Pontifícia Universidade Católica
do Rio de Janeiro

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Discretization of "four-vertex type" theorems for spatial and spherical polygons

PhD Dissertation

Dissertation presented to the Programa de Pós–graduação em Matemática, do Departamento de Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Marcos Craizer

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To my family - *Love endures all things.*

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Abstract

Pacitti Gentil,Samuel; Craizer, Marcos (Advisor). **Discretization of "four-vertex type" theorems for spatial and spherical polygons**. Rio de Janeiro, 2024. 107p. PhD Dissertation – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

The aim of this work is to study a certain class of spatial polygons and prove theorems on the minimal number of flattenings that such polygons must have. In order to do this, we investigate spherical polygons which are not contained in any closed hemisphere and deduce, among many results, that under certain hypotheses such spherical polygons have a nontrivial lower bound on the number of spherical inflections.

Keywords

Spherical Polygons; Spherical Inflections; Four Vertex Theorems.

Resumo

Pacitti Gentil,Samuel; Craizer, Marcos. **Discretização de teoremas do tipo "quatro vértices" para polígonos espaciais e esféricos**. Rio de Janeiro, 2024. 107p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

O objetivo deste trabalho é estudar uma certa classe de polígonos espaciais e provar teoremas a respeito do número mínimo de achatamentos que tais polígonos necessariamente possuem. Para tal, investigamos polígonos esféricos que não estão contidos em nenhum hemisfério fechado e deduzimos, entre vários resultados, que sob certas hipóteses tais polígonos esféricos possuem uma cota inferior não-trivial para o número de inflexões esféricas.

Palavras-chave

Polígonos Esféricos; Inflexões Esféricas; Teoremas dos Quatro Vértices.

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And further, my son, be admonished by
these: Of making many books there is no end,
and much study is wearisome to the flesh.
Let us hear the conclusion of the whole
matter: Fear God and keep His
commandments, for this is man's all.

Qoheleth, *Book of Ecclesiastes 12:12-13, NKJV.*

1
Introduction

1.1
The four-vertex theorem and its generalizations

The four-vertex theorem is a remarkable result in the differential geometry of planar curves. In order to state the theorem, we need some basic definitions: Let γ be a planar closed curve which is also *simple* (i.e., it does not have self-intersections). We say that γ is *convex* if its interior is a convex region, i.e., for each pair of points in the interior, the segment determined by the pair is also in the interior. The original four-vertex-theorem proved by Syamadas Mukhopadhyaya in 1909 (see (MUKHOPADHYAYA, 1909)) is:

Theorem 1.1 *A planar curve α of class $\mathcal{C}^3$ which is closed, simple and convex must have at least four local extremal points for its curvature κ (two of these points are local minima and the other two are local maxima). Such points are called the vertices of the curve.*

Of course, if the curve being considered contains a segment, then the curvature is constant equal to zero in this segment. Hence every point in the relative interior of this segment is a local maximum and a local minimum and therefore the four-vertex-theorem follows trivially for such a curve. For this reason we will assume that our smooth convex curves do not contain any segments (some authors call such curves *strictly convex*, but we will not follow this terminology as applied to such curves since we reserve this term to another class of curves to appear).

Since then, many mathematicians have come up with new proofs and new generalizations of this beautiful theorem. For instance, Adolph Kneser proved the theorem in 1912 (see (KNESER, 1912)) without the restriction of the curve being convex.

Another version is a result of a “global" rather than “local" nature. Recall from the theory of differential geometry that the *osculating circle* $\mathcal{C}$ at a point p of a curve γ is the circle that best approximates the curve at a given point (such a circle is said to have order of contact equal or greater than 3). We say that an osculating circle $\mathcal{C}$ at a point of a smooth curve γ is *full* if γ lies entirely inside $\mathcal{C}$, and *empty* if α lies entirely outside $\mathcal{C}$. An *extremal* circle is any full or empty circle. Hellmuth Kneser (Adolph Kneser's son) proved in 1922 (see (KNESER, 1922)) the following result:

Theorem 1.2 *Any smooth convex closed curve has at least four extremal circles (two of them being full and the other two being empty).*

Now, if a circle $\mathcal{C}$ at a point $p \in \gamma$ is full/empty, then the curvature of γ has a local minimum/maximal (but not the converse), since the curvature is the inverse of the radius of the osculating circle. Therefore the classical four-vertex theorem follows as a corollary from Kneser's result.

We can extend the previous theorem even more. Bose (see (BOSE, 1932)) proved in 1932 the next theorem:

Theorem 1.3 *Let γ be a regular planar convex curve which is generic (i.e., it is not tangent to any circle at more than 3 points). Denote by s_+ and s_- the number of full and empty osculating circles to γ, respectively. Denote by t_+ and t_- the number of full and empty circles tangent to γ at three points. Then*

$$s_+ - t_+ = s_- - t_- = 2.$$

By Theorem 1.3, $s_+ = 2 + s_- \geq 2$ and $t_+ = 2 + t_- = 2 \geq 2$, which then implies Theorem 1.2. For a proof of Theorem 1.3, see (BOSE, 1932).

Another version concerns the minimal number of points of a *spherical* (rather than planar) closed curve at which its geodesic curvature κ_g attains a maximum or a minimum (these are called the *vertices* of the spherical curve). If the curve is *convex in the spherical sense* (i.e., the curve is contained in a closed hemisphere and the interior of the curve is convex in the sense that for any two points of the interior, the spherical segment between them also lies in the interior), then the number of vertices is greater or equal to four.

One can generalize this theorem even further. In order to do this for space curves (not necessarily spherical ones), it is necessary to reformulate not only the notion of convexity for such curves, but also the notion of a vertex.

First recall from the theory of plane curves that the vertex of a curve is also the point p at which the osculating circle C has a point of contact with the curve greater or equal to 4 (see (BRUCE; GIBLIN, 1984), pp 16-37 for more details). The same phenomenon happens with spherical curves: a vertex of a spherical curve is a point p at which the *spherical osculating circle* C has a point of contact with the curve greater or equal to 4. This spherical circle C is the intersection of the sphere with a plane Π. Again, from the theory of space curves, the circle C has contact of order 4 with the curve at p if and only if the plane Π has contact of order 4 with the curve at p. But this happens if and only if the *torsion* τ of the curve vanishes at point p.

It makes sense, therefore, to define the *vertex* of a space curve (spherical or not) as a point at which its torsion τ vanishes. A vertex of a space curve is also called a *flattening* (because of the higher order of contact that the osculating plane has with the curve at such point), and we will stick to this terminology.

Now we need to come up with a good definition of convexity for space curves. Contrary to the case of planar or spherical curves, a space curve does not have an interior region. We need another way of expressing convexity for planar curves that also applies for space curves.

A first approach would be to notice that a convex curve in $\mathbb{R}^2$ is convex if and only if it is (entirely) contained in the boundary of its convex hull. Following this idea, Romero Fuster (see (ROMERO-FUSTER, 1988)) and Sedykh (see (SEDYKH, 1992)) studied such curves in $\mathbb{R}^3$ in order to derive a four-vertex type theorem. We say that a regular space closed curve is *weakly convex* if it lies on the boundary of its convex hull.

Theorem 1.4 *Any weakly convex curve has at least four flattenings.*

For a proof of theorem 1.4, see (ROMERO-FUSTER, 1988) or (SEDYKH, 1992).

Another reformulation of the notion of convexity is given by the following condition: a plane curve is convex if and only if, for any two given points of the curve, the line that passes through them intersects the curve only at these two points (the two points might be the same, in which case the tangent line intersects the curve only at this point, but with multiplicity 2). We say that a regular space curve γ is *strictly convex* if, for any two given points of γ, there is a plane intersecting the curve only at these points (if the two points are distinct, then the plane intersects the curve transversally at both points, otherwise the plane is tangent to the curve at the point). For such a curve, a result by Barner (see (BARNER, 1956)) establishes a lower bound for the number of flattenings:

Theorem 1.5 *A strictly convex space curve has at least* 4 *flattenings.*

Barner's result is more general since it is valid for certain curves defined in any projective space of dimension $n \geq 2$ and for any n-dimensional euclidean space if $n \geq 3$ is odd, but we will not delve into this subject here. For the definition of *strictly convexity for projective curves* and a proof of 1.5 in this more general context, see (BARNER, 1956) and (OVSIENKO; TABACHNIKOV, 2005).

A third way to see convexity of a plane curve is to look at the tangent vector at each point: a plane curve is convex if and only if, for each pair of (distinct) points of the curve, their respective tangent vectors do not point to the same direction. Segre proved in 1968 (see (SEGRE, 1968)) that space curves that satisfy this property must have at least four flattenings.

Theorem 1.6 *A regular space closed curve without parallel tangents with the same orientation has at least four flattenings.*

There does not seem to be a name in the literature for curves that satisfy the hypotheses of theorem 1.6. Following Uribe (see (URIBE-VARGAS, 2003)), we will call them *Segre curves*. Here is another way of viewing such curves.

Given a regular curve γ in $\mathbb{R}^3$ parametrized by arclength, its unit tangent vector γ', with its base point translated to the origin, describes a curve on the unit sphere $\mathbb{S}^2$, which is called the *tangent indicatrix* of γ. For instance, if a space curve γ is contained in a plane, its tangent indicatrix γ' is contained in a great circle. Moreover, if the space curve γ is closed and not contained in any plane, then its tangent indicatrix γ' is not contained in any closed hemisphere.

The condition for a space curve γ to be a Segre curve is then equivalent to the condition of its tangent indicatrix being embedded in $\mathbb{S}^2$, i.e., not having self-intersection nor cusps. Moreover, a space curve γ has a flattening at a point $\gamma(t_0)$ if and only if the tangent indicatrix has a geodesic inflection at the corresponding point $\gamma'(t_0)$. Therefore, theorem 1.6 follows from the following result:

Theorem 1.7 *If a spherical curve γ, not contained in a closed hemisphere, is regular, closed and simple (i.e., without self-intersections), then γ has at least four (geodesic) inflections.*

For a proof of theorem 1.7, see (SEGRE, 1968) or (GHOMI, 2013). A famous corollary of 1.7 is the following theorem, rediscovered and popularized by Arnold (see (ARNOLD, 1994)):

Theorem 1.8 *(Tennis Ball Theorem) If a spherical curve γ is regular, closed and divides the sphere in two regions with the same area, then γ has at least four inflections.*

Another corollary of theorem 1.7 is the following result due to Möbius (see (MöBIUS, 1886)):

Theorem 1.9 *If a spherical curve γ is regular, closed and symmetric with respect to the origin, then γ has at least six inflections.*

Although the three conditions (weak convexity, strictly convexity and “Segre convexity") are equivalent in the planar case (since they are equivalent to the usual notion of convexity), they will characterize different classes of curves in Euclidean space $\mathbb{R}^3$. Uribe (see (URIBE-VARGAS, 2003)) studied these classes and established relations between them. He proved that:

– strictly convex curves in $\mathbb{R}^3$ are weakly convex curves and also Segre curves. Therefore theorem 1.5 is a corollary of both theorems 1.4 and 1.6;
– there is a non-empty open set of weakly convex curves in $\mathbb{R}^3$ which are not Segre curves, and there is a non-empty open set of Segre curves in $\mathbb{R}^3$ which are not weakly convex. Therefore neither of these two notions of convexity is more general than the other.

Besides the classes of curves discussed above, there are other types (in $\mathbb{R}^3$ or in higher-dimensional Euclidean/projective spaces), but we will not consider them here.

1.2 Discrete versions of the four-vertex theorem in the plane

Another strategy to deal with these theorems (for both the planar and the spatial versions) is to consider the discrete case: instead of using smooth curves, the object of study consists of polygons. This approach simplifies considerably the problem, enables us to use induction on the number of the vertices, and makes it possible to use tools from combinatorics. Moreover, the discrete result becomes, in the limit, a smooth one, providing in this way an alternative proof of the latter. The downside of this strategy is the ambiguity of the process of discretization: there might be more than a way of doing so. Consequently, there might be discrete versions of theorems from the smooth case which are not equivalent to each other.

The literature on discrete versions of the four-vertex theorem on the plane has been growing in the last decades. In this case there are several results, each one shining a new light on different aspects of the problem. Some of these results are stated in terms of the angle differences, others as the number of certain circles defined by triples of vertices. Besides that, these theorems translate easily to theorems on *convex spherical polygons* on the sphere.

Here we understand a *plane polygon* $P = [v_1, ..., v_n]$ as a closed curve on the plane formed by a concatenation of straight line segments, each one connecting the *vertices* v_1, v_2, ..., v_n of the polygon P in a cyclic sequence. The *edges* of the polygon are the closed segments $e_i = [v_i, v_{i+1}]$, where we consider the indices $i \mod n$. Here convexity for polygons is the same as for smooth curves: a polygon P is *convex* if its interior is a convex region.

The first difficulty is terminology: *vertex* of a polygon $P = [v_1, v_2, ..., v_n]$ has the usual meaning, i.e., it refers to each of the points at which the polygon does not need to be smooth. If we want to distinguish special points of the polygon that inherit the notion of "vertex as a singular point" (since we want to discretize the smooth theory), it is better to use new terms such as *extremal vertices*. And what would be a good definition for such vertices?

The most obvious choice is to think of the curvature of the polygon as a discrete function at each vertex. A good candidate for such a discrete function is then the exterior angle at each vertex (if three vertices are collinear, the intermediate vertex would correspond to a point of curvature zero). Denote each of these exterior angles by θ_i.

Another definition for curvature of a polygon at a vertex, however, is inspired by the smooth theory of osculating circles. First notice that the notion of a osculating circle can be easily defined for a plane polygon: at each triple $\{v_{i-1}, v_i, v_{i+1}\}$ of consecutive vertices, the *osculating circle* $\mathcal{C}_i$ is the circle defined by these three vertices (i.e., the circle circumscribed to the triangle whose vertices are v_{i-1}, v_i and v_{i+1}). If r_i is the radius of the circle $\mathcal{C}_i$, define the *curvature* of P at vertex v_i as $\kappa_i := 1/r_i$. Notice that this aligns with what we would expect from the smooth theory: it is a basic but fundamental result for curves parametrized by arclength that the absolute value of the curvature at a given point is the inverse of the radius of the osculating circle at such point.

Both cyclic sequences $(\theta_1, \theta_2, ..., \theta_n)$ and $(\kappa_1, \kappa_2, ..., \kappa_n)$ encode in this way different but reasonable types of "discrete curvatures" of the polygon P. We can say that the any of these curvatures (θ or κ) has an *extremum* at v_i if is greater or equal, or less or equal, than both the curvature at v_{i-1} and v_{i+1}. In the first case we say that the curvature has a local *maximum* at v_i, and in the second case that it has a local *minimum*. Now, is it true that for a convex polygon P any of these sequences will have 4 sign changes? Unfortunately, without extra assumptions, this might be false.

But not all is lost. We say that a plane polygon $P = [v_1, v_2, ..., v_n]$ is *generic* if no four vertices of P lie on a circle. We also say that vertex v_i is *extremal* if vertices v_{i-2} and v_{i+2} lie on the same side of the circle $\mathcal{C}_i$ (i.e., they are both inside or outside the circle).

Theorem 1.10 *Every generic convex polygon* $P = [v_1, v_2, ..., v_n]$ *with (*$n \geq 4$*) has at least four extremal vertices.*

Theorem 1.10 is an analog of theorem 1.1 in the sense that in the smooth setting the osculating circle at the vertex p of the curve is such that a neighborhood of the curve around p lie on the same closed region determined by the circle. In the discrete setting, the "neighborhood" is represented by the set of vertices $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$.

Actually, a stronger result is true. We say that an osculating circle $\mathcal{C}_i$ is *full* if all other vertices are inside $\mathcal{C}_i$, and *empty* if all other vertices are outside

$\mathcal{C}_i$. An *extremal circle* is a circle which either full or empty. Now we can state the discrete analog of theorem 1.2:

Theorem 1.11 *Every generic convex polygon* $P = [v_1, v_2, ..., v_n]$ *(*$n \geq 4$*) has at least four extremal circles (two of them being full and the other two being empty).*

In order to state the most general theorem for convex polygons in the plane, we need some definitions. Given a plane polygon $P = [v_1, v_2, ..., v_n]$, let $\mathcal{C}_{ijk}$ be the circle determined by vertices v_i, v_j and v_k (where $i < j < k$). As we have already seen, the case where the vertices are consecutive is when C_{ijk} is *osculating*. We also say that $\mathcal{C}_{ijk}$ is *disjoint* if no two vertices are adjacent. The remaining circles (where only two of the vertices are adjacent) are called *intermediate*. A circle C_{ijk} is called *full* if all other vertices of P are inside, and *empty* if all other vertices of P are outside. Now we can state the discrete analog of theorem 1.3:

Theorem 1.12 *Let* $P = [v_1, v_2, ..., v_n]$ *(*$n \geq 4$*) be a planar, generic convex polygon. Let* s_+*,* t_+ *and* u_+ *be the number of full circles that are osculating, disjoint and intermediate, respectively. Let* s_-*,* t_- *and* u_- *be the number of empty circles that are osculating, disjoint and intermediate, respectively. Then the following equalities hold:*

$$s_+ - t_+ = s_- - t_- = 2,$$

$$s_+ + t_+ + u_+ = s_- + t_- + u_- = n - 2.$$

As in the smooth case, theorem 1.12 implies theorem 1.11. Notice also that while the first equalities are the same as in theorem 1.3, the others do not seem to have any smooth counterpart.

An interesting and simple proof of theorem 1.12 relies on the *Voronoi diagram* and its *cut locus* of the vertices of polygon P and can be found in Pak's book (see (PAK,)). We will obtain theorem 1.12 as a corollary of a result by Sedykh on space polygons, which will be discussed in the next chapter.

There are many other results for planar convex polygons. We will talk about them briefly, without going into the details.

Following Sedykh, we say that a planar polygon $P = [v_1, ..., v_n]$ is *good* if for any i the center of the circle C_i (defined by vertices v_{i-1}, v_i and v_{i+1}) lies in the interior of the angle $\angle(v_{i-1}v_iv_{i+1})$.

Theorem 1.13 *Let $P = [v_1, ..., v_n]$ $(n \geq 4)$ be a plane, generic, good convex polygon, and let κ_i be as defined before (inverse to the raddi of the osculating circles). Then the sequence $(\kappa_1, ..., \kappa_n)$ has at least four extrema.*

For a proof of 1.13, see (SEDYKH, 1997) (pp. 204-205).

Recall that the θ_i were defined as the exterior angles of the polygon at each vertex v_i of P. Denote by ψ_i the interior angle of the polygon at each v_i. Since equilateral convex polygons are also good, theorem 1.13 implies the following result:

Corollary 1.14 *Let $P = [v_1, ..., v_n]$ $(n \geq 4)$ be a plane, generic, equilateral convex polygon. Then the sequences $(\psi_1, ..., \psi_n)$ and $(\theta_1, ..., \theta_n)$ have at least four extrema (at the same indices).*

1.3
Convex spherical polygons

As we did in the smooth case, we can also consider four-vertex type theorems for *spherical polygons*. First define a *spherical/geodesic segment* between two points in the sphere as the geodesic path between them (i.e., the arc of the great circle defined by the points with the least length). For this we require that both points are not antipodal to each other. By a *spherical polygon* $P = [u_1, u_2, ..., u_n]$ we mean a closed spherical curve obtained by a concatenation of spherical/geodesic segments, each one connecting the *vertices* v_1, v_2, ..., v_n in a cyclic sequence (for this to make sense, we require that $v_{i+1} \neq -v_i$ for all $i \in \{1, ..., n\}$). The *edges* of P are the closed spherical segments $e_i = [v_i, v_{i+1}]$, where we consider the indices $i \mod n$.

All definitions and results for plane polygons in this section extend verbatim to the case of spherical polygons. For instance, a vertex v_i is called *extremal* if vertices v_{i-2} and v_{i+2} lie on the same region of the *osculating circle* $\mathcal{C}_i$ (i.e., the circle spanned by vertices v_{i-1}, v_i and v_{i+1}). If the spherical polygon P is *convex in the spherical sense* as a spherical closed curve (i.e., the polygon is contained in a closed hemisphere and for any two points in the interior of the polygon, the spherical segment connecting them is inside the interior), then it will have at least four extremal vertices (this is analogous to theorem 1.10).

Although we will study spherical polygons in chapters 3, 4 and 5, we shall not pursue specifically the theory of four-vertex type theorems for convex spherical polygons as defined above: these polygons are always contained in a closed hemisphere and therefore its geometry is, in a sense, “the same" as in the planar case. What we really want to understand are spherical polygons that are *not contained in any closed hemisphere* and therefore reflect better the nature of the ambient space $\mathbb{S}^2$. We will, however, need occasionally the notion of a convex spherical polygon in some of our proofs.

1.4
Discrete versions of the four-vertex theorem in space

Now we want to investigate four-vertex type theorems for space polygons. The definition of a space polygon $P = [v_1, v_2, ..., v_n]$ is the same as in the planar case, but now the ambient space is the Euclidean space $\mathbb{R}^3$.

Now, what would be a flattening for a space polygon? The two following facts suggest such a definition:

- Recall that for a smooth curve γ the flattening $\gamma(t_0) = p$ is a point at which the osculating plane has an order of contact greater than usual with curve at the point, which happens if and only if the torsion τ is zero at this point. Suppose that $\tau'(t_0) \neq 0$. Then, in a neighborhood of γ around p, *all points are on the same side of the osculating plane* at p (in contrast to the case that $\tau(t_0) \neq 0$, where the curve crosses locally the osculating plane at p);
- At the end of the previous section we have seen that, by definition, the osculating circle $\mathcal{C}_i$ at an extremal vertex v_i does not separate vertices v_{i-2} and v_{i+2}. Since the circle $\mathcal{C}_i$ is the intersection of the sphere $\mathbb{S}^2$ with a plane Π_i, this means that vertices v_{i-2} and v_{i+2} lie on the same half-space determined by the plane Π_i.

Therefore, we define a *flattening* of a space polygon $P = [v_1, ..., v_n]$ as a triple $\{v_{i-1}, v_i, v_{i+1}\}$ such that vertices v_{i-2} and v_{i+2} are on the same side of the plane Π_i spanned by the $\{v_{i-1}, v_i, v_{i+1}\}$. We call Π_i the *osculating plane* of P at v_i.

When one considers space polygons, however, one does not see as many discrete four-vertex type theorems as in the two-dimensional case. While theorems 1.4 and 1.5 have discrete versions (see (SEDYKH, 1997) and (OVSIENKO; TABACHNIKOV, 2001), respectively), discrete versions of theorems 1.6, 1.7, 1.8 and 1.9 do not appear in the literature. The sole exception seems to be an article by Panina (see (PANINA, 2010)), where she states and proves a discrete version of theorem 1.7, but does so using the original smooth theorem by Segre. *It is the main goal of this work to state and prove a discrete version of Segre's theorem 1.6 using only discrete tools, and to explore later some consequences of this theorem, which includes discrete versions of theorems 1.7, 1.8 and 1.9, as well as some generalizations and extensions.*

The rest of this work is organized as follows:

- In Chapter 2 we discuss the notion of *weak convexity for polygons* and derive four-vertex type theorems. Most of this chapter is a discrete version of the article by Romero Fuster (see (ROMERO-FUSTER, 1988)) and is influenced by the texts of Sedykh (see (SEDYKH, 1997)) and Pak (see (PAK,)). We show how the results for weakly convex polygons can be used via a stereographic projection to obtain results in the plane (for instance, to prove theorem 1.12). Up to this point there is essentially no original result here. We then complement the theory presented so far pointing out some easy but interesting corollaries that we have not found in the literature. The first fact is a generalization of a formula that relates the number of osculating and tritangent support planes of a space polygon. We also discuss the notion of *strictly convexity for polygons* and show that strictly convexity implies weak convexity (which incidentally implies a discrete version of theorem 1.5). Finally, we also discuss centrally symmetric space polygons;
- In Chapter 3 we define the notion of a *Segre polygon* and prove one of the main theorems of this work, namely, that *Segre polygons must have at least four flattenings* (this is the discrete version of theorem 1.6). In order to do this we define the notion of *discrete tangent indicatrix* of a space polygon and prove that *a spherical simple polygon which is not contained in any closed hemisphere must have at least four spherical inflections* (this is the main result of the chapter and a discrete version of theorem 1.7). We also state and prove discrete versions of theorems 1.8 and 1.9;
- In Chapter 4 we state and prove one of the most important theorems of this work on spherical polygons. It states an improvement to 6 for the lower bound on the number of inflections of a simple spherical polygon not contained in any closed hemisphere, provided that it does not have *antipodal intersections.*
- In Chapter 5 we generalize the main theorems of Chapter 3 and 4, allowing for the spherical polygon to have self-intersections and/or antipodal intersections. Here we obtain lower bounds for the numbers of inflections plus the numbers of self-intersections and antipodal intersections with “multiplicity".

2
Weakly convex polygons

2.1
Introduction

In the second section, we go over some known theorems that have been established by Romero Fuster in the smooth case (see (ROMERO-FUSTER, 1988)) and by Sedykh in both the smooth and discrete case (see (SEDYKH, 1992) and (SEDYKH, 1997), respectively). We present the material with the notion of the *Maxwell's Graph* of a space polygon, which appears in (ROMERO-FUSTER, 1988) in the smooth setting and in (GENTIL, 2020) in the discrete case. In the third section we prove a theorem for a more general class of convex polygons in the plane using the results from the second section. In the fourth section we deduce an easy but interesting theorem that unifies some theorems of the second section. In the fifth section we see some examples of the theory presented so far. In the sixth section we define the notion of *strict convexity for space polygons* and deduce a four-vertex theorem for such polygons. Finally, in the seventh section we go briefly into the subject of weakly generic polygons and centrally symmetric polygons and state a theorem that improves the results of the second section for this latter class of polygons.

2.2
The convex hull of a space polygon

Given a finite set of points $V = \{v_1, ..., v_n\}$ in $\mathbb{R}^3$, the convex hull of V, denoted by $\mathcal{H}(V)$, is the smallest convex set in $\mathbb{R}^3$ that contains V. Equivalently, it is the set of all finite positive combinations of points of V whose coefficients sum up to 1. In symbols:

$$\mathcal{H}(V) = \{\lambda_1 v_1 + ...\lambda_k v_k; v_1, ..., v_k \in V, \lambda_1, ..., \lambda_k \geq 0, \lambda_1 + ... + \lambda_k = 1\}.$$

The set $\mathcal{H}(V)$, thus defined, is a space polytope. It is known (see (ZIEGLER, 1995), p. 29) that in this case $\mathcal{H}(V)$ also has a different, but equivalent description: it is the limited intersection of a finite number of closed halfspaces, each one of them determined by an affine plane (see figure 2.1).

We say that a plane Π is a *support plane* of $\mathcal{H}(V)$ if $P \cap \Pi \neq \emptyset$ and if all the vertices of P (and therefore the convex hull itself $\mathcal{H}(P)$) are contained in one of the closed half-spaces H determined by Π. Now, it is clear that, given a set V of points in $\mathbb{R}^3$, when we consider the affine planes that determine $\mathcal{H}(V)$, we can consider only the support planes that contain at least three different vertices.

The *faces* of $\mathcal{H}(V)$ are the sets of the form $\mathcal{H}(V)\cap\Pi$, where Π is a support plane of $\mathcal{H}(V)$ that contains at least three non-collinear vertices of V.

Now, if the set $V = \{v_1, ..., v_n\}$ is the set of vertices of a space polygon $P = [v_1, ..., v_n]$, then $\mathcal{H}(P) = \mathcal{H}(V)$, since each edge of P is contained in $\mathcal{H}(V)$.

We say that a space polygon P is *generic* if no four vertices of P lie on a plane (notice that for a plane curve the *genericity condition* concerned circles instead of lines). In such case, all the faces of $\mathcal{H}(P)$ are triangles. $\mathcal{H}(P)$ is then said to be a *simplicial* polytope.

Denote by $\partial\mathcal{H}(P)$ the topological boundary of $\mathcal{H}(V)$. If P is generic, $\partial\mathcal{H}(P)$ is a triangulated PL-surface. Each triangle of this triangulation is the convex hull of vertices v_i, v_j and v_k, and therefore will be denoted by $\triangle = \triangle(i, j, k)$.

Given a polygon with n vertices, we say that two distinct integers $i < j$ are *consecutive* if $j = i + 1$ or $i = 1$ and $j = n$. There are three possibilities regarding the relative position of the indices i, j and k:

- i, j, k are consecutive. In this case, both the corresponding triangle and the plane spanned by these vertices will be called *osculating*;
- i, j, k are such that two of them are consecutive, but one of them is isolated. In this case, both the corresponding triangle and the plane spanned by these vertices will be called *bitangent*;
- i, j, k are all isolated from each other. In this case, both the corresponding triangle and the plane spanned by these vertices will be called *tritangent*.

We say that a space polygon $P = [v_1, ..., v_n]$ is *weakly convex* if it is contained on the boundary of its convex hull. In symbols:

$$P \subset \partial\mathcal{H}(P).$$

Otherwise P is *non weakly convex*. Figure 2.1 shows both a weakly convex polygon and a non weakly convex polygon.

We want to prove a result on the minimal number of osculating triangles that a weakly convex polygon must have. The proof uses a basic idea from graph theory that will appear repeatedly in this work. A graph is called a *tree* if it is connected and does not have cycles. A tree is *nontrivial* if it has at least 1 edge. A *leaf* is a vertex which is adjacent to only one vertex.

Theorem 2.1 *Every nontrivial tree has at least two leaves.*

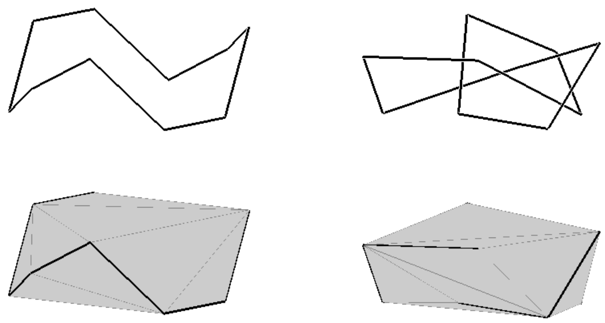

Figure 2.1: A weakly convex polygon on the left, a non-weakly convex polygon on the right, together with their respective convex hulls.

Theorem 2.2 *If v and e are the numbers of vertices and edges of a tree, respectively, then the formula equality holds:*

$$v = e + 1.$$

Theorems 2.1 and 2.2 are basic and classic results of graph theory and will not be proved here (see (BONDY; MURTY, 2008), pp. 99-100, or (HENLE, 1994), pp. 138-139).

Theorem 2.3 *A generic, weakly convex space polygon $P = [v_1, ..., v_n]$ ($n \geq 4$) has at least 4 osculating triangles.*

Proof. By hypothesis, $P \subset \partial\mathcal{H}(P)$ is generic. In particular, it is simple. Therefore, P separates $\partial\mathcal{H}(P)$ in two different regions, each one with a certain triangulation. The dual graph of each triangulated region is a tree (i.e., it does not have cycles). Since $n \geq 4$, each of these trees is nontrivial and, by theorem 2.1, has therefore at least 2 leaves, i.e., two triangles with incidence number equal to 1. In other words, each region has at least two osculating triangles. which implies the result. ■

Recall that a *flattening* of a space polygon P is a triple $\{v_{i-1}, v_i, v_{i+1}\}$ such that v_{i-2} and v_{i+2} are on the same side of the plane $\Pi(i-1, i, i+1)$. Now, if a triangle spanned by v_{i-1}, v_i and v_{i+1} is a osculating support triangle, then the respective triple of vertices is a flattening. Therefore, theorem 2.3 implies the following result.

Theorem 2.4 *A generic, weakly convex space polygon $P = [v_1, ..., v_n]$ ($n \geq 4$) has at least 4 flattenings.*

The dual graph that appears in the proof of theorem 2.3 is closely related to the so called *(discrete) Maxwell's Graph $G_M(P)$* of the polygon P. More

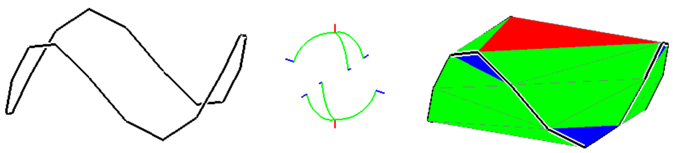

Figure 2.2: An example of a polygon P with its associated Maxwell's Graph $G_M(P)$ and its convex hull $\mathcal{H}(P)$. The triangles are represented with differents colors: red for tritangent triangles, green for bitangent triangles and blue for osculating triangles.

generally, given a space polygon P, we consider first the dual graph $G(P)$ of the triangulation of $\mathcal{H}(P) - P$: two triangles (and therefore its respective dual vertices) are considered connected to each other if the common edge is not an edge of P. Define then the *(discrete) Maxwell's Graph* $G_M(P)$ as the graph whose vertices are the vertices of $G(P)$ with incidence number equal to 1 or 3 (i.e., they are dual to osculating or tritangent triangles, respectively), while the vertices of $G(P)$ with incidence number equal to 2 (i.e., the vertices which are dual to the bitangent triangles) are considered as part of the edges of $G_M(P)$. In other words, the edges of $G_M(P)$ are dual to the sequences of bitangent triangles that connect two triangles which are osculating and/or tritangent. See figure 2.2 for an example.

Both the graph $G(P)$ and the Maxwell's graph $G_M(P)$ can be represented by spherical graphs on the sphere whose vertices are the endpoints of the normal internal vectors to the triangles that define the triangulation (i.e., the normal vectors that point to the inside of $\mathcal{H}(P)$). The following proposition collects the main features of the Maxwell's Graph (the interested reader may found a more complete discussion in (GENTIL, 2020)):

Proposition 2.5 *Let* $P = [v_1, ..., v_n]$ *(*$n \geq 4$*) be a generic space polygon. Then we have that:*

(a) $G_M(P)$ *has at most two connected components;*

(b) $G_M(P)$ *is connected if and only if* P *is weakly convex;*

(c) If $P \cap \partial\mathcal{H}(P)$ *has 1 connected component, then each component of* $G_M(P)$ *is simply connected, i.e., it is a tree.*

Proof. (a) and (b) are clear.

(c) If $G_M(P)$ had any cycles, then $P \cap \partial\mathcal{H}(P)$ would have at least two connected components (see figure 2.3 for some examples). ■

From the previous results we can derive some important numerical relations between the number of support triangles. Given a connected component

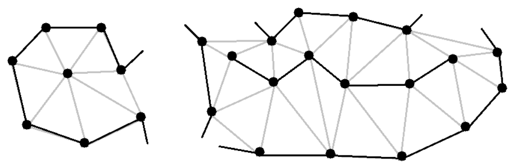

Figure 2.3: If $G_M(P)$ had any cycles, then $P \cap \partial\mathcal{H}(P)$ would have at least two components.

of $\mathcal{H}(P) - P$, denote by T_i, B_i and O_i the numbers of its tritangent, bitangent and osculating triangles, respectively (which are the same to the numbers of vertices of the respective component of $G(P)$ with incidence number equal to 3, 2 and 1, respectively), and by V_i and E_i the numbers of vertices and edges of the respective component of the Maxwell's Graph $G_M(P)$.

Proposition 2.6 *Given a generic space polygon $P = [v_1, ..., v_n]$ $(n \geq 4)$, the following equalities hold for any component of $G_M(P)$:*

$$T_i + O_i = V_i,$$

$$3T_i + O_i = 2E_i.$$

Moreover, if P is weakly convex, then the following equality holds for both components of $\mathcal{H}(P) - P$ (or, equivalently, for both components of $G_M(P)$):

$$T_i + B_i + O_i = n - 2.$$

Proof. The first equality is obvious. The second one follows from noticing that, if one counts the number of edges that are incident with each of the vertices and then adds all them up, this will give the expression on the left. On the other hand, each edge will be counted twice.

Now, for the third equality, it suffices to prove that the number of faces of any of the two connected components of $\mathcal{H}(P)$ equals $n - 2$. Denote by v, e and f the numbers of vertices, edges and faces of $\partial\mathcal{H}(P)$, respectively. Since P is generic, $\mathcal{H}(P)$ is simplicial, i.e., all of its faces are triangles. Thus $3f = 2e$, which combined with Euler's Formula $v - e + f = 2$ (applied to $\partial\mathcal{H}(P)$) and the fact that $v = n$ implies that

$$f = 2n - 4.$$

Since both components of $\mathcal{H}(P)$ have the same number of faces, the result follows. ∎

Given a generic weakly convex space polygon P, denote by O and T the total number of osculating and tritangent triangles of $\mathcal{H}(P)$, respectively (which are the same as the total number of vertices of $G_M(P)$ with incidence numbers equal to 1 and 3, respectively).

Theorem 2.7 *Given a generic weakly convex space polygon $P = [v_1, ..., v_n]$ ($n \geq 4$), the following equality holds for any connected component of $\partial\mathcal{H}(P) - P$ (and of $G_M(P)$):*

$$O_i - T_i = 2,$$

for each $i = 1, 2$. Moreover, for the two components taken into account, we have that

$$O - T = 4.$$

Proof. Since P is weakly convex, $\partial\mathcal{H}(P) - P$ (and $G_M(P)$) has two components (by Proposition 2.5(b)), both of which are trees (by Proposition 2.5(c)). By proposition 2.6 and by theorem 2.2,

$$T_i + O_i = V_i = E_i + 1,$$

$$3T_i + O_i = 2E_i.$$

Multiplying the first equality by 2, substituting $2E_i$ by $3T_i + O_i$ and simplifying, we get:

$$O_i - T_i = 2.$$

Summing on the two indices $i = 1$ and $i = 2$, we get:

$$O - T = 4.$$

■

Theorem 2.8 *Given a generic non weakly convex polygon $P = [v_1, ..., v_n]$ ($n \geq 5$), the following equality holds:*

$$O - T = 4 - 2\rho.$$

Proof. Given a non weakly convex polygon P, denote by v, e and f the number of vertices, edges and faces $\partial\mathcal{H}(P)$. By Euler's formula (applied to $\partial\mathcal{H}(P)$), we have that

$$v - e + f = 2.$$

Recall that ρ is the number of connected components of $P \cap \partial\mathcal{H}(P)$. Denote by e_P the number of edges of $\partial\mathcal{H}(P)$ which are contained in P, and by $e_\triangle$ the number of edges of $\partial\mathcal{H}(P)$ which are not contained in P.

It is not hard to see that $v - e_\rho = \rho$ and that $e_\triangle - B = E_i$. Substituting these numbers in Euler's formula we get:

$$2 = v - e + f = v - e_P - e_\triangle + T + B + O$$

$$= \rho - E_i + T + O.$$

Multiplying both sides by 2 and using proposition 2.6:

$$2\rho - 2E_i + 2T + 2O = 4$$

$$\implies 2\rho - 3T - O + 2T + 2O = 4$$

$$\implies O - T = 4 - 2\rho.$$

■

Remark 2.9 *Theorem 2.3 was proved in the smooth setting by Romero Fuster (see (ROMERO-FUSTER, 1988)) and Sedykh (see (SEDYKH, 1997)). The proof we presented for the discrete setting can be found in Pak's book (see (PAK,)). Interestingly, a different proof for weakly convex but not necessarily generic polygons was found by Sedykh (see (SEDYKH, 1997)).*

The idea of using the Maxwell's Graph of a smooth curve was used extensively in (ROMERO-FUSTER, 1988) and was (as far as we know) first adapted for the discrete case in (GENTIL, 2020). Propositions 2.5 and 2.6 and theorems 2.7 and 2.8 appear in both these works, although the discrete versions we present here are slightly more complete than the ones in (GENTIL, 2020). Moreover, our proof of theorem 2.8 is slightly different than the one presented in both (ROMERO-FUSTER, 1988) and (GENTIL, 2020).

2.3 Application to convex polygons in the plane

We can apply the previous theorems to prove theorem 1.12 from Chapter 1. This section is not essential to the subsequent sections and chapters and therefore may be skipped on a first read.

In this section we follow Sedykh's terminology (see (SEDYKH, 1997)). Given a simple generic plane polygon P, we say that a circle passing through at least one vertex of P is a *support circle* if all the vertices of the polygon lie in one of the closed regions defined by this circle (i.e., they all lie on the inside or they all lie on the outside). Notice that, if a support circle $\mathcal{C}$ passes through 3 points of P, then $\mathcal{C}$ is extremal (as defined in the previous chapter).

We say that a simple generic plane polygon P is *normal* if every two of consecutive vertices v_i and v_{i+1} lie on a support circle.

Now, if P is convex, then the straight line spanned by any pair of consecutive vertices v_i and v_{i+1} is such that all other vertices are contained in one of the closed halfplanes determined by this line. This implies that there is a support circle passing through v_i and v_{i+1}. In other words, a convex polygon is normal. Therefore, theorem 1.12 follows from the following result. Recall that for any three vertices v_i, v_j and v_k of a plane polygon P, the circle $\mathcal{C}_{ijk}$ determined by them is *osculating/intermediate/disjoint* if the number of consecutive pairs among indices i, j and k is 2, 1 or 0, respectively. $\mathcal{C}_{ijk}$ is *full/empty* if all other vertices of P are inside/outside of $\mathcal{C}_{ijk}$.

Theorem 2.10 *Let $P = [v_1, ..., v_n]$ ($n \geq 4$) be a normal generic plane polygon. Let s_+, t_+ and u_+ be the number of full circles that are osculating, disjoint and intermediate, respectively. Let s_-, t_- and u_- be the number of empty circles that are osculating, disjoint and intermediate, respectively. Then the following equalities hold:*

$$s_+ - t_+ = s_- - t_- = 2,$$

$$s_+ + t_+ + u_+ = s_- + t_- + u_- = n - 2.$$

Proof. Let $P = [v_1, ..., v_n]$ be a normal generic plane polygon. Let $P' = [v'_1, ..., v'_n]$ be a stereographic projection of P into a sphere $\mathbb{S}^2$. Recall that such a projection takes plane circles into spherical circles, which in turn equal intersections of the sphere with planes.

Since P is generic as a plane polygon, i.e., no four vertices of P lie on a plane circle, we have that no four vertices of P' lie on a spherical circle. Hence no four vertices of P' lie on the same plane, i.e., P' is generic as a space polygon. In particular, P' is simple.

Since P is normal, there is for each pair of consecutive vertices v_i and v_{i+1} a support circle $\mathcal{C}$ passing through them. The stereographic projection then maps this circle onto a circle $\mathcal{C}'$ on the sphere such that all other vertices of P' are on the same region of $\mathcal{C}'$. Since $\mathcal{C}'$ is the intersection of the sphere with a plane Π', this means that all other vertices of P' are on the same halfspace determined by Π'. In other words, P' is weakly convex when considered as a space polygon.

Now we observe that the full and empty circles $\mathcal{C}_{ijk}$ of the plane P are mapped, in a one-to-one correspondence via the stereographic projection, to the circles on the sphere that are contained in the support planes that intersect the space polygon P' at three vertices. Consequently, each plane circle $\mathcal{C}_{ijk}$ corresponds to a support triangle $\triangle(i,j,k)$ of $\mathcal{H}(P')$. Since P' is weakly convex, it separates $\partial\mathcal{H}(P') - P'$ into two regions. One of them is triangulated by triangles that correspond to full circles of P, while the other is triangulated by triangles that correspond to empty circles of P.

Moreover, we have that the osculating, intermediate and disjoint circles of the plane polygon P are mapped, in a one-to-one correspondence via the stereographic projection, to the circles on the sphere that are contained in osculating, bitangent and tritangent planes of the space polygon P', respectively. Therefore $s_+ = O_1$, $t_+ = T_1$, $u_+ = B_1$, and $s_- = O_2$, $t_- = T_2$, $u_- = B_2$.

The result now follows from theorem 2.7 and proposition 2.6. ∎

Remark 2.11 *Theorem 2.10 can be found in (PAK,) (p. 202), although the version we present here is slightly more general. It is important to note that the idea of using the stereographic projection to relate convex (or even normal) plane polygons with weakly convex space polygons can be used to deduce theorem 1.11 directly from theorem 2.3. This was actually Sedykh's original argument when proving theorem 2.3 (see (SEDYKH, 1997), pp. 201-204).*

2.4
A unified formula for weakly convex and non weakly convex polygons

There is some similarity between equations for weakly convex/non weakly convex polygons in theorems 2.7 and 2.8. Notice that the equation of the former is not a particular case of the equation of the latter, since ρ equals 1 and not 0 for weakly convex polygons. We might wonder if there is a unified formula for both equations. This is true if we consider the number of connected components of $P \cap int\mathcal{H}(P)$ instead of $P \cap \partial\mathcal{H}(P)$ (i.e., we consider the number of components of P in the topological interior of $\mathcal{H}(P)$), and not in its boundary). Denote by $\mu = \mu(P)$ the number of connected components of $P \cap int\mathcal{H}(P)$. For weakly convex polygons, $\rho = 1 \neq 0 = \mu$. For non weakly convex simple polygons, however, the next proposition implies that $\rho = \mu$.

Proposition 2.12 *If a simple polygon P in $\mathbb{R}^3$ is non weakly convex, then*

$$\mu(P) = \rho(P).$$

Proof. We may assume without loss of generality that $(v_n, v_1) \subset int\mathcal{H}(P)$ and $v_1 \in \partial\mathcal{H}(P)$. If we start moving at v_1 and going through the polygon with the usual orientation (v_2,v_3 and so on), we will pass through every connected component of both sets $P \cap \partial\mathcal{H}(P)$ and $P \cap int\mathcal{H}(P)$. Since P is simple, we will not pass through any of these connected regions twice. Moreover, since P is a closed polygonal line, these distinct components must alternate between the ones contained in the boundary and the ones contained in the interior of $\mathcal{H}(P)$, which implies that $\mu(P) = \rho(P)$. ■

Since generic polygons are also simple, proposition 2.12 is also true for weakly convex generic polygons.

Theorem 2.13 *Given a generic polygon $P = [v_1, ..., v_n]$ ($n \geq 4$), the following equality holds:*

$$O - T = 4 - 2\mu.$$

Proof. If P is weakly convex, $\mu = 0$. By theorem 2.7,

$$O - T = 4 = 4 - 2\mu.$$

If P is non weakly convex, proposition 2.12 implies that $\mu = \rho$. By theorem 2.8,

$$O - T = 4 - 2\rho = 4 - 2\mu.$$

■

An immediate consequence of theorem 2.13 is the following result:

Corollary 2.14 *Given a generic polygon $P = [v_1, ..., v_n]$ ($n \geq 4$), the following inequality holds:*

$$2\mu + O \geq 4.$$

Recall that we denote by F the number of flattenings of a polygon.

Corollary 2.15 *Given a generic polygon $P = [v_1, ..., v_n]$ ($n \geq 4$), the following inequality holds:*

$$2\mu + F \geq 4.$$

Remark 2.16 *Although theorem 2.13 and corollaries 2.14 and 2.15 are easy consequences of the theory presented so far, we have not found them in the literature. The reason we present them here is to parallel the results that will be proved in subsequent chapters for another class of polygons.*

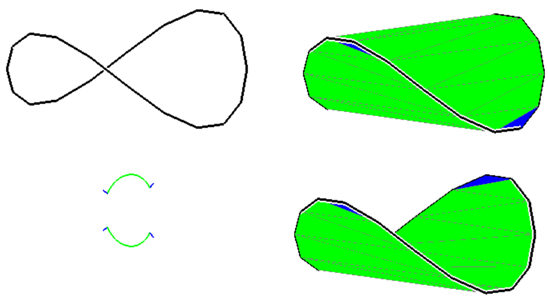

Figure 2.4: Weakly convex curve, with $C = 4$, $T = 0$ and $\mu = 0$.

2.5 Examples of generic polygons

We now present some examples of generic polygons, showing that the lower bound on corollary 2.14 cannot be improved for the values $\mu = 0, 1, 2$. We used polygons that approximate certain curves, which are by their turn given by parametric equations. For each one we give the equation together with the number of points used. For each example, there is a figure with the polygon P, its associated Maxwell's Graph $G_M(P)$ and the boundary of its convex hull $\partial H(P)$.

In order to make it easier to visualize the convex hull, we display it in two different ways: the complete image and the one without the superior part. Besides that, the support triangles of $\partial\mathcal{H}(P)$ appear with different colors: red for tritangent, green for bitangent and blue for osculating triangles. In the graph $G_M(P)$ the corresponding vertices have the same colors as their corresponding panels: red for vertices with incidence degree equal to 3, green for the edges and blue for vertices with incidence degree equal to 1.

Example 2.17 *We use the curve $\alpha : [0, 2\pi] \longrightarrow \mathbb{R}^3$, given by*

$$\alpha(t) = (\cos(t), \sin(t), \epsilon \sin(2t)),$$

where ϵ is small, but not so much (for instance, $\epsilon \in [\frac{1}{5}, \frac{3}{5}]$). The discretization has 20 points. Figure 2.4 displays the polygon (where $C = 4$ e $T = 0$).

Example 2.18 *We use the curve $\alpha : [0, 2\pi] \longrightarrow \mathbb{R}^3$, given by*

$$\alpha(t) = (2\sin(t) - \sin(2t), 2\cos(t) - \cos(2t), -\sin(2t)).$$

The discretization has 20 points and is represented in figure 2.5. It is a non weakly convex polygon, with $C = 2$, $T = 0$ e $\rho = \mu = 1$.

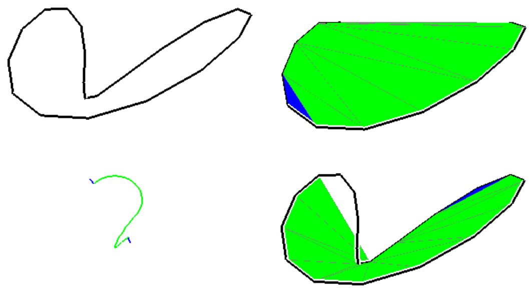

Figure 2.5: Non weakly convex polygon, with $C = 2$, $T = 0$ and $\rho = \mu = 1$.

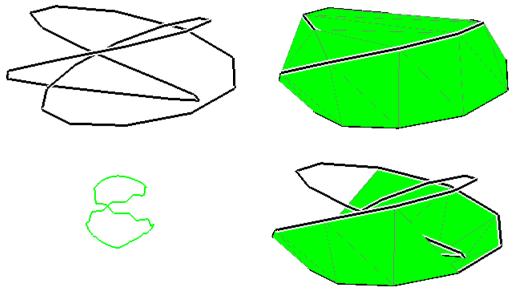

Figure 2.6: Non weakly convex polygon, with $C = 0$, $T = 0$ and $\rho = \mu = 2$.

Example 2.19 ***(Torus knot (3-2))*** *We use the curve* $\alpha : [0, 2\pi] \longrightarrow \mathbb{R}^3$, *given by*

$$\alpha(t) = ((4 + \cos(2t)) \cos(3t), (4 + cos(2t)) \sin(3t), 2 \sin(2t)).$$

The discretization has 30 points and is represented in figure 2.6. It is a non weakly convex polygon, with $O = 0$, $T = 0$ *and* $\rho = \mu = 2$. *Hence the intermediate graph* $G(P)$ *for this polygon is a cycle and the Maxwell's graph* $G_M(P)$ *is not a proper graph (it should have at least one vertex).*

2.6 Strictly convex polygons

In the previous chapter we defined the notion of *strict convexity* for space curves and stated without proof a theorem on the lower bound of flattenings that such curves must have. Now we discuss briefly the discrete counterpart of it. In the following definitions we follow almost verbatim a section of Tabachnikov and Ovsienko's article (see (OVSIENKO; TABACHNIKOV, 2001)), where a more general version of strict convexity for *projective* polygons is discussed. Here we restrict ourselves to the ambient space $\mathbb{R}^3$.

A space polygon P is said to be *transverse* to a plane Π at point $p \in P \cap \Pi$ if either p is an interior point of an edge and this edge is transverse to Π, or p

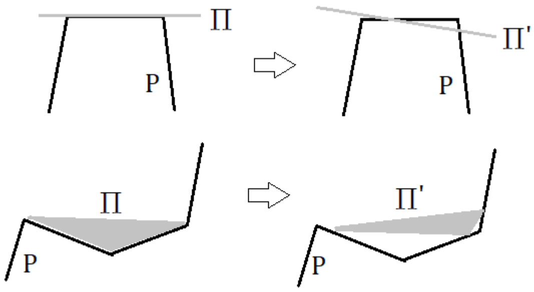

Figure 2.7: In the first case, plane Π (as seen from “above") intersects polygon P with multiplicity 2. In the second case, plane Π intersects polygon P with multiplicity 3.

is a vertex and the two edges incident do P are transverse to Π and are locally separated by Π.

A space polygon P is said to *intersect a plane* Π *with multiplicity* k if, for every plane Π' sufficiently close to Π and transverse to P, the number of points of $P \cap \Pi'$ does not exceed k and, moreover, k is achieved for some Π' (see figure 2.7).

A generic space polygon $P = [v_1, ..., v_n] \subset \mathbb{R}^3$ is said to be *strictly convex* if, for any two vertices v_i and v_j of P, there is a plane Π that contains both v_i and v_j and intersects P with multiplicity 2. This definition does not exclude the case where $v_i = v_j$.

Uribe (in (URIBE-VARGAS, 2003)) proved, in the smooth case, that strictly convex curves are also weakly convex. Here we prove the discrete version of his result, which uses the same underlying idea of his proof, but taking into account Tabachnikov and Ovsienko's formalism for intersection of polygons.

Lemma 2.20 *Let* $v_{i_1}, ..., v_{i_k}$ *be vertices of a generic polygon* P *(with* $k \leq 3$*). Then any plane* Π *passing through* $v_{i_1}, ..., v_{i_k}$ *intersects* P *with multiplicity at least* k*.*

Lemma 2.20 is actually a particular case of a result due to Ovsienko and Tabachnikov and will not be proved here (see Lemma 3.3 of (OVSIENKO; TABACHNIKOV, 2001)).

Proposition 2.21 *A strictly convex polygon is weakly convex.*

Proof. Let P be a strictly convex polygon. Let $e_i = \overrightarrow{v_i v_{i+1}}$ be any of its edges. We must show that $e_i \subset \partial\mathcal{H}(P)$. Since P is strictly convex, there is a plane Π_i passing through v_i and v_{i+1} (and therefore Π_i contains the whole edge e_i) with multiplicity 2. In particular, the plane Π_i does not intersect P elsewhere (otherwise it would have multiplicity greater or equal to 3, by lemma 2.20) and therefore is a support plane of $\mathcal{H}(P)$.

Denote by $\overline{H}$ the closed halfspace determined by Π which contains $\mathcal{H}(P)$. Since $e_i \subset \mathcal{H}(P) \subset \overline{H}$ and $e_i \subset \Pi = \partial\overline{H}$, we have that

$$e_i \subset \partial\mathcal{H}(P).$$

Since the edge e_i was chosen arbitrarily, it follows that $P \subset \partial\mathcal{H}(P)$. ■

Theorem 2.22 *A strictly convex space polygon $P = [v_1, ..., v_n]$ $(n \geq 4)$ has at least 4 flattenings.*

Proof. It follows immediately from proposition 2.21 and theorem 2.4. ■

Remark 2.23 *Theorem 2.22 is a particular case of a more general result for projective polygons, stated and proved in (OVSIENKO; TABACHNIKOV, 2001). Our proof (which is considerably shorter than the one presented in (OVSIENKO; TABACHNIKOV, 2001)) has the same strategy used by Uribe in the smooth case (see (URIBE-VARGAS, 2003)). Since we have not found this proof in the discrete form in the literature, we included it here.*

2.7 A further remark: weakly generic polygons

The results proved until now always required our polygons to be generic. This condition can, however, be weakened in the following way:

Remark 2.24 *The notion of genericity was important so that the polytope $\mathcal{H}(P)$ was simplicial (i.e., all of its faces were triangles) and thus we avoided degenerate cases. There is, however, no problem if P has 4 vertices in the same plane Π provided that Π is not a support plane of P, since it does not affect the triangulation of $\partial\mathcal{H}(P)$. An example of such polygon is given in figure 2.2.*

For this reason, we have the following definition: we say that P is weakly generic if all quadruples of coplanar vertices of P do not span a support plane of P. Therefore, all results proved in this chapter are also valid when we substitute "generic" by "weakly generic".

Before introducing an important class of weakly generic polygons, we need the following definition: we say that a set $X \subset \mathbb{R}^3$ is *centrally symmetric* to a point $x_0 \in \mathbb{R}^3$ if $x_0 + x \in X$ holds if and only if $x_0 - x \in X$. In other words, the set $X \subset \mathbb{R}^3$ is preserved under reflections through a point $x_0 \in \mathbb{R}^3$.

Consider now the particular case of X being a space polygon with $2n$ vertices. Notice that the condition that for every point/vertex of P being reflected through x_0 to another point/vertex of P implies that every vertex v_i of P is reflected through x_0 to the vertex v_{i+n}, where we consider the indices mod $2n$.

Lemma 2.25 *Let P be a centrally symmetric space polygon with at least $2n$ vertices ($2n \geq 6$). Suppose that P is not contained in any plane. Then, for any $i \neq j, j+n \in \{1, ..., 2n\}$, the plane spanned by v_i, v_{i+n}, v_j and v_{j+n} is not a support plane.*

Proof. Since P is centrally symmetric to a point x_0, the plane Π spanned by v_i, v_{i+n}, v_j and v_{j+n} contains x_0. Since P is not contained in any plane, there is at least one vertex $v_k \notin \Pi$. Because P is centrally symmetric to x_0, we have that v_{k+n} (i.e., the reflection of v_k through x_0) is on the other side of Π.

Our conclusion is that Π cannot be a support plane of P. ∎

Therefore, it makes sense to talk about centrally symmetric polygons which are weakly generic. An interesting result is the following theorem, which states an improvement on the lower bound of flattenings that a centrally polygon must have. We will prove this theorem at the end of the next chapter, using the notion of tangent indicatrix of a space polygon.

Theorem 2.26 *Let P be a weakly generic polygon with $2n$ vertices ($2n \geq 6$). If P is a centrally symmetric to a point x_0 and weakly convex, then it has at least 6 flattenings.*

Remark 2.27 *As with the results presented in Section 4, theorem 2.26 is not hard to prove (as we will see in the next chapter), and is another result that we have not found in the literature. By the same reason as before, we present it here to parallel a theorem that will be proved in the next chapter for another class of polygons.*

3
Segre polygons

3.1
Introduction

In this chapter we define the notion of a *Segre polygon* and prove the discrete analog of theorem 1.6. In order to do this, we need to define the notion of the (discrete) tangent indicatrix of a space polygon (which is a spherical polygon) and prove a discrete analog of theorem 1.7, which is the main result of this chapter.

The general strategy to prove this theorem is to use induction on the number of vertices of the spherical polygon. The most difficult and subtlest point of the induction step is to prove that there is at least one point that can be deleted from the spherical polygon so that the resulting spherical polygon still will neither be contained in any closed hemisphere nor will have self-intersections. In order to prove this fact, we also obtain some interesting results regarding spherical polygons in general using basic tools of convex geometry. In the last section we present two applications of the main theorem of the chapter: a discrete version of theorem 1.8 (a discrete Tennis Ball Theorem) and a discrete version of theorem 1.9.

3.2
Basic definitions

Recall from chapter 1 that a closed space curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$ is called a *Segre curve* if it has non-vanishing curvature and if, for any $t_1 \neq t_2 \in \mathbb{S}^1$, the tangent vectors $\gamma'(t_1)$ and $\gamma'(t_2)$ do not point to the same direction.

Now, let $P = [v_1, v_2, ..., v_n]$ be a *space polygon*, i.e., a closed polygonal line (where we consider the indices i modulo n) in $\mathbb{R}^3$. Recall that P is *generic* if it does not have 4 of its vertices on the same plane. A naive approach to discretize the notion of a Segre curve would be as follows: a "Segre polygon" should not have directed edges pointing to the same direction. Notice, however, that any generic polygon satisfies this condition: if there were $e_i = \overrightarrow{v_i v_{i+1}}$ and $e_j = \overrightarrow{v_j v_{j+1}}$ with $e_i \parallel e_j$, then the vertices v_i, v_{i+1}, v_j and v_{j+1} would be in the same plane, contradicting the genericity of the polygon.

Before we present a better approach for this problem, we recall the following definition from chapter 1: given a smooth curve γ in $\mathbb{R}^3$ with non-vanishing curvature, translate the unit tangent vector at each point of the

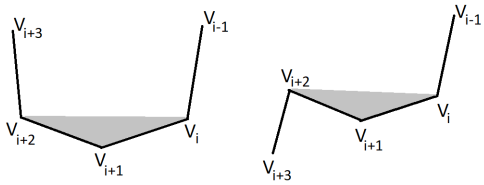

Figure 3.1: A flattening on the left, a non-flattening on the right

curve to a fixed point **0**. The endpoints of the translated vectors describe then a curve on the unit sphere $\mathbb{S}^2$. We call this curve the *tangent indicatrix* of γ.

Therefore, a Segre curve can be reformulated as a closed curve such that its tangent indicatrix is embedded in $\mathbb{S}^2$ (i.e., smooth and without self-intersections). Moreover, Theorem 1.6 now reads:

Theorem 3.1 *Let γ be a closed curve in $\mathbb{R}^3$. If its tangent indicatrix is embedded in $\mathbb{S}^2$, then γ has at least 4 flattenings.*

Definition 3.2 *Given a polygon $P = [v_1, v_2, ..., v_n]$ in $\mathbb{R}^3$, denote by u_i the unit tangent vector with the same direction of the edge e_i, i.e.,*

$$u_i = \frac{e_i}{|e_i|} = \frac{\overrightarrow{v_i v_{i+1}}}{|\overrightarrow{v_i v_{i+1}}|} = \frac{v_{i+1} - v_i}{|v_{i+1} - v_i|}.$$

We define the (discrete) tangent indicatrix of P as the closed spherical polygonal line, i.e., the spherical polygon

$$Q = [u_1, u_2, ..., u_n],$$

whose edges are the spherical segments (with minimal length) joining u_i and u_{i+1}. This definition goes back to the work of Banchoff (see (BANCHOFF, 1982)).

We can finally define the discrete counterpart of a Segre polygon:

Definition 3.3 *A polygon P is a Segre polygon if its tangent indicatrix Q does not have self-intersections.*

Definition 3.4 *A flattening of a space polygon is a triple $\{v_i, v_{i+1}, v_{i+2}\}$ such that v_{i-1} and v_{i+3} are on the same side of the plane generated by vertices v_i, v_{i+1} and v_{i+1} (see figure 3.1).*

Remark 3.5 *The previous definition implies that, if the triple $\{v_i, v_{i+1}, v_{i+2}\}$ is a flattening, then the vectors e_{i-1} and e_{i+2} point to different sides of the plane generated by $\{v_i, e_i, e_{i+1}\}$. This in turn implies that u_{i-1} and u_{i+2} are on different sides of span$\{u_i, u_{i+1}\}$ (see figure 3.2).*

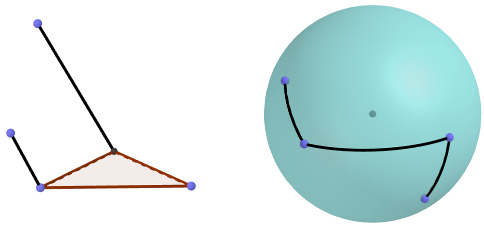

Figure 3.2: Flattening of a polygon P and the corresponding inflection of the tangent indicatrix Q.

The main goal of this chapter is to prove the following result:

Theorem 3.6 *A Segre polygon with at least 4 vertices has at least 4 flattenings.*

It is important to notice that, although this theorem states the result for Segre polygons in $\mathbb{R}^3$, its proof will work entirely within the realm of certain spherical polygons in $\mathbb{S}^2$. To get a feeling by what we mean by this, first we notice that the previous remark suggests the following definition:

Definition 3.7 *Given a spherical polygon $Q \subset \mathbb{S}^2$, a (spherical) inflection of Q is a pair $\{u_i, u_{i+1}\}$ such that u_{i-1} and u_{i+2} are in different sides of the plane spanned by $\{u_i, u_{i+1}\}$. Equivalently, u_{i-1} and u_{i+2} are in different hemispheres determined by the spherical line spanned by $\{u_i, u_{i+1}\}$.*

The condition that u_{i-1} and u_{i+2} are in different hemispheres determined by the spherical line spanned by $\{u_i, u_{i+1}\}$ is equivalent to the condition that the determinants $\epsilon_{i-1} = [u_{i-1}, u_i, u_{i+1}]$ and $\epsilon_i = [u_i, u_{i+1}, u_{i+2}] = [u_{i+2}, u_i, u_{i+1}]$ have opposite signs. Consequently, Theorem 3.6 states that, if a spherical polygon Q is the tangent indicatrix of a polygon and does not have self-intersections, then the cyclic sequence $(\epsilon_1, \epsilon_2, ..., \epsilon_n)$ has at least 4 sign changes (where each ϵ_i is defined as $[u_i, u_{i+1}, u_{i+2}]$).

3.3 The Cone Condition

It is instructive to see first the following example of spherical polygon.

Example 3.8 *Let $Q = [u_1, u_2, u_3, u_4] \subset \mathbb{S}^2$ the spherical polygon with $u_1 = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$, $u_2 = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $u_3 = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$ and $u_4 = \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.*

It is not hard to (visually) see that Q does not have any flattenings (see figure 3.3). It is also not hard to check algebraically that this is indeed the case: all determinants ϵ_1, ϵ_2, ϵ_3 and ϵ_4 are positive.

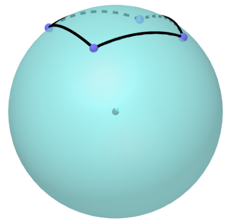

Figure 3.3: A spherical polygon without inflections.

The spherical polygon of the previous example does not have any self-intersections, but it does not have any inflections. Is that a counterexample to our result? The answer is no. The reason why this happened is because *there is no space polygon whose tangent indicatrix is* Q.

More generally, consider a spherical polygon $Q \subset \mathbb{S}^2$ contained in a closed hemisphere, but not entirely contained in a spherical line. We may assume that this hemisphere is the one above the xy-plane (rotate the sphere $\mathbb{S}^2$ for that to be the case). This implies that all vectors $u_1, u_2, ..., u_n$ (and consequently $e_1, e_2, ..., e_n$) have z-coordinate equal or greater than zero. Since Q is not entirely contained in any spherical line, at least one of the e_i, say e_n, have positive z-coordinate. Suppose that there is a space polygon $P = [v_1, v_2, ..., v_n]$ whose tangent indicatrix is Q. Denoting by $z(v_i)$ and $z(e_i)$ the z-coordinate of v_i and e_i respectively, we have

$$z(v_1) \leq z(v_1) + z(e_1) = z(v_2) \leq z(v_2) + z(e_2) = z(v_3) \leq ...,$$

since each $z(e_i)$ is equal or greater than zero. Now, because $z(e_n)$ is strictly greater than zero, we have

$$z(v_1) \leq z(v_2) \leq ... \leq z(v_{n-1}) \leq z(v_n) < z(v_n) + z(e_n) = z(v_1),$$

i.e., $z(v_1) < z(v_1)$. This contradiction implies that there is no such polygon P whose tangent indicatrix is Q. We have therefore proved

Proposition 3.9 *A necessary condition for a spherical polygon* $Q \subset \mathbb{S}^2$*, not entirely contained in a spherical line, to be the tangent indicatrix of some polygon* $P \subset \mathbb{R}^3$ *is that it cannot be contained in any closed hemisphere, i.e.,it must intersect every great circle of* $\mathbb{S}^2$*.*

It turns out that the converse of the previous proposition is also true.

Proposition 3.10 *If a spherical polygon* $Q \subset \mathbb{S}^2$*, not entirely contained in a spherical line, is not contained in any closed hemisphere (equivalently, it* *intersects every great circle of* $\mathbb{S}^2$*), then* Q *is the tangent indicatrix of some polygon* P *in* $\mathbb{R}^3$*.*

A proof of Proposition 3.10 will be provided later. In order to do so, as well to prepare the way for the proof of Theorem 3.6, it will be again useful to express the geometry of the configuration of points in terms of determinants.

Recall what we have done so far: given a polygon $P = [v_1, ..., v_n] \subset \mathbb{R}^3$, we calculated its edges $\{e_1, ..., e_n\}$ and normalized them, obtaining $\{u_1, ..., u_n\}$. Now we want to go the other way around: given $Q = [u_1, ...u_n]$, we must obtain $\{e_1, ..., e_n\}$ as an edge set of some polygon $P = [v_1, ..., v_n]$.

Notice that, since a space polygon P is closed,

$$v_1 + e_1 + e_2 + ... + e_n = v_2 + e_2 + ... + e_n = ... =$$

$$= v_n + e_n = v_1.$$

Thus $e_1 + ... + e_n = 0$, the zero vector. Conversely, if $e_1, ..., e_n$ are such that their sum is zero, then one can choose an arbitrary point $v \in \mathbb{R}^3$ and put $v_1 = v$, $v_2 = v_1 + e_1$, ..., and $v_n = v_{n-1} + e_{n-1}$. Since $v_n + e_n = v_1 + e_1 + e_2 + ... + e_{n-1} + e_n = v_1 + 0 = v_1$, we obtain a closed polygon $P = [v_1, ..., v_n]$ whose "not normalized tangent indicatrix" is the space polygon $[e_1, ..., e_n]$.

Therefore, it is easy to pass from $\{e_1, ..., e_n\}$ to $\{v_1, ..., v_n\}$. The difficult step is, given $\{u_1, ..., u_n\}$, to rescale them so that the new vectors sum up to zero. Since for each $i \in \{1, ..., n\}$ the vectors e_i and u_i point to the same direction, what we want are positive real numbers $\lambda_1, ..., \lambda_n$ such that $e_i = \lambda_i u_i$ for each $i \in \{1, ..., n\}$, and with sum

$$e_1 + ... + e_n = \lambda_1 u_1 + ... + \lambda_n u_n$$

equal to zero.

One can already see how we can use the fact of the $\{u_1, ..., u_n\}$ not be entirely contained in one hemisphere: for any u_i there must be a certain number of vectors which, for a convenient sum, cancel out the (possibly rescaled) vector u_i. At this point we introduce the following definitions:

Definition 3.11 *Given* m *vectors* $w_1, ..., w_m \in \mathbb{R}^N$*, the closed cone generated by* $\{w_1, ..., w_m\}$ *is the set defined by*

$$\overline{\mathcal{C}}(w_1, ..., w_n) = \{\lambda_1 w_1 + ... + \lambda_n w_m; \lambda_i \geq 0 \textit{ for each } i \in \{1, ..., m\}\}.$$

Similarly, the open cone generated by $\{w_1, ..., w_m\}$ *is the set defined by*

$$\mathcal{C}(w_1, ..., w_n) = \{\lambda_1 w_1 + ... + \lambda_n w_m; \lambda_i > 0 \textit{ for each } i \in \{1, ..., m\}\}.$$

Given 3 linearly independent vectors $u_2, u_3, u_4 \in \mathbb{R}^3$, any vector $u \in \mathbb{R}^3$ can be written as a unique linear combination of u_2, u_3 and u_4. If in addition u is contained in $\mathcal{C}(u_2, u_3, u_4)$, then the coefficients $\lambda_2, \lambda_3, \lambda_4$ are all positive. It is clear that in this case the plane $span\{u_2, u_3\}$ does not separate u and u_4. In terms of determinants, this means that

$$sign[u, u_2, u_3] = sign[u_4, u_2, u_3] = sign[u_2, u_3, u_4].$$

Analogously, we deduce that

$$sign[u, u_3, u_4] = sign[u_2, u_3, u_4]$$

and

$$sign[u, u_2, u_4] = sign[u_3, u_2, u_4] = -sign[u_2, u_3, u_4].$$

Conversely, it is clear that if a unit vector u satisfies the above three equations, then $u \in \mathcal{C}(u_2, u_3, u_4)$.

Now, using the notion of cone, let us examine the following situation: Suppose that we are given four unit vectors $\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, not all contained in the same closed hemisphere. Assume $\{u_2, u_3, u_4\}$ to be linearly independent.

We claim that $-u_1 \in C(u_2, u_3, u_4)$. For suppose that this is not the case, i.e., that one of the last three equations (say the first one) does not hold. If

$$sign[-u_1, u_2, u_3] \neq sign[u_2, u_3, u_4],$$

then either $[u_1, u_2, u_3] = 0$ (in which case u_1, u_2 and u_3 are in the same spherical line and therefore u_1, u_2, u_3 and u_4 are on the same closed hemisphere) or $sign[u_1, u_2, u_3] = sign[u_2, u_3, u_4]$, i.e., u_1 and u_4 are on the same side of $span\{u_2, u_3\}$, i.e., u_1, u_2, u_3 and u_4 are on the same closed hemisphere.

Similarly, assuming that one of the other two equations does not hold, one gets another contradiction.

Now, assume that the set of unit vectors $\{u_1, u_2, u_3, u_4\}$ is contained in a closed hemisphere $\overline{H}$. Denote by H' the open hemisphere which is the reflection of the open hemisphere H. Since $u_1 \in \overline{H}$ and $C(-u_2, -u_3, -u_4) \cap \mathbb{S}^2 \subset H'$, and moreover $\overline{H} \cap H' = \emptyset$, it follows that $u_1 \notin C(-u_2, -u_3, -u_4)$.

We have therefore proved:

Proposition 3.12 *Given any 4 vectors u_1, u_2, u_3, u_4 in $\mathbb{S}^2$ such that $\{u_2, u_3, u_4\}$ is linearly independent, the following conditions are equivalent:*

(a) u_1, u_2, u_3 and u_4 are not on the same closed hemisphere;

(b) $-u_1 \in \mathcal{C}(u_2, u_3, u_4)$*;*
(c) $sign[u_1, u_2, u_3] = sign[u_1, u_3, u_4] = -sign[u_1, u_2, u_4] = -sign[u_2, u_3, u_4]$.

If $Q = [u_1, u_2, u_3, u_4]$ is a spherical polygon, then the hypothesis that the points are not on the same hemisphere is equivalent to $-u_1 \in \mathcal{C}(u_2, u_3, u_4)$, which in turn is equivalent to the fact that there are positive real numbers λ_2, λ_2 and λ_3 such that $-u_1 = \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$, i.e.,

$$1 \cdot u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0.$$

Therefore, in this case we were successful at lifting the vectors $u_1, ..., u_4$ to rescaled vectors $e_1, ..., e_4$ such that their sum is zero, which in turn implies the existence of (an infinite number of) polygons P whose tangent indicatrix is Q.

An interesting and simple geometric fact that follows immediately from the previous proposition is the following:

Corollary 3.13 *Given any 4 vectors* u_1, u_2, u_3, u_4 *in* $\mathbb{S}^2$ *such that any triple of them is linearly independent, the following conditions are equivalent:*
(a) $-u_1 \in \mathcal{C}(u_2, u_3, u_4)$*;*
(b) $-u_2 \in \mathcal{C}(u_1, u_3, u_4)$*.*
(c) $-u_3 \in \mathcal{C}(u_1, u_2, u_4)$*.*
(d) $-u_4 \in \mathcal{C}(u_1, u_2, u_4)$*.*

Proof. By Proposition 3.12, all of the above conditions are equivalent to the condition that u_1, u_2, u_3 and u_4 are not on the same closed hemisphere. ■

Now we want to look at configurations with more than just 4 points in $\mathbb{S}^2$. To have an idea of what problems might arise, let us look at the following examples:

Example 3.14 *Let* $\{u_1, u_2, u_3, u_4, u_5\}$ *be a set of 5 points in the sphere* $\mathbb{S}^2$*, where* $u_2 = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$*,* $u_3 = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$*,* $u_4 = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$ *and* $u_5 = \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ *(the same points of Example 3.8, except that the indices are translated by 1).*

Depending on the position of the vector u_1*, its antipode* $-u_1$ *might be in different regions of* $\mathbb{S}^2$*. Figure 3.4 shows some of the possibilities.*

In case (a), $-u_1 \in \mathcal{C}(u_2, u_3, u_4) \cap \mathcal{C}(u_2, u_3, u_5)$*.*

In case (b), $-u_1 \in \mathcal{C}(u_2, u_3, u_4) \cap \mathcal{C}(u_3, u_5)$*.*

Finally, in case (c), $-u_1 \in \mathcal{C}(u_2, u_4) \cap \mathcal{C}(u_3, u_5)$*.*

If $-u_1$ *is not in one of these configurations, then* $-u_1 \notin \mathcal{C}(u_2, u_3, u_4, u_5)$*. Similarly as in the proof of Proposition 3.12, there are two possibilities:*

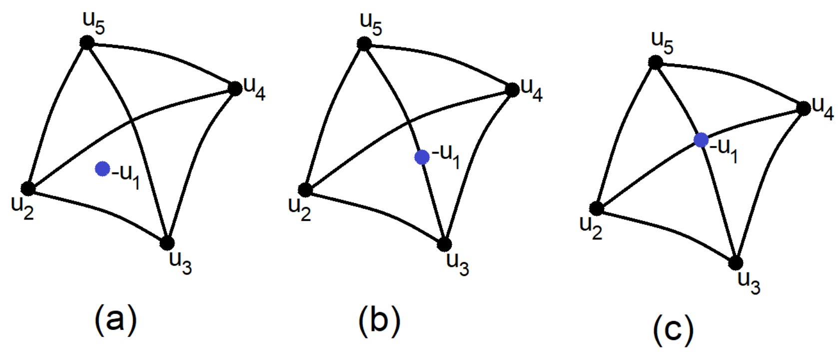

Figure 3.4: Three different cases

- $-u_1$ *is on one of the spherical edges of the quadrangular region (say,* $[u_2, u_3]$*) and then, since* u_4 *and* u_5 *are on the same side of span*$\{u_2, u_3\}$*, we have that all of the points are one the same closed hemisphere.*
- $-u_1$ *is separated by the plane span*$\{u_i, u_j\}$ *(where* i, j *are some indices of* $\{2, 3, 4, 5\}$*) from the remaining pair* $\{u_k, u_l\}$*, i.e., the points* u_1*,* u_k *and* u_l *are on the same side of the plane span*$\{u_i, u_j\}$*. In other words, all points* $u_1, ..., u_5$ *are on the same closed hemisphere.*

Therefore, for a set of points not entirely contained in a hemisphere, the three cases above are (up to symmetry) the only possibilities. Thus:

- *In (a),* $-u_1 = \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ *and* $-u_1 = \mu_2 u_2 + \mu_3 u_3 + \mu_4 u_5$*, which implies*

$$2u_1 + (\lambda_2 + \mu_2)u_2 + (\lambda_3 + \mu_3)u_3 + \lambda_4 u_4 + \mu_5 u_5 = 0.$$

- *In (b),* $-u_1 = \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ *and* $-u_1 = \mu_3 u_3 + \mu_5 u_5$*, which implies*

$$2u_1 + \lambda_2 u_2 + (\lambda_3 + \mu_3)u_3 + \lambda_4 u_4 + \mu_5 u_5 = 0.$$

- *In (c),* $-u_1 = \lambda_2 u_2 + \lambda_4 u_4$ *and* $-u_1 = \mu_3 u_3 + \mu_5 u_5$*, which implies*

$$2u_1 + \lambda_2 u_2 + \mu_3 u_3 + \lambda_4 u_4 + \mu_5 u_5 = 0.$$

In any of these three cases, we succeeded at rescaling our original unit vectors so that their new sum equals zero. Now, if these points were originally the vertices of a spherical polygon $Q = [u_1, u_2, u_3, u_4, u_5]$*, this implies the existence of (a infinite number of) polygons whose tangent indicatrix is exactly* Q*.*

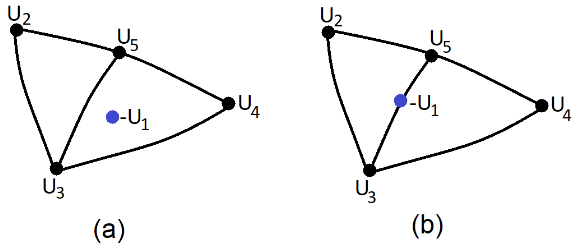

Figure 3.5: Two degenerate cases

Example 3.15 *Now let us look at another configuration, as shown in figure 3.5.*

In case (a), $-u_1 \in \mathcal{C}(u_2, u_3, u_4) \cap \mathcal{C}(u_3, u_4, u_5)$*, i.e.,* $-u_1 = \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ *and* $-u_1 = \mu_3 u_3 + \mu_4 u_4 + \mu_5 u_5$ *for positive* λ*'s and* μ*'s, which implies that*

$$2u_1 + \lambda_2 u_2 + (\lambda_3 + \mu_3)u_3 + (\lambda_4 + \mu_4)u_4 + \mu_5 u_5 = 0,$$

where all coefficients are positive.

In case (b), $-u_1 \in \mathcal{C}(u_2, u_3, u_4) \cap \mathcal{C}(u_3, u_5)$*, i.e.,* $-u_1 = \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4$ *and* $-u_1 = \mu_3 u_3 + \mu_5 u_5$ *for positive* λ*'s and* μ*'s, which implies that*

$$2u_1 + \lambda_2 u_2 + (\lambda_3 + \mu_3)u_3 + \lambda_4 u_4 + \mu_5 u_5 = 0,$$

where all coefficients are positive. These two cases are (up to symmetry) the only possibilities (if $-u_1 \notin \mathcal{C}(u_2, u_3, u_4, u_5)$*, then we derive a contradiction in the same way as we did in Example 3.14). Since we could rescale these points so that they sum to zero, we can then find a space polygon* P *whose tangent indicatrix is* $Q = [u_1, u_2, u_3, u_4, u_5]$*.*

Example 3.16 *A third type of configuration is given by figure 3.6.*

In case (a) and (b), $u_1 \in \mathcal{C}(u_2, u_3, u_4) \cap \mathcal{C}(u_2, u_3, u_5)$*.*

In case (c), $-u_1 \in \mathcal{C}(u_2, u_3, u_4) \cap \mathcal{C}(u_4, u_5)$*.*

In case (d), $-u_1 \in \mathcal{C}(u_2, u_3, u_4) \cap \mathcal{C}(u_5)$ *Notice that in this case* u_5 *is the antipode of* u_1*.*

Proceeding the same way as it was done in the previous examples, one shows that, if the u_i*'s are not entirely contained in a hemisphere, then these four cases are (up to symmetry) the only possibilities. For each case one can then obtain rescaled versions of the* u_i*'s so that they sum up to zero. Therefore there is a space polygon* P *whose tangent indicatrix is* $Q = [u_1, u_2, u_3, u_4, u_5]$*.*

As the three previous examples have shown, a certain configuration of points determines a couple of cases to consider. A little thought might convince

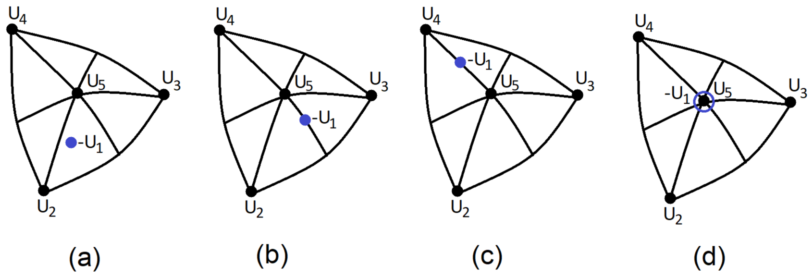

Figure 3.6: Four more cases

the reader that these three examples exhaust all possibilities for the relative position of the points u_2, u_3, u_4 and u_5 up to some permutation of the indices (the case in which u_2, u_3, u_4 and u_5 are in the same spherical line does not appear since it would then imply that all points from u_1 to u_5 would be on the same closed hemisphere).

Another important feature of what we have just done is that, given a unit vector u_1, every other unit vector from u_2, u_3, u_4 and u_5 appeared at least once as the generator of one of the cones that contained the antipode of u_1 (some of them appeared more than once, but that does not matter). This made it possible to obtain a sum with all the (rescaled) vectors.

Moreover, as we have seen in these examples, there might happen that three different points are in the same spherical line (i.e.,three vectors on the same plane). Although we could prove Proposition 3.10 in this more general case, it will be convenient to assume that this does not happen. The reason is twofold: it will make the proof considerably simpler and, as we will see later, any spherical polygon with three non-consecutive collinear vertices can be perturbed into a spherical polygon with no three collinear vertices, but with the same number of inflections.

From now on, *we assume the following typographical conventions*: given vectors u_i,u_j and u_k, we may also write $[i, j, k]$ instead of $[u_i, u_j, u_k]$. Additionally, the notation

$$[i, j, k] \simeq [a, b, c]$$

means that $sign[i, j, k] = sign[a, b, c]$. Therefore, if the determinants have opposite signs, we write

$$[i, j, k] \simeq -[a, b, c].$$

Since *we are also assuming from now on that the spherical polygons considered do not have three points in the same spherical line*, any determinant calculated using a triple of the points of the spherical polygon is nonzero. Thus, in this case, $[i, j, k] \simeq -[a, b, c]$ is equivalent to and will be written as

$[i, j, k] \not\simeq [a, b, c]$.

3.4
Some results of Convex Geometry

Before proving Proposition 3.10, we need some results of Convex Geometry. The proof of the first one reveals an interplay between conical sets and the notion of *convexity in the sphere*. For an account of these ideas, the interested reader might consult (FERREIRA; IUSEM; NéMETH, 2013).

Lemma 3.17 *Let $Q = \{u_1, u_2, ..., u_n\}$ be a finite set of points on the sphere $\mathbb{S}^2$ ($n \geq 4$), not all of them on the same closed hemisphere. Then $\overline{\mathcal{C}}(u_1, ..., u_n) = \mathbb{R}^3$.*

Proof. The proof is by induction on the number of points. For $n = 4$, then Proposition 3.12 (or equivalently, Corollary 3.13) implies that each point u_i ($i = 1, 2, 3, 4$) is such that its antipode in the open cone spanned by the other points. Since the four closed cones (each one generated by a different triple of points from the set $\{u_1, u_2, u_3, u_4\}$), restricted to the sphere, divide it into four regions, we have that $\overline{\mathcal{C}}(u_1, u_2, u_3, u_4) = \mathbb{R}^3$.

Now, assume the result true for n, and suppose we are given a set of $n+1$ points $Q = \{u_1, ..., u_n, u_{n+1}\}$, not all of them in the same closed hemisphere. If the set $Q - \{u_{n+1}\}$ is not in the same hemisphere, then by the induction hypothesis $\mathbb{R}^3 = \overline{\mathcal{C}}(u_1, ..., u_n) \subset \overline{\mathcal{C}}(u_1, ..., u_n, u_{n+1}) \subset \mathbb{R}^3$, from which the result follows.

If, however, $\{u_1, ..., u_n\}$ is in some closed hemisphere H, consider then the open region $R = \mathcal{C}(u_1, ..., u_n) \cap \mathbb{S}^2 \subset \overline{\mathcal{C}}(u_1, ..., u_n) \cap \mathbb{S}^2 \subset H$. We may assume that the vertices of the topological boundary of this region are all the u_i's of Q. For if it were not the case (say u_j is the topological interior of R), then $\overline{\mathcal{C}}(u_1, ..., u_n) = \overline{\mathcal{C}}(u_1, ..., \hat{u}_j, ..., u_n)$ and, consequently, $\overline{\mathcal{C}}(u_1, ..., u_{n+1}) = \overline{\mathcal{C}}(u_1, ..., \hat{u}_j, ..., u_{n+1})$. By the induction hypothesis applied to the set $\{u_1, ..., \hat{u}_j, ..., u_{n+1}\}$,

$$\overline{\mathcal{C}}(u_1, ..., u_{n+1}) = \overline{\mathcal{C}}(u_1, ..., \hat{u}_j, ..., u_{n+1}) = \mathbb{R}^3.$$

After labelling the indices, if necessary, we may assume that the boundary of the region R is a convex polygon with the ordering $u_1, u_2, ..., u_n$ and oriented so that the R is always on the left of the polygon. This region is, therefore, the intersection of all the open hemispheres

$$H_i = \{u \in \mathbb{S}^2; [u, u_i, u_{i+1}] > 0\},$$

for $i = 1, 2, ..., n$. Notice that $u_j \in H_i$, for all $j \neq i, i+1$.

We claim that $-u_{n+1} \in R$. For if it were not in R, then $-u_{n+1}$ would not be in at least one of the H_i. This would imply that $[-u_{n+1}, u_i, u_{i+1}] \leq 0$, i.e., $[u_{n+1}, u_i, u_{i+1}] \geq 0$, i.e., $u_{n+1} \in \overline{H}_i$. Since all other points u_j's are in $\overline{H}_i$, this implies that all points are in the closed hemisphere $\overline{H}_i$, contrary to the hypothesis.

Now, since u_{n+1} is not in any of the H_i, then the open cones of the form $\mathcal{C}(u_i, u_{i+1}, u_{n+1})$ (for all $i = 1, ..., n$) are disjoint and do not intersect the open cone $\mathcal{C}(u_1, ..., u_n)$. Moreover, we have that

$$\overline{\mathcal{C}}(u_1, ..., u_n) \cup \bigcup_{i=1}^{n} \overline{\mathcal{C}}(u_i, u_{i+1}, u_{n+1}) = \mathbb{R}^3.$$

Since the set on the left is contained in $\overline{\mathcal{C}}(u_1, ..., u_n, u_{n+1}) \subset \mathbb{R}^3$, the result follows. ∎

The following proposition is the conical version of the known Carathéodory's Theorem for convex sets (see for instance (HUG; WEIL, 2020), p. 14). The proof of the former is similar to the usual proof of the latter result.

Proposition 3.18 *Let Q be a finite set of n points in $\mathbb{R}^d$ ($n \geq d$). Let u be any point of*

$$\overline{\mathcal{C}}(Q) = \{ \textit{ finite sums of elements of the form } \lambda u; \lambda \geq 0, u \in Q\}.$$

Then there are d points $u_1, ..., u_d$ in Q and non-negative numbers $\lambda_1, ..., \lambda_d$ such that

$$u = \lambda_1 u_1 + ... + \lambda_d u_d.$$

Proof. Given $u \in \overline{\mathcal{C}}(Q)$, we have that

$$u = \lambda_1 u_1 + ... + \lambda_m u_m,$$

with $\lambda_i \geq 0$, for all $i \in \{1, ..., m\}$. Let m be the minimal number for which such a conical combination for u is possible.

We claim that $\{u_1, ..., u_d\}$ is linearly independent (from which it follows that $m \leq d$). For if it were linearly independent, then there would be $\alpha_1, ..., \alpha_m$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i u_i = 0.$$

Let $I := \{i \in \{1, ..., m\}; \alpha_i > 0\}$ (which can be assumed to be nonempty, otherwise we could work with $-\alpha_i$'s instead of α_i's). Choose $i_0 \in I$ such that

$$\frac{\lambda_{i_0}}{\alpha_{i_0}} = \min_{i \in I} \frac{\lambda_i}{\alpha_i}.$$

Hence,

$$\lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}}\alpha_i \geq 0,$$

for all $i \in I$ (notice also that this inequality always holds when $\alpha_i \leq 0$). Then, we have

$$\sum_{i=1}^{m}\left(\lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}}\alpha_i\right)u_i = \sum_{i=1}^{m}\lambda_i u_i - \frac{\lambda_{i_0}}{\alpha_{i_0}}\sum_{i=1}^{m}\alpha_i u_i = \sum_{i=1}^{m}\lambda_i u_i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \cdot 0 = u,$$

with $\lambda_i - \frac{\lambda_{i_0}}{\alpha_{i_0}}\alpha_i \geq 0$, for all $i \in \{1, ..., m\}$, and $\lambda_{i_0} - \frac{\lambda_{i_0}}{\alpha_{i_0}}\alpha_{i_0} = 0$. This contradicts minimality of m. ■

Lemma 3.19 *Let $Q = \{u_1, u_2, ..., u_n\}$ be a finite set of points on the sphere $\mathbb{S}^2$, with $n \geq 5$, not all of them on the same hemisphere. Then the set*

$$X = \{u_i \in Q; \{u_1, ..., \hat{u}_i, ..., u_n\} \textit{ is not contained on a hemisphere}\}$$

has at least $n - 3$ elements.

Proof. By Lemma 3.17, $\overline{\mathcal{C}}(u_1, ..., u_n) = \mathbb{R}^3$. In particular, $-u_1 \in \overline{\mathcal{C}}(u_1, ..., u_n)$, i.e., there are non-negative numbers λ_i $(1 \leq i \leq n)$ such that

$$-u_1 = \lambda_1 u_1 + \lambda_2 u_2 + ... + \lambda_n u_n,$$

i.e.,

$$-u_1 = \mu_2 u_2 + ... + \mu_n u_n,$$

where $\mu_i = \lambda_i/(1+\lambda_1) \geq 0$. In other words, $-u_1 \in \overline{\mathcal{C}}(u_2, ..., u_n)$. By Proposition 3.18, there are $u_i, u_j, u_k \in Q$ and non-negative numbers α_i, α_j and α_k such that

$$-u_1 = \alpha_i u_i + \alpha_j u_j + \alpha_k u_k.$$

Since we are assuming that there are no three spherically collinear points in Q, all three numbers α_i, α_j and α_k are positive, i.e., $-u_1 \in \mathcal{C}(u_i, u_j, u_k)$.

After a relabelling of the indices, if necessary, we may assume that the points u_i, u_j and u_k are u_2, u_3 and u_4.

By Proposition 3.12, each one of the points u_1, u_2, u_3 and u_4 is such that its antipode is on the open cone spanned by the other points. The respective four closed cones divide the sphere into four regions. Since we are assuming that no three points of Q are collinear, we have that any of the remaining points $u_5, ..., u_n$ is such that its antipode is contained in one and only one of the cones $C(u_1, u_2, u_3)$, $C(u_1, u_2, u_4)$, $C(u_1, u_3, u_4)$ and $C(u_2, u_3, u_4)$.

Since the antipode of any point from u_1 to u_n is in an open cone spanned by a triple from the four points u_1, u_2, u_3 or u_4, this means that all points from u_5 to u_n are not essential as a cone generator. Hence all points from u_5 to u_n are in the set X, as defined before.

Now, we just need to show that at least one of the points u_1, u_2, u_3 or u_4 is in X. Since $n \geq 5$ and no three points are (spherically) collinear, at least one of the following sets is non-empty:

$$Q_1 = \{u_i \in Q - \{u_1\}; -u_i \in C(u_2, u_3, u_4)\},$$

$$Q_2 = \{u_i \in Q - \{u_2\}; -u_i \in C(u_1, u_3, u_4)\},$$

$$Q_3 = \{u_i \in Q - \{u_3\}; -u_i \in C(u_1, u_2, u_4)\},$$

$$Q_4 = \{u_i \in Q - \{u_4\}; -u_i \in C(u_1, u_2, u_3)\}.$$

We may assume that this non-empty set is Q_1. This implies, by Proposition 3.12, that for some $i \in \{5, ..., n\}$ the points u_2, u_3, u_4 and u_i are not on the same hemisphere. This in turn implies that any of the remaining points (including u_1) is in one and only one of the four open cones spanned by each possible triple from $\{u_2, u_3, u_4, u_i\}$. Thus u_1 is not essential as a cone generator, i.e., $u_1 \in X$. ■

Remark 3.20 *The proof of Lemma 3.19 actually showed a stronger result: We could get rid of all points except four at once so that the new configuration would still not be contained in a hemisphere.*

Proof. (of Proposition 3.10) Given a spherical polygon $Q = [u_1, u_2, ..., u_n]$, we just have to show that there are positive scalars α_i such that the rescaled vectors $e_i = \alpha_i u_i$ sum up to zero.

The proof is on induction on the number of points $n \geq 4$. The case $n = 4$ is Proposition 3.12: $-u_1 \in C(u_2, u_3, u_4)$, which implies that $1 \cdot u_1 + \alpha_2 \cdot u_2 + \alpha_3 \cdot u_3 + \alpha_4 \cdot u_4 = 0$.

Now, assume the result for n points. Suppose we are given $(n+1)$ points, not all of them on the same hemisphere. By Lemma 3.19, there is at least one point (say, u_{n+1}) such that the remaining points are not on the same hemisphere. By the induction hypothesis, there are positive λ_i such that

$$\lambda_1 \cdot u_1 + \lambda_2 \cdot u_2 + ... + \lambda_n \cdot u_n = 0.$$

By the proof of Lemma 3.19, there are four points u_i, u_j, u_k and u_l (which can be assumed to be different from u_{n+1}) such that the four different cones divide the sphere in four regions. Since we assume that no three points of Q are (spherically) collinear, we have that $-u_{n+1}$ is in one of these open cones, say $C(u_i, u_j, u_k)$. Therefore

$$\mu_i \cdot u_i + \mu_j \cdot u_j + \mu_k \cdot u_k + 1 \cdot u_{n+1} = 0,$$

which, summing to the previous sum, gives

$$\sum_{m=1}^{n+1} \alpha_m u_m = 0,$$

where $\alpha_m = \lambda_m + \mu_m$ for $m = i, j$ or k, $\alpha_{n+1} = 1$ and $\alpha_m = \lambda_m$ for the remaining points. ■

The reader might be wondering why we bothered to prove Lemma 3.19, which is considerably stronger than what we actually used in the proof of Proposition 3.10. The reason why we need this result will become clear in the course of the proof of Theorem 3.6.

In order to simplify language, we introduce the following terminology:

Definition 3.21 *A set of points $Q = \{u_1, ..., u_n\} \subset \mathbb{S}^2$ $(n \geq 4)$, not in the same spherical line, is said to be balanced or in balanced position if its points are not in the same closed hemisphere. A point u_i of a balanced set is said to be essential if the set $\{u_1, ..., \hat{u}_i, ..., u_n\}$ is not balanced. Otherwise u_i is nonessential. For a spherical polygon $Q = [u_1, .., u_n]$, the same definitions apply to Q considered as a set of vertices.*

The condition of having all unit vectors $u_1, u_2, ..., u_n$ in balanced position simply means that, for each $i \in \{1, ..., n\}$, there is at least one triple of points u_j, u_k and u_l such that $-u_i \in \mathcal{C}(u_j, u_k, u_l)$. By Proposition 3.12, this is equivalent to

$$[i, j, k] \simeq [i, k, l] \not\simeq [i, j, l] \simeq [j, k, l].$$

In Lemma 3.19, however, we improved this even more: there are actually four specific points u_i, u_j, u_k and u_l such that any u_m of the remaining points has its antipode located in one and only one of the four cones spanned by these points.

The next step is, therefore, to express the fact of a spherical polygon *not having self-intersection* as a relation of signs of determinants. Looking at figure 3.7 we have some possibilities regarding the relative position of two spherical edges. For the sake of simplicity of notation we assume that one (spherical) edge is $\overrightarrow{u_1 u_2}$ and the other is $\overrightarrow{u_5 u_6}$.

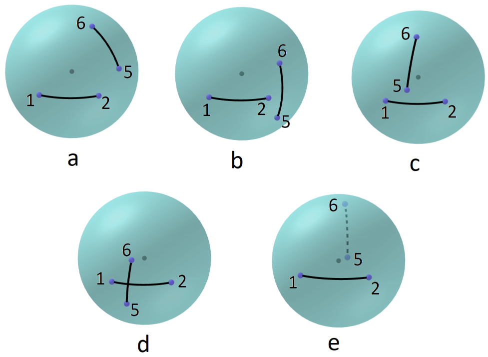

Figure 3.7: Possibilities regarding the relative positions of edges

If edges $\overrightarrow{u_1u_2}$ and $\overrightarrow{u_5u_6}$ intersect, then the spherical line spanned by an edge separates the two endpoints of the other edge (see figure 3.7(d)). In terms of determinants, this means that

$$[1,2,5] \not\simeq [1,2,6] \text{ and } [1,5,6] \not\simeq [2,5,6].$$

Notice, however, that this relation is not exclusive to the case where both edges intersect: this relation is also true if one edge intersects the antipode of the other edge (see figure 3.7(e)). In order to distinguish these two possibilities, notice that, if the edges intersect, then the spherical line spanned by u_1 and u_6 does not separate u_2 and u_5, while the spherical line spanned by u_2 and u_5 does not separate u_1 and u_6 (that would not be case if one edge intersected the antipode of the other edge). Hence

$$[1,6,2] \simeq [1,6,5] \text{ and } [2,5,1] \simeq [2,5,6].$$

i.e.,

$$[1,2,6] \simeq [1,5,6] \text{ and } [1,2,5] \simeq [2,5,6].$$

Therefore, if edges $\overrightarrow{u_1u_2}$ and $\overrightarrow{u_5u_6}$ intersect, we have that

$$[1,2,5] \simeq [2,5,6] \not\simeq [1,2,6] \simeq [1,5,6].$$

We have therefore proved

Proposition 3.22 *A spherical polygon $Q \subset \mathbb{S}^2$ has a self-intersection at edges $\overrightarrow{u_iu_{i+1}}$ and $\overrightarrow{u_ju_{j+1}}$ (where $j \neq i+1$ and $i \neq j+1$) if and only if the relation*

$$[i, i+1, j] \simeq [i+1, j, j+1] \not\simeq [i, i+1, j+1] \simeq [i, j, j+1]$$

holds.

Example 3.23 *Let* $Q = [u_1, u_2, u_3, u_4] \subset \mathbb{S}^2$ *be a spherical polygon whose vertices are not entirely contained on one hemisphere. By Proposition 3.12, this is equivalent to*

$$[1,2,3] \simeq [1,3,4] \not\simeq [1,2,4] \simeq [2,3,4],$$

i.e., the cyclic sequence $\epsilon_1 = [1,2,3]$, $\epsilon_2 = [2,3,4]$, $\epsilon_3 = [3,4,1] = [1,3,4]$ *and* $\epsilon_4 = [4,1,2] = [1,2,4]$ *has 4 sign changes. As we saw earlier, this is equivalent to the polygon Q having 4 (spherical) inflections.*

Moreover, Q does not have self-intersections. For if it had (say, between edges $\overrightarrow{u_1u_2}$ *and* $\overrightarrow{u_3u_4}$*), then Proposition 3.22 would imply*

$$[1,2,3] \simeq [2,3,4] \not\simeq [1,2,4] \simeq [1,3,4],$$

contradicting the previous determinant relations.

Our conclusion is that, for a spherical polygon Q with 4 points, not only the condition (a) of Proposition 3.12 implies the existence of 4 spherical inflections, but also the converse. Besides that, any of these two statements imply that Q does not have self-intersections.

Remark 3.24 *Our extra assumption on the spherical polygons not having three points in the same spherical line might seem redundant, since we assume the original polygon P in* $\mathbb{R}^3$ *to be generic: if its tangent indicatrix Q had three consecutive points* u_i, u_{i+1} *and* u_{n+2} *in the same spherical line, then* e_i, e_{i+1} *and* e_{i+2} *would be in the same plane, i.e., the vertices* v_i, v_{i+1}, v_{i+2} *and* v_{i+3} *would be in the same plane.*

Notice, however, that if the tangent indicatrix had three non-consecutive points in the same spherical line, say u_i, u_{i+1} *and* u_j*, that would only mean that* v_j *and* v_{j+1} *are in a plane parallel to the plane generated by* v_i, v_{i+1} *and* v_{i+2}*. This does not contradict the genericity of P.*

The justification of why we can assume Q to have this extra property rests on the following remark: given a spherical polygon without $Q \subset \mathbb{S}^2$*, we can perturb its vertices slightly so that Q will not have three (spherically) collinear vertices, but preserving at the same time not only the property of not being entirely contained in a hemisphere but also the property of not having self-intersections. Moreover, if Q does not have three consecutive collinear vertices (which is the case), this perturbation can be done without altering the state of a triple of vertices of P of being a flattening or not.*

3.5 Good vertices and proof of the Main Result

Theorem 3.6 follows from the following Main Result:

Theorem 3.25 *Let $Q = [u_1, ..., u_n] \in \mathbb{S}^2$ $(n \geq 4)$ be a spherical polygon in balanced position and without self-intersections. Then Q has at least four spherical inflections.*

The proof will need some lemmas. Given a spherical polygon Q and any of its vertices u_i, denote by $Q - u_i$ the polygon $[u_1, ..., \hat{u}_i, ..., u_n]$, obtained from Q by deleting the vertex u_i along with the edges $\overrightarrow{u_{i-1}u_i}$ and $\overrightarrow{u_iu_{i+1}}$, and adding the edge $\overrightarrow{u_{i-1}u_{i+1}}$ to connect vertices u_{i-1} and u_{i+1}.

Definition 3.26 *A spherical polygon $Q = [u_1, ..., u_n]$ is simple if it does not have self-intersections. A vertex u_i is said to be good if the spherical polygon $Q - u_i$ is simple. Otherwise u_i is said to be bad.*

Lemma 3.27 *Let $Q = [u_1, u_2, ..., u_n]$ be a balanced, simple spherical polygon, $(n \geq 4)$. Then the set*

$$Y = \{u_i \in Q; u_i \text{ is good}\}$$

has at least four elements.

Proof. Since Q is simple, it divides the sphere $\mathbb{S}^2$ into two disjoint, open regions R_1 and R_2. The fact that Q is balanced implies, by Lemma 3.17, that for any point u of $\mathbb{S}^2$, u can be expressed as a non-negative combination of vertices of Q (at most three of them, by Proposition 3.18), i.e., u in the inside of the triangle spanned by vertices u_i, u_j and u_k of Q.

Therefore, the sphere can be subdivided into triangles whose vertices are the vertices of Q. Choose any such triangulation T of the sphere whose triangles are entirely contained either in $R_1 \cup Q$ or $R_2 \cup Q$. Such triangulations always exist in this case (see figure 3.8). (For instance, for vertex u_1, connect to it all other vertices u_i such that the spherical segment $\overrightarrow{u_1u_i}$ (i.e., the segment that minimizes distance between the points) only intersects Q at u_1 and u_i; then connect u_2 to all other vertices u_i such that the spherical segment $\overrightarrow{u_2u_i}$ does not intersect Q and the previous added segments, except of course at the vertices of Q; and so on.) Notice that, since $n \geq 4$, this triangulation has at least 4 triangles.

For the triangulation T restricted to the region R_1 (denoted by T_1), consider its dual graph G_1 (a triangle $\triangle_1 \in T$ is considered a vertex and is connected to another triangle $\triangle_2$ if both have a common edge which is not in Q). Since the triangulation only uses triangles with vertices in Q, then the

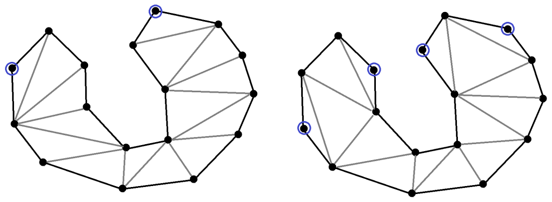

Figure 3.8: Two possible triangulations for a region determined by a spherical polygon (again, we represent such objects on the plane in order to aid visualization). Notice that different triangulations might lead to different sets of good vertices. Our argument, however, guarantees that for any triangulation there will always be at least 2 such vertices per region.

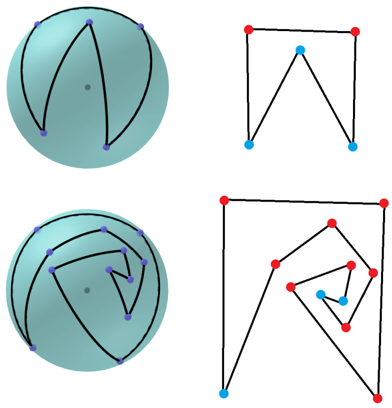

Figure 3.9: Two examples showing that the balanced position hypothesis is necessary for Lemma 3.27 to be true. On the left the spherical polygons, on the right their planar version to aid visualization. The vertices in blue are good, while the ones in red are bad.

dual graph G_1 is a *tree*, i.e., it is connected and does not have cycles. By a basic theorem of graph theory (theorem 2.1), such a graph (provided it has at least two vertices, which is the case), has at least two *leaves*, i.e., 2 vertices adjacent to only one other vertex (see figure 3.8).

In terms of the triangulation T_1, this means that there are two triangles $\triangle_1$ and $\triangle_2$ in T_1 with only one edge in the relative interior of the region R_1. For $\triangle_1$, let u_i be the vertex adjacent to the edges of $\triangle_1$ that are contained in Q. Since the edge $\overrightarrow{u_{i-1}u_{i+1}}$ of $\triangle_1$ is entirely contained in R_1, this means in particular that it does not intersect Q at any other edge, i.e., $Q - u_i$ is simple. In other words, u_i is good. By the same argument applied to $\triangle_2$, we obtain another good vertex u_j.

Proceeding analogously to the triangulation T restricted to the region R_2, we obtain other two good vertices. ∎

Figure 3.9 shows that the balanced position hypothesis on the spherical polygon is necessary, even for a large number of vertices.

Lemma 3.28 *Given a balanced, simple spherical polygon $Q = [u_1, u_2, ..., u_n]$, $n \geq 5$, there is at least one good, nonessential vertex u_i.*

Proof. By Lemma 3.19, the set

$$X = \{u_i \in Q; u_i \text{ is nonessential}\}$$

has at least $n - 3$ elements. By Lemma 3.27, the set

$$Y = \{u_i \in Q; u_i \text{ is good}\}$$

has at least four elements. Therefore, the set $X \cap Y$ has at least one element, i.e., there is at least one good, nonessential vertex. ■

Lemma 3.29 *Given a simple spherical polygon Q, let u_i be a good vertex of Q. Then the number of spherical inflections of Q is greater or equal to the number of spherical inflections of the resulting spherical polygon $Q - u_i$.*

Proof. Given $u_i \in Q$, the polygon $Q - u_i$ will be formed by deleting u_i along with the (spherical) edges $\overrightarrow{u_{i-1}u_i}$ and $\overrightarrow{u_iu_{i+1}}$ from Q, and by adding the edge $\overrightarrow{u_{i-1}u_{i+1}}$. Figure 3.10 depicts two of the many possibilities (we represent them on the plane instead of the sphere to aid visualization).

If u_i is the vertex of the conclusion of Lemma 3.28, then the situation of figure 3.10 (b) cannot happen: if at least one of the vertices u_{i-2} and u_{i+2} were in the inside of the spherical triangle formed by the vertices u_{i-1}, u_i and u_{i+1}, then Q would either have a self-intersection (which is impossible by hypothesis) or $Q - u_i$ would have a self-intersection (which is not true due to the choice of u_i).

Therefore, all possible possibilities are, up to symmetry, the ones represented in figures 3.11 and 3.12 (again, we represent these configurations on the plane instead of the sphere). Denoting by $d_i(x)$ the number of spherical inflections of Q minus the number of spherical inflections of $Q - u_i$ in configuration (x), we see that $d_i(a) = 0$, $d_i(b) = 0$, $d_i(c) = +2$, $d_i(d) = 0$, $d_i(e) = +2$, $d_i(f) = 0$, $d_i(g) = +2$, $d_i(h) = 0$, $d_i(i) = +2$ and $d_i(j) = +4$. Since all these numbers are either positive or zero, the lemma is proved. ■

Proof. (of Theorem 3.25) The proof in on induction on the number of vertices of Q. The case $n = 4$ is Example 3.23, for which the result is valid.

Assume that the result holds for spherical polygons with n points. Suppose we are given a spherical polygon Q with $n + 1$ points.

By Lemma 3.28, there is at least one point u_i such that the resulting polygon $Q - u_i$ is balanced and simple. By Lemma 3.29, the number of

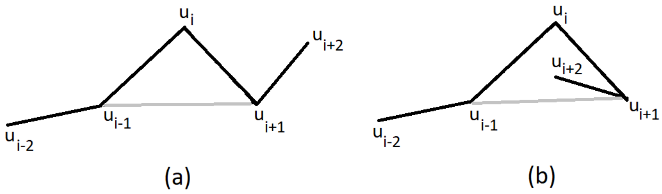

Figure 3.10: Two of many possibilities

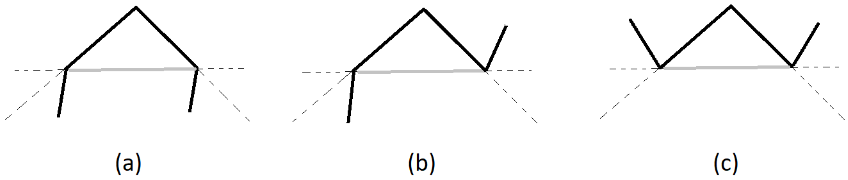

Figure 3.11: Three possible simple cases

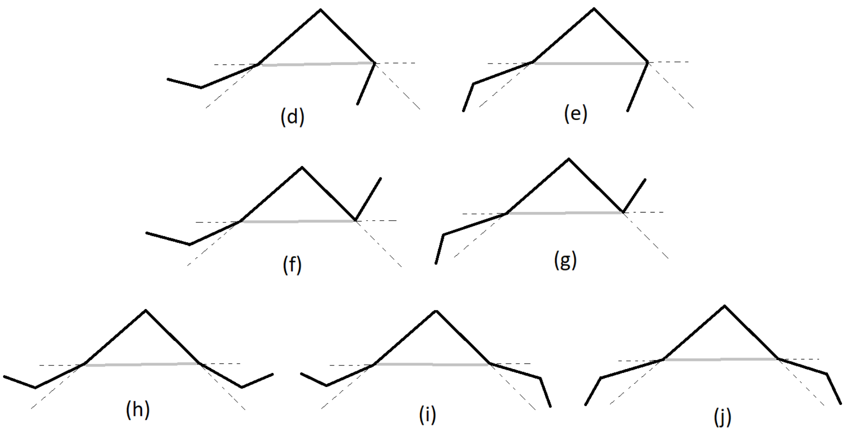

Figure 3.12: Seven possible cases in which at least one of the adjacent edges might change its condition of being an inflection or not.

inflections of Q is greater or equal to the number of inflections of $Q - u_i$. By the induction hypothesis, however, the number of inflections of $Q - u_i$ is greater or equal to four. ■

3.6 Applications to Spherical Polygons

Besides the original Segre's Theorem for spherical curves, there are in the literature other interesting results regarding smooth curves. Among these results we have the *Tennis Ball Theorem* (theorem 1.8) and a theorem by Möbius on smooth projective curves, which can be formulated in terms of spherical *centrally symmetric curves* (theorem 1.9). In (OVSIENKO; TABACHNIKOV, 2001), Ovsienko and Tabachnikov state discrete analogs of these theorems as Conjectures, adding that it would be interesting to find discrete proofs of these results. Since both of these theorems follow from Segre's Theorem, while our proof of the latter result is entirely discrete, our approach follows the outline set by Ovsienko and Tabachnikov. Before stating and proving these results, we need a preliminary remark.

Remark 3.30 *We assume the following convention: a simple spherical polygon Q which is contained in a spherical line will be considered balanced and all its edges will be considered spherical inflections. Note that Definition 3.21 does not apply here since we assumed then that the points of Q would not be in the same spherical line.*

The reason for this convention is that such a spherical polygon can always be realized as a tangent indicatrix of a planar polygon P in $\mathbb{R}^3$. Definition 3.21 could be phrased in terms of open hemispheres instead of closed ones in order to contain the planar case, but the proofs involving this alternative notion would always require some argument of perturbation of hemispheres.

Moreover, since the notion of inflection we use is related to the change of signs of the cyclic sequence of determinants, it is a way to mimic the smooth idea of the torsion going from negative to positive (or vice-versa), i.e., passing through zero. For a planar spherical polygon, all determinants $[u_i, u_{i+1}, u_{i+2}]$ are zero, hence it is reasonable to consider all edges as inflections.

As a first application of Theorem 3.25 we have the following result:

Theorem 3.31 *(Discrete Tennis Ball Theorem) If a spherical, simple polygon $Q = [u_1, ..., u_n]$ $(n \geq 4)$ divides the sphere into two regions with the same area, then Q has at least 4 spherical inflections.*

Proof. If Q is contained in a spherical line, then the result follows by Remark 3.30.

Suppose now that Q is not contained in a spherical line. Since Q is simple, it suffices by Theorem 3.25 to show that Q is balanced. If it were not balanced, then Q would be contained in a closed hemisphere H. Hence one of the two regions R_1 and R_2 determined by Q would be contained in H (say $R_1 \subset H$). Since Q is not planar, $R_1 \neq H$ and therefore $area(R_1) < area(H) = 2\pi$, contrary to hypothesis that $area(R_1) = area(R_2) = 2\pi$. ■

Corollary 3.32 *A space polygon $P = [v_1, ..., v_n]$ $(n \geq 4)$ whose tangent indicatrix divides the sphere into two regions with the same area must have at least 4 flattenings.*

Before our second application of Theorem 3.25, we need a definition:

Definition 3.33 *For a set $X \subset \mathbb{R}^d$, define $-X$ as $-X = \{-x; x \in X\}$. We say that X is centrally symmetric (to the origin) if $-X = X$.*

Proposition 3.34 *Let Q be a simple, centrally symmetric spherical polygon. Then*

(a) Q is balanced.

(b) Q divides the sphere into two regions with the same area.

Proof. (a) We may assume Q is not contained in a spherical line. If Q were not balanced, then it would be contained in a closed hemisphere H. Hence $-Q \subset -H$. Since Q is centrally symmetric, $Q = -Q \subset -H$, which implies that $Q \subset H \cap -H$, i.e., H is contained in a spherical line, contrary to our assumption.

(b) Let R_1 and R_2 be the two (connected) regions of $\mathbb{S}^2$ determined by Q. Since both $\mathbb{S}^2$ and Q are centrally symmetric and R_1 and R_2 are connected, we have that $-R_1 = R_2$ and $-R_2 = R_1$. Since the operation $-X$ on sets preserves area, the result follows. ■

The following result is a discrete analog of a theorem by Möbius (theorem 1.9). Recall that the indices of the vertices are always taken modulo the number of vertices of the polygon.

Theorem 3.35 *A simple, centrally symmetric spherical polygon Q with at least $2n$ vertices ($2n \geq 6$) has at least 6 inflections.*

Proof. We may assume that Q is not a spherical line. By Proposition 3.34 (a) and Theorem 3.25 (or also by Proposition 3.34 (b) and Theorem 3.31), Q has at least 4 inflections. Recall that, in terms of determinants, a pair $\{u_i, u_{i+1}\}$ is an inflection if and only if the determinants $[i-1, i, i+1]$ and $[i, i+1, i+2]$ have opposite signs. From this the following facts follow:

(i) Since Q is centrally symmetric (hence $u_{i+n} = -u_i$), the pair $\{u_i, u_{i+1}\}$ is an inflection if and only if $\{u_{i+n}, u_{i+n+1}\}$ is an inflection, because in both cases there will be a sign change of determinants.

(ii) Moreover, if the sign change in $\{u_i, u_{i+1}\}$ was from negative to positive (resp. from positive to negative), then the sign change in $\{u_{i+n}, u_{i+n+1}\}$ will be from positive to negative (resp. from negative to positive), by the same reason in (i).

If the inflections already obtained are

$$\{u_i, u_{i+1}\}, \{u_j, u_{j+1}\}, \{u_k, u_{k+1}\} \text{ and } \{u_l, u_{l+1}\},$$

then by fact (i) the pairs

$$\{u_{i+n}, u_{i+n+1}\}, \{u_{j+n}, u_{j+n+1}\}, \{u_{k+n}, u_{k+n+1}\} \text{ and } \{u_{l+n}, u_{l+n+1}\}$$

are also inflections. There might be some repetitions if some of the first 4 inflections are symmetric to each other. If that does not happen, then we obtain in total 8 inflections. If there is only one pair of symmetric inflections among these first ones, then we obtain in total 6 inflections. Finally, if there are two pairs of symmetric inflections among the first 4 ones, then we still have only 4 inflections. In this case, we label these inflections simply as

$$\{u_i, u_{i+1}\}, \{u_j, u_{j+1}\}, \{u_{i+n}, u_{i+n+1}\} \text{ and } \{u_{j+n}, u_{j+n+1}\},$$

with $i < j < i+n < j+n$. We may assume, without loss of generality, that in inflection $\{u_i, u_{i+1}\}$ the sign change went from positive to negative. Consequently, the sign change in $\{u_{i+n}, u_{i+n+1}\}$ goes from negative to positive (by fact (ii)). Since inflection $\{u_j, u_{j+1}\}$ happens between them (hence, changing the sign), there must be an odd extra number of inflections between $\{u_i, u_{i+1}\}$ and $\{u_{i+n}, u_{i+n+1}\}$ in order to compensate for the change. In particular, there is at least one other inflection $\{u_k, u_{k+1}\}$, with $i < k < i+n$ and $k \neq j$. By fact (i) again, edge $\{u_{k+n}, u_{k+n+1}\}$ is also an inflection (a new one). We have, thus, proved that also in this case Q has at least 6 inflections. ∎

Corollary 3.36 *A space polygon P with $2n$ vertices ($2n \geq 6$) and whose tangent indicatrix is simple and centrally symmetric must have at least 6 flattenings.*

3.7 An application to weakly generic polygons

Notice that in the proof of theorem 3.35 we actually proved the following result, which is stated here separately for reference:

Lemma 3.37 *If a centrally symmetric spherical polygon Q with at least $2n$ vertices ($2n \geq 6$) has at least 4 inflections, then it has at least 6 inflections.*

Using lemma 3.37, we can prove theorem 2.26:

Proof. (of theorem 2.26) Since P is weakly generic and weakly convex, then it has at least 4 flattenings, by remark 2.24 and theorem 2.4. Because each flattening of P corresponds to an inflection of its tangent indicatrix Q, it follows that Q has at least 4 inflections.

Moreover, since P is centrally symmetric to a point $x_0 \in \mathbb{R}^3$, Q is also centrally symmetric. By lemma 3.37, Q has at least 6 inflections. But this implies that P has at least 6 flattenings. ∎

3.8 Further remarks

Although propositions 3.12 and 3.22 are elementary, their proofs and application to our study of spherical polygons seem to be new.

On the other hand, lemmas 3.19, 3.27 and 3.28 are new. Since we have not found proposition 3.10 in the discrete case in the literature (although the corresponding result for smooth curves is known), we included it here for completeness.

Lemma 3.29 also does not appear in the literature, and by the same reason as before we included it here.

The proof of theorem 3.6 using only discrete tools is new. Consequently, the proofs of theorems 3.25, 3.31 and 3.35 and corollaries 3.32 and 3.36 are new in the sense that, since they follow from theorem 3.25, they also depend only on discrete tools.

Notice also that theorem 3.6 and corollary 3.36 parallel theorem 2.4 and theorem 2.26, respectively, in the sense that that, once one adds the hypothesis of central symmetry to the condition of “convexity" in each case, one improves the lower bound on the number of flattenings of the polygon.

4
Spherical polygons without self nor antipodal intersections

4.1
Introduction

In the previous chapter, we proved a discrete analog of a theorem by Segre. Although the curves originally considered by Segre were smooth and the ones considered in our case were polygons, both versions stated that these curves, under the hypotheses of not being contained in a closed hemisphere and not having self-intersections, must have at least four inflections.

Ghomi (in (GHOMI, 2013)) proved in the smooth setting that, under one additional condition (namely, that such curves do not have antipodal intersections), the lower bound on the number of inflections can be improved to six:

Theorem 4.1 *Let γ be a $\mathcal{C}^2$ closed spherical curve, not entirely contained in any closed hemisphere. If, for any pair of points $t \neq s \in \mathbb{S}^1$, $\gamma(t) \neq \pm\gamma(s)$ (i.e., γ does not have self nor antipodal intersections), then γ must have at least six (spherical) inflections.*

Theorem 4.1 is actually a particular case of a theorem also proved by Ghomi, which will be stated in the next chapter (but see (GHOMI, 2013)).

The main goal of this chapter is to state a discrete analog of theorem 4.1 and prove it. In order to do it, we use determinants again to express algebraically when the edge of a spherical polygon intersects the antipode of another edge. The proof of our theorem then proceeds on induction on the number of vertices of the polygon. Similarly as in the previous chapter, the most difficult step of the proof is to prove the existence of a vertex such that, after its deletion, the resulting polygon is not contained in a closed hemisphere nor has self or antipodal intersections. In this chapter we also assume that any spherical polygon being considered is such that no three of its vertices are in the same spherical line.

4.2
Antipodal intersections

Recall that, given four unit points u_i, u_j, u_k and u_l (any triple of them being linearly independent), they are said to be in *balanced position* if they are not in the same closed hemisphere. By proposition 3.12, this happens if and only if the following relation holds:

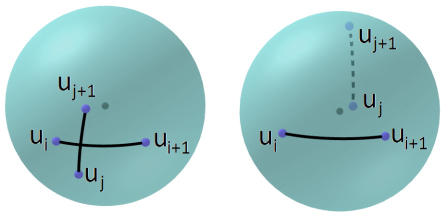

Figure 4.1: A self-intersection (left) and a antipodal intersection (right).

$$[u_i, u_j, u_k] \simeq [u_i, u_k, u_l] \not\simeq [u_i, u_j, u_l] \simeq [u_j, u_k, u_l].$$

Recall also that, by proposition 3.22, a spherical polygon $Q \subset \mathbb{S}^2$ has a self-intersection at edges $\overrightarrow{u_i u_{i+1}}$ and $\overrightarrow{u_j u_{j+1}}$ (where $j \neq i+1$ and $i \neq j+1$) if and only if the relation

$$[i, i+1, j] \simeq [i+1, j, j+1] \not\simeq [i, i+1, j+1] \simeq [i, j, j+1]$$

holds (see figure 4.1 on the left).

We will also need the notion of intersection of an edge with the antipode of another edge (see figure 4.1 on the right). We will call such an intersection as an *antipodal intersection*. In the proof of proposition 3.22 (see our previous chapter), we noticed that, given edges $\overrightarrow{u_i u_{i+1}}$ and $\overrightarrow{u_j u_{j+1}}$ (assume that $i = 1$ and $j = 5$ for the sake of simplicity of notation) such that one intersects the antipode of the other, the spherical line spanned by each one intersects the other one. This implies that

$$[1, 2, 5] \not\simeq [1, 2, 6] \text{ and } [1, 5, 6] \not\simeq [2, 5, 6].$$

However, this condition also applies to the case of ordinary intersection of edges. The difference now is that here the spherical line spanned by u_1 and u_6 separates u_2 and u_5, while the spherical line spanned by u_2 and u_5 separates u_1 and u_6. Hence

$$[1, 6, 2] \not\simeq [1, 6, 5] \text{ and } [2, 5, 1] \not\simeq [2, 5, 6],$$

i.e.,

$$[1, 2, 6] \not\simeq [1, 5, 6] \text{ and } [1, 2, 5] \not\simeq [2, 5, 6].$$

Therefore, if edge $\overrightarrow{u_1 u_2}$ intersects the antipode of $\overrightarrow{u_5 u_6}$, we have that

$$[1, 2, 5] \simeq [1, 5, 6] \not\simeq [1, 2, 6] \simeq [2, 5, 6],$$

which is equivalent to the condition of the vertices of edges $\overrightarrow{u_1 u_2}$ and $\overrightarrow{u_5 u_6}$

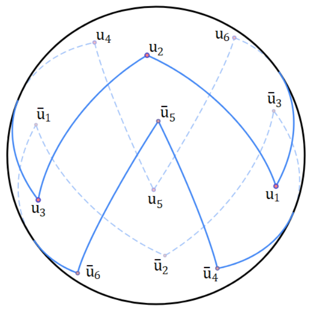

Figure 4.2: A balanced polygon Q with 6 vertices, without self nor antipodal intersections, together with its reflected polygon $\overline{Q}$. Notice that a polygon not having antipodal intersections is equivalent to this same polygon not intersecting its reflected version.

being in balanced position. We have therefore proved

Proposition 4.2 *A spherical polygon $Q \subset \mathbb{S}^2$ has a antipodal intersection at edges $\overrightarrow{u_i u_{i+1}}$ and $\overrightarrow{u_j u_{j+1}}$ (where $j \neq i+1$ and $i \neq j+1$) if and only if the following relation holds:*

$$[i, i+1, j] \simeq [i, j, j+1] \not\simeq [i, i+1, j+1] \simeq [i+1, j, j+1].$$

This, on its turn, happens if and only if the set of vectors $\{u_i, u_{i+1}, u_j, u_{j+1}\}$ is in balanced position.

4.3
Statement of theorem and idea of its proof

We can finally state the discrete analog of theorem 4.1:

Theorem 4.3 *Let $Q = [u_1, ..., u_n] \in \mathbb{S}^2$ $(n \geq 6)$ be a spherical polygon in balanced position. If Q does not have self-intersections nor antipodal intersections, then it has at least six inflections.*

Notice that the hypotheses of theorem 4.3 do not even hold for polygons with 4 or 5 vertices: since a balanced spherical polygon has 4 of its vertices in balanced position, these vertices will necessarily be consecutive in any spherical polygon with 4 or 5 vertices. For spherical polygons with 6 vertices, however, the hypotheses of theorem 4.3 can be met (see figure 4.2 for an example).

We can prove 4.3 by induction on the number $n \geq 6$ of vertices: for the case $n = 6$ we can make explicit use of propositions 3.22 and 4.2. For the induction step we need to find a vertex v with the property that the polygon $Q' = Q - \{u_i\}$ (obtained by deleting u_i and its adjacent edges and connecting its adjacent vertices with a new edge) is such that:

- Q' is not contained in any closed hemisphere (recall that in this case u_i is called *nonessential*);
- Q' does not have self-intersections (recall that in this case u_i is called *good*);
- Q' does not have antipodal intersections.

Now notice that lemma 3.29 also applies to our case: given a simple spherical polygon Q, let u_i be a good vertex of Q. Then the number of spherical inflections of Q is greater or equal to the number of spherical inflections of the resulting spherical polygon $Q' = Q - u_i$. By the induction hypothesis, Q' has at least six inflections, from which follows that Q has at least six inflections. It remains, therefore, to show that:

- the result holds for polygons with $n = 6$ vertices satisfying the hypotheses of theorem 4.3;
- there is a vertex that can be eliminated so that the resulting polygon still satisfies the hypotheses of theorem 4.3.

The base case of our induction argument is given in the following proposition:

Proposition 4.4 *Let $Q = [u_1, u_2, u_3, u_4, u_5, u_6] \in \mathbb{S}^2$ be a balanced spherical polygon without self nor antipodal intersections. Then Q has 6 inflections.*

Proof. Since Q is balanced, four of its vertices are in balanced position. Since Q does not have antipodal intersections, the indices of these vertices cannot be two pairs of consecutive indices (which includes in particular the case of four consecutive indices). Therefore the indices must be such that three of them are consecutive and the remaining one is isolated in the cyclic sequence $(1, 2, 3, 4, 5, 6)$ (for example, $\{1, 2, 3, 5\}$ or $\{2, 4, 5, 6\}$ are in principle valid, but $\{1, 2, 4, 5\}$ and $\{2, 3, 4, 5\}$ are not). After a cyclic rearrangement of the indices of Q, we may assume that the indices of these vertices are $\{1, 2, 3, 5\}$. Therefore

$$[1, 2, 3] \simeq [1, 3, 5] \not\simeq [1, 2, 5] \simeq [2, 3, 5].$$

In this case, the vertices u_1, u_2, u_3 and u_5 subdivide the sphere in four spherical regions (triangles). The vertices u_4 is such that its antipode must be in one of these four regions, i.e., u_4 must be in balanced position with three of the four other vertices. The only triple that works is $\{u_1, u_3, u_5\}$, since any other triple, being in balanced position with u_4, would then form an antipodal intersection. The fact of $\{u_1, u_3, u_4, u_5\}$ being in balanced position then implies that

$$[3,4,5] \simeq [1,3,5] \not\simeq [1,4,5] \simeq [1,3,4].$$

Again, by the same argument as with the vertex u_4, $\{u_1, u_3, u_5, u_6\}$ is in balanced position. Therefore

$$[1,3,5] \simeq [1,5,6] \not\simeq [3,5,6] \simeq [1,3,6].$$

Notice that the determinant $[1,3,5]$ appears in all cases. Suppose without loss of generality that $[1,3,5] > 0$. Hence

- $[1,3,5]$, $[1,2,3]$, $[3,4,5]$ and $[1,5,6]$ are positive;
- $[1,2,5]$, $[2,3,5]$, $[1,4,5]$, $[1,3,4]$, $[3,5,6]$ and $[1,3,6]$ are negative.

Since $[1,2,3]$, $[3,4,5]$ and $[1,5,6]$ are positive, we just need to show that $[2,3,4]$, $[4,5,6]$ and $[1,2,6]$ are negative.

Suppose by contradiction that one of these determinants, say $[1,2,6]$, is positive. From this the following two facts follow:

- $[2,5,6]$ is positive. In fact, if $[2,5,6]$ were negative, then we would have

  $$[1,2,5] \simeq [2,5,6] \not\simeq [1,2,6] \simeq [1,5,6],$$

  i.e., edges $\overrightarrow{u_1u_2}$ and $\overrightarrow{u_5u_6}$ would have a (usual) self-intersection, contrary to hypothesis.
- $[2,3,6]$ is positive. In fact, if $[2,3,6]$ were negative, then we would have

  $$[1,2,3] \simeq [1,2,6] \not\simeq [2,3,6] \simeq [1,3,6],$$

  i.e.,

  $$[2,3,6] \simeq [3,6,1] \not\simeq [2,3,1] \simeq [2,6,1],$$

  i.e., the edges $\overrightarrow{u_2u_3}$ and $\overrightarrow{u_6u_1}$ would have a (usual) self-intersection, contrary to hypothesis.

Now, since $[2,5,6]$ and $[2,3,6]$ are positive, we have that

$$[2,3,5] \simeq [3,5,6] \not\simeq [2,3,6] \simeq [2,5,6],$$

i.e., the edges $\overrightarrow{u_2u_3}$ and $\overrightarrow{u_5u_6}$ have a (usual) self-intersection. But this contradicts our hypothesis on Q.

Our conclusion then is that the determinant $[1,2,6]$ must be negative.

Now, the fact that the determinant $[1,2,6]$ is negative actually implies that $[2,3,4]$ and $[4,5,6]$ are also negative when one considers a certain symmetry between the determinants of the six-vector configuration. Notice that the bijection of the sets of vertices of Q into itself, defined by $u_i \mapsto u_{i+2} \mod 6$, preserves the sign of all determinants considered (for example, $[1,3,6] < 0$, and $[3,5,2] = [2,3,5] < 0$). Geometrically, we are now looking to polygon Q but with a cyclically different order: $Q' = [u_5, u_6, u_1, u_2, u_3, u_4]$. It follows that our previous argument (for the determinant $[1,2,6]$) works exactly the same but to the polygon Q', implying that the determinant $[4,5,6]$ is negative. Applying this bijection one more time we can use the argument again but to polygon $Q'' = [u_3, u_4, u_5, u_6, u_1, u_2]$, implying that the determinant $[2,3,4]$ is negative. ■

Now, given a spherical polygon Q in balanced position without self nor antipodal intersections, we must prove the existence of a vertex v_i such that the resulting polygon $Q' = Q - v_i$ is also in balanced position and does not have self nor antipodal intersections. An idea to prove the existence of such vertex would be similar to the idea that we used to prove the existence of a nonessential, good vertex in a simple polygon in balanced position: recall that there are always $n-3$ nonessential vertices and 4 good vertices, therefore there is at least one vertex in both subsets.

In our present case, however, there are three different features that the resulting polygon Q' must have. Therefore a counting argument such as the one before would not necessarily work as easily now. A more hopeful strategy is to consider again two subsets of the vertices of Q:

– vertices v_i such that $Q' = Q - v_i$ is balanced, i.e., nonessential vertices;

– vertices v_i such that $Q' = Q - v_i$ does not have self nor antipodal intersections. From now on, such vertices will be called *excellent*.

We know already that the number of nonessential vertices is always greater or equal to $n - 3$. And what about the minimal number of excellent vertices?

Proposition 4.5 *Let $Q = [u_1, ..., u_n]$ ($n \geq 6$) be a balanced spherical polygon, without self nor antipodal intersections. Then Q has at least 2 excellent vertices.*

Proof. Since Q does not have self nor antipodal intersections, both Q and its reflection $\overline{Q}$ through the origin subdivide the sphere into three different regions. Here it makes sense to talk about the *interior* of Q: it is one of the two regions determined by Q which does not contain the reflected polygon $\overline{Q}$.

Since Q is balanced, the interior of Q can be triangulated with vertices of Q. Since the dual graph of this triangulation (i.e., the graph whose vertices are dual to the triangles of the triangulation and whose edges are dual to the edges used in the triangulation but not contained in the polygon Q) is a nontrivial tree, it has (by theorem 2.1) at least two leaves, i.e., two triangles whose set of vertices are of the form $\{u_{i-1}, u_i, u_{i+1}\}$ and $\{u_{j-1}, u_j, u_{j+1}\}$. Since both triangles $\triangle(u_{i-1}, u_i, u_{i+1})$ and $\triangle(u_{j-1}, u_j, u_{j+1})$ are entirely contained in the interior of Q, both edges $\overrightarrow{u_{i-1}u_{i+1}}$ and $\overrightarrow{u_{j-1}u_{j+1}}$ do not intersect any other edge of Q nor $\overline{Q}$. Therefore vertices u_i and u_j are excellent, by definition. ■

At first sight we could hope to improve the lower bound on the number of nonessential or excellent vertices. A reasonable guess would be, for instance, that:

- the lower bound for nonessential vertices could be improved from $n-3$ to $n-2$;
- the lower bound for excellent vertices could be improved from 2 to 3.

In this case, by the same counting argument as before, we would find our nonessential, excellent vertex. There are, however, balanced polygons without self or antipodal intersections but with exactly $n-3$ nonessential vertices. And, among these latter polygons, some of them do not have more than 3 excellent vertices. Therefore, we must proceed differently.

Denote by $Ess(Q)$ and $Exc(Q)$ the the number of vertices of P that are essential and excellent, respectively. Let us first rephrase what we know: $Ess(Q) \leq 3$ and $Exc(Q) \geq 2$. Our strategy will be as follows: besides the cases where $Ess(Q)$ equals 0 or 1 (where the existence of a nonessential and excellent vertex follows immediately from proposition 4.5), we must study separately the cases where $Ess(Q) = 2$ and $Ess(Q) = 3$ and prove, in each case, the existence of an excellent vertex among the nonessential ones.

For what follows, we will need the following definitions. A *(closed) lune* of the sphere is the intersection of two closed hemispheres H_1 and H_2, each of them determined by distinct spherical lines (i.e., great circles) l_1 and l_2. The intersection $l_1 \cap l_2$ is a pair of antipodal points, which are called the *cusps* of the lune. The intersection of each of the spherical lines l_1 and l_2 with the boundary of the lune are called the *sides* of the lune.

Now, let $u, v, w \in \mathbb{S}^2$ be three noncollinear points (in the spherical sense). Consider the lune which has as cusps the vertices u and $\overline{u}$ and whose sides pass through v and w. We will denote such lune by $L(u; v, w)$.

Lemma 4.6 *Let L be a lune, p any of its cusps and q any point of the interior of the lune. Then the open segment $\overrightarrow{pq}$ is entirely contained in the interior of the lune.*

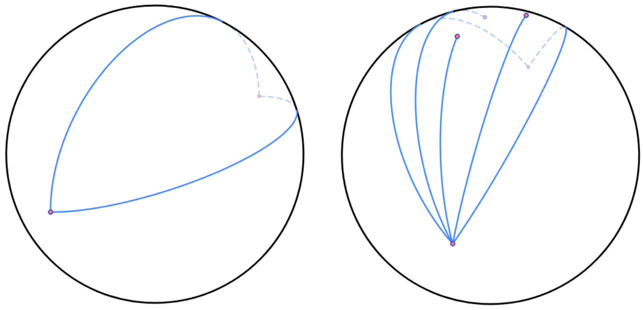

Figure 4.3: Examples of lunes. The spherical segment that connects any interior point of the lune to any of the two cusps is entirely contained in the closed lune.

Proof. Let l_1 and l_2 be the two spherical lines that determine the lune. Denote by $\overline{p}$ the antipodal point to p. Consider the line l spanned by the points p and q. If the open segment $\overrightarrow{pq}$ were not contained entirely in the interior of the lune, then it would have to intersect one of the sides of the lune, and consequently intersect one of the spherical lines (say l_1), at a point distinct from p and $\overline{p}$. But then the spherical lines l and l_1 would intersect at three different points. This would imply that the lines l and l_1 are the same, contrary to the hypothesis that q is an interior point of the lune. ■

4.4 The case where $Ess(Q) = 3$

We will first treat the case where $Ess(Q) = 3$:

Proposition 4.7 *Let $Q = [u_1, ..., u_n]$ ($n \geq 7$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) =$ 3. Then there is at least one nonessential, excellent vertex.*

For the proof of 4.7 we will need some lemmas. Since Q is balanced, there are four vertices of Q which are in balanced position. By hypothesis, exactly three of them are essential. Let u_i, u_j and u_k be these vertices (with $i < j < k$). Since all the remaining vertices are nonessential, they must all be contained in the triangular region spanned by $\overline{u}_i$, $\overline{u}_j$ and $\overline{u}_k$, i.e., the antipodes of the essential vertices. See figure 4.4. Notice that this triangular region is the intersection of any two of the three following lunes (see figure 4.5 for an example):

- the one whose cusps are u_i and $\overline{u}_i$ and whose sides pass through $\overline{u}_j$ and $\overline{u}_k$, i.e., $L(u_i; \overline{u}_j, \overline{u}_k)$;
- the one whose cusps are u_j and $\overline{u}_j$ and whose sides pass through $\overline{u}_k$ and $\overline{u}_i$, i.e., $L(u_j; \overline{u}_k, \overline{u}_i)$;

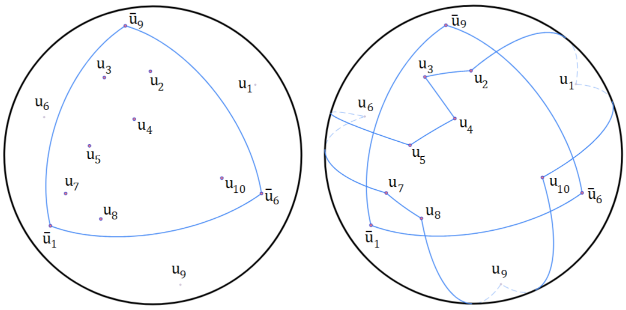

Figure 4.4: When $Ess(Q) = 3$, all nonessential vertices are in the triangular region spanned by the antipodes of the essential vertices.

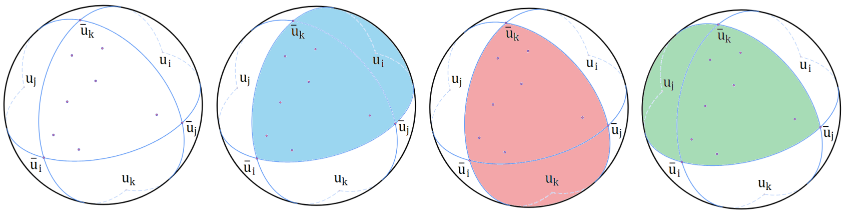

Figure 4.5: The triangle $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$ as the intersection of the lunes $L(u_i; \overline{u}_j, \overline{u}_k)$ (in blue), $L(u_j; \overline{u}_k, \overline{u}_i)$ (in green) and $L(u_k; \overline{u}_j, \overline{u}_i)$ (in red).

– and the one whose cusps are u_k and $\overline{u}_k$ and whose sides pass through $\overline{u}_i$ and $\overline{u}_j$, i.e., $L(u_k; \overline{u}_i, \overline{u}_j)$.

Lemma 4.8 *Let $Q = [u_1, ..., u_n]$ ($n \geq 6$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(Q) = 3$. Then the triple of essential vertices of Q does not have a pair of consecutive vertices.*

Proof. If there were at least one pair of consecutive vertices among the essential ones, then we would have one of the two situations:

– two of them are consecutive and the other one is isolated. After a cyclic rearrangement, we can relabel these vertices as u_1, u_2 and u_i, where $i \neq 3, n$. Since u_i must be connected to vertices u_{i-1} and u_{i+1}, which are in the lune $L(u_i; \overline{u}_1, \overline{u}_2)$, the respective edges must then be contained in this same lune, by lemma 4.6. This implies that these edges intersect the open segment $\overrightarrow{\overline{u}_1\overline{u}_2}$, which means that Q has two antipodal intersections, contrary to hypothesis (see figure 4.6 on the left for an example);

– the three vertices are consecutive. After a cyclic rearrangement, we can relabel them as u_1, u_2 and u_3. Since u_3 must be connected to vertex u_4, the corresponding edge must be (by lemma 4.6 again) entirely contained in the lune $L(u_3; \overline{u}_1, \overline{u}_2)$, and therefore intersects the open segment $\overrightarrow{\overline{u}_1\overline{u}_2}$. An analogous argument also implies that the edge connecting vertices u_1 and u_n intersects the open segment $\overrightarrow{\overline{u}_2\overline{u}_3}$. In other words, Q has two

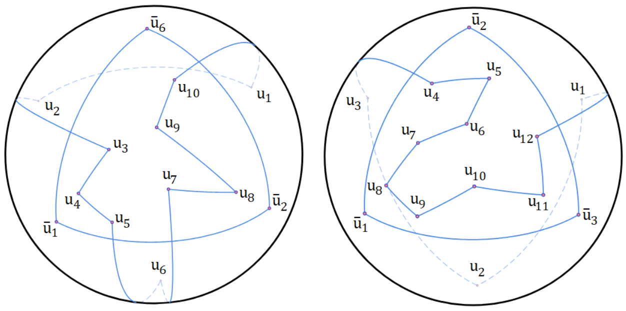

Figure 4.6: Spherical polygons where the triple of essential vertices always has at least a pair of consecutive vertices (on the left, the triple is $\{u_1, u_2, u_6\}$, while on the right it is $\{u_1, u_2, u_3\}$. In both examples, this implies at least two antipodal intersections.

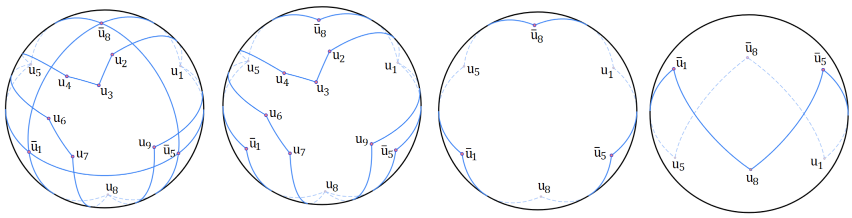

Figure 4.7: Given a polygon Q with $Ess(Q) = 3$, the region $U(Q)$ is the union of the lunes $L(u_i; \overline{u}_j, \overline{u}_k)$, $L(u_j; \overline{u}_k, \overline{u}_i)$ and $L(u_k; \overline{u}_i, \overline{u}_j)$.

antipodal intersections, contrary to hypothesis (see figure 4.6 on the right for an example).

■

Recall that the triangle $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$ is the intersection of lunes $L(u_i; \overline{u}_j, \overline{u}_k)$, $L(u_j; \overline{u}_k, \overline{u}_i)$ and $L(u_k; \overline{u}_i, \overline{u}_j)$. Denote by $U(Q)$ the union of these three lunes. Notice that the boundary of $U(Q)$ is the polygon $[u_i, \overline{u}_k, u_j, \overline{u}_i, u_k, \overline{u}_j]$, which is symmetric. Therefore the reflection of $U(Q)$ is the union of other three lunes, namely, $L(\overline{u}_i; u_j, u_k)$, $L(\overline{u}_j; u_k, u_i)$ and $L(\overline{u}_k; u_i, u_j)$ (see figure 4.7 for an example).

Lemma 4.9 *Let $Q = [u_1, ..., u_n]$ ($n \geq 7$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(Q) = 3$. Then*

(a) $Q \subset U(Q)$;

(b) the vertices u_i,u_j and u_k are in the topological boundary of $U(Q)$, while all other points of Q (including interior points of its edges) are in the topological interior of $U(Q)$.

Proof. (b) The first claim is immediate. Any other vertex of Q is in the interior of $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$ and hence in the interior of $U(Q)$. For an open edge e of Q, it either:

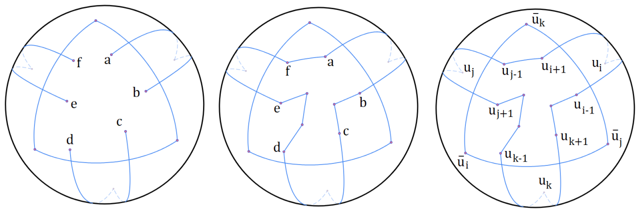

Figure 4.8: Given the vertices which are adjacent to the essential ones, the polygonal lines connecting them must be such that a is connected to f, e to d and c to b.

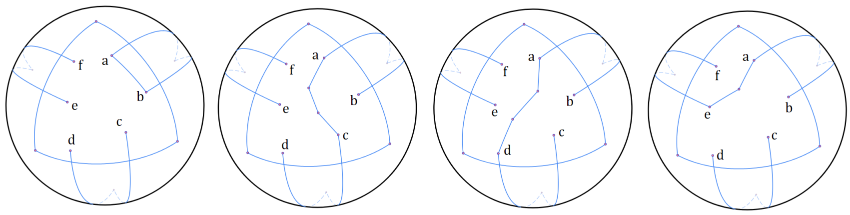

Figure 4.9: If vertex a is connected via a polygonal line to vertex b, then Q is split into two polygons. If a is connected via a polygonal line to any of the vertices c, d or e, then Q will either have a self-intersection or becomes split into two polygons again.

- has as endpoints two nonessential vertices. Hence it is contained in the interior of $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$ and therefore in the interior of $U(Q)$;
- has as endpoints a nonessential vertex and a essential vertex. Hence it is, by lemma 4.6, entirely contained in the interior of the corresponding lune and therefore in the interior of $U(Q)$.

(a) follows immediately from (b). ■

Now we can finally prove proposition 4.7:

Proof. (of proposition 4.7) By lemma 4.8, the three essential vertices are not consecutive. Denote these vertices by u_i, u_j and u_k, where $i < j < k$ and $j \neq i+1$, $k \neq j+1$. In principle, the vertices which are connected to any of the essential vertices could be connected via a polygonal line to any of the others. In figure 4.8 in the center we see a possible configuration, while in figure 4.9 any of the connecting polygonal lines lead to the resulting polygon having either a self-intersection or being split into two different polygons. Therefore these vertices are connected in the following way: the vertex adjacent to u_i which is nearest to $\overline{u}_k$ and the vertex adjacent to u_j which is nearest to $\overline{u}_k$ will be connected to each other via a polygonal line; do the same but with indices i, j and k permuted (see figure 4.8 right).

It might happen that $u_{i+1} = u_{j-1}$, $u_{j+1} = u_{k-1}$ or $u_{k+1} = u_{i-1}$. However, since Q has at least 7 vertices, there is at least one pair of consecutive vertices inside the spherical triangle $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$. We might assume without loss of

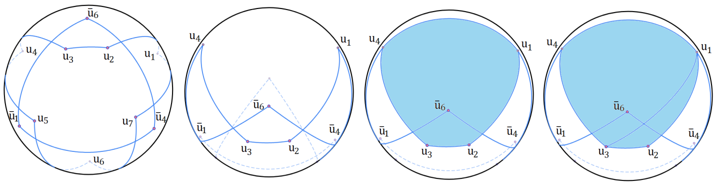

Figure 4.10: A polygon Q with 7 vertices and its corresponding polygon $\tilde{Q}$ with 4 vertices: in the given triangulation we have u_2 as an excellent vertex.

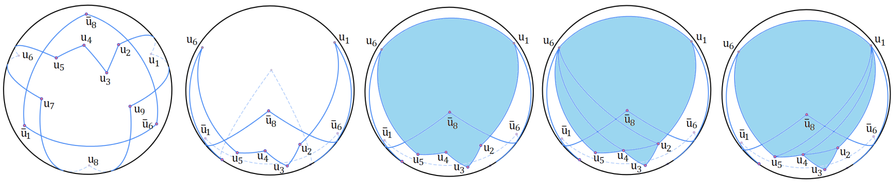

Figure 4.11: A polygon Q with 9 vertices and its corresponding polygon $\tilde{Q}$ with 6 vertices. Here we depict two different ways of triangulating the region R: in the first one we have u_3 and u_5 as excellent vertices, while in the second one we have only u_3 as an excellent vertex.

generality that u_{i+1} and u_{j-1} are distinct and that therefore the polygonal line with vertices u_i, u_{i+1}, ..., u_{j-1} and u_j has at least four vertices.

Consider then the (closed) polygon $\tilde{Q} = [u_i, u_{i+1}, ..., u_{j-1}, u_j]$ and the region R enclosed by it which contains the vertex $\overline{u}_k$. Since $\tilde{Q}$ has at least 4 vertices, any triangulation of R has at least two triangles. The dual graph of this triangulation is then a nontrivial tree and has therefore at least two leaves $\triangle_1$ and $\triangle_2$. Because all vertices u_{i+1},...,u_{j-1} are in the same triangular region, all the edges added to form the triangulation must be contained in one of the two lunes (if its endpoints are nonessential, it is inside the triangular region by convexity; if one of the endpoints is essential, the claim follows from lemma 4.6) and therefore the only triangle that contains the vertex $\overline{u}_k$ is the one with side $\overrightarrow{u_j u_i}$, while the other triangles are inside of one of the two lunes. Therefore at least one leaf of this triangulation (say $\triangle_1$) has two of its sides as edges of the original polygon Q. Figures 4.10 and 4.11 show some examples.

We claim that the intermediate vertex of $\triangle_1$ (i.e., the vertex of $\triangle_1$ connected to the edges of Q) is an excellent vertex of Q. Consider the side of $\triangle_1$ that is not an edge of Q and denote its open segment by s. It is clear that s does not intersect any other edge of Q. Now, s has as endpoints either:

– two nonessential vertices of Q and therefore is entirely contained in the interior of $U(Q)$, by the same argument in the proof of lemma 4.9(b);

– a nonessential vertex and an essential vertex of Q, in which case the same edge must be in the interior of $U(Q)$, again by the same argument in the proof of lemma 4.9(b).

Since $\overline{Q}$ satisfies the same hypotheses of Q (since it is its reflected image), lemma 4.9(a) applied to $\overline{Q}$ implies that $\overline{Q}$ is contained in $U(\overline{Q})$ (i.e., the reflection of $U(Q)$), which is disjoint from *int* $U(Q)$. Therefore s cannot intersect any edge of $\overline{Q}$.

Since s does not intersect no other edge of Q or $\overline{Q}$, the intermediate vertex of $\triangle_1$ is excellent. Because this vertex is not one of the essential ones, the proposition is proved. ■

4.5 The case where $Ess(Q) = 2$

Now we turn our attention to balanced spherical polygons without self nor antipodal intersections with $Ess(Q) = 2$. The proof of the existence of a nonessential, excellent vertex turns out to be more involved in this case.

Proposition 4.10 *Let $Q = [u_1, ..., u_n]$ ($n \geq 7$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) = 2$. Then there is at least one nonessential, excellent vertex.*

As it was the case for proposition 4.7, we will also need some lemmas for the proof of proposition 4.10. Since Q is balanced, there are four vertices of Q in balanced position. By hypothesis, exactly two of them are essential. Let u_i and u_j be the essential vertices (with $i < j$) and u_k and u_l be nonessential ones among these four vertices (the indices k and l do not hold any specific position with respect to i and j). All the nonessential vertices must be contained in the triangular regions $\triangle(\overline{u_i}, \overline{u_j}, \overline{u_k})$ and $\triangle(\overline{u_i}, \overline{u_j}, \overline{u_l})$, and each of these regions must have at least two vertices in its interior (for if one of them had only one, then this sole vertex would be a third essential vertex, contradicting our assumption). Figure 4.12 shows some examples.

Moreover, by the next lemma, the spherical convex hull of the set of nonessential vertices must be contained in the union of these two triangular regions. Figure 4.13 shows a polygon for which this latter property holds, while figure 4.14 shows a polygon for which this property does not hold.

Lemma 4.11 *Let $Q = [u_1, ..., u_n]$ ($n \geq 7$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) = 2$. Then the spherical convex hull of the set of nonessential vertices must be contained in the union of $\triangle(\overline{u_i}, \overline{u_j}, \overline{u_k})$ and $\triangle(\overline{u_i}, \overline{u_j}, \overline{u_l})$.*

In particular, the spherical convex hull of the nonessential vertices does not contain the antipodal points of the essential vertices.

Proof. If it were not the case, then the spherical convex hull of the nonessential vertices would intersect one of the following open spherical segments: $\overrightarrow{\overline{u_i}\overline{u_k}}$,

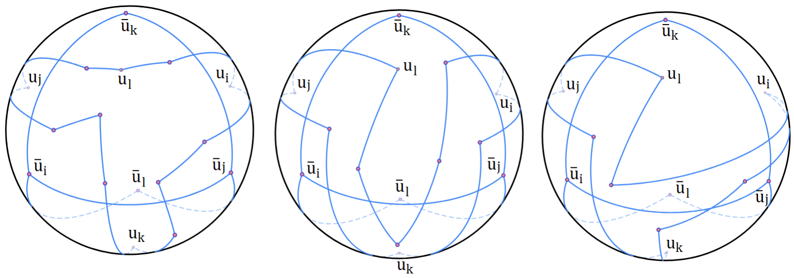

Figure 4.12: Examples of polygons for which the number of essential vertices is exactly 2.

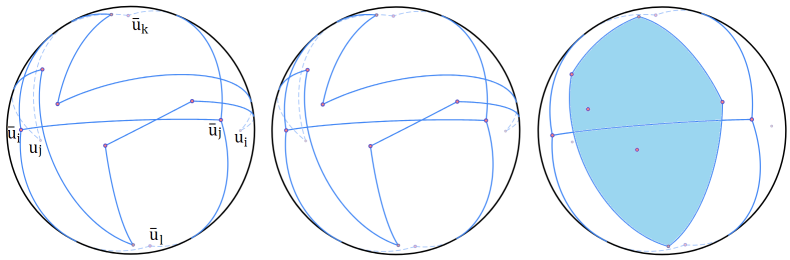

Figure 4.13: Valid spherical convex hull

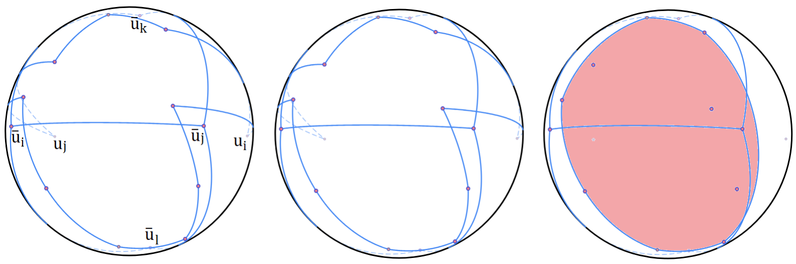

Figure 4.14: Invalid spherical convex hull

$\overrightarrow{\overline{u}_k\overline{u}_j}$, $\overrightarrow{\overline{u}_j\overline{u}_l}$ or $\overrightarrow{\overline{u}_l\overline{u}_i}$. But this would mean that a pair of nonessential vertices intersects one of these segments, say $\overrightarrow{\overline{u}_j\overline{u}_l}$, which implies that these two nonessential vertices, together with u_j and u_l, are in balanced position. In other words, u_i would be nonessential, contrary to hypothesis. ■

As in the case where $Ess(Q) = 3$, the essential vertices cannot be consecutive:

Lemma 4.12 *Let $Q = [u_1, ..., u_n]$ $(n \geq 6)$ be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) = 2$. Then the essential vertices cannot be consecutive.*

Proof. Suppose that the essential vertices u_i and u_j are consecutive. After a cyclic rearrangement, we can relabel them as u_1 and u_2. Let u_k and u_l be nonessential vertices of Q in balanced position with u_1 and u_2. There are two possibilities (see figure 4.15):

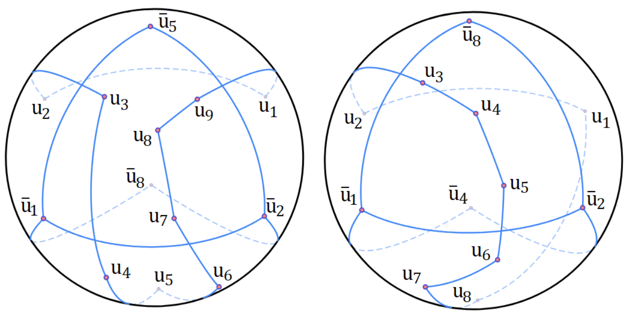

Figure 4.15: If the only 2 essential vertices were consecutive, Q would necessarily have at least one antipodal intersection.

- u_3 and u_n are in the same triangle. Without loss of generality we can consider this triangle to be $\triangle(\overline{u}_1, \overline{u}_2, \overline{u}_k)$. Since polygon Q also has vertices in the triangular region $\triangle(\overline{u}_1, \overline{u}_2, \overline{u}_l)$ and the spherical convex hull of the nonessential vertices is contained in the union of the triangular regions, it must cross the edge segment $\overrightarrow{\overline{u}_1\overline{u}_2}$ at least twice, i.e., it has at least two antipodal intersections, contrary to hypothesis (see figure 4.15 on the left for an example);
- u_3 and u_n are in different triangles. Since polygon Q must close itself and the spherical convex hull of the nonessential vertices is contained in the union of the triangular regions, it must cross the edge segment $\overrightarrow{\overline{u}_1\overline{u}_2}$ at least once, i.e., it has at least one antipodal intersection, contrary to hypothesis (see figure 4.15 on the right for an example).

■

In the different examples of figure 4.12 we see that the vertices of Q connected to u_i (i.e., u_{i-1} and u_{i+1}) or u_j (i.e., u_{j-1} and u_{j+1}) are both in the same triangle. We might wonder if it is possible for a polygon satisfying our hypothesis to have these four (or three, if $u_{i+1} = u_{j-1}$) vertices in different triangles. The next lemmas show that this cannot happen:

Lemma 4.13 *Let $Q = [u_1, ..., u_n]$ ($n \geq 6$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) = 2$. Let u_i and u_j be the essential vertices of Q. Then:*

(a) u_{i-1} and u_{i+1} are in the same triangular region.

(b) u_{j-1} and u_{j+1} are in the same triangular region.

Proof. (a) First we prove that u_{i-1} and u_{i+1} cannot be in different triangular regions.

Suppose by contradiction that one of them is in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_l)$ while the other one is in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$. Since the spherical convex hull of the nonessential

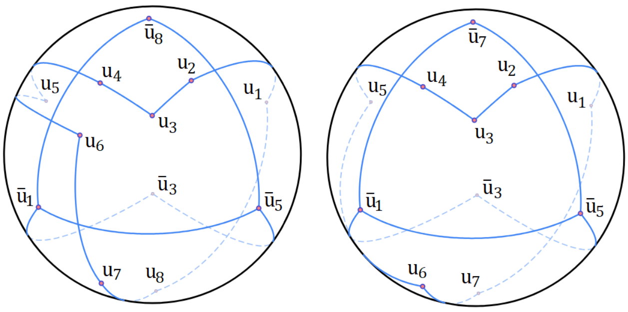

Figure 4.16: In both examples, $u_i = u_1$ and $u_j = u_5$. If u_{i-1} and u_{i+1} were in different triangles, vertex $\overline{u}_j$ would be isolated from vertices $\overline{u}_k$ and $\overline{u}_l$.

vertices of Q does not cross edge segments $\overrightarrow{\overline{u}_j\overline{u}_l}$ and $\overrightarrow{\overline{u}_j\overline{u}_k}$ and moreover Q must close itself, the vertex $\overline{u}_j$ would then be separated by Q from vertices $\overline{u}_k$ and $\overline{u}_l$, i.e., they would be in different regions delimited by the polygon Q. But because $\overline{Q}$ is a continuous curve and must connect $\overline{u}_j$ to $\overline{u}_k$ and to $\overline{u}_l$ via polygonal lines, it would necessarily intersect polygon Q at some edge, contrary to hypothesis (see figure 4.16 for some examples).

(b) The proof that vertices u_{j-1} and u_{j+1} are in the same triangular region is analogous to the proof of (a). ∎

Lemma 4.14 *Let $Q = [u_1, ..., u_n]$ ($n \geq 6$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) = 2$. Let u_i and u_j be the essential vertices of Q. Then u_{i-1}, u_{i+1}, u_{j-1} and u_{j+1} are in the same triangular region.*

Proof. Given lemma 4.13, it suffices to prove that the pair $\{u_{i-1}, u_{i+1}\}$ is in the same triangular region as $\{u_{j-1}, u_{j+1}\}$.

Suppose by contradiction that u_{i-1} and u_{i+1} are in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_l)$ while u_{j-1} and u_{j+1} are in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$. In particular, there are two polygonal lines from u_i to u_k, each of them crossing the edge segment $\overrightarrow{\overline{u}_l\overline{u}_j}$ only once (see figure 4.17 for two different examples). Consider the polygonal line not passing through u_j and denote it by $pl_{i,k} = pl(u_i, ..., u_k)$. Similarly, there are two polygonal lines from u_j to u_l, each of them crossing the edge segment $\overrightarrow{\overline{u}_i\overline{u}_k}$ only once. Consider the antipodal polygonal line of the one not passing through u_i and denote it by $\overline{pl}_{j,l} = pl(\overline{u}_j, ..., \overline{u}_l)$.

We know that $pl_{i,k}$ intersects the segment $\overrightarrow{\overline{u}_l\overline{u}_j}$ once and that $\overline{pl}_{j,l}$ intersects the segment $\overrightarrow{u_iu_k}$ once. We shall prove that $pl_{i,k}$ intersects $\overline{pl}_{j,l}$, which contradicts the hypothesis of Q not having antipodal intersections.

Concatenating the polygonal line $\overline{pl}_{j,l}$ and the segment $\overrightarrow{\overline{u}_l\overline{u}_j}$ one forms a (closed) polygon $\overline{p}_{j,l} := [\overline{u}_j, ..., \overline{u}_l]$ which intersects the segment $\overrightarrow{u_iu_k}$ twice (see figure 4.18.a).

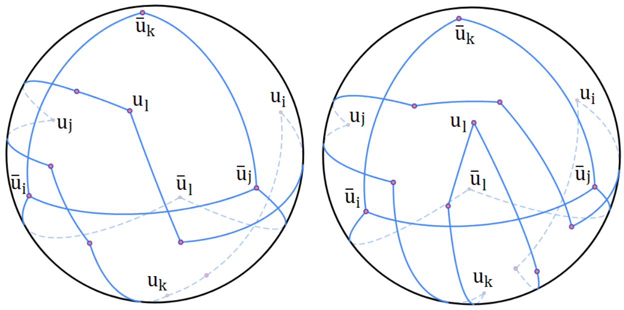

Figure 4.17: Two examples where u_{i-1} and u_{i+1} are in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_l)$ while u_{j-1} and u_{j+1} are in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$.

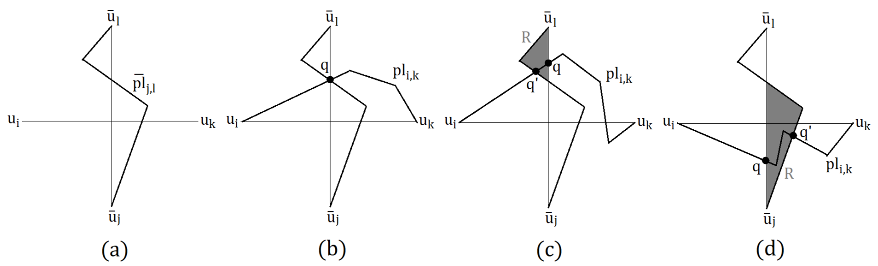

Figure 4.18: In (a), we see that the polygon $\overline{p}_{j,l}$ subdivides the sphere into different regions, one of them containing both vertices u_i and u_k in its interior. Figures (b), (c) and (d) show different possibilities regarding the polygonal line $pl_{i,k}$ and its intersections with polygon $\overline{p}_{j,l}$.

The polygon $\overline{p}_{j,l}$ is a closed continuous curve which subdivides the sphere in at least two regions (more than two if it has self-intersections). Since $\overline{p}_{j,l}$ crosses segment $\overrightarrow{u_i u_k}$ twice and, except by $\overline{u}_j$ and the edges adjacent to $\overline{u}_j$, all the polygon $\overline{p}_{j,l}$ is contained in the two triangular regions, both vertices u_i and u_k must in the same region among the regions determined by $\overline{p}_{j,l}$.

On the other hand, the polygonal line $pl_{i,k}$ must cross segment $\overrightarrow{\overline{u}_l, \overline{u}_j}$ once, say at a point q. If this point happens to be also in the polygonal line $\overline{p}_{j,l}$, we are done (see figure 4.18(b)). Suppose then that q is not in $\overline{p}_{j,l}$. Since we are dealing with segments and there are no three collinear vertices, all intersections between segments are *transversal*, which implies in our case that at the point of intersection q the curve $pl_{i,k}$ is locally separated by segment $\overrightarrow{\overline{u}_l \overline{u}_j}$. In other words, the curve $pl_{i,k}$ passes in at least two different regions determined by $\overline{p}_{j,l}$: the one that contains vertices u_i and u_k, and another one denoted by R (see figure 4.18(c) and (d)). Therefore $pl_{i,k}$ must intersect another boundary point q' of R. Since q' cannot be in $\overrightarrow{\overline{u}_l \overline{u}_j}$ (by hypothesis), it must be a point of $\overline{pl}_{j,l}$.

Similarly, assuming that u_{i-1} and u_{i+1} are in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$ while u_{j-1} and u_{j+1} are in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_l)$, we derive another contradiction. ■

Here is an idea to prove proposition 4.10: we have a geometrically similar situation as we had in the case where $Ess(Q) = 3$: although two triangular

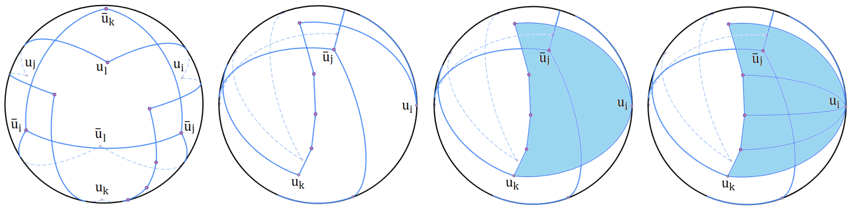

Figure 4.19: In this polygon Q, only one vertex between u_k and u_i is in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$. Consider the polygonal line from u_k to u_i and concatenate it with segment $\overrightarrow{u_i u_k}$, obtaining polygon $p_{k,i}$. Consider now the region determined by $p_{k,i}$ which contains $\overline{u}_j$. The unique triangulation on it has as leaves: the triangle whose side is the added segment $\overrightarrow{u_i u_k}$ (which does not have three consecutive vertices of the original polygon Q); and a triangle with $\overline{u}_j$ in its interior (which therefore will intersect the polygon $\overline{Q}$).

regions contain more than one vertex, one of them, say $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$ has "more parts of Q" in the sense that the polygon Q enters and exits this region more times (more precisely: the number of connected components of $Q \cap \triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$ is equal the number of connected components of $Q \cap \triangle(\overline{u}_i, \overline{u}_j, \overline{u}_l)$ plus 2). At first we could try to use the same argument as in proposition 4.7: we could triangulate a certain region enclosed by a polygon whose vertices are some essential and nonessential vertices of Q, and then obtain a leaf of the triangulation.

The problem with this approach in the case of $Ess(Q) = 2$ is that it depends for the polygon Q to have at least two consecutive vertices in $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_k)$, which might not happen here. If we try to triangulate the corresponding region as in figure 4.19, all extra edges arising from the triangulation will intersect the reflected polygon $\overline{Q}$.

Therefore, in order to prove proposition 4.10, we must choose carefully the region that we want to triangulate. First we need the following terminology. Given a balanced spherical polygon Q without self nor antipodal intersections, Q separates the sphere into two regions: we say that the region which contains $\overline{Q}$ is the *exterior of* Q, (which we denote it by $Ext(Q)$), while the other one is the *interior of* Q (which we denote by $Int(Q)$). First notice that a striking difference between the triangulations used in the proofs of proposition 4.5 and of proposition 4.7 is that in the former the *interior* of Q is triangulated, while in the latter a subset of the *exterior* of Q is triangulated. Of course, in the second proof we also had to show that the one of the leaves of the triangulation did not intersect polygon $\overline{Q}$, which was true due to the fact that the corresponding triangle was contained in $U(Q)$ and that $\overline{Q}$ was contained in $U(\overline{Q})$ (recall that it only makes sense to speak of the subset $U(Q)$ when $Ess(Q) = 3$).

In the case where $Ess(Q) = 2$, the excellent (and nonessential) vertex to be found might come from either an internal or an external triangulation. By proposition 4.5 and its proof, Q has at least 2 excellent vertices, the two of them coming from a triangulation of the interior of Q. If one of them is nonessential, we are done. Therefore we might assume that both of them are the essential vertices.

Denote by $\mathcal{H}_S(X)$ the spherical convex hull of a subset X of the sphere, provided X is contained in a closed hemisphere. Let u_i and u_j be the essential vertices of Q. In order to find an excellent vertex among the nonessential ones, we want to look at the new (simple) polygon $\tilde{Q} := Q - u_i - u_j$ (i.e., the polygon obtained deleting vertices u_i and u_j and the edges adjacent to these two vertices, and then connecting u_{i-1} to u_{i+1} and u_{j-1} to u_{j+1} by spherical segments) and its spherical convex hull $\mathcal{H}_S(\tilde{Q})$. It is instructive to see some examples. First, a lemma:

Lemma 4.15 *Let $Q = [u_1, ..., u_n]$ $(n \geq 7)$ be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) = 2$. Let u_i an u_j be essential vertices and $\tilde{Q} := Q - u_i - u_j$. Then*

$$\mathcal{H}_S(\tilde{Q}) \cap \mathcal{H}_S(\overline{\tilde{Q}}) = \emptyset.$$

Proof. Since u_i and u_j are essential, $\tilde{Q}$ is contained in a (open) hemisphere H. By the same reason, $\overline{\tilde{Q}}$ is also contained in a (open) hemisphere H' which is disjoint from H. Since $\overline{\tilde{Q}} = \tilde{\overline{Q}}$, the result follows. ∎

Example 4.16 *Let Q be the polygon of figure 4.20. In this case we have two excellent and nonessential vertices. The triangulation of $Int(\tilde{Q})$ has two leaves, which gives u_5 and u_7 as excellent vertices of the polygon $\tilde{Q}$. Since u_5 is connected through edges of the original polygon Q while u_7 is not, we can deduce only that u_5 is an excellent vertex of the original polygon Q. On the other hand, $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$ is already a triangle, which gives u_6 as the intermediate vertex of the unique leaf of the triangulation. Notice that the edges adjacent to u_6 are contained in the original polygon Q.*

Is u_6 an excellent vertex of Q? If it were not excellent, then the segment $\overrightarrow{u_5u_7}$ would intersect $\overline{Q}$ at some edge, which would imply that a vertex $\overline{u}$ of $\overline{Q}$ is in $\mathcal{H}_S(\tilde{Q})$. But this contradicts lemmas 4.11 and 4.15. Therefore, u_6 is indeed excellent.

Example 4.17 *Let Q be the polygon of figure 4.21. It has two excellent and nonessential vertices: u_5 and u_6. Both come from internal triangulations but, since they are consecutive, the triangulations used in each case must be*

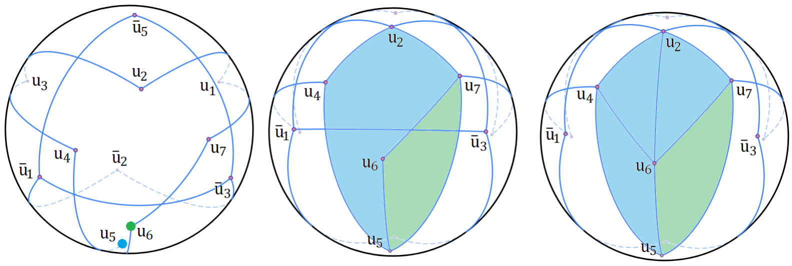

Figure 4.20: This polygon has u_5 and u_6 as excellent and nonessential vertices: u_5 (blue) can be found through an triangulation of the interior of polygon $\tilde{Q}$ (blue), while u_6 (green) comes from the triangle obtained as the intersection of the exterior of $\tilde{Q}$ with $\mathcal{H}_S(\tilde{Q})$ (green).

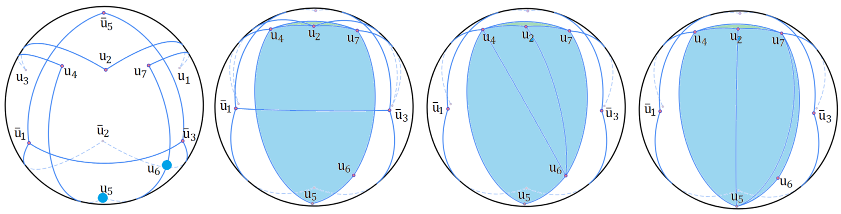

Figure 4.21: This polygon Q has u_5 and u_6 as excellent and nonessential vertices (in blue). Different triangulations are needed in order to conclude that both of them are excellent because both vertices are consecutive. On the other hand, although u_2 is the intermediate vertex of the unique triangle of the triangulation of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$, it is not an excellent vertex of Q.

different. Notice that these two triangulations also point to u_4 and u_7 as the intermediate vertices of the leaves of each triangulation, but since at least one edge adjacent to each these vertices is not of the original polygon Q, we cannot conclude whether or not they excellent vertices of Q.

On the other hand, u_2 is the intermediate vertex of the unique triangle of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$, but the two edges adjacent to it are not of the original polygon Q. Therefore we cannot conclude that u_2 is an excellent vertex of Q.

Example 4.18 *Let Q be the polygon of figure 4.22. Both vertices u_2 and u_4 are the intermediate vertices of the leaves of the internal triangulation, but since they are adjacent to edges which are not contained in the original polygon Q, we cannot conclude that they are excellent vertices of Q.*

On the other hand, vertex u_3 is the intermediate vertex of the unique triangle of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$ and, moreover, is adjacent to edges of the original polygon Q. To conclude that u_3 is indeed excellent, it suffices to see that the segment $\overrightarrow{u_2 u_4}$ does not intersect $\overline{Q}$. For if it intersected an edge of $\overline{Q}$, it would follow that that $\mathcal{H}_S(\tilde{Q})$ contains a vertex u of $\overline{Q}$ in its interior, which contradicts lemmas 4.11 and 4.15. Therefore, u_3 must be an excellent vertex of Q.

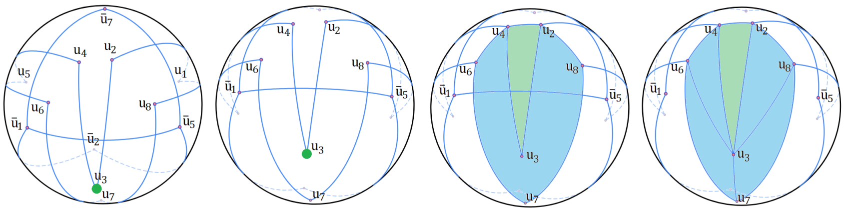

Figure 4.22: This polygon has only u_3 (in green) as an excellent vertex: it comes from the unique triangle of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$.

The previous examples show some general principles when looking for excellent and nonessential vertices of Q. When we consider the intermediate vertex of any leaf of the triangulation, it might be a vertex among u_{i-1}, u_{i+1}, u_{j-1} or u_{j+1}. In this case it is an excellent vertex of $\tilde{Q}$, although we cannot conclude in principle if it is an excellent vertex of Q. On the other hand, if the vertex is not one of these four vertices, the next lemma shows that it must be an excellent vertex of Q:

Lemma 4.19 *Let $Q = [u_1, ..., u_n]$ ($n \geq 7$) be a spherical polygon in balanced position and without self nor antipodal intersections, with $Ess(P) = 2$. Let u_i an u_j be the essential vertices and let $\tilde{Q} := Q - u_i - u_j$. Let u_t be a good vertex of $\tilde{Q}$ coming from a triangulation of either $Int(\tilde{Q})$ or of a connected component of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$. If u_t is not one of the vertices u_{i-1}, u_{i+1}, u_{j-1} and u_{j+1}, then u_t is an excellent vertex of Q.*

Proof. If u_t comes from a triangulation of $Int(\tilde{Q})$, it is excellent since in this case edge $\overrightarrow{u_{t-1}u_{t+1}}$ is contained in $Int(\tilde{Q})$ and therefore cannot intersect $\overline{Q}$.

Now, suppose that u_t comes from a triangulation of a connected component of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$. To prove that u_t is excellent, it suffices to show that the segment $\overrightarrow{u_{t-1}u_{t+1}}$ (i.e., the side of $\triangle_1$ not adjacent to u_t) does not intersect $\overline{Q}$. For if it intersected an edge of $\overline{Q}$, it would follow that that $\mathcal{H}_S(\tilde{Q})$ contains a vertex $\overline{u}$ of $\overline{Q}$ in its interior, which contradicts lemmas 4.11 and 4.15. ■

Before proving proposition 4.10, we prove a lemma which states a lower bound on the number of good vertices of a *polygon which is not in balanced position*. From now on, "convex" will always mean "spherically convex".

Lemma 4.20 *Let $Q = [u_1, ..., u_n]$ ($n \geq 4$) be a simple spherical (or even planar) polygon which is contained in a closed hemisphere. Then Q has at least 3 good vertices. More precisely:*

(a) If Q is convex, then all of its vertices are good;

(b) If Q is not convex, then it has at least 3 good vertices.

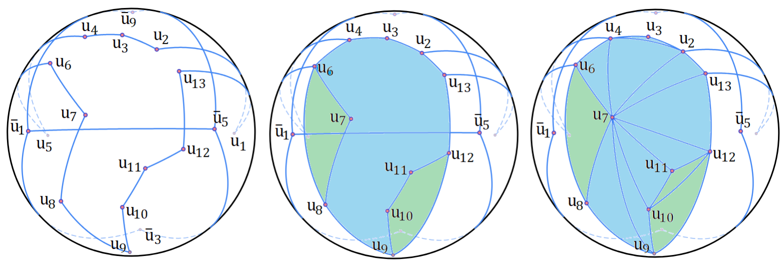

Figure 4.23: Vertices u_3 and u_8 are excellent and come from a triangulation of $Int(\tilde{Q})$. Vertices u_7 and u_{11} are also excellent, but come from a triangulation of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$. Notice that since u_6, u_8 and u_9 are also at the boundary of components of $Ext(\tilde{Q}) \cap \mathcal{H}_S(\tilde{Q})$, they can be seen as the "intermediate" vertices of leaves of these regions. However, each of them is adjacent to a segment that is not of the original polygon Q, so lemma 4.19 does not apply here.

Proof. (a) is clear: if a vertex u_i were not good, then the segment $\overrightarrow{u_{i-1}u_{i+1}}$ would intersect the polygon Q at some other edge and therefore some of its points would not be in the interior of Q, contradicting the convexity of Q.

(b) If Q is not convex, then $\mathcal{H}_S(Q) \cap Ext(Q)$ is nonempty. Consider any triangulation of $\mathcal{H}_S(Q)$: since Q has at least 4 vertices, the dual graph of $\mathcal{H}_S(Q) \cap Int(Q)$ is nontrivial and, by theorem 2.1, has at least two leaves. The intermediate vertex of each of these triangles is a good vertex.

Moreover, $\mathcal{H}_S(Q) \cap Ext(Q)$ is nonempty (it might have more than one connected component). Consider a connected component $\mathcal{C}$ of $\mathcal{H}_S(Q) \cap Ext(Q)$: it is either a triangle with vertices u_{i-1}, u_i and u_{i+1} (in which case u_i is the good vertex), or another type of poligonal region. In the latter case the dual graph of any triangulation of $\mathcal{C}$ is nontrivial and, again by theorem 2.1, has at least two leaves. Since at most one of these triangles might have as side the segment of $\partial\mathcal{H}_S(Q)$ which is not a side of Q, at least one of the leaves has two sides contained in Q. The vertex of Q which is adjacent to these sides is therefore good. ■

Now we can finally prove proposition 4.10:

Proof. (of proposition 4.10) Let p be the polygonal line from u_{i+1} to u_{j-1}.

Step 1: First we prove the proposition in the case where $i + 2 < j$, i.e., where p is not a sole point (see figure 4.24.a). Here we proceed analogously to the proof of the proposition for $Ess(Q) = 3$: we consider the (closed) polygon $[u_i, u_{i+1}, ..., u_{j-1}, u_j]$ and denote by R the region enclosed by it which contains the vertex $\overline{u}_k$. Triangulating the region R, we obtain again two leaves $\triangle_1$ and $\triangle_2$, one of them (say $\triangle_1$) having as intermediate vertex a nonessential vertex (see figure 4.24.b). It must be excellent: for if it were not the case, then the edge of $\triangle_1$ which is not contained in Q (which we denote by e) would intersect an edge of $\overline{Q}$ (which we denote by $\overline{f}$), in which case there would be

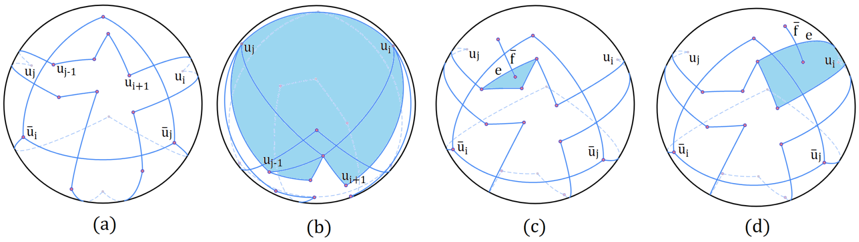

Figure 4.24: If the polygonal line p is not a sole point, the polygon $[u_i, u_{i+1}, ..., u_{j-1}, u_j]$ defines a region R whose triangulation has at least two leaves. In this example, the intermediate vertices of each leaf of the triangulation are excellent vertices of Q.

two possibilities:

– The endpoints of e are two nonessential vertices (see figure 4.24.c). Since e intersects $\overline{f}$, one of the endpoints of $\overline{f}$ is inside of $\triangle_1$ and therefore cannot be $\overline{u}_i$ nor $\overline{u}_j$, i.e., it must be a nonessential vertex of $\overline{Q}$. But then we would have four points of Q in balanced position (namely, the endpoints of e and f), at most one of them being essential. But this contradicts our hypothesis of $Ess(Q) = 2$;

– One endpoint of e is nonessential and the other is essential (see figure 4.24.d). We can assume without loss of generality that the latter point is u_i. Again, since e intersects $\overline{f}$, one of the endpoints of $\overline{f}$ is inside of $\triangle_1$ and therefore cannot be $\overline{u}_i$ nor $\overline{u}_j$, i.e., it must be a nonessential vertex of $\overline{Q}$. The other endpoint cannot be $\overline{u}_i$ nor $\overline{u}_j$, since in this case $\overline{f}$ would not intersect e. Therefore, we would have four points of Q in balanced position (namely, the endpoints of e and f), with only one of them being essential. But this contradicts our hypothesis of $Ess(Q) = 2$.

Step 2: Now we assume (for the rest of the proof) that $i + 2 = j$, i.e., the polygonal line p is the sole point $u_{i+1} = u_{j-1}$ (see figure 4.25). Denote this point by u_h. Denote by p' the polygonal line from u_{j+1} to u_{i-1}.

Let $\tilde{Q} := Q - u_i - u_j$, where u_i and u_j are the essential vertices of Q (see figure 4.25). This new polygon is equal to the concatenation of the polygonal line p, the segment $\overrightarrow{u_{j-1}u_{j+1}}$, the polygonal line p', and the segment $\overrightarrow{u_{i-1}u_{i+1}}$. Moreover, $\tilde{Q}$ is a simple polygon with at least 5 vertices.

If $\tilde{Q}$ is convex, then all of its vertices are good (by lemma 4.20.a) and therefore at least two of its vertices (namely, the ones different from u_{i-1}, u_h and u_{j+1}) are also excellent vertices of the original polygon Q, by lemma 4.19.

Now, if $\tilde{Q}$ is not convex, then it has at least 3 good vertices by lemma 4.20.b. If one of these vertices is different from u_{i-1}, u_h and u_{j+1}, then it is an excellent vertex of Q, by lemma 4.19.

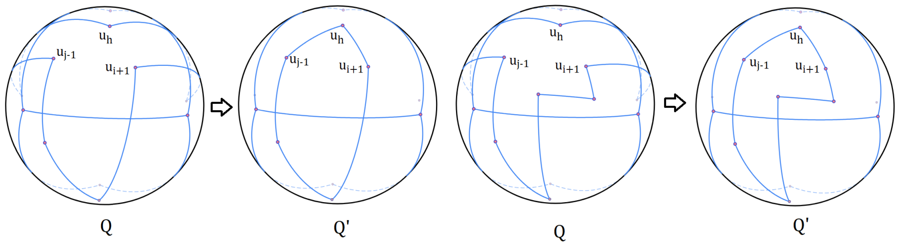

Figure 4.25: Two examples of a polygon Q together with the corresponding $\tilde{Q} := Q - u_i - u_j$.

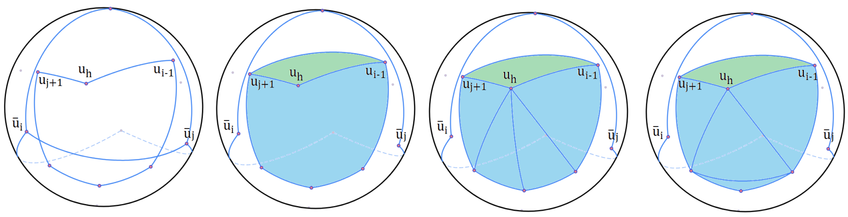

Figure 4.26: An example of a polygon $\tilde{Q}$ such that the triangulation of $\mathcal{H}_S(\tilde{Q})$ gives only u_{i-1}, u_h and u_{j+1} as good vertices of $\tilde{Q}$. Notice that all edges which were added to form the internal triangulation must be adjacent to u_h. Moreover, we can "flip" a pair of consecutive triangles in order to find another good vertex of $\tilde{Q}$.

Step 3: Suppose now (for the rest of the proof) that the set of good vertices of $\tilde{Q}$ (relative to the given triangulation) is precisely $\{u_{i-1}, u_h, u_{j+1}\}$ (see figures 4.26 and 4.27). By the proof of lemma 4.20.b, two of them are the intermediate vertices of leaves of a triangulation of the interior of $\tilde{Q}$, while the other is the intermediate vertex of a leaf of a triangulation of $\mathcal{H}_S(\tilde{Q}) \cap Ext(\tilde{Q})$. Since u_{i-1}, u_h and u_{j+1} are consecutive in $\tilde{Q}$, the only possibility is that u_h is the intermediate vertex of the leaf of a triangulation of $\mathcal{H}_S(\tilde{Q}) \cap Ext(\tilde{Q})$.

Now, since u_{i-1}, u_h and u_{j+1} are (in this order) consecutive in $\tilde{Q}$, we have that all edges which were added to form the internal triangulation of $\tilde{Q}$ must have u_h as one of their endpoints: since $\triangle(u_{i-2}, u_{i-1}, u_h)$ and $\triangle(u_h, u_{j+1}, u_{j+2})$ are the only leaves of the internal triangulation, all added edges must have as endpoints both a vertex in the polygonal line between u_{i-1} and u_{j+1} (in this order, relative to polygon $\tilde{Q}$) and a vertex in the polygonal line between u_{j+1} and u_{i-1} (in this order, relative also to polygon $\tilde{Q}$).

Moreover, we have that the polygon $\tilde{Q}$ is *star-shaped* with respect to u_h: given any point of $\tilde{Q}$ (including edge points), the segment connecting it to u_h does not intersect the exterior of $\tilde{Q}$.

Therefore, the triangles

$$\triangle(u_h, u_{j+1}, u_{j+2}), \triangle(u_h, u_{j+2}, u_{j+3}), \ldots \triangle(u_h, u_{i-3}, u_{i-2}), \triangle(u_h, u_{i-2}, u_{i-1})$$

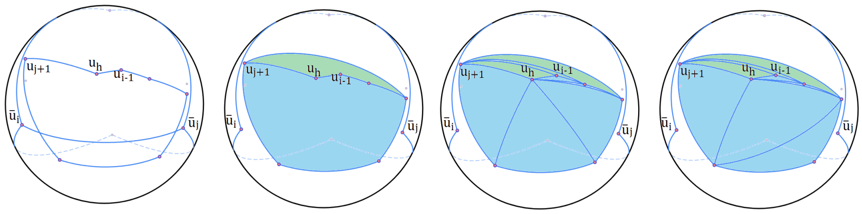

Figure 4.27: An example of a polygon $\tilde{Q}$ such that the triangulation of $\mathcal{H}_S(\tilde{Q})$ gives only u_{i-1}, u_h and u_{j+1} as good vertices of $\tilde{Q}$. Notice that all edges which were added to form the internal triangulation must be adjacent to u_h. This example also shows that, besides the pairs of consecutive triangles of type (a) (which can be “flipped"), type (b) pairs may also appear (which happens at vertices u_h, u_{i-3}, u_{i-2} and u_{i-1}).

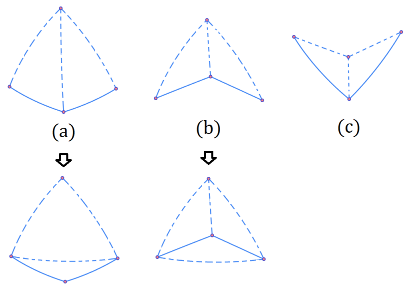

Figure 4.28: Three possibilities regarding a pair of consecutive triangles of the internal triangulation of $\tilde{Q}$.

are all the triangles of the internal triangulation of $\tilde{Q}$. We want to examine each pair $\mathcal{P}$ of consecutive triangles of this sequence ($\triangle(u_h, u_{t-1}, u_t)$ and $\triangle(u_h, u_t, u_{t+1})$, for each $j+1 < t < i-1$). The four vertices of $\tilde{Q}$ which define them (i.e., u_h, u_{t-1}, u_t and u_{t+1}) might be either in convex position or not. Because $\tilde{Q}$ is star-shaped with respect to u_h, there are in principle only three possibilities (see figure 4.28):

(a) the four points are in convex position and such that u_h, u_{t-1}, u_t and u_{t+1} are in the counterclockwise order in $\partial\mathcal{H}_S(\mathcal{P})$;

(b) u_t is in the interior of $\mathcal{H}_S(\mathcal{P})$ and u_h, u_{t-1} and u_{t+1} are in the counterclockwise order in $\partial\mathcal{H}_S(\mathcal{P})$;

(c) u_h is in the interior of $\mathcal{H}_S(\mathcal{P})$ and u_{t-1}, u_t and u_{t+1} are in the counterclockwise order in $\partial\mathcal{H}_S(\mathcal{P})$;

In case (a), we can just “flip" the pair of triangles, i.e., switch the triangulation of $\mathcal{P}$ (and therefore change also the triangulation of the interior of $\tilde{Q}$). In this case, we have that vertex u_t is a good vertex of $\tilde{Q}$.

In case (b), we notice that, since $\tilde{Q}$ is star-shaped with respect to u_h, we have that the triangle $\triangle(u_{t-1}, u_t, u_{t+1})$ does not have vertices in its interior and, therefore, the vertex u_t is a good vertex of $\tilde{Q}$.

It remains to show that case (c) cannot happen for every pair of triangles $\mathcal{P}$ of the internal triangulation of $\tilde{Q}$. We have two cases to consider: the polygonal line p' has either 4 vertices or more than 4 vertices.

– If the polygonal line p' from u_{j+1} to u_{i-1} has exactly 4 vertices (see figure 4.29), then $\tilde{Q}$ is a pentagon and, moreover, vertices u_{j+2} and u_{i-2} are consecutive and are both in the triangular region $\triangle(\overline{u}_i, \overline{u}_j, \overline{u}_l)$.

  Here, it is impossible for vertex u_h to be inside of $\triangle(u_{j+2}, u_{i-2}, u_{i-1})$: since u_h is adjacent to u_i in the original polygon Q, it must be "on the left" of the oriented segment $\overrightarrow{u_{i-1}u_i}$ (i.e., the determinant $[u_{i-1}, u_i, u_h]$ is positive). On the other hand, any point of the interior of $\triangle(u_{j+2}, u_{i-2}, u_{i-1})$ must be "on the right" of $\overrightarrow{u_{i-1}u_i}$ (i.e., the corresponding determinant is negative), because points u_{j+2} and u_{i-2} are also on the right of $\overrightarrow{u_{i-1}u_i}$.

  Analogously, one proves that $\triangle(u_{j+1}, u_{j+2}, u_{i-2})$ cannot have u_h in its interior.

  The conclusion is that in the case of $\tilde{Q}$ being a pentagon, only cases (a) and (b) can happen, which implies the existence of at least two good vertices of $\tilde{Q}$, both of them different from u_h, u_{j+1} and u_{i-1}. Therefore, they are excellent vertices of Q, by lemma 4.19.

– If the polygonal line p' from u_{j+1} to u_{i-1} has more than 4 vertices (see figure 4.30), then $\tilde{Q}$ is a polygon with at least 6 vertices, whose interior is triangulated with at least 4 triangles. Therefore, there are at least 3 pairs of consecutive triangles $\mathcal{P}_1$,..., $\mathcal{P}_m$ ($m \geq 3$). Assume that there are no pairs of triangles of type (b). In this case (see figure 4.30), vertex u_h cannot be in the interior of all triangles of the form $\triangle(u_{j+1}, u_{j+2}, u_{j+3})$, ..., $\triangle(u_{i-3}, u_{i-2}, u_{i-1})$, since some of them have disjoint interiors (for instance, triangles $\triangle(u_{j+1}, u_{j+2}, u_{j+3})$ and $\triangle(u_{i-3}, u_{i-2}, u_{i-1})$).

  Therefore, there must be at least one pair $\mathcal{P}$ of triangles of type (a) or (b). This implies the existence of at least one good vertex of $\tilde{Q}$, which is different from u_h, u_{j+1} and u_{i-1}. Hence it is also an excellent vertex of Q, by lemma 4.19.

■

Since we proved propositions 4.7 and 4.10, theorem 4.3 is also proved. An immediate consequence of theorem 4.3 is the following result:

Corollary 4.21 *Let $P = [v_1, ..., v_n]$ ($n \geq 6$) be a space polygon whose tangent indicatrix does not self nor antipodal intersections. Then P must have at least 6 flattenings.*

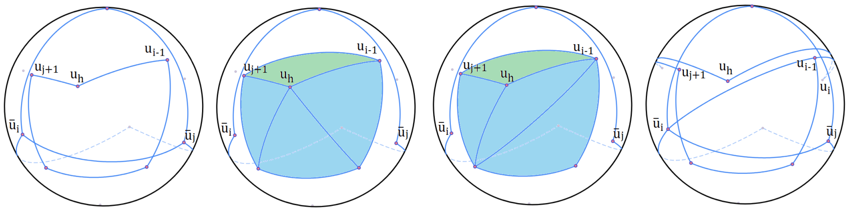

Figure 4.29: If $\tilde{Q}$ is a pentagon and does not have pairs of type (b), we have that vertex u_h cannot be inside of $\triangle(u_{j+2}, u_{i-2}, u_{i-1})$, since both sets are separated by the spherical line spanned by u_{i-1} and u_i. Analogously, u_h cannot be inside of $\triangle(u_{j+1}, u_{j+1}, u_{i-2})$.

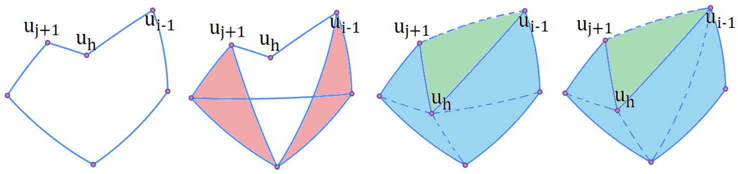

Figure 4.30: If $\tilde{Q}$ has 6 or more vertices and does not have pairs of type (b), then the situation is depicted above: there will be triangles formed by consecutive vertices of p' whose interiors are disjoint. Even if we move u_h, it cannot be inside all of these triangles.

4.6 Further remarks

As with propositions 3.12 and 3.22, the characterization of antipodal intersections given by proposition 4.2 and its application to our study of spherical polygons seem to be new.

Propositions 4.4, 4.5, 4.7 and 4.10 are new.

Theorem 4.3 and corollary 4.21 are new in the discrete setting.

Except for figures 4.1 and 4.18, all figures depicted in this chapter were made using the app *Spherical Easel* (see (DICKINSON; DULIMARTA,)).

5
Spherical polygons with self and antipodal intersections

5.1
Introduction

In this chapter we prove some generalizations of the main results on spherical polygons from the previous chapters. Recall that theorems 1.7 and 3.25 stated that a closed smooth spherical curve and a spherical polygon, respectively, without self-intersections and not entirely contained in any hemisphere must have at least four inflections. If such curve / polygon is symmetric, then this lower bound can be improved to six (the smooth case is theorem 1.9 and the discrete case is theorem 3.35).

In the previous chapter, we saw that this lower bound on the number of inflections may be improved for nonsymmetric curves / polygons when the given curve / polygon does not have antipodal intersections (in the smooth case, this result was theorem 4.1 due to Ghomi, while the discrete case was theorem 4.3 from the previous chapter).

Ghomi (in (GHOMI, 2013)) extended all these smooth results by allowing the curves to have singularities (i.e., cusps) and/or double points (i.e., self and antipodal intersections). Then he obtains the following results:

Theorem 5.1 *Given a $\mathcal{C}^2$ closed spherical curve γ, not entirely contained in any closed hemisphere, let S be the number of singular points of γ, I be the number of its inflections, and D the number of pairs of points $t \neq s \in \mathbb{S}^1$ where $\gamma(t) = \pm\gamma(s)$. Then*

$$2(D + S) + I \geq 6.$$

Theorem 5.2 *Let γ be a spherical curve under the same conditions and notations of the previous theorem. If $D^+(\leq D)$ denotes the number of pairs of points $t \neq s \in \mathbb{S}^1$ where $\gamma(t) = \gamma(s)$, then*

$$2(D^+ + S) + I \geq 4.$$

Furthermore, if γ is symmetric, i.e., $-\gamma = \gamma$, then

$$2(D^+ + S) + I \geq 6.$$

Figure 5.1: A cusp in the smooth/discrete setting.

Note that in the particular case where $D^+ = S = 0$, theorem 5.2 is theorem 1.7. In the particular case where $D = S = 0$, 5.1 is theorem 4.1.

We have already proved the discrete version of Theorem 5.2 in the case where $D^+ = S = 0$ (theorem 3.25) and Theorem 5.1 in the case where $D = S = 0$ (theorem 4.3). It remains, therefore, to state Theorems 5.1 and 5.2 in the discrete setting and then prove them.

Our strategy to prove these theorems consists of a “cutting-and-pasting" procedure: at the point of self-intersection we delete the two edges and reconnect its adjacent vertices in such a way that we obtain a new polygon, which is free from self-intersections. We then prove that the number of inflections of the original polygon plus 2 (which refers to the self-intersection counted twice) is greater or equal than the number of inflections of the resulting polygon, for which the lower bound is known.

5.2
Cusps in the discrete setting and statement of theorems

We still have to define the notion of a “discrete singular point". In the smooth case, the singular point is a cusp, i.e., a point at which the curve fails to be differentiable. Therefore it makes sense to define a “discrete cusp" as a pair of consecutive inflections (see figure 5.1).

Therefore, since a discrete cusp already counts as two inflections, it is enough to state theorems 5.1 and 5.2 without the explicit number S of discrete cusps of the polygon.

Given a spherical polygon Q, let D^+ the number of double points (i.e., self-intersections) of the polygon, D^- be the number of its antipodal-double points (i.e., antipodal intersections), $D := D^+ + D^-$ and I be the number of its inflections. Now we can state the discrete analogs of theorems 5.1 and 5.2.

Theorem 5.3 *Let $Q = [u_1, ..., u_n] \in \mathbb{S}^2$ $(n \geq 6)$ be a spherical polygon, not entirely contained in a closed hemisphere. Then*

$$2D + I \geq 6.$$

Theorem 5.4 *Let $Q = [u_1, ..., u_n] \in \mathbb{S}^2$ $(n \geq 4)$ be a spherical polygon, not entirely contained in a closed hemisphere. Then*

$$2D^+ + I \geq 4.$$

Theorem 5.5 *Let $Q = [u_1, ..., u_{2n}] \in \mathbb{S}^2$ $(2n \geq 6)$ be a spherical polygon. If Q is symmetric, then*

$$2D^+ + I \geq 6.$$

We will first prove theorem 5.4. Notice that for $D^+ \geq 2$ the result follows trivially, while for $D^+ = 0$ the result follows from Theorem 3.25. Therefore, it suffices to consider the case where $D^+ = 1$. The strategy to prove the theorem in this case will be following: we eliminate the intersecting edges and reconnect the adjacent vertices in such a way that the resulting polygon Q' does not have self-intersections (we might need to add some more vertices to the polygon P before this operation, in order to avoid new self-intersections). We then show that the number $2D^+ + I = 2 + I$ is greater or equal than the number I' of inflections of Q', which in its turn is greater or equal than 4, by Theorem 3.25.

5.3 Eliminating self-intersections

Suppose that we are given a spherical (or even planar) polygon $Q = [u_1, u_2, ..., u_n]$ (with the usual orientation) with only one self-intersection, which happens at edges $\overrightarrow{u_i u_{i+1}}$ and $\overrightarrow{u_j u_{j+1}}$ (see figure 5.2 - left). Suppose that no other vertex is in the spherical region spanned by $\{u_i, u_{i+1}, u_j, u_{j+1}\}$, i.e., $\mathcal{C}(u_i, u_{i+1}, u_j, u_{j+1}) \cap \mathbb{S}^2$ (in the case of a planar polygon, this region would be the convex hull of points u_i, u_{i+1}, u_j and u_{j+1}).

After deleting edges $\overrightarrow{u_i u_{i+1}}$ and $\overrightarrow{u_j u_{j+1}}$, there are two different ways of reconnecting these pairs of vertices (see figure 5.2):

- form an edge from u_i to u_{j+1} and an edge of from u_j to u_{i+1}. The result will be two new polygons $Q_1 = [u_i, u_{j+1}, u_{j+2}, ..., u_{i-1}]$ and $Q_2 = [u_j, u_{i+1}, u_{i+2}, ..., u_{j-1}]$, both of them with the orientation induced by that of Q;
- form an edge from u_i to u_j and an edge from u_{i+1} to u_{j+1}, and reverse the orientation from vertex u_{i+1} to u_j. This will result in only one polygon $Q' = [u_i, u_j, u_{j-1}, ..., u_{i+2}, u_{i+1}, u_{j+1}, u_{j+2}, ..., u_{i-1}]$.

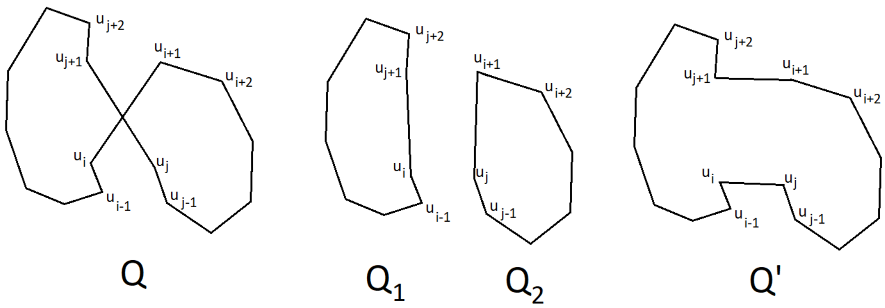

Figure 5.2: Two possible ways of eliminating a intersection: only one way results in a connected polygon.

Both possibilities do not cause new self-intersections because we are assuming that no other vertex is in the region spanned by $\{u_i, u_{i+1}, u_j, u_{j+1}\}$. More precisely, denoting by w the self-intersection of edges $\overrightarrow{u_i u_{i+1}}$ and $\overrightarrow{u_j u_{j+1}}$, we see that:

- Q_1 and Q_2 do not have self-intersections because no other vertex is in the regions spanned by $\{u_i, u_{j+1}, w\}$ and $\{u_j, u_{i+1}, w\}$;
- Q' does not have self-intersections because no other vertex is in the regions spanned by $\{u_i, u_j, w\}$ and $\{u_{i+1}, u_{j+1}, w\}$.

Since we will need only the case where we obtain polygon Q', our hypothesis might be narrowed down to the case in which no other vertex is in the regions spanned by $\{u_i, u_j, w\}$ and $\{u_{i+1}, u_{j+1}, w\}$.

Now we need the following lemma. Recall that D^+ and I denote respectively the number of self-intersections and the number of inflections of Q , while D'^+ and I' denote respectively the number of self-intersections and the number of inflections of Q'.

Lemma 5.6 *Let Q be a spherical (or planar) polygon with one self-intersection w and such that no other vertex is in the regions spanned by $\{u_i, u_j, w\}$ and $\{u_{i+1}, u_{j+1}, w\}$. Let Q' be the spherical (or planar) polygon obtained as above. Then*

$$2D^+ + I \geq 2D'^+ + I'.$$

Lemma 5.6 does not apply to a polygon Q which does not satisfy the hypothesis concerning the regions spanned by $\{u_i, u_j, w\}$ and $\{u_{i+1}, u_{j+1}, w\}$ (see figure 5.3). We will see later how to deal with such polygons.

Since Q has only one self-intersection and Q' has none, we have that $D^+ = 1$ and $D'^+ = 0$. Therefore it suffices to show that $2 + I$ is greater than or equal to I', i.e., that $2 + I - I'$ is greater than or equal to zero. First

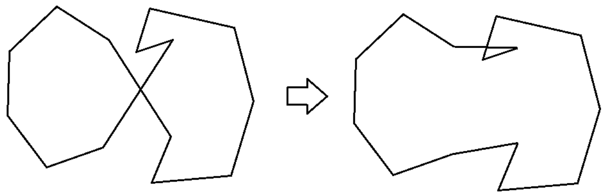

Figure 5.3: For the original polygon Q we have $D^+ = 1$ and $I = 2$, while for the resulting polygon Q' we have $D'^+ = 1$ and $I' = 4$.

notice that an edge does not change its condition of being an inflection or not if we reverse its orientation (since $[u_{i+2}, u_{i+1}, u_i] = (-1) \cdot [u_i, u_{i+1}, u_{i+2}]$ and $[u_{i+3}, u_{i+2}, u_{i+1}] = (-1) \cdot [u_{i+1}, u_{i+2}, u_{i+3}]$). Hence the operation of reversing the orientation of the polygon from vertex u_{i+1} to u_j does not increase nor decrease the number of inflections.

Therefore, in order to determine the number $2 + I - I'$, it is enough to study what happens to the edges near the intersection:

- two of the original edges disappear (namely, $\overrightarrow{u_i u_{i+1}}$ and $\overrightarrow{u_j u_{j+1}}$). These edges might be inflections, in which case they would contribute to the number I;
- two new edges appear (namely, $\overrightarrow{u_i u_j}$ and $\overrightarrow{u_{i+1} u_{j+1}}$). These edges might be inflections, in which case they would contribute to the number I';
- the adjacent edges $\overrightarrow{u_{i-1} u_i}$, $\overrightarrow{u_{i+1} u_{i+2}}$, $\overrightarrow{u_{j-1} u_j}$ and $\overrightarrow{u_{j+1} u_{j+2}}$ might also change their condition of being inflections or not, since we are altering one of the vertices that are adjacent to each one of these edges. Therefore theses edges might contribute either to I or I' (or both).

The condition of any of these edges to be an inflection or not depends on the position of the adjacent vertices u_{i-1}, u_{i+2}, u_{j-1} and u_{j+2}. We call these vertices *external*. The vertices u_i, u_{i+1}, u_j or u_{j+1} are *internal*. Likewise, the edges whose endpoints are internal vertices ($\overrightarrow{u_i u_{i+1}}$, $\overrightarrow{u_j u_{j+1}}$, $\overrightarrow{u_i u_j}$ and $\overrightarrow{u_{i+1} u_{j+1}}$) are called *internal edges*, while the remaining adjacent edges ($\overrightarrow{u_{i-1} u_i}$, $\overrightarrow{u_{i+2} u_{i+1}}$, $\overrightarrow{u_j u_{j-1}}$ and $\overrightarrow{u_{j+1} u_{j+2}}$) are called *external edges*.

Each external vertex might be in one of four different regions. For instance, vertex u_{i-1} might be in one of the two regions determined by the line spanned by $\overrightarrow{u_i u_{i+1}}$ and in one of the two regions determined by the line spanned by $\overrightarrow{u_i u_j}$, which gives us four regions in total (see figure 5.4). Now, if u_{i-1} is not in any of the gray regions as in the figure 5.4 (a), then u_{i+1} and u_j will be on the same region determined by the line spanned by $\overrightarrow{u_{i-1} u_i}$. This implies that edge $\overrightarrow{u_{i-1} u_i}$ does not alter its condition of being an inflection or not after the cut-and-connect process, and therefore its contribution to the

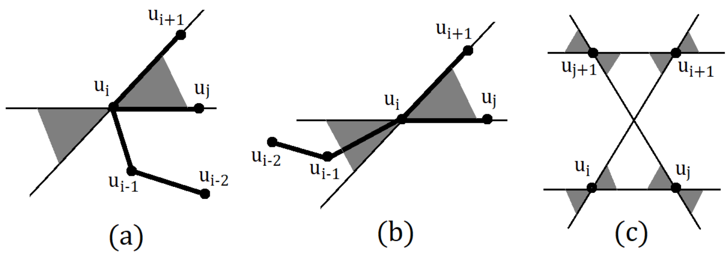

Figure 5.4: In (a), edge $\overrightarrow{u_{i-1}u_i}$ is not an inflection in P nor in P'. In (b), edge $\overrightarrow{u_{i-1}u_i}$ is not an inflection in P, but it is an inflection in P'.

number $2+I-I'$ is zero. If, however, u_{i-1} is in one of the gray regions, then we must also consider subcases in which $\overrightarrow{u_{i-1}u_i}$ goes from being an ordinary edge to being an inflection, and vice-versa (in which case its contribution to the number $2+I-I'$ would be -1 resp. $+1$). Figure 5.4(c) shows simultaneously the gray regions for all internal vertices.

Figure 5.5 shows 4 possibilities (among many others). In case (a), $\overrightarrow{u_iu_{i+1}}$ and $\overrightarrow{u_{i+1}u_{j+1}}$ are inflections, while $\overrightarrow{u_ju_{j+1}}$ and $\overrightarrow{u_iu_j}$ are not. This gives us $2+I-I' = 2+0 = 2$.

In case (b), $\overrightarrow{u_ju_{j+1}}$ and $\overrightarrow{u_{i+1}u_{j+1}}$ are inflections, while $\overrightarrow{u_iu_{i+1}}$ and $\overrightarrow{u_iu_j}$ are not, which means that these four edges contribute zero to the number $2+I-I'$. Now, in this case we must also look at edges $\overrightarrow{u_{i-1}u_i}$ and $\overrightarrow{u_{j-1}u_j}$: both of them are originally inflections but become ordinary edges. Therefore $2+I-I' = 2+2 = 4$.

A similar analysis shows that for case (c) the number $2+I-I'$ equals 0. Notice that configurations (b) and (c) are the same except for the vertices u_{i-2} and u_{j-2}, which in each case determine if edges $\overrightarrow{u_{i-1}u_i}$ and $\overrightarrow{u_{j-1}u_j}$ cease to be or become inflections. This, on its turn, determine a change in the number $2+I-I'$. Hence, in a certain sense, case (c) is a “worse" situation than case (b) because its corresponding number $2+I-I'$ is less than that of (b), although it does not contradict the conclusion of Lemma 5.6. Since we want to simplify the number of configurations to be analyzed, we will assume that if any of the external vertices u is in a gray region, then the adjacent vertex to u which is not internal is positioned in such a way that the corresponding external edge goes from being an ordinary edge to becoming an inflection, i.e., it will contribute -1 to the number $2+I-I'$ (that is, since we want to prove that the number $2+I-I'$ is equal or greater than zero, we are already considering the “worst-case scenario").

For case (d) the number $2+I-I'$ equals -2. This does not contradict the validity of Lemma 5.6, since in this case vertex u_{j+2} is in the forbidden region and therefore the configuration does not satisfy the hypotheses.

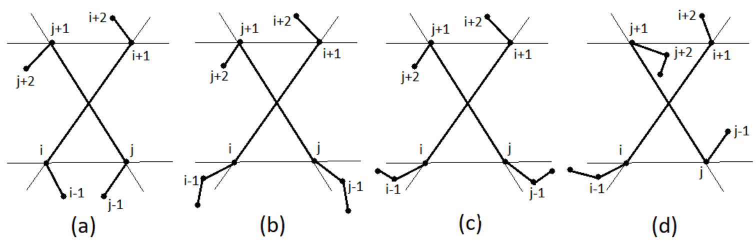

Figure 5.5: Four different cases

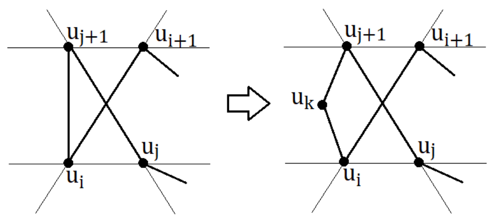

Figure 5.6: If vertices u_{j+1} and u_i are consecutive, one can add an intermediate vertex without altering the numbers I and I'.

Another important fact is that vertices u_{j+1} and u_i (or u_{i+1} and u_j) might be consecutive. In this case, we can just add another vertex u_k between u_{j+1} and u_i as in figure 5.6: just delete edge $\overrightarrow{u_{j+1}u_i}$ and add the new ones $\overrightarrow{u_{j+1}u_k}$ and $\overrightarrow{u_ku_i}$. This can be done without altering the numbers I and I' provided that one places vertex u_k as in figure 5.6: edges $\overrightarrow{u_iu_{i+1}}$, $\overrightarrow{u_ju_{j+1}}$, $\overrightarrow{u_iu_j}$ and $\overrightarrow{u_{i+1}u_{j+1}}$ do not change their condition of being or not being an inflection, while edges $\overrightarrow{u_{j+1}u_i}$ (the deleted one), $\overrightarrow{u_{j+1}u_k}$ and $\overrightarrow{u_ku_i}$ (the new ones) are not inflections. Since the number $2+I+I'$ of the new polygon is equal to the corresponding number of the original polygon, we can therefore assume that our polygons do not have self-intersections of the type of figure 5.6-left.

We could prove Lemma 5.6 by looking at all $3^4 = 81$ possible configurations (each of the 4 external vertices might be in 3 different regions) and checking that the number $2+I-I'$ is always greater or equal to zero. One can notice further that many of these configurations are symmetric (by reflections) to each other. For example, in figure 5.7 the configurations (a),(b),(c) and (d) can be obtained from the others by reflecting "horizontally" and "vertically", and inflections are reflected into each other. Hence the number $2+I-I' = 2+0 = 2$ is the same for these 4 cases. In figure 5.7 cases (e) and (f) can also be obtained from one another by a reflection, and cases (g) and (h) can also be reflected into each other.

Therefore the number of configurations to be considered can be decreased. The final number of cases is 27 and is depicted in figure 5.8. Since in all the cases the number $\gamma = 2+I-I'$ is greater or equal than 0, lemma 5.6 is proved.

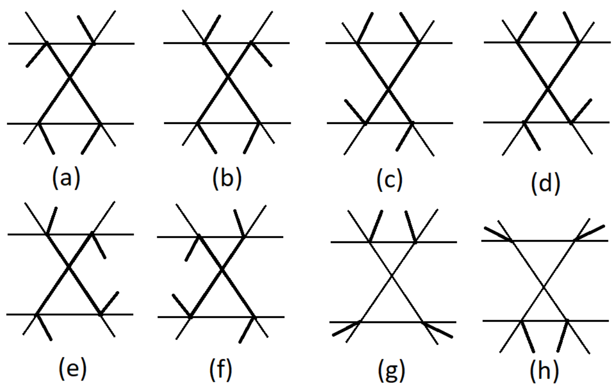

Figure 5.7: Cases (a),(b),(c) and (d) can be reflected one into the other, and for all of them the number $2 + I - I'$ is the same, equal to 2. Cases (e) and (f) can be reflected into each other as well, and their number $2 + I - I'$ is the same, equal to 0. Cases (g) and (h) can also be reflected into each other, and their number $2 + I - I'$ is the same, equal to 0 (recall that we assume the "worst-case scenario", i.e., the external edges in the gray regions goes from ordinary edges to becoming inflections).

A more algebraic approach to checking that the number $2+I-I'$ is always greater or equal to zero may be found in the following (optional) section.

5.4 An elementary algebraic approach to study all the possible configurations

Instead of depicting all the possible configurations, we can do a more synthetic approach: we noticed before that the location of the external vertices suffices to discover the number $\gamma = 2 + I - I'$ (recall that for each vertex in a gray region we assume an extra contribution of -1 to γ). Each of these vertices does not work alone: the location of pairs of them is what determines which edges (from $\overrightarrow{u_i u_{i+1}}$, $\overrightarrow{u_j u_{j+1}}$, $\overrightarrow{u_i u_j}$ and $\overrightarrow{u_{i+1} u_{j+1}}$) are inflections or not. This suggests that it might be possible to assign to each external vertex a signed number that, all of them considered together, gives us a formula that computes the number γ. If we succeed at finding such a formula, it will be easier for a computer to check all the possible configurations. If we are lucky, however, it might be possible to prove easily (by elementary algebra) that such a formula always gives zero or positive numbers.

The idea to deduce a formula that gives the number γ is to associate to each external vertex a pair $(x_k, y_k) \in \{\pm 1\}^2$. At each internal vertex two different lines intersect. The corresponding external vertex might be in one of the two regions determined by one line, and in one of the regions determined by the other line. From now on we relabel (for simplicity of notation) each index of the internal vertices as in figure 5.9 (c). If the external vertex corresponding

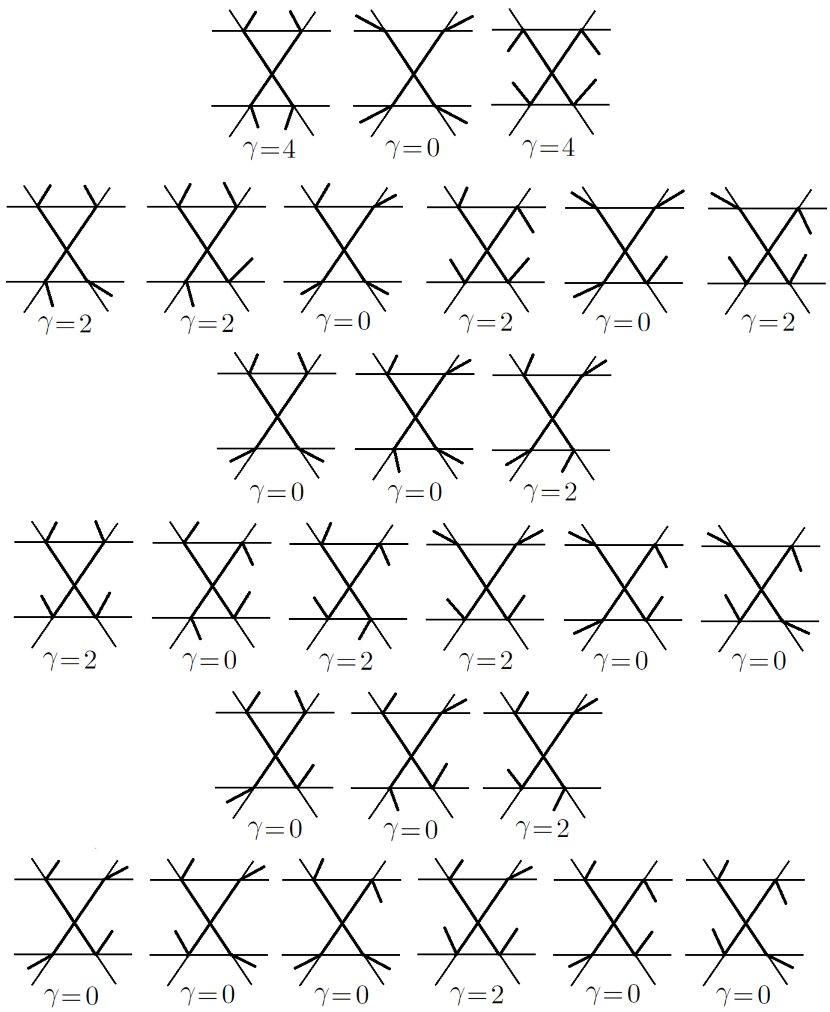

Figure 5.8: The 27 possible types of configurations regarding the location of the external vertices. In all of the cases, the number $\gamma = 2 + I - I'$ is greater or equal to 0.

to the internal vertex u_k is in a region with plus respectively minus sign in figure 5.9 (a), then put x_k equal to $+1$ respectively -1. If the external vertex corresponding to the internal vertex u_k is in a region with a plus respectively minus sign in figure 5.9 (b), then put y_k equal to $+1$ respectively -1. See figure 5.10 for an example. Once this is done, the following facts follow:

- edge $\overrightarrow{u_4 u_1}$ is an inflection if and only if x_1 and x_4 have the same sign, and edge $\overrightarrow{u_3 u_2}$ is an inflection if and only if x_2 and x_3 have the same sign;
- edge $\overrightarrow{u_2 u_1}$ is an inflection if and only if y_1 and y_2 have opposite signs, and edge $\overrightarrow{u_3 u_4}$ is an inflection if and only if y_3 and y_4 have opposite signs;
- the edge connecting the internal and external vertices at u_i is changes its condition of being an inflection if and only if it is in one of the gray regions, which on its turn happens if and only if x_i and y_i have the same sign;

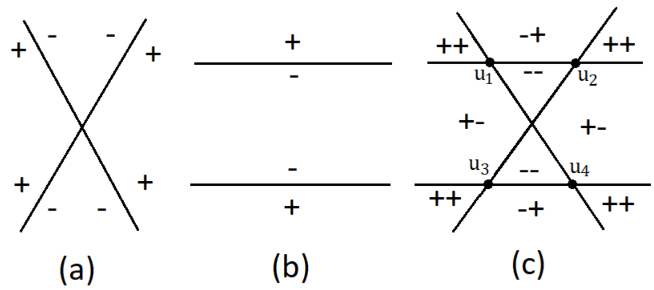

Figure 5.9: Assigning a pair of signed numbers to each region

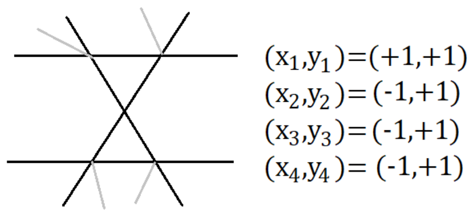

Figure 5.10: An example of configuration with the respective values of x_k's and y_k's.

Although the assigning was a bit arbitrary, it still conveys the symmetry we are looking for as figure 5.7 shows. Now comes the trick: we want to find a formula in terms of the x_k's and the y_k's that gives the number $\gamma = 2 + I - I'$. The previous facts imply the following numerical relations, respectively:

– We want an expression that gives 1 if edge $\overrightarrow{u_4u_1}$ is an inflection, and 0 if it is not. It is not hard to see that $\frac{x_1x_4+|x_1x_4|}{2}$ does the work. Similarly, for the edge $\overrightarrow{u_3u_2}$, the expression $\frac{x_2x_3+|x_2x_3|}{2}$ gives 1 when this edge is an inflection, and 0 when it is not. Therefore the contribution of inflections among the internal edges to the number I is equal to

$$\frac{x_1x_4 + x_2x_3 + |x_1x_4| + |x_2x_3|}{2} = \frac{x_1x_4 + x_2x_3 + 2}{2} = \frac{x_1x_4 + x_2x_3}{2} + 1.$$

– We want an expression that gives 1 if edge $\overrightarrow{u_2u_1}$ is an inflection, and 0 if it is not. Such an expression is $\frac{-y_1y_2+|y_1y_2|}{2}$. Similarly, for the edge $\overrightarrow{u_3u_4}$, the expression $\frac{-y_3y_4+|y_3y_4|}{2}$ gives 1 when this edge is an inflection, 0 when it is not. Therefore the contribution of inflections among the internal edges to the number I' is equal to

$$\frac{-y_1y_2 - y_3y_4 + |y_1y_2| + |y_3y_4|}{2} = \frac{-y_1y_2 - y_3y_4 + 2}{2} = -\frac{y_1y_2 + y_3y_4}{2} + 1.$$

– Assuming the “worst-case scenario" concerning the change of the conditions of the external edges, we want an expression that gives 1 if the external edge at u_k is an inflection, and 0 when it is not (for all $k \in \{1, 2, 3, 4\}$). Such an expression is $\frac{x_k y_k + |x_k y_k|}{2}$. Therefore, the contribution of inflections among the external edges to the number I' is equal to

$$\frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + |x_1y_1| + |x_2y_2| + |x_3y_3| + |x_4y_4|}{2}$$

$$= \frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + 4}{2}$$

$$= \frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4}{2} + 2.$$

Our conclusion is that the number $\gamma = 2 + I - I'$ is equal to

$$2 + \frac{x_1x_4 + x_2x_3}{2} + 1 - (-\frac{y_1y_2 + y_3y_4}{2} + 1 + \frac{x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4}{2} + 2)$$

$$= \frac{x_1x_4 + x_2x_3 + y_1y_2 + y_3y_4 - x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4}{2}.$$

Notice that this formula is invariant by permuting the indices by (12)(34) (which is related by the “y-axis reflection" symmetry) and by (13)(24) (which is related by the “x-axis reflection" symmetry). Therefore the formula gives the same number for configurations that can be obtained one from another by reflections (as we saw in the examples of figure 5.7).

It remains to prove that the number γ is always positive or equal to zero, provided that the external vertices are not in the forbidden regions. By the previous discussion, it suffices to prove the following lemma:

Lemma 5.7 *If* $(x_k, y_k) \neq (-1, -1)$ *for all* $k \in \{1, 2, 3, 4\}$*, then* 2γ*, given by*

$$x_1x_4 + x_2x_3 + y_1y_2 + y_3y_4 - x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4,$$

is always greater or equal to zero.

Proof. Consider the sequence $(x_1y_1, x_2y_2, x_3y_3, x_4y_4)$. There are five possibilities regarding the number of $+1$'s in the sequence:

- 4 of them. The hypothesis implies then that all x_k's and y_k's are equal to 1, and therefore 2γ is equal to 0.
- 3 of them. The hypothesis implies then that exactly one number among the x_k's and y_k's is equal to -1, which then implies that 2γ is equal to 0.
- 2 of them. This immediately implies that the last four terms of the expression of 2γ add up to 0. Moreover, by hypothesis, this implies that

exactly two numbers among the x_k's and y_k's are equal to -1. This in turn implies that at least two of the first four terms of the expression of 2γ are equal to 1, which suffices for the entire expression of 2γ to be greater or equal to 0.

– 1 of them. This immediately implies that the last four terms of the expression of 2γ add up to 2. Moreover, by hypothesis, this implies that exactly five numbers among the x_k's and y_k's are equal to $+1$. This in turn implies that at least one of the first four terms of the expression of 2γ is equal to 1, which suffices for the entire expression of 2γ to be greater of equal to 0.

– 0 of them. This immediately implies that the last four terms of the expression of 2γ add up to 4, which suffices for the entire expression of 2γ to be greater or equal to 0.

■

Lemma 5.6 now follows immediately from lemma 5.7.

5.5 Proofs of theorems

Now we can prove the promised theorems.

Proof. (of Theorem 5.4) It suffices to prove the theorem when $D^+ = 1$. If the polygon Q satisfies at its self-intersection the hypothesis of Lemma 5.6 and the corresponding polygon Q' is obtained as indicated, then $2D^+ + I = 2 + I \geq 2D'^+ + I' = 0 + I' = I'$, which is greater or equal to 4, since Q' satisfies the hypotheses of Theorem 3.25.

If, however, one (or even both) of the regions determined by $\{u_i, u_j, w\}$ and $\{u_{i+1}, u_{j+1}, w\}$ contain in its interior other vertices of Q (which would then imply that the new polygon would have a self-intersection), then we construct and intermediate polygon before "cutting-and-pasting". We put

$$\epsilon = \frac{1}{2}\min\{|w - u_i|_{\mathbb{S}^2}; i \in \{1, ..., n\}\}$$

(where we denote by $|\cdot|_{\mathbb{S}^2}$ the spherical distance) and denote by $S_\epsilon(w)$ the circle with radius ϵ centered at w. We then add new vertices to the polygon Q in the following way (see figure 5.11):

– If the region determined by $\{u_i, u_j, w\}$ has some other vertex in its interior, then add to Q the new vertices $u_{i+\frac{1}{3}} := \overrightarrow{u_i w} \cap S_\epsilon(w)$ and $u_{j+\frac{1}{3}} := \overrightarrow{u_j w} \cap S_\epsilon(w)$, with the ordering induced by the usual ordering of rational numbers $\mod n$;

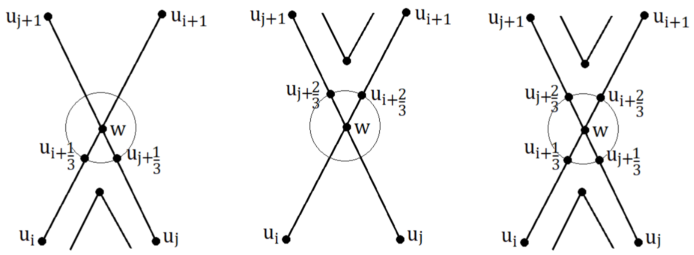

Figure 5.11: Adding vertices

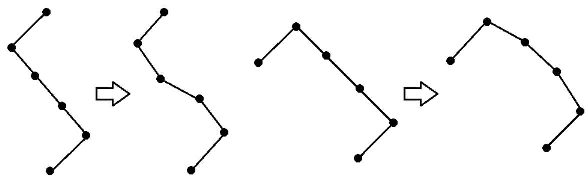

Figure 5.12: Perturbing a polygon with three consecutive vertices so that the number of its inflections will still be the same.

– if the region determined by $\{u_{i+1}, u_{j+1}, w\}$ has some other vertex in its interior, then add to Q the new vertices $u_{i+\frac{2}{3}} := \overrightarrow{wu_{i+1}} \cap S_\epsilon(w)$ and $u_{j+\frac{2}{3}} := \overrightarrow{wu_{j+1}} \cap S_\epsilon(w)$, with the ordering induced by the usual ordering of rational number $\mod n$.

Denote the newly obtained polygon by Q'. Perturb the vertices of Q' slightly so that it does not have three consecutive collinear vertices nor new self-intersections, and such that the number of inflections of Q' is the same as the original polygon Q (see figure 5.12 for examples). This can always be done. Continue denoting such polygon by Q'. Now, the new polygon Q' satisfies the hypotheses of Lemma 5.6, with $I' = I$ and $D'^+ = D^+$. Therefore we can construct from Q' a new polygon Q'' without self-intersections as in the discussion before Lemma 5.6 (in the case of figure 5.11 on the right, for instance, the internal vertices would be $u_{i+\frac{1}{3}}$, $u_{i+\frac{2}{3}}$, $u_{j+\frac{1}{3}}$ and $u_{j+\frac{2}{3}}$). By the latter result we have:

$$2D^+ + I = 2D'^+ + I' \geq 2D''^+ + I'' = 0 + I'' \geq 4,$$

where the last inequality holds by Theorem 3.25.

■

Now we prove Theorem 5.5:

Proof. (of Theorem 5.5) If the original polygon Q is symmetric, then D^+ is always even. If $D^+ \geq 4$, the result follows immediately. If $D^+ = 0$, then we apply theorem 3.35. Now, if $D^+ = 2$, then we can apply the same procedure as we did in this section: construct from Q a new polygon Q' through "cutting-and-pasting" at each of the two self-intersections (adding new vertices if necessary), so that Q' does not have self-intersections. Notice that each step of this procedure is invariant by reflection on the origin, hence Q' is also symmetric. Therefore

$$2D^+ + I \geq I' \geq 6,$$

where the first inequality holds applying Lemma 5.6 twice and the second inequality holds by Theorem 3.35. ■

We still need to prove theorem 5.3. Recall that D^+ is the number of self-intersections of Q, D^- is the number of antipodal intersections of Q and $D = D^+ + D^-$.

Proof. (of theorem 5.3) If for a given spherical polygon Q not contained in any hemisphere the number D^- equals 1, then

$$2D + I = 2(D^+ + D^-) + I = 2D^- + 2D^+ + I = 2 + 2D^+ + I \geq 2 + 4 = 6,$$

where the inequality holds by theorem 5.4. If, however, $D^- = 0$, then we still need to prove that in this case the number $2D^+ + I$ is greater or equal to 6. Now, Q either has no self-intersections or has at least one. In the former case we apply theorem 4.3, while in the latter case we proceed as we did in the previous section: at the self-intersection we cut-and-paste the edges of Q such that the resulting polygon Q has no self-intersections and no antipodal intersections. In case we need to add new vertices before cutting-and-pasting, we must take now as ϵ the number

$$\epsilon = \frac{1}{2}\min\{|w \pm u_i|_{\mathbb{S}^2}; i \in \{1, ..., n\}\}$$

(where we denote again by $|\cdot|_{\mathbb{S}^2}$ the spherical distance), so that the proof of theorem 5.4 also works here. The conclusion is that $2D^+ + I$ is greater or equal to I', where I' is the number of inflections of the resulting polygon Q', which does not have self-intersections nor antipodal intersections. By theorem 4.3, this number is greater or equal to 6. ■

5.6 Further remarks

The "cutting-and-pasting" approach used around the double points (i.e., intersection points) is also used in the smooth setting by Ghomi in (GHOMI, 2013) to prove theorems 5.1 and 5.2. In our case, however, the separate study of the 27 cases (up to symmetry) around the intersection point to prove lemma 5.6 is necessary since we are dealing with (spherical) polygons. The elementary algebraic argument of section 5.4 and the proof of lemma 5.7 are new.

In the smooth setting, theorems 5.1 and 5.2 are due to Ghomi (see (GHOMI, 2013)). From these theorems, he also considers corresponding results for space curves: self/antipodal intersections in the tangent indicatrix correspond to pairs of parallel tangents with the same/inverse orientation in the original space curve. Taking into account the number of such pairs, he obtains inequalities for space curves which are analogous to the inequalities of theorems 5.1 and 5.2.

As far as we know, the proofs of theorems 5.3, 5.4 and 5.5 in the discrete setting are new. Notice also that these theorems, although stated for spherical polygons, can be thought as applied to tangent indicatrices of space polygons. Since generic space polygons do not have "pairs of parallel tangents", one can define pairs of *parallel vertices with the same/inverse orientation* of a space polygon P as pairs of vertices v_i and v_j such that the tangent indicatrix Q of P has self/ antipodal intersections between spherical edges $\overrightarrow{u_{i-1}u_i}$ and $\overrightarrow{u_{j-1}u_j}$. Taking into account the number of such pairs, we can obtain the following inequalities for space polygons which are analogous to the inequalities of theorems 5.3 and 5.4. Denote by T^+ and T^- the numbers of pairs of parallel vertices with the same/inverse orientation of a space polygon, and put $T := T^+ + T^-$. Denote also by F the number of flattenings of a space polygon.

Theorem 5.8 *Let $P = [v_1, ..., v_n] \in \mathbb{R}^3$ $(n \geq 6)$ be a space polygon. Then*

$$2T + F \geq 6.$$

Theorem 5.9 *Let $P = [v_1, ..., v_n] \in \mathbb{R}^3$ $(n \geq 6)$ be a space polygon. Then*

$$2T^+ + F \geq 4.$$

Notice also that theorem 5.9 parallels corollary 2.15 in the sense that the notion of "convexity" in each case can be slightly weakened and, still, the polygon still admits a lower bound for the the number of flattenings plus twice a "measure of nonconvexity" of the polygon.

On the other hand, we are not aware of a result on weakly convex polygons which is analogous to theorem 5.8, even when $T = 0$ (in which case theorem 5.8 reduces to corollary 4.21).

6
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