Pontifícia Universidade Católica
do Rio de Janeiro

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Law and Order(ing): Providing a Natural Deduction System and Non-monotonic Reasoning to an Intuitionistic Description Logic

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Informática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Informática.

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Abstract

Pinto de Alkmim, Bernardo; Hermann Haeusler, Edward (Advisor); Nalon, Cláudia (Co-Advisor). **Law and Order(ing): Providing a Natural Deduction System and Non-monotonic Reasoning to an Intuitionistic Description Logic**. Rio de Janeiro, 2023. 138p. Doctorate Thesis – Departamento of Informatics, Pontifícia Universidade Católica do Rio de Janeiro.

The intuitionistic description logic iALC was created to model and reason over the domain of Law based on Kelsenian Jurisprudence [1]. Over the past decade, this logic has been used in several ways to either model norms or formalise legal reasoning [2, 3, 4, 5, 6, 7, 8, 9, 10]. In this work we intend to complement previous research done with this logic by filling some gaps found while working with it.

The first gap occurs in iALC needing an intuitive way to explain reasoning for non-logicians. It has a sound and complete (concerning intuitionistic conceptual models [3]) Sequent Calculus (SC) [6] that has seen less usage than expected due to its non-intuitive way of presenting a proof. We present a (*quasi*-)normalising, sound and complete (w.r.t. TBox validity for intuitionistic conceptual models) Natural Deduction (ND) System to cover this difficulty in explaining SC to non-logicians, especially those in the domain of Law, which are essential to us. We do not achieve full normalisation due to a kind of derivation which cannot be normalised - aside from this exception, the rest of the system can provide uniform derivations.

The second gap is being unable to deal with non-monotonic reasoning (NMR). Usually, one considers monotonic reasoning, in which, if one can conclude something from a set of premises, there is no way to add another premise to avoid said conclusion. This is not the case in a court of law, for instance, in which different parties aim to convince a judge or jury of opposite consequences by adding different premises to the case itself. We provide an exploratory investigation of an extension of iALC to deal with NMR to represent legal reasoning in aspects of the Law, such as the judicial process, which is non-monotonic by nature. We present desirable properties and a possible application of such a system via a case study.

We explain further the motivation for both the ND system and the NMR extension and the decisions taken for both.

Keywords

iALC; Intuitionistic Logic; Description Logic; Legal Reasoning; Natural Deduction; Non-monotonic Reasoning.

Resumo

Pinto de Alkmim, Bernardo; Hermann Haeusler, Edward; Nalon, Cláudia. **Lei e Ordenação: Adicionando Dedução Natural e Mecanismos de Raciocínio Não Monotônico a uma Lógica Descritiva Intuicionista**. Rio de Janeiro, 2023. 138p. Tese de Doutorado – Departamento de Informática, Pontifícia Universidade Católica do Rio de Janeiro.

A lógica descritiva intuicionista iALC foi criada para modelar e raciocinar sobre o domínio de Leis baseada na Jurisprudência Kelseniana [1]. No decorrer da década anterior, essa lógica foi usada de diversas maneiras para modelar normas ou formalizar raciocínio jurídico [2, 3, 4, 5, 6, 7, 8, 9, 10]. Neste trabalho pretendemos complementar trabalhos anteriores ralizados com essa lógica ao preencher algumas lacunas encontradas enquanto trabalhando com ela.

A primeira lacuna ocorre por iALC não ter um modo intuitivo de explicar raciocínio nela realizado para pessoas fora do domínio da Lógica. Ela tem um Cálculo de Sequentes (CS) [6] correto e completo (com respeito a modelos conceituais intuitionistas [3]) que tem sido menos usado que o desejado, e isso se dá em grande parte devido à maneira pouco intuitiva com que CS representa provas. Apresentamos um sistema de Dedução Natural (DN) correto e completo e com (*quasi-*)normalização para compensar por essa dificuldade em explicar CS para não-lógicos, especialmente os do domínio legal, essenciais para nossa pesquisa. Normalização completa não é possível devido a um tipo de derivação - tirando essa exceção, o resto do sistema gera derivações uniformes.

A segunda lacuna envolve não poder lidar com raciocínio não-monotônico (RNM). Em geral, utiliza-se raciocínio monotônico, no qual, se é possível concluir algo de um conjunto de premissas, não há como acrescentar outra premissa de modo a evitar a conclusão prévia. Isso não é o caso em um julgamento legal, por exemplo, no qual lados opostos buscam convencer um juiz ou júri de consequências opostas ao adicionar premissas diferentes ao caso em questão. Propomos uma investigação de caráter exploratório em busca de uma extensão de iALC para lidar com RNM a fim de representar raciocínio jurídico em outras facetas da Lei como o processo judicial, que é não-monotônico por natureza. Apresentamos propriedades desejadas e uma possível aplicação de um sistema assim via um estudo de caso.

Detalhamos mais a motivação tanto para o sistema de DN quanto a extensão de RNM, assim como as decisões tomadas ao criar cada um.

Palavras-chave

iALC; Lógica Intuicionista; Lógica Descritiva; Raciocínio Legal; Dedução Natural; Raciocínio Não-monotônico.

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Bells and whistles
Are all you need
Plus a comprehensive knowledge of the law!

Bobby Flanagan, *Bells and Whistles - Schmigadoon.*

1
Introduction

In this thesis we complement previous work done with the logic iALC [2, 3, 4, 5, 6, 7, 8, 9, 10], an intuitionistic description logic tailored to model and reason over the domain of Law based on Kelsenian Jurisprudence [1]. Over the past decade, iALC has been used to deal with the modelling of norms and legal reasoning. This thesis aims to fill some gaps faced when working with this logic in the past.

By having logic as a formal way to represent reasoning, we aim with our work to aid those taking part in legal reasoning via providing ways to verify if their reasoning follows in a sound manner. This usually happens in the processes of legislation and adjudication, and our works helps those involved in finding antinomies and inconsistencies in reasoning, facilitating these processes, and possibly optimising their duration. We have no intention of substituting human agents in the related domains.

There are two main components to the thesis: a sound and complete Natural Deduction (ND) system for TBox validity w.r.t. intuitionistic conceptual models [3] which has a limited notion of normalisation, and a proposal of augmentation of iALC with Non-monotonic Reasoning (NMR) called ĩALC, based mainly on the KLM axioms [11, 12], but with an intuitionistic conditional semantics based on [13, 14, 15, 16].

The first part is an improvement of what was published in [17], which contained an error in the soundness proof of one rule (namely, $\exists e$) and is now corrected. It consists of a labelled ND system for iALC, which was used first in [10], despite having yet to be fully formally introduced. Here, we present it formally - especially the motivation for using labels - and prove its main properties: soundness, completeness, and (*quasi-*) normalisation, as well as give an example of its application. We do not achieve full normalisation due to a kind of derivation in this system, which cannot be normalised. Aside from this exception, the rest of the system can provide uniform derivations. Further discussion on this specific topic is in Section 3.3.2.

The second part is work in progress, and the results shown here consist of a *desiderata* for an extension of iALC that would deal with non-monotonicity in legal reasoning, which we name ĩALC. We based our investigation mostly on [18, 19, 20, 14, 15, 21, 22]. In general, $\widetilde{\text{i}}$ALC is iALC with a new concept-forming operator that acts as non-monotonic entailment, encompassing certain types of non-monotonic legal processes. One of these processes is the judicial process, which occurs in a court of law when two opposing parties try to convince a third party (a judge or jury) of each of their arguments. During this process, each side tries to prevent any possibility that the judge or jury may side with the opposing party by reaching an unwanted conclusion. The judge or jury, during this process, may side with the first or the second party in an alternating matter depending on each of their arguments, and, as these sides are in opposition to one another, each of the *possible* conclusions to which they arrive are in direct contradiction to one another.

1.1 Motivation

This research project intends to fill specific gaps perceived when working with iALC in the past few years. In the following, we explicit the problems or weaknesses we encountered in the previously proposed formalisations, motivating our proposal.

1.1.1 Previous Work with iALC

iALC is an intuitionistic description logic originally introduced in [2] to model and reason about Law. The intuitionistic aspect of it allows for better fitting into Kelsenian Jurisprudence [1], an important doctrine in the domain of Law, which is especially important in negating concepts in description logic, as explained in Section 2.1. The motivation for it being a description logic lies in avoiding Jørgensen's Dilemma and several paradoxes encountered in deontic logics, as these logics give the same definition of validity to declarative and normative sentences alike. In Section 2.4.1, we introduce both the dilemma and the paradoxes and show how iALC avoids them in further detail.

iALC has a Sequent Calculus (SC) [6] which is sound and complete w.r.t. intuitionistic conceptual models. However, in [7, 9, 8, 10], even though the SC was already formalised, some form or other of ND was already being informally utilised in order to explain better what happens in the logic. We believe ND facilitates the conception of a formal derivation as a proof that is desired, especially considering non-logicians. This work led to the publication of [17], a ND system to formalise this usage of more explainable, intuitive reasoning. The ND system in Chapter 3 improves the one in [17], correcting a mistake in the soundness proof of rule $\exists e$.

ND was first provided by Jaśkowiski in [23] upon observing that mathematicians do not make use of axiomatic systems to prove theorems but use higher forms of deduction and assumptions in natural language. Gentzen, who proposed a similar system [24] to that of Jaśkowiski, also defended this. According to Gentzen, SC is a meta-calculus for ND, with no commitment to explainability or verisimilitude to mathematical reasoning.

We believe that this motivation for reflecting the logical form of reasoning mathematicians use is enough to consider ND a good tool to represent formal arguments to people, whether they are experts or not. Although we are aware of empirical studies in explainability of logic and formal ways of reasoning [25, 26, 27, 28, 29], this work focuses on the formal constructions of the logic. However, with this work, we want to enable future empirical studies with iALC in the different legal processes to which it can be applied.

1.1.2
Expanding the Scope: Non-monotonic Reasoning

NMR is a research field in which the goal is to model kinds of reasoning that are non-monotonic, i.e. given a set of premises Γ and a conclusion α and $\Gamma \models \alpha$, it is not always the case that α will still be true by adding any premise γ to Γ. It may be the case that $\Gamma \cup \gamma \not\models \alpha$. For instance, if we have a bird, we may conclude that it flies, i.e. $b \to f$. However, if it is also a penguin, then f is no longer the case: $b \wedge p \to \neg f$, even though we know all penguins are birds. Thus, the bird can fly or not, depending on its status as a penguin or general bird. This kind of reasoning happens in many domains of human discourse and is essential to Legal Reasoning in some legal processes, such as the judicial process.

In a court of law, different parties present law-based arguments - sustained by different pieces of evidence and statements from parties involved in the cases in question - against one another in front of a judge or a jury to convince them of their theses. During this process, both parties present new evidence and arguments that aim to change the conclusions to which the judge would arrive. Faithful modelling of this process is not naturally monotonic: since it is based on adding new information, the conclusion may change.

In our point of view, this situation is better represented in logic rather than rhetoric [30], because we wish to establish formal relations between that which is said in the court of law in order to represent the Law itself. In rhetoric, the main focus deviates from that.

Even though we show this situation in an argumentation-centred environment, in this work we chose not to increment iALC with Argumentation Theory or Argumentation Frameworks [31]. Instead, we increment the expressiveness of the logic itself. We provide an overview on Argumentation Frameworks in the end of Section 4.4.1.

The current formalisation of iALC deals only with monotonic reasoning. However, it would be interesting to deal with aspects of law other than modelling and legislation, such as the judicial process stated above and the juridical process. ĩALC aims to fill this gap.

In this work, we increment the language of iALC and add some way to reason on it via a set of axioms based on the KLM axioms [11, 12]. Systems based on these axioms usually consider a classical basis instead of an intuitionistic one, so in order to adapt them to our logic, we have further extended our investigation by considering the works in [13, 14, 15]. We then discuss semantic constraints based on these intuitionistic systems and their relevance to ĩALC, especially considering its intended use to formalise law.

1.2 Organisation of Chapters

This document is organised as follows. Chapter 2 introduces the components and legal basis for logic iALC. Chapter 3 shows the ND System we created and the decisions we took in its design (especially considering the use of labels), as well as proofs of terminating (*quasi*-)normalisation, soundness and completeness with regards to TBox validity following intuitionistic conceptual models [3]. Chapter 4 provides motivation and background to apply non-monotonic reasoning to legal reasoning. Chapter 5 presents our ongoing research on adding NMR to iALC and indicates our main inspirations and a discussion of related semantic restrictions. Finally, in Chapter 6, we summarise our contribution and discuss future work.

2
The Logic iALC

In this Section, we will introduce the legal context as a motivation for using logic iALC and what Intuitionistic Logic and Description Logic (DL) are before presenting the logic itself. iALC is a DL of intuitionistic semantics created to deal with Law and legal reasoning [2, 3, 4, 5, 6, 7, 8, 9, 10].

We will then introduce its core concepts and basis in the legal context, provide its syntax and semantics, give an example of modelling, and briefly discuss the complexity of TBox satisfiability (the usual kind of satisfiability in DLs).

2.1
Related Works in Legal Reasoning

Reasoning processes, specifically in the domain of Law, are called *legal reasoning*. Legal reasoning is defined as an argument (or set of arguments) that aims to show whether a claim or decision is or is not justified based on legal text and principles.

For those in the domain of Law, legal reasoning is usually taught having only practice in mind, utilising analogies over case studies, which lacks formalisation. To formally represent legal reasoning, it is necessary to have some representation of the reasoning itself and a way to represent the norms themselves. Logic is a natural candidate for this task, as we can consider some proof procedure for the reasoning. However, the level of formalisation the field of Logic requires is usually detached from the syllabus of Law schools. For those in Law, more concrete methodologies are not introduced, even though the practice of law has a solid foundation in logical syllogisms and argumentation theory. The practical usage of these results by those in the field of Law itself - application of formal representation of Law in proper legal cases or legislation, for instance - ends up left behind, underlining a gap between the result of these studies and the area which would benefit from them effectively. There are a few exceptions, however, such as [32], in which Khan et al. convert legal texts to a set of rules of classical propositional logic based on a set of laws of the USA that deal with medicine via tools for relational modelling and a decision support system specialised for this domain.

Walker [33] states that works that aim to establish methodologies are rare and not well-known for those in Law, being applied primarily to areas of knowledge that use Law as a *subject*. Walker also proposes a formalisation of legal reasoning based on rules, i.e. deductions, emphasising his search for a logic that may represent this kind of reasoning faithfully. An attractive characteristic of this work is the structure at which the author arrives to resolve a judicial case. He divides this structure into two parts: *pieces of evidence* and *legal rules*. For the pieces of evidence, he utilises what he calls *instantiated plausibility schemata*, i.e. instances which may or may not follow what is written in the law and end up acting as hypotheses of deductive reasoning. For the legal rules, he utilises *implication trees*, which, as the name suggests, are deduction trees restricted to classical implication over a particular formalisation of legal text in deontic logic (since the norms become propositions).

The general schema Walker presents is relatively simple and insufficient at capturing the entirety of legal reasoning, as it is limited to monotonic reasoning and deontic logic. However, it can be generalised to more complex systems, which may be more accurate in expressing all the details of Law and legal reasoning.

Still, legal reasoning is not a monolithic phenomenon and is divided into several legal processes [34], e.g. legal regulation, lawmaking, implementation, enforcement, and interpretation. Some of those are non-monotonic, as is the judicial process (or process of *adjudication*), which represents what happens in a court of law - which calls for a more appropriate way to model them. We discuss these aspects in Chapters 4 and 5 in further detail. For now, we will consider only monotonic aspects of legal reasoning.

2.1.1
Modelling Law and (Monotonic) Legal Reasoning

Deontic logic is the usual logic for representing and reasoning about laws, i.e. normative sentences, and it consists of adding some form of modality to propositions to reason about obligations, permissions, and impossibilities, for instance.

Mally made the first proposal of a formal system for deontic logic in [35]. Mally used the operator $!p$ to represent *p is obligatory* and provided an axiomatic system. However, this system faces several inconsistencies and is considered unacceptable by most deontic logicians nowadays. Mally himself noticed some redundancies and inconsistencies in some of the theorems he listed. One critic of his work was Menger, who in [36] stated that the operator introduced did not offer any new meaning to a classical proposition since it allowed for the equivalence $p \leftrightarrow !p$. There have been some attempts to alter Mally's original formalism, but all have some form of inconsistency or loss of expressiveness. For example, many of Mally's inconsistent theorems are proven by changing the material implication to a *relevant* implication [37]. However, some other essential ones are not anymore, such as $p \rightarrow !p$. By changing to intuitionistic logic [38], some other theorems are not derivable, e.g. $!(p \vee q) \rightarrow (!p \vee !q)$.

The most widely used formalism for deontic logic is Standard Deontic Logic (SDL) [39], by von Wright. SDL is basically the modal logic **KD**, i.e. it extends classical propositional logic with the operator $\Box$ (with $\Box p$ meaning that p is obligatory), and its dual $\Diamond$ (where $\Diamond p$ is defined as $\neg\Box\neg p$) and it has axioms **K** and **D**:

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \qquad (\mathbf{K})$$

$$\Box p \rightarrow \Diamond p \qquad (\mathbf{D})$$

as well as the rule of *necessitation*:

$$\text{If } p \text{ is a theorem, then so is } \Box p \qquad (\mathbf{N})$$

Even SDL faces some ontological issues, however. In [40], Hansen, Pigozzi, and van der Torre present ten of these problems, mainly in the form of paradoxes as well as Jørgensen's Dilemma [41], which was one of the criticisms to Mally's system that persisted to SDL. Jørgensen's Dilemma deals with the problem of considering norms as propositions, i.e. giving them a truth-value, according to the classical notion of what a truth-value is. When one considers a norm, say, *remove the book from the table*, the notion of validity of classical logic only applies to whether or not one obeys the norm, i.e. the book was removed from the table, represented by a declarative sentence. The dilemma is whether or not arguments containing normative sentences can be considered valid. Let us see an example with two premises and a conclusion:

Arrest all criminals!

Jack is a criminal.

Therefore, *arrest Jack.*

The validity of an argument is dependent upon the validity of the premises leading to the validity of the conclusion, but with normative sentences, this notion is not adequately defined. However, in deontic logic (either Mally's or a flavour of SDL), declarative and normative sentences coexist with the same notion of validity. Thus, we must decide if we **a**) expand our notion of validity of a sentence/formula to allow these different types of sentences to be evaluated under the same notion, or **b**) accept that there is no true *deontic logic* i.e. logic *of* norms with our notion of validity over declarative sentences.

In [42], Hansen shows that some problems still happen when dealing with multimodal deontic logic, which gives a temporal aspect to obligations. In [43], Maranhão concentrates on von Wright's vision of deontic logic in his works after [44], in which he states that *there is no logic of norms*, and that deontic logic is actually about normative entailment. Maranhão then links this vision to Wittgenstein's therapeutic method [45, 46] by stating that what happens in deontic logic are *misleading analogies* and *temptations*. Maranhão states, as well, that dyadic, temporal or even deontic logics based on a logic of actions may solve some of the paradoxes, but all of them have in common the presence of Jørgensen's Dilemma. From this view, the question arises that, given this aspect of deontic logic, it is relevant not to have norms act as propositions, separating such distinct concepts. A formalism in which this happens is description logics (DLs), one of the foundations for iALC. In Section 2.3, we introduce DLs in more detail.

As for the paradoxes, one that happens in SDL is Ross' Paradox [47], which deals directly with entailment on normative statements. In declarative statements, an inference such as "*The sky is blue. Thus, the sky is blue or the walls are painted white.*" is entirely valid. When applied to norms, though, this inference may lose meaning. For example, in "*It is mandatory that the letter be sent. Thus, it is mandatory that the letter be sent or burned.*". In this case, the realisation of the second part of the disjunction implies the impossibility of realisation of the first, thus making the original norm lose meaning - in a way, we end up with the formula $\Box p \to \Box(p \vee \neg p)$. This situation makes us question what a valid entailment in SDL should mean because, even though structurally from the premise we derive the conclusion and it is valid in the logic, we cannot appropriately translate the actual meaning of the sentence back to natural language.

Another well-known paradox that happens in SDL is Prior's Paradox [48]. In his work, Prior pointed out some deontic parallels to classical propositional entailments, from which $\Box \neg p \to \Box(p \to q)$, the analogous to $\neg p \to (p \to q)$ (*ex falso sequitur quodlibet* - from falsehood, anything follows), stands out. This entailment means that doing something forbidden obligates one to do anything else. For instance, if a person robs a bank, it commits them to engage in arson. Prior states that this goes against the ordinary notion of commitment.

There are also some paradoxes grouped as the Contrary-to-Duty (CTD) paradoxes. One of them, the Chisholm Paradox, is presented in [6]. The Chisholm Paradox consists of a group of four premises. Firstly, a norm states that a particular fact or action is mandatory, $\Box p$. The second norm indicates what must happen if the first norm is followed, $\Box(p \to q)$. There is, however, a third norm which considers the possibility of p not happening, and in such case, q must not happen: $\neg p \to \Box\neg q$. Finally, there is a fact (which, in SDL, gets the same semantic treatment as norms) stating that p was not fulfilled, i.e. $\neg p$. From the first and second premises, via the axiom **K** and modus ponens, we conclude $\Box q$. From the third and fourth premises, via modus ponens, we arrive at $\Box\neg q$, generating the paradox, as both q and $\neg q$ are mandatory.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box(p \to q) \to (\Box p \to \Box q)}\,\mathbf{K} \quad \Box(p \to q)}{\Box p \to \Box q} \quad \Box p}{\Box q} \quad \dfrac{\neg p \to \Box\neg q \quad \neg p}{\Box\neg q}}{\Box q \wedge \Box\neg q}\,Adj$$

This problem is not related to the inconsistency of the norms and statements but to their representation and use. In general, problems such as the paradoxes mentioned earlier and Jørgensen's Dilemma happen in SDL due to a vague conceptualisation of what it means to be a logic that deals with norms. This situation is especially noticeable in the choice of the axiom **K**, where the entailment from the antecedent $\Box(p \to q)$, which is an expression of conditional obligation, to the consequent $\Box p \to \Box q$, which expresses itself an entailment between two obligations, was no longer considered adequate even by von Wright himself after acknowledging Prior's Paradox. In iALC, due to it being a description logic, there is no attempt at assigning truth values to norms, and we avoid the paradoxes and the dilemma. We give some examples in Section 2.4.1.

Now, we introduce the components of iALC before introducing the logic itself.

2.2 Intuitionistic Logic

The usual notion of validity (called Classical) involves two truth values, classical implication (or maybe some other variant), the usual propositional operators ($\neg$ for negation, $\wedge$ for conjunction, and $\vee$ for disjunction) and a few principles or rules of inference. For example, the principle of non-contradiction, formalised as $\neg(\varphi \wedge \neg\varphi)$, double negation elimination, formalised as $\neg\neg\varphi \to \varphi$, and the principle of the excluded middle, formalised as $\varphi \vee \neg\varphi$. From these principles, one has, for instance, that any formula must have a truth value, be it true or false, and there is no room for uncertainty. Usually, one increments this language via modalities (for Modal Logics) or quantifiers (for First-order Logic), but all formulas must still have an assignment of either true or false.

However, some of these principles allow for non-constructive proofs, whose results may not be considered enough by some, especially considering proofs of existence or disjunction. Let us take a common example of a proof of existence:

Claim. *There exist two irrational numbers p and q such that p^q is rational.*

Proof. Let us focus on $\sqrt{2}^{\sqrt{2}}$. This number is either rational or not. If it is, then $p = \sqrt{2}$ and $q = \sqrt{2}$. If it is not, then we may use $p = \sqrt{2}^{\sqrt{2}}$ and $q = \sqrt{2}$, since $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}.\sqrt{2}} = 2$, which is rational. $\square$

We proved here that there do exist two such numbers. However, one may ask oneself: What value does p have? Which one is the answer? One may expect that proofs of existence show that something exists by presenting that which exists itself, not just by stating that its existence is imperative. Intuitionistic logic aims to serve as a basis for this kind of constructive questioning.

We want no note here that there is, in fact, a constructive proof for this claim, which uses the Gelfond–Schneider theorem [49] (in French). Basically, one takes $p = \sqrt{2}$ and $q = 2log_2 3$.

Brouwer first envisioned intuitionistic logic in [50, 51] as a way to represent logic constructively. It was further developed by Heyting in his works, mainly [52]. Brouwer envisioned it as having classical logic but removing the principle of the excluded middle or double negation elimination (which are equivalent - having any of them present would revert the logic to classical). This removal diminishes the number of formulas considered valid in intuitionistic propositional logic when compared to classical. We still have, however, the principle of non-contradiction, $\neg(\varphi \wedge \neg\varphi)$, and *ex falso quodlibet*. We no longer have *reductio ad absurdum* (if we arrive at a contradiction having $\neg\varphi$ as a premise, we may conclude φ) since it is equivalent to having double negation or the excluded middle in terms of derivability.

Let us call the intuitionistic propositional logic **IPL**. Formulas in **IPL** have the usual propositional operators: $\wedge$, $\vee$, and $\rightarrow$. We also introduce $\perp$ as a symbol representing absurdity. We can use $\neg\varphi$ as an abbreviation for $\varphi \rightarrow \perp$ from this definition.

In **IPL**, we can have no proof of φ, but it may not be the case that there is a proof of $\neg\varphi$. From this, we see that we can prove fewer formulas in intuitionistic logic than in classical logic - which seems reasonable since we can no longer produce non-constructive proofs and some formulas require some form of non-constructibility.

One example is that there is no correspondence between an implication of two formulas and its contra-positive, i.e. there is no proof of $(\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)$. Only one of the directions still follows intuitionistically; namely $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$, which is verifiable via the following derivation in Natural Deduction for **IPL** in the style of Gentzen [24, 53]:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]^3 \quad [\varphi \to \psi]^1}{\psi} \quad [\neg\psi]^2}{\bot}}{\neg\varphi}\,3}{(\neg\psi \to \neg\varphi)}\,2}{(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)}\,1}{}$$

As for the invalidity of $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$, we will then assume that it is an intuitionistic tautology and see where it leads us. Let us assume that ψ is a formula for which the contra-positive in question is valid between it and any other formula φ. Since φ can be any formula, let $\varphi = \neg\bot$. Then, $\neg\varphi = \neg\neg\bot = \bot$, since $\bot$ is a formula for which the double negation works even intuitionistically, as $\neg\bot$ is $\bot \to \bot$, a tautology, thus making $\neg\neg\bot$ an absurd, $\bot$. From there, we have $(\neg\neg\psi) \to (\neg\bot \to \psi)$, which is equivalent to $(\neg\neg\psi) \to (\psi)$. Thus, since ψ is any formula, we derived the principle of double negation from the contra-positive, arriving at classical logic.

It is important to note that when one assumes a principle or axiom in a logic, one means it works for *any* formula. An axiom can be absent from a system, yet we can still apply the rule expressed by the axiom to specific formulas, as in the previous paragraph.

Despite being more strict, intuitionistic logic has great applicability and connection to concepts in computing and meta-mathematics. By the Curry-Howard Isomorphism [54], there is an equivalence between types and formulas in intuitionistic logic and algorithms and proofs in intuitionistic logic.

Intuitionistic logic also has a rich semantics of possible worlds, introduced by Kripke in [55]. Let us consider **IPL**. An interpretation is $\mathcal{I} = \langle W, \leq \rangle$, where W is a set of objects called *worlds*, and $\leq$ is a reflexive and transitive binary relation on worlds. Let us also consider $\models$, a satisfaction relation between worlds and formulas in **IPL**, and V, a valuation function that assigns propositional symbols to subsets of W, indicating the set of worlds that satisfy such formula. We also have that, for any $w, w' \in W$ and formula φ, $w \models \varphi$ and $w \leq w'$ imply $w' \models \varphi$. This property is called the persistency, heredity, or even monotonicity property. Let φ and ψ be any formulas in **IPL**, p represent a propositional symbol, and $\forall w, w' \in W$:

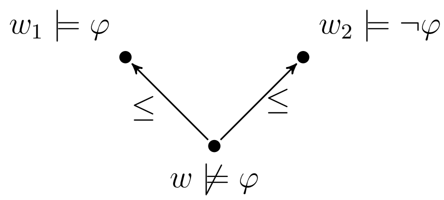
$$
\begin{aligned}
&w \models \varphi \text{ iff } w \in V(p)\\
&w \models \varphi \wedge \psi \text{ iff } w \models \varphi \text{ and } w \models \psi\\
&w \models \varphi \vee \psi \text{ iff } w \models \varphi \text{ or } w \models \psi\\
&w \models \varphi \rightarrow \psi \text{ iff } (w \leq w' \text{ and } w' \models \varphi \text{ implies } w' \models \psi)\\
&w \models \neg\varphi \text{ iff } (w \leq w' \text{ implies } w' \not\models \varphi)\\
&w \not\models \bot
\end{aligned}
$$

We can see negation as a particular case of the implication, $\varphi \rightarrow \bot$.

One may think of a world as a group of information present at a given time. We say that, given an interpretation $\mathcal{I}$, the world w satisfies the formula φ via $\mathcal{I}, w \models \varphi$. We see any worlds to which w is related as extensions of what is in w, containing what is satisfied in w or possibly more - but not less, due to the heredity property, showing the monotonicity of **IPL**.

The main difference from classical logic semantics lies in that, in intuitionistic logic, we consider φ to be true only when there is a proof of it, i.e. if it can *be constructed*. From this notion, we can no longer utilise classical implication. Intuitionistic implication $\varphi \rightarrow \psi$ now states that if there is a proof of φ, then there is a proof of ψ, which is different to the classical implication, which can be defined as $\neg\varphi \vee \psi$ - notice how, in the classical sense, we can have neither φ nor ψ and still have proof of $\varphi \rightarrow \psi$ to be true, since not having φ is enough.

To show an example of a model for **IPL**, we can have, in an interpretation $\mathcal{I}$, a world w at which φ is not satisfied, and w can be related to two worlds, w_1 and w_2 such that $\mathcal{I}, w_1 \models \varphi$ and $\mathcal{I}, w_2 \models \neg\varphi$.



Notice how different w and w_2 are w.r.t. φ satisfiability: w_2 cannot be related via $\leq$ to any φ-world, whereas w is related to the φ-world w_1. This separation is not present in the semantics of classical propositional logic and shows how intuitionistic logic differentiates between not having φ and having $\neg\varphi$. In classical logic, the definition of (for any world w) $w \models \neg\varphi$ is $w \not\models \varphi$.

2.3 Description Logic

Description Logics (DLs) [56] are a set of logics which deal primarily with the formalisation of knowledge representation (KR) and ontologies. In DLs, we generally deal with concepts, roles and individuals. Individuals are part of a non-empty domain. Concepts group individuals via the properties they hold. Roles can be seen as relations between individuals. Each DL consists of two main components (of finite size): a terminological box (TBox) and an assertion box (ABox). The TBox contains relations between concepts given a particular domain, whereas the ABox consists of assertions of individuals in the knowledge base. DLs vary in complexity and expressiveness depending on how they represent and reason over the domains. Unless stated otherwise, DLs are assumed to have **classical** operators.

To better prepare the reader for iALC, we will summarise the DL $\mathcal{ALC}$, to which our logic is more closely related. $\mathcal{ALC}$ stands for *$\mathcal{A}$ttributive $\mathcal{L}$anguage with $\mathcal{C}$omplex concept negation*, and has enough expressiveness to be applied to many situations without implying big leaps in complexity - its concept satisfiability is **PSPACE**-complete [57].

In the language of $\mathcal{ALC}$, there are three main sets: a finite set of atomic concept names N_C, a finite set of role names N_R, and a finite set Δ for the domain. Complex concepts can be built via the following operators: $\neg$ (complement), $\sqcap$ (concept conjunction), $\sqcup$ (concept disjunction), $\forall$ (universal restriction), and $\exists$ (existential restriction). Let A be an atomic concept, α and β concepts, and R a role. Then, complex concepts are formed in $\mathcal{ALC}$ by the following grammar:

$$\alpha,\beta ::= A \mid \bot \mid \top \mid \neg\alpha \mid \alpha \sqcap \beta \mid \alpha \sqcup \beta \mid \exists R.\alpha \mid \forall R.\alpha$$

With the language of $\mathcal{ALC}$ in hand, we can now move on to the semantics to understand what each component means.

An interpretation for $\mathcal{ALC}$ is a structure $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ of individuals called a domain, and an interpretation function $\cdot^{\mathcal{I}}$ mapping each role name $R \in N_R$ to a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, concept name $A \in N_C$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and individual a to $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The interpretation $\mathcal{I}$ is lifted from atomic concepts to arbitrary concepts as follows:

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
(\alpha \sqcap \beta)^{\mathcal{I}} &= \alpha^{\mathcal{I}} \cap \beta^{\mathcal{I}} \\
(\alpha \sqcup \beta)^{\mathcal{I}} &= \alpha^{\mathcal{I}} \cup \beta^{\mathcal{I}} \\
(\neg\alpha)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus \alpha^{\mathcal{I}} \\
(\forall R.\alpha)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y((x,y) \in R^{\mathcal{I}} \Rightarrow y \in \alpha^{\mathcal{I}})\} \\
(\exists R.\alpha)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \exists y((x,y) \in R^{\mathcal{I}} \wedge y \in \alpha^{\mathcal{I}})\}
\end{aligned}$$

As for other constructions, $\mathcal{ALC}$ has subsumptions between concepts (or concept inclusion), which are the statements present in the TBox, and are stated as $\alpha \sqsubseteq \beta$, meaning $\neg\alpha \sqcup \beta$. Their meaning is that each individual in α is also in β, i.e. $\alpha^{\mathcal{I}} \subseteq \beta^{\mathcal{I}}$, given an interpretation $\mathcal{I}$. The reader may also notice the similarities between subsumption and classical entailment. Not surprisingly, when changing to an intuitionistic point of view with iALC in Section 2.4, subsumption will face one of the main changes from $\mathcal{ALC}$.

Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a knowledge base, consisting in a TBox $\mathcal{T}$ and an ABox $\mathcal{A}$. Given $\mathcal{K}$ and an interpretation $\mathcal{I}$, we say that $\mathcal{I}$ models $\mathcal{K}$, notated as $\mathcal{I} \models \mathcal{K}$ when:

$$\mathcal{I} \models \mathcal{K} \text{ if and only if } \mathcal{I} \models \mathcal{T} \text{ and } \mathcal{I} \models \mathcal{A}$$

It models $\mathcal{T}$ when:

$$\begin{aligned}
&\mathcal{I} \models \mathcal{T} \text{ if and only if } \forall \Phi \in \mathcal{T}, \mathcal{I} \models \Phi \\
&\mathcal{I} \models \alpha \sqsubseteq \beta \text{ if and only if } \alpha^{\mathcal{I}} \subseteq \beta^{\mathcal{I}}
\end{aligned}$$

$\mathcal{ALC}$ also has assertions in the ABox, which are of the forms $a : \alpha$ and aRb. And it models $\mathcal{A}$ when:

$$\begin{aligned}
&\mathcal{I} \models \mathcal{A} \text{ if and only if } \forall \Phi \in \mathcal{A}, \mathcal{I} \models \Phi \\
&\mathcal{I} \models a : \alpha \text{ if and only if } a^{\mathcal{I}} \in \alpha^{\mathcal{I}} \\
&\mathcal{I} \models aRb \text{ if and only if } (a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}
\end{aligned}$$

Let us see an example. Imagine that we have a KB aiming to represent singers. There are specific categorisations for each type of singer and some relevant business information.

There are types of singers, such as those classically trained, which usually sing operatic repertoire, and pop singers, which sing genres called *popular*, such as rock, jazz, R&B, among others. Singers can be solo artists or sing in choirs. A soloist can sing with others, but choir singers cannot sing alone; otherwise, they would be solo artists. There are many types of singers depending on the kind of instrument their voice is and what sounds it can produce. A classical tenor, for instance, is a classically trained singer who can perform any *usual* operatic (or similar) tenor repertoire.

Let us assume that we consider tenor repertoire to be just the following two arias: *Every Valley shall be Exalted*, by Handel, and *O del mio Dolce Ardor*, by Gluck. Nicolai is a classically trained singer who can sing both - and no arias for other voice types since, usually, classical singers are so specialised that they cannot sing in an optimised manner repertoire for other voice types. Hence, he can be considered a classical tenor. However, despite being trained in a popular singing style, Sydney is a singer in a choir who can adequately perform the aria by Gluck - yet not the one by Handel. He is not considered a classical tenor. Carrie is another singer in Sydney's choir.

In order to represent each object in the domain, we will have to name them accordingly, as well as the concepts and roles. Then, we have:

$$\Delta = \{nicolai,\ handelaria,\ gluckaria,\ sydney,\ carrie\}$$
$$N_C = \{Singer,\ Choir,\ Solo,\ Classical,\ Pop,\ ClaTenor,\ TenorRep\}$$
$$N_R = \{singsWith,\ canSing\}$$

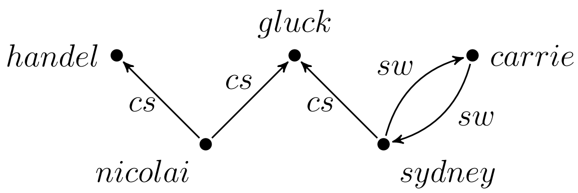
Now, we must show the relations between the different concepts and how the individuals relate to them and one another. For the TBox $\mathcal{T}$, we have:

$$\mathcal{T} = \left\{ \begin{array}{c} Singer \sqsubseteq Classical \sqcup Pop \\ Singer \sqsubseteq Choir \sqcup Solo \\ Choir \sqsubseteq \exists singsWith.Singer \\ ClaTenor \sqsubseteq Classical \sqcap \forall canSing.TenorRep \end{array} \right\}$$

And for the ABox $\mathcal{A}$, we have:

$$\mathcal{A} = \left\{ \begin{array}{c} nicolai : ClaTenor \\ handel : TenorRep \\ gluck : TenorRep \\ nicolai \quad canSing \quad handel \\ nicolai \quad canSing \quad gluck \\ sydney : Choir \\ sydney : Pop \\ sydney \quad canSing \quad gluck \\ carrie : Choir \\ carrie \quad singsWith \quad sydney \\ sydney \quad singsWith \quad carrie \end{array} \right\}$$

Notice a clear distinction between the categorisations of this example in the TBox and the *specific* information in the ABox. Now, for a model in an interpretation $\mathcal{I}$, we have:



where roles $canSing$ and $singsWith$ had their names shortened to cs and sw, respectively. Concept interpretations are:

$$\begin{aligned} Singer^{\mathcal{I}} &= \{nicolai^{\mathcal{I}}, sydney^{\mathcal{I}}, carrie^{\mathcal{I}}\} \\ Solo^{\mathcal{I}} &= \{nicolai^{\mathcal{I}}, sydney^{\mathcal{I}}\} \\ Choir^{\mathcal{I}} &= \{sydney^{\mathcal{I}}, carrie^{\mathcal{I}}\} \\ Pop^{\mathcal{I}} &= \{sydney^{\mathcal{I}}\} \\ Classical^{\mathcal{I}} &= \{nicolai^{\mathcal{I}}\} \\ TenorRep^{\mathcal{I}} &= \{handel^{\mathcal{I}}, gluck^{\mathcal{I}}\} \\ ClaTenor^{\mathcal{I}} &= \{nicolai^{\mathcal{I}}\} \end{aligned}$$

Role interpretations are:

$$\begin{aligned} canSing^{\mathcal{I}} &= \{\langle nicolai^{\mathcal{I}}, handel^{\mathcal{I}}\rangle, \langle nicolai^{\mathcal{I}}, gluck^{\mathcal{I}}\rangle, \langle sydney^{\mathcal{I}}, gluck^{\mathcal{I}}\rangle\} \\ singsWith^{\mathcal{I}} &= \{\langle sydney^{\mathcal{I}}, carrie^{\mathcal{I}}\rangle, \langle carrie^{\mathcal{I}}, sydney^{\mathcal{I}}\rangle\} \end{aligned}$$

The reader may notice that this model satisfies the TBox. If we kept the same model and removed the pair $\langle nicolai^{\mathcal{I}}, handel^{\mathcal{I}}\rangle$ from $canSing^{\mathcal{I}}$, for instance, it would no longer satisfy it, since the concept interpretation $ClaTenor^{\mathcal{I}}$ would then need to be empty, as we would no longer have a classically-trained tenor able to sing every aria in the tenor repertoire, i.e. we would not satisfy the subsumption $ClaTenor \sqsubseteq Classical \sqcap \forall canSing.TenorRep$.

We can introduce iALC directly now that we have established the principles behind it.

2.4
iALC

The logic iALC, first introduced in [2] and further developed in [3, 4, 6], is an intuitionistic description logic with nominals created to represent and reason about legal knowledge. It has a Sequent Calculus [6], sound and complete for intuitionistic conceptual models and was used to answer multiple-choice questions from the Brazilian Bar Exam in [8], to formalise Brazilian Law in [9], and to model and deal with problems involving trust, privacy and transparency in knowledge graphs in [10], showing its adequacy to deal with legal representation and reasoning.

The primary basis for iALC is Kelsenian Jurisprudence [1]. For Kelsen, legal systems are made by different *individual laws* or *norms* organised hierarchically, and they can never contradict norms that precede them, all the way to the so-called *ground-norms*, which form the basis of any legal system. We can consider an individual law as a simple norm stating that *murder is a crime*. A ground norm would be a fundamental principle which gives this law a reason to exist, such as *the life of a person cannot be treated lightly, and they must be allowed to live with dignity*, in this case.

iALC is a logic *on* norms, where norms are not propositions but individuals instead. A single norm is what the authors call a Valid Legal Statement (VLS). They also have a precedence relationship derived from the hierarchy of individual laws of Kelsenian Jurisprudence. As an example, take Brazilian Law or other similar systems. The system has the Constitution, comprised of general principles that must precede every other (usually more specific) law based on or created after it. With this notion, one can create different tiers of laws to avoid legal *antinomies* (contradictions between two or more laws) with *ground-norms* that precede them.

iALC models Kelsenian Jurisprudence due to its intuitionistic aspect. Intuitionistic logic differs from classical logic, and it is constructive by nature [52]. In representing the domain of Law, this allows, for instance, a model representing two distinct legal systems, one in which the death penalty is not allowed and the other in which it is (referencing the same concept of death penalty). In a regular (i.e. Classical) DL, this cannot be represented as directly as in its intuitionistic counterpart when modelling since a Kripke model for a Classical Logic collapses all worlds (the legal individuals) to the same one, causing a contradiction by allowing and prohibiting the death penalty at the same time. Representing in a classical DL would force us to find a more conflated way to insert Legal Individuals into the logic at the risk of losing legibility and, even worse, soundness. In the case of the death penalty, one would need a concept *DeathPenaltyBrazil* and another *DeathPenaltyTexas* as well as one or more structural rules connecting both concepts in one manner or another (and would have to find another way to model VLSs, instead of directly having them being worlds in the model). By having this intuitionistic perception, we better translate the foundations of Private International Law into the modelling, connecting VLSs who are part of different legal systems via the same concept, for example, *DeathPenalty*, without having to resort to many workarounds. Legal precedence is also defined constructively in the logic through the VLSs and does not need additional formulations in the model.

For the language of iALC, let α and β be concepts, A be an atomic concept, R be an atomic role, δ be any formula, and x be a nominal. We describe iALC formulas by the following grammar:

$$\delta ::= \alpha \mid x : \alpha$$

where the concepts α, β are given by the following grammar:

$$\alpha,\beta ::= A \mid \bot \mid \top \mid \neg\alpha \mid \alpha \sqcap \beta \mid \alpha \sqcup \beta \mid \alpha \sqsubset\!\!\!\!\!\!- \beta \mid \exists R.\alpha \mid \forall R.\alpha$$

Formulas have a restricted use of nominals as nominal assertions are not concept constructors.

As given by the above definition, the main difference from the grammar of ALC is that, in iALC, we have $\sqsubset\!\!\!\!\!\!-$ as a concept-forming operator representing intuitionistic entailment. The need for this being explicit is due to the impossibility of differentiating classical and intuitionistic logics via $\neg$ and $\sqcap$ alone [58]. This characterisation was defined and explained in [3]: since iALC has intuitionistic semantics, this entailment operator is analogous to the intuitionistic implication. This construction allows for an abstraction of the precedence relation $\preceq$ (a pre-order on worlds, as in the Kripke semantics for intuitionistic logics) in TBox reasoning.

In previous work in iALC, the symbol utilised for the entailment in a concept level was $\sqsubseteq$, which also represented intuitionistic entailment in the TBox. Due to confusion with classical DLs, we chose to utilise $\multimap$ instead.

In iALC, there are four main sets: a finite set of atomic concept names N_C, a finite set of role names N_R, a finite set of nominals N_N, and a finite set of individuals Δ.

A constructive interpretation of iALC is a structure $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \preceq \rangle$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ of entities in which each entity represents a legal individual (a valid legal statement); a refinement pre-ordering $\preceq$ on $\Delta^{\mathcal{I}}$, i.e. a reflexive and transitive relation (a pre-order), and an interpretation function $\cdot^{\mathcal{I}}$ mapping each role name $R \in N_R$ to a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and atomic concept $A \in N_C$ to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ which is closed under refinement, i.e. $w \in A^{\mathcal{I}}$ and $w \preceq w'$ implies $w' \in A^{\mathcal{I}}$. We will also refer to this last property as the *heredity rule*, as it applies to any concept, not only the atomic ones.

The interpretation $\mathcal{I}$ is lifted from atomic concepts to arbitrary concepts as follows:

$$\begin{aligned}
\top^{\mathcal{I}} &= \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &= \varnothing \\
(\alpha \sqcap \beta)^{\mathcal{I}} &= \alpha^{\mathcal{I}} \cap \beta^{\mathcal{I}} \\
(\alpha \sqcup \beta)^{\mathcal{I}} &= \alpha^{\mathcal{I}} \cup \beta^{\mathcal{I}} \\
(\neg \alpha)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y, x \preceq y \Rightarrow y \notin \alpha^{\mathcal{I}}\} \\
(\forall R.\alpha)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y (x \preceq y \Rightarrow \forall z ((y,z) \in R^{\mathcal{I}} \Rightarrow z \in \alpha^{\mathcal{I}}))\} \\
(\exists R.\alpha)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y (x \preceq y \Rightarrow \exists z ((y,z) \in R^{\mathcal{I}} \wedge z \in \alpha^{\mathcal{I}}))\} \\
(\alpha \multimap \beta)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y, (x \preceq y \wedge y \in \alpha^{\mathcal{I}} \Rightarrow y \in \beta^{\mathcal{I}})\}
\end{aligned}$$

The logic follows the semantics of **IK** [59], where the structures $\mathcal{I}$ are models for iALC if they satisfy two frame conditions (let R be a role, and w_1 and w_2, worlds):

F1 if $w_1 \preceq w_1'$ and $w_1 R w_2$ then $\exists w_2'.w_1' R w_2'$ and $w_2 \preceq w_2'$; and

F2 if $w_2 \preceq w_2'$ and $w_1 R w_2$ then $\exists w_1'.w_1' R w_2'$ and $w_1 \preceq w_1'$.

We can see them as conditions for completing the following diagrams, respectively:

$$\begin{tikzcd} w_2 \arrow[r, dotted, "\preceq"] & w_2' \\ w_1 \arrow[u, "R"] \arrow[r, "\preceq"'] & w_1' \arrow[u, dotted, "R"'] \end{tikzcd} \qquad \begin{tikzcd} w_2 \arrow[r, "\preceq"] & w_2' \\ w_1 \arrow[u, "R"] \arrow[r, dotted, "\preceq"'] & w_1' \arrow[u, dotted, "R"'] \end{tikzcd}$$

Given our interpretations of the universal and existential role restrictions on concepts, these frame conditions are necessary to maintain heredity on role relations.

Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a knowledge base (KB). An interpretation $\mathcal{I}$ satisfies the TBox $\mathcal{T}$ when:

$$\mathcal{I} \models \alpha \sqsubseteq \beta \text{ if and only if } \forall w \in \Delta^{\mathcal{I}}, w \in (\alpha \sqsubseteq \beta)^{\mathcal{I}}$$
$$\mathcal{I} \models \mathcal{T} \text{ if and only if } \forall \Phi \in \mathcal{T}, \mathcal{I} \models \Phi$$

And the ABox $\mathcal{A}$ when:

$$\mathcal{I} \models x : \alpha \text{ if and only if } \forall x_1 (x^{\mathcal{I}} \preceq x_1^{\mathcal{I}} \Rightarrow x_1^{\mathcal{I}} \in \alpha^{\mathcal{I}})$$
$$\mathcal{I} \models xRy \text{ if and only if } \forall x_1 (\forall y_1 (x^{\mathcal{I}} \preceq x_1^{\mathcal{I}} \wedge y^{\mathcal{I}} \preceq y_1^{\mathcal{I}} \Rightarrow (x_1^{\mathcal{I}}, y_1^{\mathcal{I}}) \in R^{\mathcal{I}}))$$
$$\mathcal{I} \models \mathcal{A} \text{ if and only if } \forall \Phi \in \mathcal{A}, \mathcal{I} \models \Phi$$

The definitions for the ABox assertions stem from the heredity condition and the frame conditions **F1** and **F2** - concepts and role relations are propagated to the $\preceq$-successors.

Then, finally, if $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$, then $\mathcal{I} \models \mathcal{K}$ and we say that $\mathcal{I}$ is a model of $\mathcal{K}$.

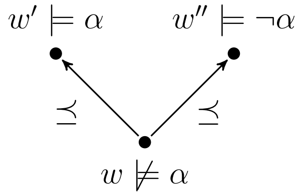
The interpretation for formulas in $\mathcal{A}$ with nominals, such as $(x : \alpha)^{\mathcal{I}}$ (where α is any concept), can be written as $x^{\mathcal{I}} \models_{\mathcal{I}} \alpha$, where $\models_{\mathcal{I}}$ is the usual satisfaction relation of Kripke models, which we can alternatively write as $\mathcal{I}, x \models \alpha$ as well.

If this happens for any entity in the model, then we can say $\models_{\mathcal{I}} \alpha$ (or $\mathcal{I} \models \alpha$). If it is the case for any interpretation, then we can say $\models \alpha$, meaning that α is valid, i.e. is a non-empty concept no matter which interpretation we utilise.

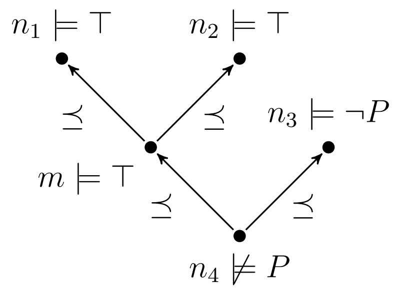
Finally, let Γ be a set of iALC-formulas and δ, an iALC-formula. We write that δ is a logical consequence of Γ as $\Gamma \models \delta$, meaning that, for every interpretation in which every formula in Γ is satisfied must also be an interpretation in which δ is satisfied.

2.4.1 Comparing iALC to Other Logics for Law

iALC avoids both the inconsistencies of Mally's formalism of deontic logic as well as the paradoxes and Jørgensen's Dilemma, faced by SDL and its variations, since in iALC norms are not propositions, but legal individuals - and therefore related to worlds in a Kripke model - instead. We can conceive an interpretation $\mathcal{I}$ having a world w such that $w \not\models \alpha$, for a concept α, preceding both a world w' such that $w' \models \neg\alpha$ and w'' such that $w'' \models \alpha$ without contradictions between the norms, for, intuitionistically, $\not\models \alpha$ differs from $\models \neg\alpha$, all the while maintaining the heredity of the semantics.



For the example of the Chisholm Paradox, we will follow the presentation in [6]. In iALC, each norm is a valid legal statement (VLS). Semantically, for iALC, each VLS is interpreted as a possible world in a Kripke model. Let us recall that the Chisholm Paradox consists of a group of four premises - expressed here in the language of SDL. Firstly, a norm states that a particular fact or action is mandatory, $\Box p$. The second norm indicates what must happen if the first norm is followed, $\Box(p \to q)$. There is, however, a third norm which considers the possibility of p not happening, and in such case, q must not happen: $\neg p \to \Box\neg q$. Finally, there is a fact (which, in SDL, gets the same semantic treatment as norms) stating that p was not fulfilled, $\neg p$. In iALC, on the other hand, the first premise becomes simply an individual, n_1, such that $n_1 : \top$ (we do not impose restrictions on this individual), since laws do not have their existence conditioned to anything but their own promulgation under a Kelsenian [1] point of view. The same happens for the second norm, which becomes the individual n_2, such that $n_2 : \top$. Since a Kripke model for iALC is a Heyting algebra, it is a lattice, and there must be a *meet* of these two worlds, m, such that $m : \top$. Intuitively, this individual states that doing what norms 1 and 2 state is obligatory. The third premise is trickier. As iALC does not represent the kind of implication in this premise, it becomes a nominal n_3 such that $n_3 : \neg P$, where the individual represents $\Box\neg q$ and P is a concept representing a mock legal document stating that $\neg p$ happened in the world of this norm. The last premise, $\neg p$, is represented by n_4 such that $n_4 \not\models P$ (notice how we do not have the construct ":" for nominals), which also acts as the meet of individuals m and n_3.



Due to the heredity rule, if $x : \alpha$, then $\forall x'(x \preceq x' \Rightarrow x' : \alpha)$. With the model above, we show no paradox in representing this situation in iALC.

2.4.2 Modelling in iALC: an Example

We will now show an example of a model of a situation that may occur involving the Law using iALC. This example will be revisited and expanded in Sections 3.4 and 5.2.

In the following situation, the central Law involved is Law 8906 [60] (in Portuguese), which states the rights and duties of attorneys in Brazil, as well as what the OAB - *Ordem dos Advogados do Brasil* (Order of Attorneys of Brazil) - is or is not entitled to do.

> John is a Law student in Brazil and is an intern at a Law firm named Firm Attorneys at Law. There, he is supervised by an Attorney named Anna. Anna can practice Law because she passed her Bar Exam and possesses an OAB registration number, as per Article 3 of Law 8906:
>
> *Practicing Law in Brazilian territory and the title of Attorney are restricted to those with a registration in the Order of Attorneys of Brazil (OAB).*[1]
>
> John, however, is still a student and cannot practice Law by himself since he only has a temporary student registration to OAB and is not an Attorney. There is an exception, though: interns at a Law firm are allowed to practice Law when supervised directly and working alongside an Attorney, as per the second paragraph of the same article:

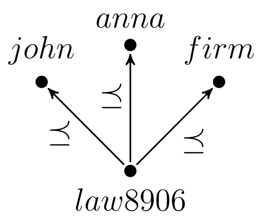
[1] Translated from the Portuguese *O exercício da atividade de advocacia no território brasileiro e a denominação de advogado são privativos dos inscritos na Ordem dos Advogados do Brasil (OAB).*

> *A regularly enrolled intern can practice the actions described in Article 1 alongside an attorney and under their responsibility.*[2]
>
> The referenced Article 1 lists all the actions exclusive to attorneys.
>
> In order to enrich his internship experience, Anna decided to supervise and work with John on a simple divorce case she received. The client, Mary, was initially reluctant to accept having him on the case due to her lack of knowledge of the full extent of the Law and feared they were up to something illegal - she was confident that interns could not practice Law under any circumstances.
>
> Anna then promptly showed her the referring article, which Mary understood and with which she complied.

Firstly, we will represent the legal documents pertaining to each person or institution or legal text in this example by a VLS: the text of Law 8906 will be represented by VLS $law8906$, John's documentation will be represented by $john$, Anna's by $anna$ and the Law Firm's by $firm$.

Law 8906 is an ordinary law, so it must precede any specific document referring to specific individuals due to legal ordering [61]. We give more detail on legal ordering in Section 4.5. Then, we have that:

$$\Delta = \{law8906,\ john,\ anna,\ firm\}$$
$$N_C = \{Attorney,\ PracticeLaw,\ Intern,\ LawFirm\}$$
$$N_R = \{SupervisedBy,\ EmployeeAt\}$$



Since $\preceq$ is reflexive, we omitted the self-arrows to avoid polluting the model visually.

For the TBox $\mathcal{T}$, we have:

$$\mathcal{T} = \left\{ \begin{array}{c} Attorney \mathrel{-\!\!\dashv} PracticeLaw \\ (Intern \sqcap \exists SupervisedBy.\exists EmployeeAt.LawFirm) \mathrel{-\!\!\dashv} \\ \hfill PracticeLaw \end{array} \right\}$$

[2]From the Portuguese *O estagiário de advocacia, regularmente inscrito, pode praticar os atos previstos no art. 1º, na forma do regimento geral, em conjunto com advogado e sob responsabilidade deste.*

And as for the ABox $\mathcal{A}$, we have:

$$\mathcal{A} = \left\{ \begin{array}{c} john : Intern \\ anna : Attorney \\ firm : LawFirm \\ john \quad SupervisedBy \quad anna \\ anna \quad EmployeeAt \quad firm \end{array} \right\}$$

From the precedence between VLSs, the entailments in $law8906$ will be present in all the other VLSs. This situation represents the (individual) law of a higher position in the hierarchy ruling over those in the same legal system. We discuss legal hierarchy and ordering in Section 4.5.

The reader may notice that we do not have the entailment $Intern \multimap \neg PracticeLaw$ in the TBox nor the assertion $law8906 : Intern \multimap \neg PracticeLaw$ in the ABox, even though the related text is still present in Law 8906. Keeping them in our model would create an inconsistency together with $(Intern \sqcap \exists SupervisedBy.\exists EmployeeAt.LawFirm) \multimap PracticeLaw$ and $law8906 : (Intern \sqcap \exists SupervisedBy.\exists EmployeeAt.LawFirm) \multimap PracticeLaw$, so they were removed. This choice is not ideal, as it diminishes the link between the legal text and our representation in iALC since these kinds of normative sentences expressing exceptions occur frequently. In Chapter 5, we discuss further how we intend to deal with these inconsistencies.

We could remedy this by creating an ordering on the different articles and paragraphs of Law 8906 via our precedence operator $\preceq$. However, this would be an executive decision, not a legally based one, based on the posterior analysis of the contents of each article since both have the same hierarchical position, legally speaking. Choosing this option opens a precedent of manually ordering each article and paragraph of each law instead of simply having a law be the conjunction of each of its textual components. Deciding this order is not viable nor wanted - we wish to obey the legal structures.

In Section 3.4, we create a derivation in our ND system for this situation, still with the omission of $Intern \multimap \neg PracticeLaw$ and $law8906 : Intern \multimap \neg PracticeLaw$ in the base. Further, in Section 5.2, we show how we should model - and expand - this situation in ĩALC as a motivation for nonmonotonicity in legal contexts.

2.4.3
Complexity in iALC

iALC is decidable regarding satisfiability, and the complexity of satisfiability and derivability problems in iALC are **PSPACE**-complete [4, 5]. The authors divide the proof of satisfiability into two parts: the upper and lower bounds. The proof of the upper bound is via a 2-person game of polynomial size of the sequent whose satisfiability one wishes to verify. One of the players has a winning strategy if and only if the proposed sequent is satisfiable. The existence of winning strategies in this scenario can be implemented by **PTIME** Alternating Turing Machines, which are implemented by **PSPACE** regular Turing Machines. Ladner provides the lower bound [62], proving that the provability problem is **PSPACE**-complete for modal logics K, T and S4 and their fusions and **coNP**-complete for S5. Finally, as **PSPACE** equals **coPSPACE**, we conclude that the provability of iALC using the sequent calculus in [6] is also **PSPACE**-complete.

3
The Natural Deduction System

As previously stated, we believe that Natural Deduction (ND) is a much more reasonable way than Sequent Calculus (SC) to present the inner workings of iALC to others, especially considering an interdisciplinary environment, which is the case for the intended application of iALC to legal reasoning.

The authors in [6] formalised a SC for iALC. In [7, 9, 8, 10], even though the SC was already formalised and functioning, in order to better explain what happens in the logic - especially in the case studies in [7, 9, 8] - some form or other of ND was already being utilised. The venues in which the authors published these works were interdisciplinary, not involving only those in the field of study of logic. Thus, to non-logicians, the examples had to be shown in an informal ND form for better understanding since the established SC lacked explainability in these cases. We believe ND facilitates the conception of a formal derivation as a proof that is desired. In [17], we provided a ND system for iALC, which contained an error in the soundness proof of one rule. The system presented here is a sound, complete, and (*quasi*-)normalising update of that system to formalise the explanations of reasoning in iALC in interdisciplinary settings. We do not achieve full normalisation due to a kind of derivation that cannot be normalised - aside from this exception, the rest of the system can provide uniform derivations.

Natural Deduction is a type of deduction system that aims to represent the structure of the operators of the language with its rules. Jaśkowski [23] and Gentzen [24] developed two systems for ND, which have differences in the presentation of rules but are equivalent in expressiveness: Gentzen's has tree-shaped derivations whereas Jaśkowski's produces linear derivations. In this work, we follow Gentzen's approach due to the history of iALC with tree-like derivations produced with Sequent Calculus.

Rules in ND aim to resemble the meaning of the operators in the language, and we split them into introduction and elimination rules. In Gentzen's system $\mathcal{N}$ [24], for instance, the rule $\wedge i$ (introduction of the conjunction operator $\wedge$) *introduces* the conjunction of two formulas in the conclusion. Given propositional formulas φ and ψ, we have the rule:

$$\frac{\varphi \quad \psi}{\varphi \land \psi} \land i$$

which we can view as if we have that φ is a theorem and ψ is a theorem, then their conjunction $\varphi \land \psi$ is also a theorem, effectively introducing the operator in the conclusion. The elimination rule for $\land$ has a similar notion; however, we eliminate the operator from the premise instead:

$$\frac{\varphi \land \psi}{\varphi} \land e_1 \qquad \frac{\varphi \land \psi}{\psi} \land e_2$$

The first rule above represents that one can conclude that φ is a theorem from $\varphi \land \psi$ being a theorem, and the second rule indicates that ψ is a theorem from $\varphi \land \psi$ being a theorem. This division into two rules happens due to $\mathcal{N}$ being a single-conclusion system (as is ours). One eliminates the $\land$ operator from the premise with these rules.

One of the characteristics that make ND interesting from a proof-theoretic point of view is the notion of sub-derivations, i.e. if one considers a derivation Π of $\Gamma \vdash_{ND} \alpha$ (from formulas in Γ to formula α) that has more than one rule being applied, it is possible to obtain derivations Π' of Π which are strictly smaller - and Π' may depend on certain hypotheses created for the sake of argumentation. This situation can be seen in rule $\to i$. This rule indicates that we can conclude that φ implies ψ when we assume φ and *obtain* ψ via a sequence of applications of rules:

$$\frac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi} \to i$$

We can, then, discharge this hypothesis of φ since it was made for the sake of the argument (the brackets around the formula represent the discharge).

For example, let us consider the derivation Π', which is a derivation of $\Gamma \cup \{\varphi\} \vdash_{\mathcal{N}} \psi$:

$$\begin{matrix}\varphi \quad \Gamma \\ \vdots \\ \psi\end{matrix}$$

We see that, from assuming φ, we can eventually conclude ψ, so we can further conclude that φ implies ψ and no longer need to assume explicitly φ.

Let Π represent the following derivation of $\Gamma \vdash_{\mathcal{N}} (\varphi \to \psi)$, ending in an application of $\to$ introduction and concluding $\varphi \to \psi$:

$$\dfrac{\begin{array}{cc}[\varphi] & \Gamma \\ \vdots & \\ \psi & \end{array}}{\varphi \to \psi} \to i$$

In this case, we say that the former derivation, Π', is a sub-derivation of the latter, Π, since Π has one extra rule application *below* the conclusion of Π'. In Section 3.3.3, we provide a formal account of a sub-derivation for the normalisation proof.

From the initial works in ND, systems for different kinds of logics were created [63, 64, 65, 66]. The following section presents the works related to the system described in this chapter.

3.1 Related Works

In [53], Prawitz provided a normalisation proof for minimal, intuitionistic and classical first-order logic and second-order and modal logic. We base our normalisation proof mainly on this work, with a few characteristics inspired by [63, 64, 65].

In his doctoral thesis [63], Alex Simpson proposed several different proof systems for intuitionistic modal logics (known as **IK**). We base our ND system mainly on the one he presented in the fourth chapter. Most of the structure of the rules and the notation utilised by us is based on his work - the main differences occurring in the interactions of our rules with ABox assertions - which are necessary to consider with DLs - through the use of labels, based on the work of Rademaker in [66].

Based on Simpson's approach, Rademaker presented in [66] a ND system for description logic $\mathcal{ALC}$, which mostly followed Simpson's rules, with some restrictions due to some limitations of $\mathcal{ALC}$ when compared to the logics with which Simpson dealt. The differences appear in the elimination and introduction rules for the existential and universal restrictions on concepts since the roles between objects in $\mathcal{ALC}$ models do not have to interact with an accessibility relation needed in intuitionistic logic. His system, however, uses labels to encompass the context of restrictions between concepts. In our system, the structure of the elimination and introduction rules for the existential and universal restrictions follows those of Simpson, using labels to give needed context.

In [64], Andou provided a normalisation proof for first-order classical logic in which the language had disjunction and existential quantification as primitives - which differs from Prawitz's proof, where he derives these operators from conjunction, negation and the universal quantification. Even though our logic is intuitionistic, this work provided an elegant way of organising the proof into different lemmas.

In [65], Medeiros showed an error in [53] for the normalisation of S4 modal logic and presented a proof for a new normalisation procedure for S4. The error consisted of the reduction around the modal operator $\Box$ not respecting the restriction needed on assumptions for the $\Box$ introduction rule. She fixes this mistake by changing the shape of the $\Box$ introduction rule - the new system added several dischargeable assumptions to the rule, correctly preserving the restriction needed. This work provided an interesting account of the interaction of modalities with normalisation. It also inspired how we deal with the universal and existential restrictions in iALC, which are modalities.

In [17], we provided a previous version of this system, which contained an error in the soundness proof of rule $\exists e$:

$$\frac{xRy \quad x : \exists R.\alpha}{y : \alpha^{\exists R}} \ \exists e$$

We now have updated this rule to:

$$\frac{x : (\exists R.\alpha)^{L} \qquad \begin{array}{c}[y : \alpha^{\exists Rx, L}][xRy] \\ \vdots \\ z : \beta^{L'}\end{array}}{z : \beta^{L'}} \ \exists e$$

which required a change in the label system and a few rules, namely *dist*, *chng* and *join*. We will explain these rules in further detail when we introduce the system directly.

The system defined in the following sections is a (*quasi*-) normalising, sound and complete system for TBox validity w.r.t. intuitionistic conceptual models [3]. We do not achieve full normalisation due to a kind of derivation that cannot be normalised. Aside from this exception, the rest of the system can provide uniform derivations. It presents labels as a way to internalise TBox reasoning assertions present in the ABox and provide context in the middle of the derivation.

3.2 The System

The ND system we developed is aimed at TBox validity and uses Gentzen-style derivations, based mainly on [53, 63]. Concepts have labels to allow for the mixing of nominals and existential/universal restrictions in a TBox-only style of reasoning, even though role assertions belong in the ABox.

We expand the language with labels for our ND system, giving the context of universal and existential restrictions in concepts. Let α an iALC concept, R a role, and x a nominal. We then extend the grammar for concepts of iALC with lists of labels (denoted by L):

$$\alpha ::= \alpha^L$$
$$L ::= \exists Rx, L \mid \forall Rx, L \mid \varnothing$$

where $\varnothing$ is an empty list of labels. Given a role R and a nominal x, a label is a pair $(\circ R, x)$, usually denoted simply $\circ Rx$, where $\circ$ can be either $\forall$ or $\exists$, indicating that the concept in question can be further restricted universally or existentially via R to nominal x. For instance, if we have $y : \alpha^{\exists Rx}$ (for a nominal y), then we also have xRy and $x : (\exists R.\alpha)$ in the ABox. Intuitively, one can see this as an indication that the assertion on y is connected to the assertion on x. This connection is usually only seen one way: if we have the assertion on the existential restriction $x : (\exists R.\alpha)$ and xRy, then we have that $y : \alpha$ (for some y in this case, as this is an existential restriction). However, without labels, we do not have the other way around - we do not know, at first, if there is an existential or universal restriction. Labels provide a path to follow when reading derivations in the ND system.

Due to the constraints imposed by the logic itself, we do not present labels inside conjunctions or disjunctions of concepts, e.g., in $(\alpha^{L_1} \sqcap \beta^{L_2})^L$, it is always the case that $L_1 = L_2 = \varnothing$, so we do not write explicitly $\varnothing$ in lists of labels inside these operators. Regarding the $\multimap$ operator, i.e. $(\alpha_1^{L_1} \multimap \alpha_2^{L_2})^L$, we have two cases: either $L_1 = L_2 = \varnothing$ or $L = \varnothing$. The rules in our calculus reflect all of these limitations, and the interactions between the concept constructors of iALC and the lists of labels respect these constraints by construction.

Formally, we divide the interpretation of formulas with labelled concepts of the syntax of the ND system (from here on, labelled formulas) into two cases: one, the standard case, where α is a concept of any shape other than $\alpha_1^{L_1} \multimap \alpha_2^{L_2}$ (with nonempty L_1 and L_2), and the second case, this exceptional situation. Let R, R_1 and R_2 be roles, L, L_1 and L_2 be lists of labels, x and y

be nominals and α, α_1 and α_2 be concepts:

$$\begin{array}{lll}\mathcal{I} \models y : \alpha^{\varnothing} & \text{iff} & \mathcal{I} \models y : \alpha \\ \mathcal{I} \models y : \alpha^{\exists Rx,L} & \text{iff} & \mathcal{I} \models xRy \text{ and } \mathcal{I} \models x : (\exists R.\alpha)^L \\ \mathcal{I} \models y : \alpha^{\forall Rx,L} & \text{iff} & \mathcal{I} \models xRy \text{ implies } \mathcal{I} \models x : (\forall R.\alpha)^L\end{array}$$

where α is not of the shape $\alpha_1^{L_1} \multimapinv \alpha_2^{L_2}$ (L_1 and L_2 non-empty). The remaining case is:

$$\mathcal{I} \models y : (\alpha_1^{L_1} \multimapinv \alpha_2^{L_2})^{\varnothing} \quad \text{iff} \quad \mathcal{I} \models y : \alpha_1^{L_1} \text{ implies } \mathcal{I} \models y : \alpha_2^{L_2}$$

This definition follows the semantics of iALC due to the heredity condition and frame conditions **F1** and **F2**.

Finally, we define the *labelled version* of iALC formulas:

Definition 3.1 (Labelled version of an iALC formula)**.** *Let δ be an iALC formula of the form $x : \alpha$ for a nominal x and a concept α. We call $x : \alpha^{\varnothing}$ the labelled version of δ.*

Definition 3.2 (Labelled version of sets of iALC formulas)**.** *Let Γ be a set of iALC formulas. We call Γ' the labelled version of Γ when every formula $\delta' \in \Gamma'$ is the labelled version of a formula $\delta \in \Gamma$, in a one-to-one correspondence.*

3.2.1 The Rules

Table 3.1 contains all the rules of the calculus. We present rules as usual, where the premises for each rule are over the bar, and its (single) conclusion lies below. The derivations in ND have a tree-like form by connecting the deduction steps given by the applied rules in the derivation. Rules $\multimapinv i$, $\sqcup e$, $\exists e$, $\forall i$ and *Gen* have formulas surrounded by brackets, indicating that the rule in question can *discharge* these assumptions, i.e. the conclusion of the derivation does not depend on them explicitly anymore, this assumption was made internally in the derivation mechanism itself.

Let α, β and γ be concepts, x, y and z be meta-variables for nominals, and R be a role. L and L' are meta-variables that represent (possibly empty) lists of labels. Labels represent universal or existential restrictions on concepts, made *implicit* by the introduction rules - the associated elimination rules make them *explicit*. If any concept α^L (or β etc.) in the rules has shape $\alpha_1^{L_1} \multimapinv \alpha_2^{L_2}$, then the usual constraints on the interactions between labels and $\multimapinv$ apply: either $L_1 = L_2 = \varnothing$, or $L = \varnothing$.

Table 3.1: Rules of the ND System

$$\dfrac{\begin{array}{c}\left[x : \alpha^{L}\right]\\ \vdots \\ x : \beta^{L'}\end{array}}{x : (\alpha^{L} \relbar\!\!\dashv \beta^{L'})^{\varnothing}} \relbar\!\!\dashv i \qquad \dfrac{x : \alpha^{L} \quad x : (\alpha^{L} \relbar\!\!\dashv \beta^{L'})^{\varnothing}}{x : \beta^{L'}} \relbar\!\!\dashv e$$

$$\dfrac{x : \alpha^{L^{(\forall)}} \quad x : \beta^{L^{(\forall)}}}{x : (\alpha \sqcap \beta)^{L^{(\forall)}}} \sqcap i \qquad \dfrac{x : (\alpha \sqcap \beta)^{L^{(\forall)}}}{x : \alpha^{L^{(\forall)}}} \sqcap e_1 \qquad \dfrac{x : (\alpha \sqcap \beta)^{L^{(\forall)}}}{x : \beta^{L^{(\forall)}}} \sqcap e_2$$

$$\dfrac{x : \alpha^{L^{(\exists)}}}{x : (\alpha \sqcup \beta)^{L^{(\exists)}}} \sqcup i_1 \qquad \dfrac{x : \beta^{L^{(\exists)}}}{x : (\alpha \sqcup \beta)^{L^{(\exists)}}} \sqcup i_2 \qquad \dfrac{x : (\alpha \sqcup \beta)^{L^{(\exists)}} \quad \begin{array}{c}\left[x : \alpha^{L^{(\exists)}}\right]\\ \vdots \\ z : \gamma^{L'}\end{array} \quad \begin{array}{c}\left[x : \beta^{L^{(\exists)}}\right]\\ \vdots \\ z : \gamma^{L'}\end{array}}{z : \gamma^{L'}} \sqcup e$$

$$\dfrac{xRy \quad y : \alpha^{\exists Rx, L}}{x : (\exists R.\alpha)^{L}} \exists i \qquad \dfrac{x : (\exists R.\alpha)^{L} \quad \begin{array}{c}[y : \alpha^{\exists Rx, L}][xRy]\\ \vdots \\ z : \beta^{L'}\end{array}}{z : \beta^{L'}} \exists e^{*^1}$$

$$\dfrac{\begin{array}{c}[xRy]\\ \vdots \\ y : \alpha^{\forall Rx, L}\end{array}}{x : (\forall R.\alpha)^{L}} \forall i^{*^2} \qquad \dfrac{xRy \quad x : (\forall R.\alpha)^{L}}{y : \alpha^{\forall Rx, L}} \forall e$$

$$\dfrac{x : \bot^{L^{(\exists)}}}{z : \alpha^{L'}} \; efq \qquad \dfrac{\begin{array}{c}[xRy]\\ \vdots \\ y : \alpha^{L}\end{array}}{y : \alpha^{L, \forall Rx}} \; Gen^{*^2} \qquad \dfrac{y : (\alpha^{\varnothing} \relbar\!\!\dashv \beta^{\varnothing})^{L^{(\forall)}, \forall Rx}}{y : (\alpha^{L^{(\forall)}, \forall Rx} \relbar\!\!\dashv \beta^{L^{(\forall)}, \forall Rx})^{\varnothing}} \; dist^{*^2}$$

$$\dfrac{xRy \quad y : (\alpha^{L, \forall Rx} \relbar\!\!\dashv \beta^{L', \forall Rx})^{\varnothing}}{y : (\alpha^{L, \exists Rx} \relbar\!\!\dashv \beta^{L', \exists Rx})^{\varnothing}} \; chng^{*^2} \qquad \dfrac{y : (\alpha^{L^{(\forall)}, \exists Rx} \relbar\!\!\dashv \beta^{L^{(\forall)}, \forall Rx})^{\varnothing}}{y : (\alpha^{\varnothing} \relbar\!\!\dashv \beta^{\varnothing})^{L^{(\forall)}, \forall Rx}} \; join^{*^2}$$

$L^{(\forall)}$ indicates that L contains only labels of universal restrictions.
$L^{(\exists)}$ indicates that L contains only labels of existential restrictions.
$*^1 : x \neq y$, $z \neq y$ and z does not appear in any undischarged assumption.
$*^2 : x \neq y$ and y does not appear in any undischarged assumption.

Rules for introduction and elimination of $\neg$ ($\neg i$ and $\neg e$) are derived from rules $\sqsubset\!\!\!\!\!\!\!\!\;\, \rightarrow i$ and $\rightarrow e$ in the case of $\beta = \bot$, since $\neg\alpha$ is defined as $\alpha \rightarrow \bot$.

Rules $\exists i$, $\exists e$, $\forall i$, and $\forall e$ require an assertion xRy (or that one can be assumed) to be applied. These rules are what allow our calculus to use the labels as restrictions. Even though this is a TBox-centered calculus, we assume that there is an adjacent ABox extension that introduces these role assertions. We add them to each rule's premises explicitly to avoid missing information from the final derivation itself. Having an auxiliary structure to a derivation in ND is counterproductive to the main objective of using ND itself - having better explainability of the proof.

Rule *Gen*, much like rule $\Box I$ in [63], has the explicit discharged assumption of xRy as a convenience - it is not needed for the rule to be sound but serves as a way to show its behaviour explicitly. We show this situation in Section 3.3.4, where we utilise this rule to show that we have necessitation. *Gen* adds a label to the *end* of the list of labels.

Rules *dist*, *chng*, and *join* manipulate labels around the $\rightarrow$ operator, which has strict requirements when considering labels since we deal with a DL. They stand for *label distribution*, *label change*, and *label join*, respectively. The usual constraints to the interactions between $\rightarrow$ and the labels apply. These rules manipulate labels on the *end* of their respective lists of labels.

If the previously mentioned rules (*Gen*, *dist*, *chng*, and *join*) manipulated labels at the *beginning* of the list of labels instead of the end, they would no longer be sound. A more structural way to look at this phenomenon is to notice that the *last* label in a list of labels of a labelled formula will be the *outmost* restriction operator in the associated iALC formula.

Take rule *Gen*, for instance. If it added a label to the beginning of the list $L = \{\exists Rz\}$, we would have $\{\forall Rx, \exists Rz\}$. Having $y : \alpha^L$ as a premise, we would end up with $y : \alpha^{\forall Rx, \exists Rz}$, which would mean that, assuming xRy and zRx, from $z : \exists R.\alpha$, we would arrive at $z : \exists R.\forall R.\alpha$, requiring R to represent an Euclidean relation - something we do not want, as our goal is to avoid forcing any properties on roles used in the rules. By adding the label to the end, we do not need to *insert* the universal restriction through any other - it is added as the most external role restriction no matter what labels are in L.

Even though there are constraints on the lists of labels of rules $\sqcap i$, $\sqcap e$, $\sqcup i$, and $\sqcup e$ (as well as efq), we are still able to prove **IK** theorems due to rules *dist*, *join* and *chng*. For instance, theorem $\exists R.(\alpha \sqcap \beta) \rightarrow \exists R.\alpha$, containing $\sqcap$, as well as $\exists R$, which have limited direct interaction in our system, can be proven via (we omit empty lists of labels for clarity):

$$
\dfrac{\dfrac{[x:\exists R.(\alpha\sqcap\beta)]^1 \qquad \dfrac{[xRy]^2 \qquad \dfrac{[y:(\alpha\sqcap\beta)^{\exists Rx}]^2 \qquad \dfrac{[xRy]^2 \qquad \dfrac{\dfrac{[y:(\alpha\sqcap\beta)^{\forall Rx}]^3}{y:\alpha^{\forall Rx}}\,\sqcap e}{y:(\alpha\sqcap\beta)^{\forall Rx}\mathrel{-\!\!\neg}\alpha^{\forall Rx}}\,{\mathrel{-\!\!\neg}}\,i(3)}{y:(\alpha\sqcap\beta)^{\exists Rx}\mathrel{-\!\!\neg}\alpha^{\exists Rx}}\,chng}{y:\alpha^{\exists Rx}}\,{\mathrel{-\!\!\neg}}\,e}{x:\exists R.\alpha}\,\exists i}{x:\exists R.\alpha}\,\exists e(2)}{x:\exists R.(\alpha\sqcap\beta)\mathrel{-\!\!\neg}\exists R.\alpha}\,{\mathrel{-\!\!\neg}}\,i(1)
$$

3.3 Main Properties

In this section, we show that essential properties expected of ND systems hold in the system proposed for iALC. Our primary focus is to show that these properties still hold when dealing with labels and the rules that introduce and eliminate existential and universal restrictions.

3.3.1 Soundness

Before proving soundness itself, we first provide some useful lemmas. Throughout this section, let R and R_{1-n} be roles, x_{1-n}, x, y and z be nominals, α, β and γ be concepts, and $\mathcal{I}$ any interpretation. We write $xR_1yR_2z\ldots R_nz'$ to abbreviate xR_1y and yR_2z etc.

Lemma 3.1. *Let R_{1-n} be roles, x_{1-n} and x be nominals, α and β be concepts, and $\mathcal{I}$ any interpretation. If we assume $x_nR_nx_{n-1}R_{n-1}\ldots x_1R_1x$, then having both $\mathcal{I}\models x_n:\forall R_n\ldots\forall R_1.\alpha$ and $\mathcal{I}\models x_n:\forall R_n\ldots\forall R_1.\beta$ (with only universal restrictions in each) is equivalent to having $\mathcal{I}\models x_n:\forall R_n\ldots\forall R_1.(\alpha\sqcap\beta)$ (with only universal restrictions).*

Proof. The proof follows by induction on the amount n of chained roles. The base case, where $n=0$, indicates the case of having $\mathcal{I}\models x_n:\alpha$ and $\mathcal{I}\models x_n:\beta$ being equivalent to $\mathcal{I}\models x_n:(\alpha\sqcap\beta)$, which follows directly by the definition of the semantics of $\sqcap$ as the conjunction of the interpretations of α and β.

We split the inductive case into two:

($\Rightarrow$) From the definition of the universal restriction operator, we have that, $\forall x'_n$ such that $x_n\preceq x'_n$, $\forall x_{n-1}$, if it is the case that $x'_nR_nx_{n-1}$, then $x_{n-1}:\forall R_{n-1}\ldots\forall R_1.\alpha$, as well as $x_{n-1}:\forall R_{n-1}\ldots\forall R_1.\beta$. By the frame conditions **F1** and **F2**, for every x'_n (assuming $x_nR_nx_{n-1}$) there must be a x'_{n-1} such that $x_{n-1}\preceq x'_{n-1}$ for which we have $x'_nR_nx'_{n-1}$, as well as, for every x_{n-1} there must be a x'_{n-1} for which we have $x'_nR_nx'_{n-1}$.

So, for each x'_{n-1} it is the case that $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \ldots \forall R_1.\alpha$, as well as $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \ldots \forall R_1.\beta$. By the inductive hypothesis, we then have $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \ldots \forall R_1.(\alpha \sqcap \beta)$. Since we have $x'_n R_n x'_{n-1}$ for any x'_n such that $x_n \preceq x'_n$, per the definition of the semantics of the universal restriction, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.(\alpha \sqcap \beta)$.

($\Leftarrow$) Using analogous reasoning to the previous case, for each x'_{n-1} it is the case that $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \ldots \forall R_1.(\alpha \sqcap \beta)$. By the inductive hypothesis, we have $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \ldots \forall R_1.\alpha$, as well as $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \ldots \forall R_1.\beta$. Since we have $x'_n R_n x'_{n-1}$ for any x'_n such that $x_n \preceq x'_n$, per the definition of the semantics of the universal restriction, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\alpha$ and $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\beta$. □

Lemma 3.2. *Let R_{1-n} be roles, x_{1-n} and x be nominals, α and β be concepts, and $\mathcal{I}$ any interpretation. If we assume $x_n R_n x_{n-1} R_{n-1} \ldots x_1 R_1 x$, then having $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\alpha$ or $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\beta$ (with only existential restrictions in each) is equivalent to having $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.(\alpha \sqcup \beta)$ (with only existential restrictions).*

Proof. The proof follows by induction on the amount n of chained roles. The base case, where $n = 0$, indicates the case of having $\mathcal{I} \models x_n : \alpha$ or $\mathcal{I} \models x_n : \beta$ being equivalent to $\mathcal{I} \models x_n : (\alpha \sqcup \beta)$, which follows directly by the definition of the semantics of $\sqcup$ as the disjunction of the interpretations of α and β.

We split the inductive case into two:

($\Rightarrow$) From the definition of the existential restriction operator, we have that, $\forall x'_n$ such that $x_n \preceq x'_n$, $\exists x_{n-1}$ such that $x'_n R_n x_{n-1}$ and $x_{n-1} : \exists R_{n-1} \ldots \exists R_1.\alpha$, or $x_{n-1} : \exists R_{n-1} \ldots \exists R_1.\beta$. By the frame conditions **F1** and **F2**, for every x'_n there must be a x'_{n-1} such that $x_{n-1} \preceq x'_{n-1}$ for which we have $x'_n R_n x'_{n-1}$, as well as, there must be a x'_{n-1} for which we have $x'_n R_n x'_{n-1}$.

So, for each x'_{n-1} we have $x'_n R_n x'_{n-1}$ and it is the case that $\mathcal{I} \models x'_{n-1} : \exists R_{n-1} \ldots \exists R_1.\alpha$, or $\mathcal{I} \models x'_{n-1} : \exists R_{n-1} \ldots \exists R_1.\beta$. By the inductive hypothesis, we then have $\mathcal{I} \models x'_{n-1} : \exists R_{n-1} \ldots \exists R_1.(\alpha \sqcup \beta)$. Since we have $x'_n R_n x'_{n-1}$ for any x'_n such that $x_n \preceq x'_n$, per the definition of the semantics of the existential restriction, we have $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.(\alpha \sqcup \beta)$.

($\Leftarrow$) Using analogous reasoning to the previous case, for each x'_{n-1} we have $x'_n R_n x'_{n-1}$ and it is the case that $\mathcal{I} \models x'_{n-1} : \exists R_{n-1} \ldots \exists R_1.(\alpha \sqcup \beta)$. By the inductive hypothesis, we have $\mathcal{I} \models x'_{n-1} : \exists R_{n-1} \ldots \exists R_1.\alpha$, or $\mathcal{I} \models x'_{n-1} : \exists R_{n-1} \ldots \exists R_1.\beta$. Since we have $x'_n R_n x'_{n-1}$ for any x'_n such that $x_n \preceq x'_n$, per the definition of the semantics of the existential restriction, we have $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\alpha$ or $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\beta$. □

Lemma 3.3. *Let R_{1-n} be roles, x_{1-n} and x be nominals, α be a concept, and $\mathcal{I}$ be any interpretation. If we assume $x_n R_n x_{n-1} R_{n-1} \dots x_1 R_1 x$, then $\mathcal{I} \models x_n : \exists R_n \dots \exists R_1.\bot$ (with only existential restrictions) implies $\mathcal{I} \models x_n : \bot$.*

Proof. The proof follows by induction on the amount n of chained roles. We will consider $n = 1$ as our base case - $n = 0$ is trivial. We, then, have $\mathcal{I} \models x_1 : \exists R_1.\bot$. By the definition of the existential restriction, it must be the case that, $\forall x'_1$ such that $x_1 \preceq x'_1$, $\exists y$ such that $x'_1 R_1 y$ and $y : \bot$. However, $\bot^{\mathcal{I}} = \varnothing$, so there cannot exist such a y. Furthermore, since $\preceq$ is reflexive, we know that there is at least $x_1 \preceq x_1$. So, $x_1^{\mathcal{I}}$ cannot belong to any set, i.e. $\mathcal{I} \models x_1 : \bot$.

For the inductive case, we have $\mathcal{I} \models x_n : \exists R_n \dots \exists R_1.\bot$. By definition of the existential restriction, $\forall x'_n$ such that $x_n \preceq x'_n$, $\exists x_{n-1}$ such that $x'_n R_n x_{n-1}$ and $x_{n-1} : \exists R_{n-1} \dots \exists R_1.\bot$. By **F1** and **F2**, for every x'_n there must be an x'_{n-1} such that $x_{n-1} \preceq x'_{n-1}$, for which we have $x'_n R_n x'_{n-1}$.

So, for each x'_{n-1} we have $x'_n R_n x'_{n-1}$ as well as $\mathcal{I} \models x'_{n-1} : \exists R_{n-1} \dots \exists R_1.\bot$. By the inductive hypothesis, $\mathcal{I} \models x'_{n-1} : \bot$. From $x'_n R_n x'_{n-1}$, we have $\mathcal{I} \models x'_n : \exists R_n.\bot$. From the base case, we have $\mathcal{I} \models x'_n : \bot$ for any x'_n such that $x_n \preceq x'_n$. Thus, $\mathcal{I} \models x_n : \bot$. □

Lemma 3.4. *Let R_{1-n} be roles, x_{1-n} and x be nominals, α and β be concepts, and $\mathcal{I}$ be any interpretation. If we assume $x_n R_n x_{n-1} R_{n-1} \dots x_1 R_1 x$, then $\mathcal{I} \models x_n : \forall R_n \dots \forall R_1.(\alpha \multimap \beta)$ (with only universal restrictions) if and only if $\mathcal{I} \models x_n : \forall R_n \dots \forall R_1.\alpha \multimap \forall R_n \dots \forall R_1.\beta$.*

Proof. The proof follows by induction on the amount n of chained roles. The base case, $n = 0$, is trivial.

Now, we assume $x_n R_n x_{n-1} R_{n-1} \dots x_1 R_1 x$ for the inductive case.

($\Rightarrow$) From the definition of the universal restriction operator, we have $\forall x'_n$ such that $x_n \preceq x'_n$, it is the case that, $\forall x'_{n-1}$ such that $x'_n R_n x'_{n-1}$, $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \dots \forall R_1.(\alpha \multimap \beta)$. From the definition of $\mathcal{I} \models x_n R_n x_{n-1}$, we have that $x'_n R_n x'_{n-1}$ for all $\preceq$-successors x'_n of x_n and x'_{n-1} of x_{n-1}. From the inductive hypothesis, $\mathcal{I} \models x'_{n-1} : \forall R_{n-1} \dots \forall R_1.\alpha \multimap \forall R_{n-1} \dots \forall R_1.\beta$.

Since we also have that $x'_n R_n x'_{n-1}$, we have that, from the definition of $\multimap$, that for all x''_n such that $x'_n \preceq x''_n$ and all x''_{n-1} such that $x'_{n-1} \preceq x''_{n-1}$, if $\mathcal{I} \models x''_{n-1} : \forall R_{n-1} \dots \forall R_1.\alpha$, then $\mathcal{I} \models x''_{n-1} : \forall R_{n-1} \dots \forall R_1.\beta$, which fits the criteria for, if $\mathcal{I} \models x'_n : \forall R_n.\forall R_{n-1} \dots \forall R_1.\alpha$, then $\mathcal{I} \models x'_n : \forall R_n.\forall R_{n-1} \dots \forall R_1.\beta$. This leads us to $\mathcal{I} \models x_n : \forall R_n.\forall R_{n-1} \dots \forall R_1.\alpha \multimap \forall R_n.\forall R_{n-1} \dots \forall R_1.\beta$, since $x_n \preceq x'_n$.

($\Leftarrow$) From the definition of the universal restriction operator, we have $\forall x'_n$ such that $x_n \preceq x'_n$, it is the case that, $\forall x'_{n-1}$ such that $x'_n R_n x'_{n-1}$, $\mathcal{I} \models x'_{n-1}$:

$\forall R_{n-1}\ldots\forall R_1.\alpha \to \forall R_{n-1}\ldots\forall R_1.\beta$. From the definition of $\mathcal{I} \models x_n R_n x_{n-1}$, we have that $x'_n R_n x'_{n-1}$ for all $\preceq$-successors x'_n of x_n and x'_{n-1} of x_{n-1}. From the inductive hypothesis, $\mathcal{I} \models x'_{n-1} : \forall R_{n-1}\ldots\forall R_1.(\alpha \to \beta)$.

Since we also have that $x'_n R_n x'_{n-1}$, we have that, from the definition of $\to$, that for all x''_n such that $x'_n \preceq x''_n$ and all x''_{n-1} such that $x'_{n-1} \preceq x''_{n-1}$, if $x''_n R_n x''_{n-1}$, then $\mathcal{I} \models x''_{n-1} : \forall R_{n-1}\ldots\forall R_1.(\alpha \to \beta)$, which fits precisely the criteria required by the definition of the universal restriction on R_n, leading us to $\mathcal{I} \models x'_n : \forall R_n \ldots \forall R_1.(\alpha \to \beta)$. Since $x_n \preceq x'_n$, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.(\alpha \to \beta)$.

□

Theorem 3.1 (Soundness)**.** *Let δ be an iALC formula, Γ a set of iALC formulas, and δ' and Γ', the labelled versions of δ and Γ, respectively. Then, $\Gamma' \vdash_{ND} \delta'$ implies $\Gamma \models \delta$.*

Proof. The proof follows by induction on the size of the derivation, focusing on the last formula applied.

For the base case, $\delta \in \Gamma$. Let us assume that we have $\mathcal{I} \models \delta'$, given any interpretation $\mathcal{I}$, for any formula $\delta' \in \Gamma$. Since $\delta \in \Gamma$, we have $\mathcal{I} \models \delta$.

For the inductive cases, we will consider derivations ending in the different rules of our system, assume that the inductive hypothesis works for the sub-derivations above them (as they have a smaller size than the derivation in question), and show that the rules preserve soundness, given any interpretation $\mathcal{I}$ - we will not assume anything else on $\mathcal{I}$ other than the assumptions needed in each step.

In the subsequent derivations, sub-derivations will be named Π_i, where $i > 0$, each having their own corresponding set of assumptions Γ_i.

– $\sqcap i$

$$\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ x : \alpha^{L^{(\forall)}} & x : \beta^{L^{(\forall)}}\end{array}}{x : (\alpha \sqcap \beta)^{L^{(\forall)}}}\ \sqcap i$$

We have $\Gamma_1 \vdash_{ND} x : \alpha^L$ and $\Gamma_2 \vdash_{ND} x : \beta^L$. Then, by inductive hypothesis, we have $\Gamma_1 \models x : \alpha^L$ and $\Gamma_2 \models x : \beta^L$. We wish to prove $\Gamma_1 \cup \Gamma_2 \models x : (\alpha \sqcap \beta)^L$.

First, suppose $\mathcal{I} \models \Gamma_1$ and $\mathcal{I} \models \Gamma_2$. Then, we also have $\mathcal{I} \models x : \alpha^L$ and $\mathcal{I} \models x : \beta^L$, since $\Gamma_1 \models x : \alpha^L$ and $\Gamma_2 \models x : \beta^L$. Let $L = \langle \forall R_1 x_1, \ldots, \forall R_n x_n \rangle$ (only universal restrictions). Then, by assuming $x_n R_n x_{n-1} \ldots x_1 R_1 x$, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\alpha$ and $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\beta$. From Lemma 3.1, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.(\alpha \sqcap \beta)$, which is precisely $\mathcal{I} \models x : (\alpha \sqcap \beta)^L$.

– $\sqcap e$

$$\dfrac{\begin{array}{c}\Pi_1\\ x:(\alpha\sqcap\beta)^{L^{(\forall)}}\end{array}}{x:\alpha^{L^{(\forall)}}}\sqcap e_1 \qquad \dfrac{\begin{array}{c}\Pi_1\\ x:(\alpha\sqcap\beta)^{L^{(\forall)}}\end{array}}{x:\beta^{L^{(\forall)}}}\sqcap e_2$$

We have $\Gamma_1 \vdash_{ND} x : (\alpha \sqcap \beta)^L$. Then, by inductive hypothesis, we have $\Gamma_1 \models x : (\alpha \sqcap \beta)^L$. We wish to prove $\Gamma_1 \models x : \alpha^L$.

We start by assuming $\mathcal{I} \models \Gamma_1$. Then, we have $\mathcal{I} \models x : (\alpha \sqcap \beta)^L$. Let $L = \langle \forall R_1 x_1, \ldots, \forall R_n x_n \rangle$ (only universal restrictions). Then, by assuming $x_n R_n x_{n-1} \ldots x_1 R_1 x$, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.(\alpha \sqcap \beta)$. From Lemma 3.1, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\alpha$ and $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\beta$, leading us to having $\mathcal{I} \models x : \alpha^L$, covering $\sqcap e_1$, and $\mathcal{I} \models x : \beta^L$, covering $\sqcap e_2$.

– $\sqcup i$

$$\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha^{L^{(\exists)}}\end{array}}{x:(\alpha\sqcup\beta)^{L^{(\exists)}}}\sqcup i_1 \qquad \dfrac{\begin{array}{c}\Pi_1\\ x:\beta^{L^{(\exists)}}\end{array}}{x:(\alpha\sqcup\beta)^{L^{(\exists)}}}\sqcup i_2$$

Here follows the proof only for $\sqcup i_1$, as the reasoning for $\sqcup i_2$ is analogous, by switching $x : \alpha^L$ for $x : \beta^L$.

We have $\Gamma_1 \vdash_{ND} x : \alpha^L$. By inductive hypothesis, $\Gamma_1 \models x : \alpha^L$. We wish to prove $\Gamma_1 \models x : (\alpha \sqcup \beta)^L$.

Start by assuming $\mathcal{I} \models \Gamma_1$. This leads us to $\mathcal{I} \models x : \alpha^L$. Let $L = \langle \exists R_1 x_1, \ldots, \exists R_n x_n \rangle$ (only existential restrictions). Then, we have $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\alpha$. From Lemma 3.2, we have $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.(\alpha \sqcup \beta)$, which is the same as $\mathcal{I} \models x : (\alpha \sqcup \beta)^L$.

– $\sqcup e$

$$\dfrac{\begin{array}{c}\Pi_1\\ x:(\alpha\sqcup\beta)^{L^{(\exists)}}\end{array}\quad \begin{array}{c}\left[x:\alpha^{L^{(\exists)}}\right]\\ \Pi_2\\ z:\gamma^{L'}\end{array}\quad \begin{array}{c}\left[x:\beta^{L^{(\exists)}}\right]\\ \Pi_3\\ z:\gamma^{L'}\end{array}}{z:\gamma^{L'}}\sqcup e$$

We have $\Gamma_1 \vdash_{ND} x : (\alpha \sqcup \beta)^L$, $\Gamma_2 \cup \{x : \alpha^L)\} \vdash_{ND} z : \gamma^{L'}$ and $\Gamma_3 \cup \{x : \beta^L\} \vdash_{ND} z : \gamma^{L'}$. Then, by inductive hypothesis, $\Gamma_1 \models x : (\alpha \sqcup \beta)^L$, $\Gamma_2 \cup \{x : \alpha^L\} \models z : \gamma^{L'}$ and $\Gamma_3 \cup \{x : \beta^L\} \models z : \gamma^{L'}$. We want to prove $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \models z : \gamma^{L'}$.

We assume $\mathcal{I} \models \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Then, we have $\mathcal{I} \models \Gamma_1$, which leads to $\mathcal{I} \models x : (\alpha \sqcup \beta)^L$. Let $L = \langle \exists R_1 x_1, \ldots, \exists R_n x_n \rangle$ (only existential restrictions). Then, we have $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.(\alpha \sqcup \beta)$. From Lemma 3.2, we have $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\alpha$ or $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\beta$, i.e. $\mathcal{I} \models x : \alpha^L)$ or $\mathcal{I} \models x : \beta^L$.

Consider the first case: $\mathcal{I} \models x : \alpha^L$. We have $\mathcal{I} \models \Gamma_2$. Then, $\mathcal{I} \models z : \gamma^{L'}$, for we have $\Gamma_2 \cup \{x : \alpha^L\} \models z : \gamma^{L'}$.

Now for the second case: $\mathcal{I} \models x : \beta^L$. We have $\mathcal{I} \models \Gamma_3$. Then, $\mathcal{I} \models z : \gamma^{L'}$, for we have $\Gamma_3 \cup \{x : \beta^L\} \models z : \gamma^{L'}$.

– $\multimap i$

$$\dfrac{\begin{array}{c}\left[x : \alpha^L\right] \\ \Pi_1 \\ x : \beta^{L'}\end{array}}{x : (\alpha^L \multimap \beta^{L'})^{\varnothing}} \multimap i$$

We have $\Gamma_1 \cup \{x : \alpha^L\} \vdash_{ND} x : \beta^{L'})$. Then, by inductive hypothesis, $\Gamma_1 \cup \{x : \alpha^L\} \models x : \beta^{L'}$. We wish to prove $\Gamma_1 \models x : (\alpha^L \multimap \beta^{L'})^{\varnothing}$.

We assume $\mathcal{I} \models \Gamma_1$ and $\mathcal{I} \models x : \alpha^L$. Hence, $\mathcal{I} \models x : \beta^{L'}$, since $\Gamma_1 \cup \{x : \alpha^L\} \models x : \beta^{L'}$. This fits precisely $\mathcal{I} \models x : (\alpha^L \multimap \beta^{L'})^{\varnothing}$.

$\neg i$ is a special case of this rule, where $\beta = \bot$.

– $\multimap e$

$$\dfrac{\begin{array}{c}\Pi_1 \\ x : \alpha^L\end{array} \quad \begin{array}{c}\Pi_2 \\ x : (\alpha^L \multimap \beta^{L'})^{\varnothing}\end{array}}{x : \beta^{L'}} \multimap e$$

We have $\Gamma_1 \vdash_{ND} x : \alpha^L$ and $\Gamma_2 \vdash_{ND} x : (\alpha^L \multimap \beta^{L'})^{\varnothing}$. By inductive hypothesis, $\Gamma_1 \models x : \alpha^L$ and $\Gamma_2 \models x : (\alpha^L \multimap \beta^{L'})^{\varnothing}$. We wish to prove $\Gamma_1 \cup \Gamma_2 \models x : \beta^{L'}$.

We assume $\mathcal{I} \models \Gamma_1$ and $\mathcal{I} \models \Gamma_2$, leading us to $\mathcal{I} \models x : \alpha^L$ and $\mathcal{I} \models x : (\alpha^L \multimap \beta^{L'})^{\varnothing}$, respectively. From the definition of $\mathcal{I} \models x : (\alpha^L \multimap \beta^{L'})^{\varnothing}$ and the fact that we have $\mathcal{I} \models x : \alpha^L$, we may conclude $\mathcal{I} \models x : \beta^{L'}$.

$\neg e$ is a special case of this rule, where $\beta = \bot$.

– $\forall i$

$$\dfrac{\begin{array}{c}[xRy] \\ \Pi_1 \\ y : \alpha^{\forall Rx, L}\end{array}}{x : (\forall R.\alpha)^L} \forall i$$

$x \neq y$ and y does not appear in any formula in Γ_1.

We have that $\Gamma_1 \cup \{xRy\} \vdash_{ND} y : \alpha^{\forall Rx,L}$. By the inductive hypothesis, we have $\Gamma_1 \cup \{xRy\} \models y : \alpha^{\forall Rx,L}$. We want to prove $\Gamma_1 \models x : (\forall R.\alpha)^L$.

We start by assuming $\mathcal{I} \models \Gamma_1 \cup \{xRy\}$. Then, we have $\mathcal{I} \models y : \alpha^{\forall Rx,L}$. Since $\mathcal{I} \models xRy$ and y does not appear in any formula in Γ_1, we have $\mathcal{I} \models x : (\forall R.\alpha)^L$.

– $\forall e$

$$\dfrac{xRy \quad \begin{matrix}\Pi_1 \\ x : (\forall R.\alpha)^L\end{matrix}}{y : \alpha^{\forall Rx,L}}\ \forall e$$

We have that $\Gamma_1 \vdash_{ND} x : (\forall R.\alpha)^L$. By the inductive hypothesis, we have $\Gamma_1 \models x : (\forall R.\alpha)^L$. We want to prove $\Gamma_1 \cup \{xRy\} \models y : \alpha^{\forall Rx,L}$.

We start by assuming $\mathcal{I} \models \Gamma_1$. Then, we have $\mathcal{I} \models x : (\forall R.\alpha)^L$. We also have $\mathcal{I} \models xRy$. Then, we arrive at $\mathcal{I} \models y : \alpha^{\forall Rx,L}$ by definition, since $\mathcal{I} \models xRy$ propagates this relation to all the $\preceq$-successors of x and y.

– $\exists i$

$$\dfrac{xRy \quad \begin{matrix}\Pi_1 \\ y : \alpha^{\exists Rx,L}\end{matrix}}{x : (\exists R.\alpha)^L}\ \exists i$$

We have that $\Gamma_1 \vdash_{ND} y : \alpha^{\exists Rx,L}$. By the inductive hypothesis, we have $\Gamma_1 \models y : \alpha^{\exists Rx,L}$. We want to prove $\Gamma_1 \cup \{xRy\} \models x : (\exists R.\alpha)^L$.

We start by assuming $\mathcal{I} \models \Gamma_1$. Then, we have $\mathcal{I} \models y : \alpha^{\exists Rx,L}$. We also have $\mathcal{I} \models xRy$. Then, we arrive at $\mathcal{I} \models x : (\exists R.\alpha)^L$, since $\mathcal{I} \models xRy$ propagates this relation to all the $\preceq$-successors of x and y.

– $\exists e$

$$\dfrac{\begin{matrix}\Pi_1 \\ x : (\exists Rx.\alpha)^L\end{matrix} \qquad \begin{matrix}[y : \alpha^{\exists Rx,L}][xRy] \\ \Pi_2 \\ z : \beta^{L'}\end{matrix}}{z : \beta^{L'}}\ \exists e$$

$x \neq y$, $z \neq y$ and z does not appear in any formula in Γ_2.

We have that $\Gamma_1 \vdash_{ND} x : (\exists Rx.\alpha)^L$ and $\Gamma_2 \cup \{y : \alpha^{\exists Rx,L}, xRy\} \vdash_{ND} z : \beta^{L'}$. Then, by inductive hypothesis, $\Gamma_1 \models x : (\exists Rx.\alpha)^L$ and $\Gamma_2 \cup \{y : \alpha^{\exists Rx,L}, xRy\} \models z : \beta^{L'}$. We want to prove $\Gamma_1 \cup \Gamma_2 \models z : \beta^{L'}$.

We assume $\mathcal{I} \models \Gamma_1 \cup \Gamma_2 \cup \{y : \alpha^{\exists Rx,L}, xRy\}$. Then, we have $\mathcal{I} \models \Gamma_1$, which leads to $\mathcal{I} \models x : (\exists Rx.\alpha)^L$. Then, $\mathcal{I} \models y' : \alpha^{\exists Rx,L})$ and $\mathcal{I} \models xRy'$, for some y'. From our assumptions, we have $\mathcal{I} \models y : \alpha^{\exists Rx,L}$ and $\mathcal{I} \models xRy$. We have $\mathcal{I} \models \Gamma_2$. Then, $\mathcal{I} \models z : \beta^{L'}$, for we have $\Gamma_2 \cup \{y : \alpha^{\exists Rx,L}, xRy\} \models z : \beta^{L'}$.

– efq

$$\dfrac{\begin{matrix}\Pi_1 \\ x : \bot^{L^{(\exists)}}\end{matrix}}{z : \alpha^{L'}}\ efq$$

We have $\Gamma_1 \vdash_{ND} x : \bot^L$. By inductive hypothesis, we have $\Gamma_1 \models x : \bot^L$. We want to prove $\Gamma_1 \models z : \alpha^{L'}$.

We assume $\mathcal{I} \models \Gamma_1$. Then, we have $\mathcal{I} \models x : \bot^L$. Let $L = \langle \exists R_1 x_1, \ldots, \exists R_n x_n \rangle$ (only existential restrictions). Then, we have $\mathcal{I} \models x_n : \exists R_n \ldots \exists R_1.\bot$. If it is the case that $\Gamma_1 \not\models z : \alpha^{L'}$, then $\exists w$ such that $\mathcal{I}, w \models x_n : \exists R_n \ldots \exists R_1.\bot$ and $\mathcal{I}, w \not\models z : \alpha^{L'}$. By Lemma 3.3, we have $\mathcal{I} \models x_n : \bot$, so we have $\mathcal{I}, w \models x_n : \bot$, which is impossible since $\bot^{\mathcal{I}} = \varnothing$ and x_n cannot be an element of an empty set (nor can any of its $\preceq$-successors). Thus, $\Gamma_1 \models z : \alpha^{L'}$.

– *Gen*

$$\dfrac{\begin{array}{c}[xRy]\\ \Pi_1\\ y : \alpha^L\end{array}}{y : \alpha^{L,\forall Rx}}\ Gen$$

y is not present in any formula of Γ_1 and $x \neq y$. We have that $\Gamma_1 \cup \{xRy\} \vdash_{ND} y : \alpha^L$. Then, by inductive hypothesis, $\Gamma_1 \cup \{xRy\} \models y : \alpha^L$. We want to prove $\Gamma_1 \models y : \alpha^{L,\forall Rx}$.

We assume $\mathcal{I} \models \Gamma_1$ and $\mathcal{I} \models xRy$. Then, we have $\mathcal{I} \models y : \alpha^L$. Due to the heredity condition, $\forall y'$ such that $y \preceq y'$, it is also the case that $y' : \alpha^L$. From $\mathcal{I} \models xRy$, we have that $x'Ry'$ for any x' such that $x \preceq x'$. This fits precisely the criteria for $\mathcal{I} \models x : \forall R.(\alpha^L)$. Thus, we arrive at, if $\mathcal{I} \models xRy$, then $\mathcal{I} \models x : \forall R.(\alpha^L)$, i.e. $\mathcal{I} \models y : \alpha^{L,\forall Rx}$.

– *dist*

$$\dfrac{\begin{array}{c}\Pi_1\\ y : (\alpha^{\varnothing} \multimap \beta^{\varnothing})^{L(\forall),\forall Rx}\end{array}}{y : (\alpha^{L(\forall),\forall Rx} \multimap \beta^{L(\forall),\forall Rx})^{\varnothing}}\ dist$$

y is not present in any formula of Γ_1 and $x \neq y$. We have that $\Gamma_1 \vdash_{ND} y : (\alpha^{\varnothing} \multimap \beta^{\varnothing})^{L,\forall Rx}$. Then, by inductive hypothesis, $\Gamma_1 \models y : (\alpha^{\varnothing} \multimap \beta^{\varnothing})^{L,\forall Rx}$. We want to prove $\Gamma_1 \models y : (\alpha^{L,\forall Rx} \multimap \beta^{L,\forall Rx})^{\varnothing}$.

We assume $\mathcal{I} \models \Gamma_1$. Then, we have $\mathcal{I} \models y : (\alpha^{\varnothing} \multimap \beta^{\varnothing})^{L(\forall),\forall Rx}$. Let $n = size(L)$. Then, if $\mathcal{I} \models x_n R_n x_{n-1} \ldots x_1 R_1 x R y$, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\forall R.(\alpha \multimap \beta)$.

Since L only contains universal restrictions, by Lemma 3.4 and by assuming $\mathcal{I} \models x_n R_n x_{n-1} \ldots x_1 R_1 x R y$, we have $\mathcal{I} \models x_n : \forall R_n \ldots \forall R_1.\forall R.\alpha \multimap \forall R_n \ldots \forall R_1.\forall R \beta$. This is precisely $\mathcal{I} \models y : (\alpha^{L,\forall Rx} \multimap \beta^{L,\forall Rx})^{\varnothing}$.

– *chng*

$$\dfrac{\begin{array}{cc} & \Pi_1 \\ xRy & y : (\alpha^{L,\forall Rx} \rightarrowtail \beta^{L',\forall Rx})^{\varnothing} \end{array}}{y : (\alpha^{L,\exists Rx} \rightarrowtail \beta^{L',\exists Rx})^{\varnothing}} \; chng$$

y is not present in any formula of Γ_1 and $x \neq y$. We have that $\Gamma_1 \vdash_{ND} y : (\alpha^{L,\forall Rx} \rightarrowtail \beta^{L',\forall Rx})^{\varnothing}$. Then, by inductive hypothesis, $\Gamma_1 \models y : (\alpha^{L,\forall Rx} \rightarrowtail \beta^{L',\forall Rx})^{\varnothing}$. We want to prove $\Gamma_1 \cup \{xRy\} \models y : (\alpha^{L,\exists Rx} \rightarrowtail \beta^{L',\exists Rx})^{\varnothing}$.

We start by assuming $\mathcal{I} \models \Gamma_1$. Then, we have $\mathcal{I} \models y : (\alpha^{L,\forall Rx} \rightarrowtail \beta^{L',\forall Rx})^{\varnothing}$, i.e. if we have $\mathcal{I} \models xRy$ (which we do have), then we have $\mathcal{I} \models x : \forall R.(\alpha^{L}) \rightarrowtail \forall R.(\beta^{L'})$. This means that, $\forall x'$ such that $x \preceq x'$, it is the case that, if $\mathcal{I} \models x' : \forall R.(\alpha^{L})$, then $\mathcal{I} \models x' : \forall R.(\beta^{L'})$. From the definition of the universal restriction, we have that, if $\forall x''$ such that $x' \preceq x''$, it is the case that, $\forall y'$, if $x''Ry'$, then $\mathcal{I} \models y' : \alpha^{L}$, then we also have $\forall x''$ such that $x' \preceq x''$, it is the case that, $\forall y'$, if $x''Ry'$, then $\mathcal{I} \models y' : \beta^{L'}$.

Since xRy propagates to all $\preceq$-successors of x and y, we do have $x''Ry''$ for (at least) some y'' such that $y \preceq y''$ leading to $\mathcal{I} \models y'' : \alpha^{L}$ - which in itself leads us to $\mathcal{I} \models y'' : \beta^{L'}$, for this y'', since it must happen for all of the $\preceq$-successors y' of y. So, there exists a y'' for which we have $x''Ry''$, $\mathcal{I} \models y'' : \alpha$ and $\mathcal{I} \models y'' : \beta^{L'}$ - and we fit the criteria for, if $\mathcal{I} \models x' : \exists R.(\alpha^{L})$, then $\mathcal{I} \models x' : \exists R.(\beta^{L'})$, since this happens for any x'' such that $x' \preceq x''$.

Thus, since this happens for all x' such that $x \preceq x'$, we have $\mathcal{I} \models x : \exists R.(\alpha^{L}) \rightarrowtail \exists R.(\beta^{L'})$, and also have $\mathcal{I} \models xRy$ from our assumptions. So, we have $\mathcal{I} \models y : (\alpha^{L,\exists Rx} \rightarrowtail \beta^{L',\exists Rx})^{\varnothing}$.

– *join*

$$\dfrac{\begin{array}{c} \Pi_1 \\ y : (\alpha^{L(\forall),\exists Rx} \rightarrowtail \beta^{L(\forall),\forall Rx})^{\varnothing} \end{array}}{y : (\alpha^{\varnothing} \rightarrowtail \beta^{\varnothing})^{L(\forall),\forall Rx}} \; join$$

y is not present in any formula of Γ_1 and $x \neq y$. We have that $\Gamma_1 \vdash_{ND} y : (\alpha^{L,\exists Rx} \rightarrowtail \beta^{L,\forall Rx})^{\varnothing}$. Then, by inductive hypothesis, $\Gamma_1 \models y : (\alpha^{L,\exists Rx} \rightarrowtail \beta^{L,\forall Rx})^{\varnothing}$. We want to prove $\Gamma_1 \models y : (\alpha^{\varnothing} \rightarrowtail \beta^{\varnothing})^{L,\forall Rx}$.

We assume $\mathcal{I} \models \Gamma_1$. Then, we have $\mathcal{I} \models y : (\alpha^{L,\exists Rx} \rightarrowtail \beta^{L,\forall Rx})^{\varnothing}$, i.e. if it is the case that $\mathcal{I} \models xRy$, then $\mathcal{I} \models x : \exists R.(\alpha^{L}) \rightarrowtail \forall R.(\beta^{L})$. Thus, we assume $\mathcal{I} \models xRy$.

From $\mathcal{I} \models x : \exists R.(\alpha^L) \rightarrow \forall R.(\beta^L)$, we have, $\forall x'$ such that $x \preceq x'$, if $\mathcal{I} \models x' : \exists R.(\alpha^L)$, then $\mathcal{I} \models x' : \forall R.(\beta^L)$. If we had $\mathcal{I} \models x' : \exists R.(\alpha^L)$, it would be the case that, $\forall x''$ such that $x' \preceq x''$, there must be a y_1' for which $x''Ry_1'$ and $\mathcal{I} \models y_1' : (\alpha^L)$, whereas, for $\mathcal{I} \models x' : \forall R.(\beta^L)$, for any y_2', if $x''Ry_2'$, then $\mathcal{I} \models y_2' : (\beta^L)$.

Since xRy propagates to all $\preceq$-successors of x and y, we have $x'Ry_1'$. However, for all of the y_2' for which $x'Ry_2'$, it is also the case that $\mathcal{I} \models y_2' : (\beta^L)$, including y_1', for which we know that $\mathcal{I} \models y_1' : (\alpha^L)$. So, for all $\preceq$-successors y' of y such that $x'Ry'$, it is the case that $\mathcal{I} \models y' : (\alpha^L) \rightarrow (\beta^L)$. From Lemma 3.4, it is also the case that $\mathcal{I} \models y' : (\alpha \rightarrow \beta)^L$, since L contains only universal restrictions. Then, by the definition of $\rightarrow$ and the frame conditions, $\mathcal{I} \models x' : \forall R.((\alpha \rightarrow \beta)^L)$ for all $\preceq$-successors x' of x, leading to $\mathcal{I} \models x : \forall R.((\alpha \rightarrow \beta)^L)$, due to heredity. Thus, if $\mathcal{I} \models xRy$, then $\mathcal{I} \models x : \forall R.((\alpha \rightarrow \beta)^L)$, i.e. $\mathcal{I} \models y : (\alpha^{\varnothing} \rightarrow \beta^{\varnothing})^{L, \forall Rx}$.

□

3.3.2 A Note on Normalisation in our System

Before proceeding with the proof of normalisation on our system, we indicate to the reader that there is a case in which the normalisation procedure would not preserve soundness. Then, in this specific case, we impose restrictions in the normalisation operation to be applied (called a *reduction*) to avoid this undesired consequence. Thus, we have an instance of a *detour* which our normalisation procedure cannot reduce. Hence, we call it a *quasi*-normalisation or a partial normalisation of the system.

Since the explanation of this specific situation requires a few notions of our normalisation process - those of *detour*, *reduction*, *quasi-deduction*, among others -, we explain it in further detail in Example 3.5, which is right after the definition of a $dist - join$-reduction, inside Definition 3.20. By that point, we will have explained all the needed vocabulary.

3.3.3 Normalisation

Normalisation is crucial to show that the sub-formula principle holds in our system. This principle states that every step in a derivation may contain only sub-formulas of the conclusion or premises, which is extremely important to do proof search efficiently. We will base our normalisation proof on the ones presented in [53, 64, 65, 67].

From here on, we consider concepts that do not explicitly present a list of labels to have an empty list, avoiding visual clutter.

The following definitions are needed to reach normalisation.

Definition 3.3 (*Top-formula*). *Let δ be an iALC formula in a derivation Π. δ is called a top-formula of Π when there are no rule applications for which δ is a conclusion in Π.*

Definition 3.4 (*End-formula*). *Let δ be an iALC formula in a derivation Π. δ is called the end-formula of Π when there are no rule applications for which δ is a premise in Π.*

The end-formula is unique because we have a single-conclusion system.

Definition 3.5 (*Formulas immediately above and below*). *Let δ be the end-formula in derivation Π as the consequence of an application of rule ρ. Let $\delta_1, \ldots, \delta_n$ be the premises of ρ. Then, δ is immediately below each δ_i, for $i \leq n$, and each of the δ_i's is immediately above δ.*

Definition 3.6 (*Side-connected formulas*). *Let δ be the end-formula in derivation Π as the consequence of an application of rule ρ. Let $\delta_1, \ldots, \delta_n$ be the premises of ρ. Then, δ_i is side-connected with δ_j, for $i, j \leq n$.*

Definition 3.7 (*Major premise*). *The major premise is the premise of an elimination rule which contains the operator to be eliminated. Elimination rules are $\sqcap e$, $\sqcup e$, $\sqsupset e$, $\forall e$, $\exists e$, and $\neg e$. We also consider the premise of chng containing the $\sqsupset$ operator as its major premise.*

Definition 3.8 (*Minor premise*). *Other premises in the rules mentioned in Definition 3.7, if existent, are called minor premises.*

We do not divide the premises into major and minor in non-elimination rules.

Definition 3.9 (*Thread*). *In a derivation, a thread is a sequence of formulas starting with one of the top-formulas and ending in the end-formula of the derivation. Every formula in a thread must be immediately above the next one in the sequence (i.e. given a thread $\delta_1, \ldots, \delta_n$ of size n, every formula δ_i, for $i < n$, must be one of the premises of the application of a rule ρ which has δ_{i+1} as its conclusion).*

Example 3.1 (A thread). *In the following derivation, there are two threads, namely $x : \alpha \sqcap \beta$, $x : \beta$, $x : \beta \sqcap \alpha$ (on the left side) and $x : \alpha \sqcap \beta$, $x : \alpha$, $x : \beta \sqcap \alpha$ (on the right side).*

$$\dfrac{\dfrac{x : \alpha \sqcap \beta}{x : \beta}\,\sqcap e_2 \qquad \dfrac{x : \alpha \sqcap \beta}{x : \alpha}\,\sqcap e_1}{x : \beta \sqcap \alpha}\,\sqcap i$$

So far, we have been talking about discharging formulas, but we still need to define how we decide which rule to discharge a particular formula if more than one rule is eligible. We first provide an account of how each rule discharges a formula.

Definition 3.10 (*Discharge*)**.** *Let δ be a top-formula in a derivation Π and τ be the thread that begins with δ. We say that δ is discharged in Π at δ' by an application ρ' of rule ρ if and only if δ' is a formula occurrence δ_1 in τ such that one of the following conditions holds:*

1. *ρ is $\multimap i$, δ_1 is the premise of ρ' and has the form $x : \beta^{L'}$ for some x, β and L', δ has the form $x : \alpha^L$ for some α and L, and the consequence of ρ' has the form $x : \alpha^L \multimap \beta^{L'}$;*

2. *ρ is $\neg i$, δ_1 is the premise of ρ' and has the form $x : \bot^L$ for some x and L', δ has the form $x : \alpha^L$ for some α and L, and the consequence of ρ' has the form $x : \alpha^L \multimap \bot^{L'}$;*

3. *ρ is $\sqcup e$, δ has the form $x : \alpha^L$ for some x, α and L, the major premise of ρ' has the form $x : (\alpha \sqcup \beta)^L$ or $x : (\beta \sqcup \alpha)^L$ for some β, and δ_1 is the first or second minor premise of ρ', respectively;*

4. *ρ is $\exists e$, δ has the form $y : \alpha^{\exists Rx,L}$ or the form xRy for some x, y, R, α and L, the major premise of ρ' has the form $x : (\exists R.\alpha)^L$, δ_1 is the minor premise of ρ' and has the form $z : \beta^{L'}$ for some z, β and L';*

5. *ρ is $\forall i$, δ_1 is the premise of ρ' and has the form $y : \alpha^{\forall Rx,L}$ for some x, y, R, α and L, δ has the form xRy, and the consequence of ρ' has the form $x : (\forall R.\alpha)^L$.*

6. *ρ is Gen, δ_1 is the premise of ρ' and has the form $y : \alpha^L$ for some x, y, R and α, δ has the form xRy, and the consequence of ρ' has the form $y : \alpha^{L,\forall Rx}$.*

Before defining formally a derivation, it is necessary to define how we deal with hypothesis discharge on a general formula tree, i.e. a tree made of formulas that does not necessarily follow the structure of the rules of our calculus. Thus, we define a quasi-derivation.

Definition 3.11 (*Quasi-derivation*)**.** *Π is a quasi-derivation if it is a formula tree such that, if δ is a formula occurrence in Π and $\delta_1, \ldots, \delta_n$ are formula occurrences immediately above δ in Π in their order from left to right, then the following is an application of a rule of our calculus:*

$$\frac{\delta_1 \qquad \dots \qquad \delta_n}{\delta}$$

A quasi-derivation consists of a formula tree respecting the structure we need. However, a quasi-derivation does not assign hypothesis discharge to formulas. For example, let us consider that we want to have a derivation of the formula $\delta = x : \alpha \multimapinv (\beta \multimapinv \alpha)$ from an empty set of premises $\Gamma = \varnothing$. The following is a quasi-derivation of δ from Γ since we apply all the rules according to our calculus. However, it is not a derivation of δ from Γ, for we depend actually on $\Gamma \cup \{x : \alpha\}$ (it is, in fact, a derivation of δ from $\Gamma \cup \{x : \alpha\}$). In this example, the formula $x : \alpha$, however, could be discharged by the application of $\multimapinv i(1)$, which would upgrade the quasi-derivation to a derivation since the dependence would rely only on Γ because discharged hypotheses are considered part of the structure of the derivation; we do not need to consider them premises anymore.

$$\cfrac{\cfrac{x : \alpha}{x : \beta \multimapinv \alpha}\ \multimapinv i(2)}{x : \alpha \multimapinv (\beta \multimapinv \alpha)}\ \multimapinv i(1)$$

The way we deal with hypothesis discharge is via a discharge-function.

Definition 3.12 (*Discharge-function*)**.** *A discharge-function $\mathcal{F}$ for a quasi-derivation Π is a function from a set of top-formulas in Π that assigns to a formula δ either δ itself or a formula occurrence in Π below δ.*

Let $\mathcal{F}$ be a discharge-function for a quasi-derivation Π. We say that top-formula δ in Π is discharged with respect to $\mathcal{F}$ at δ' if $\mathcal{F}(\delta) = \delta'$.

Definition 3.13 (*Dependence*)**.** *There are two ways to talk about formula dependence. Let $\mathcal{F}$ be a discharge-function for a quasi-derivation Π, and δ and δ' be two iALC formulas.*

- *δ' is said to depend on the top-formula δ w.r.t. $\mathcal{F}$ if δ' belongs to the thread τ in Π that begins with δ and δ is not discharged with respect to $\mathcal{F}$ at a formula occurrence above δ' in τ.*
- *δ' is said to depend on the formula δ w.r.t. $\mathcal{F}$ if δ' depends on a top-formula which is an occurrence of δ.*

Definition 3.14 (*Regular Discharge-function*)**.** *$\mathcal{F}$ is a regular discharge-function for a quasi-derivation Π if the following conditions apply:*

1. *nominals in an application of $\forall i$, Gen, dist, chng, join (y) or $\exists e$ (z) do not occur in any assumption on which the premise of this application depends; and*

2. *$\mathcal{F}(\delta)$ is a premise δ_1 in an application ρ' of a rule ρ satisfying one of the conditions in Definition 3.10.*

Then, employing a regular discharge-function and a quasi-derivation, we can define a derivation formally.

Definition 3.15 (*Derivation*)**.** *Let Π be a quasi-derivation, δ a formula and Γ a set of formulas. Then, Π is a derivation of δ from Γ if δ is the end-formula of Π and there exists a regular discharge-function $\mathcal{F}$ for Π such that the end-formula of Π depends only on formulas of Γ with respect to $\mathcal{F}$.*

Definition 3.16 (*Branch*)**.** *A branch in a derivation is a sequence $\delta_1, \ldots, \delta_n$ of formula occurrences such that:*

1. *δ_1 is a top-formula that is not discharged by an application of $\sqcup e$ or $\exists e$;*
2. *δ_i, for all $i < n$, is not a minor premise of an application of $\multimap e$ and:*
   (a) *if δ_i is not a major premise of $\sqcup e$ or $\exists e$, then δ_{i+1} occurs immediately below δ_i;*
   (b) *if δ_i is a major premise of $\sqcup e$ or $\exists e$, then δ_{i+1} is a top-formula discharged by the same application of $\sqcup e$ or $\exists e$;*
3. *δ_n is either a minor premise of an application of $\multimap e$ or $\neg e$, or a major premise of an application of $\sqcup e$ or $\exists e$ that does not discharge any top-formula, or the end formula of the derivation.*

Defining a branch is helpful to understand the behaviour of a derivation around applications of $\sqcup e$ or $\exists e$, since, in these rules, we should see the major premise as happening before or *above* discharged top-formula(s) in the sub-derivation(s) above the minor premise(s).

Example 3.2 (A branch - left side)**.** *In the following example, the formulas in bold form a branch. i.e. the sequence $\boldsymbol{x : \alpha \sqcup \beta}$, $\boldsymbol{x : \alpha}$, $\boldsymbol{x : \alpha \sqcap \beta}$, $\boldsymbol{x : \alpha \sqcap \beta}$:*

$$\dfrac{\boldsymbol{x : \alpha \sqcup \beta} \quad \dfrac{[\boldsymbol{x : \alpha}]^1 \quad x : \beta}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcap i \quad \dfrac{x : \alpha \quad [x : \beta]^1}{x : \alpha \sqcap \beta}\sqcap i}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcup e(1)$$

Example 3.3 (Another branch - right side)**.** *In this example, we highlight the branch on the right side, namely $\boldsymbol{x : \alpha \sqcup \beta}$, $\boldsymbol{x : \beta}$, $\boldsymbol{x : \alpha \sqcap \beta}$, $\boldsymbol{x : \alpha \sqcap \beta}$:*

$$\dfrac{\boldsymbol{x : \alpha \sqcup \beta} \quad \dfrac{[x : \alpha]^1 \quad x : \beta}{x : \alpha \sqcap \beta}\sqcap i \quad \dfrac{x : \alpha \quad [\boldsymbol{x : \beta}]^1}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcap i}{\boldsymbol{x : \alpha \sqcap \beta}}\sqcup e(1)$$

Notice that the disjunction always appears before a corresponding discharged formula. A branch generalises the concept of a thread, encompassing derivations with $\sqcup e$ or $\exists e$ accurately.

Definition 3.17 (*Detour*)**.** *We characterise a detour by one of the following cases:*

- *a formula occurrence that is the conclusion of an introduction rule and the major premise of an elimination rule;*
- *a formula occurrence that is the conclusion of rule efq and either the major premise of an elimination rule or rule chng, or the premise of Gen, dist or join;*
- *a formula that is the premise of an application of dist whose conclusion is the major premise of $\rightharpoondown e$, followed by $\rightharpoondown i$ and, finally, an application of join, whose conclusion is the same formula as the premise of the previous application of dist (will be shown in the $dist - join$ reduction, in Definition 3.20);*
- *a formula that is the major premise of an application of chng whose conclusion is the major premise of $\neg e$ which has an application of efq below it, introducing the same formula as the major premise of the previous application of chng (will be shown in the $chng - efq$ reduction, in Definition 3.20).*

The special detours $dist - join$ and $chng - efq$ happen in our system due to the behaviour of the label-arranging rules with one another and only occur in certain specific kinds of derivations - hence the elaborate shape of their detours, which consist of more than two rules applied in sequence.

Our goal, in the end, will be to remove such detours via *reductions* and arrive at so-called *normal* derivations to show that the system has normalisation.

Example 3.4 (A detour with $\sqcap$)**.** *Let Π_1 and Π_2 be sub-derivations. We give an example of a detour below where the list of labels is empty:*

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ x : \alpha\end{matrix} \quad \begin{matrix}\Pi_2 \\ x : \beta\end{matrix}}{x : (\alpha \sqcap \beta)}\,\sqcap i}{x : \alpha}\,\sqcap e_1$$

By introducing and eliminating a conjunction, $x : \alpha$ occurs twice in the same branch of the derivation. This sequence creates an unnecessary step, as we make no progress in the derivation.

Definition 3.18 (*Maximum Formula*)**.** *A formula δ in a derivation Π is a maximum formula if:*

1. *δ is the conclusion of an application of an introduction rule and the major premise of an application of an elimination rule of the same operator;*
2. *δ is the conclusion of an application of $\sqcup e$ or $\exists e$, as well as the major premise of an application of an elimination rule;*
3. *in a thread τ, δ is the conclusion of an application of dist or the conclusion of a lower application of join (in a dist $-$ join reduction, we consider both formulas to be the maximum formulas);*
4. *δ appears twice on a thread τ, first as the major premise of an application of chng, and after as the conclusion of an application of efq.*

Maximum formulas are precisely those that represent a detour.

In order to evaluate how a reduction works in simplifying a derivation, it is necessary to have some measure. We then introduce the concept of the degree of a formula, indicating its size by the number of occurrences of logical operators. Not surprisingly, the name *maximum formula* is related to this measurement.

Definition 3.19 (*Degree of a Formula*)**.** *The degree of a formula is as follows:*

$$
\begin{aligned}
&deg(C) = 0, \text{ for an atomic concept } C\\
&deg(\bot) = 0\\
&deg(x : \alpha) = deg(\alpha)\\
&deg(\alpha \sqcup \beta) = deg(\alpha) + deg(\beta) + 1\\
&deg(\alpha \sqcap \beta) = deg(\alpha) + deg(\beta) + 1\\
&deg(\alpha \rightarrow\!\!\!\!\!\dashv \beta) = deg(\alpha) + deg(\beta) + 1\\
&deg(\neg\alpha) = deg(\alpha) + 1\\
&deg(\exists R.\alpha) = deg(\alpha) + 1\\
&deg(\forall R.\alpha) = deg(\alpha) + 1\\
&deg(\alpha^L) = deg(\alpha) + size(L)\\
&deg(xRy) = 0
\end{aligned}
$$

Now, we will see how to remove these unwanted detours.

A reduction is an operation on derivations that removes maximum formulas locally. The reductions for the operators are extensions of the usual Prawitz reductions, as well as our instances of $dist - join$ and $chng - efq$. Besides these, we have permutation operations to arrange a $\sqcup$-maximum or $\exists$-maximum segment to be appropriately reduced.

As a quick note on reductions, they usually copy whole parts of subderivations (thus possibly increasing the size of the derivation itself). However, they do not increase the complexity of the formulas therein. First, we will show some primary cases.

Definition 3.20 (*Reduction*)**.** *We say a derivation* Π *reduces to another,* Π'*, when we apply zero or more reductions to maximum formulas of* Π*, and it becomes equal to* Π'*. We denote by* $\Pi \rhd \Pi'$ *the reduction from* Π *to* Π'*.*

For all the following reductions, let R be a role, α, β and γ concepts, x, y and z VLSs, and L and L' lists of labels.

$\sqcap$-reduction Generalising the labels from the derivation from Example 3.4, we have:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ x : \alpha^L\end{matrix} \quad \begin{matrix}\Pi_2 \\ x : \beta^L\end{matrix}}{x : (\alpha \sqcap \beta)^L}\,\sqcap i}{\begin{matrix}x : \alpha^L \\ \Pi_3\end{matrix}}\,\sqcap e_1 \qquad \rhd \qquad \begin{matrix}\Pi_1 \\ x : \alpha^L \\ \Pi_3\end{matrix}$$

This way, the proof goes directly through $x : \alpha^L$ without introducing and eliminating an unused operator. The reduction utilises $\sqcap e_1$, but $\sqcap e_2$ is analogous, the only difference being the presence of $x : \beta^L$ and Π_2 instead of $x : \alpha^L$ and Π_1 in Π'.

$\sqsupset$-reduction This derivation has a maximum formula whose main operator is $\sqsupset$:

$$\dfrac{\begin{matrix}\Pi_1 \\ x : \alpha^L\end{matrix} \quad \dfrac{\begin{matrix}[x : \alpha^L] \\ \Pi_2 \\ x : \beta^{L'}\end{matrix}}{x : (\alpha^L \sqsupset \beta^{L'})}\,\sqsupset i}{\begin{matrix}x : \beta^{L'} \\ \Pi_3\end{matrix}}\,\sqsupset e \qquad \rhd \qquad \begin{matrix}\Pi_1 \\ x : \alpha^L \\ \Pi_2 \\ x : \beta^{L'} \\ \Pi_3\end{matrix}$$

Note that, in this case, there can be multiple occurrences of the hypothesis discharge over Π_2, so this reduction ends up copying Π_1 possibly many times, increasing the (horizontal) size of the derivation itself, but not its complexity since we removed the maximum formula. We show a way to measure this complexity when introducing the *degree* of a formula. For now, we can consider that a maximum formula of a derivation (the one to be reduced) has the highest degree of any of the formulas in the derivation, so upon its removal, the degree of the (new) maximum formula of the (new) derivation cannot increase from the degree of the previous one.

If $x : \alpha^L$ does not appear above Π_2, then the situation is similar to the case of $\sqcap$-reduction:

$$
\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha^L\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ x:\beta^{L'}\end{array}}{x:(\alpha^L \mathbin{-\!\!\dashv} \beta^{L'})}\,{\mathbin{-\!\!\dashv}}i}{\begin{array}{c}x:\beta^{L'}\\ \Pi_3\end{array}}\,{\mathbin{-\!\!\dashv}}e
\qquad \rhd \qquad
\begin{array}{c}\Pi_2\\ x:\beta^{L}\\ \Pi_3\end{array}
$$

This situation occurs as well with $\neg e$-reduction and $\sqcup$-reduction, since their rules are the ones involving hypothesis discharge, and the dependence on the *dischargeable* formula is not always present, i.e. this formula may not be a top-formula above the application of the rule that can discharge it.

$\neg$-reduction The case of $\neg$-reduction via $\neg i$ and $\neg e$ is analogous, as negation is but a particular case of $\mathbin{-\!\!\dashv}$:

$$
\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha^L\end{array} \quad \dfrac{\begin{array}{c}[x:\alpha^L]\\ \Pi_2\\ x:\bot^{L'}\end{array}}{x:\alpha^L \mathbin{-\!\!\dashv} \bot^{L'}}\,\neg i}{\begin{array}{c}x:\bot^{L'}\\ \Pi_3\end{array}}\,\neg e
\qquad \rhd \qquad
\begin{array}{c}\Pi_1\\ x:\alpha^L\\ \Pi_2\\ x:\bot^{L'}\\ \Pi_3\end{array}
$$

If $x : \alpha^L$ does not appear above Π_2, then the situation is similar to the case of $\mathbin{-\!\!\dashv}$-reduction:

$$
\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha^L\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ x:\bot^{L'}\end{array}}{x:\alpha^L \mathbin{-\!\!\dashv} \bot^{L'}}\,\neg i}{\begin{array}{c}x:\bot^{L'}\\ \Pi_3\end{array}}\,\neg e
\qquad \rhd \qquad
\begin{array}{c}\Pi_2\\ x:\bot^{L'}\\ \Pi_3\end{array}
$$

$\forall$-reduction For this reduction, $x \neq y$ and y does not occur in undischarged assumptions of Π_1:

$$
\dfrac{xRy \quad \dfrac{\begin{array}{c}[xRy]\\ \Pi_1\\ y:\alpha^{\forall Rx,L}\end{array}}{x:(\forall R.\alpha)^L}\,\forall i}{\begin{array}{c}y:\alpha^{\forall Rx,L}\\ \Pi_2\end{array}}\,\forall e
\qquad \rhd \qquad
\begin{array}{c}\Pi_1\\ y:\alpha^{\forall Rx,L}\\ \Pi_2\end{array}
$$

$\sqcup$-reduction This operator, much like $\sqcap$, has two possible reductions, but both are equivalent. We will show only the one involving $\sqcup i_1$. The case of $\sqcup i_2$ has $x : \beta^L$ instead of $x : \alpha^L$ in both Π (as the premise of $\sqcup i$) and Π'.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x : \alpha^L\end{matrix}}{x : (\alpha \sqcup \beta)^L}\sqcup i_1 \quad \begin{matrix}[x : \alpha^L]\\ \Pi_2\\ z : \gamma^{L'}\end{matrix} \quad \begin{matrix}[x : \beta^L]\\ \Pi_3\\ z : \gamma^{L'}\end{matrix}}{\begin{matrix}z : \gamma^{L'}\\ \Pi_4\end{matrix}}\sqcup e \qquad \rhd \qquad \begin{matrix}\Pi_1\\ x : \alpha^L\\ \Pi_2\\ z : \gamma^{L'}\\ \Pi_4\end{matrix}$$

As with $\sqcap$, we can add multiple copies of Π_1. If $x : \alpha^L$ does not appear above Π_2, then the reduction is:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x : \alpha^L\end{matrix}}{x : (\alpha \sqcup \beta)^L}\sqcup i_1 \quad \begin{matrix}\Pi_2\\ z : \gamma^{L'}\end{matrix} \quad \begin{matrix}[x : \beta^L]\\ \Pi_3\\ z : \gamma^{L'}\end{matrix}}{\begin{matrix}z : \gamma^{L'}\\ \Pi_4\end{matrix}}\sqcup e \qquad \rhd \qquad \begin{matrix}\Pi_2\\ z : \gamma^{L'}\\ \Pi_4\end{matrix}$$

$\exists$-reduction For this reduction, $x \neq y$, $z \neq y$, and z does not occur in undischarged assumptions of Π_2:

$$\dfrac{\dfrac{xRy \quad \begin{matrix}\Pi_1\\ y : \alpha^{\exists Rx,L}\end{matrix}}{x : (\exists R.\alpha)^L}\exists i \quad \begin{matrix}[y : \alpha^{\exists Rx,L}][xRy]\\ \Pi_2\\ z : \beta^{L'}\end{matrix}}{\begin{matrix}z : \beta^{L'}\\ \Pi_3\end{matrix}}\exists e \qquad \rhd \qquad \begin{matrix}\Pi_1\\ y : \alpha^{\exists Rx,L}\\ \Pi_2\\ z : \beta^{L'}\\ \Pi_3\end{matrix}$$

The assumptions of xRy above Π_2 can be discharged or not as they do not interfere with the reduction since Π depends on xRy regardless (see the left side of the derivation).

As was the case with $\sqcap$ and $\sqcup$, multiple copies of Π_1 can be added. If $y : \alpha^{\exists Rx,L}$ does not appear above Π_2, then the reduction is:

$$\dfrac{\dfrac{xRy \quad \begin{matrix}\Pi_1\\ y : \alpha^{\exists Rx,L}\end{matrix}}{x : (\exists R.\alpha)^L}\exists i \quad \begin{matrix}[xRy]\\ \Pi_2\\ z : \beta^{L'}\end{matrix}}{\begin{matrix}z : \beta^{L'}\\ \Pi_3\end{matrix}}\exists e \qquad \rhd \qquad \begin{matrix}\Pi_2\\ z : \beta^{L'}\\ \Pi_3\end{matrix}$$

***efq*-reduction** Rule efq can serve as a generalised introduction rule. Suppose we introduce the major premise of any elimination rule ρ_e via an application of efq. In that case, we might apply efq to directly conclude the conclusion of ρ_e itself. In the following derivation, we assume ρ_e to have one minor premise. However, it can have none or even multiple minor premises - efq-reduction eliminates the parts of the derivation associated with them in any case. In the following derivation, let δ_1 be the major premise of ρ_e:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x : \bot^L\end{matrix}}{\delta_1}\,efq \quad \begin{matrix}\Pi_2\\ \delta_2\end{matrix}}{\begin{matrix}\delta_3\\ \Pi_3\end{matrix}}\,\rho_e \qquad \rhd \qquad \dfrac{\begin{matrix}\Pi_1\\ x : \bot^L\end{matrix}}{\begin{matrix}\delta_3\\ \Pi_3\end{matrix}}\,efq$$

Prawitz [53] considers this reduction a *permutative* reduction since it does not deal directly with an introduction rule. We categorise it alongside the usual reductions due to their similarity in argumentation - we do not deal with sequences of the same formula, as is the case with the hidden detours that may occur around $\sqcup$ and $\exists$ operators.

$dist - join$**-reduction** The rules on labels $dist$ and $join$ can produce unwanted detours when mixed with the rules for operator $\multimap$.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ y : \alpha^{L,\forall Rx}\end{matrix} \quad \dfrac{\begin{matrix}\Pi_2\\ y : (\alpha \multimap \beta)^{L,\forall Rx}\end{matrix}}{y : \alpha^{L,\forall Rx} \multimap \beta^{L,\forall Rx}}\,dist}{y : \beta^{L,\forall Rx}}\,\multimap e}{y : \alpha^{L,\exists Rx} \multimap \beta^{L,\forall Rx}}\,\multimap i}{\begin{matrix}y : (\alpha \multimap \beta)^{L,\forall Rx}\\ \Pi_3\end{matrix}}\,join \qquad \rhd \qquad \begin{matrix}\Pi_2\\ y : (\alpha \multimap \beta)^{L,\forall Rx}\\ \Pi_3\end{matrix}$$

There cannot be any occurrences of y in undischarged assumptions of Π_2. L must contain only universal restrictions. We also need that assumptions of Π_2 not be discharged by the depicted application of $\multimap i$, as this would make the derivation dependant on more assumptions after the reduction - effectively turning it into a quasi-derivation. We show this via an example, which serves as an argument as to why our normalisation procedure considers a less strict notion of normalisation, i.e. it normalises derivations *up to a certain degree*, as indicated in Section 3.3.2.

Example 3.5 (An Exception to Normalisation)**.** *This example shows a derivation of* $y : (\alpha \multimap \beta)^{\forall Rx}$ *from the set of assumptions* $\Gamma = \{y : \alpha^{\forall Rx}, y : \alpha^{\exists Rx} \multimap (\alpha \multimap \beta)^{\forall Rx}\}$*. The assumption of* $y : \alpha^{\exists Rx}$ *is discharged by the application of* $\multimap i$ *depicted. We notice a candidate detour for a* $dist - join$ *reduction,* *but we will show that, in this case, we cannot apply the reduction.*

$$\dfrac{\dfrac{\dfrac{y:\alpha^{\forall Rx} \qquad \dfrac{\dfrac{[y:\alpha^{\exists Rx}]^1 \quad y:\alpha^{\exists Rx} \dashv (\alpha \dashv \beta)^{\forall Rx}}{y:(\alpha \dashv \beta)^{\forall Rx}}\dashv e}{y:\alpha^{\forall Rx} \dashv \beta^{\forall Rx}}\, dist}{y:\beta^{\forall Rx}}\dashv e}{y:\alpha^{\exists Rx} \dashv \beta^{\forall Rx}}\dashv i(1)}{y:(\alpha \dashv \beta)^{\forall Rx}}\, join$$

Suppose we allow $dist - join$ reduction to be applied to this derivation in which the application of $\dashv i$ discharges assumptions above the application of $dist$. In that case, we arrive at the following derivation:

$$\dfrac{y:\alpha^{\exists Rx} \quad y:\alpha^{\exists Rx} \dashv (\alpha \dashv \beta)^{\forall Rx}}{y:(\alpha \dashv \beta)^{\forall Rx}}\dashv e$$

This derivation, however, is no longer a derivation from Γ to the conclusion since we add a new assumption - $y:\alpha^{\exists Rx}$. It is, in fact, a derivation from $\Gamma \cup \{y:\alpha^{\exists Rx}\}$ to the conclusion, but only a quasi-derivation from Γ to the conclusion.

Due to derivations such as the one from Example 3.5, we call this process a *quasi*-normalisation. Thus, we add the restriction to $dist - join$ reduction to avoid losing soundness.

$chng - efq$**-reduction** The rule on labels $chng$ can produce unwanted detours when mixed with the rules for operator $\dashv$ (in this case, rule $\neg e$, which implies $\beta = \bot$) and rule efq.

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ y:\alpha^{L,\exists Rx}\end{array} \qquad \dfrac{xRy \quad \begin{array}{c}\Pi_2\\ y:\alpha^{L,\forall Rx} \dashv \bot^{L',\forall Rx}\end{array}}{y:\alpha^{L,\exists Rx} \dashv \bot^{L',\exists Rx}}\, chng}{y:\bot^{L',\exists Rx}}\neg e}{y:\alpha^{L,\forall Rx} \dashv \bot^{L',\forall Rx}}\, efq \\ \Pi_3 \end{array} \quad \rhd \quad \begin{array}{c}\Pi_2\\ y:\alpha^{L,\forall Rx} \dashv \bot^{L',\forall Rx}\\ \Pi_3\end{array}$$

There cannot be any occurrences of y in undischarged assumptions of Π_2. L' consists of only existential restrictions.

After a reduction, we change the regular discharge-function $\mathcal{F}$ of Π to a function $\mathcal{F}'$, which is an adaptation of $\mathcal{F}$ with the necessary modifications because certain assumptions in Π disappear and that Π' may contain many sub-derivations of the same form, as is the case with, for instance, $\dashv$-reduction if there is more than one discharged hypothesis by the application of the rule to be removed.

There are, however, situations in which the detour is not apparent, so two kinds of *permutation* are needed.

$\sqcup$-permutation This operator may *hide* a maximum formula, since δ can be the major premise of another elimination rule. Let us, then, define $\sqcup$-permutation, which allows for a rearrangement of the derivation around $\sqcup e$. Let ρ_e be an elimination rule with major premise δ_1, minor premise δ_2 and conclusion δ_3. We assume that ρ_e has only one minor premise; the argument is analogous with more premises, as the minor premises will be propagated accordingly. If there are no minor premises, this holds as well.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x:(\alpha\sqcup\beta)^L\end{array} \quad \begin{array}{c}[x:\alpha^L]\\ \Pi_2\\ \delta_1\end{array} \quad \begin{array}{c}[x:\beta^L]\\ \Pi_3\\ \delta_1\end{array}}{\delta_1}\sqcup e \qquad \begin{array}{c}\Pi_4\\ \delta_2\end{array}}{\begin{array}{c}\delta_3\\ \Pi_5\end{array}}\rho_e$$

This derivation is changed via **$\sqcup$-permutation** to:

$$\dfrac{\begin{array}{c}\Pi_1\\ x:(\alpha\sqcup\beta)^L\end{array} \qquad \begin{array}{c}[x:\alpha^L]\\ \dfrac{\begin{array}{c}\Pi_2\\ \delta_1\end{array} \quad \begin{array}{c}\Pi_4\\ \delta_2\end{array}}{\delta_3}\rho_e\end{array} \qquad \begin{array}{c}[x:\beta^L]\\ \dfrac{\begin{array}{c}\Pi_3\\ \delta_1\end{array} \quad \begin{array}{c}\Pi_4\\ \delta_2\end{array}}{\delta_3}\rho_e\end{array}}{\begin{array}{c}\delta_3\\ \Pi_5\end{array}}\sqcup e$$

With this derivation in hand, there may be an introduction rule in either Π_2 or Π_3 which creates an *unseen* detour with ρ_e, since ρ_e is an elimination rule and there may be other detours in the sub-derivations. It is important to note that this permutation does not increase the degree of the derivation since the maximum formula, be it $x : (\alpha \sqcup \beta)^L$ or any of the δ's, does not increase in complexity. The branches increase in the number of formulas, however.

It is important to note that this derivation makes the major premise of the depicted application of ρ_e depend on $x : \alpha^L$ on the left side and $x : \beta^L$ on the right side, as these assumptions are only discharged in the $\sqcup e$ below it. This new dependence, however, does not interfere with the restrictions of the rules ρ_e may be - the only restrictions on assumptions exist in $\forall i$ (an introduction rule, which ρ_e cannot be) and in the *minor* premise of $\exists e$ rule. So, we still have a derivation from the same set of assumptions to the same conclusion.

The same kind of unseen detour happens with $\exists e$.

$\exists$-permutation In a similar fashion to the previous elimination rule, $\exists e$ may hide detours. Let ρ_e be an elimination rule with major premise δ_1, minor premise δ_2 and conclusion δ_3. We assume that ρ_e has only one minor premise;

the argument is analogous with more premises, as the minor premises will be propagated accordingly. If there are no minor premises, this holds as well.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x:(\exists R.\alpha)^L\end{array} \quad \begin{array}{c}[y:\alpha^{\exists Rx,L}][xRy]\\ \Pi_2\\ \delta_1\end{array}}{\delta_1}\exists e \qquad \begin{array}{c}\Pi_3\\ \delta_2\end{array}}{\begin{array}{c}\delta_3\\ \Pi_4\end{array}}\rho_e$$

This derivation is changed via **$\exists$-permutation** to:

$$\dfrac{\begin{array}{c}\Pi_1\\ x:(\exists R.\alpha)^L\end{array} \quad \begin{array}{c}[y:\alpha^{\exists Rx,L}][xRy]\\ \dfrac{\begin{array}{c}\Pi_2\\ \delta_1\end{array} \quad \begin{array}{c}\Pi_3\\ \delta_2\end{array}}{\delta_3}\rho_e\end{array}}{\begin{array}{c}\delta_3\\ \Pi_4\end{array}}\exists e$$

In this case, the unseen detour may lie in Π_2.

Similar to $\sqcup$-permutation, we add new assumptions to the major premise of ρ_e, and it does not interfere with the restrictions of our rules. So, we still have a derivation from the same set of assumptions to the same conclusion.

With these permutations, we define *segments* and how they are related.

Example 3.6 (An unseen detour with $\sqcup$-permutation)**.** *In this example, let ρ_e be an application of $\sqcap e_1$, δ_m be a formula introduced by $\sqcap i$ (and eliminated by ρ_e). Let δ_1 and δ_2 be formulas, and Π_{1-5} be sub-derivations.*

$$\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \dfrac{\begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta_1\end{array} \quad \begin{array}{c}[x:\alpha]\\ \Pi_3\\ \delta_2\end{array}}{\delta_m}\sqcap i \quad \begin{array}{c}[x:\beta]\\ \Pi_4\\ \delta_m\end{array}}{\begin{array}{c}\dfrac{\delta_m}{\delta_1}\sqcap e_1\\ \Pi_5\end{array}}\sqcup e$$

After an application of $\sqcup$-permutation, this derivation becomes:

$$\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array} \quad \dfrac{\dfrac{\begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta_1\end{array} \quad \begin{array}{c}[x:\alpha]\\ \Pi_3\\ \delta_2\end{array}}{\delta_m}\sqcap i}{\delta_1}\sqcap e_1 \quad \begin{array}{c}[x:\beta]\\ \Pi_4\\ \dfrac{\delta_m}{\delta_1}\sqcap e_1\end{array}}{\begin{array}{c}\delta_1\\ \Pi_5\end{array}}\sqcup e$$

Then, after the detour around the $\sqcap$ operator becomes apparent, this derivation is reduced via $\sqcap$-reduction to:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha\sqcup\beta\end{array}\quad\begin{array}{c}[x:\alpha]\\ \Pi_2\\ \delta_1\end{array}\quad\begin{array}{c}[x:\beta]\\ \Pi_4\\ \dfrac{\delta_m}{\delta_1}\sqcap e_1\end{array}}{\delta_1}\sqcup e\\ \Pi_5\end{array}$$

Example 3.7 (An unseen detour with $\exists$-permutation)**.** *In this example, let ρ_e be an application of $\sqcap e_1$, δ_m be a formula introduced by $\sqcap i$ (and eliminated by ρ_e). Let δ_1 and δ_2 be formulas, and Π_{1-4} be sub-derivations.*

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\exists R.\alpha\end{array}\quad\dfrac{\begin{array}{c}[y:\alpha^{\exists Rx}][xRy]\\ \Pi_2\\ \delta_1\end{array}\quad\begin{array}{c}[y:\alpha^{\exists Rx}][xRy]\\ \Pi_3\\ \delta_2\end{array}}{\delta_m}\sqcap i}{\dfrac{\delta_m}{\delta_1}\sqcap e_1}\exists e\\ \Pi_4\end{array}$$

After an application of $\exists$-permutation, this derivation becomes:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\exists R.\alpha\end{array}\quad\dfrac{\dfrac{\begin{array}{c}[y:\alpha^{\exists Rx}][xRy]\\ \Pi_2\\ \delta_1\end{array}\quad\begin{array}{c}[y:\alpha^{\exists Rx}][xRy]\\ \Pi_3\\ \delta_2\end{array}}{\delta_m}\sqcap i}{\delta_1}\sqcap e_1}{\delta_1}\exists e\\ \Pi_4\end{array}$$

Then, after the detour around the $\sqcap$ operator becomes apparent, this derivation is reduced via $\sqcap$-reduction to:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ x:\exists R.\alpha\end{array}\quad\begin{array}{c}[y:\alpha^{\exists Rx}][xRy]\\ \Pi_2\\ \delta_1\end{array}}{\delta_1}\exists e\\ \Pi_4\end{array}$$

As seen in the second case of Definition 3.18, it is necessary to generalise the notion of maximum formula via defining what a *segment* and a *maximum segment* are, as there may be *hidden* detours due to applications of $\sqcup e$ or $\exists e$.

Definition 3.21 (*Segment*)**.** *A segment σ in a derivation Π is a sequence δ_1, $\ldots$, δ_n of consecutive formulas in a branch on Π such that:*

1. *δ_1 is not the consequence of an application of $\sqcup e$ or $\exists e$;*

2. *δ_i, for all $i < n$, is a minor premise of an application of $\sqcup e$ or $\exists e$; and*

3. *δ_n is not a minor premise of an application of $\sqcup e$ or $\exists e$.*

Since the formulas are consecutive in the branch containing the segment, all formulas are immediately above the next one (apart, obviously, from the last formula in the segment).

One can see that a segment consists of a sequence of occurrences of the same formula,[1] since all but the last of them are minor premises of applications of $\sqcup e$ or $\exists e$ and, in these rules, the conclusion must be the same as the minor premise(s). As δ_n is the conclusion of the final application of $\sqcup e$ or $\exists e$, it is also the same formula as the others.

Example 3.8 (A segment)**.** *Let Π_{1-5} and Π' be sub-derivations. An example of a segment is given below by the formulas in bold (i.e. $\boldsymbol{x:\alpha}$, $\boldsymbol{x:\alpha}$, $\boldsymbol{x:\alpha}$). For simplicity, we assume that no hypothesis discharge relative to the applications of rules shown was made in this derivation.*

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:\alpha_1\sqcup\beta_1\end{matrix}\quad \dfrac{\begin{matrix}\Pi_2\\ x:\exists R.\alpha_2\end{matrix}\quad \dfrac{\begin{matrix}\Pi_3\\ x:\alpha\sqcap\beta\end{matrix}}{\boldsymbol{x:\alpha}}\sqcap e_1}{\boldsymbol{x:\alpha}}\exists e\quad \begin{matrix}\Pi_4\\ x:\alpha\end{matrix}}{\boldsymbol{x:\alpha}}\sqcup e\quad \begin{matrix}\Pi_5\\ x:\alpha \relbar\!\!\dashv \beta\end{matrix}}{\begin{matrix}x:\beta\\ \Pi'\end{matrix}}\relbar\!\!\dashv e$$

In this example, at least one other segment is starting in Π_4 or, if it does not end in an application of $\sqcup e$ or $\exists e$, in the formulas shown below. Nothing stops the possibility of even another segment in any of the sub-derivations.

An important consideration is that, as $\sqcup e$ has two minor premises of equal shape, a derivation containing this rule always spawns at least two segments, which can then be further subdivided depending on the number of applications of this rule and how they are spaced within the derivation.

Definition 3.22 (*Length of a segment*)**.** *Let $\sigma = \delta_1, \ldots, \delta_n$ be a segment in a derivation. We call n the length of σ.*

Definition 3.23 (*Maximum Segment*)**.** *A maximum segment is a segment σ which begins with an application of an introduction rule (i.e. $\sqcap i$, $\sqcup i$, $\relbar\!\!\dashv i$, $\neg i$, $\forall i$ or $\exists i$) or rule efq and ends with a major premise of an elimination rule.*

Since the formula δ_n in σ is the major premise of an elimination rule, if the rule that has δ_1 (which is the same as δ_n) was an introduction rule for the corresponding operator, then we have a *hidden detour*.

One can see that a maximum formula is equivalent to a maximum segment of size 1.

[1]Alternatively: *formulas of the same shape* [53].

Example 3.9 (A maximum segment). *Let Π_{1-6} and Π' be sub-derivations. An example of a maximum segment is given below by the formulas in bold (i.e. $\boldsymbol{x : \neg\alpha}$, $\boldsymbol{x : \neg\alpha}$, $\boldsymbol{x : \neg\alpha}$), where the lists of labels are empty:*

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:\alpha_1\sqcup\beta_1\end{matrix}\quad \dfrac{\begin{matrix}\Pi_2\\ x:\alpha_2\sqcup\beta_2\end{matrix}\quad \dfrac{\begin{matrix}\Pi_5\\ x:\bot\end{matrix}}{\boldsymbol{x:\neg\alpha}}\neg i\quad \begin{matrix}\Pi_4\\ x:\neg\alpha\end{matrix}}{\boldsymbol{x:\neg\alpha}}\sqcup e\quad \begin{matrix}\Pi_3\\ x:\neg\alpha\end{matrix}}{\boldsymbol{x:\neg\alpha}}\sqcup e\quad \begin{matrix}\Pi_6\\ x:\alpha\end{matrix}}{\begin{matrix}x:\bot\\ \Pi'\end{matrix}}\neg e$$

In this example, there may be other maximum segments - even of greater length and degree than the one above.

We also generalise the concept of the degree of a formula for a segment:

Definition 3.24 (*Degree of a Segment*). *The degree of a segment $\sigma = \delta_1, \ldots, \delta_n$ is defined as: $deg(\sigma) = deg(\delta_n)$.*

As the formulas in a segment are all the same, it does not matter which we choose.

For the cases of $dist - join$ and $chng - efq$ detours, we create a special analogue of segments called *sections*:

Definition 3.25 (*$dist-join$ Section*). *A section σ is a sequence $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ of consecutive formulas such that δ_1 is the premise of an application of $dist$, δ_5 is the conclusion of an application of $join$, and $\delta_1 = \delta_5$. A $dist - join$ Section has length 5.*

Definition 3.26 (*$chng - efq$ Section*). *A section σ is a sequence $\delta_1, \delta_2, \delta_3, \delta_4$ of consecutive formulas such that δ_1 is the major premise of an application of $chng$, δ_4 is the conclusion of an application of efq, and $\delta_1 = \delta_4$. A $chng - efq$ Section has length 4.*

Definition 3.27 (*Degree of a Section*). *The degree of a section $\sigma = \delta_1, \ldots, \delta_n$ (either $dist - join$ or $chng - efq$) is defined as: $deg(\sigma) = deg(\delta_2)$.*

The second formula characterises the detour in question in any of the two possible sections since it was introduced without necessity.

Since every section contains a detour, they trivially are *maximum* sections, so we omit this characterisation.

Definition 3.28 (*Degree of a Derivation*). *The degree of a derivation Π is defined as $deg(\Pi) = max\{deg(\sigma) : \sigma$ is a maximum segment or section in $\Pi\}$.*

This definition shows the intuition behind the terms *maximum formula* and *maximum segment.*

Definition 3.29 (*Index of a Derivation*)**.** *The index of a derivation* Π *is a pair* $i(\Pi) = \langle d, l\rangle$*, where* l *is the sum of the lengths of the maximal segments and sections of* Π *with degree* d*. If there are no maximal segments and sections, then* $i(\Pi) = \langle 0, 0\rangle$*.*

We compare indexes of derivations in lexicographical order.

With these definitions in hand, we can then evaluate derivations in terms of detours.

Definition 3.30 (*Critical Derivation*)**.** *A derivation* Π *is called critical when*

1. Π *ends with an application of an elimination rule* ρ_e *and the major premise* δ *of* ρ_e *is in a maximum segment* σ*, or* Π *ends in a section* σ*; and*

2. $deg(\Pi) = deg(\sigma)$*; and*

3. *for all* Π' *sub-derivations of* Π*,* $deg(\Pi') < deg(\sigma) = deg(\Pi)$*.*

With this definition, we know that the primary detour to be reduced lies (at first) in the conclusion of the critical derivation.

Definition 3.31 (*Normal Derivation*)**.** *A derivation* Π *is called normal when* $i(\Pi) = \langle 0, 0\rangle$*.*

We will refer to simple derivations using the following notation: Π/δ represents a derivation Π with δ as a conclusion. Conversely, δ/Π represents a derivation Π with δ as a premise. As stated previously, there may be more copies of δ in the actual derivation Π.

Before reaching normalisation, we take a couple of preliminary steps.

Lemma 3.5. *Let* Π_1/δ *and* δ/Π_2 *be two derivations in the ND system such that* $deg(\Pi_1) = n_1$ *and* $deg(\Pi_2) = n_2$*. Then,* $deg(\Pi_1/[\delta]/\Pi_2) = max\{deg(\delta), n_1, n_2\}$*.*

Proof. Directly from Definition 3.28: the maximum formula in this derivation is either in Π_1, in Π_2 or is δ itself. □

Lemma 3.6. *Let* δ *be a formula in iALC,* Γ *be a set of formulas,* $\Delta \subseteq \Gamma$*, and* Π *be a critical derivation of* $\Gamma \vdash_{ND} \delta$*. Then,* Π *reduces to a derivation* Π' *of* $\Delta \vdash_{ND} \delta$ *such that* $i(\Pi') < i(\Pi)$*.*

Proof. The proof follows by induction on the index $\langle d, l\rangle$ of Π. We consider first the base cases where Π contains only maximum segments of length 1, i.e. maximum formulas, and sections. Then, we apply the appropriate reductions from Definition 3.20 to each detour, starting with the upper-leftmost maximum segments and sections in Π. By upper-leftmost segment or section, we mean that there is no maximum segment or section of the same degree above it or contains a formula occurrence side-connected with the last formula occurrence of the segment in question, which has a connection with Π being a critical derivation: no sub-derivation above the upper-leftmost segment of Π has degree equal to d. The same applies to sections as well. In other words, by always choosing the upper-leftmost segment, we ensure that there will be no increases in the index by a hidden detour. The following are the base cases (we will consider $size(L) = n$):

– $\sqcap$-reduction

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ x : \alpha^L\end{matrix} \quad \begin{matrix}\Pi_2 \\ x : \beta^L\end{matrix}}{x : (\alpha \sqcap \beta)^L}\sqcap i}{\begin{matrix} x : \alpha^L \\ \Pi_3 \end{matrix}}\sqcap e_1 \qquad \rhd \qquad \begin{matrix}\Pi_1 \\ x : \alpha^L \\ \Pi_3\end{matrix}$$

We have $deg(\Pi) = deg(x : (\alpha \sqcap \beta)^L)$ (according to Item 3 of Definition 3.30), which equals $deg(\alpha)+deg(\beta)+1+n$. When reduced to Π', we have, as per Lemma 3.5, that $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_3), deg(x : \alpha^L)\}$. If it is equal to $deg(\Pi_1)$, then it is strictly smaller because of item 4 of Definition 3.30. If it is $deg(x : \alpha^L) = deg(\alpha^L)$, then it is strictly smaller than $deg(\Pi) = deg(\alpha) + deg(\beta) + 1 + n$. If it is equal to $deg(\Pi_3)$, then it is less or equal to $deg(x : (\alpha \sqcap \beta)^L)$. If d remains the same, we reduce l by 1, so $i(\Pi') < i(\Pi)$.

If we instead reduce via $\sqcap e_2$ and $x : \beta^L$ to $\Pi_2/(x : \beta^L)/\Pi_3$, the reasoning is the same since this is a critical derivation and we consider its upper-leftmost detour.

– $\multimap$-reduction (and $\neg$-reduction)

$$\dfrac{\begin{matrix}\Pi_1 \\ x : \alpha^L\end{matrix} \quad \dfrac{\begin{matrix}[x : \alpha^L] \\ \Pi_2 \\ x : \beta^{L'}\end{matrix}}{x : (\alpha^L \multimap \beta^{L'})}\multimap i}{\begin{matrix} x : \beta^{L'} \\ \Pi_3 \end{matrix}}\multimap e \qquad \rhd \qquad \begin{matrix}\Pi_1 \\ x : \alpha^L \\ \Pi_2 \\ x : \beta^{L'} \\ \Pi_3\end{matrix}$$

We have $deg(\Pi) = deg(x : \alpha^L \mathrel{-\!\!\dashv} \beta^{L'})$, which equals $deg(\alpha) + deg(\beta) + 1 + n + n'$, where $size(L') = n'$. When reduced to Π', $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(\Pi_3), deg(x : \alpha^L), deg(x : \beta^{L'})\}$. If it is equal to $deg(\Pi_1)$ or $deg(\Pi_2)$, it is strictly smaller. On the other hand, if it is either $deg(x : \alpha^L) = deg(\alpha^L)$ or $deg(x : \beta^{L'}) = deg(\beta^{L'})$ it will be smaller than $deg(\Pi)$ as well. If it is equal to $deg(\Pi_3)$, then it is less or equal to $deg(x : \alpha^L \mathrel{-\!\!\dashv} \beta^{L'})$. If d remains the same, we reduce l by 1, so $i(\Pi') < i(\Pi)$ in any case.

If this derivation is instead reduced to $\Pi_2/(x : \beta^{L'})/\Pi_3$ i.e. if $x : \alpha^L$ does not appear discharged above Π_2, the argument remains the same.

Since we define $\neg$ as a particular case of $\mathrel{-\!\!\dashv}$, the proof for $\neg$-reduction follows directly.

– $\forall$-reduction

$$\begin{array}{c}\dfrac{xRy \qquad \dfrac{\begin{array}{c}[xRy]\\ \Pi_1\\ y:\alpha^{\forall Rx,L}\end{array}}{x:(\forall R.\alpha)^L}\,\forall i}{y:\alpha^{\forall Rx,L}}\,\forall e\\ \Pi_2\end{array} \qquad \rhd \qquad \begin{array}{c}\Pi_1\\ y:\alpha^{\forall Rx,L}\\ \Pi_2\end{array}$$

We have $deg(\Pi) = deg(x : (\forall R.\alpha)^L)$, which equals $deg(\alpha) + 1 + n$. With Π', we have that $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(y : \alpha^{\forall Rx,L})\}$. If $deg(\Pi') = deg(\Pi_1)$, then it is strictly smaller. If $deg(\Pi') = deg(y : \alpha^{\forall Rx,L}) = deg(\alpha^{\forall Rx,L}) = deg(\alpha) + 1 + n$, then it is equal to $deg(\Pi)$. If it is equal to $deg(\Pi_2)$, then it is less or equal to $deg(x : (\forall R.\alpha)^L)$. If d remains the same, we reduce l by 1, so $i(\Pi') < i(\Pi)$ anyway.

– $\sqcup$-reduction

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x:\alpha^L\end{array}}{x:(\alpha\sqcup\beta)^L}\,\sqcup i_1 \qquad \begin{array}{c}[x:\alpha^L]\\ \Pi_2\\ z:\gamma^{L'}\end{array} \qquad \begin{array}{c}[x:\beta^L]\\ \Pi_3\\ z:\gamma^{L'}\end{array}}{z:\gamma^{L'}}\,\sqcup e\\ \Pi_4\end{array} \qquad \rhd \qquad \begin{array}{c}\Pi_1\\ x:\alpha^L\\ \Pi_2\\ z:\gamma^{L'}\\ \Pi_4\end{array}$$

We have $deg(\Pi) = deg(x : (\alpha\sqcup\beta)^L)$, which equals $deg(\alpha)+deg(\beta)+1+n$. When turned to Π', $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_2), deg(\Pi_4), deg(z : \gamma^{L'}), deg(x : \alpha^L))\}$. The argument for $deg(\Pi_1)$, $deg(\Pi_2)$ or $deg(z : \gamma^{L'})$ is the same due to Π being a critical derivation. If $deg(\Pi') = deg(x : \alpha^L) = deg(\alpha^L)$, then it is strictly smaller than $deg(\Pi)$. If it is equal to $deg(\Pi_4)$, then it is less or equal to $deg(\Pi)$. If d remains the same, we reduce l by 1, so $i(\Pi') < i(\Pi)$.

The argument remains the same if $x : \alpha^L$ does not appear discharged above Π_2.

The case for $\sqcup i_2$ through $x : \beta^L$ is precisely the same since this is a critical derivation, and we choose the upper-leftmost detour.

– $\exists$-reduction

$$\dfrac{\dfrac{xRy \quad \begin{matrix}\Pi_1 \\ y : \alpha^{\exists Rx,L}\end{matrix}}{x : (\exists R.\alpha)^L}\,\exists i \qquad \begin{matrix}[y : \alpha^{\exists Rx,L}][xRy] \\ \Pi_2 \\ z : \beta^{L'}\end{matrix}}{z : \beta^{L'}}\,\exists e \quad \begin{matrix} \\ \Pi_3\end{matrix} \qquad \triangleright \qquad \begin{matrix}\Pi_1 \\ y : \alpha^{\exists Rx,L} \\ \Pi_2 \\ z : \beta^{L'} \\ \Pi_3\end{matrix}$$

We have $deg(\Pi) = deg(x : (\exists R.\alpha)^L)$, which equals $deg(\alpha) + 1 + n$. When turned to Π', $deg(\Pi') = \max\{deg(\Pi_1), deg(\Pi_2), deg(\Pi_3), deg(y : \alpha^{\exists Rx,L}), deg(z : \beta^{L'})\}$. If it is equal to that of any of the sub-derivations or $deg(z : \beta^{L'})$, then, due to it being a critical derivation, the degree is strictly smaller. If it is equal to $deg(y : \alpha^{\exists Rx,L})$, then it is the same. If d remains the same, we reduce l by 1, so $i(\Pi') < i(\Pi)$.

The argument remains the same if $y : \alpha^{\exists Rx,L}$ does not appear discharged above Π_2.

– efq-reduction

$$\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ x : \bot^L\end{matrix}}{\delta_1}\,efq \qquad \begin{matrix}\Pi_2 \\ \delta_2\end{matrix}}{\delta_3}\,\rho_e \quad \begin{matrix} \\ \Pi_3\end{matrix} \qquad \triangleright \qquad \begin{matrix}\Pi_1 \\ \dfrac{x : \bot^L}{\delta_3}\,efq \\ \Pi_3\end{matrix}$$

Let δ_1 be the maximum formula in this derivation and the major premise of ρ_e. Thus, $deg(\Pi) = deg(\delta_1)$. $deg(\Pi') = max\{deg(\Pi_1), deg(\Pi_3), deg(x : \bot^L), deg(\delta_3)\}$. In any case, $deg(\Pi') < d$, since δ_1 was the maximum formula.

In these cases, where there are only maximum segments of length 1, it is plain to see that once we remove one of these segments, l is reduced by 1, and we show in each case how d never increases. If $l = 1$, the removed detour was the last one since it was on a critical derivation, and Π' must have a degree strictly lower than Π. Thus, $i(\Pi') < i(\Pi)$ in either situation.

For the exceptional cases of $dist - join$-reduction and $chng - efq$-reduction, we show that d in each case is either the same or smaller, but we always reduce l, thus reducing the index of the derivation.

– $dist - join$-reduction

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\Pi_1\\ y : \alpha^{L,\forall Rx}\end{matrix} \quad \dfrac{\begin{matrix}\Pi_2\\ y : (\alpha \mathrel{-\!\!\dashv} \beta)^{L,\forall Rx}\end{matrix}}{y : \alpha^{L,\forall Rx} \mathrel{-\!\!\dashv} \beta^{L,\forall Rx}}\, dist}{y : \beta^{L,\forall Rx}}\, {-\!\!\dashv} e}{y : \alpha^{L,\exists Rx} \mathrel{-\!\!\dashv} \beta^{L,\forall Rx}}\, {-\!\!\dashv} i}{\begin{matrix} y : (\alpha \mathrel{-\!\!\dashv} \beta)^{L,\forall Rx}\\ \Pi_3\end{matrix}}\, join
\qquad \triangleright \qquad
\begin{matrix}\Pi_2\\ y : (\alpha \mathrel{-\!\!\dashv} \beta)^{L,\forall Rx}\\ \Pi_3\end{matrix}
$$

$deg(\Pi) = deg(y : \alpha^{L,\forall Rx} \mathrel{-\!\!\dashv} \beta^{L,\forall Rx}) = deg(\alpha) + deg(\beta) + 2n + 3$, since it is the maximum formula in the derivation. Thus, $deg(\Pi') = max\{deg(\Pi_2), deg(\Pi_3), deg(y : (\alpha \mathrel{-\!\!\dashv} \beta)^{L,\forall Rx})\}$. In any case, $deg(\Pi') < d$, since $deg(y : (\alpha \mathrel{-\!\!\dashv} \beta)^{L,\forall Rx}) = deg(\alpha) + deg(\beta) + n + 2 < deg(\Pi)$, and the sub-derivations have strictly smaller degree. We also reduce l by 5 since the derivation has one less section.

– $chng - efq$-reduction

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1\\ y : \alpha^{L,\exists Rx}\end{matrix} \quad \dfrac{\begin{matrix}\Pi_2\\ xRy \quad y : \alpha^{L,\forall Rx} \mathrel{-\!\!\dashv} \bot^{L',\forall Rx}\end{matrix}}{y : \alpha^{L,\exists Rx} \mathrel{-\!\!\dashv} \bot^{L',\exists Rx}}\, chng}{y : \bot^{L',\exists Rx}}\, \neg e}{\begin{matrix} y : \alpha^{L,\forall Rx} \mathrel{-\!\!\dashv} \bot^{L',\forall Rx}\\ \Pi_3\end{matrix}}\, efq
\qquad \triangleright \qquad
\begin{matrix}\Pi_2\\ y : \alpha^{L,\forall Rx} \mathrel{-\!\!\dashv} \bot^{L',\forall Rx}\\ \Pi_3\end{matrix}
$$

$deg(\Pi) = deg(y : \alpha^{L,\exists Rx} \mathrel{-\!\!\dashv} \bot^{L',\exists Rx}) = deg(\alpha) + deg(\beta) + 3 + n + n'$ (where $size(L') = n'$), since it is the maximum formula in the derivation. Thus, $deg(\Pi') = max\{deg(\Pi_2), deg(\Pi_3), deg(y : \alpha^{L,\forall Rx} \mathrel{-\!\!\dashv} \bot^{L',\forall Rx})\}$. It is strictly smaller than $deg(\Pi)$ if it is equal to any sub-derivations. If it is equal to $deg(y : \alpha^{L,\forall Rx} \mathrel{-\!\!\dashv} \bot^{L',\forall Rx} = deg(\alpha) + deg(\beta) + 3 + n + n'$, then d is the same. However, l is, in either case, reduced (by 4) since there is one less section in the derivation.

Now, we move on to the cases where l consists of segments of length greater than 1. Our objective here is to shorten each of these segments with $\sqcup$-permutations and $\exists$-permutations until they reach length 1, and we can apply the base cases. We will, then, show that applying a permutation lowers the index of Π, first, if the last formula occurrence below the segment is an application of $\sqcup e$, and after, if it is an application of $\exists e$. We will assume that δ_1 is the maximum formula of the critical derivation.

– $\sqcup$-permutation

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:(\alpha\sqcup\beta)^L\end{matrix}\quad \begin{matrix}[x:\alpha^L]\\ \Pi_2\\ \delta_1\end{matrix}\quad \begin{matrix}[x:\beta^L]\\ \Pi_3\\ \delta_1\end{matrix}}{\delta_1}\sqcup e \quad \begin{matrix}\Pi_4\\ \delta_2\end{matrix}}{\begin{matrix}\delta_3\\ \Pi_5\end{matrix}}\rho_e$$

This derivation is changed via **$\sqcup$-permutation** to:

$$\dfrac{\begin{matrix}\Pi_1\\ x:(\alpha\sqcup\beta)^L\end{matrix}\quad \dfrac{\begin{matrix}[x:\alpha^L]\\ \Pi_2\\ \delta_1\end{matrix}\quad \begin{matrix}\Pi_4\\ \delta_2\end{matrix}}{\delta_3}\rho_e \quad \dfrac{\begin{matrix}[x:\beta^L]\\ \Pi_3\\ \delta_1\end{matrix}\quad \begin{matrix}\Pi_4\\ \delta_2\end{matrix}}{\delta_3}\rho_e}{\begin{matrix}\delta_3\\ \Pi_5\end{matrix}}\sqcup e$$

For each application of $\sqcup$-permutation, the maximum segment in question (be it on the left or the right minor premises) has its length reduced by 1, as the application of the elimination rule ρ_e swaps place in the respective branch with the lowest application of $\sqcup e$ of the segment, thus staying above it, *breaking* the preexisting segment. The degree of the derivation remains the same since the degree of the segment does not change.

– $\exists$-permutation

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ x:(\exists R.\alpha)^L\end{matrix}\quad \begin{matrix}[y:\alpha^{\exists Rx,L}][xRy]\\ \Pi_2\\ \delta_1\end{matrix}}{\delta_1}\exists e \quad \begin{matrix}\Pi_3\\ \delta_2\end{matrix}}{\begin{matrix}\delta_3\\ \Pi_4\end{matrix}}\rho_e$$

This derivation is changed via **$\exists$-permutation** to:

$$\dfrac{\begin{matrix}\Pi_1\\ x:(\exists R.\alpha)^L\end{matrix}\quad \dfrac{\begin{matrix}[y:\alpha^{\exists Rx,L}][xRy]\\ \Pi_2\\ \delta_1\end{matrix}\quad \begin{matrix}\Pi_3\\ \delta_2\end{matrix}}{\delta_3}\rho_e}{\begin{matrix}\delta_3\\ \Pi_4\end{matrix}}\exists e$$

For each application of ∃-permutation, the segment in question has its length reduced by 1, as the application of the elimination rule ρ_e swaps place in the respective branch with the lowest application of $\sqcup e$ of the segment, thus staying above it, *breaking* the preexisting segment. The degree remains the same.

Since we always reduce l without increasing d, we show that there is always a way to reduce a critical derivation Π to a derivation Π' such that $i(\Pi') < i(\Pi)$. □

Now, we can reach normalisation.

Theorem 3.2 (Normalisation). *Let* Π *be a derivation of* $\Gamma \vdash_{ND} \delta$*. Then,* Π *reduces to a normal derivation* Π' *of* $\Delta \subseteq \Gamma \vdash_{ND} \delta$*.*

Proof. The proof follows directly from Lemma 3.6 by applying the permutations to remove maximum segments and reductions to remove sections and maximum formulas inductively over the index of Π.

We repeat this procedure for each maximum segment or section σ throughout the derivation (always choosing the upper-leftmost one) until they have length 1 (or are eliminated, in the case of sections), which is then covered by our base cases.

By repeating the application of permutations, we create a new segment as more and more applications of $\sqcup e$ or $\exists e$ stack below ρ_e. However, this new segment is (a) of lower degree than the original maximum segment, as its formula is no longer the maximum formula in question, and (b) of strictly lower length than the original maximum segment, as it is limited by the number of applications of $\sqcup e$ or $\exists e$ in the original segment. Thus, it does not increase the index of the derivation.

It is important to note that once we remove the maximum formulas of degree d from a derivation with index $\langle d, l \rangle$, we generate a new derivation with index $\langle d', l' \rangle$, where $d' < d$. However, there may be the case where there is one - or more - maximum segment σ' with length greater than 1 (or even a section), so the permutations of Lemma 3.6 have to be reapplied for the new maximum segments and sections of this new derivation. We then repeat this process until the index of the derivation becomes $\langle 0, 0 \rangle$, i.e. it is normal. □

Corollary 3.1 (Termination). *The normalisation process is terminating since it is done inductively on the index of the derivation.*

Corollary 3.2 (Sub-concept Principle). *Let* Π *be a normal derivation of* $\Gamma \vdash_{ND} \delta$*, where* Γ *is a set of labelled formulas and* δ*, a labelled formula. Then, every unlabelled concept occurrence in* Π *is a sub-concept of unlabelled concepts either in* δ *or in formulas of* Γ*.*

From normalisation, one arrives at the sub-concept principle (our version of the sub-formula principle, as we deal with TBox reasoning) following similar steps Prawitz took in [53]. The main difference is considering the unlabelled version of concepts instead of the labelled ones since labels have specific interactions that may require some kind of overhead in derivations, such as the example presented in Section 3.2.1.

It is important to note that we provided *weak* normalisation, i.e. we apply our normalisation procedure following a particular order. Strongly normalising systems have a more general procedure, in which the order of application of reductions is not relevant, as they have unification, so they arrive at *the* normal form, whichever it may be.

We also remind the reader of the specific restriction to the $dist - join$ detour to consider this system normalising. Without it, we only achieve a kind of *quasi*-normalisation at the expense of soundness.

3.3.4 Completeness

Theorem 3.3 (Completeness)**.** *Let δ be an iALC formula, Γ a set of iALC formulas, and δ' and Γ', the labelled versions of δ and Γ, respectively. Then, $\Gamma \models \delta$ implies $\Gamma' \vdash_{ND} \delta'$.*

Proof. In [4, 6], the authors provide a Hilbert system that implements TBox reasoning for iALC as per [63, 68, 59, 69] consisting in:

$$
\begin{array}{rl}
\text{(IPL)} & \text{all axioms of intuitionistic propositional logic} \\
(\forall\text{K}) & (\forall R.(\alpha \sqsubset \beta)) \sqsubset (\forall R.\alpha \sqsubset \forall R.\beta) \\
(\exists\text{K}) & (\forall R.(\alpha \sqsubset \beta)) \sqsubset (\exists R.\alpha \sqsubset \exists R.\beta) \\
\text{(DIST)} & \exists R.(\alpha \sqcup \beta) \sqsubset (\exists R.\alpha \sqcup \exists R.\beta) \\
\text{(DIST0)} & \exists R.\bot \sqsubset \bot \\
\text{(DISTm)} & (\exists R.\alpha \sqsubset \forall R.\beta) \sqsubset \forall R.(\alpha \sqsubset \beta) \\
\text{(Nec)} & \text{If } \alpha \text{ is a theorem, then } \forall R.\alpha \text{ is a theorem too.} \\
\text{(MP)} & \text{If } \alpha \text{ and } \alpha \sqsubset \beta \text{ are theorems,} \\
 & \text{then } \beta \text{ is a theorem too.}
\end{array}
$$

In [69], the authors prove that this system is sound and complete for TBox reasoning. So, to prove the completeness of our ND system, all we have to do is prove each of these axioms. Axioms in **(IPL)** are easily proven since all substitution instances of IPL theorems can be proven in our ND system using only propositional rules considering empty lists of labels. **(MP)** is covered by our $\multimap e$ rule, and **(Nec)**, by the following derivation (assuming empty lists of labels where there are no explicit lists):

$$\dfrac{\dfrac{\begin{matrix}[xRy]\\ \Pi_1\\ y:\alpha\end{matrix}}{y:\alpha^{\forall Rx}}\,Gen}{x:\forall R.\alpha}\,\forall i$$

We, then, only have to prove the remaining five axioms.

1. **(∀K)** $\boldsymbol{x:\forall R.(\alpha\multimap\beta)\multimap(\forall R.\alpha\multimap\forall R.\beta)}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[xRy]^3\quad [x:\forall R.\alpha]^2}{y:\alpha^{\forall Rx}}\,\forall e \qquad \dfrac{\dfrac{[xRy]^3\quad [x:\forall R.(\alpha\multimap\beta)]^1}{y:(\alpha\multimap\beta)^{\forall Rx}}\,\forall e}{y:\alpha^{\forall Rx}\multimap\beta^{\forall Rx}}\,dist}{y:\beta^{\forall Rx}}\,\multimap e}{x:\forall R.\beta}\,\forall i(3)}{x:(\forall R.\alpha\multimap\forall R.\beta)}\,\multimap i(2)}{x:\forall R.(\alpha\multimap\beta)\multimap(\forall R.\alpha\multimap\forall R.\beta)}\,\multimap i(1)$$

2. **(∃K)** $\boldsymbol{x:\forall R.(\alpha\multimap\beta)\multimap(\exists R.\alpha\multimap\exists R.\beta)}$

$$\Pi_1 : \quad \dfrac{[xRy]^3 \qquad \dfrac{\dfrac{[xRy]^3\quad [x:\forall R.(\alpha\multimap\beta)]^1}{y:(\alpha\multimap\beta)^{\forall Rx}}\,\forall e}{y:\alpha^{\forall Rx}\multimap\beta^{\forall Rx}}\,dist}{y:\alpha^{\exists Rx}\multimap\beta^{\exists Rx}}\,chng$$

$$\dfrac{\dfrac{\dfrac{[x:\exists R.\alpha]^2 \qquad \dfrac{[xRy]^3 \qquad \dfrac{[y:\alpha^{\exists Rx}]^3\quad \Pi_1}{y:\beta^{\exists Rx}}\,\multimap e}{x:\exists R.\beta}\,\exists i}{x:\exists R.\beta}\,\exists e(3)}{x:(\exists R.\alpha\multimap\exists R.\beta)}\,\multimap i(2)}{x:\forall R.(\alpha\multimap\beta)\multimap(\exists R.\alpha\multimap\exists R.\beta)}\,\multimap i(1)$$

A critical observation on this derivation (w.r.t. normalisation) is that it does not contain a detour, even if we have a $\exists i$ followed by $\exists e$, because we introduce $x:\exists R.\beta$ and eliminate $x:\exists R.\alpha$ - $x:\exists R.\beta$ is the minor premise of $\exists e$.

3. **(DIST)** $\boldsymbol{x:\exists R.(\alpha\sqcup\beta)\multimap(\exists R.\alpha\sqcup\exists R.\beta)}$

$$\Pi_1 : \quad \dfrac{\dfrac{[xRy]^2 \quad [y : \alpha^{\exists Rx}]^3}{x : \exists R.\alpha}\,\exists i}{x : \exists R.\alpha \sqcup \exists R.\beta}\,\sqcup i_1$$

$$\Pi_2 : \quad \dfrac{\dfrac{[xRy]^2 \quad [y : \beta^{\exists Rx}]^3}{x : \exists R.\beta}\,\exists i}{x : \exists R.\alpha \sqcup \exists R.\beta}\,\sqcup i_2$$

$$\dfrac{\dfrac{[x : \exists R.(\alpha \sqcup \beta)]^1 \qquad \dfrac{[y : (\alpha \sqcup \beta)^{\exists Rx}]^2 \quad \Pi_1 \quad \Pi_2}{x : \exists R.\alpha \sqcup \exists R.\beta}\,\sqcup e(3)}{x : \exists R.\alpha \sqcup \exists R.\beta}\,\exists e(2)}{x : \exists R.(\alpha \sqcup \beta) \dashv (\exists R.\alpha \sqcup \exists R.\beta)}\,\dashv i(1)$$

4. **(DIST0)** $\boldsymbol{x : \exists R.\bot \dashv \bot}$

$$\dfrac{\dfrac{[x : \exists R.\bot]^1 \qquad \dfrac{[y : \bot^{\exists Rx}]^2}{x : \bot}\,efq}{x : \bot}\,\exists e(2)}{x : (\exists R.\bot \dashv \bot)}\,\dashv i(1)$$

5. **(DISTm)** $\boldsymbol{x : (\exists R.\alpha \dashv \forall R.\beta) \dashv \forall R.(\alpha \dashv \beta)}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[xRy]^2 \qquad \dfrac{\dfrac{[xRy]^2 \quad [y : \alpha^{\exists Rx}]^3}{x : \exists R.\alpha}\,\exists i \qquad [x : \exists R.\alpha \dashv \forall R.\beta]^1}{x : \forall R.\beta}\,\dashv e}{y : \beta^{\forall Rx}}\,\forall e}{y : \alpha^{\exists Rx} \dashv \beta^{\forall Rx}}\,\dashv i(3)}{y : (\alpha \dashv \beta)^{\forall Rx}}\,join}{x : \forall R.(\alpha \dashv \beta)}\,\forall i(2)}{x : (\exists R.\alpha \dashv \forall R.\beta) \dashv \forall R.(\alpha \dashv \beta)}\,\dashv i(1)$$

□

3.4 Reasoning in ND: an Example

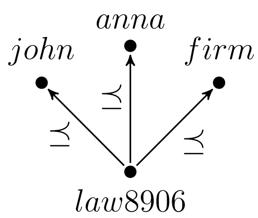
We will continue to work with the example in Section 2.4.2 but apply the ND system for iALC to formalise the reasoning in question. Firstly, let us recall the names for nominals, concepts and roles, and the TBox and ABox:

$$\Delta = \{law8906,\ john,\ anna,\ firm\}$$
$$N_C = \{Attorney,\ PracticeLaw,\ Intern,\ LawFirm\}$$
$$N_R = \{SupervisedBy,\ EmployeeAt\}$$

$$\mathcal{T} = \left\{ \begin{array}{c} Attorney \dashv PracticeLaw \\ (Intern \sqcap \exists SupervisedBy.\exists EmployeeAt.LawFirm) \dashv \\ \hfill PracticeLaw \end{array} \right\}$$

$$\mathcal{A} = \left\{ \begin{array}{c} john : Intern \\ anna : Attorney \\ firm : LawFirm \\ john \quad SupervisedBy \quad anna \\ anna \quad EmployeeAt \quad firm \end{array} \right\}$$

And the precedence between our VLSs:



We aim to reason over this KB and conclude that John can Practice Law, i.e. $john : PracticeLaw$. However, since our ND system reasons over the TBox, we need to adjust the ABox. The reader may recall that we can only swap between nominals in formulas via the rules $\forall e$, $\forall i$, $\exists e$, and $\exists i$. So, we must have a way to connect $john$ to any other nominal relevant to the derivation via universal or existential restrictions.

For instance, from the following assertions:

$$anna : Attorney \qquad firm : LawFirm \qquad anna\ EmployeeAt\ firm$$

We have that $anna : \exists EmployeeAt.LawFirm$, from the semantic definition of an existential restriction. This process also happens for:

$$john : Intern \qquad anna : Attorney \qquad john\ SupervisedBy\ anna$$

Which yields $john : \exists SupervisedBy.Attorney$. However, since we have $anna : \exists EmployeeAt.LawFirm$ as well, we can refine this assertion further to $john : \exists SupervisedBy.\exists EmployeeAt.LawFirm$.

From this, we can provide additional context via labels to previous assertions: $anna : \exists EmployeeAt.LawFirm$ becomes the assertion $anna : (\exists EmployeeAt.LawFirm)^{\exists SupervisedBy}$, since, in this situation, the concept $\exists EmployeeAt.LawFirm$ can be restricted further via $SupervisedBy$. The same applies to $firm : LawFirm$, which becomes $firm : LawFirm^{\exists EmployeeAt, \exists SupervisedBy}$ - meaning it can be restricted via $EmployeeAt$, and after via $SupervisedBy$.

It is also important to note that all the entailments present in $law8906$ will be passed by heredity to the other VLSs in this example, as it precedes all of them, especially $john$. So, from $law8906 \preceq john$ we have that $john : (Intern \sqcap \exists SupervisedBy.\exists EmployeeAt.LawFirm) \multimap PracticeLaw$.

This adaptation allows us to *focus* the reasoning of this situation on the individual $john$, which is directly relevant to the conclusion.

Then, with these adaptations in hand, we may show the derivation with the following changes: we shortened $SupervisedBy$ to S, $EmployeeAt$ to E, $LawFirm$ to $LFirm$ and $PracticeLaw$ to $PracLaw$, and the nominals to their initials.

$$\dfrac{\dfrac{jSa \quad \dfrac{aEf \quad f : LFirm^{\exists Ea, \exists Sj}}{a : (\exists E.LFirm)^{\exists Sj}}\ \exists i}{j : \exists S.\exists E.LFirm}\ \exists i \qquad j : \exists S.\exists E.LFirm \multimap PracLaw}{j : PracLaw}\ \multimap e$$

4
Motivation and Background for NMR

This chapter investigates what properties would be required of a logic that deals with NMR in a legal context. The motivation for this research arose from limitations in how iALC deals with law modelling and reasoning in specific contexts.

In Logic, the property of **monotonicity** is usually wanted (and assumed). This property states that given a set of premises Γ and a conclusion φ, if it is already the case that $\Gamma \models \varphi$, then there is nothing (for example, another set of premises Δ) we can add to Γ in order not to have φ as a conclusion anymore, i.e. it is not the case that $\Gamma \cup \Delta \not\models \varphi$. Monotonicity provides a sense of *safety* to a deduction, meaning that once something is seen as true or proven to be true, it cannot lose such status anymore, no matter what new premises may arise. If we, instead, take some premises away, it is always possible to not have the same conclusion anymore - monotonicity is concerned with adding information. Several logical formalisms consider monotonicity since they are mainly concerned with preserving truth deductively since we can see Logic as the study of sound argumentation. Notable examples are classical and intuitionistic logics and many systems based on them, including most DLs.

However, not all situations in which one encounters *reasoning*, one encounters monotonicity. For instance, if we have a bird, we may conclude that it flies, i.e. $b \to f$. However, if it is also a penguin, then f is no longer the case: $b \wedge p \to \neg f$, even though we know all penguins are birds. Thus, the bird can fly or not, depending on its status as a penguin or general bird.

Another example, more contextualised in the main application of iALC, is the judicial process. In a court of law, there are three main agents: one side, say, A, another, B, and a judge or jury. The main goals of sides A and B are to convince the judge or jury of their thesis via pieces of evidence, legal precedence and rules of the legal system in which the court takes place. However, it is essential to note that each side's thesis directly negates the other: they cannot coexist. So, depending on how the argumentation and presentation of evidence for each side proceeds, the judge/jury may be tempted to keep changing the conclusion at which they may arrive.

Example 4.1 (Non-monotonicity in a Legal Setting)**.** *Using the language of iALC, one may have a situation where person X murders person Y, but X acted in self-defence, as Y was a threat to X's well-being. Then, attorney A, going against X in a court of law, states that there is a particular law indicating that murder is a crime; thus, the murderer must be sent to jail, which could be represented in the TBox of iALC by* $Murderer \sqsupset InJail$*. However, attorney B, defending X, claims that a particular article in a particular legal text (of the same hierarchy as the law used for A's claim) states that even a homicide can be forgiven if the act was made in self-defence, namely* $Murderer \sqcap SelfDecence \sqsupset \neg InJail$*. Since these laws have no direct precedence, we have a contradiction in the knowledge base. We have, then, some options to handle this exceptional case of acting in self-defence: we either force a non-legal precedence between these laws - which we do not want since we want to reflect faithfully what is present in the law - or we remove A's claim from the TBox - which we also do not want, for we must represent all that is relevant to the case in order to let the judge reach a justified conclusion. A way to represent both norms, the one with the exception* ***and*** *the one without it simultaneously, must be found.*

Even though we show this situation in an argumentation-centred environment, in this work we chose not to increment iALC with Argumentation Theory or Argumentation Frameworks [31]. Instead, we increment the expressiveness of the logic itself. We provide an overview on Argumentation Frameworks in the end of Section 4.4.1.

In order to represent knowledge and reasoning, monotonicity is not necessarily desired depending on context. Non-monotonic logics aim to fill in this gap. In these logics, one usually adds an operator (unary or binary) to a logical language to represent defeasibility in the semantics. The unary operator is best seen in works centred on modelling NMR via an auto-epistemic logic [70, 71] - for which there is an intuitionistic version [72] as well. The binary operator happens in systems based on Lewis' counterfactuals [73], which deal with defeasibility related to possibilities of facts had their circumstances been different.

In [13], Gabbay showed that having an unary or a binary operator leads to equivalent expressiveness. In the following, we will consider only binary operators. There is notable work with intuitionistic conditional logics - utilising this binary operator to represent a non-monotonic entailment - in the literature [14, 15, 16]. Second, because, even though there are works in DLs with either the unary operator [18] or the binary [21, 22], both end up having to work around the fact that classical subsumption is not related directly to a constructforming operator in the language. Since we already have $\sqsubseteq$, it is a natural pathway to define an operator $\sqsubset\!\!\sim$ to form a defeasible TBox (or DTBox, for short).

The following section explains how these logics - called *conditional logics* - work.

4.1 Conditional Logics

The first works on conditional logics were [74] by Adams, [75] by Stalnaker, and [73] by Lewis, in which the authors investigated the kinds of semantics of conditionals. From these works, many others arose, especially with connections to modal logic [76, 77] and NMR [11, 12]. In [78], there is an overview of the initial works in this field.

Conditional logics consist of a conservative extension of (usually) classical logic with a binary operator on formulas $>$ representing this new conditional, which leads to different logics depending on constraints imposed on its semantics. One may read $\varphi > \psi$ as *usually/normally/typically, φ implies ψ*. There are systems (e.g. **V**, **VW**, and **VC** [73]) based on Lewis' concept of *minimal change*, i.e. $\varphi > \psi$ is true in a (Kripke) φ-world (a world in which φ is true), then the worlds which are more similar to it (with minimal changes) are also ψ-worlds. Another is system **C2** by Stalnaker, based on a similar principle: the value of $\varphi > \psi$ in a world depends on the value of ψ in the closest φ-world, i.e. the one which differs minimally from it. The mentioned systems treat these worlds as *states of inquiry*, stating that if a state s is related to s', s' has more factual knowledge than s. Thus, in this section, we will refer to them as states to better reflect this notion.

Since $>$ is a binary operator, it is necessary to represent accessibility between states of inquiry *modulo* a proposition. Thus, we need a ternary relation to represent its semantics under certain constraints. For instance, if we have a state s satisfying $\varphi > \psi$, we want to have a way to represent the ψ-states s' related to s which believe φ as well. The reader may find definitions of ternary relations in systems based on Montague-Scott semantics [79, 80, 81], such as those by Chellas [76] and Segerberg [77], which extend Kripke semantics. Section 5.1 provides more details on how this ternary relation works.

In [82], Friedman and Halpern examine the complexity of satisfiability of different conditional logics based on the constraints they must have. The complexity results themselves are not our focus in analysing this work, but rather the constraints of different kinds of conditional logics, which the authors present in a general manner. The constraints for the accessibility relation are (for any state s in the set of states and any conditional $\varphi > \psi$ satisfied in s):

- **Normality**

  The relation must not be empty for s, i.e. s is related to at least one φ-state. Almost all applications have this constraint.

- **Reflexivity**

  s, being a φ-state, must be related to itself, i.e. it is also a normal/expected/usual φ-state. It is easy to see that this constraint implies normality.

- **Centering**

  Also known as the *smoothness* condition [11], this states that s is a minimal φ-state to which s is related. This property implies reflexivity.

  Depending on the system, the notion of *strict* centering is utilised, in which s should be **the** minimal φ-state to which s is related. Both notions (having *a* or *the* minimal φ-state) are equivalent w.r.t. computational complexity.

- **Uniformity**

  The relation is independent of s, i.e. for any φ-state s' to which s is related, s and s' access the same φ-states.

- **Absoluteness**

  The relation is independent of φ, i.e. there is no need to restrict the relation to φ-states. This property implies Uniformity.

- **Connectedness**

  All states are comparable according to this relation.

Most applications have at least the **Centering** property, usually varying in the presence of the latter three. It is important to note that the presence or absence of **Uniformity** or **Absoluteness** impacts the complexity of satisfiability for conditional logics the most.

One aspect of Friedman and Halpern's work is that the underlying monotonic conditional is not assumed to be material implication - meaning that the properties found relate directly to the new conditional $>$ added to the language, not to any assumptions on the original logic itself.

As stated, some works did not consider extensions of classical logic but were all based on Lewis's work. We will focus here on those with intuitionistic logic as a basis. In [13], Gabbay introduced an intuitionistically-based system, which Fischer Servi further refined in [14] and used as a basis for [15]. Ciardelli and Liu [16] proposed a set of constraints that birthed several intuitionistic versions of systems from Lewis and Stalnaker. Even though the systems of Fischer Servi, and Ciardelli and Liu are not based on the same sets of axioms or semantic constraints for operator $>$, one characteristic all these systems have - and agree that an intuitionistic conditional system must have - is the lack of the **Absoluteness** property, since having it would collapse the underlying reasoning into a classical one. Thus, these systems represent the semantic constraints with a ternary relation contextualised by propositions in a Chellas-Segerberg style.

The semantics of the works of Fischer Servi are the most interesting to be considered in Chapter 5, for she introduces a set of axioms that is more compact and resembles more closely the KLM axioms, which have spawned much more work in (classical) DLs. Her axioms and rules connect us more directly to existing work, especially regarding comparisons of expressiveness and complexity.

4.2 The KLM Framework

The KLM axioms/framework were developed in [11] by Kraus, Lehmann and Magidor and expanded upon in [12] by Lehmann and Magidor. In their work, they present properties for a consequence relation $|\!\sim$ (inspired by operator $>$) in different systems, based on a preferential approach, i.e. there is a preference relation (a pre-order) $\prec$ on worlds/states. $w \prec v$ means that w is more preferred, i.e. more plausible/normal than v. If α and β are formulas, then the pair $\alpha \mathrel{|\!\sim} \beta$ (if α, then normally β, or β is a plausible consequence of α) is called a conditional assertion. A consequence relation is a set of conditional assertions. The authors of the KLM approach believe that leaving this conditional as a meta-logical notion leads to the right language to represent non-monotonicity.

The reader may notice that this accessibility relation is binary. Indeed, in [12], Lehmann and Magidor state that *"[...] we take the view that the truth of a conditional assertion is necessary, i.e. does not depend on the state of the world"*, which, in terms of [82], means that their non-monotonic accessibility relation assumes the property of absoluteness - something we do not assume in either of our systems, as it would make them lose their intuitionistic refinement. We leave proofs of how this behaves for future work. The relation $\prec$ of KLM assumes all of the properties in Friedman and Halpern's article, including Connectedness.

In the KLM approach, the three central systems presented are **C** for cumulative logic, **P** for preferential logic, and **R** for rational logic, written here in order of expressiveness. Each system imposes particular properties on $\mid\!\sim$, and the authors show the adequate models for each. The authors state that system **C** is not expressive enough due to its inability to compare some kinds of conditional assertions. For example, this system cannot compare *birds that are penguins normally do not fly* ($b \wedge p \mid\!\sim \neg f$) and *normally, for a bird, if it is a penguin, then it does not fly* ($b \mid\!\sim p \rightarrow \neg f$). The other systems presented can compare each assertion and indicate which one is stronger or can be applied to one situation and not to another. Thus, we will present here just **P** and **R**. The organisation of the presentation of these systems is based on that of Chafik in [83], which presented clear and straightforward explanations of both.

Before introducing system **P**, it is essential to recall that the main property we do not wish for $\mid\!\sim$ to have is monotonicity, i.e. if, for any α, γ and β in the adequate logical language $\mathcal{L}$, we have $\alpha \mid\!\sim \beta$, then we also have $\alpha \wedge \gamma \mid\!\sim \beta$. If we want $\mid\!\sim$ to be defeasible, we want the possibility that there is a γ that, when present, stops us from being able to conclude β. In a more practical sense, a reasoner would have to be able to retract consequences based on new facts they may come upon - with a certain degree of *caution*, as to minimise belief revision every time a new fact appears. In the bird example from the beginning of this chapter, one may assume the following statements (given $\rightarrow$ as a monotonic implication, which in this case can be material or intuitionistic without loss of generality): *penguins are birds*, i.e. $p \rightarrow b$, *birds normally fly*, i.e. $b \mid\!\sim f$, and *penguins normally do not fly, i.e.* $p \mid\!\sim \neg f$. It is easy to see that we do not wish for the adverb *normally* to rule over every instance of the consequence relation, i.e. behave monotonically, as it would create a contradiction. We, then, need to allow exceptions.

The authors present the properties for $\mid\!\sim$ as axioms and rules in Gentzen style for meta-logical assertions with $\mid\!\sim$ - and, in some rules, with the monotonic notion of logical consequence $\models$. System **P** has the following properties for $\mid\!\sim$ in order to reason (let $\alpha, \beta, \gamma \in \mathcal{L}$):

1. Reflexivity

$$\alpha \mid\!\sim \alpha$$

   α is normally inferred from itself, i.e. if we have α, then it is plausible to have α.

2. Left Logical Equivalence

$$\frac{\models \alpha \leftrightarrow \beta \quad \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$$

If two formulas/sentences are equivalent, then what is typically inferred from one is also typically inferred from the other.

3. Right Weakening
$$\frac{\models \alpha \rightarrow \beta \quad \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}$$
Logical consequences of what is plausibly concluded are also plausible consequences.

4. Cut
$$\frac{\alpha \wedge \beta \mid\!\sim \gamma \quad \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma}$$
If α and β have γ as a plausible consequence and β is usually a consequence of having α, then α is enough to usually conclude γ.

5. Cautious Monotonicity
$$\frac{\alpha \mid\!\sim \beta \quad \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$$
Unlike (general) monotonicity, this rule states that if β is a plausible conclusion from α, it is also consistent as a premise for other plausible conclusions of α. In the bird example, a penguin would probably not be considered a typical bird (α), but maybe a canary would, and if Tweety is a canary (β), then they could still be considered able to fly (γ). Adams [74] considers this to be an essential property for many systems of NMR.

6. Or
$$\frac{\alpha \mid\!\sim \gamma \quad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}$$
This property separates **P** from the less expressive **C**. It states that if γ can be plausibly concluded from both α and β, then it can be plausibly inferred from their disjunction.

From these properties, there are a few others that can be derived:

– Ent
$$\frac{\models \alpha \rightarrow \beta}{\alpha \mid\!\sim \beta}$$
Although the authors did not name this property, naming it will be relevant when we start to present ĩALC, the language we devise in Chapter 5. This name is short for *entailment* since it gives a direct and

isolated relation between the monotonic and non-monotonic entailment relations.

This property is derived from Reflexivity and Right Weakening, and it means that if α implies β, then α usually implies β, i.e. the monotonic entailment is at least as strong as the non-monotonic consequence.

– And

$$\frac{\alpha \mid\sim \beta \quad \alpha \mid\sim \gamma}{\alpha \mid\sim \beta \wedge \gamma}$$

And is derived from Cautious Monotonicity and Cut. It states that the conjunction of plausible consequences of α is itself a plausible consequence of α.

– S

$$\frac{\alpha \wedge \beta \mid\sim \gamma}{\alpha \mid\sim (\beta \rightarrow \gamma)}$$

S is derived from Reflexivity, Right Weakening, Left Logical Equivalence and Or. This property indicates that if from α and β one can plausibly infer γ, then it is plausible to infer from just α that β implies γ. It is important to note that the contrary, i.e.

$$\frac{\alpha \mid\sim (\beta \rightarrow \gamma)}{\alpha \wedge \beta \mid\sim \gamma}$$

would lead to the following statement (if we consider α to be a tautology): if $\boldsymbol{true} \mid\sim (\beta \rightarrow \gamma)$, then $\beta \mid\sim \gamma$. The antecedent indicates that it is always normal to assume that β implies γ, whereas the consequent indicates that *if β, then normally γ*. If one considers β to be normally false, the antecedent is automatically true, whereas the consequent states that in an exceptional case where β is true, then γ is as well. The antecedent may as well not say anything in case β is true. In the case of birds, one may accept that *it is normal to assume that penguins fly* (since penguins are not typical birds, this monotonic implication, namely $\mathbf{true} \mid\sim p \rightarrow f$, is vacuously true in most instances) but be reluctant to accept that *a penguin normally flies* ($p \mid\sim f$), indicated by the consequent. If one considers the monotonic implication to be material implication, i.e. $\beta \rightarrow \gamma \equiv \neg\beta \vee \gamma$, this becomes clearer, as one can see this as *it is normal to assume that birds are either not penguins or that they fly* ($\mathbf{true} \mid\sim \neg p \vee f$). It is, then, important not to represent this direction of reasoning, staying with the direction presented in property S.

A consequence relation $\mid\!\sim$ is called preferential if it can be deduced from a set of conditionals Δ via the rules of system **P**. In order to define its semantics, one needs a preferential model, which consists of a non-empty set of states of inquiry S (where each $s \in S$ is a state in a certain universe of reference) and a strict partial order (asymmetric, irreflexive and transitive) $\prec \subseteq S \times S$. $\prec$ must also satisfy *smoothness*, i.e. S is smooth if $s \in S$ is a minimal φ-state to which s is related. First, we need to define *minimal states*:

Definition 4.1 (*Minimal State*)**.** *Let $S' \subseteq S$ and $s \in S'$. s is said to be minimal in S' if there is no $t \in S'$ such that $t \prec s$.*

We also define the set $min_{\prec}(S')$:

$$min_{\prec}(S') = \{s \in S' \mid s \text{ is minimal in } S'\}$$

Definition 4.2 (*Smooth*)**.** *Let $S' \subseteq S$ and $\prec$ be a strict partial order on S. We say that S' is smooth if, forall $t \in S'$, either there is $s \in S'$ such that $s \in min_{\prec}(S')$ and $s \prec t$, or $t \in min_{\prec}(S')$.*

From this definition, we may now define preferential models.

Definition 4.3 (*Preferential Model*)**.** *Let $\mathcal{L}$ be a set of atomic propositions and $\mathcal{U}$ be a universe of reference. A preferential model $\mathcal{P} = \langle S, l, \prec \rangle$ is a tuple where S is a set of states, $l : S \to \mathcal{U}$ assigns to each state s a world $u \in \mathcal{U}$, and the relation $\prec \subseteq S \times S$ is a strict partial order satisfying the smoothness condition.*

We can then compare states: if $s \prec t$, one considers s more normal/-plausible/preferred than t. We proceed by defining satisfiability in preferential models. We say that a state $s \in S$ **satisfies** $\alpha \in \mathcal{L}$ ($s \models\!\!\!\equiv \alpha$) if its corresponding world $l(s)$ satisfies α, i.e. $l(s) \Vdash \alpha$. A preferential model $\mathcal{P} = \langle S, l, \prec \rangle$ satisfies α ($\mathcal{P} \models\!\!\!\equiv \alpha$) if $s \models\!\!\!\equiv \alpha$ for all $s \in S$.

An important definition as well is that of α-states, which is the set of states that satisfy α: $[\![\alpha]\!]^{\mathcal{P}} = \{s \in S \mid s \models\!\!\!\equiv \alpha\}$. Now, we can compare via $\prec$ the states that satisfy a given sentence α, being ordered from least to most normal. For instance, $min_{\prec}([\![\alpha]\!]^{\mathcal{P}})$ represents the set of most plausible α-states w.r.t. $\prec$. We may now define the smoothness condition of well-formed formulas:

Definition 4.4 (*Smoothness condition*)**.** *A preferential model $\mathcal{P} = \langle S, l, \prec \rangle$ satisfies the smoothness condition if for all $\alpha \in \mathcal{L}$, $[\![\alpha]\!]^{\mathcal{P}}$ is smooth.*

With this condition, one can find the preferred states to any statement α w.r.t. $\prec$ - one can see it as stating that, for any α such that $[\![\alpha]\!]^{\mathcal{P}} \neq \varnothing$, then $min_{\prec}([\![\alpha]\!]^{\mathcal{P}}) \neq \varnothing$ too.

The authors [11] defined $\mid\!\sim_{\mathcal{P}}$, the non-monotonic consequence for preferential models. Formally, this is stated as $\mathcal{P} \models \alpha \mid\!\sim_{\mathcal{P}} \beta$ if it is the case that $min_{\prec}([\![\alpha]\!]^{\mathcal{P}}) \subseteq [\![\beta]\!]^{\mathcal{P}}$, indicating that the most preferred α-states are also β-states.

Thus, Kraus et al. proved the representation theorem for preferential relations in [11]:

Theorem 4.1 (*Representation theorem for preferential relations*)**.** *A consequence relation* $\mid\!\sim$ *is preferential if it is defined by a preferential model* $\mathcal{P}$.

System **P** is a robust and intuitive system for NMR [83]. However, it does not offer a solution to certain kinds of situations. Let us now consider an example of the so-called Nixon-diamond, which is a situation created around some observations: (a) *typically, Quakers are pacifists*, (b) *typically, Republicans are not pacifists*, and (c) *Nixon is both a Quaker and a Republican.* Let us consider the language $\mathcal{L}$ of propositional classical logic and the meta-logical consequence relation $\mid\!\sim$ to be in system **P**. In this system, then, what defeasible conclusions can we reach? Is it reasonable to assume that Nixon is a pacifist or not?

Let us consider the following propositional formulas: q is *Nixon is a Quaker*, r is *Nixon is a Republican*, and p is *Nixon is a pacifist.*

From observation (a), we have $q \mid\!\sim p$. From (b), we have $r \mid\!\sim \neg p$. Since Nixon is both, we wish to have a conditional with $q \wedge r$ as an antecedent; however, it is not present. This example highlights one limitation of system **P**, which is that of *non-pertinence*, i.e. it cannot reason adequately based on that which is not present in the knowledge base. In this case, as we have cautious monotonicity, we can only conclude $q \wedge r \mid\!\sim \gamma$, for some γ, if we have either $q \mid\!\sim r$ or $r \mid\!\sim q$ explicitly in the base, as to have a warrant that one can be plausibly inferred from the other. In [12], Lehmann and Magidor provide a system called **R** (for the rational closure of **P**) to deal with this type of situation by extending **P** with an additional rule and changing the semantics of $\prec$ accordingly.

System **R** has the axioms and rules of system **P** plus **rational monotonicity**:

$$\frac{\alpha \mid\!\sim \gamma \qquad \alpha \not\mid\!\sim \neg\beta}{\alpha \wedge \beta \mid\!\sim \gamma}$$

This rule states that if γ is a plausible conclusion from α and it is not the case that $\neg\beta$ is plausible from α, one assumes that, then, it is the case that β does not usually contradict α, and can then conclude that β can be added to α and still have γ as a conclusion.

Definition 4.5 (*Rational consequence*)**.** *Let $\alpha, \beta \in \mathcal{L}$ and Δ be a set of conditionals. $\alpha \mid\!\sim \beta$ is a rational consequence relation if it can be deduced from Δ using rules of system **R**.*

For the Nixon diamond example, we now have a conditional base $\Delta = \{q \mid\!\sim p, r \mid\!\sim \neg p\}$, to which we apply rules of **R** in order to compute the closure Δ^C. We start with $\Delta^C = \Delta$. Since $q \mid\!\sim p \in \Delta^C$ and $q \mid\!\sim \neg r \notin \Delta^C$, we have $r \wedge q \mid\!\sim p$, which is added to Δ^C. However, we can extend Δ^C another way: from $r \mid\!\sim \neg p$ and $r \mid\!\sim \neg q \notin \Delta^C$, we can add $r \wedge q \mid\!\sim \neg p$ to Δ^C, which renders it inconsistent, for we already have $r \wedge q \mid\!\sim p$. As we can see, this closure cannot have all formulas deduced, so it is not unique. We can have, from this situation, the following consistent closures:

$$\Delta_1^C = \{q \mid\!\sim p, r \mid\!\sim \neg p, r \wedge q \mid\!\sim p\}$$
$$\Delta_2^C = \{q \mid\!\sim p, r \mid\!\sim \neg p, r \wedge q \mid\!\sim \neg p\}$$
$$\Delta_3^C = \{q \mid\!\sim p, r \mid\!\sim \neg p\}$$

Lehmann and Magidor proposed a method for reaching a unique *rational* closure, denoted by Δ^{RC}, which can be seen as a minimal closure that completes the set. This procedure is done by assigning a rank to each sentence to decide which ones are the most or least exceptional to Δ. Then, these ranks are minimised, i.e. each of them is interpreted as plausibly as possible, according to Δ. We assert $\alpha \mid\!\sim \beta \in \Delta^{RC}$ if, having α, one would rather conclude β than $\neg\beta$, i.e. the ranking of $\alpha \wedge \beta$ is less than that of $\alpha \wedge \neg\beta$. First, they define what it means to call a formula or sentence exceptional:

Definition 4.6 (*Exceptionality of sentences*)**.** *Let $\alpha, \beta \in \mathcal{L}$ and Δ be a set of conditionals. Formula α is said to be exceptional for Δ if **true** $\mid\!\sim \neg\alpha$ can be inferred from Δ. Assertion $\alpha \mid\!\sim \beta$ is said exceptional for Δ if α is exceptional for Δ. The set of exceptional conditionals is denoted $E(\Delta)$.*

With a finite starting base Δ, we can always define a non-increasing sequence of subsets of Δ such that $\Delta_0 = \Delta$ and $\Delta_i = E(\Delta_{i-1})$, for all $i > 0$. This way, a set in this sequence contains the exceptional conditionals of the previous subset until we reach a fixpoint (which can be empty) since Δ is finite. We can consider this fixpoint the end of the sequence, as the rest of it is no longer adding any new information. These sets are used to define the ranks of sentences.

Definition 4.7 (*Rank of sentences*)**.** *Let $\alpha \in \mathcal{L}$ and Δ be a set of conditionals. We say that the rank of α is i, i.e. $rank(\alpha) = i$, if $i \in \mathbb{N}$ is the smallest number for which α is not exceptional for Δ_i. If α is exceptional for all Δ_i, we say it has no rank via $rank(\alpha) = \infty$.*

Thus, $rank(\alpha \mid\!\sim \beta) = rank(\alpha)$, and $\Delta_i \backslash \Delta_{i-1}$ is the set of conditionals of rank i or higher. If $rank(\alpha) = \infty$, then $\alpha \mid\!\sim \beta$ is also said to have no rank.

In our Nixon diamond example, our initial conditional base is $\Delta = \{q \mid\!\sim p, r \mid\!\sim \neg p\}$. We construct the sequence with $\Delta_0 = \Delta$. We have $rank(q) = 0$, since $\boldsymbol{true} \mid\!\sim \neg q$ is not true in Δ_0, as well as $rank(r) = 0$, for the same reason. This leads us to $rank(q \mid\!\sim p) = 0$ and $rank(r \mid\!\sim \neg p) = 0$. Via rational monotonicity, however, from having $r \mid\!\sim \neg p$ and not having $r \mid\!\sim \neg q$ we have $r \wedge q \mid\!\sim \neg p$, but we also have $r \wedge q \mid\!\sim p$ from having $q \mid\!\sim p$ and not having $q \mid\!\sim \neg r$, which generates a conflict in Δ_0. So, $\boldsymbol{true} \mid\!\sim \neg(r \wedge q)$ in Δ_0, and $r \wedge q$ is exceptional in Δ_0. Since $\Delta_1 = E(\Delta_0) = \varnothing$, and we can stop this process.

If, instead, we consider that being a Quaker trumps being a Republican regarding peace, then the starting base would be $\Delta_0 = \{q \mid\!\sim p, r \mid\!\sim \neg p, r \wedge q \mid\!\sim p\}$, we have the same reasoning, except that, now, $\Delta_1 = E(\Delta_0) = \{r \wedge q \mid\!\sim p\}$, since this conditional was exceptional in Δ_0. In Δ_1, we cannot infer $\boldsymbol{true} \mid\!\sim \neg(r \wedge q)$, thus $rank(r \wedge q \mid\!\sim p) = rank(r \wedge q) = 1$. $\Delta_2 = E(\Delta_1) = \varnothing$, and the process stops.

As for the third possible closure, i.e. $\Delta_0 = \{q \mid\!\sim p, r \mid\!\sim \neg p, r \wedge q \mid\!\sim \neg p\}$, the reasoning is analogous to the previous one, with the difference that $\Delta_1 = \{r \wedge q \mid\!\sim \neg p\}$. We then have $rank(r \wedge q \mid\!\sim \neg p) = rank(r \wedge q) = 1$. $\Delta_2 = \varnothing$ and the process stops.

We need to define the rational closure to show that all three situations will reach the same set.

Definition 4.8 (*Rational closure*)**.** *Let $\alpha \in \mathcal{L}$ and Δ be a set of conditionals. The rational closure of Δ is:*

$$\Delta^{RC} = \{\alpha \mid\!\sim \beta \mid rank(\alpha) < rank(\alpha \wedge \neg\beta) \textit{ or } rank(\alpha) = \infty\}$$

Lehmann and Magidor showed in [12] that this closure is unique and minimal w.r.t. all possible rational extensions of Δ.

Using this definition, in all three situations, one arrives at $\Delta^{RC} = \{q \mid\!\sim p, q \mid\!\sim \neg p\}$. It is important to note that we have $\boldsymbol{true} \mid\!\sim \neg(r \wedge q)$, which means that we consider those who are both Quakers and Republicans exceptions, as well as $r \mid\!\sim \neg q$ and $q \mid\!\sim \neg r$. Unless we add some explicit preference between being a Quaker and being a Republican to our base, we cannot reach a conclusion. This situation is, however, expected: it is not *rational* to conclude anything in this case - but it is worth noting that we now were able to reason why, differently from system **P**, where this was not possible. In Section 4.5, we see how, in a legal context, most situations will have innate precedence, which makes the knowledge bases adapt well to rational entailment.

As for the semantics, we have ranked models, which extend preferential models.

Definition 4.9 (*Modular ordering*)**.** *Let $\prec$ be a strict partial order on S, and $s, s' \in S$. $\prec$ is said to be modular if there is a ranking function $r : S \to \mathbb{N}$ such that if $r(s) < r(s')$, then $s \prec s'$.*

A modular relation is defined via this ranking function that ranks states according to their preference. Thus, given s and s', it is always the case that either $s \prec s'$, $s' \prec s$ or they are equally preferable and $r(s) = r(s')$.

With this, we can define a *ranked model* $\mathcal{R} = \langle S, l, \prec \rangle$ as a preferential model for which $\prec$ is modular.

These models create tiers/layers of states, starting with those most preferable (and for which no state precedes them). Each layer is analogous to the previous definitions of a sequence of subsets of a base Δ that create the ranking of the conditionals therein.

The satisfiability of rational relations is similar to that of preferential relations. The authors defined $\mid\!\sim_{\mathcal{R}}$, the non-monotonic consequence relation for rational models. Formally, this is stated as $\mathcal{R} \models \alpha \mid\!\sim_{\mathcal{R}} \beta$ if it is the case that $min_{\prec}(\llbracket \alpha \rrbracket^{\mathcal{R}}) \subseteq \llbracket \beta \rrbracket^{\mathcal{R}}$, indicating that the most preferred α-states are also β-states. Thus, they provide a representation theorem:

Theorem 4.2 (*Representation theorem for rational relations*)**.** *A consequence relation $\mid\!\sim$ is rational if it is defined by a ranked model $\mathcal{R}$.*

The semantics for a rational closure of a conditional base was also defined. Let $\mathcal{P} = \langle S, l, \prec \rangle$ be a preferential model of a base Δ. We can build a ranked model $\mathcal{P}_{RC} = \langle S_{RC}, l_{RC}, \prec_{RC} \rangle$ such that $S_{RC} = S$ and $l_{RC} = l$. $\prec_{RC}$ is a total order that completes $\prec$. Each state $s \in S_{RC}$ has a rank $r(s)$, indicating the longest ascending chain having s as the minimal element. Lehmann and Magidor showed that any conditional $\alpha \mid\!\sim \beta \in \Delta^{RC}$ satisfies it.

Theorem 4.3 (*Rational closure of preferential models*)**.** *Let $\mathcal{P} = \langle S, l, \prec \rangle$ be a preferential model of a base Δ. Let Δ^{RC} be its rational closure and $\mathcal{P}_{RC} = \langle S_{RC}, l_{RC}, \prec_{RC} \rangle$ the ranked model extending $\mathcal{P}$. We have that, if $\mathcal{P}_{RC} \models \alpha \mid\!\sim \beta$, then $\alpha \mid\!\sim \beta \in \Delta^{RC}$.*

With this in hand, we may introduce how the KLM Framework is used in conditional DLs.

4.3 Conditional Logics in DLs

In DLs, the KLM approach is present in the so-called *defeasible* DLs, which extend DLs with this notion of plausibility or typicality present in the works in [11, 12]. There are two main approaches to this extension: either via adding an unary operator on concepts, which shows in subsumptions in the TBox, or a Defeasible TBox (DTBox for short), containing defeasible subsumptions. The former is done in $\mathcal{ALC}+\mathbf{T}$, by Giordano et al. [18] and $\mathcal{ALC}\bullet$, by Fernandes et al. [84]. In this approach, the unary operator on concepts (either $\mathbf{T}$ or $\bullet$) represents more typical or less exceptional concepts. In contrast, the latter is developed by Britz et al. [85, 86, 87, 88, 89, 21, 90, 22, 91, 92, 93, 94] and focused in introducing defeasible subsumption as well as comparing it to the preferential consequence $\mid\!\sim$ of the KLM approach and relating to modal alternatives. We focus on works based on $\mathcal{ALC}$ to better connect to what we intend to do when extending iALC.

Gabbay, in [13], showed that both approaches yield the same expressiveness, so here we will show only those on which we based our work for ĩALC: the works by Britz et al. [89, 21, 22, 87, 94]. The authors define defeasible subsumptions for $\mathcal{ALC}$.

Defeasible subsumption is a binary relation $\sqsubsetsim$ on concepts. A statement $C \sqsubsetsim D$ reads as *normally, individuals of C are in D*. A KB $\mathcal{K}$ has three components: a TBox $\mathcal{T}$, an ABox $\mathcal{A}$, and a DTBox $\mathcal{D}$, which consists of defeasible subsumptions. It is, then, a triple $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{A} \rangle$.

Let $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \prec^{\mathcal{I}} \rangle$ be a preferential DL interpretation, where $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ is a regular DL interpretation as given in Section 2.3, and $\prec^{\mathcal{I}}$ is a strict partial order. The satisfiability relation defined in Section 2.3 is extended to deal with defeasible subsumptions. A defeasible subsumption $C \sqsubsetsim D$ ($\mathcal{I} \Vdash C \sqsubsetsim D$) is satisfied if $min_{\prec^{\mathcal{I}}}(C^{\mathcal{I}}) \subseteq D^{\mathcal{I}}$. This definition means that $C \sqsubsetsim D$ is true if all of the minimal individuals of C are also in D w.r.t. $\prec^{\mathcal{I}}$. The definition of a minimal individual is similar to that of Kraus et al. (for states), which leads to the definition of the set of minimal individuals of a concept:

$$min_{\prec}(C^{\mathcal{I}}) = \{x \in C^{\mathcal{I}} \mid \text{there is no } y \in C^{\mathcal{I}} \text{ such that } y \prec^{\mathcal{I}} x\}$$

The semantics for the operators of $\mathcal{ALC}$ are the same as given in Section 2.3.

A defeasible subsumption is *preferential* if it satisfies the following axioms (which are similar to the meta-logical axioms of system $\mathbf{P}$):

1. Reflexivity

$$\alpha \underset{\sim}{\sqsubset} \alpha$$

2. Left Logical Equivalence

$$\frac{\alpha \equiv \beta \quad \alpha \underset{\sim}{\sqsubset} \gamma}{\beta \underset{\sim}{\sqsubset} \gamma}$$

3. Right Weakening

$$\frac{\alpha \sqsubseteq \beta \quad \gamma \underset{\sim}{\sqsubset} \alpha}{\gamma \underset{\sim}{\sqsubset} \beta}$$

4. Cut

$$\frac{\alpha \wedge \beta \underset{\sim}{\sqsubset} \gamma \quad \alpha \underset{\sim}{\sqsubset} \beta}{\alpha \underset{\sim}{\sqsubset} \gamma}$$

5. Cautious Monotonicity

$$\frac{\alpha \underset{\sim}{\sqsubset} \beta \quad \alpha \underset{\sim}{\sqsubset} \gamma}{\alpha \wedge \beta \underset{\sim}{\sqsubset} \gamma}$$

6. Or

$$\frac{\alpha \underset{\sim}{\sqsubset} \gamma \quad \beta \underset{\sim}{\sqsubset} \gamma}{\alpha \vee \beta \underset{\sim}{\sqsubset} \gamma}$$

It also must satisfy the following axiom of Consistency:

$$\neg(\top \underset{\sim}{\sqsubset} \bot)$$

To show that defeasible subsumption captures the notion of defeasibility when reasoning about DL KBs, Britz et al. provided the following definition:

Definition 4.10 ($\mathcal{I}$-*induced defeasible subsumption*)**.** *Let* $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \prec^{\mathcal{I}} \rangle$ *be a preferential interpretation. Then* $\underset{\sim}{\sqsubset}_{\mathcal{I}} = \{(C^{\mathcal{I}}, D^{\mathcal{I}}) \mid \mathcal{I} \Vdash C \underset{\sim}{\sqsubset} D\}$ *is the defeasible subsumption induced by* $\mathcal{I}$.

Similarly to system **P**, there is a correspondence between preferential subsumptions and defeasible subsumptions induced by preferential interpretations, in which $\prec$ is a strict partial order that satisfies smoothness.

Theorem 4.4 (*Representation theorem for preferential subsumption*)**.** *Let* C *and* D *be any concepts. A defeasible subsumption* $\underset{\sim}{\sqsubset}$ *is preferential if there is a preferential interpretation* $\mathcal{I}$ *such that* $C^{\mathcal{I}} \underset{\sim}{\sqsubset}_{\mathcal{I}} D^{\mathcal{I}}$ *if and only if* $\mathcal{I} \Vdash C \underset{\sim}{\sqsubset} D$.

Britz et al. showed one crucial result for DLs in [94].

Lemma 4.1. *For every preferential interpretation* $\mathcal{P}$*, and any concepts* C *and* D*,* $\mathcal{P} \Vdash C \sqsubseteq D$ *if and only if* $\mathcal{P} \Vdash C \sqcap \neg D \underset{\sim}{\sqsubset} \bot$.

This lemma is an essential relation between the subsumptions in the TBox and the defeasible subsumptions in the DTBox.

Defeasible subsumption can also be similarly interpreted by rational interpretations to system **R**, i.e. by adding the rational monotonicity rule to the others:

$$\frac{\alpha \sqsubsetsim \gamma \qquad \neg(\alpha \sqsubsetsim \neg\beta)}{\alpha \wedge \beta \sqsubsetsim \gamma}$$

Then, one can define a ranked interpretation analogously to system **R** [11, 12]. A ranked interpretation is a preferential interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \prec^{\mathcal{I}} \rangle$ is a preferential interpretation where $\prec^{\mathcal{I}}$ is modular, as per Definition 4.9 - except with individuals of the domain instead of states. The induced subsumption behaves in the same manner as the preferential one:

Definition 4.11 ($\mathcal{I}$*-induced defeasible subsumption*)**.** *Let* $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \prec^{\mathcal{I}} \rangle$ *be a ranked interpretation. Then* $\sqsubsetsim_{\mathcal{I}} = \{(C^{\mathcal{I}}, D^{\mathcal{I}}) \mid \mathcal{I} \Vdash C \sqsubsetsim D\}$ *is the defeasible subsumption induced by* $\mathcal{I}$*.*

Finally, Britz et al. showed that rational subsumption can be represented by ranked DL interpretations.

Theorem 4.5 (*Representation theorem for rational subsumption*)**.** *Let* C *and* D *be any concepts. A defeasible subsumption* $\sqsubsetsim$ *is rational if there is a ranked interpretation* $\mathcal{I}$ *such that* $C^{\mathcal{I}} \sqsubsetsim_{\mathcal{I}} D^{\mathcal{I}}$ *if and only if* $\mathcal{I} \Vdash C \sqsubsetsim D$*.*

In [94], rational closure is presented in further detail.

We now present an overview of works on NMR to situate conditional logics, the KLM approach and the works by Britz et al. in the literature.

4.4
Related Works

This section is divided into two main parts: general works concerned with modelling NMR and works discussing NMR in a legal setting.

4.4.1
Modelling NMR

As stated in Section 1.1.2, NMR is not confined to the world of law; it is a broad field with different formalisations and kinds of applications.

One of the earlier works in formal NMR are logics for counterfactuals, proposed by Lewis in [73]. Counterfactuals are subjunctive conditionals in which one states facts that could be if circumstances were different, e.g. "*Had I studied more, I would have passed the test.*" He proposes a variably strict conditional to deal with these types of sentences by having some form of *proximity* of consequent, a kind of measure of verisimilitude. This formalism is the primary basis of conditional logics. We discussed it in Section 4.1.

Another formalism is circumscription, proposed by MacCarthy in [95], which aims to formalise the common sense assumption that things are as expected unless some exception occurs. In his work, McCarthy considered a problem of missionaries and cannibals: three missionaries and three cannibals on one river bank; they must cross the river using a boat that can only take two. However, cannibals must never outnumber the missionaries on either bank. The problem is not to find a sequence of steps to reach the goal but rather to exclude conditions that are not explicitly stated. For instance, the solution *walk a mile north and cross the river on the bridge* is intuitively not valid because the statement of the situation does not mention a bridge. On the other hand, we do not exclude the existence of this bridge, either. The nonexistence of the bridge is a consequence of the implicit assumption that the statement of the problem contains everything relevant to its solution. Explicitly stating that a bridge does not exist is not a solution to this problem, as there are many other exceptional conditions that we should exclude.

Circumscription is concerned with *minimal* solutions or models. When considering its applicability on a DL, the main drawback of this approach is the burden on the ontology engineer to make appropriate decisions related to the (circumscriptive) fixing and varying concepts. There is also the question of establishing the priority of defeasible subsumption statements since it is essential to find which ones are the most relevant to each situation and must be included in the minimal model. Such choices can significantly affect the conclusions drawn from the system and can easily lead to counter-intuitive inferences. Circumscription, as proposed by MacCarthy, also has a problem when dealing with disjunction and certain kinds of rationality in reasoning [11, 12] of not being able to utilise absent information to reach certain conclusions. For example, the consequence relation defined by circumscription does not always satisfy the rule of *negation rationality*, which is satisfied by rational relations. In [96], a method called *curbing* is presented with a **PSPACE**-complexity algorithm to find minimal models for first-order finitely axiomatisable theories. In [97], the authors present results on the complexity of Circumscription in DLs by minimising predicates via circumscription patterns.

There is also a well-known formalism in NMR called default logic [98], by Reiter. In default logic, specific rules called *defaults* indicate that a particular conclusion can be made from a premise, given some non-contradictory justifications. For instance, given a bird (premise), if one assumes that it is a regular bird (justification), it is reasonable to conclude (by default) that it flies. Let a default theory consist of W, a set of formulas and D, a set of default rules. Then, an extension E of (D, W) is the smallest set of formulas containing W, closed under classical consequence and applicable rules in D. Depending on (D, W), it is possible to arrive at an infinite number of extensions.

Reiter's work is well-established in the world of NMR and is the basis for several others. In [70], Marek and Truszczyński introduce the concept of iterative expansion of default theories, relating them to default extensions that connect auto-epistemic logic to default logic. This concept is interesting as it is independent of the syntax utilised for the base logic. There are also several different Sequent Calculi (SC) for different kinds of default logic: [71] for default logic with *credulous* reasoning, [99] for *sceptical* reasoning, and [100] for a default logic with an intuitionistic notion of entailment - differing from other accounts of default logic, which consider classical implication. Credulous reasoning occurs when one agent accepts a set of beliefs from a given theory, even if their certainty is not proved. Sceptical reasoning is the opposite: the agent does not accept new information as reasonable unless given some degree of credibility. In [101], the authors introduce DML, a modal logic for subjective default reasoning obtained via adding a binary modal operator. DML generalises the syntax and semantics of other default logics via filter-based models. The proof systems presented for DML are sound and complete regarding semantic constraints analogous to those required by rules LLE, RW and And of the KLM Framework [11, 12].

In [102], Pollock introduces a framework for defeasible reasoning to represent ways defeasibility appears in discourse - in a way, generalising circumscription. Defeasible reasoning can be considered convincing, compelling, or reasonable but not deductively valid. For instance, *there are some dark clouds in the sky today, so I will take my umbrella when I go out* expresses a reasonable argument based on previous experiences with rain based on how the sky looks. This reasoning, however, does not guarantee that it will rain later and that there will be no need to take an umbrella when leaving the house. Defeasible reasoning was also explored in [103], in which the authors explain priorities and NMR as an extension of logic programming and present their proofs in a dialectical style.

Considering the question of credulous *versus* sceptical semantics, in [104], the authors show that it is possible to have a mixture of credulity and scepticism when providing semantics for ordered logic programs. From this arises the question: for the domain of law, *how* credulous/sceptical should one be? According to Prakken in [105], credulous reasoning makes little sense when compared to legal reasoning, depending on the process.

Another formalism is abductive inference [106], which aims to define the *best* conclusion to different problems. It requires, however, a very rich and thorough meta-theory in order to define what would be the *best* conclusion to derive from each situation.

Defeasible Logic [107] is an attempt at representing defeasible reasoning to unify the research done at the time. In defeasible logic, propositions are divided into strict rules, which represent facts that are direct consequences of others; defeasible rules, for facts that typically are consequences of others; and undercutting defeaters, which specify exceptions to rules. This approach is similar to that of Britz et al. to separate the TBox and the DTBox [89, 21, 22, 87, 94] but not in a DL setting.

In [108], the authors present IDL, an intuitionistic interpretation of propositional default Logic that characterises the modal logic S4F in terms of minimal models by encoding some variations of default logic in a non-monotonic version of S4F. Logic S4F is modal logic S4 plus axiom F, i.e. $\varphi \wedge \Diamond\Box\psi \rightarrow \Box(\Diamond\varphi \vee \psi)$. They also conclude that models for intuitionistic default logic are a particular case of models of this non-monotonic version of S4F. In [109], Donini provides complexity results for a uniform Tableaux system for several non-monotonic modal logics, including S4F, for which the complexity of the Tableau is Π_2^p-complete.

In [110], the authors provide a tableaux calculus for propositional intuitionistic default logic with operational semantics, which is sound and complete and terminates. They also implemented it and compared it to other provers for formulas of IPL. Due to the lack of robust benchmarks for non-monotonic provers based on intuitionistic logic, they could only provide tests for a couple of case examples, whose results were promising nonetheless.

There are also ways to model NMR that do not involve (necessarily) logic, one of such being Argumentation Theory (AT). This distinction, however, does not prohibit adding logic to models of AT. This collaboration of approaches has, in general, richer results regarding representation [111]. One of the better-known formalisations of AT consists of argumentation frameworks, first idealised by Dung in [31]. An argumentation framework is a pair $AF = \langle A, R \rangle$, consisting in a set A of arguments (we can see each argument as a black box, but it is not uncommon for them to have an underlying logical language) and a set R of relations between these arguments, consisting basically of attacks. An argument a attacks b when it is a rebuttal (contradicts the conclusion) or an undercut (contradicts one of the premises) of the other. There are more subdivisions, depending on what kind of interaction we wish to consider. An argument can be considered acceptable when it is either (a) not attacked by any other argument or (b) any argument that attacks it is attacked by another accepted argument - in this case, the argument is said to be defended. In this article, the author shows that argumentation can be viewed as a particular form of logic programming with negation as failure, introducing a general method for generating meta-interpreters for argumentation systems and applying this approach to default logic. Based on Dung's Argumentation Frameworks, several other works arose [112, 113, 111, 114, 115, 116, 117, 118, 119, 120].

One attractive characteristic in common in all of the works involving Argumentation Frameworks is that they all agree that there are two components to argumentation: argument construction (a *monotonic* process) and argument evaluation (a *non-monotonic* process). Argument construction can be seen as the gathering of knowledge in order to build an argument to be presented. In contrast, argument evaluation involves comparing arguments and, in the language of AT, having them *attack* and *defend* from one another. AT, however, is more concerned with the interaction between arguments than their representation, which strays from our focus with this work.

For the case of description logics specifically, there are several works in NMR, many in the area of *Belief Revision* [121] (and, by consequence, to *Ontology Repair* as well [122, 123]). Belief Revision is concerned with changing beliefs to consistently add new information, a non-monotonic process closely related to changing ontologies for description logics. Works with NMR in description logics are not restricted to these areas, though [124, 18, 125, 19, 22, 126].

In [124], the authors provide operational-semantics tableaux systems and other proof methods for all KLM Logics (DLs based upon the KLM axioms, namely **R**, **P**, **CL** and **C**) and give complexity results. Their work spans both rational and preferential models. They prove that validity for **R** and **CL** is **coNP**-complete (Lehmann and Magidor had already proven validity for **P** to be **coNP**-complete as well in [12]) and, for **C**, is hyper-exponential.

In [18], the authors extend $\mathcal{ALC}$ with a typicality operator on concepts, $\mathbf{T}$, in order to reason on exceptions to inheritance. A concept $\mathbf{T}C$ contains the most typical elements of the interpretation of concept C, i.e. the least exceptional elements regarding the TBox's subsumptions. The global satisfiability for the $\mathcal{ALC}+\mathbf{T}$ KB is **EXPTIME**.

In [22], the authors provide theoretical foundations for extending DLs with non-monotonic features via preferential and rational subsumptions. They utilise intuitive semantics and link their results to KLM axioms, analysing each type of entailment. They conclude that rational entailment is **EXPTIME**-complete and that the complexity of global satisfiability for a defeasible DL is no worse than the complexity for $\mathcal{ALC}$, **PSPACE**-complete [56, 127].

In [84], the authors present an extension of $\mathcal{ALC}$ with an unary typicality operator on concepts, namely $\mathcal{ALC}\bullet$, a terminating connection calculus. This calculus is sound and complete concerning a DL version of preferential semantics [11].

Letia and Groza [128] apply a fuzzy extension of the description logic $\mathcal{SHIF}(\mathcal{D})$ to deal with the modelling of imprecise arguments and to bridge the gap between man-made arguments and those made by software agents. They use Fuzzy logic to create measures of the precision of imprecise terms used in natural language, i.e. *very*, *more-or-less*, *slightly.* A limitation of this approach is that the definition of imprecision varies from term to term and on the situation in which they are used, leading to ontologies that lack the generality of application.

In [20], the authors provide a contextual rational closure for a defeasible extension of $\mathcal{ALC}$ with role-given contextualisation through adding different contextualised non-monotonic operators - one for universal restriction, another for existential restriction and a third one for subsumption. Via the closure method, they can generalise the non-monotonic operators and relate them to the KLM axioms. This process provides a more refined way of viewing the relations between DL roles and precedence relations derived from the non-monotonic operators.

The following section provides an overview of works involving NMR in legal reasoning.

4.4.2 Non-monotonic Legal Reasoning

Before mentioning the related works in this section, it is essential to categorise different legal systems. Two of the central legal systems in the world are *civil law*, used in most of mainland Europe and other countries such as Brazil, and *common law*, used in the UK and the US, for example. Other kinds are *customary law* and *Muslim law*, for instance. In common law systems, the primary sources of the law are statutes created by judges in cases known as precedents, and in civil law, the primary sources are codes created by the legislature.

Since we wish to model legal reasoning, it is vital to state the relevance of NMR in it. There is much debate about whether legislating is a non-monotonic process [129]. However, it is widely accepted that adjudication is naturally non-monotonic, as well as the representation of legal rules in the way they deal with exceptions to their premises [130, 131, 132, 129].

Even though most researchers in law do not attempt to formalise reasoning according to [33], there have been several works relating legal reasoning to NMR, which we will present in this section. Legal systems are essential to society, and ensuring that they do not allow for contradictions or unfaithful interpretations directly affects people's lives. Most argue that this type of reasoning better reflects the intricate relationships between legal rules and principles and the cases to which they are applied *in lieu* of monotonic reasoning with deductive rules based solely on material implication - or variations thereof.

In [131], Sartor advocates in favour of ordering in legal norms to solve conflicts (called *antinomies*) and proposes a model for reasoning with ordered defaults over deontic logic. He states that it is necessary in a legal context to reason from premises that seem incompatible at first due to some reasons: defeasibility of legal norms, dynamics of normative systems (which change over time and precedence is given to most recent norms), concurrence of multiple legal sources, and semantic indeterminacy of legal language. Although some may say there is always a *correct* way to interpret laws, this notion of correctness changes from person to person and case to case.

Regarding the defeasibility of legal norms, Sartor gives the following example: killing an assaulter is a homicide, but it also represents a defence of the life of the assaulted person. In cases like this, there must be a way to distinguish between the different and possibly incompatible *prima facie* evaluations of this situation. Thus, he refers to works of Ross in [133, 134], where he defines *prima facie* (or conditional) duty as a way of referring to an act that would be a proper duty (e.g. keeping a promise) if not for its morally ambiguous nature. Ross further states in his works:

> *Moral intuitions are not principles by the immediate application of which our duty in particular circumstances can be deduced. They state [. . .]* ***prima facie*** *obligations. We are not obliged to do what is only* ***prima facie*** *obligatory.*

Sartor also claims that there are different categories of legal consequences: logical, grounded, plausible and justified. So-called *logical* consequences are derived from an inconsistent base utilising *ex falso quodlibet*. Grounded consequences are those derived from a consistent base by unidirectional inference rules. Plausible consequences are those whose arguments in favour are not *worse* than any argument to the contrary. Finally, justified consequences are those whose every argument in favour is *better* than every argument to the contrary. These notions of being better or worse are related to attacks on arguments. The distinction between plausible and justified consequences is essential and depends very closely on how the precedence between norms is defined. More on this precedence is present in Section 4.5, where we explain how we intend to represent most orderings in iALC, and ĩALC in Chapter 5.

Ross states that there are two qualifications for normative systems, namely consistency and determinacy, and claims that consistency is not a reasonable objective in representing legal knowledge, per his definition of consistency. He defines it as the logical consistency of $\Sigma \cup C$, where Σ is the legal system as a set of norms and C is any possible case. Ross calls a system consistent if and only if there are only plausible or justified consequences from any set $\Sigma \cup C$. He then defines that $\Sigma \cup C$ is determinate if and only if there are only justified consequences for it - and this should be the goal of representing a normative system, an ideal to pursue. Near the end, there is an argument in favour of *paracompleteness* to separate obligations from permissions in deontic logic - indicating that intuitionistic logic is a good fit for the legal context since it does not have the principle of the excluded middle, i.e. it has paracomplete calculi [50, 51].

In [135], Lawsky presents many decisions one needs to take into account when applying NMR to legal reasoning by modelling rule-based legal reasoning in the context of *common law* systems utilising default logic to solve problems in statutory law in the processes of law interpretation and adjudication. Lawsky states that legal reasoning is defeasible, especially when one considers a case being evaluated by a judge, indicating that non-monotonic logics are a much better tool to deal with defeasible reasoning than monotonic ones, even if not *necessary* to model defeasible reasoning - having a monotonic logic to model defeasible reasoning would imply frequent changes to the knowledge bases, however. Non-monotonic logics do so in a manner more faithful to how legal text is represented (all legal text relevant to each case should be considered to the case in question) and how evaluation of legal rules and principles is done. She argues there is less *overhead* when dealing with exceptions.

In [132], Hage describes a model of legal reasoning based on rules, principles and goals and presents a logic suited to it. Principles and goals are defeasible concepts that express a legal system's fundamental ideas, whereas rules sometimes summarise the behaviour of principles and goals. We can use both in legal reasoning, but they have different roles.

For instance, let us assume that there is a legal rule stating that the rent contract be continued with the new owner of a house whose ownership is transferred. The legal goal of this rule is to protect the tenants' rights. Then, we assume that a family rents a house, which is sold. From the goal and the principle that contracts are only in force between contracting parties, we have a reason for the rule's applicability to the case. If a house is sold, the rule may be applied by analogy (since a sale is a case of transfer of ownership), so there must be a way to keep the goal while also adhering to the principle. Hence, the continuation of the contract.

Hage also states that legal reasoning is much more complex than it might seem (as is, for instance, modelled by Walker in [33]), and a logic that considers the complexity of legal reasoning must be non-monotonic to reflect that. One of the reasons is that rules are not necessarily statements. Statements have a *word to world* fit in the sense that they state what we must follow. In contrast, rules and principles have a *world to word* fit in that they represent a specific characteristic that happens according to a particular situation (or a group of situations) but allow for exceptions or weighing of similarity of different situations in their applicability.

In [129], Bayón criticises the state of debate of defeasibility in legal reasoning in the literature, arguing that the arguments in favour of it being defeasible need to be more formal. He then presents three arguments in favour of it regarding procedural defeasibility, incomplete knowledge and handling of exceptions in legal knowledge representation, showing how they are not convincing enough and providing ways to increment them.

One interesting point Bayón makes is that legal norms can be seen conditionally, i.e. they have some conditions that must be met to reach a legal consequence. These conditions involve the presence of *positive conditions* and the absence of *exceptions*. Suppose one considers the absence of exceptions to be part of the antecedent. In that case, to justifiably apply a norm to a case, one must prove that all positive conditions apply and that no exception is present - as the reader may notice, the absence of an exception is not the same as proving it does not apply. There is debate on whether representing this kind of reasoning is non-monotonic. However, this situation must have at least a *paracomplete* representation to differentiate both. This argument favours representing law with intuitionistic logic, as shown in Section 2.2 since it has a paracomplete calculus.

Bayón finalises by disagreeing that every form of legal reasoning is non-monotonic. However, he agrees that adjudication (i.e. the judicial process) is, indeed, non-monotonic. So, even if not all legal reasoning is non-monotonic, some form of defeasibility in law is warranted.

There are works concerned with the formalisation of legal reasoning into logic. Pertierra et al. [136] developed a project to facilitate the representation and checking of laws before passing their text to a formal language. The languages of choice are some variations of default logic, focusing on problems found in statutes and tax regulations of British Law. Their work - as one may expect from their usage of the non-monotonic default logic - deals mainly with laws that cancel or update others.

Verheij [137] formalises legal arguments using cases and rules, applying them to actual cases in Dutch Law. His methods utilise Machine Learning, but our focus when analysing this work lies in the modelling. Verheij formalises the two parts: the case models and the arguments themselves via specific rules. For the case models, he utilises a preferential relation. Arguments are divided into two parts, namely *coherent and conclusive arguments*, represented by classical validity on formulas and *presumptively valid arguments*, represented via a non-monotonic notion of entailment based on the rules in the KLM axioms [11, 12]. He then introduces the rules of interaction between arguments and case models, which vary depending on the case. They are divided into rules for supporting arguments and rules for attacking other arguments.

Before finishing this chapter, we will discuss the concept of legal order, how iALC deals with it, and how this has some limitations when considering non-monotonicity.

4.5 Legal Order

A legal system has different categories of individual laws, varying by importance and applicability. If it were to be the case that all laws presented the same degree of importance, it would be challenging to legislate - and to adjudicate, as well - as to avoid the so-called *real antinomies*, i.e. antinomies for which there is no way to decide which of the laws in contradiction would be the correct one or the one to follow.

As per Bobbio in [61], there are three main ways of solving legal antinomies to stop them from becoming real antinomies: hierarchically (more fundamental laws have priority - *lex superior derogat legi inferiori*), chronologically (posterior laws have priority - *lex posterior derogat legi priori*), or via speciality (more specific laws have priority - *lex specialis derogat legi generali*). Sartor, in [130], introduces even a fourth one, namely *hermeneutically* (more plausible interpretations of laws have priority over less plausible ones), which happens much more directly in adjudication rather than in legislation. These methods provide laws with structural precedence via legal ordering, which diminishes the chance of real antinomies happening and diminishes the scope of these real antinomies to those laws of the same tier.

These methods of solving antinomies generate a structure and a priority in legal texts. Thus, hierarchical layers of laws arise in each legal system, depending on which ones are the most fundamental, posterior or specific, and follow a certain pre-order - represented by the semantics of iALC. Brazilian Law, for instance, has seven of such layers of laws (in a free translation based on work in [9], which is in Portuguese). It is worth remembering that the Brazilian law system is based on civil law, so the legislated code has a strong notion of hierarchy.

1. Fundamental Laws

   These are the most important laws and must precede all others. Here lie the Federal Constitution, the constitutions of each state, complementary laws, and constitutional amendments, among others.

2. Supra-legal Laws

   International treaties form supra-legal laws, being hierarchically inferior only to Brazilian fundamental laws. These laws are positioned below the fundamental laws to not interfere with Brazilian sovereignty.

3. Ordinary Laws

   In this category are most Brazilian laws, among resolutions, provisional measures, and international treaties of lesser importance. For instance, Law 8906, used in the examples of Sections 2.4.2 and 3.4, lies here.

4. Propositions

   Here are the proposals and projects of laws, such as projects of constitutional amendments (PECs) and legislative decree projects. Upon approval by the authorities which are responsible for it, they become laws of the corresponding categories.

5. Governing Laws

   Internal regulations, decrees, normative instructions, and ordinances form governing laws.

6. Collective Infra-legal Laws

   They refer to collective labour agreements and conventions.

7. Private Infra-legal Laws

   Here lie contracts, representing the biggest yet lowest-positioned tier of Brazilian laws.

The reader may notice a distinction between individual laws in Tiers 1 to 4 and those in Tiers 5 to 7. Those in the first four tiers refer to general rules and principles, whereas those in the tiers below always refer to persons/objects/(real) individuals. Non-surprisingly, in iALC, laws in Tiers 1–4 are usually represented by TBox statements and those in Tiers 5–7 by ABox statements.

In iALC, we can simulate legal order in the semantics, but this does not stop the KB from being contradictory in the presence of exceptions. Let us recall Example 4.1. Person X murders person Y, but X acts in self-defence, as Y threatens X's well-being at that moment. Then, attorney A, going against X in a court of law, states that there is a particular law indicating that murder is a crime; thus, the murderer must be sent to jail, which, now, in iALC, could be represented in the TBox as $Murderer \sqsubseteq InJail$. However, attorney B, defending X, claims that a particular article in the Consitution, i.e. a more fundamental law, which must precede any ordinary law, states that acts can be forgiven if made in self-defence, namely $SelfDecence \sqsubseteq \neg InJail$. We now see clearly that this exceptional case has a degree of precedence over the previous one. Attorney B states that $x : Murderer \sqcap SelfDefence$, where x is the VLS for the legal documents concerning the status of their client, X. Then, there is a contradiction in the KB, so the solution is to remove $Murderer \sqsubseteq InJail$ from the base, which is stated in the legal text and is relevant to the judge's decision. So, we must allow for defeasible representation.

In Chapter 5, we will discuss how we introduce defeasible representation to the language of iALC.

5
ĩALC

In this chapter, we analyse the desired properties discussed in Chapter 4 and indicate how they should be present in an extension of iALC, establishing their adequacy in legal reasoning and representation. We investigate how to take legal text as defeasible in its representation, as it is the most direct way to deal with exception-handling in the legal text itself, without resorting to removing rules from the knowledge base or forcing orderings that do not represent legal ordering faithfully.

Given the context and motivation for NMR in Legal Reasoning, we propose to expand iALC to a non-monotonic logic called ĩALC. We mainly add a new concept-forming binary operator to the language, denoted by $\rightsquigarrow\!\!\!\dashv$, to represent non-monotonic entailment. The role this operator has in ĩALC is similar to that of the DLs developed by Britz et al. [85, 86, 87, 88, 89, 21, 90, 22, 91, 92, 93], which utilise a similar conditional operator - albeit, only as a subsumption $\sqsubsetsim$ - based on the KLM Framework. ĩALC is different because we are still within an intuitionistic point of view, so there are a few relations between the existing monotonic kernel and this non-monotonic addition about which we should be concerned. The intuitionistic semantics for this logic was mostly inspired by the semantics of conditional logics given by Gabbay in [13] and Fischer Servi in [14, 15], as well as by Ciardelli and Liu in [16].

By adding this operator, we will then give motivation as to why this is adequate to deal with legal representation and reasoning, as well as discuss which properties would be better for the task at hand, mainly motivating the need for rational monotonicity.

5.1
Syntax and Desired Semantics

Let us start by adding $\rightsquigarrow\!\!\!\dashv$ to the list of concept-forming operators (let A represent an atomic concept):

$$\alpha,\beta ::= A \mid \bot \mid \top \mid \neg\alpha \mid \alpha \sqcap \beta \mid \alpha \sqcup \beta \mid \alpha \multimapinv \beta \mid \exists R.\alpha \mid \forall R.\alpha \mid \alpha \rightsquigarrow\!\!\!\dashv \beta$$

The grammar for formulas remains as before, in Section 2.4.

An interpretation for ĩALC is, now, not just a structure with a pre-order on VLSs $\preceq$, but with a ternary relation ϱ as well. This relation is not necessarily related to $\preceq$; it represents VLSs that are less exceptional than others given specific contexts. This notion is relevant to reason non-monotonically in the logic. The semantics of $\preceq$ remains unchanged.

This new relation, ϱ, aims to intuitively represent reasoning under presumptions, as was the case with conditional logics. For instance, given the concept $\alpha \rightsquigarrow \beta$, we want it to represent that it is *normal* or *expected* that α may typically lead to β at this point, without giving away the possibility that it may well be the case that there are situations where we have both α and $\neg\beta$. We wish to say that, for a VLS x, to assert $x : \alpha \rightsquigarrow \beta$ is to say that for any α-world y connected to x via ϱ, i.e. $x\varrho_\alpha y$, it is assumed that $y : \beta$ as well, meaning that y is one of the typical possibilities of α from the point of view of x.

To define the semantics, we need to specify which properties this new ternary relation should have. These properties depend, however, on which logic we intend to represent. As per Section 4.5, we illustrate how it is not enough to fully represent the domain of law without rational monotonicity, motivating us to increment it with rational monotonicity to achieve this goal.

In ĩALC, we intend to give an intuitionistic version of the rational logic presented in [22]. We provide the same axioms (including axiom **Cons**, which is necessary for DLs) and a few extra needed to deal with the intuitionistic basis. First, we motivate the need for rationality by considering the axioms without rational monotonicity (RM):

1. (Cons) $\neg(\top \rightsquigarrow \bot)$
2. (Refl) $\alpha \rightsquigarrow \alpha$
3. (LLE) $\alpha \rightarrow \beta$, $\beta \rightarrow \alpha$ and $\alpha \rightsquigarrow \gamma$ imply $\beta \rightsquigarrow \gamma$
4. (RW) $\alpha \rightsquigarrow \beta$ and $\beta \rightarrow \gamma$ imply $\alpha \rightsquigarrow \gamma$
5. (Or) $\alpha \rightsquigarrow \gamma$ and $\beta \rightsquigarrow \gamma$ imply $\alpha \sqcup \beta \rightsquigarrow \gamma$
6. (CC) $\alpha \rightsquigarrow \beta$ and $\alpha \sqcap \beta \rightsquigarrow \gamma$ imply $\alpha \rightsquigarrow \gamma$
7. (CM) $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \gamma$ imply $\alpha \sqcap \beta \rightsquigarrow \gamma$
8. (S) $\alpha \sqcap \beta \rightsquigarrow \gamma$ implies $\alpha \rightsquigarrow (\beta \rightarrow \gamma)$
9. (ANN) $\alpha \rightsquigarrow \bot$ implies $\neg\alpha$
10. (CEM$_i$) $(\alpha \rightsquigarrow \beta) \sqcup \neg(\alpha \rightsquigarrow \beta)$

Rules (Cons), (Refl), (LLE), (RW), and (CM) represent Consistency, Reflexivity, Left Logical Equivalence, Right Weakening, and Cautious Monotonicity, respectively.

Rule (CC) stands for Cautious Cut and is similar to rule Cut in [11]. We call it **Cautious** Cut to separate it from a more generalised form of Cut present in monotonic systems of conditionals [138], which would be given in the language of ĩALC by:

– (Cut) $\alpha \sqcap \beta \rightsquigarrow\!\!\dashv \gamma$ and $\beta' \rightsquigarrow\!\!\dashv \beta$ imply $\alpha \sqcap \beta' \rightsquigarrow\!\!\dashv \gamma$

This version collapses into monotonicity due to the presence of (LLE) and (CM), i.e. from (Cut), (LLE) and (CM) one derives:

– (Monotonicity) $\alpha \longrightarrow\!\!\!\!\dashv \beta$ and $\beta \rightsquigarrow\!\!\dashv \gamma$ imply $\alpha \rightsquigarrow\!\!\dashv \gamma$

This rule is something we do not want, as it defeats the whole purpose of NMR.

Rule (Or) is especially interesting from an intuitionistic point of view because in [11], the authors state that there is a reading in which the meaning implied by this rule is *essentially void* - basically, by considering β to be $\neg\alpha$. By rule (Or), one concludes that $\alpha \sqcup \neg\alpha \rightsquigarrow\!\!\dashv \gamma$, which does not add any new information in a classical setting, but it does so intuitionistically, for we do not have the principle of the excluded middle.

(S) is a rule that shows an interaction between $\longrightarrow\!\!\!\!\dashv$ and $\rightsquigarrow\!\!\dashv$. It is present since it is not derivable intuitionistically from (Refl), (Or), (RW), and (LLE), differently from the classical **P** [11], since one needs to assume the principle of the excluded middle for the monotonic entailment in order to derive it from the other axioms. Since $\longrightarrow\!\!\!\!\dashv$ is intuitionistic, we need (S) explicitly.

As for the new rules, we have (ANN), which stands for *Absurdity is Never Normal* [14]. This rule states that if a normal individual of concept α is non-existent, then we can assume that α leads to an absurdity - if we cannot find an usual individual of α, then there must be none. We also have (CEM$_i$), which stands for *Conditional Excluded Middle* - the subscript i is there to indicate the intuitionistic basis and to differentiate it from rule CEM in [138], which takes on a different form, that can lead to a classical collapse in the presence of the other axioms. Rule (CEM$_i$) indicates that conditional assertions are either typical or not, i.e. we have the principal of the excluded middle for them. This rule reflects the notion that, in reasoning, defeasible assumptions can - and, in some cases, *must* - be made without fear of inconsistency. It is relevant because we do not have the excluded middle generalised for any formula, and we also wish to reason with a lack of information - the reader may already perceive our will to integrate rational monotonicity into the logic.

Aside from the previously stated rules, we also have the following derived ones:

– (And) $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \gamma$ imply $\alpha \rightsquigarrow \beta \sqcap \gamma$

  This rule is derived from (CC) and (CM). It states that if β and γ are plausibly derivable from α, then so is their conjunction. Notice how none of these rules ((CC) and (CM)) interact with the monotonic entailment $\multimap$ - a vital sign that an intuitionistic base does not interfere with these rules.

– (Ent) $\alpha \multimap \beta$ implies $\alpha \rightsquigarrow \beta$

  This rule is derived from (Refl) and (RW) and is important to relate $\multimap$ and $\rightsquigarrow$.

We consider the example given at the end of Section 4.5: a murder happened but in self-defence. If we try to model it only with the axioms stated above, *murderers should be sent to jail* would be $Murderer \rightsquigarrow InJail$ in the DTBox, *acts made in self-defence are justifiable* would be $SelfDecence \rightsquigarrow \neg InJail$ in the DTBox, and *X murdered in self-defence* would be $x : Murderer \sqcap SelfDefence$ in the ABox (the assertion stays the same, as it represents a factual situation). In this case, even though we have a different ordering on laws that focuses on representing defeasible entailments without generating contradiction in the KB, there is no way to properly order these statements, as we would have both $x : Murder \rightsquigarrow InJail$ and $x : SelfDefence \rightsquigarrow \neg InJail$. We wish to conclude $Murderer \sqcap SelfDefence \rightsquigarrow \neg InJail$, even though there is no direct relation to this statement in this situation, and this happens by giving priority to $SelfDecence \rightsquigarrow \neg InJail$ over $Murderer \rightsquigarrow InJail$, and reasoning with this missing information. We need, then, rational monotonicity.

With the current axioms, we would avoid the possible contradictions and $Murderer \rightsquigarrow InJail$ and $SelfDecence \rightsquigarrow \neg InJail$ would be equally preferable. However, in adjudication, one must make a decision. In a similar fashion to the rational closure of the KLM Framework in [12], we propose an expansion by adding rational monotonicity to the rules:

– (RM) $\alpha \rightsquigarrow \beta$ and $\neg(\alpha \rightsquigarrow \neg\gamma)$ imply $\alpha \sqcap \gamma \rightsquigarrow \beta$

With this, we should have the *minimum reasonable* set of rules to represent defeasible reasoning faithfully [19, 12]. Lawsky [135] corroborates this notion in the context of legal reasoning.

In the example of the previous section, since we do not have $SelfDefence \rightsquigarrow \neg Murder$ (we cannot say that acts that happen in self-defence are not murders), it is reasonable to assume that $SelfDefence \sqcap Murder \rightsquigarrow \neg InJail$ (the fact that an act was a murder is less critical to the law than the fact that the act was in self-defence), reflecting the modularity of the law, as the Constitution precedes an ordinary law. Thus, the axioms we need are the ones previously listed plus (RM).

We will present the semantic constraints needed for ĩALC. As expected, we want to follow the semantics proposed by Fischer Servi [14, 15], based on CS-frames involving a ternary relation ϱ for which the set $\{y \in \Delta | (x, \alpha, y) \in \varrho\}$ represents the normal α-states, from the point of view of state x. We name it ϱ instead of R to avoid confusing the reader with our roles in the fragment related to DL in iALC.

The following semantic constraints are precisely those in [15] for logic φ, an intuitionistic analogue for **R**. Their adequacy to ĩALC is still under investigation, and we leave further formalisation for future work. However, we believe they provide a solid basis for our intended semantics.

In formal terms, we have that an interpretation for ĩALC must be an intuitionistic preferential interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \preceq, \varrho \rangle$, where $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \preceq \rangle$ is an usual iALC interpretation, imbued with sets for nominals, concept names, N_C, and role names, N_R. $\varrho \subseteq \Delta^{\mathcal{I}} \times 2^{\Delta^{\mathcal{I}}} \times \Delta^{\mathcal{I}}$ is a ternary relation which has the following properties (let α, β and γ be any concepts, and for any $x, y, z \in \Delta^{\mathcal{I}}$):

1. if $(x, \alpha^{\mathcal{I}}, y) \in \varrho$, then $y \in \alpha^{\mathcal{I}}$;
2. if $\alpha^{\mathcal{I}} \neq \varnothing$, then $\exists x', y' \in \Delta^{\mathcal{I}}$ such that $(x', \alpha^{\mathcal{I}}, y') \in \varrho$;
3. if $(x, \alpha^{\mathcal{I}}, y) \in \varrho$, $y \preceq z$, and $z \in \beta^{\mathcal{I}}$, then $(x, (\alpha \sqcap \beta)^{\mathcal{I}}, z) \in \varrho$;
4. if $(x, (\alpha \sqcup \beta)^{\mathcal{I}}, y) \in \varrho$, then $(x, \alpha^{\mathcal{I}}, y) \in \varrho$ or $(x, \beta^{\mathcal{I}}, y) \in \varrho$;
5. if $(x, \alpha^{\mathcal{I}}, y) \in \varrho$ implies $y \in \beta^{\mathcal{I}}$ for every y, then $(x, (\alpha \sqcap \beta)^{\mathcal{I}}, y) \in \varrho$ implies $(x, \alpha^{\mathcal{I}}, y) \in \varrho$ for every y;
6. if $x \preceq y$, then $(x, \alpha^{\mathcal{I}}, z) \in \varrho$ if and only if $(y, \alpha^{\mathcal{I}}, z) \in \varrho$;
7. if $(x, (\alpha \sqcap \gamma)^{\mathcal{I}}, y) \in \varrho$ and if there is z such that $(x, \alpha^{\mathcal{I}}, z) \in \varrho$ and $z \notin (\neg\gamma)^{\mathcal{I}}$, then $(x, \alpha^{\mathcal{I}}, y) \in \varrho$.

The semantics of concepts behaves in the same way as in iALC, with the addition of the interpretation of $\rightsquigarrow$:

$$(\alpha \rightsquigarrow \beta)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}},\ \text{if } (x, \alpha^{\mathcal{I}}, y) \in \varrho, \text{ then } y \in \beta^{\mathcal{I}}\}$$

Satisfiability in ĩALC would be the same as in iALC, i.e. if we have $x : \alpha$, we interpret this as $\mathcal{I} \models x : \alpha$. Since $\alpha \rightsquigarrow \beta$ is a concept, we have that $x : \alpha \rightsquigarrow \beta$ is interpreted as $\mathcal{I} \models x : \alpha \rightsquigarrow \beta$. However, since TBox and ABox satisfiability is still the same, we need to add this new kind of information somewhere in the KB. Thus, we extend the KB with a DTBox (Defeasible TBox) $\mathcal{D}$, containing defeasible entailments of ĩALC:

$$\mathcal{I} \models \alpha \rightsquigarrow \beta \text{ if and only if } \forall x \in \Delta^{\mathcal{I}}, x \in (\alpha \rightsquigarrow \beta)^{\mathcal{I}}$$
$$\mathcal{I} \models \mathcal{D} \text{ if and only if } \forall \Phi \in \mathcal{D}, \mathcal{I} \models \Phi$$

Finally, let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}, \mathcal{D} \rangle$ be a *defeasible* KB. Then, if $\mathcal{I} \models \mathcal{T}$, $\mathcal{I} \models \mathcal{A}$, and $\mathcal{I} \models \mathcal{D}$ then $\mathcal{I} \models \mathcal{K}$, and we say that $\mathcal{I}$ is a model of $\mathcal{K}$.

Finally, as for the relations between the DTBox and the TBox, let us recall that, in the works by Britz et al. [21, 22], having $\alpha \sqcap \neg\beta \mathrel{\underset{\sim}{\sqsubset}} \bot$ in the DTBox is equivalent to having $\alpha \sqsubseteq \beta$ in the TBox. However, this is only a true equivalence in a classical setting. In [15], Fischer Servi proposes a different equivalence in an intuitionistic setting, which, after translation to a DL language, becomes $\alpha \multimap \beta$ being equivalent to having $\alpha \rightsquigarrow \neg\beta$ not be derivable anywhere in the model (given an interpretation $\mathcal{I}$) - which is something we still aim to study further.

As per Fischer Servi [14, 15], the rules presented previously satisfy the semantic constraints for ϱ. The first constraint states that, from the point of view of x, if y is a typical α-state, it is an α-state. Rule (Refl) satisfies this condition. The second states that if there is an α-state, there must be a normal α-state. Rule (ANN) satisfies this condition. The third constraint states that all β-states monotonically preceded (via $\preceq$) from a typical α-state are also $(\alpha \sqcap \beta)$-states, i.e. they are also normal β-states. Rule (S) satisfies this condition. The fourth constraint indicates that $(\alpha \sqcup \beta)$-states are either α-states or β-states. Rule (Or) satisfies this condition. The fifth constraint states that if every normal α-state is a β-state, then β does not interfere with the normality of α. Rule (And) satisfies this condition. The sixth constraint indicates that normality persists in related states. Rule (LLE) satisfies this condition. Finally, the seventh constraint indicates that normal α-states that are not γ-states allow for the assumption that γ does not interfere with α. Rule (RM) satisfies this condition. We leave formal proofs for future work.

One crucial difference this has from the KLM approach is that we do not assume absoluteness, i.e. we may consider the triple $(x, \alpha^{\mathcal{I}}, y) \in \varrho$ as a relation $x\varrho_{\alpha}y$, in which we must index by α indicating that x is related to y in the sense that x considers y to be a normal α-state - there is no notion of *general* normality in the intuitionistic approaches in the literature [14, 15, 16], as it would cause a classical collapse.

Thus, adopting the vocabulary of [82], the properties of ϱ are centering (which implies reflexivity and normality) and uniformity. This categorisation has **EXPTIME** complexity for propositional logic, which may be equal or worse for ĩALC.

Since these works revolve mostly around propositional logic, there is still a question about how ϱ interacts with our roles of DL. For instance, in iALC, the frame conditions **F1** and **F2** are in the semantics relating $\preceq$ to roles, though we do not believe they could be translated to relation ϱ.

The following section presents a possible application of ĩALC.

5.2 Modeling in ĩALC: Expanding further on the Example

In this section, we expand and discuss the situation given in Sections 2.4.2 and 3.4 to reach the closest possible to a real case situation in which we show NMR to be an essential part of the process, highlighting properties we wish that ĩALC should be able to deal with it.

Mary went to Anna and John's firm to be represented in a divorce case made against her by her (ex-)husband, Bob. He took the children to his mother's house and wants full custody of them after being physically assaulted by Mary. However, she states that the assault was in ***defence of others****, for Bob, in a drunken rage, yelled at one of their sons, 11-year-old Nathan, and tried to hit him. She then went over to defend him from his father, resulting in the assault. Desperate, she wishes to have full custody to protect the children from future outbursts of violence from their father.*

Mary is not formally employed, so she asked to be represented ***pro bono*** *- the reason why she accepted being represented by a team with an intern under supervision. Bob convinced her to stop working when their first child was born, and since then, he has done everything to convince her she would not be capable of re-entering the job market - leaving her with no economic means to provide for herself and her children.*

Mary confides to Anna and John that Bob has a history of patrimonial, moral and psychological violence against her and the children and that she tried to denounce him to the authorities a few times, but since Bob is a police officer, they covered for him, and nothing happened. There was never a previous physical assault on his part, which is why she is so nervous about the possibility of losing the kids. Eventually, she can make contact with her eldest child, a 17-year-old girl named Carla, who said she would side with Mary in anything *her mother needed, even testify for Mary about what happened on the night of the physical violence since she was present, as well as testify about previous situations of verbal abuse from Bob towards Mary, her brothers and herself.*

Upon hearing her story, Anna and John searched for the relevant laws to the case in question, and they arrived at three: the Constitution [139] (in Portuguese) itself, the Civil Code (Law 10406) [140] (in Portuguese), and Maria da Penha Law (Law 11340) [141] (in Portuguese and English). The Civil Code is a set of norms establishing people's rights and duties, assets, and private relations. Maria da Penha Law was created in 2006 and classifies domestic violence as one of the forms of human rights violation. It alters the Penal Code and makes it possible to arrest aggressors in the act or to have them preventively arrested when they threaten the woman's physical integrity. It also provides for new protection measures for women under life threats, such as removing the aggressor from the home and prohibiting him from physically coming close to the victim and her children.

From the Constitution, we have Article 227, which states (in a free translation from Portuguese): *it is the duty of family, society and the State to ensure with utmost priority the right of life, healthcare, nourishment, education, leisure, professionalisation, culture, dignity, respect, freedom, and familiar and communal coexistence to children and adolescents, as well as saving them from any form of negligence, discrimination, exploitation, violence, cruelty and oppression.* This article serves as a basic justification for the negligence of Bob to their kids and can be formalised as $\exists familyOf.Minor \rightsquigarrow \exists ensureRights.Minor$ in the DTBox and $(mary, nathan) : ensureRights$, $(mary, carla) : ensureRights$, $bob : \exists familyOf.Minor$ (yet, we have neither $(bob, nathan) : ensureRights$ nor $(bob, carla) : ensureRights$) in the ABox, for example. The full list will be in the ABox below).

From the Civil Code, we have Article 188, which states (free translation): *the following are not illicit acts: (i) those practised in self-defence or while regularly exercising a granted right, or (ii) the deterioration or destruction of an object of others, or injury caused to others directly, in order to remove impending danger.* In this article, there is also the first paragraph: *in the case of (ii), the act will be considered legitimate only when the circumstances render it absolutely necessary, not exceeding the limits of the indispensable for the removal of danger.* This article serves to justify Mary's act of physical assault on Bob. We formalise it as $PhysicalV \rightsquigarrow \neg Justified$ (where $PhysicalV$ stands for an act of physical *violence*), and $PhysicalV \sqcap SelfDefence \rightsquigarrow Justified$ in the DTBox.

We also have Article 228 from the Civil Code (free translation): *cannot be admitted as witnesses: (i) those under 16 years old; [...] (v) spouses, ascendants, descendants and relatives up to third degree of any of the parts, by either consanguinity or affinity, however for the evidence of facts only they may know, the judge may allow for their testimony.* This yields $Under16 \rightsquigarrow \neg Witness$, $\exists familyOf.Accused \rightsquigarrow \neg Witness$ and $\exists familyOf.Accused \sqcap Allowed \rightsquigarrow Witness$ in the DTBox. The last sentence of this article is the most important one since it is up to the judge to accept or not for a relative to testify (in our case, for Mary's daughter Carla to testify against her father), highlighting the dependency of the adjudication process to a certain degree of the will of judges. In this example, we will omit this notion and place the consequence of accepting Carla's testimony directly into the ABox through $carla : Witness$.

From Maria da Penha Law, we have Article 7, which states: *the forms of domestic and family violence against women, are, among others: (i) physical violence [...]; (ii) psychological violence, understood as any behaviour that causes emotional damage and reduction of self-esteem or that harms and disturbs full development or that aims at degrading or controlling the woman's actions, behaviours, beliefs and decisions, by means of threat, embarrassment, humiliation, manipulation, isolation, constant surveillance, constant pursuit, insult, blackmail, ridiculing, exploitation and limitation of the right to come and go or any another means that causes damage to the woman's psychological health and self-determination; (iii) sexual violence [...]; (iv) patrimonial violence, understood as any behaviour that constitutes retention, subtraction, partial or total destruction of the woman's objects, working instruments, personal documents, property, assets and economic rights or resources, including those intended to satisfy her needs; (v) moral violence, understood as any behaviour that constitutes slander, defamation or insult.* This helps categorise the violence suffered by Mary during her marriage with Bob, even if not physical, and can be formalised as $Violence \sqsubseteq PhysicalV \sqcup PsychologicalV \sqcup SexualV \sqcup PatrimonialV \sqcup MoralV$ in the TBox.

We also have Article 10, which refers to the lack of assistance Mary received from authorities and serves as further justification for the escalation of the situation into physical violence: *in case of imminent or actual domestic and family violence against women, the police authority that learns of the occurrence shall immediately adopt the appropriate legal measures. The provision in the heading of this article applies to failure to comply with the urgent protective measure that has been determined.*, which shows how law enforcement failed to protect Mary from abuse in her relationship. We can formalise this as

$Vulnerable \rightsquigarrow\!\!\!\!\mid SelfDefence$ in the DTBox and $mary : Vulnerable$ in the ABox, indicating that, when there is no protection for a victim of violence, acts they may take can be seen as in self-defence - the reader may notice that we can only make this last assertion when the testimony given by Carla can be accepted in court since there is no legal document stating her vulnerability due to the lack of success at pressing charges against her husband at the police department.

Then, for this situation, we have:

$$\Delta = \left\{ \begin{array}{c} constitution,\ civilcode,\ dapenha, \\ mary,\ bob,\ nathan,\ carla \end{array} \right\}$$

$$N_C = \left\{ \begin{array}{c} Minor,\ PhysicalV,\ Justified,\ SelfDefence, \\ Witness,\ Accused,\ Allowed,\ Violence,\ PsychologicalV, \\ SexualV,\ PatrimonialV,\ MoralV,\ Vulnerable \end{array} \right\}$$

$$N_R = \{familyOf,\ ensuresRights\}$$

$$\preceq = \left\{ \begin{array}{cc} \langle constitution, civilcode \rangle & \langle constitution, dapenha \rangle \\ \langle civilcode, mary \rangle & \langle civilcode, bob \rangle \\ \langle civilcode, nathan \rangle & \langle civilcode, carla \rangle \\ \langle dapenha, mary \rangle & \langle dapenha, bob \rangle \\ \langle dapenha, nathan \rangle & \langle dapenha, carla \rangle \end{array} \right\}$$

We omit the reflexive and transitive arrows of $\preceq$ to avoid clutter in the model.

$$\mathcal{T} = \left\{ \begin{array}{c} Under16 \longrightarrow\!\!\!\!\mid Minor \\ Violence \longrightarrow\!\!\!\!\mid PhysicalV \sqcup PsychologicalV \sqcup SexualV \sqcup \\ PatrimonialV \sqcup MoralV \end{array} \right\}$$

$$\mathcal{D} = \left\{ \begin{array}{c} \exists familyOf.Minor \rightsquigarrow\!\!\!\!\mid \exists ensureRights.Minor \\ PhysicalV \rightsquigarrow\!\!\!\!\mid \neg Justified \\ PhysicalV \sqcap SelfDefence \rightsquigarrow\!\!\!\!\mid Justified \\ Under16 \rightsquigarrow\!\!\!\!\mid \neg Witness \\ \exists familyOf.Accused \rightsquigarrow\!\!\!\!\mid \neg Witness \\ \exists familyOf.Accused \sqcap Allowed \rightsquigarrow\!\!\!\!\mid Witness \\ Vulnerable \rightsquigarrow\!\!\!\!\mid SelfDefence \end{array} \right\}$$

$$\mathcal{A} = \left\{ \begin{array}{cc} carla : Minor & nathan : Under16 \\ mary : PhysicalV & (mary, carla) : familyOf \\ (mary, nathan) : familyOf & (bob, carla) : familyOf \\ (bob, nathan) : familyOf & carla : Witness \\ mary : Vulnerable & (mary, nathan) : ensureRights \\ (mary, carla) : ensureRights & \end{array} \right\}$$

This case has many different aspects, but here, we will focus on the central situation, highlighting the need for a *particular* type of NMR. The main argument here is to let the judge infer that $mary : SelfDefence$, i.e. Mary acted in self-defence, leading to deciding in her favour. To reach this conclusion, however, some intermediate steps are needed. For instance, we must conclude that she is justified in acting as she did.

In order to reach $mary : Justified$, we need to consider $mary : PhysicalV$, $mary : PhysicalV \sqcap SelfDefence \leadsto Justified$ (which stems from the precedence $civilcode \preceq mary$) and $mary : Vulnerable$. This last part is essential, but it depends entirely on her daughter's testimony in court, which the judge may accept or not. We can only consider Mary to be in a vulnerable position if we understand that she tried to report Bob formally. However, no formal record was made, so the legality of this argument depends only on how what Carla states is perceived.

However, we do not have a direct relation between the normality of physical assaults and people in a situation of vulnerability; it may well be the case that somebody resorts to physical assault when needed if there is a lack of support by the State to keep on living. This relation would translate as $\neg(Vulnerable \leadsto \neg PhysicalV)$, which makes sense according to the presumption of innocence - one cannot assume at first that a violent act was not performed by a mother in a situation of vulnerability, especially considering a delicate situation such as this, given the precedence in cases of domestic violence. Thus, we need a way to deal with the lack of information in the KB, which comes with our rule (RM) in conjunction with (CEM$_i$) and (S).

Since we do not have $Vulnerable \leadsto \neg PhysicalV$, through (CEM$_i$) we have $\neg(Vulnerable \leadsto \neg PhysicalV)$. From $Vulnerable \leadsto SelfDefence$ and $\neg(Vulnerable \leadsto \neg PhysicalV)$, through (RM), we conclude that $Vulnerable \sqcap PhysicalV \leadsto SelfDefence$, i.e. an act of physical violence performed when vulnerable is considered to be in self-defence. Furthermore, we have from the Civil Code that $PhysicalV \sqcap SelfDefence \leadsto Justified$. From (S), we arrive at $SelfDefence \rightsquigarrow (PhysicalV \rightarrow Justified)$. In an organised derivation, we have:

1. $Vulnerable \rightsquigarrow SelfDefence$ from DTBox
2. $\neg(Vulnerable \rightsquigarrow \neg PhysicalV)$ from (CEM$_i$)
3. $Vulnerable \sqcap PhysicalV \rightsquigarrow SelfDefence$ from (RM) 1,2
4. $Vulnerable \rightsquigarrow (PhysicalV \rightarrow SelfDefence)$ from (S) 3
5. $PhysicalV \sqcap SelfDefence \rightsquigarrow Justified$ from DTBox
6. $SelfDefence \rightsquigarrow (PhysicalV \rightarrow Justified)$ from (S) 5

Since $Vulnerable \rightsquigarrow \neg(PhysicalV \rightarrow SelfDefence)$ cannot be derivable in this situation due to the lack of the principle of the excluded middle for the monotonic fragment, we can conclude that $Vulnerable \rightarrow (PhysicalV \rightarrow SelfDefence)$. The same happens with $SelfDefence \rightsquigarrow \neg(PhysicalV \rightarrow Justified)$, letting us conclude $SelfDefence \rightarrow (PhysicalV \rightarrow Justified)$, leading us via modus ponens to $m : Justified$. Thus, Mary was justified in her act, for she was acting in self-defence due to being in a vulnerable situation.

The reader may notice that this situation *requires* that the assertion $mary : Vulnerable$ be present in the KB in order to conclude this case, as the whole argument for the decisions lies in considering Carla's testimony relevant or not to the case, indicating that the construction of the KB is dependant on the decisions of the judge of the case in question.

Further details from this KB can be considered as well for the whole case: the relations between defending a family member and acts in self-defence, parental negligence by Bob as an aggravation, his connections to the police station in which Mary tried to report him and how that indicates his untrustworthiness, among others. Many (if not all) involve some form of defeasibility since we tailored this situation to deal with exceptions to highlight the possibilities of NMR in Law. A more direct situation in which the forms of violence would come only from Bob could be solved even monotonically, but that does not happen necessarily in every case.

Many other legal texts are relevant even to this fictional case, for instance, the Criminal Code itself, which would enrich the KB. However, we kept it simple and focused on justification for rational monotonicity.

6
Conclusion

In this thesis, we presented work on expanding theoretical tools to the intuitionistic description logic iALC to cover specific gaps encountered in its usage. Such work consisted of two parts: a (*quasi-*)normalising, sound, and complete ND System regarding intuitionistic conceptual models [3] in order to better work with iALC in interdisciplinary environments (especially with those in the area of Law), and a proposal for a non-monotonic expansion of iALC, namely $\tilde{\text{i}}$ALC, involving rational logic, in order to represent legal reasoning better in processes such as the judicial process, which is non-monotonic, as well as motivating for non-monotonicity in legal *modelling*.

By having logic as a formal way to represent reasoning, we aim with our work to aid those taking part in legal reasoning via providing ways to verify if their reasoning follows in a sound manner. We have no intention of substituting human agents in the related domains.

This work is interdisciplinary, and we hope to have given a thoughtful account of the domain of Law and the reasoning processes therein from the point of view of logic. Nevertheless, we know the many possibilities for future work left open, w.r.t. applicability, theoretical foundations, formal details of formalisation, and implementation. Following the primary division of this thesis into two, the possibilities for future work are also divided into future work regarding the ND system and NMR.

For ND, we have a few possibilities, both theoretical and practical, such as effectively establishing complexity results for proof search and implementing the system in either an interactive style (e.g. in Coq,[1] Isabelle,[2] or other interactive theorem prover), or in an automated style. When one considers the possible *end-user* of this system (and its implementations), one could consider someone working in Law - or any norm-related field - which may need aid in formalising reasoning over different laws and situations, either in a (judicial) case or in checking for antinomies while legislating. For this user, it would be necessary to have two steps: first, the creation of the KB - establishing relations between laws, modelling what is said in each law, instantiating adequately each

[1] `https://coq.inria.fr/`, accessed on 09/07/2023.
[2] `https://isabelle.in.tum.de/`, accessed on 09/07/2023.

related VLS -, and second, the application of the reasoning itself. Another possibility for the system is to create an adaptation of it in order to reach full normalisation. Finally, we wish to further test the explainability of the ND system in empirical studies with subjects from the domain of Law.

As for NMR, we need first to provide a correct set of semantic constraints for ĩALC, as well as define proper models for it. Then, we will be able to establish proofs of soundness and completeness of the axiomatic systems according to these types of models. However, to make these proofs, some more specific decisions have to be made still, such as how to adequately set interactions between ϱ, $\preceq$ and the roles R existent in iALC, which would be akin to defining conditions such as the frame conditions **F1** and **F2** of Section 2.4 - which we do not want to have interacting with $\leadsto$ since they maintain monotonicity of roles. There is also the question of expressiveness of nesting of $\leadsto$, because it is a concept-forming operator, leading us to concepts of the form $\alpha \leadsto (\beta \leadsto \gamma)$, for instance. We also must define the interactions between the TBox and the DTBox to relate the logic to a defeasible knowledge base accurately. These decisions will require further study in conditional logics of different kinds, intuitionistic, classical or even other monotonic bases, such as the works of Weiss [142, 143], which have a similar but different axiomatisation to the works of Gabbay and Fischer Servi. After these results, we can establish the complexity of satisfiability of the axioms concerning their respective models and compare ĩALC to its classical DL counterpart w.r.t. complexity and expressiveness. Afterwards, it will then be possible to implement algorithms for a reasoner in these logics, as to reach more easily a possible end-user who may be an expert in Law but not necessarily in logic or computing, primarily to provide aid in solving severe cases such as the one presented in Section 5.2, as well as a more user-friendly reasoning system than an axiomatic system - perhaps a ND system.

There is, of course, future work and discussion not directly related to the ND system or NMR specifically but to the representation of Law in general, which is still relevant. As stated in Section 4.5, there is also a *temporal* component to relating laws to one another. This temporal aspect can be simulated artificially in iALC if one considers a legal system to be *frozen* in time. However, it may be necessary to compare different states of legal systems - something that happens in legislation. It would be interesting to represent how a KB in iALC could adapt to legal changes over time. This aspect focuses much more on the consistency and scalability aspect of a KB rather than representing knowledge with a specific purpose in mind - which was the case of this current document and previous work with iALC [7, 9, 8, 10]. Perhaps solving how to represent time - or even *changes over time* to a KB in iALC - could lead to more answers in this respect.

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