$ *be an absolutely integrable function. We have*

- *A measurable subset* $A \subset \Omega$ *is said to be* T-invariant *if* $T^{-1}(A) = A$.
- *A function* ϕ *is a* T-invariant function *if* $\phi \circ T = \phi$, *for* μ*-almost every point.*

Definition A.15 *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A measure* μ *on* Ω *is called* σ-finite *if there is a countable family of subsets* $A_1, A_2, \dots$ *of* Ω *such that* $\mu(A_i) < \infty$ *for every* $i \in \mathbb{N}$ *and*

$$\Omega = \bigcup_{i=1}^{\infty} A_i.$$

Definition A.16 *An* T*-invariant measure* μ *is called* ergodic *if* $\mu(A) = 1$ *or* $\mu(A) = 0$ *for every* T*-invariant subset* $A \subset \Omega$*. If the measure* μ *is* σ*-finite, then it is ergodic if* $\mu(A) = 0$ *or* $\mu(\Omega \setminus A) = 0$.

The following theorem is the most important result of ergodic theory, states that given an preserving dynamical system, the time average of an observable along the orbit converge everwhere. This result is formulated as follows.

Theorem A.17 (Birkhoff) *Let* $(\Omega, \mathcal{F}, \mu, T)$ *be a measure-preserving dynamical system. Then, for every integrable function* $\phi\colon \Omega \to \mathbb{R}$*, the limit*

$$\widehat{\phi}(\xi) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(\xi))$$

exists for μ*-almost every* $\xi \in \Omega$*. Moreover:*

- $\widehat{\phi}$ *is a* T*-invariant function, and* $\int \widehat{\phi}\, d\mu = \int \phi\, d\mu$,
- *if* μ *is ergodic, then*

$$\widehat{\phi}(\xi) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(T^j(\xi)) = \int \phi\, d\mu, \quad \textit{for } \mu\textit{-almost every } \xi \in \Omega.$$

Let $M = X^{\mathbb{N}}$ be a product space where X is some probability space, endowed with the product measure $\mu = \rho^{\mathbb{N}}$ and N a measurable space. Consider the one-step skew product

$$F\colon M \times N \to M \times N, \quad F(x, v) \stackrel{\text{def}}{=} (\sigma(x), f_x(v)), \tag{A-6}$$

where $f_x\colon N \to N$ are measurable functions.

Definition A.18 *Let $F\colon M \times N \to M \times N$ be a skew product as in* (A-6)*. A non-zero measure ν on N is called F-*stationary *if*

$$\nu(A) = \int \nu(f_x^{-1}(A))\, d\nu, \quad \textit{for every measurable set } A \subset N.$$

Proposition A.19 *Let $F\colon M \times N \to M \times N$ be a skew product as in* (A-6)*. A measure ν on N is F-stationary if, and only if, the product measure $\mu \times \nu$ on $M \times N$ is F-invariant.*

B
Schwarzian derivative

In this chapter, we study some properties of the *Schwarzian derivative*. Let $f\colon I \to I$ be a C^3-diffeomorphism on the interval $I = [0,1]$. The *Schwarzian derivative* of any C^3-diffeomorphism is defined by

$$Sf(y) \overset{\text{def}}{=} \frac{f'''(y)}{f''(y)} - \frac{3}{2}\left(\frac{f''(y)}{f'(y)}\right)^2.$$

We introduce the following notation for Sf:

- If $Sf(y) < 0$ for every y in a dense subset of I, we write $Sf \prec 0$;
- If $Sf(y) > 0$ for every y in a dense subset of I, we write $Sf \succ 0$; and
- If $Sf(y) = 0$ for every y in a dense subset of I, we write $Sf \equiv 0$.

The following results state some properties of the Schwarzian derivative (see [9] for more information).

Proposition B.1 *The Schwarzian derivative has the following properties:*

(a) *The sign of Sf is preserved under composition: given a C^3-diffeomorphism $f\colon I \to I$ with $Sf \prec 0$ and a C^3-diffeomorphism $g\colon I \to I$ with $Sg \prec 0$, then $S(g \circ f) \prec 0$;*

(b) *$Sf \prec 0$ if and only if $Sf^{-1} \succ 0$;*

(c) *$Sf \prec 0$ if and only if $\phi(y) = \frac{1}{\sqrt{|f'(y)|}}$ is strictly convex (in other words, if and only if $\phi'(y)$ is increasing); and*

(d) *If $Sf \equiv 0$ if and only if f is a fractional linear map $f(x) = \frac{ax+b}{cx+d}$ with $ad - bc \neq 0$ and $cx + d \neq 0$.*

Definition B.2 *Given $f\colon I \to I$ a diffeomorphism, a fixed point $y = f(y) \in I$ is called hyperbolic attractor if $|f'(y)| < 1$ and hyperbolic repeller if $|f'(y)| > 1$.*

Lemma B.3 *Let $f\colon I \to I$ be a C^3-diffeomorphism with $Sf \prec 0$, then f it can only have at most three fixed point. If f has three fixed points, then the middle one must be hyperbolic repeller and the other two be hyperbolicc attractors.*

Proof First, we show that it cannot have four fixed points. Suppose there are four points $y_1 < y_2 < y_3 < y_4$ that are fixed points. Then, by the Mean Value Theorem on the intervals (y_1, y_2), (y_2, y_3), and (y_3, y_4), there exist points in each interval with $f'(y) = 1$. Therefore, the function $\varphi(y) = \frac{1}{\sqrt{|f'(y)|}}$ is constant, which contradicts $Sf \prec 0$ since the function φ is strictly convex.

Now, suppose there are three fixed points $y_1 < y_2 < y_3$. Due to the strict convexity of φ, there exists $\alpha \in (y_1, y_2)$ and $\beta \in (y_2, y_3)$ such that $\varphi(\alpha) = \varphi(\beta) = 1$. Then, by convexity, we have

$$\varphi(y_2) < 1 < \varphi(y_1) \quad \text{and} \quad \varphi(y_2) < 1 < \varphi(y_3).$$

Thus, by the definition of φ and Definition B.2, we conclude that:

$$\begin{aligned} 1 < |f'(y_2)| &\quad \text{so } y_2 \text{ is strictly a repeller;} \\ 1 > |f'(y_1)| &\quad \text{so } y_1 \text{ is strictly an attractor; and} \\ 1 > |f'(y_3)| &\quad \text{so } y_3 \text{ is strictly an attractor,} \end{aligned}$$

ending the proof of lemma ■

Definition B.4 *The cross-ratio of four distinct real numbers is defined by*

$$\rho(y_1, y_2, y_3, y_4) \stackrel{\text{def}}{=} \frac{(y_3 - y_1)(y_4 - y_2)}{(y_2 - y_1)(y_4 - y_3)}. \tag{B-1}$$

Note that $\rho(y_1, y_2, y_3, y_4) > 1$ when $y_1 < y_2 < y_3 < y_4$. We say that a map f increases the cross-ratio if

$$\rho(f(y_1), f(y_2), f(y_3), f(y_4)) > \rho(y_1, y_2, y_3, y_4) \quad \text{when } y_1 < y_2 < y_3 < y_4.$$

Lemma B.5 *Consider $f\colon I \to I$ to be a C^3-diffeomorphism. We have the following:*

(i) *f increases the cross-ratio if and only if $Sf \prec 0$;*

(ii) *f decreases the cross-ratio if and only if $Sf \succ 0$; and*

(iii) *f preserves the cross-ratio if and only if $Sf \equiv 0$.*

Proof We prove only item (i), the other items are similar, and their proofs are hence omitted. Suppose $Sf < 0$ on a dense open set. Given the points $y_1 < y_2 < y_3 < y_4$, after composing f with a fractional linear transformation, we can assume that f has three fixed points y_1, y_2, y_4. If $Sf \prec 0$, then y_1, y_4 are attractors, and y_2 is a repeller by Lemma B.3. Therefore, since there cannot be a fixed point between y_2 and y_4, we conclude that f moves every intermediate point to the right. Then $y_3 < f(y_3)$, and we have

$$\begin{aligned}\rho(y_1, y_2, y_3, y_4) &< \rho(y_1, y_2, f(y_3), y_4) \\ &= \rho(f(y_1), f(y_2), f(y_3), f(y_4)).\end{aligned}$$

We conclude that f increases the cross-ratio of the points $y_1, y_2, f(y_3)$, and y_4.

On the other hand, if Sf is non-negative on a dense open set, we have the following cases:

(i) Sf is strictly positive. Then, by item (b) of Proposition B.1, $Sf^{-1} \prec 0$, so f^{-1} increases the cross-ratio, and therefore, f decreases the cross-ratio;

(ii) Sf is identically zero. Then, by item (d) of Proposition B.1, f is a fractional linear map, and it preserves the cross-ratio within the interval.

We conclude that f does not increase the cross-ratio. Therefore, f increases the cross-ratio if and only if $Sf \prec 0$. ■

Lemma B.6 *If $Sf \prec 0$ for a diffeomorphism f that preserves the orientation of an interval $J = [a, b]$, then $f'(a)f'(b) < 1$.*

Proof First, suppose that $f'(a) = f'(b)$. If $f'(a)f'(b) \geq 1$, then, since $Sf \prec 0$, the function φ is convex, and we have $\varphi(y) < 1$ for $y \in (a, b)$. Therefore, $f'(y) > 1$ for all $y \in (a, b)$. However, this is impossible since if $f'(y) > 1$, then $|f(b) - f(a)| > |b - a|$ and $f\colon J \to J$. Thus, we conclude that $f'(a)f'(b) < 1$. For the general case, let $r\colon J \to J$ be a reflection that interchanges a and b, defined by $r(y) = b - (y - a)$. Consider an auxiliary function $g = r \circ f \circ r$. Then, $Sg \prec 0$ with $g'(a) = f'(b)$ and $g'(b) = f'(a)$. Therefore, the composition $f \circ g$ satisfies $S(f \circ g) \prec 0$ and the derivatives

$$(f \circ g)'(a) = (f \circ g)'(b) = f'(a)f'(b).$$

By the initial argument, we have $f'(a)f'(b) < 1$, ending the proof. ■

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