Pontifícia Universidade Católica
do Rio de Janeiro

Miguel Beltran Walker Ureña

Regularity Theory for Nonlinear Partial Differential Equations

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor : Prof. Boyan Slavchev Sirakov
Co-advisor: Prof. Yannick Sire

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Abstract

Walker Ureña, Miguel Beltran; Sirakov, Slavchev Boyan (Advisor); Sire, Yannick (Co-Advisor). **Regularity Theory for Nonlinear Partial Differential Equations**. Rio de Janeiro, 2023. 63p. Tese de doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

We first examine L^p-viscosity solutions to fully nonlinear elliptic equations with bounded measurable ingredients. By considering $p_0 < p < d$, we focus on gradient-regularity estimates stemming from nonlinear potentials. We find conditions for local Lipschitz-continuity of the solutions and continuity of the gradient. We survey recent breakthroughs in regularity theory arising from (nonlinear) potential estimates. Our findings follow from – and are inspired by – fundamental facts in the theory of L^p-viscosity solutions, and results in the work of Panagiota Daskalopoulos, Tuomo Kuusi and Giuseppe Mingione (DKM2014). In the second part we prove partial regularity of weakly stationary weighted harmonic maps with free boundary data on a cone. As a starting point we take a look at the interior partial regularity theory for intrinsic energy minimising fractional harmonic maps from Euclidean space into smooth compact Riemannian manifolds for fractional powers strictly between zero and one. Intrinsic fractional harmonic maps can be extended to weighted harmonic maps, so we prove partial regularity for locally minimising harmonic maps with (partially) free boundary data on half-spaces, fractional harmonic maps then inherit this regularity.

Keywords

Fully nonlinear equations; Viscosity solutions; Potential estimates; Gradient-regularity estimates; Harmonic maps; Free boundary; Partial regularity;

Resumo

Walker Ureña, Miguel Beltran; Sirakov, Slavchev Boyan; Sire, Yannick. **Teoria da regularidade para equações diferenciais parciais não lineares**. Rio de Janeiro, 2023. 63p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Primeiro examinamos soluções de viscosidade L^p para equações elípticas totalmente não lineares com ingredientes de fronteira mensuráveis. Ao considerar $p_0 < p < d$, focamos nas estimativas da regularidade dos gradientes derivadas de potenciais não lineares. Encontramos condições para Lipschitz-continuidade local das soluções e continuidade do gradiente. Examinamos avanços recentes na teoria da regularidade decorrentes de estimativas potenciais (não lineares). Nossas descobertas decorrem de – e são inspiradas por – fatos fundamentais na teoria de soluções de L^p-viscosidade, e resultados do trabalho de Panagiota Daskalopoulos, Tuomo Kuusi e Giuseppe Mingione (DKM2014). Na segunda parte provamos a regularidade parcial de mapas harmônicos com peso fracamente estacionários com dados de fronteira livre em um cone. Como ponto de partida, damos uma olhada na teoria da regularidade parcial interior para mapas harmônicos fracionários de minimização de energia intrínseca do espaço euclidiano em variedades Riemannianas compactas e suaves para potências fracionárias estritamente entre zero e um. Mapas harmônicos fracionários intrínsecos podem ser estendidos para mapas harmônicos com peso, então provamos regularidade parcial para mapas harmônicos minimizantes locais com dados de fronteira (parcialmente) livres em meios-espaços, mapas harmônicos fracionários então herdam essa regularidade.

Palavras-chave

Equações totalmente não lineares; Soluções de viscosidade; Estimativas potenciais; Estimativas de regularidade de gradiente; Mapas harmônicos; Fronteira livre; Parcial regularidade;

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List of Abbreviations

In what follows we put forward a list of notations used throughout this text. This is mostly based on [(E2010), Appendix A.3].

Basic Notation:

(i) The d-dimensional Euclidean space is

$$\mathbb{R}^d = \{(x_1, x_2, \dots, x_d) \,|\, x_i \in \mathbb{R},\ \forall i = 1, 2, \dots, d\},$$

with norms

$$\|(x_1, x_2, \dots, x_d)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$$

and

$$\|(x_1, x_2, \dots, x_d)\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_d|\}\,.$$

(ii) Open ball of radius r and center x_0:

$$B_r(x_0) = \{x \in \mathbb{R}^d \,|\, \|x - x_0\| < r\} \quad \text{and} \quad B_r = B_r(0).$$

(iii) Lebesgue measure of a set $\mathcal{O}$ is denoted $|\mathcal{O}|$ or $\mathcal{L}(\mathcal{O})$.

Function Spaces and their norms: Let $\mathcal{O} \subseteq \mathbb{R}^d$.

(i) $\mathcal{C}(\mathcal{O}) = \{\, u : \mathcal{O} \to \mathbb{R} \,|\, u \text{ continuous} \,\}$.

$\mathcal{C}(\overline{\mathcal{O}}) = \{\, u : \mathcal{O} \to \mathbb{R} \,|\, u \text{ uniformly continuous} \,\}$.

If $\mathcal{O}$ is a bounded set, $C(\overline{\mathcal{O}})$ is a Banach space with

$$\|u\|_{\mathcal{C}(\overline{\mathcal{O}})} = \sup_{x \in \mathcal{O}} |u(x)|.$$

(ii) $\mathcal{C}^k(\mathcal{O}) = \{\, u : \mathcal{O} \to \mathbb{R} \,|\, u \text{ is } k\text{-times continuously differentiable} \,\}$.

$\mathcal{C}^k(\overline{\mathcal{O}}) = \{\, u \in \mathcal{C}^k(\mathcal{O}) \,|\, D^{\boldsymbol{\alpha}} u \text{ is uniformly continuous for all } |\boldsymbol{\alpha}| \le k \,\}$ where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$ and

$$D^{\boldsymbol{\alpha}} u = \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{\alpha_1} \cdots \partial x_1^{\alpha_d}}\ , \ |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_d.$$

If $\mathcal{O}$ is a bounded set, $C^k(\overline{\mathcal{O}})$ is a Banach space with

$$\|u\|_{\mathcal{C}^k(\overline{\mathcal{O}})} = \sup_{|\boldsymbol{\alpha}|\leq k} \sup_{x\in\mathcal{O}} |D^{\boldsymbol{\alpha}} u(x)|$$

We also denote,

$$\mathcal{C}^\infty(\mathcal{O}) = \bigcap_{k=0}^{\infty} \mathcal{C}^k(\mathcal{O}) \quad \text{and} \quad C^\infty(\overline{\mathcal{O}}) = \bigcap_{k=0}^{\infty} C^k(\overline{\mathcal{O}}).$$

(iii) We denote $\mathcal{C}_c(\mathcal{O})$, $\mathcal{C}_c^k(\mathcal{O}), \ldots$, the spaces comprised of $\mathcal{C}(\mathcal{O})$, $\mathcal{C}^k(\mathcal{O}), \ldots$ functions with compact support.

(iv) If $0 < \gamma \leq 1$, $\mathcal{C}^{0,\gamma}(\overline{\mathcal{O}})$ is the space of Hölder continuous functions with exponent γ, or γ^{th}-Hölder continuous. That is,

$$\mathcal{C}^{0,\gamma}(\overline{\mathcal{O}}) = \left\{ u \in C(\overline{\mathcal{O}}) \,|\, \exists C < \infty,\ |f(x) - f(y)| \leq C|x-y|^\gamma,\ \forall x, y \in \mathcal{O} \right\}.$$

If $\gamma \neq 1$ we can write $\mathcal{C}^\gamma = C^{0,\gamma}$, $\mathcal{C}^{0,1}$ is also called Lipschitz space.

In this case we have the seminorm

$$[u]_{\gamma,\mathcal{O}} = [u]_{C^{0,\gamma}(\overline{\mathcal{O}})} = \sup_{\substack{x,y\in\overline{\mathcal{O}} \\ x\neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\gamma} \right\},$$

and the γ^{th}-Hölder norm

$$\|u\|_{C^{0,\gamma}(\overline{\mathcal{O}})} = \|u\|_{C(\overline{\mathcal{O}})} + [u]_{\gamma,\mathcal{O}}.$$

In general, $\mathcal{C}^{k,\gamma}(\overline{\mathcal{O}})$ is the γ^{th}-Hölder space

$$\mathcal{C}^{k,\gamma}(\overline{\mathcal{O}}) = \left\{ u \in \mathcal{C}(\overline{\mathcal{O}}) \,|\, \|u\|_{C^{k,\gamma}(\overline{\mathcal{O}})} < \infty \right\},$$

where

$$\|u\|_{C^{k,\gamma}(\overline{\mathcal{O}})} = \sum_{|\boldsymbol{\alpha}|\leq k} \|D^{\boldsymbol{\alpha}} u\|_{C(\overline{\mathcal{O}})} + \sum_{|\boldsymbol{\alpha}|=k} [D^{\boldsymbol{\alpha}} u]_{\gamma,\mathcal{O}}.$$

The γ^{th}-Hölder Spaces are also Banach Spaces.

(v) $L^p(\mathcal{O}) = \left\{ u : \mathcal{O} \to \mathbb{R} \,|\, u \text{ is Lebesgue measurable and } \|u\|_{L^p(\mathcal{O})} < \infty \right\}$, where for $1 \leq p < \infty$

$$\|u\|_{L^p(\mathcal{O})} = \left(\int_{\mathcal{O}} |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

$L^\infty(\mathcal{O}) \;=\; \Big\{ u : \mathcal{O} \to \mathbb{R} \,|\, u \text{ is Lebesgue measurable and } \|u\|_{L^\infty(\mathcal{O})} < \infty \Big\}$, where

$$\|u\|_{L^\infty(\mathcal{O})} = \operatorname*{ess\,sup}_{x\in\mathcal{O}} |u(x)| = \inf \Big\{ C \in \mathbb{R} \;\Big|\; |f(x)| \le C \text{ a.e. on } \mathcal{O} \Big\}.$$

(vi) p-BMO$(\mathcal{O})$ is the p^{th}-bounded mean oscillation space of functions $f \in L^1_{\text{loc}}(\mathcal{O})$, with norm

$$\|f\|_{\mathrm{BMO}(\mathcal{O})} = \sup_{B_r(x)\subset\mathcal{O}} \left\{ \left(⨍_{B_r(x)} |f(y) - \langle f\rangle_{x,r}|^p \, dy \right)^{1/p} \right\} < \infty,$$

where $\langle f\rangle_{x,r}$ is the average value of f in $B_r(x)$:

$$\langle f\rangle_{x,r} = \langle f\rangle_{B_r(x)} = ⨍_{B_r(x)} f(y)\,dy = \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)\,dy.$$

We also denote $\langle f\rangle = \langle f\rangle_{0,1}$ and $m(f)(x) = \sup_{r>0} \langle f\rangle_{x,r}$.

(vii) $W^{k,p}(\mathcal{O}) \;=\; \Big\{ u \in L^p(\mathcal{O}) \,|\, D^{\boldsymbol{\alpha}} u \in L^p(\mathcal{O}), \forall \boldsymbol{\alpha} \in \mathbb{N}^d \text{ s.t. } |\boldsymbol{\alpha}| \le k \Big\}$ is the Sobolev space.

We have the norm

$$\|u\|_{W^{k,p}}(\mathcal{O}) = \left[\sum_{0\le|\boldsymbol{\alpha}|\le k} \|D^{\boldsymbol{\alpha}} u\|^p_{L^p(\mathcal{O})} \right]^{1/p} \qquad \text{in the case } 1 \le p < \infty,$$

and we have

$$\|u\|_{W^{k,\infty}(\mathcal{O})} = \max_{0\le|\boldsymbol{\alpha}|\le k} \|D^{\boldsymbol{\alpha}} u\|_{L^\infty(\mathcal{O})}.$$

We also denote $H^k = W^{k,2}$.

(viii) $W^{k,p}_0(\mathcal{O})$ is the closure of $C^\infty_c(\mathcal{O})$ in $W^{k,p}(\mathcal{O})$, so

$$W^{k,p}_0(\mathcal{O}) = \Big\{ u \in W^{k,p}(\mathcal{O}) \;\Big|\; \exists\, (u_n)_{n\in\mathbb{N}} \text{ s.t. } u_n \in C^\infty_c(\mathcal{O}) \text{ and } \|u_n - u\|_{W^{k,p}(\mathcal{O})} \to 0 \Big\}.$$

1 Introduction

In this thesis, we study two types of problems. The first one is driven by a fully nonlinear elliptic partial differential equation (PDE), whereas the second one concerns fractional harmonic maps whose target is a Lipschitz manifold. The former is presented in Chapter 2 and is based on the joint work (PW2023) with Edgard Pimentel. Here, we examine L^p-viscosity solutions for PDE with bounded-measurable ingredients. We focus on gradient-regularity estimates.

The second model, which we develop in the Chapter 3, is mostly inspired by (AHL2017) and (RM2022). It focuses on the regularity analysis of weighted harmonic solutions on a half-space associated with an extension of a fractional harmonic equation when the target is a cone. The strategy used on this second problem relies on monotonicity formulas, compactness and energy decay for minimizers of a modified Ericksen energy.

1.1 Main Results

In Chapter 2, we study the regularity of L^p-viscosity solutions to

$$F\Big(D^2u, Du, u, x\Big) = f \quad \text{in} \quad \Omega, \tag{1-1}$$

where $F : \mathcal{S}(d) \times \mathbb{R}^d \times \mathbb{R} \times \Omega \setminus \mathcal{N} \to \mathbb{R}$, is a uniformly elliptic operator with bounded-measurable ingredients, and $f \in L^p(\Omega)$ for $p > p_0$. Here, $\Omega \subset \mathbb{R}^d$ is an open and bounded domain, $\mathcal{N}$ is a null set, $S(d) \sim \mathbb{R}^{\frac{d(d+1)}{2}}$ is the space of symmetric matrices, and $d/2 < p_0 < d$ is the exponent such that the Aleksandrov-Bakelman-Pucci (ABP) estimate is available for elliptic equations with right-hand side in L^p, for $p > p_0$.

We extend the gradient potential estimates reported in (DKM2014) to operators with bounded-measurable coefficients depending explicitly on lower-order terms. Our analysis heavily relies on properties of L^p-viscosity solutions (CCKS1996, S1997); see also (WN2009).

Our first main result concerns the Lipschitz-continuity of L^p-viscosity solutions to (1-1) and reads as follows.

Theorem 1.1 (Lipschitz continuity) *Let $u \in C(\Omega)$ be an L^p-viscosity solution to* (1-1)*. Suppose Assumptions A1 and A2 are in force. Then, for every*

$q > d$, there exists a constant $\theta^* = \theta^*(d, \lambda, \Lambda, p, q)$ such that if Assumption A3 holds with $\theta \equiv \theta^*$, one has

$$|Du(x)| \leq C \left[\mathbf{I}_p^f(x,r) + \left(\fint_{B_r(x)} |Du(y)|^q \, \mathrm{d}y \right)^{\frac{1}{q}} \right]$$

for every $x \in \Omega$ and $r > 0$ with $B_r(x) \subset \Omega$, for some universal constant $C > 0$.

The potential estimate in Theorem 1.1 builds upon Święch's $W^{1,q}$-estimates to produce uniform estimates in $B_{1/2}$. In fact, by taking $d < q < p^*$ in Theorem 1.1, with

$$p^* := \frac{pd}{d-p}, \quad \text{and} \quad d^* = +\infty,$$

one finds $C = C(d, \lambda, \Lambda, p)$ such that

$$\|Du\|_{L^\infty(B_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right).$$

Our second main result establishes gradient-continuity for the L^p-solutions to (1-1) and provides an explicit modulus of continuity for the gradient. It reads as follows.

Theorem 1.2 (Gradient continuity) *Let $u \in C(\Omega)$ be an L^p-viscosity solution to* (1-1)*. Suppose Assumptions A1 and A2 are in force. Suppose further that $\mathbf{I}_p^f(x,r) \to 0$ as $r \to 0$, uniformly in x. There exists $0 < \theta^* \ll 1$ such that, if Assumption A3 holds for $\theta \equiv \theta^*$, then Du is continuous. In addition, for $\Omega' \Subset \Omega'' \Subset \Omega$, and any $\delta \in (0,1]$, one has*

$$|Du(x) - Du(y)| \leq C \left(\|Du\|_{L^\infty(\Omega')} |x-y|^{\alpha(1-\delta)} + \sup_{x \in \Omega} \mathbf{I}_p^f \left(x, 4|x-y|^\delta \right) \right),$$

for every $x, y \in \Omega'$, where $C = C(d, p, \lambda, \Lambda, \omega, \Omega', \Omega'')$ and $\alpha = \alpha(d, p, \lambda, \Lambda)$.

The strategy to prove Theorems 1.1 and 1.2 combines fundamental facts in L^p-viscosity theory to show that a solution to (1-1) also solves an equation of the form

$$\tilde{F}(D^2u, x) = \tilde{f} \quad \text{in} \quad \Omega,$$

where $\tilde{F}$ and $\tilde{f}$ meet the conditions required in (DKM2014). In particular, the Lorentz borderline condition for gradient-continuity follows as a corollary.

Corollary 1 (Borderline gradient-regularity) *Let $u \in C(\Omega)$ be an L^p-viscosity solution to* (1-1)*. Suppose Assumptions A1 and A2 are in force.*

Suppose further $f \in L^{d,1}(\Omega)$*. There exists* $0 < \theta^* \ll 1$ *such that, if Assumption A3 holds for* $\theta \equiv \theta^*$*, then* Du *is continuous.*

We organize the remainder of this part as follows. Section 2.2 presents some context on potential estimates, briefly describing their motivation and mentioning recent breakthroughs. We detail our first main assumptions in Section 2.3.1, whereas Section 2.3.2 gathers preliminary material. The proofs of Theorems 1.1 and 1.2 are the subject of Section 2.4.

The theory developed on the Chapter 3 studies the regularity of intrinsic harmonic maps related with fractional harmonic maps.

Given $s \in (0,1)$ and $v : \mathbb{R}^d \to \mathbb{R}^m$, the fractional Laplace operator $\Delta^s u$ is defined such that

$$(-\Delta)^s v(x) = p.v. \left(\gamma_{d,s} \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x-y|^{d+2s}} \, dy \right)$$

where

$$\gamma_{d,s} = s 2^{2s} \pi^{-d/2} \frac{\Gamma\left(\frac{d+2s}{2}\right)}{\Gamma(1-s)},$$

thus the maps v such that $(-\Delta)^s v = 0$ are called fractional s-harmonic functions. Those maps, on Riemannian manifolds with free or constrained boundary were introduced by Da Lio and Rivière (DLR2011), who proved that they are smooth on domains of one dimension. In recent years related problems has been increasingly analyzed.

In particular, Caffarelli and Silvestre (CS2007) extend fractional harmonic maps to weighted harmonic maps over the half space, mapping the Dirichlet boundary condition to the Neumann condition.

In this part we focus on maps targeted on a semicone over $\mathbb{R}^m$

$$\mathbb{C}_k := \left\{ y \in \mathbb{R}^m \,/\, y_m = |k-1|^{1/2} \, |y'| \right\} \tag{1-2}$$

being $y = (y_1, \ldots, y_m) = (y', y_m)$ and $y' = (y_1, \ldots, y_{m-1})$. The cone $\mathbb{C}_k$ is just a simple connected Lipchitz target that is not compact, but eventually we take in account that away the origin, $B_R(0) \cap \mathbb{C}_k$ is a Lipchitz submanifold.

Consider a bounded admissible open subset $\Omega \subseteq \mathbb{R}^{d+1}_+$, we denote

$$H^1_s(\Omega; \mathbb{C}_k) = \left\{ u : \Omega \to \mathbb{C}_k \,:\, u \in L^2_s(\Omega) \text{ and } \nabla u \in L^2_s(\Omega) \right\}$$

the weighted Sobolev space, where

$$L^2_s(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) \,:\, x_{d+1}^{(1-2s)/2} \, |u| \in L^2(\Omega) \right\}.$$

On $H^1_s(\Omega; \mathbb{C}_k)$, the weighted energy is defined as

$$\mathbf{E}_s(u,\Omega) := \frac{1}{2}\int_\Omega x_{d+1}^a |\nabla u(x)|^2\,dx \tag{1-3}$$

where $x_{d+1} > 0$, $s \in (0,1)$ and $a = 1-2s$.

We say that u is a minimizing weighted harmonic map in Ω with respect to the partially free boundary condition $u(\overline{\Omega}) \subseteq \mathbb{C}_k$ if

$$\mathbf{E}_s(u,\Omega) \leq \mathbf{E}_s(w,\Omega)$$

for all $w \in H_s^1(\Omega;\mathbb{C}_k)$ such that $w(x) \in \mathbb{C}_k$ a.e. in Ω and $\operatorname{spt}(w-v) \subseteq \overline{\Omega} = \Omega \cup \partial^0\Omega$, being

$$\partial^0\Omega := \left\{x \in \partial\Omega \cap \partial\mathbb{R}_+^{d+1} : B_r^+(x) \subseteq \Omega \text{ for some } r > 0\right\}.$$

The Euler-Lagrange equation derived is

$$\int_\Omega x_{d+1}^a\, \nabla u(x) \bullet \nabla\zeta(x)\,dx = 0 \tag{1-4}$$

for all $\zeta \in H_s^1(\Omega;\mathbb{C}_k)$ such that $\zeta \in T_u\mathbb{C}_k$ a.e. $\partial^0\Omega$ and such that $\operatorname{spt}(\zeta) \subseteq \overline{\Omega}$ and denoting $T_u\mathbb{C}_k$ the tangent space to $\mathbb{C}_k$ at u. In other words, the Neumann-type problem

$$\begin{cases} \operatorname{div}\left(x_{d+1}^a \nabla u\right) = 0 \text{ in } \Omega \\ x_{d+1}^a \dfrac{\partial u}{\partial \nu} \perp T_u\mathbb{C}_k \text{ in } H_s^1(\partial^0\Omega;\mathbb{C}_k), \end{cases} \tag{1-5}$$

in a weakly sense, is fulfilled.

As is described on (MSW2018) and (MPS2021), this u can be seen as an extension of a fractional s-harmonic map $v \in \hat{H}^s(\omega;\mathbb{R}^m)$ (the set of functions with finite fractional energy) as

$$u(x) = v^e(x) = \gamma_{d,s}\int_{\mathbb{R}^d} \frac{x_{d+1}^{2s} v(z)}{\left(|x'-z|^2 + x_{d+1}^2\right)^{\frac{d+2s}{2}}}\,dz. \tag{1-6}$$

Since for every $v \in \hat{H}^s(\omega;\mathbb{R}^m)$ we have, given $\omega \subseteq \mathbb{R}^d$ a bounded open set with Lipschitz boundary,

$$\left\langle (-\Delta)^s v, \varphi \right\rangle_\omega = \left\langle x_{d+1}^a \frac{\partial u}{\partial \nu}, \Phi \right\rangle_{\partial^0\Omega}$$

for all $\varphi \in H_{00}^s(\omega;\mathbb{R}^m)$ (elements of compact support from $H^s(\omega;\mathbb{R}^m)$) and $\Phi \in H_0^1$ such that $\Phi\Big|_{\mathbb{R}^d\times\{0\}} = \phi$, u solves the Dirichlet problem

$$\begin{cases} \operatorname{div}\left(x_{d+1}^a \nabla u\right) = 0 & \text{in } \Omega \\ u = v & \text{on } \partial^0\Omega. \end{cases} \tag{1-7}$$

Related regularity theory is developed on (MPS2021), where is proved a called ε-regularity theorem (Small Energy Hölder Regularity). It first proves regularity for v (MPS2021, Theorem 4.1), and then the regularity of his extension $u = v^e$ is a corollary (MPS2021, Corollary 4.2), but with the condition $u(\Omega) \subseteq \mathbb{S}^m$.

A weighted harmonic function u with weight s is defined as a solution of

$$\operatorname{div}\left(x_{d+1}^{1-2s} \nabla u\right) = 0,$$

that we simply call it s-harmonic function, general regularity results were presented and proved in (R2018, Section 2). In (AHL2017) the authors develop regularity theory in the case $s = 1/2$ and with target on a whole cone. We assume $s \in (0,1)$ and took mostly inspiration of (R2018, Roberts 2018) and (RM2022, Roberts-Moser 2022) to develop some specific regularity results and finally prove a ϵ-regularity result (Theorem 1.3).

The (R2018, Sections 3 and 4) takes as target a generalized smooth compact Riemmanian manifold, studying minimizers of the analogous of the energy (1-3), also called intrinsic fractional harmonic maps, been the critical points of an energy whose first variation is a Dirichlet to Neuman map for the harmonic map problem on a half-space (1-5).

Like in (AHL2017), the motivation on take $\mathbb{C}_k$ as our target are the Erickson's model suggested in 1985 and his potential applications on the study of defects in liquid crystals. As is described on (E1991) and (AHL2017), if $\Omega \subset \mathbb{R}^3$, that previous model minimizes the energy $\int_\Omega X(s,n)\,dx$, where

$$X(s,n) = s^2 W(n) + \kappa_5 |\nabla s|^2 + \kappa_6 |\nabla s \bullet n|^2 + \psi(s),$$

with $W(n)$ equal to

$$\kappa_1 |\operatorname{div} n|^2 + \kappa_2 |n \bullet \operatorname{curl} n|^2 + \kappa_3 |n \times \operatorname{curl} n|^2 + (\kappa_1 + \kappa_4)\left[\operatorname{tr}(\nabla n)^2 - (\operatorname{div} n)^2\right],$$

and a C^2-potential ψ such that

$$\lim_{s \to -1/2} \psi(s) = +\infty, \quad \lim_{s \to 1} \psi(s) = 1, \quad \psi(0) = 0$$

and has a minimum at some $s^* \in (0,1)$. Next (L1989) and (L1991) studied the case $\kappa_1 = \kappa_2 = \kappa_3 = 1$, $\kappa_4 = \kappa_6 = 0$ and $\kappa_5 = 6$ and relate the minimizing on the par $(s,n) \in \mathbb{R} \times \mathbb{R}^2$ with a minimizing harmonic map into a cone, recasting the maps

$$u = (|k-1|^{1/2} s, sn)$$

and

$$X(s,n) = |\nabla u| + \psi(k^{-1/2}|u|),$$

so the image of u lie in the cone $\mathbb{C}_k$ defined in (1-2).

Depending on the smallness of the weighted density function

$$\mathbf{\Theta}_s(u, x_0, r) := \frac{1}{r^{d+a-1}} \, \mathbf{E}_s\Big(u, B_r^+(x_0)\Big), \tag{1-8}$$

we will prove the following regularity result:

Theorem 1.3 (A result on ϵ-regularity) *Let $d \geq 2$, $a \in (-1,1)$ and $u \in H_s^1(\mathbb{R}_+^{d+1}; \mathbb{C}_k)$ be a minimizing of $\mathbf{E}_s(u, \Omega)$. Suppose $B_R^+(x_0)$ satisfies $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \Omega$. There exists an $\varepsilon = \varepsilon(d, \|u\|_{L^2(\Omega)}, a) > 0$ and a $\theta = \theta\left(d, \|u\|_{L^2(\Omega)}, a\right) \in (0,1)$ such that if*

$$\mathbf{\Theta}_s(u, x_0, R) \leq \varepsilon,$$

then $u \in C^{0,\gamma}\left(\overline{B_{\theta R}^+(x_0)}\right)$ for some $\gamma = \gamma\left(d, \|u\|_{L^2(\Omega)}, a\right) \in (0,1)$. In particular,

$$|u(x_1) - u(x_2)| \leq C \, \mathbf{\Theta}_s(u, x_0, R)^{\frac{1}{2}} \, |x_1 - x_2|^\gamma$$

for every $x_1, x_2 \in B_{\theta R}^+(x_0)$ and a constant $C = C\left(d, \|u\|_{L^2(\Omega)}, a\right)$.

In order to prove it, we present a sequence of lemmas organized as follows; in the Section 3.2 we prove important monotonicity formulas around the density function defined as (1-8), which are the main support of the estimates that allow us prove Theorem 1.3. Next, in the Section 3.3, the first collection of previous useful estimates and the Section 3.4 is focused on the principal energy decay lemmas, from which we derive almost directly the Hölder regularity, these lemmas include the so-called energy decay lemmas, highlighting Lemmas 3.17 and 3.18 as the main ones corresponding with border and interior decay respectively. Finally, Proposition 1.3 is proved in Section 3.5.

2
Potential estimates for fully nonlinear elliptic equations with bounded ingredients

2.1
Preliminaries

The regularity theory for viscosity solutions to (1-1) is a delicate matter. Indeed, the first result in this realm is the so-called Krylov-Safonov theory. It states that, if $u \in C(B_1)$ is a viscosity solution to

$$F(D^2u) \leq 0 \leq G(D^2u) \quad \text{in} \quad B_1 \tag{2-1}$$

and F and G are (λ, Λ)-elliptic operators, then $u \in C^{\alpha}_{\text{loc}}(B_1)$, for some $\alpha \in (0,1)$ depending only on d, λ and Λ. In addition, one derives an estimate of the form

$$\|u\|_{C^{\alpha}(B_{1/2})} \leq C \|u\|_{L^{\infty}(B_1)},$$

where $C = C(d, \lambda, \Lambda)$ (KS1980). Indeed, the regularity result in the Krylov-Safonov theory concerns inequalities of the form

$$a_{ij}(x)\partial^2_{ij}u \leq 0 \leq b_{ij}(x)\partial^2_{ij}u \tag{2-2}$$

where the matrices $A := (a_{ij})^d_{i,j=1}$ and $B := (b_{ij})^d_{i,j=1}$ are uniformly elliptic, *with the same ellipticity constants*. The transition of those inequalities to (2-1) comes from the fundamental theorem of calculus. Indeed, notice that if $F(0) = G(0) = 0$, we get

$$\int_0^1 \frac{d}{dt}F(tD^2u)\mathrm{d}t = F(D^2u) \leq 0 \leq G(D^2u) = \int_0^1 \frac{d}{dt}G(tD^2u)\mathrm{d}t.$$

By computing the derivatives above with respect to the variable t and setting

$$a_{ij}(x) := \int_0^1 D_M F(tD^2u)\mathrm{d}t \quad \text{and} \quad b_{ij}(x) := \int_0^1 D_M G(tD^2u)\mathrm{d}t,$$

one notices that a solution to (2-1) also satisfies (2-2).

If we replace the inequality in (2-1) with the equation

$$F(D^2u) = 0 \quad \text{in} \quad B_1 \tag{2-3}$$

and require F to be a (λ, Λ)-elliptic operator, solutions become of class $C^{1,\alpha}$ with estimates. Once again, $\alpha \in (0,1)$ depends only on the dimension and the ellipticity (TN1988, CC1995). Finally, if we require F to be uniformly elliptic *and convex* (or concave) viscosity solutions to (2-3) are of class $C^{2,\alpha}$, with estimates. This is known as the Evans-Krylov theory, developed independently in the works of Lawrence C. Evans (E1982) and Nikolai Krylov (K1982).

The analysis of operators with variable coefficients, in the context of non-homogeneous problems first appeared in the work of Luis Caffarelli (C1989). In that paper, the author considers the equation

$$F(D^2u, x) = f \quad \text{in} \quad B_1 \tag{2-4}$$

and requires $F(M, x)$ to be uniformly elliptic. The fundamental breakthrough launched in (C1989) concerns the connection of the variable coefficients operator with its fixed-coefficients counterpart. To be more precise, the author introduces an *oscillation measure* $\beta(x, x_0)$ defined as

$$\beta(x, x_0) := \sup_{M \in S(d)} \frac{|F(M, x) - F(M, x_0)|}{1 + \|M\|}.$$

Different smallness conditions on this quantity yield estimates in distinct spaces. It includes estimates in $C^{1,\alpha}$, $W^{2,p}$ and $C^{2,\alpha}$-spaces. Of course, further conditions on the source term f must hold. In particular, it is critical that $f \in L^p(B_1)$, for $p > d$.

An interesting aspect of this theory concerns the continuity hypotheses on the data of the problem. For instance, the regularity estimates do not depend on the continuity of f. Meanwhile, the notion of C-viscosity solution requires f to be defined everywhere in the domain, as it depends on pointwise inequalities (CL1993, CEL1984, CIL1992). Hence, asking f to be merely a measurable function in some Lebesgue space is not compatible with the theory. See the last paragraph before Theorem 1 in (C1989).

In (CCKS1996), the authors propose an L^p-viscosity theory, recasting the notion of viscosity solutions in an *almost-everywhere* sense. In that paper, the authors examine (1-1) and suppose the ingredients of the problem are in L^p, for $p > p_0$. The quantity $d/2 < p_0 < d$ appeared in the work of Eugene Fabes and Daniel Stroock (FS1984). It stems from the improved integrability of the Green function for (λ, Λ)-linear operators.

In (E1993), and before the formalization of L^p-viscosity solutions, the quantity p_0 appeared in the context of Sobolev regularity. In that paper, Luis Escauriaza resorted to the improved integrability of the Green function from (FS1984) to extend Caffarelli's $W^{2,p}$-regularity theory to the range $p_0 < p < d$.

For that reason, p_0 is referred to in the literature as Escauriaza's exponent.

A fundamental study of the regularity theory for L^p-viscosity solutions to (1-1) appeared in (S1997). Working merely under uniform ellipticity, the author proves regularity results for the gradient of the solutions. In case $p > d$, solutions are of class $C^{1,\alpha}$. Here, the smoothness degree depends on the Krylov-Safonov exponent, and on the ratio d/p. However, in case $p_0 < p \leq d$, solutions are only in $W^{1,q}$, where $q \to \infty$ as $p \to d$.

The findings in (S1997) highlight an important aspect of the theory, namely: the smoothness of Du, in the range $p_0 < p < d$, is a very delicate matter. It is known that $C^{1,\alpha}$-regularity is not available in this context.

A program that successfully accessed this class of information is the one in (DKM2014). Through a modification in the linear Riesz potential, tailored to accommodate the p-integrability of the data, the authors produce potential estimates for the L^p-viscosity solutions to (2-4). Ultimately, those estimates yield a modulus of continuity for the gradient of the solutions.

In addition to uniform ellipticity, the results in (DKM2014) require an average control on the oscillation of $F(M, x)$. It also assumes $f \in L^p(\Omega)$ for $p_0 < p < d$. Under these conditions, the authors prove a series of potential estimates. Those lead to local boundedness and (an explicit modulus of) continuity for Du. Also, a borderline condition in Lorentz spaces follows: if $f \in L^{d,1}(\Omega)$, then Du is continuous. Besides providing new, fundamental developments to the regularity theory of fully nonlinear elliptic equations, the arguments in (DKM2014) are pioneering in taking to the non-variational setting a class of methods available before only for problems in the divergence form.

2.2
Potential estimates: from the Poisson equation to fully nonlinear problems

Potential estimates are natural in the context of linear equations for which a representation formula is available. For instance, let $\mu \in L^1(\mathbb{R}^d)$ be a measure and consider the Poisson equation

$$-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^d. \tag{2-5}$$

It is well-known that u can be represented through the convolution of μ with the appropriate Green function. In case $d > 2$, we have

$$u(x) = C \int_{\mathbb{R}^d} \frac{\mu(y)}{|x-y|^{d-2}} \mathrm{d}y, \tag{2-6}$$

where $C > 0$ depends only on the dimension.

Now, recall the β-Riesz potential of a Borel measure $\mu \in L^1(\mathbb{R}^d)$ is given by

$$I_\beta^\mu(x) := \int_{\mathbb{R}^d} \frac{\mu(y)}{|x-y|^{d-\beta}} \mathrm{d}y.$$

Hence, the representation formula (2-6) allows us to write $u(x)$ as the 2-Riesz potential of μ. Immediately one infers that

$$|u(x)| \leq C \left| I_2^\mu(x) \right|,$$

obtaining a *potential* estimate for u. By differentiating (2-6) with respect to an arbitrary direction $e \in \mathbb{S}^{d-1}$, one concludes

$$|Du(x)| \leq C \left| I_1^\mu(x) \right|.$$

That is, *the representation formula* available for the solutions to the Poisson equation yields potential estimates for the solutions.

This reasoning collapses if (2-5) is replaced with a nonlinear equation lacking representation formulas. Then a *fundamental question* arises: it concerns the availability of potential estimates for (nonlinear and inhomogeneous) problems for which representation formulas are not available.

The first answer to that question appears in the works of Tero Kilpeläinen and Jan Malý (KM1992), and Neil Trudinger and Xu-Jia Wang (TNWX2002), where the authors produce potential estimates for the solutions of p-Poisson type equations. Taking this approach a notch up, and accounting for potential estimates *for the gradient* of solutions, one finds the contributions of Giuseppe Mingione (M2018, M2014, M2011, M2011a), Frank Duzaar and Giuseppe Mingione (DM2011, DM2010, DM2010a, DM2009), and Tuomo Kuusi and Giuseppe Mingione (KM2018, KM2016, KM2016a, KM2014, KM2014a, KM2014b, KM2014c, KM2013, KM2013a, KM2012, KM2012a, KM2011). Of particular interest to the present thesis is the analysis of potential estimates in the fully nonlinear setting, due to Panagiota Daskalopoulos, Tuomo Kuusi, and Giuseppe Mingione (DKM2014). More recent contributions appeared in the works of Cristiana De Filippis (CDF2022) and Cristiana De Filippis and Giuseppe Mingione (FM2021, FM2022). See also the works of Cristiana De Filippis and collaborators (FS2022, FP2022).

In (DM2011) the authors examine an equation of the form

$$-\operatorname{div} a(x, Du) = \mu \quad \text{in} \quad \Omega, \tag{2-7}$$

where $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, and $\mu \in L^1(\Omega)$ is a Radon measure with finite mass. Here, $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies natural conditions, regarding growth, ellipticity, and continuity. Those conditions involve an inhomogeneous exponent $p \geq 2$, concerning the behavior of $a = a(x, z)$ on z. An oversimplification yields

$$a(x, z) = |z|^{p-2} z,$$

for $p > 2$, turning (2-7) into the degenerate p-Poisson equation. In that paper, the authors resort to the Wolff potential $\mathbf{W}^{\mu}_{\beta,p}$, defined as

$$\mathbf{W}^{\mu}_{\beta,p}(x, R) := \int_0^R \frac{1}{r^{\frac{d-\beta p}{p-1}}} \left(\int_{B_r(x)} \mu(y) \mathrm{d}y \right)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r},$$

for $\beta \in (0, d/p]$. Their main result is a pointwise estimate for the gradient of the solutions to (2-7). It reads as

$$|Du(x)| \leq C \left[\fint_{B_R(x)} |Du(y)| \, \mathrm{d}y + \mathbf{W}^{\mu}_{\frac{1}{p},p}(x, 2R) \right], \tag{2-8}$$

whenever $B_R(x) \subset \Omega$, and $R > 0$ is bounded from above by some universal quantity depending also on the data of the problem; see (DM2011, Theorem 1.1). A remarkable consequence of this estimate is a Lipschitz-continuity criterium for u obtained solely in terms of the Wolff potential of μ. Indeed, if $\mathbf{W}^{\mu}_{\frac{1}{p},p}(\cdot, R)$ is essentially bounded for some $R > 0$, every $W^{1,p}_0$-weak solution to (2-7) would be locally Lipschitz continuous. We notice the nonlinear character of the Wolff potential suits the growth conditions the authors impose on $a(x, z)$, as it scales accordingly under Lipschitz geometries.

The findings in (DM2011) also respect a class of very weak solutions, known as solutions obtained by limit of approximations (SOLA); see (BG1989, BG1992). This class of solutions is interesting because, among other things, it allows us to consider functions in larger Sobolev spaces. Indeed, for $2 - 1/d < p < d$ one can prove the existence of a SOLA $u \in W^{1,1}_0(\Omega)$ to

$$\begin{cases} -\Delta_p u = \mu & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

In addition, $u \in W^{1,q}_0(\Omega)$ with estimates, provided $q > 1$ such that

$$1 < q < \frac{d(p-1)}{d-1}.$$

When it comes to the proof of (2-8), the arguments in (DM2011) are very involved. However, one notices a fundamental ingredient. Namely, a *decay rate* *for the excess* of the gradient with respect to its average. Indeed, the authors prove there exist $\beta \in (0,1]$ and $C \geq 1$ such that

$$\fint_{B_r(x)} |Du(y) - (Du)_{r,x}|\,\mathrm{d}y \leq C\left(\frac{r}{R}\right)^{\beta} \fint_{B_R(x)} |Du(y) - (Du)_{R,x}|\,\mathrm{d}y, \qquad \text{(2-9)}$$

for every $0 < r < R$ with $B_R(x) \subset \Omega$. Here,

$$(Du)_{\rho,x} := \fint_{B_\rho(x)} Du(z)\mathrm{d}z.$$

See (DM2011, Theorem 3.1). An important step in the proof of (2-9) is a measure alternative, depending on the fraction of the ball B_r in which the gradient is larger than, or smaller than, some radius-dependent quantity.

Although the Wolff potential captures the inhomogeneous and nonlinear aspects of $a = a(x,z)$, a natural question concerns the use of linear potentials in the analysis of (2-7).

Indeed, in (M2011) the author supposes $a(x,z)$ to satisfy

$$\begin{cases} \lambda|\xi|^2 \leq \langle \partial_z a(x,z)\xi, \xi\rangle\,, \\ |\partial_z a(x,z)| + |a(x,0)| \leq C, \\ |a(x,z) - a(y,z)| \leq K|x-y|^{\alpha}(1+|z|), \end{cases} \qquad \text{(2-10)}$$

for every $x, y \in \Omega$, $z \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^d$, for some $C, \lambda > 0$, and $\alpha \in (0,1]$. Under these natural conditions, he derives a gradient bound in terms of the (linear) localized Riesz potential $\mathbf{I}^{\sigma}_{\beta}(x,R)$, defined as

$$\mathbf{I}^{\sigma}_{\beta}(x,R) := \int_0^R \frac{1}{r^{d-\beta}}\left(\int_{B_r(x)} \sigma(y)\mathrm{d}y\right)\frac{\mathrm{d}r}{r},$$

for a measure $\sigma \in L^1(\Omega)$, and $\beta \in (0,1]$, whenever $B_R(x) \subset \Omega$.

Indeed, the main contribution in (M2011) is the following: under (2-10), solutions to (2-7) satisfy

$$|Du(x)| \leq C\left[\fint_{B_R(x)} |Du(y)|\mathrm{d}y + \mathbf{I}^{\mu}_{1}(x,2R) + K\left(\mathbf{I}^{|Du|}_{\alpha}(x,2R) + R^{\alpha}\right)\right], \qquad \text{(2-11)}$$

where $C > 0$ depends on the data in (2-10). In case $a = a(z)$ does not depend on the spatial variable, $K \equiv 0$ and (2-11) recovers the usual potential estimate.

A further consequence of potential estimates is in unveiling the borderline conditions for C^1-regularity of the solutions to (2-7). See (DM2010); see also (C2011) for related results. More precisely, the intrinsic connection between Lorentz spaces and the nonlinear Wolff potentials unlocks the minimal conditions on the right-hand side μ that ensures continuity of Du.

In (DM2010), the authors impose p-growth, ellipticity, and continuity conditions on $a = a(x,z)$, and derive minimal requirements on μ to ensure that $u \in C^1(\Omega)$ (DM2010, Theorem 3); see also (DM2010, Theorem 9) for the vectorial counterpart of this fact.

They prove that if $\mu \in L^{d,\frac{1}{p-1}}_{\mathrm{loc}}(\Omega)$, then Du is continuous in Ω. To get this fact, one first derives an estimate for the Wolff potential $\mathbf{W}^{\mu}_{\frac{1}{p},p}(x,R)$ in terms of the $(d, 1/(p-1))$-Lorentz norm of μ. It follows from averages of decreasing rearrangements of μ. See (DM2010, Lemma 2). Then one notices that such control implies

$$\mathbf{W}^{\mu}_{\frac{1}{p},p}(x,R) \to 0$$

uniformly in $x \in \Omega$, as $R \to 0$; see (DM2010, Lemma 3).

The previous (very brief) panorama of the literature suggests that whenever $a = a(x,z)$ satisfies natural conditions – concerning p-growth, ellipticity, and continuity – potential estimates are available for the solutions to (2-7). Those follow through Wolff and (linear) Riesz potentials. Furthermore, this approach comes with a borderline criterion on μ for the differentiability of solutions. However, these developments appear in the variational setting, closely related to the notion of weak distributional solutions.

Potential estimates in the non-variational case are the subject of (DKM2014). In that paper, the authors examine fully nonlinear elliptic equations

$$F(D^2u, x) = f \quad \text{in} \quad \Omega, \tag{2-12}$$

where F is uniformly elliptic and $f \in L^p(B_1)$. In this context, the appropriate notion of solution is the one of L^p-viscosity solution (CCKS1996). Technical aspects of the theory – including its very definition – rule out the case where $f \in L^1(\Omega)$, regardless of the dimension $d \geq 2$. Instead, the authors work in the range $p_0 < p < d$, where $d/2 < p_0 < d$ is the exponent associated with the Green's function estimates appearing in (FS1984).

The consequences of potential estimates for fully nonlinear equations are remarkable. In fact, if $f \in L^p(\Omega)$ with $p > d$, solutions to (2-12) are known to be of class $C^{1,\alpha}$, with $\alpha \in (0,1)$ satisfying

$$\alpha < \min\left\{\alpha_0, 1 - \frac{d}{p}\right\},$$

where $\alpha_0 \in (0,1)$ is the exponent in the Krylov-Safonov theory available for $F = 0$; see (S1997). It is also known that $C^{1,\alpha}$-regularity is no longer available for (2-12) in case $p < d$. The fundamental question arising in this scenario concerns the *regularity of Du in the Escauriaza range $p_0 < p < d$.*

In (S1997), the author imposes an oscillation control on $F(M,\cdot)$ with respect to its fixed-coefficients counterpart and proves regularity estimates for the solutions in $W^{1,q}(\Omega)$, for $p_0 < p < d$, for every

$$q < p^* := \frac{pd}{d-p},$$

with $d^* := +\infty$. Meanwhile, the existence of a gradient in the classical sense, or any further information on its degree of smoothness, was not available in the $p < d$ setting.

In (DKM2014) the authors consider L^p-viscosity solutions to (2-12), with $f \in L^p(\Omega)$, for $p_0 < p < d$. In this context, they prove the local boundedness of Du in terms of a p-variant of the (linear) Riesz potential. In addition, the authors derive continuity of the gradient, with an explicit modulus of continuity. Finally, they obtain a borderline condition on f, once again involving Lorentz spaces. In fact, if $f \in L^{d,1}(\Omega)$, then $u \in C^1(\Omega)$.

The reasoning in (DKM2014) involves the excess of the gradient vis-a-vis its average and a decay rate for this quantity. However, in the context of viscosity solutions, energy estimates are not available as a starting point for the argument. Instead, the authors cleverly resort to Święch's $W^{1,q}$-estimates and prove a decay of the excess at an initial scale. An involved iteration scheme builds upon the natural scaling of the operator and unlocks the main building blocks of the argument.

2.3 Technical preliminaries and main assumptions

This section details our assumptions and gathers basic notions and facts used throughout this thesis. We start by putting forward the former.

2.3.1 Main assumptions

For completeness, we proceed by defining the extremal Pucci operators $\mathcal{P}^{\pm}_{\lambda,\Lambda} : S(d) \to \mathbb{R}$.

Definition 2.1 (Pucci extremal operators) *Let $0 < \lambda \leq \Lambda$. For $M \in S(d)$ denote with $\lambda_1, \ldots, \lambda_d$ its eigenvalues. We define the Pucci extremal operator $\mathcal{P}^{+}_{\lambda,\Lambda} : S(d) \to \mathbb{R}$ as*

$$\mathcal{P}^{+}_{\lambda,\Lambda}(M) := \Lambda \sum_{\lambda_i > 0} \lambda_i + \lambda \sum_{\lambda_i < 0} \lambda_i.$$

Similarly, we define the Pucci extremal operator $\mathcal{P}^{-} : S(d) \to \mathbb{R}$ as

$$\mathcal{P}^{-}_{\lambda,\Lambda}(M) := \lambda \sum_{\lambda_i > 0} \lambda_i + \Lambda \sum_{\lambda_i < 0} \lambda_i.$$

A 1 (Structural condition) *Let $\omega : [0,+\infty) \to [0,+\infty)$ be a modulus of continuity, and fix $\gamma > 0$. We suppose the operator F satisfies*

$$\begin{aligned}\mathcal{P}^-_{\lambda,\Lambda}(M-N) - \gamma|p-q| - \omega(|r-s|) &\leq F(M,p,r,x) - F(N,q,s,x)\\ &\leq \mathcal{P}^+_{\lambda,\Lambda}(M-N) + \gamma|p-q| + \omega(|r-s|),\end{aligned}$$

for every (M,p,r) and (d,q,s) in $S(d)\times\mathbb{R}^d\times\mathbb{R}$, and every $x\in\Omega\setminus\mathcal{N}$. Also, $F = F(M,p,r,x)$ is non-decreasing in r and $F(0,0,0,x) = 0$.

Our next assumption sets the integrability of the right-hand side f.

A 2 (Integrability of the right-hand side) *We suppose $f \in L^p(B_1)$, for $p > p_0$, where $d/2 < p_0 < d$ is the exponent such that the ABP maximum principle holds for solutions to uniformly elliptic equations $F = f$ provided $f \in L^p$, with $p > p_0$.*

We continue with an assumption on the oscillation of F on x. To that end, consider

$$\beta(x,y) := \sup_{M\in S(d)\setminus\{0\}} \frac{|F(M,0,0,x) - F(M,0,0,y)|}{\|M\|}.$$

We proceed with a smallness condition on $\beta(\cdot,y)$, uniformly in $y\in B_1$.

A 3 (Oscillation control) *For every $y\in\Omega$, we have*

$$\sup_{B_r(y)\subset\Omega} \fint_{B_r(y)} \beta(x,y)^p \mathrm{d}x \leq \theta^p,$$

where $0<\theta\ll 1$ is a small parameter we choose further in the paper.

We close this section with a remark on the modulus of continuity ω appearing in Assumption A1. For any $v\in C(B_1)\cap L^\infty(B_1)$ we notice that $\omega(|v(x)|) \leq C$ for some $C>0$, perhaps depending on the L^∞-norm of v. Hence

$$\left(\int_{B_1} \omega(|v(x)|)^p \mathrm{d}x\right)^{\frac{1}{p}} \leq C.$$

This information will be useful when estimating certain quantities in L^p-spaces appearing further in the work.

2.3.2 Preliminaries

In the sequel, we introduce the basics of L^p-viscosity solutions, mainly focusing on the properties we use in our arguments. We start with the definition of L^p-viscosity solutions for (1-1).

Definition 2.2 (L^p-viscosity solution) *Let $F = F(M, p, r, x)$ be nondecreasing in r and $f \in L^p(B_1)$ for $p > d/2$. We say that $u \in C(\Omega)$ is an L^p-viscosity subsolution to $F = f$ if for every $\phi \in W^{2,p}_{\text{loc}}(\Omega)$, $\varepsilon > 0$ and open subset $\mathcal{U} \subset \Omega$ such that*

$$F(D^2\phi(x), D\phi(x), \phi(x), x) - f(x) \geq \varepsilon$$

almost everywhere in $\mathcal{U}$, then $u - \phi$ cannot have a local maximum in $\mathcal{U}$. We say that $u \in C(\Omega)$ is an L^p-viscosity supersolution to $F = f$ if for every $\phi \in W^{2,p}_{\text{loc}}(\Omega)$, $\varepsilon > 0$ and open subset $\mathcal{U} \subset \Omega$ such that

$$F(D^2\phi(x), D\phi(x), \phi(x), x) - f(x) \leq -\varepsilon$$

almost everywhere in $\mathcal{U}$, then $u - \phi$ cannot have a local minimum in $\mathcal{U}$. We say that $u \in C(\Omega)$ is an L^p-viscosity solution to $F = f$ if it is both an L^p-sub and an L^p-supersolution to $F = f$.

Although the definition of L^p-viscosity solutions requires $p > d/2$, the appropriate range for the integrability of the data is indeed $p > p_0 > d/2$, as most results in the theory are available only in this setting. See, for instance, (CCKS1996). For further reference, we recall a result on the twice-differentiability of L^p-viscosity solutions.

Lemma 2.1 (Twice-differentiability) *Let $u \in C(\Omega)$ be an L^p-viscosity solution to* (1-1). *Suppose Assumptions A1 and A2 are in force. Then u is twice differentiable almost everywhere in Ω. Moreover, its pointwise derivatives satisfy the equation almost everywhere in Ω.*

For the proof of Lemma 2.1, see (CCKS1996, Theorem 3.6). In what follows, we present a lemma relating L^p-viscosity solutions to $F = f$ with equations governed by the extremal Pucci operators.

Lemma 2.2 *Suppose Assumption A1 and A2 and are in force. Suppose further that $u \in C(\Omega)$ is twice differentiable almost everywhere in Ω. Then u is an L^p-viscosity subsolution* [resp. supersolution] *of* (1-1) *if and only if* *i. we have*

$$F(D^2u(x), Du(x), u(x), x) \leq f(x)$$
$$[\text{resp. } F(D^2u(x), Du(x), u(x), x) \geq f(x)]$$

almost everywhere in Ω*, and*

ii. whenever $\phi \in W^{2,p}_{\text{loc}}(\Omega)$ *and* $u - \phi$ *has a local maximum* [resp. minimum] *at* x^**, then*

$$\operatorname*{ess\,lim\,inf}_{x \to x^*} \left(\mathcal{P}^- \left(D^2(u-\phi)(x) \right) - \gamma \left| D(u-\phi)(x) \right| \right) \geq 0$$
$$[\text{resp.} \quad \operatorname*{ess\,lim\,sup}_{x \to x^*} \left(\mathcal{P}^+ \left(D^2(u-\phi)(x) \right) + \gamma \left| D(u-\phi)(x) \right| \right) \leq 0].$$

For the proof of Lemma 2.2, we refer the reader to (S1997, Lemma 1.5). We are interested in a consequence of Lemma 2.2 that allows us to relate the solutions of $F(D^2u, Du, u, x) = f$ with the equation $F(D^2u, 0, 0, x) = \tilde{f}$, for some $\tilde{f} \in L^p(\Omega)$. This is the content of the next corollary.

Corollary 2 *Let* $u \in C(\Omega)$ *be an* L^p*-viscosity solution to* (1-1). *Suppose A1 and A2 hold. Define* $\tilde{f} : \Omega \to \mathbb{R}$ *as*

$$\tilde{f}(x) := F(D^2u(x), 0, 0, x).$$

If $\tilde{f} \in L^p(\Omega)$*, then* u *is an* L^p*-viscosity solution of*

$$F(D^2u, 0, 0, x) = \tilde{f} \quad in \quad \Omega. \tag{2-13}$$

Proof. We only prove that u is an L^p-viscosity subsolution to (2-13), as the case of supersolutions is analogous. Notice the proof amounts to verify the conditions in items *i.* and *ii.* of Lemma 2.2.

Because u solves (1-1) in the L^p-viscosity sense, Lemma 2.1 implies it is twice differentiable almost everywhere in Ω. Hence, the definition of $\tilde{f}$ ensures

$$F(D^2u(x), 0, 0, x) \leq \tilde{f}(x)$$

almost everywhere in Ω, which verifies item *i.* in Lemma 2.2.

To address item *ii.*, we resort to Lemma 2.2 in the opposite direction. Let $\phi \in W^{2,p}_{\text{loc}}(\Omega)$ and suppose $x^* \in \Omega$ is a point of maximum for $u - \phi$. Since u is an L^p-viscosity solution to (1-1), that lemma ensures that

$$\operatorname*{ess\,lim\,inf}_{x \to x^*} \left(\mathcal{P}^- \left(D^2(u-\phi)(x) \right) - \gamma \left| D(u-\phi)(x) \right| \right) \geq 0.$$

Therefore, item *ii.* also follows and the proof is complete. ∎

We also use the truncated Riesz potential of f. In fact, we consider its L^p-variant, introduced in (DKM2014). To be precise, given $f \in L^p(\Omega)$, we define its (truncated) Riesz potential $\mathbf{I}_p^f(x,r)$ as

$$\mathbf{I}_p^f(x,r) := \int_0^r \left(\fint_{B_\rho(x)} |f(y)|^p \mathrm{d}y \right)^{\frac{1}{p}} \mathrm{d}\rho.$$

In case $p = 1$ we recover the usual truncated Riesz potential.

We proceed by stating Theorems 1.2 and 1.3 in (DKM2014).

Proposition 2.1 (Daskalopoulos-Kuusi-Mingione I) *Let $u \in C(\Omega)$ be an L^p-viscosity solution to*

$$F(D^2u, x) = f \quad in \quad B_1.$$

Suppose Assumptions A1 and A2 are in force. Then there exists θ_1 such that, if Assumption A3 holds for $\theta \equiv \theta_1$, one has

$$|Du(x)| \leq C \left[\mathbf{I}_p^f(x,r) + \left(\fint_{B_r(x)} |Du(y)|^q \, \mathrm{d}y \right)^{\frac{1}{q}} \right]$$

for every $x \in \Omega$ and $r > 0$ with $B_r(x) \subset \Omega$, for some universal constant $C > 0$.

Proposition 2.2 (Daskalopoulos-Kuusi-Mingione II) *Let $u \in C(\Omega)$ be an L^p-viscosity solution to*

$$F(D^2u, x) = f \quad in \quad \Omega.$$

Suppose Assumptions A1 and A2 are in force. Suppose further that $\mathbf{I}_p^f(x,r) \to 0$ as $r \to 0$, uniformly in x. Then there exists θ_2 such that, if Assumption A3 holds for $\theta \equiv \theta_2$, Du is continuous. In addition, for $\Omega' \Subset \Omega'' \Subset \Omega$, and any $\delta \in (0,1]$, one has

$$|Du(x) - Du(y)| \leq C \left(\|Du\|_{L^\infty(\Omega'')} |x-y|^{\alpha(1-\delta)} + \sup_{z \in \{x,y\}} \mathbf{I}_p^f \left(z, 4|x-y|^\delta \right) \right),$$

for every $x, y \in \Omega'$, where $C = C(d, p, \lambda, \Lambda, \gamma, \omega, \Omega', \Omega'')$ and $\alpha = \alpha(d, p, \lambda, \Lambda)$.

For the proofs of Propositions 2.1 and 2.2, we refer the reader to (DKM2014, Theorem 1.3). We close this section by including (S1997, Święch)'s $W^{1,p}$-regularity result.

Proposition 2.3 ($W^{1,q}$-regularity estimates) *Let $u \in C(\Omega)$ be an L^p-viscosity solution to* (1-1). *Suppose Assumptions A1 and A2 are in force.*

There exists $0 < \overline{\theta} \ll 1$ such that, if Assumption A3 holds with $\theta \equiv \overline{\theta}$, then $u \in W^{1,q}_{\rm loc}(\Omega)$ for every $1 < q < p^*$, where

$$p^* := \frac{pd}{d-p}, \qquad \textit{and} \qquad d^* = +\infty.$$

Also, for $\Omega' \Subset \Omega$, there exists $C = C(d, \lambda, \Lambda, \gamma, \omega, q, \mathrm{diam}(\Omega'), \mathrm{dist}(\Omega', \partial\Omega))$ such that

$$\|u\|_{W^{1,q}(\Omega')} \leq C \left(\|u\|_{L^\infty(\partial\Omega)} + \|f\|_{L^p(\Omega)}\right).$$

The former result plays an important role in our argument since it allows us to relate the operator $F(M, p, r, x)$ with $F(M, 0, 0, x)$.

2.4 Proof of Theorems 1.1 and 1.2

In the sequel, we detail the proofs of Theorems 1.1 and 1.2. Resorting to a covering argument, we work in the unit ball B_1 instead of Ω. As we described before, the strategy is to show that L^p-viscosity solutions to (1-1) are also L^p-viscosity solutions to

$$G(D^2u, x) = g \quad \text{in} \quad B_1.$$

Then verify that $G : S(d) \times B_1 \setminus \mathcal{N} \to \mathbb{R}$ and $g \in L^p(B_1)$ are in the scope of (DKM2014). More precisely, satisfying the conditions in Theorems 1.2 and 1.3 in that paper. We continue with a proposition.

Proposition 2.4 *Let $u \in C(B_1)$ be an L^p-viscosity solution to* (1-1). *Suppose Assumptions A1 and A2 are in force. Suppose further that Assumption A3 holds with $\theta \equiv \overline{\theta}$, where $\overline{\theta}$ is the parameter from Proposition 2.3. Then u is an L^p-viscosity solution for*

$$F(D^2u, 0, 0, x) = \tilde{f} \quad \textit{in} \quad B_{9/10},$$

where $\tilde{f} \in L^p_{\rm loc}(B_1)$ and there exists $C > 0$ such that

$$\left\|\tilde{f}\right\|_{L^p(B_{9/10})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right).$$

Proof. We split the proof into two steps.

Step 1 - We start by applying Proposition 2.3 to the L^p-viscosity solutions to (1-1). By taking θ in Assumption A3 such that $\theta \equiv \overline{\theta}$, we get $u \in W^{1,q}_{\rm loc}(B_1)$ and

$$\|Du\|_{L^q(B_{9/10})} \leq C \left(\|u\|_{L^\infty(\partial B_1)} + \|f\|_{L^p(B_1)}\right), \tag{2-14}$$

for some universal constant $C > 0$. Moreover, because u is an L^p-viscosity solution to (1-1), Lemma 2.1 ensures it is twice-differentiable almost everywhere in B_1. Define $\tilde{f} : B_1 \to \mathbb{R}$ as

$$\tilde{f}(x) := F(D^2u(x), 0, 0, x).$$

Step 2 - Resorting once again to Lemma 2.1, we get that

$$\tilde{f}(x) = F(D^2u(x), 0, 0, x) - F(D^2u(x), Du(x), u(x), x) + f(x),$$

almost everywhere in B_1. Ellipticity implies

$$\left|\tilde{f}(x)\right| \leq \gamma \left|Du(x)\right| + \omega\left(\left|u(x)\right|\right) + \left|f(x)\right|,$$

for almost every $x \in B_1$. Using (2-14), and noticing that one can always take $q > p$, we get $\tilde{f} \in L^p_{\text{loc}}(B_1)$, with

$$\left\|\tilde{f}\right\|_{L^p(B_{9/10})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right),$$

for some universal constant $C > 0$, also depending on p. A straightforward application of Corollary 2 completes the proof. ■

Proposition 2.4 is the main ingredient leading to Theorems 1.1 and 1.2. Once it is available, we proceed with the proof of those theorems.

Proof.[Proof of Theorem 1.1] For clarity, we split the proof into two steps.

Step 1 - Because of Proposition 2.4, we know that an L^p-viscosity solution to (1-1) is also an L^p-viscosity solution to

$$\tilde{F}(D^2u, x) = \tilde{f} \quad \text{in} \quad B_{9/10},$$

where

$$\tilde{F}(M, x) := F(M, 0, 0, x),$$

and $\tilde{f}$ is defined as in Proposition 2.4. To conclude the proof, we must ensure that $\tilde{F}$ satisfies the conditions in Proposition 2.1.

Step 2 - One easily verifies that $\tilde{F}$ satisfies a (λ, Λ)-ellipticity condition, inherited from the original operator F. It remains to control the oscillation of $\tilde{F}(M, x)$ vis-a-vis its fixed-coefficient counterpart, $\tilde{F}(M, x_0)$, for $x_0 \in B_{9/10}$.

Because

$$\tilde{F}(M, x) - \tilde{F}(M, x_0) = F(M, 0, 0, x) - F(M, 0, 0, x_0),$$

one may take $\theta \equiv \theta_1$ in Assumption 3 to ensure that $\tilde{F}$ satisfies the conditions in Proposition 2.1. Taking

$$\theta^* := \min\left(\theta_1,\ \overline{\theta}\right)$$

and applying Proposition 2.1 to u, the proof is complete. ■

The proof of Theorem 1.2 follows word for word the previous one, except for the choice of $\theta^* := \min\left(\theta_2,\ \overline{\theta}\right)$, and is omitted.

3
Intrinsic s-Harmonic maps with free boundary

3.1
Preliminaries

Lemma 3.1 (First variation of the energy) *Let $u : \Omega \to \mathbb{C}_k$ be a minimizer for* (1-3). *Then u satisfies*

$$\operatorname{div}\left(x_{d+1}^a u Du\right) = x_{d+1}^a |Du|^2 \tag{3-1}$$

in the distributional sense. In addition, let $\xi \in C_0^\infty(\Omega, \mathbb{R}^{d+1})$ be such that $\xi(x', 0) \subset \mathbb{R}^d \times \{0\}$. Then,

$$0 = a \int_\Omega x_{d+1}^{a-1} \xi_{d+1} |Du|^2 \mathrm{d}x + \int_\Omega x_{d+1}^a \left(|Du|^2 \operatorname{div}\xi - 2DuD\xi \cdot Du\right) \mathrm{d}x. \tag{3-2}$$

Proof. The proof is standard and follows along the general lines of (S1996, Chapter 2). We consider appropriate variations of u and explore its minimality. First, let $\varphi \in C_0^\infty(\Omega)$; hence, $u_t := (1 + t\varphi)u$ maps Ω into $\mathbb{C}_k$, for $|t| \ll 1$. The minimality of u yields

$$\int_\Omega x_{d+1}^a \left[\varphi |Du|^2 + u \left\langle Du, D\varphi \right\rangle\right] \mathrm{d}x = 0,$$

which is tantamount to (3-1).

To verify (3-2), we start with a vector field $\xi \in C_0^\infty(\Omega, \mathbb{R}^{d+1})$ satisfying $\xi(x', 0) \subset \mathbb{R}^d \times \{0\}$. For $0 < |t| \ll 1$, and $x \in \Omega$, we define

$$\Psi_t(x) := x + t\xi(x).$$

Denote with Φ_t the inverse of Ψ_t and consider the variation

$$u_t := u \circ \Phi_t.$$

Once again, the minimality of u gives

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \int_\Omega (x_{d+1} + t\xi_{d+1})^a |Du_t|^2 \mathrm{d}x \\ &= a \int_\Omega x_{d+1}^{a-1} \xi_{d+1} |Du|^2 \mathrm{d}x + \int_\Omega x_{d+1}^a \left(|Du|^2 \operatorname{div}\xi - 2DuD\xi \cdot Du\right) \mathrm{d}x \end{aligned}$$

and concludes the proof. ■

Remark 1 *In (3-2), the notation $DuD\xi \cdot Du$ stands for the inner product of $DuD\xi$ and Du as vectors in $\mathbb{R}^{d+m}$. That is,*

$$DuD\xi \cdot Du = \sum_{i=1}^{m} \sum_{j,k=1}^{d+1} \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} \frac{\partial \xi_j}{\partial x_k}.$$

3.2 Monotonicity Formulas

Remark 2 *In order to extend the case $\mathbb{R}^{d+1}_+$ to a general case in $\mathbb{R}^{d+1}$ we can argue through an extended function $\tilde{u} : \mathbb{R}^{d+1} \to \mathbb{C}_k$ from $u : \mathbb{R}^{d+1}_+ \to \mathbb{C}_k$, for example taking his even reflection with respect to $\partial\mathbb{R}^{d+1}_+$*

$$\tilde{u}(x', x_{d+1}) := \begin{cases} u(x', x_{d+1}), & \text{if } x_{d+1} \geq 0 \\ u(x', -x_{d+1}), & \text{if } x_{d+1} < 0 \end{cases} \tag{3-3}$$

In this case $\tilde{u}$ still being s-harmonic and minimizes

$$\mathbf{E}_s\Big(u, \widetilde{\Omega}\Big) = \frac{1}{2} \int_{\widetilde{\Omega}} |x_{d+1}|^a |\nabla u(x)|^2 \, dx = 2\, \mathbf{E}_s\Big(u, \Omega\Big)$$

where $\widetilde{\Omega} = \Omega \cup \partial^0\Omega \cup \Big\{(x', -x_{d+1})/x \in \Omega\Big\}$.

Lemma 3.2 *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a bounded open set and suppose $u \in H^1_s(\Omega; \mathbb{C}_k)$ is a minimiser of $\mathbf{E}_s(u, \Omega)$. If $x_0 \in \Omega$ and $r \in (0, \operatorname{dist}(x_0, \partial\Omega))$, the following identity holds:*

$$\frac{d}{dr}\left[\frac{\mathbf{E}_s\Big(u, B_r(x_0)\Big)}{r^{d+a-1}}\right] = \frac{1}{r^{d+a-1}} \int_{\partial B_r(x_0)} |x_{d+1}|^a \left|\frac{\partial u}{\partial |x - x_0|}\right|^2 dS_x - \frac{a|x_0^{(d+1)}|}{2r^{d+a}} \int_{B_r(x_0)} |x_{d+1}|^{a-1} |\nabla u|^2 \, dx. \tag{3-4}$$

Proof. From (3-2) we have

$$\int_{B_r(x_0)} \sum_{i,j=0}^{d+1} |x_{d+1}|^a \Big(|\nabla u|^2 \delta_{i,j} - 2(\partial_i u \bullet \partial_j u) \Big) \partial_j X_i + a \int_{B_r(x_0)} |x_{d+1}|^{a-1} |\nabla u|^2 x_{d+1} = 0. \tag{3-5}$$

Like in Simon (S1996), if $\zeta \in C^\infty_c(B_r(x_0))$ and $\vec{a} = (a^{(1)}, \ldots, a^{(d)})$

$$\int_{B_r(x_0)} \sum_{j=1}^{d} a^{(j)} \partial_j \zeta + \Psi = 0 \implies \int_{B_r(x_0)} \sum_{j=1}^{d} a^{(j)} \partial_j \zeta = -\Psi = \int_{\partial B_r(x_0)} \eta \bullet \vec{a}\zeta - \Psi. \tag{3-6}$$

Set $\tilde{x}_0 = \Big(x_0',\, |x_0^{(d+1)}| \Big)$, then

$$\begin{aligned}
&\int_{B_r(x_0)} \sum_{i,j=0}^{d+1} |x_{d+1}|^a \Big(|\nabla u|^2 \delta_{i,j} - 2(\partial_i u \bullet \partial_j u) \Big) \partial_j X_i \\
&\qquad = \int_{\partial B_r(x_0)} \sum_{i,j=1}^{d+1} \frac{x_j - \tilde{x}_0^{(j)}}{r} \Big[|x_{d+1}|^a \Big(|\nabla u|^2 \delta_{i,j} - 2(\partial_i u \bullet \partial_j u) \Big) \Big] X_i \\
&\qquad\quad - a \int_{B_r(x_0)} |x_{d+1}|^{a-1} |\nabla u|^2 x_{d+1}.
\end{aligned}$$

If $X = x - \tilde{x}_0$

$$\begin{aligned}
&(d-1) \int_{B_r(x_0)} |x_{d+1}|^a |\nabla u|^2 \, dx \\
&\quad = \int_{B_r(x_0)} \sum_{i,j=1}^{d+1} |x_{d+1}|^a \Big(|\nabla u|^2 \delta_{i,j} - 2(\partial_i u \bullet \partial_j u) \Big) \delta_{i,j} \, dx \\
&\quad = \int_{\partial B_r(x_0)} \sum_{i,j=0}^{d+1} \frac{x_j - \tilde{x}_0^{(j)}}{r} \Big[\, |x_{d+1}|^a \Big(|\nabla u|^2 \delta_{i,j} - 2(\partial_i u \bullet \partial_j u) \Big) \Big] (x_i - \tilde{x}_0^{(i)}) \, dS_x \\
&\qquad\qquad - a \int_{B_r(x_0)} |x_{d+1}|^{a-1} \Big(x_{d+1} - \tilde{x}_0^{(d+1)} \Big) |\nabla u|^2 \, dx \\
&\quad = \int_{\partial B_r(x_0)} \left[\sum_{i=1}^{d+1} \frac{\left| x_i - \tilde{x}_0^{(i)} \right|^2}{r} |x_{d+1}|^a |\nabla u|^2 - 2r \sum_{i,j=0}^{d+1} \Big(\frac{x_j - \tilde{x}_0^{(j)}}{r} \partial_i u \Big) \bullet \Big(\frac{x_j - \tilde{x}_0^{(j)}}{r} \partial_j u \Big) \right] dS_x \\
&\qquad\qquad - a \int_{B_r(x_0)} |x_{d+1}|^a |\nabla u|^2 \, dx + a \tilde{x}_0^{(d+1)} \int_{B_r(x_0)} |x_{d+1}|^{a-1} |\nabla u|^2 \, dx \\
&\quad = \int_{\partial B_r(x_0)} |x_{d+1}|^a \left[\frac{|x - x_0|^2}{r} |\nabla u|^2 - 2r \left| \frac{x - x_0}{r} \bullet \nabla u \right|^2 \right] dS_x \\
&\qquad\qquad - a \int_{B_r(x_0)} |x_{d+1}|^a |\nabla u|^2 \, dx + a \tilde{x}_0^{(d+1)} \int_{B_r(x_0)} |x_{d+1}|^{a-1} |\nabla u|^2 \, dx \\
&\quad = r \int_{\partial B_r(x_0)} |x_{d+1}|^a \left[|\nabla u|^2 - 2 \left| \frac{\partial u}{\partial |x - x_0|} \right|^2 \right] dS_x \\
&\qquad\qquad - a \int_{B_r(x_0)} |x_{d+1}|^a |\nabla u|^2 + a \tilde{x}_0^{(d+1)} \int_{B_r(x_0)} |x_{d+1}|^{a-1} |\nabla u|^2 \, dx
\end{aligned}$$

Then, we have that

$$\begin{aligned}
(d + a - 1) \int_{B_r(x_0)} |x_{d+1}|^a |\nabla u|^2 = r \int_{\partial B_r(x_0)} |x_{d+1}|^a \left[|\nabla u|^2 - 2 \left| \frac{\partial u}{\partial |x - x_0|} \right|^2 \right] \\
+ a \tilde{x}_0^{(d+1)} \int_{B_r(x_0)} |x_{d+1}|^{a-1} |\nabla u|^2
\end{aligned} \tag{3-7}$$

and, since

$$\frac{d}{dr} \mathbf{E}_s \Big(u, B_r(x_0) \Big) = \frac{1}{2} \int_{\partial B_r(x_0)} |x_{d+1}|^a |\nabla u|^2 \, dS_x$$

We get

$$\begin{aligned}\frac{d}{dr}\left[\frac{\mathbf{E}_s\Big(u, B_r(x_0)\Big)}{r^{d+a-1}}\right] &= \frac{-(d+a-1)}{2r^{d+a}}\int_{B_r(x_0)} |x_{d+1}|^a|\nabla u|^2\,dx \\ &\quad + \frac{1}{2r^{d+a-1}}\int_{\partial B_r(x_0)} |x_{d+1}|^a|\nabla u|^2\,dS_x \\ &= \frac{-r}{2r^{d+a}}\int_{\partial B_r(x_0)} |x_{d+1}|^a\left[\,|\nabla u|^2 - 2\left|\frac{\partial u}{\partial|x-x_0|}\right|^2\,\right] \\ &\quad - \frac{a\tilde{x}_0^{(d+1)}}{2r^{d+a}}\int_{B_r(x_0)} |x_{d+1}|^{a-1}|\nabla u|^2\,dx \\ &\quad + \frac{1}{2r^{d+a-1}}\int_{\partial B_r(x_0)} |x_{d+1}|^a|\nabla u|^2\,dS_x\end{aligned}$$

Then we get the equation (3-4). ■

Remark 3 *By integrating the formula (3-4), and assuming $x_0 \in \partial\mathbb{R}^{d+1}_+$ we get*

$$\frac{\mathbf{E}_s\Big(u, B_\rho(x_0)\Big)}{\rho^{d+a-1}} - \frac{\mathbf{E}_s\Big(u, B_\sigma(x_0)\Big)}{\sigma^{d+a-1}} = \int_{B_\rho(x_0)\setminus B_\sigma(x_0)} \frac{x^a_{d+1}\left|\nabla u \bullet \eta\right|^2}{|x-x_0|^{d+a-1}}\,dx. \tag{3-8}$$

Now we have the following proposition, that corresponds with (R2018, Lemma 4.5) and (MPS2021, Proposition 2.17):

Proposition 3.3 *Let $\Omega \subseteq \mathbb{R}^{d+1}_+$ be a bounded open set and suppose $u \in H^1_s(\Omega;\mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u,\Omega)$. For every $x_0 \in \partial^0\Omega$ we have*

$$\mathbf{\Theta}_s(u,x_0,\rho) - \mathbf{\Theta}_s(u,x_0,\sigma) = \int_{B^+_\rho(x_0)\setminus B^+_\sigma(x_0)} \frac{x^a_{d+1}\left|\nabla u \bullet \eta\right|^2}{|x-x_0|^{d+a-1}}\,dx \tag{3-9}$$

for every $0<\sigma<\rho<\operatorname{dist}(x_0,\Omega^c)$, the limit

$$\mathbf{\Theta}_s(u,x_0) := \lim_{r\downarrow 0}\mathbf{\Theta}_s(u,x_0,r) \tag{3-10}$$

exists, and the function $\mathbf{\Theta}_v : \partial^0\Omega \to [0,\infty)$ is upper semicontinuous. In addition, for every $x_0 \in \partial^0\Omega$,

$$\mathbf{\Theta}_s(u,x_0,\rho) - \mathbf{\Theta}_s(u,x_0) = \int_{B^+_\rho(x_0)} \frac{x^a_{d+1}\left|\nabla u \bullet \eta\right|^2}{|x-x_0|^{d+a-1}}\,dx \tag{3-11}$$

for every $0<\rho<\operatorname{dist}(x_0,\Omega^c)$.

Proof. It follows directly from Lemma 3.2, Remark 3 and using a reflection argument like in Remark 2. ■

Another monotonicity result is corresponding to (R2018, Lemma 4.7):

Lemma 3.4 *Let $\Omega \subseteq \mathbb{R}^{d+1}_+$ be a bounded open set and suppose $u \in H^1_s(\Omega; \mathbb{C}_k)$ is a minimiser of $\mathbf{E}_s(u, \Omega)$. Fixing a ball $B_{R_0}(y)$ with $\overline{B_{R_0}(y)} \subset \Omega$, we have*

$$e^{a\rho\xi}\rho^{1-d}\,\mathbf{E}_s\Big(u, B_\rho(y)\Big) - e^{a\sigma\xi}\sigma^{1-d}\,\mathbf{E}_s\Big(u, B_\sigma(y)\Big) \geq \int_{B_\rho(y)\setminus B_\sigma(y)} \frac{e^{a|x-y|\xi}\, x^a_{d+1}\left|\nabla u \bullet \eta\right|^2}{|x-x_0|^{d-1}}\,dx \tag{3-12}$$

for every $0 < \sigma < \rho < R_0$, where $\xi = \Big(y_{d+1} - R_0\Big)^{-1}$.

Proof. We have that (the equation (3-4))

$$\frac{d}{dr}\left[\frac{\mathbf{E}_s\Big(u, B_r(y)\Big)}{r^{d+a-1}}\right] = \int_{\partial B_r(y)} \frac{x^a_{d+1}\,|\nabla u \bullet \eta|}{r^{d+a+1}}\,dx - \frac{a y_{d+1}}{2r^{d+a}}\int_{B_r(y)} x^{a-1}_{d+1}\,|\nabla u|^2\,dx \tag{3-13}$$

$$\begin{aligned}
\frac{d}{dr}\left[\frac{e^{ar\xi}}{r^{d-1}}\,\mathbf{E}_s\Big(u, B_r(y)\Big)\right] &= \frac{d}{dr}\left[e^{ar\xi}r^a\,\frac{\mathbf{E}_s\Big(u, B_r(y)\Big)}{r^{d+a-1}}\right] \\
&= \Big(ae^{ar\xi}r^{a-1} + a\xi\, e^{ar\xi}r^a\Big)\,\frac{\mathbf{E}_s\Big(u, B_r(y)\Big)}{r^{d+a-1}} \\
&\quad + e^{ar\xi}r^a\left(\int_{\partial B_r(y)} \frac{x^a_{d+1}\,|\nabla u \bullet \eta|}{r^{d+a+1}}\,dx - \frac{a y_{d+1}}{2r^{d+a}}\int_{B_r(y)} x^{a-1}_{d+1}\,|\nabla u|^2\,dx\right) \\
&= \int_{\partial B_r(y)} \frac{e^{ar\xi}\,x^a_{d+1}\,|\nabla u \bullet \eta|}{r^{d+1}}\,dx \\
&\quad + \frac{ae^{ar\xi}}{r^d}\left[(1+r\xi)\,\mathbf{E}_s\Big(u, B_r(y)\Big) - \frac{y_{d+1}}{2}\int_{B_r(y)} x^{a-1}_{d+1}\,|\nabla u|^2\,dx\right]
\end{aligned}$$

So,

$$\frac{d}{dr}\left[\frac{e^{ar\xi}}{r^{d-1}}\,\mathbf{E}_s\Big(u, B_r(y)\Big)\right] = \int_{\partial B_r(y)} \frac{e^{ar\xi}\,x^a_{d+1}\,|\nabla u \bullet \eta|}{r^{d+1}}\,dx + \frac{ae^{ar\xi}}{r^d}\,\Psi(y,r) \tag{3-14}$$

where

$$\Psi(y,r) = (1+r\xi)\,\mathbf{E}_s\Big(u, B_r(y)\Big) - \frac{y_{d+1}}{2}\int_{B_r(y)} x^{a-1}_{d+1}\,|\nabla u|^2\,dx$$

We have that

$$\begin{aligned}
\frac{y_{d+1}}{2}\int_{B_r(y)} x^{a-1}_{d+1}|\nabla u|^2\,dx &\leq \frac{1}{2}\int_{B_r(y)} \Big(x^a_{d+1} + r x^{a-1}_{d+1}\Big)\,|\nabla u|^2\,dx \\
&= \mathbf{E}_s\Big(u, B_r(y)\Big) + \frac{r}{2}\int_{B_r(y)} x^{a-1}_{d+1}|\nabla u|^2\,dx \\
&< \mathbf{E}_s\Big(u, B_r(y)\Big) + r\xi\,\mathbf{E}_s\Big(u, B_r(y)\Big) \\
&\qquad - \frac{r\xi(R_0 - r)}{2}\int_{B_r(y)} x^{a-1}_{d+1}|\nabla u|^2\,dx
\end{aligned}$$

because $\xi^{-1} = y_{d+1} - R_0 < x_{d+1} + r - R_0$, then $\Psi(y, y) > 0$. So, in the case $a > 0$ we obtain from (3-14) that

$$\frac{d}{dr}\left[\frac{e^{ar\xi}}{r^{d-1}}\,\mathbf{E}_s\Big(u, B_r(y)\Big)\right] > \int_{\partial B_r(y)} \frac{e^{ar\xi}\, x_{d+1}^a\, |\nabla u \bullet \eta|}{r^{d+1}}\, dx \tag{3-15}$$

Similarly,

$$\begin{aligned}\frac{y_{d+1}}{2}\int_{B_r(y)} x_{d+1}^{a-1}|\nabla u|^2\, dx &\geq \frac{1}{2}\int_{B_r(y)} \left(x_{d+1}^a - r x_{d+1}^{a-1}\right)|\nabla u|^2\, dx \\ &= \mathbf{E}_s\Big(u, B_r(y)\Big) - \frac{r}{2}\int_{B_r(y)} x_{d+1}^{a-1}|\nabla u|^2\, dx \\ &> \mathbf{E}_s\Big(u, B_r(y)\Big) - r\xi\, \mathbf{E}_s\Big(u, B_r(y)\Big) \\ &\quad + \frac{r\xi(R_0 - r)}{2}\int_{B_r(y)} x_{d+1}^{a-1}|\nabla u|^2\, dx\end{aligned}$$

what implies

$$\begin{aligned}\Psi(y, r) &< 2r\xi\, \mathbf{E}_s\Big(u, B_r(y)\Big) - \frac{r\xi(R_0 - r)}{2}\int_{B_r(y)} x_{d+1}^{a-1}|\nabla u|^2\, dx \\ &< \left[r\xi\,(y_{d+1} + r) - \frac{r\xi\,(R_0 - r)}{2}\right]\int_{B_r(y)} x_{d+1}^{a-1}|\nabla u|^2\, dx \\ &= r\xi\left(y_{d+1} + \frac{3r}{2} - \frac{R_0}{2}\right) \\ &< 0\end{aligned}$$

because $\xi < 0$ and $y_{d+1} + \frac{3r}{2} - \frac{R_0}{2} > 0$. Then, in the case $a < 0$ we also have from (3-14) the inequality (3-15). At the end we conclude (3-12) by integration from σ to ρ the inequality (3-15).

■

Lemma 3.5 *Let $\Omega \subseteq \mathbb{R}_+^{d+1}$ be a bounded open set and suppose $u \in H_s^1(\Omega; \mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u, \Omega)$. For every $x_0 \in \partial^0\Omega$ we have*

$$\frac{d}{dr}\left[\frac{1}{r^{d+a}}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x\right] = \frac{4}{r^{d+a}}\,\mathbf{E}_s\Big(u, B_r^+(x_0)\Big). \tag{3-16}$$

Proof. Let $\tilde{u} : \Omega \cup \partial^0\Omega \cup \{(x', -x_{d+1})/x \in \Omega\} \to \mathbb{C}_k$ the odd reflection of u respect to $\partial\mathbb{R}_+^{d+1}$, then

$$\frac{d}{dr}\left[\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x\right] = \frac{d}{dr}\left[\frac{r^d}{2}\int_{\partial B_1} \left(x_{d+1}^a |\tilde{u}|^2\right)\Big|_{x_0 + rx} dS_x\right]$$

$$= \frac{n}{r}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x + \frac{1}{2}\int_{\partial B_r(x_0)} \nabla\Big(x_{d+1}^a|\tilde u|^2\Big)\bullet \eta\, dS_x$$

$$= \frac{n}{r}\int_{\partial B_r(x_0)} x_{d+1}^a |u|^2\, dS_x + \frac{1}{2}\int_{\partial B_r(x_0)} \left[\begin{pmatrix} 0 \\ \vdots \\ 0 \\ a x_{d+1}^{a-1}\end{pmatrix} |u|^2 + x_{d+1}^a \nabla |u|^2\right]\bullet \eta\, dS_x$$

$$= \frac{n}{r}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x$$

$$+ \frac{1}{2}\int_{\partial B_r(x_0)} \left[a x_{d+1}^{a-1}\left(x_{d+1} - x_0^{(d+1)}\right)\frac{|\tilde u|^2}{r} + x_{d+1}^a \nabla|\tilde u|^2 \bullet \eta \right] dS_x$$

$$= \frac{n}{r}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x + \frac{a}{2r}\int_{\partial B_r(x_0)} x_{d+1}^a |\tilde u|^2\, dS_x + \frac{1}{2}\int_{\partial B_r(x_0)} x_{d+1}^a \nabla|\tilde u|^2\bullet\eta\, dS_x$$

By taking $\phi_k \in C_0^\infty(B_r(x_0))$ the approximation of the characteristic function in $B_r(x_0)$ and using (3-1), we have

$$\begin{aligned}
\int_{B_r^+(x_0)} x_{d+1}^a |\nabla u|^2\, dx &= \lim_{k\to\infty}\int_{B_r^+(x_0)} x_{d+1}^a |\nabla u|^2 \phi_k\, dx \\
&= \lim_{k\to\infty}\frac{1}{2}\int_{B_r^+(x_0)} \operatorname{div}\left(x_{d+1}^a \nabla |u|^2\right)\phi_k\, dx \\
&= \lim_{k\to\infty}\left[-\frac{1}{2}\int_{B_r^+(x_0)} \left\langle x_{d+1}^a \nabla|u|^2, \nabla\phi_k \right\rangle dx + \frac{1}{2}\int_{\partial B_r^+(x_0)} x_{d+1}^a \phi_k \nabla|u|^2\bullet\eta\, dS_x \right] \\
&= \lim_{k\to\infty}\left[\frac{1}{2}\int_{\partial B_r^+(x_0)} x_{d+1}^a \phi_k\, \nabla|u|^2\bullet\eta\, dS_x \right] \\
&= \frac{1}{2}\int_{\partial B_r^+(x_0)} x_{d+1}^a\, \nabla|u|^2\bullet\eta\, dS_x \\
&= \frac{1}{4}\int_{\partial B_r(x_0)} x_{d+1}^a\, \nabla|\tilde u|^2\bullet\eta\, dS_x
\end{aligned} \tag{3-17}$$

Then

$$\frac{d}{dr}\left[\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x\right] = \frac{d+a}{r}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x + 2\int_{B_r^+(x_0)} x_{d+1}^a |\nabla u|^2\, dx$$

and next, we get (3-16) by

$$\begin{aligned}
&\frac{d}{dr}\left[\frac{1}{r^{d+a}}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x\right] \\
&\quad= -\frac{d+a}{r^{d+a+1}}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x + \frac{1}{r^{d+a}}\frac{d}{dr}\left[\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2\, dS_x\right] \\
&\quad= \frac{2}{r^{d+a}}\int_{B_r^+(x_0)} x_{d+1}^a |\nabla u|^2\, dx.
\end{aligned}$$

■

Lemma 3.6 *Let $\Omega \subseteq \mathbb{R}^{d+1}_+$ be a bounded open set and suppose $u \in H^1_s(\Omega; \mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u, \Omega)$. For every $x_0 \in \partial^0\Omega$ we have*

$$\frac{d}{dr}\,\mathbf{N}_s(u, x_0, r) \geq 0, \tag{3-18}$$

where

$$\mathbf{N}_s(u, x_0, r) \int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x = 2r\mathbf{E}_s\Big(u, B_r^+(x_0)\Big). \tag{3-19}$$

Proof. From (3-19) we have

$$\log \mathbf{N}_s(u, x_0, r) = \log\left(\frac{2\mathbf{E}_s\Big(u, B_r^+(x_0)\Big)}{r^{d+a-1}}\right) - \log\left(\frac{1}{r^{d+a}}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x\right)$$

Using Lemmas 3.2 and 3.5 we obtain the identities

$$\begin{aligned}\frac{d}{dr}\log\left(\frac{2\mathbf{E}_s\Big(u, B_r^+(x_0)\Big)}{r^{d+a-1}}\right) &= \frac{1}{r^{d+a-1}}\int_{\partial B_r^+(x_0)} x_{d+1}^a \,|\,\nabla u \bullet \eta|^2 \, dS_x \left(\frac{\mathbf{E}_s\Big(u, B_r^+(x_0)\Big)}{r^{d+a-1}}\right)^{-1} \\ &= \frac{2r}{\mathbf{N}_s(u, x_0, r)}\int_{\partial B_r^+(x_0)} x_{d+1}^a \,|\,\nabla u \bullet \eta|^2 \, dS_x \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x\right)^{-1}\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dr}\log\left(\frac{1}{r^{d+a}}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x\right) &= \frac{4\mathbf{E}_s\Big(u, B_r^+(x_0)\Big)}{r^{d+a}}\left(\frac{1}{r^{d+a}}\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x\right)^{-1} \\ &= \frac{2\mathbf{N}_s(u, x_0, r)}{r},\end{aligned}$$

so, we get

$$\frac{d}{dr}\,\mathbf{N}_s(u, x_0, r) = 2r\int_{\partial B_r^+(x_0)} x_{d+1}^a \,|\,\nabla u \bullet \eta|^2 \, dS_x \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x\right)^{-1} - \frac{2\mathbf{N}_s^2(u, x_0, r)}{r} \tag{3-20}$$

Because (3-17) from Lemma 3.5's proof

$$\int_{B_r^+(x_0)} x_{d+1}^a |\nabla u|^2 \, dx = \frac{1}{2}\int_{\partial B_r^+(x_0)} x_{d+1}^a \,\nabla |u|^2 \bullet \eta \, dS_x = \int_{\partial B_r(x_0)} x_{d+1}^a \, u \nabla u \bullet \eta \, dS_x$$

By Cauchy-Schwarz inequality

$$\begin{aligned}4\mathbf{E}_s^2\Big(u, B_r^+(x_0)\Big) &= \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a \, u \nabla u \bullet \eta \, dS_x\right)^2 \\ &\leq \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a \,|\nabla u \bullet \eta|^2 dS_x\right)\left(\int_{\partial B_r^+(x_0)} x_{d+1}^a \,|u|^2 dS_x\right)\end{aligned}$$

then

$$2r \int_{\partial B_r^+(x_0)} x_{d+1}^a \, |\nabla u \bullet \eta|^2 \, dS_x \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x \right)^{-1}$$
$$\geq 8r \mathbf{E}_s^2\Big(u, B_r^+(x_0)\Big) \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a \, |u|^2 dS_x \right)^{-2}$$
$$= \frac{2\mathbf{N}_s^2(u, x_0, r)}{r}$$

So, we conclude

$$\frac{d}{dr} \mathbf{N}_s(u, x_0, r) \geq \frac{2\mathbf{N}_s^2(u, x_0, r)}{r} - \frac{2\mathbf{N}_s^2(u, x_0, r)}{r} = 0$$

■

3.3 Useful Estimates

Lemma 3.7 *Let $\Omega \subseteq \mathbb{R}_+^{d+1}$ be a bounded open set and suppose $u \in H_s^1(\Omega; \mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u, \Omega)$. For every $x_0 \in \partial^0\Omega$ we have*

$$\left(\frac{r}{R}\right)^{d+a+2\mathbf{N}_s(u,x_0,R)} \leq \frac{\int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x}{\int_{\partial B_R^+(x_0)} x_{d+1}^a |u|^2 \, dS_x} \leq \left(\frac{r}{R}\right)^{d+a+2\mathbf{N}_s(u,x_0,r)} \tag{3-21}$$

where $0 < r < R$ and $B_R(x_0)^+ \subset \Omega$.

Proof. Like we see in Lemma 3.6's proof

$$\frac{d}{dr} \log \left(\frac{1}{r^{d+a}} \int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x \right) = \frac{2\mathbf{N}_s(u, x_0, r)}{r}$$

then, by integration

$$\frac{R^{d+a} \int_{\partial B_r^+(x_0)} x_{d+1}^a |u|^2 \, dS_x}{r^{d+a} \int_{\partial B_R^+(x_0)} x_{d+1}^a |u|^2 \, dS_x} = \exp \int_r^R \frac{2\mathbf{N}_s(u, x_0, s)}{s} \, ds$$

We have, because the monotonicity of $\mathbf{N}_s(u, x_0, s)$ with respect to s (Lemma 3.6)

$$2\mathbf{N}_s(u, x_0, r) \log\left(\frac{R}{r}\right) \leq \int_r^R \frac{2\mathbf{N}_s(u, x_0, s)}{s} \, ds \leq 2\mathbf{N}_s(u, x_0, R) \log\left(\frac{R}{r}\right),$$

which implies the inequality (3-21).

■

Lemma 3.8 *Let $\Omega \subseteq \mathbb{R}^{d+1}_+$ be a bounded open set and suppose $u \in H^1_s(\Omega;\mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u,\Omega)$ with respect to the boundary data $u_0 \in H^s(\partial\Omega;\mathbb{C}_k)$. If there is $M \geq 1$ such that $|u_0| \leq M$, then $|u| \leq M$.*

Proof. Set $\Pi_M : \mathbb{C}_k \to \mathbb{C}_{k,M}$ the Lipchitz retraction of $\mathbb{C}_k$ to the set $\mathbb{C}_{k,M} := \{y \in \mathbb{C}_k : |y| \leq M\}$, defined as identity on $\mathbb{C}_{k,M}$ and sending each point $y \in \mathbb{C}_k \setminus \mathbb{C}_{k,M}$ to the point of intersection between $\partial\mathbb{C}_k$ and the geodesic line that connect y to the vertex 0.

Note that, for each $\xi \in T_x\mathbb{C}_k$ with $x \in \mathbb{C}_k \setminus \mathbb{C}_{k,M}$, $|d\Pi_M(\xi)| < |\xi|$, then

$$\begin{aligned}
\mathbf{E}_s(\Pi_M u,\Omega) &= \frac{1}{2}\int_{\Omega\cap\{|u|>M\}} x^a_{d+1}|\nabla(\Pi_M u)|\,dx + \frac{1}{2}\int_{\Omega\cap\{|u|\leq M\}} x^a_{d+1}|\nabla(\Pi_M u)|\,dx \\
&= \frac{1}{2}\int_{\Omega\cap\{|u|>M\}} x^a_{d+1}|d\Pi_M(\nabla u))|\,dx + \frac{1}{2}\int_{\Omega\cap\{|u|\leq M\}} x^a_{d+1}|\nabla u|\,dx \\
&< \frac{1}{2}\int_{\Omega\cap\{|u|>M\}} x^a_{d+1}|\nabla u|\,dx + \frac{1}{2}\int_{\Omega\cap\{|u|\leq M\}} x^a_{d+1}|\nabla u|\,dx \\
&= \mathbf{E}_s(u,\Omega)
\end{aligned}$$

but $(\Pi_M u)\Big|_{\partial\Omega} = \Pi_M(u_0) = u_0$, so we are contradicting the minimality of u, unless $u \leq M$. ■

Lemma 3.9 *Let $n \geq 2$ and $h \in H^1_s(B^+_r(x_0);\mathbb{C}_k)$ a s-harmonic function. It holds that*

$$\int_{B^+_r(x_0)} x^a_{d+1}|\nabla h|^2 dx \leq C\left(\int_{\partial B^+_r(x_0)} x^a_{d+1}|\nabla_{\tan} h|^2 dS_x\right)^{1/2}\left(\int_{\partial B^+_r(x_0)} x^a_{d+1}|h-\xi|^2\,dS_x\right)^{1/2} \tag{3-22}$$

for some $C = C(d,a)$ and any constant $\xi \in \mathbb{R}^d$.

Proof. Because integration by parts and the s-harmonicity of h, we have that

$$\begin{aligned}
\int_{B^+_r(x_0)} x^a_{d+1}|\nabla h|^2 dS_x &= \int_{B^+_r(x_0)} \nabla(h-\xi)\bullet\left(x^a_{d+1}\nabla h\right)dS_x \\
&= \int_{\partial B^+_r(x_0)} (h-\xi)\bullet\left(x^a_{d+1}\nabla h\bullet\nu\right)dS_x \\
&\quad - \int_{B^+_r(x_0)} (h-\xi)\bullet\operatorname{div}\left(x^a_{d+1}\nabla h\right)dS_x \\
&= \int_{\partial B^+_r(x_0)} x^a_{d+1}(h-\xi)\bullet(\nabla h\bullet\nu)\,dS_x,
\end{aligned}$$

and then, by Cauchy-Schwartz and Hölder inequalities

$$\begin{aligned}
\int_{B^+_r(x_0)} x^a_{d+1}|\nabla h|^2 dS_x &= \int_{\partial B^+_r(x_0)} x^a_{d+1}(h-\xi)\bullet(\nabla h\bullet\nu)\,dS_x \\
&\leq \int_{\partial B^+_r(x_0)} x^a_{d+1}|h-\xi|\,|\nabla h\bullet\nu|dS_x \\
&\leq \left(\int_{\partial B^+_r(x_0)} x^a_{d+1}|\nabla h\bullet\nu|^2 dS_x\right)^{1/2}\left(\int_{\partial B^+_r(x_0)} x^a_{d+1}|h-\xi|^2 dS_x\right)^{1/2}
\end{aligned}$$

Since there is C such that

$$\int_{\partial B_r^+(x_0)} x_{d+1}^a |\nabla h \bullet \nu|^2 dS_x \le C \int_{\partial B_r^+(x_0)} x_{d+1}^a |\nabla_{\tan} h|^2 dS_x,$$

we get (3-22).

∎

Lemma 3.10 *Let $\Omega \subseteq \mathbb{R}_+^{d+1}$ be a bounded open set and suppose $u \in H_s^1(\Omega; \mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u, \Omega)$ with $\|u\|_{L^\infty(\partial\Omega)} < \infty$. There is a constant C depending on d, k and $\|u\|_{L^\infty(\partial\Omega)}$ and there exists $w \in H_s^1(B_r(x_0); \mathbb{C}_k)$, an extension of $u\Big|_{\partial B_r^+(x_0)}$ to $B_r^+(x_0)$ satisfying:*

$$\int_{B_r^+(x_0)} x_{d+1}^a |\nabla w|^2\, dx \le C \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a |\nabla_{\tan} u|^2 dS_x \right)^{1/2} \left(\int_{\partial B_r^+(x_0)} x_{d+1}^a |u - \xi|^2 dS_x \right)^{1/2},$$

for any $\xi \in \mathbb{R}^m$ and $B_r^+(x_0) \subset \Omega$ with $x_0 \in \partial\mathbb{R}_+^{d+1}$.

Proof.

Since $x \mapsto x_{d+1}^a$ is an A_2-weight, (HKM2006, Theorem 3.17) allow us to choose $h \in H_s^1(B_r^+(x_0); \mathbb{C}_k)$ a s-harmonic function with $h\Big|_{\partial B_r^+(x_0)} = u$.

Now we claim that there exists $w \in H_s^1(B; \mathbb{C}_k)$ such that $w = h$ on $\partial B_r^+(x_0)$ and $C = C(d, |u|_{L^\infty(\partial\Omega)})$ such that

$$\int_{B_r^+(x_0)} x_{d+1}^a |\nabla w|^2\, dx \le C \int_{B_r^+(x_0)} x_{d+1}^a |\nabla h|^2\, dx \tag{3-23}$$

Set $\mathbb{C}_{k,M} = \{y \in \mathbb{C}^k \text{ s.t } |y| \le M\}$, for some $M \ge 1$ such that $\|u\|_{L^\infty(\partial\Omega)} \le M$. Because of (H1987, Lemma 6.1), there exists a compact 0-dimensional Lipschitz set $X \subset \mathbb{R}^{d+1}$ and a locally Lipschitz retraction $P : \mathbb{R}^{d+1} \setminus X \to \mathbb{C}_{k,M}$ such that

$$\int_B |\nabla P(x)|^2\, dx < \infty \quad \text{for any ball } B \subset \mathbb{R}^{d+1}.$$

Note that there is $\tau > 0$ depending on k such that the projection $\Pi(y)$ is unique under the condition $\operatorname{dist}(y, \mathbb{C}_{k,M}) < \tau$. Let $\mathbb{B}$ an open ball containing $\mathbb{C}_{k,M} \cup X$ and defines $P_z(x) := P(x - z)$ for $z \in B_\sigma$ when $\sigma < \inf\{\tau, \operatorname{dist}(\mathbb{C}_{k,M}, \partial\mathbb{B})\}$. We also need to assume that $B_\sigma(h(x)) \subset \mathbb{B}$ for almost every $x \in B_r^+(x_0)$ by taking σ small enough or increasing $\mathbb{B}$.

Since h is smooth on Ω, we may apply Sard's theorem to see that the compositions $P_z \circ h \in H_s^1(B_r^+(x_0); \mathbb{C}_{k,M})$ for almost all z. Using Fubini's theorem, the chain rule and matrix sub-multiplicativity, we can get

$$\begin{aligned}\int_{B_\sigma}\int_{B_r^+(x_0)} & x_{d+1}^a \left|\nabla(P_z\circ h)(x)\right|^2 dxdz \\ &\le \int_{B_r^+(x_0)} x_{d+1}^a \left|\nabla h(x)\right|^2 \int_{B_\sigma} \left|\nabla P_z\big(h(x)\big)\right|^2 dzdx \\ &= \int_{B_r^+(x_0)} x_{d+1}^a \left|\nabla h(x)\right|^2 \int_{B_\sigma\left(h(x)\right)} \left|\nabla P(y)\right|^2 dydx \\ &\le \int_{\mathbb{B}} \left|\nabla P(y)\right|^2 dy \int_{B_r^+(x_0)} x_{d+1}^a \left|\nabla h(x)\right|^2 dx.\end{aligned}$$

We can choose a $z_0 \in B_\sigma$ such that

$$|B_\sigma| \int_{B_r^+(x_0)} x_{d+1}^a \left|\nabla(P_{z_0}\circ h)(x)\right|^2 dx \le 2\int_{B_\sigma}\int_{B_r^+(x_0)} x_{d+1}^a \left|\nabla(P_z\circ h)(x)\right|^2 dxdz.$$

Then, taking a constant $\widetilde{C} \ge 2|B_\sigma|^{-1} \int_{\mathbb{B}} \left|\nabla P(y)\right|^2 dy$,

$$\int_{B_r^+(x_0)} x_{d+1}^a \left|\nabla(P_{z_0}\circ h)(x)\right|^2 dx \le \widetilde{C} \int_{B_r^+(x_0)} x_{d+1}^a \left|\nabla h(x)\right|^2 dx.$$

So, lets take

$$w = \left(P_{z_0}\Big|_{\mathbb{C}_{k,M}}\right)^{-1} \circ P_{z_0} \circ h \in H_s^1(B_r^+(x_0); \mathbb{C}_{k,M}),$$

clearly $w\Big|_{\partial B_r^+(x_0)} = h\Big|_{\partial B_r^+(x_0)} = u\Big|_{\partial B_r^+(x_0)}$, thus, using inverse function theorem, (3-23) follows by taking

$$C = \widetilde{C} \cdot \sup_{z\in B_\sigma} \operatorname{Lip}\left(P_z\Big|_{\mathbb{C}_{k,M}}\right)^{-1}$$

At the end, the result follows by combining the estimation (3-23) and Lemma 3.9.

■

Lemma 3.11 *Let $\Omega \subseteq \mathbb{R}_+^{d+1}$ be a bounded open set and suppose $u \in H_s^1(\Omega; \mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u,\Omega)$.For any compact $K \subset \Omega$, there exists a constant D_0 depending on $\|u\|_{L^\infty(\partial\Omega)}$, K and Ω such that*

$$\int_K x_{d+1}^a |\nabla u|^2 \le D_0.$$

Proof. By the compactness of K, scaling and translation, it suffices to prove the estimate for $\Omega = B_1$ and $K = \overline{B_{1-\delta}}$ for a fixed $\delta \in (0,1)$.

By Lemma 3.8, there is $R > 0$ such that $u(B_1) \subseteq B_R \cap \mathbb{C}_k$.

Since, for almost every $r \in (0,1)$, holds that

$$\frac{d}{dr}\int_{B_r^+} x_{d+1}^a |\nabla u|^2\,dx = \int_{\partial B_r^+} x_{d+1}^a |\nabla u|^2\,dx$$

so, because the minimality of u and taken $\xi = 0$ from Lemma 3.10, and taking $M = M(\|u\|_{L^\infty(\partial\Omega)})$ from Lemma 3.8 we have

$$\begin{aligned} D(r) := \int_{B_r^+} x_{d+1}^a |\nabla u|^2\,dx &\le C\left(\frac{d}{dr}\int_{B_r^+} x_{d+1}^a |\nabla u|^2\,dx \cdot \int_{\partial B_r^+} x_{d+1}^a |u|^2 dS_x\right)^{1/2} \\ &\le CM\psi(d,a)\sqrt{r^{d+a}}\left(\frac{d}{dr}\int_{B_r^+} x_{d+1}^a |\nabla u|^2\,dx\right)^{1/2} \\ &= CM\psi(d,a)\sqrt{r^{d+a}}\,D'(r)^{1/2} \end{aligned}$$

then

$$\frac{(CM\psi(d,a))^{-2}}{r^{d+a}} \le \frac{D'(r)}{D(r)^2}$$

so, by integrating from $(1-\delta)$ to 1, we get

$$\frac{-\left(CM\psi(d,a)\right)^{-2}}{d+a-1}\left(1 - \frac{1}{(1-\delta)^{d+a-1}}\right) \le \frac{-1}{D(1)} + \frac{1}{D(1-\delta)} \le \frac{1}{D(1-\delta)}$$

then

$$\int_K x_{d+1}^a |\nabla u|^2 = D(1-\delta) \le (d+a-1)\left(CM\psi(d,a)\right)^2 \frac{(1-\delta)^{d+a-1}}{1-(1-\delta)^{d+a-1}} = D_0$$

So the result has been prove. ■

Lets denote

$$\overline{u}_{B_r^+(y),s} = \left(\int_{B_r^+(y)} x_{d+1}^a\,dx\right)^{-1}\int_{B_r^+(y)} x_{d+1}^a u(x)\,dx \tag{3-24}$$

Lemma 3.12 (Caccioppoli-type inequality) *Let $u \in H_s^1(B_R^+;\mathbb{C}_k)$ is a minimizer of $\mathbf{E}_s(u, B_R^+)$. Then, for every $\lambda \in (0,1)$, there exists a constant C depending on k such that*

$$\mathbf{\Theta}_s\left(u, x_0, r\right) \le \lambda\,\mathbf{\Theta}_s\left(u, x_0, 2r\right) + \frac{C}{\lambda^2}\mathbf{W}_s(u, x_0, 2r), \tag{3-25}$$

where $x_0 \in \partial\mathbb{R}_+^{d+1}$, $B_{2r}(x_0) \subset B_R^+$ and

$$\mathbf{W}_s(u, x_0, \rho) := \left(\int_{B_\rho^+(x_0)} x_{d+1}^a\,dx\right)^{-1}\int_{B_\rho^+(x_0)} x_{d+1}^a \left|u(x) - \overline{u}_{B_\rho^+(x_0),s}\right|^2\,dx.$$

Proof.

From Lemma 3.10, and using the Cauchy's inequality

$$AB \leq \frac{1}{2}\left(\delta A^2 + \frac{1}{\delta}B^2\right)$$

with $\delta = \lambda\rho$ with $\mu \in (0,1)$, we have that there is a constant C depending on k and $\|u\|_{L^\infty_s(\partial\Omega)}$ and there exists $w \in H^1_s(B^+_\rho(x_0);\mathbb{C}_k)$, an extension of $u\big|_{\partial B^+_r(x_0)}$ to $B^+_r(x_0)$ satisfying:

$$\begin{aligned}\frac{1}{\rho^{d+a-1}}\int_{B^+_\rho(x_0)} x^a_{d+1}|\nabla w|^2\, dS_x \leq \frac{\lambda}{2\rho^{d+a-2}}&\int_{\partial B^+_\rho(x_0)} x^a_{d+1}|\nabla_{\tan} u|^2 dS_x\\ &+\frac{C}{2\lambda^2\rho^{d+a}}\int_{\partial B^+_\rho(x_0)} x^a_{d+1}|u-\xi|^2 dS_x,\end{aligned}$$

Note that

$$\int_r^{2r}\int_{\partial B^+_\rho(x_0)} x^a_{d+1}|\nabla u|^2\, dS_x\, d\rho = \int_{B^+_{2r}(x_0)\setminus B^+_r(x_0)} x^a_{d+1}|\nabla u|^2\, dx$$

so, there is $\rho_1 \in (r,2r)$ such that

$$\int_{\partial B^+_{\rho_1}(x_0)} x^a_{d+1}|\nabla u|^2\, dS_x = \frac{1}{r}\int_{B^+_{2r}(x_0)\setminus B^+_r(x_0)} x^a_{d+1}|\nabla u|^2\, dx \leq \frac{1}{r}\int_{B^+_{2r}(x_0)} x^a_{d+1}|\nabla u|^2\, dx,$$

in a similar way, we have that exists $\rho_2 \in (r,2r)$ such that

$$\int_{\partial B^+_{\rho_2}(x_0)} x^a_{d+1}|u-\xi|^2\, dS_x \leq \frac{1}{r}\int_{B^+_{2r}(x_0)} x^a_{d+1}|u-\xi|^2\, dx,$$

Taking in account that $\nabla_{\tan} u = \nabla u - (\nabla u)(\nu\otimes\nu)$, we have that

$$\int_{\partial B^+_{\rho_1}(x_0)} x^a_{d+1}|\nabla_{\tan} u|^2\, dS_x \leq 2\int_{\partial B^+_{\rho_1}(x_0)} x^a_{d+1}|\nabla u|^2\, dS_x,$$

now, setting $\rho = \min(\rho_1,\rho_2)$, and using the monotonicity of $\mathbf{\Theta}$ and the minimality of u, we get

$$\begin{aligned}\mathbf{\Theta}_s(u,x_0,r) &\leq \frac{1}{\rho^{d+a-1}}\int_{\partial B^+_\rho(x_0)} x^a_{d+1}|\nabla w|^2\, dS_x\\ &\leq \frac{\lambda}{r^{d+a-1}}\int_{B^+_{2r}(x_0)} x^a_{d+1}|\nabla u|^2 dS_x + \frac{C}{2\lambda^2 r^{d+a+1}}\int_{B^+_{2r}(x_0)} x^a_{d+1}|u-\xi|^2 dS_x\\ &\leq \lambda\mathbf{\Theta}_s(u,x_0,2r) + \frac{C}{\lambda^2}\mathbf{W}_s(u,x_0,2r)\end{aligned}$$

by taken $\xi = \overline{u}_{B^+_{2r}(x_0),s}$ and the fact that

$$\int_{B^+_{2r}(x_0)} x^a_{d+1}\, dx = (2r)^{d+a+1}\psi(d,a)$$

so we also replace C by $\dfrac{C \cdot \psi(d,a)}{2^{d+a}}$.

$\blacksquare$

Lemma 3.13 *Let $u_i \in H^1_s(B_1;\mathbb{C}_k)$ be a sequence of minimizing maps of $\mathbf{E}_s(u,B_1)$, such that $u_i \in L^\infty_s(\partial B_1)$ and u_i converges to u_∞ weakly in $H^1_s(B_1;\mathbb{C}_k)$. Then u_∞ is a minimizing map of $\mathbf{E}_s(u,B_1)$, and u_i converge to u_∞ strongly in $H^1_s(B_1;\mathbb{C}_k)$.*

Proof. Set $\mathbf{W}_s$ like in Lemma 3.12, using the weighted Sobolev-Poincaré Inequality (See (HKM2006), it is easy to verify that x^a_{d+1} is an A_2-weight).

$$\mathbf{W}_s(u_i,x_0,2r)^{1/2} \le C(d)\cdot r\,\mathbf{W}_s\Big(|\nabla u_i|^p,x_0,2r\Big)^{1/p}$$

where $p = \dfrac{2n}{2+d} \in [1,2]$, then from the Caccioppoli-type inequality (3-25), for any $\lambda \in (0,1)$ we have

$$\mathbf{\Theta}_s\left(u_i,x_0,r\right) \le \lambda\,\mathbf{\Theta}_s\left(u_i,x_0,2r\right) + \frac{C(d,a)r^2}{\lambda^2}\,\mathbf{W}_s\Big(|\nabla u_i|^p,x_0,2r\Big)^{2/p},$$

Take into account that, since $x_0 \in \partial\mathbb{R}^{d+1}$,

$$\int_{B^+_\rho(x_0)} x^a_{d+1}\,dx = \rho^{d+a+1}\psi(d,a),$$

where

$$\psi(d,a) = \int_{[0,\pi]^d} \frac{\sin^{d-1}(\varphi_1)\sin^{d-2}(\varphi_2)\dots\sin(\varphi_{d-1})}{d+a+1}\cos^a(\varphi_1)\,d\varphi_1\,d\varphi_2\dots\,d\varphi_n$$

so, dividing by $r^2\psi(d,a)$ we have

$$\begin{aligned}\mathbf{W}_s\left(|\nabla u_i|^2,x_0,r\right) &\le 2^{d+a+1}\lambda\,\mathbf{W}_s\left(|\nabla u_i|^2,x_0,2r\right) + \frac{C(d,a)}{\lambda^2\psi(d,a)}\,\mathbf{W}_s\Big(|\nabla u_i|^p,x_0,2r\Big)^{2/p}\\ &= \theta\,\mathbf{W}_s\left(|\nabla u_i|^2,x_0,2r\right) + \frac{\widetilde{C}}{\theta^2}\,\mathbf{W}_s\Big(|\nabla u_i|^p,x_0,2r\Big)^{2/p},\end{aligned}$$

where $\theta = \lambda/2^{d+a+1} \in (0,1)$.

From (G1983, Chapter V, Proposition 1.1), that inequality allows us to conclude that $|\nabla u_i|$ are equibounded in L^q_{loc} for some $q>2$.

Let $w \in H^1_s(B^+_1,\mathbb{C}_k)$ be an arbitrary map with boundary value $w = u$ on ∂B^+_1 and choose η a smooth cut-off function such that, given $\delta > 0$ $\eta \equiv 1$ on $B_{1-\delta}$, $\eta = 0$ on ∂B^+_1 and $|\nabla\eta| \le \delta^{-1}$ on the set $A_\delta = B^+_1 \setminus B^+_{1-\delta}$. Now set in B_1 the map

$$v_j = (1-\eta)u_j + \eta w$$

Arguing like in Lemma 3.10's proof, we can obtain a map $w_j \in H^1_s(B_1, \mathbb{C}_k)$ such that

$$\begin{aligned} \int_{A_\delta} x^a_{d+1}|\nabla w_j|^2\,dx \le C\int_{A_\delta} x^a_{d+1}|\nabla v_j|^2\,dx, \\ w_j = v_j = w \quad \text{on } \partial B^+_{1-\delta}, \\ w_j = v_j = u_j \quad \text{on } \partial B^+_1, \\ w_j = w \quad \text{on } B^+_{1-\delta}, \end{aligned} \tag{3-26}$$

where C depends on k uniform in j.

Because semicontinuity of the energy and the minimality of u_j, for j large enough, we have that

$$\int_{B_1^+} x^a_{d+1}|\nabla u|^2\,dx - \frac{\epsilon}{2} \le \int_{B_1^+} x^a_{d+1}|\nabla u_j|^2\,dx \le \int_{B_1^+} x^a_{d+1}|\nabla w_j|^2\,dx, \tag{3-27}$$

for a given $\epsilon > 0$. Then, using (3-26)

$$\int_{B_1^+} x^a_{d+1}|\nabla u|^2\,dx - \frac{\epsilon}{2} \le \int_{B_{1-\delta}} x^a_{d+1}|\nabla w_j|^2\,dx + C\int_{A_\delta} x^a_{d+1}|\nabla w_j|^2\,dx. \tag{3-28}$$

We also have

$$\begin{aligned} \int_{A_\delta} x^a_{d+1}|\nabla v_j|^2\,dx &= \int_{A_\delta} x^a_{d+1}\Big|\nabla(u_j - \eta(u_j - w))\Big|^2\,dx \\ &= \int_{A_\delta} x^a_{d+1}\Big|\nabla u_j - \eta\nabla(u_j - w) - (u_j - w)\nabla\eta\Big|^2\,dx \\ &\le 2\int_{A_\delta} x^a_{d+1}\left(|1-\eta|^2|\nabla u_j|^2 + |\eta|^2|\nabla w|^2 + |u_j - w|^2\Big|\nabla\eta\Big|^2\right)\,dx \\ &= c_1\int_{A_\delta} x^a_{d+1}|\nabla u_j|^2 + c_2\int_{A_\delta} x^a_{d+1}|\nabla w|^2 + c_3\delta^{-2}\int_{A_\delta} x^a_{d+1}|u_j - w|^2 \end{aligned}$$

Because the Hölder inequality and the equiboundness of ∇u_j, we have

$$\int_{A_\delta} |\nabla u_j|^2\,dx \le \left(\int_{A_\delta} |\nabla u_j|^q\,dx\right)^{2/q} |A_\delta|^{1-2/q} \le b_1|A_\delta|^{1-2/q}$$

using Young inequality and a varient of Poincaré inequality

$$\begin{aligned} \int_{A_\delta} x^a_{d+1}|u_j - w|^2\,dx &\le 2\int_{A_\delta} x^a_{d+1}|u_j - u|^2 + 2\int_{A_\delta} x^a_{d+1}|u - w|^2\,dx \\ &\le 2\int_{A_\delta} x^a_{d+1}|u_j - u|^2 + 2b_2\delta^2\int_{A_\delta} x^a_{d+1}|\nabla(u - w)|^2\,dx \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \int_{A_\delta} x^a_{d+1}|\nabla v_j|^2\,dx = C_1|A_\delta|^{1-2/q} &+ C_2\delta^{-2}\int_{A_\delta} x^a_{d+1}|u_j - u|^2 \\ &+ C_3\delta^{-2}\int_{A_\delta} x^a_{d+1}|\nabla w|^2 + C_4\delta^{-2}\int_{A_\delta} x^a_{d+1}|\nabla(u - w)|^2 \end{aligned}$$

Since $u_j \to u$ strongly in $L^2_s(B_1^+)$, and choosing δ small enough such that

$$\limsup_{j\to\infty}\int_{A_\delta} x_{d+1}^2|\nabla v_j|^2\,dx \le \frac{\epsilon}{2C}, \tag{3-29}$$

thus we get from (3-28) that

$$\int_{B_1^+} x_{d+1}^2|\nabla u|^2\,dx \le \int_{B_1^+} x_{d+1}^2|\nabla w|^2+\epsilon$$

Since w and ϵ are arbitrary, we conclude that u is minimizer.

Now, we can in order to prove the strong convergence, let $w=u$, thus

$$\begin{aligned}
&\int_{B_1^+} x_{d+1}^2|\nabla u_j-\nabla u|^2\,dx \\
&\quad=\int_{B_1^+} x_{d+1}^2|\nabla u_j|^2\,dx+\int_{B_1^+} x_{d+1}^2|\nabla u|^2\,dx-2\int_{B_1^+} x_{d+1}^2\nabla u_j\bullet\nabla u\,dx\\
&\quad\le\int_{B_1^+} x_{d+1}^2|\nabla u_j|^2\,dx+\int_{B_1^+} x_{d+1}^2|\nabla u|^2\,dx-2\int_{B_1^+} x_{d+1}^2|\nabla u|^2\,dx+\frac{\epsilon}{2}\\
&\quad\le\int_{B_1^+} x_{d+1}^2|\nabla u_j|^2\,dx-\int_{B_1^+} x_{d+1}^2|\nabla u|^2\,dx+\frac{\epsilon}{2}
\end{aligned}$$

By the minimimality of u_j and taking $w=u$ in (3-26) we get

$$\int_{B_1^+} x_{d+1}^2|\nabla u_j|^2\,dx \le \int_{B_1^+} x_{d+1}^2|\nabla u|^2\,dx + C\int_{A_\delta} x_{d+1}^2|\nabla v_j|^2\,dx,$$

then, taking again a small enough δ such that (3-29) holds, we have

$$\int_{B_1^+} x_{d+1}^2|\nabla u_j-\nabla u|^2\,dx \le C\int_{A_\delta} x_{d+1}^2|\nabla v_j|^2\,dx+\frac{\epsilon}{2}\le\epsilon.$$

We conclude that the convergence is strong in $H_s^1(B_1^+)$

■

3.4
Energy Decay Lemmas

Lemma 3.14 (Morrey's Decay Lemma) *Suppose* $u\in W^{1,p}\left(B_R(x_0)\right),\beta>0,\gamma\in(0,1]$ *are constants, and*

$$\rho^{p-1-d}\int_{B_r(y)}|\nabla u|^p\le\beta^p\left(\frac{\rho}{R}\right)^{p\gamma},\quad \forall y\in B_{R/2}\left(x_0\right),\rho\in(0,R/2].$$

Then $u\in C^{0,\gamma}\left(\overline{B}_R\left(x_0\right)\right)$ *and*

$$|u(x_1)-u(x_2)|\le C\beta\left(\frac{|x_1-x_2|}{R}\right)^\gamma,\quad \forall x_1,x_2\in B_{R/2}\left(x_0\right),$$

where C *depends only on* d.

Proof. (M2005, Lemma 2.1) ■

Remark 4 *In the Lemma 3.14 we can replace $B_R(x_0)$ by $B_R^+(x_0)$ through a reflection argument; that is replacing u by his even reflection in $\partial\mathbb{R}^{d+1}_+$ like in Remark 2.*

The next lemma is based on (GT2001, Lemma 8.23)

Lemma 3.15 *Let ω be non-decreasing function on an interval $(0, R]$ satisfying, for all $r \leq R$, the inequality*

$$\omega(\theta r) \leq \kappa\, \omega(r)$$

where $\gamma > 0$ and $0 < \theta < 1$. Then , there exists $\gamma \in (0,1)$ depending on (θ, κ) such that

$$\omega(r) \leq \frac{1}{\kappa}\left(\frac{r}{R}\right)^{\gamma} \omega(R).$$

Proof. Fix any $r_1 \in [r, R]$ and note that for every $m \in \mathbb{N}$

$$\omega(\theta^m r_1) \leq \kappa\omega(\theta^{m-1} r_1) \leq \cdots \leq \kappa^m \omega(r_1) \leq \kappa^m \omega(R)$$

There is m such that

$$\theta^m r_1 \leq r \leq \theta^{m-1} r_1$$

Hence, taking $\gamma_1 = \log(\kappa)/\log(\theta)$

$$\omega(r) \leq \omega(\theta^{m-1} r_1) \leq \kappa^{m-1}\omega(R) \leq \frac{1}{\kappa}\,\theta^{m\gamma_1}\omega(R) \leq \frac{1}{\kappa}\left(\frac{r}{r_1}\right)^{\gamma_1}\omega(R)$$

At the end we can choose $\mu > 0$ and $r_1 = R^{\mu}\, r^{1-\mu} \in (r, R)$ such that $\gamma = \mu\,\gamma_1 < 1$, so

$$\omega(r) \leq \frac{1}{\kappa}\left(\frac{r}{r_1}\right)^{\gamma_1}\omega(R) = \frac{1}{\kappa}\left(\frac{r}{R}\right)^{\gamma}\omega(R)$$

■

Lemma 3.16 *Let $u \in H^1_s(\mathbb{R}^{d+1}_+; \mathbb{C}_k)$ be a minimizing of $\mathbf{E}_s(u, \Omega)$. Suppose $B_R^+(x_0)$ is a half-ball with $\overline{\partial^0 B_R^+(x_0)} \subset \Omega$ and suppose $B_\rho(y) \subset B_R^+(x_0)$ such that $y \in B^+_{R/3}(x_0)$ with $y_{d+1} \geq 2\rho$, then there is a constant $C = C(d)$ such that*

$$\rho^{1-d}\int_{B_\rho(y)} |\nabla u|^2 \leq C\,\mathbf{\Theta}_s(u, x_0, R).$$

Proof.

Noticing that for any $x \in B_\rho(y)$

$$\frac{1}{2}\, y^a_{d+1} \leq x^a_{d+1} \leq 6\, y^a_{d+1}$$

and using the monotonicity result from Lemma 3.4 we have that

$$\begin{aligned}\rho^{1-d}\int_{B_\rho(y)}|\nabla u(x)|^2\,dx &\leq \frac{2}{y_{d+1}^a}\rho^{1-d}\int_{B_\rho(y)}x_{d+1}^a|\nabla u(x)|^2\,dx\\ &\leq \frac{2}{y_{d+1}^a}\frac{e^{ay_{d+1}\xi/2}}{e^{a\rho\xi}}\left(\frac{y_{d+1}}{2}\right)^{1-d}\int_{B_{\frac{y_{d+1}}{2}}(y)}x_{d+1}^a|\nabla u(x)|^2\,dx\end{aligned}$$

where $\xi = (y_{d+1} - y_{d+1}/2)^{-1} = 2/y_{d+1}$, then

$$\rho^{1-d}\int_{B_\rho(y)}|\nabla u(x)|^2\,dx \leq 2^{a+1}e^2\left(\frac{y_{d+1}}{2}\right)^{1-d-a}\int_{B_{\frac{y_{d+1}}{2}}(y)}x_{d+1}^a|\nabla u(x)|^2\,dx \tag{3-30}$$

Since $B_{\frac{y_{d+1}}{2}}(y) \subset B_{\frac{3y_{d+1}}{2}}(y', 0)$ and $y_{d+1} \leq |y - x_0| \leq \frac{R}{3}$, we get, by using the monotonicity result from Proposition 3.3, that

$$\begin{aligned}\rho^{1-d}\int_{B_\rho(y)}|\nabla u(x)|^2\,dx &\leq \frac{2^{a+2}e^2}{3^{1-d-a}}\mathbf{\Theta}_s\left(u, (y', 0), \frac{3y_{d+1}}{2}\right)\\ &\leq \frac{2^{a+2}e^2}{3^{1-d-a}}\mathbf{\Theta}_s\left(u, (y', 0), \frac{R}{2}\right)\end{aligned}$$

we also have that $B_{R/2}(y', 0) \subseteq B_R(x_0)$, so

$$\rho^{1-d}\int_{B_\rho(y)}|\nabla u(x)|^2\,dx \leq \frac{2^{a+2}e^2}{6^{1-d-a}}\mathbf{\Theta}_s\left(u, x_0, R\right)$$

Taking $C(d) = \dfrac{48e^2}{6^{1-d}}$ we get the result. ∎

The following Energy Decay Result is based on (R2018, Lemma 4.20) and (AHL2017, Claim of equation 2.27).

Lemma 3.17 *Let $u \in H_s^1(\mathbb{R}_+^{d+1}; \mathbb{C}_k)$ be a minimizing of $\mathbf{E}_s(u, \Omega)$. Suppose $B_R^+(x_0)$ satisfies $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \Omega$. There exists an $\varepsilon_0 = \varepsilon_0\left(d, \|u\|_{L^2(\Omega)}, a\right) > 0$ and a $\theta_0 = \theta_0\left(d, \|u\|_{L^2(\Omega)}, a\right) \in (0, 1/4)$ such that if*

$$\mathbf{\Theta}_s\left(u, x_0, R\right) \leq \varepsilon_0,$$

then

$$\mathbf{\Theta}_s\left(u, y, \theta_0\rho\right) \leq \frac{1}{2}\mathbf{\Theta}_s\left(u, y, \rho\right),$$

for every $B_\rho^+(y) \subset B_R^+(x_0)$ with $y \in B_{R/2}(x_0) \cap \partial\mathbb{R}_+^{d+1}$ and $\rho \leq R/2$.

Proof.

Lets prove by contradiction, considering a sequence of minimizers $\{u_i\}$ such that

$$|u_i| \leq \|u\|_{L^\infty(\partial B_\rho(x_0))}, \tag{3-31}$$

$$\mathbf{\Theta}_s\left(u_i, y, \rho\right) = \varepsilon_i \to 0, \text{ as } i \to \infty \tag{3-32}$$

but, for some $\tilde{\theta}$

$$\mathbf{\Theta}_s\left(u_i, y, \tilde{\theta}\rho\right) > \frac{1}{2}\,\mathbf{\Theta}_s\left(u_i, y, \rho\right) \tag{3-33}$$

If there exists a $C > 0$ such that

$$\int_{\partial B^+_\rho(y)} x^a_{d+1}|u_i|^2\,dS_x \leq C\,\mathbf{\Theta}_s\left(u_i, y, \rho\right)$$

Because or Lemma 3.5, we have that

$$\frac{d}{dr}\left[\frac{1}{r^{d+a}}\int_{\partial B^+_r(x_0)} x^a_{d+1}|u_i|^2\,dS_x\right] = \frac{4}{r}\,\mathbf{\Theta}_s\Big(u_i, B^+_r(x_0)\Big). \tag{3-34}$$

and also using monotonicity of $\mathbf{\Theta}_s\left(u_i, y, \rho\right)$, we get

$$\begin{aligned}
4\mathbf{\Theta}_s\Big(u_i, B^+_{\tilde{\theta}\rho}(x_0)\Big)\int_{\tilde{\theta}\rho}^{\rho}\frac{dr}{r} &\leq \int_{\tilde{\theta}\rho}^{\rho}\frac{4}{r}\,\mathbf{\Theta}_s\Big(u, B^+_r(x_0)\Big)\,dr\\
&= \frac{1}{\rho^{d+a}}\int_{\partial B^+_\rho(x_0)} x^a_{d+1}|u_i|^2\,dS_x - \frac{1}{\left(\tilde{\theta}\rho\right)^{d+a}}\int_{\partial B^+_{\tilde{\theta}\rho}(x_0)} x^a_{d+1}|u_i|^2\,dS_x\\
&\leq C\,\mathbf{\Theta}_s\left(u_i, y, \rho\right)
\end{aligned}$$

then

$$\Theta_s\Big(u_i, B^+_{\tilde{\theta}\rho}(x_0)\Big) \leq \frac{C}{4\log(1/\tilde{\theta})}\,\Theta_s\Big(u_i, B^+_{\tilde{\theta}}(x_0)\Big),$$

but when $0 \leq \tilde{\theta} \leq e^{-C/2}$ we contradict (3-33), so $\int_{\partial B^+_\rho(y)} x^a_{d+1}|u_i|^2\,dS_x$ does not decay as fast as $\mathbf{\Theta}_s\left(u_i, y, \rho\right)$.

Using the scaling invariance of the target $\mathbb{C}_k$, we can replace u_i by

$$\left(\int_{\partial B^+_\rho(x_0)} x^a_{d+1}|u_i|^2\,dS_x\right)^{-1/2} u_i$$

to assume

$$\int_{\partial B^+_\rho(x_0)} x^a_{d+1}|u_i|^2\,dS_x = 1,$$

in that case (3-32) and (3-33) still hold.

Let $\overline{u_i} = \overline{u_i}_{B^+_\rho(x_0),s}$, then

$$\begin{aligned}
0 \leq \int_{B^+_\rho(x_0)} x^a_{d+1}\left(|u_i(x)|^2 - |\overline{u_i}|^2\right)dx &= \int_{B^+_\rho(x_0)} x^a_{d+1}|u_i(x) - \overline{u_i}|^2\,dx\\
&\leq C(d)\rho^2\int_{B^+_\rho(x_0)} x^a_{d+1}|\nabla u_i|^2\,dx \qquad (3\text{-}35)\\
&\leq C(d)\rho^{d+a+1}\,\mathbf{\Theta}_s\left(u_i, x_0, \rho\right),
\end{aligned}$$

because the weighted Poincaré inequality (See (HKM2006, 1.4), it is easy to verify that x^a_{d+1} is an A_2-weight). Here we may work with an absolutely continuous representative of u_i, which we do not relabel. Then, using Lemma

3.7

$$\begin{aligned}\int_{B_\rho^+(x_0)} x_{d+1}^a|u_i|^2dx &\geq \int_0^\rho \left(\frac{r}{\rho}\right)^{d+a+2\mathbf{N}_s(u,x_0,\rho)} \int_{\partial B_\rho^+(x_0)} x_{d+1}^a|u_i|^2\,dS_x\,dr\\ &= \frac{\rho}{1+d+a+2\mathbf{N}_s(u,x_0,\rho)}\int_{\partial B_\rho^+(x_0)} x_{d+1}^a|u_i|^2\,dS_x,\end{aligned}$$

where

$$\mathbf{N}_s(u,x_0,\rho) = 2\rho\,\mathbf{E}_s\Big(u,B_\rho^+(x_0)\Big)\left(\int_{\partial B_\rho^+(x_0)} x_{d+1}^a|u_i|^2\,dS_x\right)^{-1},$$

and then

$$\int_{B_\rho^+(x_0)} x_{d+1}^a|u_i|^2dx \geq \frac{\rho}{1+d+a+2\rho^{d+a}\,\mathbf{\Theta}_s(u_i,x_0,\rho)}$$

next, we get

$$\begin{aligned}|\overline{u_i}|^2 &= \left(\int_{B_\rho^+(x_0)} x_{d+1}^a dx\right)^{-1}\left(\int_{B_\rho^+(x_0)} x_{d+1}^a|u_i|^2dx - \int_{B_\rho^+(x_0)} x_{d+1}^a\left(|u_i|^2-|\overline{u_i}|^2\right)dx\right)\\ &\geq \frac{\psi(d,a)}{\rho^{d+a+1}}\left(\frac{\rho}{d+a+2\rho^{d+a}\,\mathbf{\Theta}_s(u_i,x_0,\rho)} - C(d)\rho^{d+a+1}\,\mathbf{\Theta}_s\left(u_i,x_0,\rho\right)\right)\\ &= \frac{\psi(d,a)}{(d+a)\rho^{d+a}+2\rho^{2d+2a}\,\mathbf{\Theta}_s(u_i,x_0,\rho)} - C(d)\psi(d,a)\,\mathbf{\Theta}_s\left(u_i,x_0,\rho\right).\end{aligned}$$

Since $\mathbf{\Theta}_s\left(u_i,x_0,\rho\right)\to 0$, there is $\eta>0$ such that, for i large enough, $|\overline{u_i}|>\eta$.

From the weighted Jensen‘s inequality and Lemma 3.7

$$\begin{aligned}|\overline{u_i}|^2 &\leq \left(\int_{B_\rho^+(x_0)} x_{d+1}^a dx\right)^{-1}\int_{B_\rho^+(x_0)} x_{d+1}^a|u_i|^2dx\\ &\leq \frac{\psi(d,a)}{\rho^{d+a+1}}\int_0^\rho \left(\frac{r}{\rho}\right)^{d+a+2\mathbf{N}_s(u,x_0,r)}\int_{\partial B_\rho^+(x_0)} x_{d+1}^a|u_i|^2\,dS_x\,dr\\ &\leq \frac{\psi(d,a)}{\rho^{d+a}}\end{aligned}$$

Let $\tilde{u}_i$ be the minimizer of $\operatorname{dist}(\overline{u_i},\mathbb{C}_k)$, since

$$\begin{aligned}\operatorname{dist}^2\left(\overline{u_i},\mathbb{C}_k\right) &\leq \left(\int_{B_\rho^+(x_0)} x_{d+1}^a dx\right)^{-1}\int_{B_\rho^+(x_0)} x_{d+1}^a|u_i(x)-\overline{u_i}|^2dx\\ &\leq C(d)\rho^2\left(\int_{B_\rho^+(x_0)} x_{d+1}^a dx\right)^{-1}\int_{B_\rho^+(x_0)} x_{d+1}^a|\nabla u_i|^2dx\\ &\leq C(d,a)\,\mathbf{\Theta}_s\left(u_i,x_0,\rho\right),\end{aligned}$$

we have that

$$|\overline{u_i}-\tilde{u}_i|^2 \leq C(d,a)\,\mathbf{\Theta}_s\left(u_i,x_0,\rho\right) \tag{3-36}$$

Passing to a subsequence if necessary, and because (3-32), there is a constant u_* such that $\overline{u_i}$ and $\tilde{u}_i$, both converge to u_* and $|u_*| > \eta > 0$. We also may assume, by scaling and translations, that $x_0 = 0$ and $\rho = 1$.

Now, let $r_i = |u_* - \tilde{u}_i|$ and define

$$U_i := \begin{cases} (u_i - u_*)\, \mathbf{\Theta}_s\,(u_i, 0, 1)^{-1/2} & \text{If } \lim_{i\to\infty}(r_i/\varepsilon_i) < \infty. \\ (u_i - \tilde{u}_i)\, \mathbf{\Theta}_s\,(u_i, 0, 1)^{-1/2} & \text{If } \lim_{i\to\infty}(r_i/\varepsilon_i) < \infty \text{ is not true.} \end{cases}$$

It is clear that $\|\nabla U_i\|_{L^2_s(B_1^+)} = \sqrt{2}$.

When $\lim_{i\to\infty}(r_i/\varepsilon_i) < \infty$, we follow from (3-35) and (3-36) that

$$\begin{aligned} \|U_i\|_{L^2_s(B_1^+)} &= \mathbf{\Theta}_s\,(u_i, 0, 1)^{-1/2}\, \|u_i - u_*\|_{L^2_s(B_1^+)} \\ &\le \mathbf{\Theta}_s\,(u_i, 0, 1)^{-1/2} \left(\|u_i - \overline{u_i}\|_{L^2_s(B_1^+)} + \|\overline{u_i} - \tilde{u}_i\|_{L^2_s(B_1^+)} + r_i\, |B_1^+|^{1/2} \right) \\ &\le C(d)^{1/2} + C(d,a)^{1/2}\, |B_1^+|^{1/2} + \frac{r_i}{\varepsilon_i}\, |B_1^+|^{1/2} \varepsilon_i^{1/2} < \infty. \end{aligned}$$

In the other case,

$$\begin{aligned} \|U_i\|_{L^2_s(B_1^+)} &= \mathbf{\Theta}_s\,(u_i, 0, 1)^{-1/2}\, \|u_i - \tilde{u}_i\|_{L^2_s(B_1^+)} \\ &\le \mathbf{\Theta}_s\,(u_i, 0, 1)^{-1/2} \left(\|u_i - \overline{u_i}\|_{L^2_s(B_1^+)} + \|\overline{u_i} - \tilde{u}_i\|_{L^2_s(B_1^+)} \right) \\ &\le C(d)^{1/2} + C(d,a)^{1/2}\, |B_1^+|^{1/2} < \infty, \end{aligned}$$

Hence in any case, by Rellich Compactness, Lemma (R2018, Lemma 2.5), passing to a subsequence if necessary, that $U_i \rightharpoonup U_\infty$ weakly in H^1_s, converge strongly in $L^2_s(B_1^+)$ and satifies:

$$\|\nabla U_\infty\|_{L^2_s(B_1^+)} \le \liminf_{i\to\infty} \|\nabla U_i\|_{L^2_s(B_1^+)} = \sqrt{2}$$

Besides, we claim that U_∞ maps B_1 to $T_{u_*}\mathbb{C}_k$ almost everywhere which is in a hyperplane, as $|u_*| > \eta > 0$.

To prove this claim, first note by Egorov's Theorem, since U_i and U_∞ are bounded a.e., that for every $\delta > 0$, there exists $E_\delta \subset B_1^+$ witch $|B_1^+ \setminus E_\delta| < \delta$ such that U_i convergence uniformly on E_δ.

We also have that, there is a sequence of maps $W_i : B_1^+ \setminus E_\delta \to T_{\tilde{u}_i}\mathbb{C}_k$ such that $|U_i - W_i| = \gamma_i \to 0$. Since $\tilde{u}_i$ and u_* are bounded uniformly away from 0, there exists a sequence map $\tau_i : T_{\tilde{u}_i}\mathbb{C}_k \to T_{u_*}\mathbb{C}_k$ such that τ_i converge to the identity, as $i \to \infty$. Setting $\widetilde{W}_i := \tau_i \circ W_i \to T_{\tilde{u}_i}\mathbb{C}_k$, we have

$$\begin{aligned} |\widetilde{W}_i - U_\infty| &\le |\widetilde{W}_i - W_i| + |W_i - U_i| + |U_i - U_\infty| \\ &\le |\tau_i - id|\, |U_i| + \gamma_i + |U_i - U_\infty| \end{aligned}$$

So, U_∞ is the convergence value of $\widetilde{W}_i$, then U_∞ maps $B_1^+ \setminus E_\delta$ to $T_{u_*}\mathbb{C}_k$. Next, since δ is arbitrary the claim holds.

Another last claim is that, U_∞ must be a vector valued s-harmonic function and U_i satisfy the Caccioppoli-type inequality with the same uniform constant as u_i. Thus, we can repeat the argument from Lemma 3.13 to say that U_∞ is also a minimizer and the convergence is strong in $H^1_{s,loc}(B_1^+)$. We have the following Caccioppoli-type inequality

$$\mathbf{\Theta}_s\left(U_\infty, x_0, \tilde{\theta}\right) \leq \lambda\, \mathbf{\Theta}_s\left(U_\infty, x_0, 2\tilde{\theta}\right) + \frac{C}{\lambda^2}\,\mathbf{W}_s(U_\infty, x_0, 2\tilde{\theta}),$$

As in (RM2022, Lemma 5.1), we have that there is $\gamma > 0$ such that

$$\mathbf{W}_s(U_\infty, x_0, 2\tilde{\theta}) = \frac{\psi(d,a)}{(2\tilde{\theta})^{d+a+1}} \int_{B^+_{2\tilde{\theta}}} x_{d+1}^a |U_\infty(x) - \overline{U_\infty}|^2\, dx \leq \psi(d,a)\, C(d) \left(2\tilde{\theta}\right)^\gamma$$

So we obtain, by taking $\lambda = \tilde{\theta}^{\gamma/3}$, that

$$\mathbf{\Theta}_s\left(U_\infty, 0, \tilde{\theta}\right) \leq C\tilde{\theta}^{\gamma/3},$$

where C is depending on (d,a). For i large enough, we obtain

$$\mathbf{\Theta}_s\left(U_i, 0, \tilde{\theta}\right) = \frac{1}{2\tilde{\theta}^{d+a-1}} \int_{B^+_{\tilde{\theta}}} x_{d+1}^a |\nabla U_i(x)|^2\, dx < \frac{1}{4} + 2C\tilde{\theta}^{\gamma/3} \leq \frac{1}{2} = \frac{1}{2}\,\mathbf{\Theta}_s(U_\infty, 0, 1)$$

for any $\tilde{\theta} \leq (8C)^{-3/\gamma}$, which contradicts (3-33). ■

Lemma 3.18 *Let $u \in H^1_s(\mathbb{R}^{d+1}_+; \mathbb{C}_k)$ be a minimizing of $\mathbf{E}_s(u, \Omega)$. Suppose $B_R^+(x_0)$ is a half-ball such that $R \leq 1$ and $\overline{\partial^0 B_R^+(x_0)} \subset \Omega$. There exists $\varepsilon_1 > 0$, $\theta_1 \geq 2$ and $C > 0$ depending on d and $\|u\|_{L^2(\Omega)}$ such that if*

$$\mathbf{\Theta}_s(u, x_0, R) \leq \varepsilon_1,$$

then

$$\rho^{1-d} \int_{B_\rho(y)} |\nabla u|^2\, dx \leq C \left(\frac{\rho}{r}\right)^\gamma r^{1-d} \int_{B_r(y)} |\nabla u|^2\, dx$$

on every $B_r(y) \subset B_R^+(x_0)$ such that $y \in B^+_{R/3}(x_0)$ with $y_{d+1} \geq \theta_1 r$, $0 < \rho \leq r$ and some $\gamma = \gamma\left(d, \|u\|_{L^2(\Omega)}\right) \in (0,1)$.

Proof. Let suppose that, $\mathbf{\Theta}_s(u, x_0, R) \leq \varepsilon_1$ for an ε_1 to be chosen and let ε like in (R2018, Lemma 4.12); that's ε such that, if $v \in H^1(\Omega/\mathbb{C}_k)$ minimizing of

$$E_{\tilde{g}}(u, B_1) := \int_{B_1} |\nabla u|^2_{\tilde{g}} \sqrt{\det \tilde{g}}\, dx$$

satisfying

$$\sum_{i,j,k} |\partial_k \tilde{g}_{i,j}| < \varepsilon \quad \text{and} \quad \int_{B_1} |\nabla v| \, dx \leq \varepsilon,$$

then v is Hölder continuous in $B_{1/2}$ and

$$|v(x_1) - v(x_2)| \leq C|x_1 - x_2|^\gamma, \ \forall x_1, x_2 \in B_{1/2}, \tag{3-37}$$

where ε, C and $\gamma \in (0,1)$ are depending on d and $\|u\|_{L^2(\Omega)}$.

According to (R2018, Section 1), the Dirichlet energy $\mathbf{E}_s(u,\Omega)$ is the same energy corresponding to the metric $g = x^b\delta$ where $b = \frac{2a}{d-1}$ and δ is the Euclidean metric. Let $\tilde{g} = \delta \cdot (1 + ry_{d+1}^{-1}x_{d+1})^b = y_{d+1}^{-b} g(rx+y)$, then

$$\begin{aligned} E_{\tilde{g}}(v, B_1) = \int_{B_1} |\nabla v|_{\tilde{g}}^2 \sqrt{\det \tilde{g}} \, dx &= \frac{1}{2}\int_{1/2} (1 + ry_{d+1}^{-1}x_{d+1})^a |\nabla v|^2 \, dx \\ &= \frac{r^{-1-d}}{2} \int_{B_r(y)} z_{d+1}^a \left| \nabla v \left(\frac{z-y}{r} \right) \right|^2 \, dz. \end{aligned}$$

There exists constants c and C depending only on d such that

$$c \leq (1 + ry_{d+1}^{-1}x_{d+1})^b \leq C$$

So we can choose $\widetilde{C}$ independently of b such that

$$\sum_{k=1}^{d+1} |\partial_k \left(1 + ry_{d+1}^{-1}x_{d+1}\right)^b| = ry_{d+1}^{-1} \left|1 + ry_{d+1}^{-1}x_{d+1}\right|^{b-1} \leq \widetilde{C} r y_{d+1}^{-1},$$

so, $\sum_{i,j,k} |\partial_k \tilde{g}_{i,j}| < \varepsilon$ by taking $\theta_1 \geq \max\{2, (d+1)\widetilde{C}\varepsilon_1^{-1}\}$.

If we define $u_{r,y}(x) := u(rx+y)$ for $x \in B_1$, $u_{r,y} \in H^1(B_1;\mathbb{C}_k)$ and noting that $E_{\tilde{g}}(u_{r,y}, B_1) = r^{1-d}E_{\tilde{g}}(u, B_r(y)) = \mathbf{E}_s(u, B_r(y))$, we have that $u_{r,y}$ is $E_{\tilde{g}}$-minimizing, and from Lemma 3.16, there is $C(d)$ such that

$$\int_{B_1} |\nabla u_{r,y}(x)|^2 \, dx = r^{1-d} \int_{B_r(y)} |\nabla u|^2 \leq C(d)\, \mathbf{\Theta}_s(u, x_0, R) \leq \varepsilon,$$

by taking $\varepsilon_1 < \varepsilon / C(d)$, then we can follow that $v_{r,y}$ is Hölder continuous in $B_{1/2}$.

Re-scaling implies that u is Hölder continuous in $B_{r/2}(y)$. From (R2018, Lemma 4.14) , there is $C(d,k)$ such that

$$\sup_{B_{1/2}} |\nabla u_{r,y}|_{\tilde{g}}^2 \leq C(d,k) r^{-1-d} \int_{B_1} |\nabla u_{r,y}|_{\tilde{g}}^2 \sqrt{\det(\tilde{g})} \, dx$$

then, after a change of variables

$$r^2 \sup_{B_{r/2}} x_{d+1}^a |\nabla u|^2 \leq C(d,k) r^{1-d} \int_{B_r(y)} x_{d+1}^a |\nabla u|_{\tilde{g}}^2 \, dx$$

Then, for any $\sigma \in (0, 1/2]$

$$(\sigma r)^{1-d} \int_{B_{\sigma r}(y)} x_{d+1}^a |\nabla u|^2 \, dx \leq C(d,k)\sigma^2 r^{1-d} \int_{B_r(y)} x_{d+1}^a |\nabla u|^2 \, dx$$

By calling $C = \frac{4}{C(d,k)}$, we get

$$e^{a\sigma r\xi}(\sigma r)^{1-d} \, \mathbf{E}_s\Big(u, B_{\sigma r}(y)\Big) \leq \frac{1}{C} \, e^{ar\xi} r^{1-d} \, \mathbf{E}_s\Big(u, B_r(y)\Big)$$

Letting $\omega(\rho) := e^{a\rho\xi}\rho^{1-d} \, \mathbf{E}_s\Big(u, B_\rho(y)\Big)$ for $\xi = \Big(y_{d+1} - r\Big)^{-1}$ and $\rho \leq r$. In that case, Lemma 3.4 says that ω is non-decreasing, also it satisfies $\omega(\sigma r) \leq C^{-1}\omega(r)$, from Lemma 3.15, there exists $\gamma \in (0, 1)$ depending on d and k such that

$$\omega(\rho) \leq C \left(\frac{\rho}{r}\right)^\gamma \omega(r)$$

that concludes the proof.

■

3.5 Proof of Theorem 1.3

Proof. [Proof of Theorem 1.3] Set $\varepsilon = \min(\varepsilon_0, \varepsilon_1)$, where ε_0 and ε_1 are the numbers from Lemmas 3.17 and 3.18 respectively.

By the Lemma 3.17, there is $\theta_0 \in (0, 1/4)$ depending on d, $\|u\|_{L^2(\Omega)}$ and a such that

$$\mathbf{\Theta}_s\left(u, y, \theta_0\rho\right) \leq \frac{1}{2} \, \mathbf{\Theta}_s\left(u, y, \rho\right),$$

for every $B_\rho^+(y) \subset B_R^+(x_0)$ such that $y \in B_{R/2}(x_0) \cap \partial\mathbb{R}_+^{d+1}$ and $\rho \leq R/2$, so because the monotonicity of $\mathbf{\Theta}_s(u, y, \rho)$ on ρ from Proposition 3.3, we get by the Lemma 3.15 that

$$\mathbf{\Theta}_s\left(u, y, \rho\right) \leq 2\left(\frac{2\rho}{R}\right)^{\gamma_0} \mathbf{\Theta}_s\left(u, y, R/2\right)$$

for some $\gamma_0 \in (0, 1)$ depending on d, $\|u\|_{L^2(\Omega)}$ and a.

Note that, if $y_{d+1} = 0$ then

$$\int_{B_\rho^+(y)} x_{d+1}^{-a} \, dx = \rho^{1+d-a}\psi(d, a),$$

where

$$\psi(d, a) = \int_{[0,\pi]^d} \frac{\sin^{d-1}(\varphi_1)\sin^{d-2}(\varphi_2)\ldots\sin(\varphi_{d-1})}{(1 + d - a)\cos^a(\varphi_1)} \, d\varphi_1 \, d\varphi_2 \ldots \, d\varphi_n$$

Then

$$\rho^{-2n}\int_{B_\rho^+(y)} x_{d+1}^{-a}\,dx\int_{B_\rho^+(y)} x_{d+1}^{a}|\nabla u|^2\,dx = \psi(d,a)\,\boldsymbol{\Theta}_s\,(u,y,\rho)$$
$$\leq 2^{\gamma_0+1}\,\psi(d,a)\left(\frac{\rho}{R}\right)^{\gamma_0}\boldsymbol{\Theta}_s\,(u,y,R) \tag{3-38}$$

We also have, from Lemma 3.18, that there is $\theta_1 \geq 2$, $\gamma_1 \in (0,1)$ and $C > 0$ depending on d and $\|u\|_{L^\infty(\Omega)}$ such that

$$\rho^{1-d}\int_{B_\rho(y)} |\nabla u|^2 \leq C\left(\frac{\rho}{r}\right)^{\gamma_1} r^{1-d}\int_{B_r(y)}|\nabla u|^2, \quad 0\leq\rho\leq r$$

on any $B_r(y)\subset B_R^+(x_0)$ such that $y\in B_{R/3}^+(x_0)$ with $y_{d+1}\geq\theta_1 r$.

In this case, note that

$$\frac{1}{2}\,y_{d+1}^a \leq x_{d+1}^a \leq 6\,y_{d+1}^a, \quad x\in B_\rho(y)$$

On the one hand

$$\rho^{-2n}\int_{B_\rho(y)} x_{d+1}^{-a}\,dx\int_{B_\rho(y)} x_{d+1}^{a}|\nabla u|^2\,dx \leq 12\,\rho^{1-d}|B_1|\int_{B_\rho(y)}|\nabla u|^2\,dx$$

and, on the other hand

$$r^{1-d}\int_{B_r(y)}|\nabla u|^2\,dx \leq C(d)\,\boldsymbol{\Theta}_s(u,x_0,R)$$

because the Lemma 3.16, then

$$\rho^{-2n}\int_{B_\rho(y)} x_{d+1}^{-a}\,dx\int_{B_\rho(y)} x_{d+1}^{a}|\nabla u|^2\,dx \leq C_1\left(\frac{\rho}{R}\right)^{\gamma_1}\Theta_s(u,x_0,R) \tag{3-39}$$

where

$$C_1 = 12\,|B_1|\,C(d)\left(\frac{R}{r}\right)^{\gamma_1}$$

is depending on d and $\|u\|_{L^2(\Omega)}$.

Note that, by Hölder's Inequality, and taking $\gamma = \frac{1}{2}\min(\gamma_1,\gamma_2)$, the equations (3-38) and (3-39) produce

$$\rho^{-d}\int_B|\nabla u| \leq \left(\rho^{-2n}\int_B x_{d+1}^{-a}\,dx\int_B x_{d+1}^a|\nabla u|^2\,dx\right)^{1/2}$$
$$\leq C(d,a)\,\Theta_s(u,x_0,R)^{\frac{1}{2}}\left(\frac{\rho}{r}\right)^{\gamma},$$

where $B = B_\rho^+(y)\subset B_R^+(x_0)$ such that $y\in B_{\theta_2 R}(x_0)\cap\partial\mathbb{R}_+^{d+1}$ with $\rho\leq\theta_2 R$ or $B = B_\rho(y)\subset B_R^+(x_0)$ such that $y\in B_{\theta_2 R}^+(x_0)$ with $y_{d+1}\geq\theta_1\rho$.

The Lemma 3.14 and Remark 4 conclude the proof. ■

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