Pontifícia Universidade Católica
do Rio de Janeiro

Leonardo Gonçalves de Oliveira

Moderate deviations of triangle counts in sparse random graphs.

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática, do Departamento de Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Simon Griffiths

Rio de Janeiro
September 2022

Pontifícia Universidade Católica
do Rio de Janeiro

Leonardo Gonçalves de Oliveira

Moderate deviations of triangle counts in sparse random graphs.

Thesis presented to the Programa de Pós–graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática. Approved by the Examination Committee:

Prof. Simon Griffiths
Advisor
Departamento de Matemática – PUC-Rio

Dr. José Diego Alvarado Morales
Instituto de Matemática e Estatística – USP

Prof. Leandro Pinto Rodrigues Pimentel
Instituto de Matemática – UFRJ

Prof. Marcos Craizer
Departamento de Matemática – PUC-Rio

Prof. Robert David Morris
Instituto de Matemática Pura e Aplicada – IMPA

Prof. Roberto Imbuzeiro Moraes Felinto de Oliveira
Instituto de Matemática Pura e Aplicada – IMPA

Rio de Janeiro, 28th September2022

Leonardo Gonçalves de Oliveira

Bachelor's degree in Mathematics by Universidade Estadual de Londrina (UEL) in 2014. Master's degree in Mathematics by Instituto de Matemática Pura e Aplicada (IMPA) in 2017.

Bibliographic data

Gonçalves de Oliveira, Leonardo

Moderate deviations of triangle counts in sparse random graphs. / Leonardo Gonçalves de Oliveira; advisor: Simon Griffiths. – 2022.

122 f: il. color. ; 30 cm

Tese (doutorado) - Pontifícia Universidade Católica do Rio de Janeiro, Departamento de Matemática, 2022.

Inclui bibliografia

1. Matemática – Teses. 2. Combinatória – Teses. 3. Probabilidade – Teses. 4. Desvios Moderados. 5. Martingais. 6. Grafos Aleatórios. 7. Desigualdades de Concentração. I. Griffiths, Simon. II. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Matemática. III. Título.

CDD: 620.11

To Antônio and Rosa, my grandparents
and the first teachers of my life.

Acknowledgments

I would like to first thank my mother, Flávia. She has supported my dreams since I was a kid and she is my biggest inspiration in life. I am sure that without her help I would not be able to be a doctor. I also would like to thank my father, Dirceu, who has also supported me through my life and career.

I would like to thank my wife, Laryssa. She has been with me for the past seven years during the best and the worst moments of my life. Her help was fundamental for me to keep going through my academic career and I am the luckiest person alive to have her by my side.

I also would like to thank my grandparents Antônio and Rosa. They were very important in my life and I am sure they would be proud to know that I am becoming a doctor.

I also would like to thank my mother-in-law Fátima who is a second mother to me. She has been helping me since the first day we met.

I would also like to thank my advisor Simon for all his support. He has been very kind and generous as an advisor and working with him was a pleasure. His help was very important during the preparation of this thesis.

I would also like to thank José for the many conversations that we had and for his support during my time as a PhD student.

Also, I would like to thank all my friends at IMPA and PUC-RIO: Jennifer, João, Wenxiang, Matheus, Daniel and all other people who offered me help or even a friendly conversation during my time as a student.

I would also like to thank my professors from the Math Olympiads and my undergraduate period at UEL: Nelson, Ana Lúcia, Túlio, Neuza, Silvia and so many others who always encouraged me to keep going in my career.

I would also like to thank my therapists, Nayara and Evlyn, for their help with my struggles during my time as a PhD student.

I would also like to thank Saints Cosmas and Damian for always answering my prayers and helping me in my life, including the preparation of this thesis.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de NívelSuperior - Brasil (CAPES) - Finance Code 001

Abstract

Gonçalves de Oliveira, Leonardo; Griffiths, Simon (Advisor). **Moderate deviations of triangle counts in sparse random graphs.**. Rio de Janeiro, 2022. 122p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

In the first part of this thesis, we study the deviation of the number of triangles with respect to its mean in both the random graph models $G(n, m)$ and $G(n, p)$. We focus on the case where the random graph is sparse, in which the edge density goes to zero as the number of vertices increases to infinity. Also, our focus is in the case of moderate deviations, i.e., those of order in between the standard deviation and the mean. In addition, we derive the same kind of results for cherries (paths of length two). In the second part of this thesis, we study Freedman's inequality. This inequality gives bounds on the probability of the deviation of a bounded martingale using its conditional variance. In our work, we obtain a strengthening of Freedman's inequality, under additional symmetry conditions on the increments of the martingale process.

Keywords

Moderate Deviations; Martingales; Random Graphs; Concentration Inequalitie.

Resumo

Gonçalves de Oliveira, Leonardo; Griffiths, Simon. **Desvios moderados do número de triângulos em grafos aleatórios esparsos.**. Rio de Janeiro, 2022. 122p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Na primeira parte dessa tese, estudamos o desvio no número de triângulos com respeito à média em ambos os modelos de grafos aleatórios $G(n, m)$ e $G(n, p)$. Focamos no caso em que o grafo aleatório é esparso, no qual a densidade de arestas vai para zero quando o número de vértices cresce para o infinito. Nosso foco também reside no caso de desvios moderados, i.e., aqueles cuja ordem está entre o desvio padrão e a média. Além disso, também derivamos o mesmo tipo de resultado para cerejas (caminhos de comprimento dois). Na segunda parte dessa tese, estudamos a desigualdade de Freedman. Essa desigualdade fornece limitantes para a probabilidade de desvio de um martingal limitado usando sua variância condicional. No nosso trabalho, obtemos uma versão mais forte da desigualdade de Freedman, impondo condições adicionais de simetria nos incrementos do processo martingal.

Palavras-chave

Desvios Moderados; Martingais; Grafos Aleatórios; Desigualdades de Concentração.

Table of contents

List of figures

List of tables

How's it going to end?

Truman Burbank, *The Truman Show.*

1
Introduction

A basic problem in Probability Theory is the following: given a random variable X, what is the probability that X deviates from its expectation, $\mathbb{E}[X]$, by a value $a > 0$? Another popular way to define this question is to write the deviation $a = \delta\sigma$ where σ is the standard deviation of X and δ is a positive parameter. One of the simplest answers to this question is given by Chebyshev's inequality, which says that

$$\mathbb{P}(|X - \mathbb{E}[X]| \geqslant \delta\sigma) \leqslant \frac{1}{\delta^2}.$$

Although Chebyshev's inequality is best possible, it should not be surprising that we could obtain better bounds under some additional restrictions on X.

In many applications, X may be a sum of n random variables X_i with $n \to \infty$ and we would like to answer the deviation question asymptotically. Note that in this setting X is actually a sequence that depends on n. Maybe the most famous result about sequences of random variables is the Central Limit Theorem which approximates the distribution of X by a normal distribution.

When talking about deviations, we shall split them into three categories: we say that a deviation is small if it is of order of magnitude up to the order of the standard deviation; deviations of order of magnitude between the standard deviation and the mean are called moderate; deviations of order of magnitude at least that of the mean are called large.

The inequalities that give bounds on deviations are called concentration inequalities. We shall present some of the most famous in Chapter 2.

1.1
Some definitions and results on Probabilistic Combinatorics

We shall now present some general definitions and results from Graph Theory. A graph G is composed of a set of vertices $V(G)$ and a set of edges $E(G)$ where each edge e is a non-ordered pair of vertices. A graph G is bipartite if there are two disjoint sets $A, B \subseteq V(G)$ such that every edge $e \in E(G)$ has an endpoint in A and another in B. We say that a graph is complete (or a clique) on n vertices if it has all possible edges. We denote such a graph by

K_n and we also call it by In this work, we often refer to triangles ($\triangle$) which is another way to call a K_3. We also frequently use cherries ($\wedge$), which are graphs with three vertices and exactly two edges.

We say that G' is a subgraph of G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. We also say that G' is an induced subgraph of G if it contains all edges of G with both endpoints in $V(G')$. We shall write $G' = G[V']$ for this graph, where $V' = V(G')$.

Given a vertex v in a graph G, we define the neighbourhood of v, $N(v)$, as the set of vertices w such that $vw \in E(G)$. The size of $N(v)$ is the degree of v in G, which we denote by $d_v(G)$. Also, the codegree of a pair of vertices v, w in G, denoted by $d_{uw}(G)$, is given by the size of $N(v) \cap N(w)$.

Given an integer k, we write $[k] = \{1, 2, \ldots, k\}$. A k-colouring on the vertices of a graph G is a map $c : V(G) \to [k]$ and each label in $[k]$ is called a colour. Such a vertex colouring is proper if no two adjacent vertices share the same color. The chromatic number of G, $\chi(G)$ is the minimum number of colours in a proper colouring of G. By using a greedy colouring it is easy to see that $\chi(G) \leqslant \Delta(G) + 1$ where $\Delta(G)$ is the maximum degree of G.

In an analogous way, we define a k-colouring on the edges of G as a map $c : E(G) \to [k]$. Also, an edge colouring is proper if no two edges which share a vertex have the same colour. The edge chromatic number of G, $\chi'(G)$ is the minimum number of colours in a proper edge colouring of G. Clearly, we must have $\chi'(G) \geqslant \Delta(G)$ as all edges incindent to the same vertex must have different colours.

An hypergraph H is composed of a set of vertices $V(H)$ and a set of hyper-edges $E(H)$ where each hyper-edge e is a subset of $V(H)$. If every hyper-edge has the same size k, H is said to be k-uniform. In particular, every graph is a 2-uniform hypergraph.

Let us now introduce the concept of random graph. We use two models in this work: $G(n, p)$ and $G(n, m)$. The random graph model $G(n, p)$ is the graph which has n vertices and in which each possible edge is chosen to be in the graph independently with probability $p \in (0, 1)$. Surprisingly, this is not exactly a graph but a probability distribution on the set of all graphs of n vertices. However, we may treat this as a graph without serious problems. Now, in the model $G(n, m)$ the graph is chosen uniformly among all graphs with n vertices and m edges. Sometimes we may also consider the Erdős-Rényi random process $G_i, i = 0, 1, \ldots, m$. This process is generated by starting with the empty graph and add edges $e_1, e_2, \ldots, e_m$ one at a time uniformly among all possible edges except the ones that have already been chosen. Another way to generate the process G_i is to consider a random uniform permutation

$\{e_1, \ldots, e_N\}$of the edges of K_n and then consider G_i as the graph with edge set $\{e_1, \ldots, e_i\}$. It is easily verified that $G_i \sim G(n,i)$ for all $i \leqslant m$.

In Probabilistic Combinatorics, it is relatively common to use martingales and our work is heavily based on them. For example, suppose we have a function $f(G)$. Then we can define a martingale X_i in which we expose the first i vertices and their internal edges and take the conditional expectation of $f(G)$ with that partial information. This is called the vertex exposure martingale. Shamir and Spencer [1] used this martingale together with Hoeffding-Azuma concentration inequality to prove that

$$\mathbb{P}\left(|\chi(G(n,p)) - \mathbb{E}\left[\chi(G(n,p))\right]| > \lambda\sqrt{n-1}\right) \leqslant 2e^{-\lambda^2/2}$$

where $\chi(G)$ is the chromatic number of G.

In our work, we also define a specific martingale related to the number of triangles in a random graph. We then use Freedman's concentration inequality to obtain bounds on this variable.

A great reference for more on the extensive field of Probabilistic Combinatorics is The Probabilistic Method, by Alon and Spencer [2]. The discussion about $\chi(G(n,p))$ above is based on this reference.

1.2 Deviations on triangle counts

In the first part of this thesis, we study moderate deviations on the number of triangles in sparse random graphs.

We consider two models of random graphs: the model $G(n,p)$, already defined, and the model $G(n,m)$ in which the graph is chosen uniformly among all graphs with n vertices and m edges. We shall always consider $n \to \infty$ and the parameters p and m may depend on the value of n. Our focus is on sparse graphs, i.e., when the edge density goes to 0 as $n \to \infty$. Moreover, our results apply mainly to moderate deviations, although some of them also apply to large deviations.

Let $N_\triangle(G)$ be the number of isomoprhic copies of triangles in the graph G. In $G_{n,p}$ this variable has expected value $p^3(n)_3$ where $(n)_k = n(n-1)\ldots(n-k+1)$. We would like to understand the behaviour of

$$r(\delta,p,n) \,:=\, -\log \mathbb{P}\left(N_\triangle(G_p) \,>\, (1+\delta)p^3(n)_3\right) .$$

We note that this corresponds to large deviation when $\delta > 0$ is a fixed constant.

The first bounds for $r(\delta,p,n)$ were found in 2001 by Vu [3] and in 2005 by Kim and Vu [4] and by Janson, Oleszkiewvicz and Rucinski [5]. However, the upper and lower bounds were not of the same order. The correct order was found independently in 2012, by Chatterjee [6] and by De Marco and Khan [7]. They proved that

$$c(\delta)p^2n^2\log(1/p) \leqslant r(\delta,p,n) \leqslant C(\delta)p^2n^2\log(1/p)$$

for some constants $c(\delta), C(\delta)$.

The next natural problem is to determine the behaviour of $r(\delta,p,n)$ when $n \to \infty$ and $p = p(n)$ changes according to n. Since our work focus on the sparse case (meaning that $p \ll 1$), we refer the reader to the survey of Chatterjee [8] for other results on the dense case. Now, on the sparse case we have

$$r(\delta,p,n) = (1+o(1))\min\left\{\frac{\delta^{2/3}}{2},\frac{\delta}{2}\right\}p^2n^2\log(1/p)$$

if $n^{-1/2} \ll p \ll 1$ and

$$r(\delta,p,n) = (1+o(1))\frac{\delta^{2/3}}{2}p^2n^2\log(1/p)$$

if $n^{-1} \ll p \ll n^{-1/2}$.

The proof of the first identity above goes back to the work of Chatterjee and Dembo [9] and Lubetzky and Zhao [10] who proved this identity for $p \geqslant n^{-1/42}\log n$. Then the regime where the first equality holds was gradually extended to $p \gg n^{-1/2}(\log n)^2$ through the work of Eldan [11], Cook and Dembo [12] and Augeri [13]. Finally, Harel, Mousset and Samotij [14] recently completed the proof of both identities above. They also provided an expression for the value of $r(\delta,p,n)$ when $p^2n \to c \in \mathbb{R}$.

Up to this point, we have discussed only results related to large deviations of the number of triangles in $G(n,p)$. It is also worth to mention the existence of central limit theorems for triangle counts. In 1988, Ruciński [15] proved that subgraph counts of $G_{n,p}$ are normally distributed. In 1990, Janson [16] proved a functional central limit theorem for subgraph counts both in $G_{n,p}$ and $G_{n,m}$.

As we previously mentioned, our focus is in the study of moderate deviations, which have order between the standard deviation and the mean. For the dense case again, the expression for the asymptotics of the deviation probability was found by Féray, Méliot and Nikeghbali [17]. Their expression works whenever $p \in (0,1)$ is constant and $n^{-1} \ll \delta_n \ll n^{-1/2}$. For the sparse case, Döring and Eichelsbacher [18] proved in 2009 that

$$r(\delta,p,n) = \frac{-\delta_n^2 pn^2}{36(1-p)} + o(\delta_n^2 pn^2)$$

if $p^{-1/2}n^{-1} \ll \delta_n \ll p^7$.

It might be the case that having a certain structure in the random graph, such as a star, hub or clique is the most likely way to achieve deviation on the number of triangles. Whenever this happens, we say that there is localisation. If this is not the case, the deviation is more likely caused by the presence of extra edges in the random graph, with no specific structure.

In this thesis, we present two results for deviation on triangle counts on $G(n,p)$. The first one extends the result from Döring and Eichelsbacher [18] to the whole non-localised region.

Theorem 1.1. *Let* $n^{-1/2}\log n \ll p \ll 1$ *and let* δ_n *be a sequence satisfying*

$$p^{-1/2}n^{-1} \;\ll\; \delta_n \;\ll\; p^{3/4}(\log n)^{3/4}\,,\, n^{-1/3}(\log n)^{2/3} \;+\; p\log(1/p)\,.$$

Then

$$r(\delta_n,p,n) \;=\; (1+o(1))\frac{\delta_n^2 pn^2}{36}\,.$$

Our second result presents the order of magnitude of $r(\delta,p,n)$ for the localised region.

Theorem 1.2. *Let* $n^{-1/2}\log n \ll p \ll 1$ *and let* δ_n *be a sequence satisfying*

$$p^{3/4}(\log n)^{3/4}\,,\, n^{-1/3}(\log n)^{2/3} \;+\; p\log(1/p) \;\leqslant\; \delta_n \;\leqslant 1\,.$$

Then

$$r(\delta_n,p,n) \;=\; \Theta(1)\min\{\delta_n^{2/3}p^2n^2\log n, \delta_n^{1/2}pn^{3/2}\log n \;+\; \delta_n p^2n^2\log(1/p)\}\,.$$

We also study triangle count deviations in $G(n,m)$. In fact, the main characteristic of our method is to prove results about deviations in $G(n,m)$ and then deduce results for the $G(n,p)$ model. This is essentially the same approach taken by Goldschmidt, Griffiths and Scott [19], although their results are best possible only for dense graphs. We shall use their notation throughout this thesis. In this context, we write $N = \binom{n}{2}$ and $t = m/N$. Note that t represents the edge density in $G_m \sim G(n,m)$. In G_m we have

$$\mathbb{E}\left[N_\triangle(G_m)\right] \;=\; \frac{(m)_3(n)_3}{(N)_3}\,.$$

As the value above is of order t^3n^3, we define the analogue of $r(\delta,p,n)$ for G_m as

$$r(\delta,t,n) \;=\; -\log\mathbb{P}\left(N_\triangle(G_m) > (1+\delta)t^3n^3\right)\,.$$

In the dense case, when $t \in (0,1)$ is constant, Goldschmidt, Griffiths and Scott [19] proved that

$$r(\delta, t, n) = (1 + o(1)) \frac{-\delta_n^2 n^3}{12t^3(1-t)^2(2t+1)}$$

whenever $n^{-3/2} \ll \delta_n \ll n^{-1}$. Now, for the sparse case, they found that

$$r(\delta, t, n) = (1 + o(1)) \frac{-\delta_n^2 t^3 n^3}{12}$$

whenever $n^{-1/2} \log n < t \ll 1$ and $t^{-3/2} n^{-3/2} < \delta < t^2 n^{-1}$.

Our goal here is to extend the last result above for $n^{-1/2} \log n < t \ll 1$ and $t^{3/2} n^{-3/2} < \delta_n < t^{-3/2}$. Note that at the smallest value of δ_n we have a deviation of order of magnitude of the standard deviation. Moreover, the largest value of δ_n gives the largest possible deivation, $t^{3/2} n^3$.

We shall state the results for $G(n, m)$ in terms of the associated rate, which means that we will ask how large a deviation has probability at most e^{-b} for some b. This is specially useful for the localised region, where one can think about the type of deviation that can be produced for that "price" in various different ways. In the rest of this thesis, we always consider $b = b(n)$ to be a sequence. We write $\ell := \log(1/t)$. Our result states that there are four causes of deviations on triangle counts in $G(n, m)$, as we shall present now.

Normal: In this region, the most likely cause for deviation is just the addition of new edges. We will consider a martingale with increments of order of magnitude $t^3 n^3$. This will imply, using Freedman's inequalities that a deviation of order

$$\text{NORMAL}(b, t) := b^{1/2} t^{3/2} n^{3/2}$$

has probability e^{-b}.

Star: Consider a star with degree d. One should expect that this star is involved in $\Theta(d^2 t)$ triangles and this should occur with probability approximately $t^d = e^{-d\ell}$. Therefore, this may cause a deviation of order

$$\text{STAR}(b, t) := \frac{b^2 t}{\ell^2} 1_{b \leqslant n\ell}$$

with probability e^{-b}. Note that this only makes sense for $d < n$.

Hub: Suppose that there are k vertices with degree of order n. These vertices will be involved in $\Theta(ktn^2)$ triangles and this should occur with probability approximately $t^{kn} = e^{-kn\ell}$. Therefore, this may cause a deviation of order

$$\text{HUB}(b, t) := \frac{btn}{\ell} 1_{b \geqslant n\ell}$$

with probability e^{-b}.

Clique: A clique of k vertices creates $\Theta(k^3)$ triangles and occurs with probability approximately $t^{k^2} = e^{-k^2\ell}$. Therefore, this may cause a deviation of order

$$\text{CLIQUE}(b,t) := \frac{b^{3/2}}{\ell^{3/2}}$$

with probability e^{-b}.

We also define

$$M(b,t) := \max\{\text{NORMAL}(b,t), \text{STAR}(b,t), \text{HUB}(b,t), \text{CLIQUE}(b,t)\}.$$

Our main theorem for $G(n,m)$ shows that, up to a multiplicative constant, $M(b,t)$ is the triangle count deviation which has probability e^{-b} across a large range of t and b. We let $\text{DEV}_\triangle(b,t)$ to be the minimal value of a such that

$$\mathbb{P}\left(N_\triangle(G_m) > \mathbb{E}\left[N_\triangle(G_m)\right] + a\right) \leqslant e^{-b}.$$

Theorem 1.3. *There exist absolute constants c, C such that the following holds. For all $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant tn^2\ell$ we have*

$$cM(b,t) \leqslant DEV_\triangle(b,t) \leqslant CM(b,t).$$

In particular, the previous theorem says that $r(\delta,t,n) = \Theta(-\delta^2 t^3 n^3)$ in the normal region, $r(\delta,t,n) = \Theta(\delta^{1/2} t n^{3/2}\ell)$ in the star region, $r(\delta,t,n) = \Theta(\delta t^2 n^2 \ell)$ in the hub region and $r(\delta,t,n) = \Theta(\delta^{2/3} t^2 n^2 \ell)$ in the clique region.

Let us present a figure to summarize the different causes of triangle deviation in each region. Consider $t = n^\gamma$ and $\delta = n^\theta$. For each $\gamma \in (-1/2, 0)$ we obtain results for deviations between the order of magnitude of the standard deviation, $t^{3/2}n^{3/2}$, and the order of magnitude of the largest possible deviation, $t^{3/2}n^3$. Recalling that we consider deviations of size $\delta t^3 n^3$, our main result consider values of $\theta \in (-3/2 - 3\gamma/2, -3\gamma/2)$.

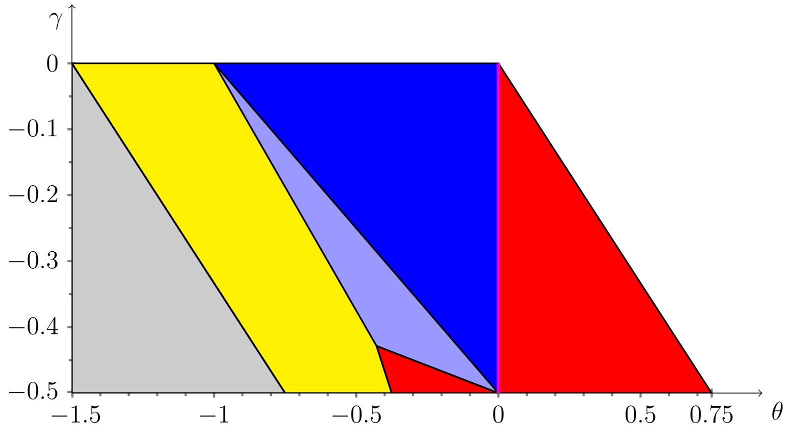

Figure 1.1: The grey regions corresponds to very small deviations, with order of magnitude smaller than the order of the standard deviation. The purple line corresponds to the traditional large deviation results. The other colours cover the regions from our main result, with each representing a different "cause" for the deviation. In the yellow region: Good luck without a structural cause. In the light blue region: A star. In the dark blue region: A hub. In the red regions: A clique.

Our method is also applied to cherry counts in random graphs. This is simpler than the triangle case, as there are only three regimes: normal, star and hub. The respective functions for cherry count deviations are:

$$\begin{aligned} \text{NORMAL}_\wedge(b,t) &:= b^{1/2} t n^{3/2} \\ \text{STAR}_\wedge(b,t) &:= \frac{b^2}{\ell^2} 1_{b<n\ell} \end{aligned}$$

and

$$\text{HUB}_\wedge(b,t) := \frac{bn}{\ell} 1_{b \geqslant n\ell}.$$

As in the triangle case, we let

$$M_\wedge(b,t) = \max\{\text{NORMAL}_\wedge(b,t), \text{STAR}_\wedge(b,t), \text{HUB}_\wedge(b,t)\}.$$

and define $\text{DEV}_\wedge(b,t)$ as the minimal value of a such that

$$\mathbb{P}\left(N_\wedge(b,t) > \mathbb{E}\left[N_\wedge(b,t)\right] + a\right) \leqslant e^{-b}.$$

The next theorem shows that, up to a multiplicative constant, $M_\wedge(b,t)$ is the cherry count deviation in $G(n,m)$ which has probability e^{-b} across a large range of t and b.

Theorem 1.4. *There exist absolute constants c, C such that the following holds. Suppose that $2n^{-1}\log n \leqslant t \leqslant 1/2$ and that $3\log n \leqslant b \leqslant tn^2\ell$. Then*

$$cM_\wedge(b,t) \leqslant DEV_\wedge(b,t) \leqslant CM_\wedge(b,t)\,.$$

1.3 Freedman's inequality

In the second part of this thesis, we obtain a strenghtening of the well-known Freedman's inequality. This is a concentration inequality for martingales, i.e., an inequality about the probability of deviation of a martingale. Perhaps the most famous of these types of inequalities is Hoeffding-Azuma [20,21], which says the following: if $(S_i)_{i=0}^m$ is a martingale with bounded increments $(X_i)_{i=1}^m$ such that $|X_i| \leqslant c_i$ for some real numbers c_i then, for $a > 0$ we have

$$\mathbb{P}\left(S_m - S_0 > a\right) \leqslant \exp\left(\frac{-a^2}{2\sum_{i=1}^m c_i^2}\right).$$

Note that the bound given by this inequality depends on $||X_i||_\infty^2$. On the other hand, Freedman [22] proved a inequality in which the bounds depend on the conditional variance of the increments, $\mathbb{E}\left[X_i^2|\mathcal{F}_{i-1}\right]$. We assume that $|X_i| \leqslant R$ for some constant R and let

$$T_n := \sum_{i=1}^n \mathbb{E}\left[X_i^2|\mathcal{F}_{i-1}\right].$$

Then, for $a, b > 0$ we have

$$\mathbb{P}\left(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n\right) \leqslant \exp\left(\frac{-a^2}{2b + aR}\right).$$

The intuition behind Freedman's inequality is that the sum of conditional variances of the increments works as a way of measure the time necessary to cross a certain height $a > 0$. In our result, which we state below, we impose an additional symmetry condition on the increments X_i.

Theorem 1.5. *Let $m \in \mathbb{N}$. Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. Suppose that there exists $R \in \mathbb{R}$ so that $|X_i| \leqslant R$ a.s. for all i. Assume that, for each i, there are real numbers ε_i so that $|\mathbb{E}\left[X_i^3|\mathcal{F}_{i-1}\right]| \leqslant \varepsilon_i$. If $0 < a \leqslant 2b$ then*

$$\mathbb{P}\left(S_n - S_0 \geqslant a \textit{ and } T_n \leqslant b \textit{ for some } n \leqslant m\right) \leqslant \exp\left(\frac{-a^2}{2b} + \frac{\xi a^3}{6R^3b^3} + \frac{a^4}{12R^2b^3}\right)$$

where $\xi = \sum_{i=1}^{m} \varepsilon_i$.

Freedman [22] also proved a corresponding lower bound for his inequality, but in a slightly different way. It says that, for all $a, b > 0$, we have

$$\mathbb{P}\left(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n\right) \geqslant \frac{1}{2}\exp\left(\frac{-(1+4\delta)a^2}{2b}\right) - \mathbb{P}\left(T_\infty < b\right).$$

where $\delta > 0$ is minimal such that $b/a > 9R\delta^{-2}$ and $a^2/b > 16\delta^{-2}\log(64\delta^{-2})$.

We also obtain a strenghtening of the lower bound of Freedman's inequality, imposing the same symmetry condition to get the upper bound.

Theorem 1.6. *Let* $m \in \mathbb{N}$. *Let* S_i *be a martingale with increments* X_i *with respect to a filtration* $\mathcal{F}_i$. *Suppose that* $|X_i| \leqslant 1$ *a.s. for all* i *and that there are* ε_i *so that* $|\mathbb{E}\left[X_i^3|\mathcal{F}_{i-1}\right]| \leqslant \varepsilon_i$. *If* $2 < a \leqslant b/8$, $a/b < \min\{\varepsilon_i/3 : 1 \leqslant i \leqslant 3m\}$ *and* $8b < a^2$ *then*

$$\mathbb{P}\left(S_n - S_0 \geqslant a \textit{ and } T_n \leqslant b \textit{ for some } n \leqslant m\right) \geqslant \frac{1}{2}\exp\left(\frac{-(1+\eta)a^2}{2b}\right) - \mathbb{P}\left(T_m < b\right).$$

where $\gamma = \sum_{i=1}^{3m} \varepsilon_i$ *and* $0 < \eta < 1/16$ *is minimal such that* $b^2/a > 36\gamma\eta^{-1}$, $b^2/a^2 > 108\eta^{-2}$ *and* $a^2/b > 180\eta^{-2}\log(90\eta^{-2})$.

Let us give an example that shows that our result is actually stronger than the original inequality. Consider a martingale S_i with increments X_i with respect to a filtration $\mathcal{F}_i$ such that $\mathbb{E}\left[X_i^2|\mathcal{F}_{i-1}\right] = \sigma^2$ and that $\mathbb{E}\left[X_i^3|\mathcal{F}_{i-1}\right] = 0$ for all i. Also, suppose that $|X_i| \leqslant 1$ for all i. Let $b > 0$ be a real number. Applying our version of Freedman's inequality with $a = b^{3/4}$ and $m = cb$ for positive constant c gives

$$\mathbb{P}\left(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n \leqslant m\right) \leqslant \exp\left(\frac{-b^{3/2}}{2b} + \frac{\xi}{6b^{3/4}} + \frac{1}{12}\right)$$

Choosing $\varepsilon_i = 3a/b$ (as this condition is necessary for the lower bound), we have $\xi = 3cb^{3/4}$ and so the upper bound above is at most

$$\exp\left(\frac{-b^{1/2}}{2} + C\right)$$

for some $C > 0$. On the other hand, the original Freedman's inequality gives

$$\mathbb{P}\left(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n\right) \leqslant \exp\left(\frac{-b^{3/2}}{2b(1+b^{-1/4})}\right)$$

We note that the exponential above satisfy

$$\exp\left(\frac{-b^{3/2}}{2b(1+b^{-1/4})}\right) \geqslant \exp\left(\frac{-b^{1/2}}{2} + \frac{b^{1/4}}{6} - \frac{1}{2}\right)$$

which is larger than the exponential given by Theorem 1.5.

1.4 Layout of the thesis

We now give a brief description of the content of each chapter in this thesis. In Chapter 2, we introduce some basic notations. We also present the definitions of conditional expectation and martingale along with some results. We also recall some concentration inequalities, as Azuma-Hoeffding and Freedman. We finish this chapter presenting another inequalities that will be used later on in the thesis.

In Chapter 3, we give a brief summary of the different bounds on triangle counts deviations that may be obtained through different methods. We also show how to obtain two of these bounds using well-known concentration inequalities.

In Chapter 4, we present a setup of our problem on cherry and triangle counts deviations. We introduce the different notations that will be used throughout this thesis. We also give a martingale representation for the deviations. We conclude this chapter describing briefly our method.

In Chapter 5, we present results about degrees in $G(n,m)$. We establish bounds on the number of vertices of large degree. We also bound the sum of squares of degress in $G(n,m)$ and their deviations. We use these results to bound the conditional variance of the martingale increments $X_\wedge(G_i)$.

In Chapter 6, we present similar results about codegrees in $G(n,m)$. In particular, we bound the number of vertices with large codegree deviation in $G(n,m)$. We also bounds the sum of squares of codegree deviations in $G(n,m)$. We finish this chapter using these results to bound the conditional variance of the martingale increments $X_\triangle(G_i)$.

In Chapter 7, we prove the upper bound of Theorem 1.4 which is about cherry counts deviations. This works as a warm-up for the triangle counts deviations. Although similar ideas are used in both proofs, the proof for cherries requires less work.

In Chapter 8, we prove the upper bound of Theorem 1.3 which is our main result about triangle deviations in $G(n,m)$. We truncate the martingale increments and use Freedman's inequality for the truncated part. Then we use our results from Chapter 5 and 6 to bound the non-truncated part.

In Chapter 9, we prove the lower bound of Theorem 1.4and Theorem 1.3. In the normal regime, we use the converse Freedman's inequality. In the other regimes, we present an explicit construction that gives the required bound.

In Chapter 10, we prove Theorem 1.1 and Theorem 1.2 which is our maing result about triangle deviations in $G(n, p)$.

Finally, we prove both Theorem 1.5 and Theorem 1.6 in Chapter 11. These are versions of Freedman's inequalities with additional symmetry conditions on the martingale increments.

2
Preliminaries

In this chapter we present some basic definitions and results that will be useful troughout this thesis.

2.1
Asymptotic notation

In this thesis, we often deal with sequences that depend on a parameter n with $n \to \infty$. Let us establish some notations. Let $f(n)$ and $g(n)$ be functions. We write $f = o(g)$ or $f \ll g$ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0\,.$$

We also write $f = O(g)$ if there exists a constant $C > 0$ such that

$$|f(n)| \leqslant C|g(n)|$$

for n large enough. We write $f = \Omega(g)$ if there exists a constant $c > 0$ such that

$$|f(n)| \geqslant c|g(n)|\,.$$

for n large enough. Finally, we write $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$ simultaneously, i.e., if there are constants $c, C > 0$ such that

$$cg(n) \leqslant f(n) \leqslant Cg(n)\,.$$

2.2
Martingales

Let us state the definition of conditional expectation.

Definition 2.1. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space, $\mathcal{F} \subseteq \mathcal{G}$ be a sigma-algebra and $X \in \mathcal{G}$ with $\mathbb{E}\left[|X|\right] < \infty$. The conditional expectation of X given $\mathcal{F}$, denoted by $\mathbb{E}\left[X|\mathcal{F}\right]$, is any $\mathcal{F}$-measurable random variable Y such that

$$\int_A XdP = \int_A YdP$$

for all $A \in \mathcal{F}$.

Clearly, $\mathbb{E}[X|\mathcal{F}]$ is integrable as $\mathbb{E}[|\mathbb{E}[X|\mathcal{F}]|] \leqslant \mathbb{E}[|X|]$. Moreover, $\mathbb{E}[X|\mathcal{F}]$ always exists and it is unique almost surely (the proof of this fact can be found in [23]). Below we give additional properties of the conditional expectation, which proofs can also be found in [23].

Theorem 2.2. *Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and $\mathcal{F} \subseteq \mathcal{G}$ be a sigma-algebra. Also, let $X \in \mathcal{G}$ and $Y \in G$. The following holds.*

(i) (Linearity) If $a \in \mathbb{R}$, $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$ then $\mathbb{E}[aX + Y|\mathcal{F}] = a\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}]$ a.s.

(ii) (Monotonicity) If $\mathbb{E}[|X|] < \infty$, $\mathbb{E}[|Y|] < \infty$ and $X \leqslant Y$ a.s. then $\mathbb{E}[X|\mathcal{F}] \leqslant \mathbb{E}[Y|\mathcal{F}]$ a.s.

(iii) (Monotonous Convergence Theorem) If X_n is a sequence such that $X_n \geqslant 0$ and $X_n \uparrow X$ with $\mathbb{E}[X] < \infty$ then $\mathbb{E}[X_n|\mathcal{F}] \uparrow \mathbb{E}[X|\mathcal{F}]$ a.s.

(iv) (Jensen's Inequality) If f is a convex function, $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|f(X)|] < \infty$ then $f(\mathbb{E}[X|\mathcal{F}]) \leqslant \mathbb{E}[f(X)|\mathcal{F}]$ a.s.

(v) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both sigma-algebras such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{G}$ then $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_1]] = \mathbb{E}[X|\mathcal{F}_1] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_2]|\mathcal{F}_1]$.

(vi) If $X \in \mathcal{F}$, $\mathbb{E}[|Y|] < \infty$ and $\mathbb{E}[|XY|] < \infty$ then $\mathbb{E}[XY|\mathcal{F}] = X\mathbb{E}[Y|\mathcal{F}]$.

We are now in position to define a martingale.

Definition 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_n$ be a filtration, i.e., a sequence of sigma-algebras contained in $\mathcal{F}$ such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all n. A martingale with respect to $\mathcal{F}_n$ is a sequence of random variables X_n such that

(i) $\mathbb{E}[|X_n|] < \infty$,

(ii) $X_n \in \mathcal{F}_n$ and

(iii) $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ for all n.

Moreover, if we change condition (iii) to $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leqslant X_n$ or $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geqslant X_n$, the sequence X_n is called a supermartingale or submartingale, respectively.

The following result is a direct consequence of the definition above.

Theorem 2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_n$ be a filtration.*

(i) If X_n is a supermartingale with respect to $\mathcal{F}_n$ and $n > m$ then $\mathbb{E}[X_n|\mathcal{F}_m] \leqslant X_m$.

(ii) If X_n is a submartingale with respect to $\mathcal{F}_n$ and $n > m$ then $\mathbb{E}[X_n|\mathcal{F}_m] \geqslant X_m$.

(iii) If X_n is a martingale with respect to $\mathcal{F}_n$ and $n > m$ then $\mathbb{E}[X_n|\mathcal{F}_m] = X_m$.

A natural example of a martingale arises in the following situation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider a sequence of random variables X_n. A natural filtration is defined by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ with $X_0 = 0$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Suppose that $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = 0$ and define $S_n = X_1 + X_2 + \cdots + X_n$. Then S_n is a martingale, since $\mathbb{E}[S_n|\mathcal{F}_{n-1}] = S_{n-1} + \mathbb{E}[X_n|\mathcal{F}_{n-1}]$. Clearly if we change the assumption to $\mathbb{E}[X_n|\mathcal{F}_{n-1}] \leqslant 0$ or $\mathbb{E}[X_n|\mathcal{F}_{n-1}] \geqslant 0$ then S_n is a supermartingale or a submartingale, respectively.

Throughout the rest of this section, we always consider a filtration $\mathcal{F}_n$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that a sequence of variables H_n is predictable (with respect to $\mathcal{F}_n$) if $H_n \in \mathcal{F}_{n-1}$ for all n. We also write

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1})$$

Theorem 2.5. *Let X_n be a supermartingale (or a submartingale) and H_n be a bounded predictable sequence with $H_n \geqslant 0$ for all n. Then $(H \cdot X)_n$ is a supermartingale (or a submartingale).*

We say that a random variable N is a stopping time if $\{N = n\} \in \mathcal{F}_n$ for all n. Using the theorem above with $H_n = 1_{\{N \geqslant n\}}$, we obtain the following result.

Theorem 2.6. *If N is a stopping time and X_n is a supermartingale (or a submartingale), then $X_{N \wedge n}$ is a supermartingale (or a submartingale).*

We are now ready to present the martingale convergence theorem.

Theorem 2.7. *If X_n is a submartingale with $\sup \mathbb{E}[X_n^+] < \infty$ then X_n converges a.s. to a limit X as $n \to \infty$. Moreover, $\mathbb{E}[|X|] < \infty$.*

Many times in this thesis we are more interested in the following special case.

Theorem 2.8. *If $X_n \geqslant 0$ is a supermartingale then X_n converges a.s. to a limit X as $n \to \infty$. Moreover, $\mathbb{E}[X] \leqslant \mathbb{E}[X_0]$.*

Finally, the next property will also be useful in this thesis.

Theorem 2.9. *Let N be a stopping time with $N \leqslant k$ a.s. for some $k \in \mathbb{R}$.*

(i) If X_n is a submartingale then $\mathbb{E}[X_0] \leqslant \mathbb{E}[X_N] \leqslant \mathbb{E}[X_k]$.

(ii) If X_n is a supermartingale then $\mathbb{E}[X_0] \geqslant \mathbb{E}[X_N] \geqslant \mathbb{E}[X_k]$.

2.3 Concentration inequalities

In this section we present some important concentration inequalities that will be used later on in this thesis.

We begin with the following Chernoff bounds which are derived from [24].

Theorem 2.10. *Let X be a binomial or hypergeometric random variable and let $\mu = \mathbb{E}[X]$. Then, for all $a > 0$, we have*

$$\mathbb{P}(X \geqslant \mu + a) \leqslant \exp\left(\frac{-a^2}{2\mu + 2a/3}\right) \tag{2-1}$$

and

$$\mathbb{P}(X \leqslant \mu - a) \leqslant \exp\left(\frac{-a^2}{2\mu}\right). \tag{2-2}$$

For $\theta \geqslant e$ we have

$$\mathbb{P}(X \geqslant \theta\mu) \leqslant \exp(-\theta\mu(\log(\theta) - 1)). \tag{2-3}$$

Consequently, for $j \geqslant 3$ and any $\nu \geqslant \mu$ we have

$$\mathbb{P}\left(X \geqslant 2^j\nu\right) \leqslant \exp(-j2^{j-2}\nu). \tag{2-4}$$

There are some useful concentration inequalities in the context of martingales. The following is called Hoeffding-Azuma inequality.

Theorem 2.11. *Let $(S_i)_{i=0}^m$ be a martingale with bounded increments $(X_i)_{i=1}^m$. Suppose that $|X_i| \leqslant c_i$ for all $i \leqslant m$. Then, for all $a > 0$, we have*

$$\mathbb{P}(S_m - S_0 > a) \leqslant \exp\left(\frac{-a^2}{2\sum_{i=1}^m c_i^2}\right). \tag{2-5}$$

If the increments are not close to the supremum the bound above is not as good as the following, which is called Freedman's inequality. This appeared first on [22].

Let S_i be a sequence of random variables, $\mathcal{F}_i$ be a filtration and $X_i := S_i - S_{i-1}$ for $i \geqslant 1$. If $\mathbb{E}[X_n|\mathcal{F}_{n-1}] = 0$ for each $i \geqslant 1$ then S_i is a martingale with increments X_i with respect to the filtration $\mathcal{F}_i$. For such a martingale we also define

$$T_n := \sum_{i=1}^n \mathbb{E}\left[X_i^2|\mathcal{F}_{i-1}\right]$$

for $n \in \mathbb{N} \cup \{+\infty\}$.

Theorem 2.12. *Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. Suppose that there exists $R \in \mathbb{R}$ so that $|X_i| \leqslant R$ a.s. for all i. For every $a, b > 0$, we have*

$$\mathbb{P}\left(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n\right) \leqslant \exp\left(\frac{-a^2}{2b + aR}\right).$$

Remark 2.13. It is possible to replicate the proof of Theorem 2.12 for the case where S_i is a supermartingale. We shall use this later on in this thesis.

There is also a lower bound provided by Freedman's inequality. For this, we let $a > 0$ and define τ_a to be the minimum value of n so that $S_n - S_0 > a$ (with $\tau_a = \infty$ if this event does not happen). Then, let

$$W_a := T_{\tau_a} = \sum_{n=1}^{\tau_a} \mathbb{E}\left[X_n^2 | \mathcal{F}_{n-1}\right].$$

We note that W_a is the total conditional variance that the process requires to cross over the value a, if it crosses. Otherwise, W_a is just the total conditional variance of the process. Therefore, the upper bound of Freedman's inequality implies that

$$\mathbb{P}\left(W_a < b\right) \leqslant \exp\left(\frac{-a^2}{2b + aR}\right).$$

Theorem 2.14. *Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. Suppose that there exists $R \in \mathbb{R}$ so that $|X_i| \leqslant R$ a.s. for all i and let W_a be defined as above. For every $a, b > 0$, we have*

$$\mathbb{P}\left(W_a < b\right) \geqslant \frac{1}{2} \exp\left(\frac{-(1+4\delta)a^2}{2b}\right).$$

where $\delta > 0$ is minimal such that $b/a > 9R\delta^{-2}$ and $a^2/b > 16\delta^{-2}\log(64\delta^{-2})$.

Note that the lower bound of Freedman's inequality implies that

$$\mathbb{P}\left(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n\right) \geqslant \frac{1}{2} \exp\left(\frac{-(1+4\delta)a^2}{2b}\right) - \mathbb{P}\left(T_\infty < b\right).$$

We may frequently apply both Freedman's inequalities to finite martingales. To see that this is possible for a finite martingale $(S_i)_{i=0}^m$, one may define $S_i = S_m$ for all $i > m$.

We shall also use a corollary of Freedman's inequality applied to the $G(n, m)$ setting. Let $\mathcal{G}_{n,m}$ be the family of graphs with n vertices and m edges.

We say that two graphs G, G' are adjacent if there exists a pair $e, e' \in E(K_n)$ so that $G = G' \setminus \{e\} \cup \{e'\}$. Given a function $\psi : E(K_n) \to \mathbb{R}^+$ we say that a function $f : \mathcal{G}_{n,m} \to \mathbb{R}$ is ψ-Lipschitz if for every adjacent pair of graphs $G, G' \in \mathcal{G}_{n,m}$ we have

$$\left| f(G) - f(G') \right| \leqslant \psi(e) + \psi(e')$$

where $G \triangle G' = \{e, e'\}$.

Theorem 2.15. *Let $G_m \sim G(n,m)$ with $t = m/N$ where $N = n(n-1)/2$. Given $\psi : E(K_n) \to \mathbb{R}$ and a ψ-Lipschitz function $f : \mathcal{G}_{n,m} \to \mathbb{R}$, we have*

$$\mathbb{P}\left(f(G_m) - \mathbb{E}\left[f(G_m)\right] \geqslant a\right) \leqslant \exp\left(\frac{-a^2}{24t||\psi||^2 + 6a\psi_{max}}\right)$$

for all $a \geqslant 0$, where $||\psi||^2 = \sum_{e \in E(K_n)} \psi(e)^2$ and $\psi_{max} := \max_e \psi(e)$. Furthermore, the same bound holds for $\mathbb{P}\left(f(G_m) - \mathbb{E}\left[f(G_m)\right] \leqslant -a\right)$.

Proof. We may assume that $m \leqslant N/2$, as we can see f as a function of the complimentary graph $G_m^c \sim G_{N-m}$ for $m > N/2$. Also, the "Furthermore" statement follows by applying the inequality to $-f$.

Let $e_1, \ldots, e_N$ be a uniform random ordering of the edges of K_n and for each i define G_i to be the graph with edges $\{e_1, \ldots, e_i\}$. This generates the Erdős-Rényi random process. To prove our result, we consider the martingale

$$Z_i := \mathbb{E}\left[f(G_m)|G_i\right]$$

with $i = 0, 1, \ldots, m$. We shall prove later that

$$\left| Z_i - Z_{i-1} \right| \leqslant \psi(e_i) + \frac{2}{N} \sum_{e \in E(K_n)} \psi(e)\,. \tag{2-6}$$

This inequality gives

$$\left| Z_i - Z_{i-1} \right| \leqslant 3\psi_{\max}$$

and

$$\begin{aligned}
\mathbb{E}\left[(Z_i - Z_{i-1})^2 | G_{i-1}\right] &\leqslant \mathbb{E}\left[\left(\psi(e_i) + \frac{2}{N} \sum_{e \in E(K_n)} \psi(e)\right)^2 | G_{i-1}\right] \\
&\leqslant 2\mathbb{E}\left[\psi(e_i)^2 | G_{i-1}\right] + \frac{8}{N^2}\left(\sum_{e \in E(K_n)} \psi(e)\right)^2 \\
&\leqslant \frac{4}{N} \sum_{e \in E(K_n)} \psi(e)^2 + \frac{8}{N} \sum_{e \in E(K_n)} \psi(e)^2 \\
&= \frac{12}{N} ||\psi||_2^2\,.
\end{aligned}$$

where the last inequality above follows from the definition of conditional expectation. Summing the inequality above over $i \leqslant m$ we obtain

$$\sum_{i=1}^{m} \mathbb{E}\left[(Z_i - Z_{i-1})^2 | G_{i-1}\right] \leqslant \frac{12m}{N}||\psi||_2^2 = 12t||\psi||_2^2 .$$

We may now apply the upper bound of Freedman's inequality (Theorem 2.12) with $\beta = 12t||\psi||_2^2$ and $R = 3\psi_{\max}$ to deduce that

$$\begin{aligned} \mathbb{P}\left(f(G_m) - \mathbb{E}\left[f(G_m)\right] \geqslant a\right) &= \mathbb{P}\left(Z_m - Z_0 \geqslant a\right) \\ &\leqslant \exp\left(\frac{-a^2}{24t||\psi||_2^2 + 6a\psi_{\max}}\right), \end{aligned}$$

as required.

We may now prove (2-6), i.e., that we have $|Z_i - Z_{i-1}| \leqslant \psi(e_i) + 2\sum_{e \in E(K_n)} \psi(e)/N$ for every sequence of edges $(e_1, \ldots, e_i)$. By definition, we can write Z_{i-1} and Z_i as sums over the choices of the edges up to e_m. Indeed, we have

$$Z_i = \frac{1}{(N-i)_{m-i}} \sum_{e_{i+1},\ldots,e_m} f(G_i \cup \{e_{i+1}, \ldots, e_m\})$$

and

$$Z_{i-1} = \frac{1}{(N-i+1)_{m-i+1}} \sum_{f_i, f_{i+1},\ldots,f_m} f(G_{i-1} \cup \{f_i, f_{i+1}, \ldots, f_m\})$$

where both sums above are over sequences of distinct edges of K_n disjoint from those already selected. We would like to pair up the terms in such a way that the graphs $G_i \cup \{e_{i+1}, \ldots, e_m\}$ and $G_{i-1} \cup \{f_i, f_{i+1}, \ldots, f_m\}$ are either equal or adjacent. The obvious problem is that the sums do not even have the same number of terms. We introduce a dummy edge g in the first sum, which may be any edge of $K_n \setminus \{e_1, \ldots, e_{i-1}\}$. We have

$$Z_i = \frac{1}{(N-i+1)_{m-i+1}} \sum_{e_{i+1},\ldots,e_m; g} f(G_i \cup \{e_{i+1}, \ldots, e_m\}) .$$

We may now pair up terms of the two summations on a one-to-one basis. Let $\mathcal{S}$ be the sequences of edges allowed in the above summation and $\mathcal{T}$ the sequences allowed in the summation for Z_{i-1}. We define a bijection $\phi : \mathcal{S} \to \mathcal{T}$ as follows:

$$\phi(e_{i+1}, \ldots, e_m; g) := \begin{cases} (e_i, e_{i+1}, \ldots, e_m) & \text{if } g = e_i \\ (e_{i+1}, \ldots, e_{j-1}, e_i, e_{j+1}, \ldots, e_m) & \text{if } g \in \{e_{i+1}, \ldots, e_m\} \\ (g, e_{i+1}, \ldots, e_m) & \text{if } g \notin \{e_i, \ldots, e_m\} \end{cases}$$

Note that in all cases the graphs $G_{i-1} \cup \{e_i, \ldots, e_m\}$ and $G_{i-1} \cup \phi(e_{i+1}, \ldots, e_m; g)$ are either equal or adjacent (with symmetric difference $\{e_i, g\}$). Now, by the triangle inequality,

$$\begin{aligned}\left|Z_i - Z_{i-1}\right| &\leqslant \frac{1}{(N-i+1)_{m-i+1}} \sum_{e_{i+1},\ldots,e_m;g} \left|f(G_{i-1} \cup \{e_i, \ldots, e_m\}) - f(G_{i-1} \cup \phi(e_{i+1}, \ldots, e_m; g))\right| \\ &\leqslant \frac{1}{(N-i+1)_{m-i+1}} \sum_{e_{i+1},\ldots,e_m;g} \psi(e_i) + \psi(g) \\ &\leqslant \psi(e_i) + \frac{1}{N-i+1} \sum_{g \in E(K_n)} \psi(g) \\ &\leqslant \psi(e_i) + \frac{2}{N} \sum_{g \in E(K_n)} \psi(g).\end{aligned}$$

This completes the proof. □

2.4 Another useful inequalities

In this section, we present some other inequalities that will be used later on in this thesis.

The following is a basic lemma from Measure Theory called Fatou's lemma. Its proof can be found in [23].

Lemma 2.16. *Let X_n be a sequence of non-negative random variables. Then*

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[X_n\right].$$

The next result compares probabilities of sum of Bernoulli random variables (not necessarily independent) with binomial random variables. This seems a fairly standard result that may be found in [25], for example.

Lemma 2.17. *Let $X_1, \ldots, X_m$ be Bernoulli random variables such that for each $1 \leqslant i \leqslant m$ we have $\mathbb{P}(X_i = 1 | X_1, \ldots, X_{i-1}) \leqslant p_i$. Let $Y_1, \ldots, Y_m$ be independent Bernoulli random variables such that $\mathbb{P}(Y_i = 1) = p_i$ for all $1 \leqslant i \leqslant m$. If $X = \sum_{i=1}^m X_i$ and $Y = \sum_{i=1}^m Y_i$ then $\mathbb{P}(X \geqslant k) \leqslant \mathbb{P}(Y \geqslant k)$ for all $k \in \{0, 1, \ldots, m\}$.*

We also use the following inequality.

Proposition 2.18. *Let $d \in \mathbb{N}$, let $r \geqslant d^{1/2}$ and let $\beta \geqslant 1$. Then,*

$$\sum_{x \in \mathbb{Z}^d : \|x\| \geqslant r} \exp\left(-\beta \|x\|^2\right) \leqslant (8\pi)^{d/2} e^{-\beta(r - d^{1/2})^2/2}.$$

The proof of this proposition uses the following definition. We shall use the euclidean norm in the rest of this section.

Definition 2.19. A function $F : \mathbb{R}^d \to \mathbb{R}$ is *radial* if the value of $F(x)$ depends only on $\|x\|$. If F is radial, let $F_{rad} : \mathbb{R}_0^+ \to \mathbb{R}$ be the function such that $F_{rad}(r) = F(x)$ for all x with $\|x\| = r$.

Given a subset $\mathcal{L} \subseteq \mathbb{Z}^d$ we say that $\mathcal{L}$ has the *non-zero* property if all coordinates x_i of all $x \in \mathcal{L}$ are non-zero.

We prove the following auxiliary proposition.

Proposition 2.20. *Let $r \in \mathbb{R}$ and let $F : \mathbb{R}^d \to \mathbb{R}^+$ be a continuous, integrable, radial function for which F_{rad} is non-increasing. Also, let $\mathcal{L}$ be a subset of $\mathbb{Z}^d$ with the non-zero property. Then,*

$$\sum_{x \in \mathcal{L}: \|x\| \geqslant r} F(x) \leqslant \int_{A(r-d^{1/2})} F(u)du$$

where $A(r) := \mathbb{R}^d \setminus B(0, r)$.

Proof. We assume, without loss of generality, that $\|x\| \geqslant r$ for all $x \in \mathcal{L}$. For each $x \in \mathcal{L}$, we let x_- be the point obtained by reducing the absolute value of each coordinate by 1.

Now, each x define an open cube C_x such that $y \in C_x$ if each coordinate of y is between the corresponding coordinates in x and x_-. By the definition of x_- each cube has volume 1. Moreover, if $x \neq x'$ then the cubes C_x and $C_{x'}$ are disjoint. We also note that x is the point with largest norm in C_x. Since F_{rad} is non-increasing, we have that

$$\sum_{x \in \mathcal{L}} F(x) = \sum_{x \in \mathcal{L}} \int F(u) 1_{C_x}(u)du = \int F(u) 1_{\bigcup_{x \in \mathcal{L}} C_x}(u)du\,.$$

Since the diameter of a unit cube is $d^{1/2}$ and $\|x\| \geqslant r$ for all $x \in \mathcal{L}$, every point in the union has norm at least $r - d^{1/2}$. The required result follows by monotonicity. □

Before we prove Proposition 2.18 we need to deal with the fact that not all $\mathcal{L} \subseteq \mathbb{Z}^d$ have the non-zero property. In this case, we partition the set depending on the subset $S \subseteq [d] = \{1, 2, \ldots, d\}$ of non-zero coordinates, obtaining the following corollary.

Corollary 2.21. *Let $r \in \mathbb{R}$ and let $F : \mathbb{R}^d \to \mathbb{R}^+$ be a continuous, integrable, radial function for which F_{rad} is non-increasing. Then*

$$\sum_{x \in \mathbb{Z}^d: \|x\| \geqslant r} F(x) \leqslant 2^d \max_{S \subseteq [d]} \int_{A_S(r-d^{1/2})} F_S(u)du$$

where $A_S(r) := \mathbb{R}^S \setminus B(0,r)$ *and* F_S *is the restriction of* F *to* $\mathbb{R}^S$.

Proof of Proposition 2.18. We shall apply this result for the function $F(x) = \exp(-\beta \|x\|^2)$, for $\beta \geqslant 1$. Note that it is not difficult to bound $\int_{A(r)} \exp(-\beta \|u\|^2) du$. Indeed, we shall use that for $u \in \mathbb{R}^d$ with $\|u\| \geqslant r$ we have $\|u\|^2 \geqslant r^2/2 + (u_1^2 + \ldots + u_d^2)/2$. Thus,

$$\begin{aligned}\int_{A(r)} \exp(-\beta \|u\|^2) &\leqslant e^{-\beta r^2/2} \int_{A(r)} \exp(-\beta(u_1^2 + \cdots + u_d^2)/2) \\ &\leqslant e^{-\beta r^2/2} \int_{\mathbb{R}^d} \exp(-(u_1^2 + \cdots + u_d^2)/2) du \\ &\leqslant e^{-\beta r^2/2} \prod_{i=1}^{d} \int \exp(-u_i^2) du_i \\ &= (2\pi)^{d/2} e^{-\beta r^2/2}.\end{aligned}$$

Note that in the last two lines we used Fubini and the identity $\int e^{-y^2/2} dy = \sqrt{2\pi}$.

It is clear that the same calculation in an s-dimensional subspace would give the upper bound $(2\pi)^{s/2} e^{-\beta r^2/2}$. The required bound,

$$\sum_{x \in \mathbb{Z}^d : \|x\| \geqslant r} \exp\left(-\beta \|x\|^2\right) \leqslant (8\pi)^{d/2} e^{-\beta(r - d^{1/2})^2/2}$$

now follows from these estimates and Corollary 2.21. This completes the proof of Proposition 2.18. □

3
Some bounds on triangle deviations

In this chapter, we present some bounds on triangle deviations that can be found using famous concentration inequalities, such as Hoeffding-Azuma and Kim-Vu. After comparing some different bounds, we shall see that the bound given by our method is better. The proofs of all these inequalities are found in [26].

We recall that

$$r(\delta, p, n) := -\log \mathbb{P}\left(N_{\triangle}(G_p) > (1+\delta)p^3(n)_3\right)$$

where $G_p \sim G(n,p)$.

3.1
Comparing different methods

Let us present a brief comparison of lower bounds for $r(\delta, p, n)$ using different concentration inequalities. We present the following table, which may be found in [26]. An entry F on this table means that there are constants $c_1 \geqslant 0$ and $c_2 > 0$ such that $\mathbb{P}\left(N_{\triangle}(G_p) > (1+\delta)p^3(n)_3\right) \leqslant O(n^{c_1}) \exp\left(-c_2 F\right)$.

Table 3.1: Lower bounds of $r(\delta, p, n)$ using different inequalities.

Method	Lower bound
Azuma-Hoeffding	$\delta^2 p^6 n^2$
Talagrand	$\delta^2 p^5 n^2$
Kim-Vu	$\delta^{1/3} p^{1/6} n^{1/3}$
Vu	$\delta^2 p n^2$

As we shall see in the next section, Hoeffding-Azuma inequality is Theorem 2.11 [20, 21]. We also have Talagrand inequality which also requires a Lipschitz condition as well as other technical one. This can be found in [27]. In Section 3.3, we present Kim-Vu inequality which appeared in [28]. There is another version of Kim-Vu inequality, more general and more technical, presented in [28]. In the table, the entry corresponding to Vu comes from an inequality proved in [3], which is derived from this general Kim-vu inequality together with an induction.

3.2
Using Hoeffding-Azuma inequality

Here we deduce a lower bound for $r(\delta, p, n)$ using Hoeffding-Azuma inequality (Theorem 2.11). We use a martingale known as the edge-exposure martingale, which is obtained as follows: consider an ordering of all N pairs in K_n and for each $0 \leqslant i \leqslant N$, let $X_i := \mathbb{E}\left[N_\triangle(G_p)|Y_1, \ldots, Y_i\right]$ where each Y_i assumes value 1 if $e_i \in G_p$ and 0 otherwise; also, we note thar $X_0 = \mathbb{E}\left[N_\triangle(G_p)\right]$ and $X_N = N_\triangle(G_p)$. Now, consider the filtration $\mathcal{F}_i = \sigma(Y_1, \ldots, Y_i)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Then $(X_i)_{i=0}^N$ is a martingale with respect to this filtration. Moreover, $|X_i - X_{i-1}| \leqslant 6(n-2)$, since it is at most the number of triangles in the complete graph K_n containing the edge e_i. From Theorem 2.11 applied to the martingale X_n we get

$$\mathbb{P}\left(N_\triangle(G_p) > (1+\delta)p^3(n)_3\right) \leqslant \exp\left(\frac{-\delta^2 p^6 (n)_3^2}{72N(n-2)^2}\right) \leqslant \exp\left(\frac{-\delta^2 p^6 n^2}{72}\right).$$

In particular, Hoeffding-Azuma gives us

$$r(\delta, p, n) \geqslant \frac{\delta^2 p^6 n^2}{72}.$$

3.3
Using Kim-Vu inequality

In this section we present the Kim-Vu inequality [28] and then we use it to deduce a lower bound for $r(\delta, p, n)$.

Let H be a weighted hypergraph with $V(H) = [N]$ for some natural N. Each edge e has some weight $\omega(e)$ and it has at most k vertices for some fixed integer $k > 2$. We also have N numbers $p_i \in [0,1]$ and N independent random variables t_i that could be of two types: either t_i is a Bernoulli variable with expected value p_i or t_i is equal p_i with probability 1. Consider a polynomial

$$Y_H = \sum_{e \in E(H)} \omega(e) \prod_{s \in e} t_s.$$

If e is empty, we let $\prod_{s \in e} t_s = 1$.

In our context of triangle counts in $G(n,p)$, we can think of H as a 3-uniform hypergraph in which $V(H) = E(K_n)$ and a triple $e_1, e_2, e_3 \in E(K_n)$ is an edge of H if and only if it is a triangle in K_n. In this case, every edge of H has weight $w(e) = 1$ and each variable t_i is Bernoulli with expectation p, representing whether or not the edge e_i is in $G(n,p)$. Under this setting, Y_H represents the number of triangles in $G(n,p)$. Note that Y_H does not differ between isomorphic copies of the same triangle, thus $Y_H = N_\triangle(G_p)/6$.

In order to state Kim-Vu inequality, we need to define truncated subhypergraphs. For each $A \subseteq V(H)$, we define H_A as follows: $V(H_A) = V(H) \setminus A$

$$E(H_A) = \{B \subseteq V(H_A) : B \cup A \in E(H)\}$$

and $\omega(B) = \omega(B \cup A)$ for $B \in E(H_A)$. We then let

$$Y_{H_A} = \sum_{e, A \subseteq e} \omega(e) \prod_{i \in e \setminus A} t_i \,.$$

Now let $E_i(H) = \max_{A \subseteq V(H), |A|=i} E(Y_{H_A})$. This should be seen as the maximum effect that a group of i variables can have in the value of Y_H, in average. Finally, we let $E^*(H) = \max_{i \geqslant 0} E_i(H)$ and $E'(H) = \max_{i \geqslant 1} E_i(H)$. We now state Kim-Vu inequality.

Theorem 3.1. *Let H be an hypergraph under the setting defined above. Then, for any $\lambda > 1$,*

$$\mathbb{P}\left(|Y_H - \mathbb{E}\left[Y_H\right]| > a_k (E^*(H)E'(H))^{1/2}\lambda^k\right) = O(\exp(-\lambda + (k-1)\log n))$$

where $a_k = 8^k k!^{1/2}$.

Let us go back to the context of triangles in $G(n,p)$. Let $A \subseteq E(K_n)$. If $|A| = 3$, the graph H_A is empty and thus $Y_{H_A} = 1$. Therefore, $E_3(H) = 1$. If $|A| = 2$, say $A = \{uv, uw\}$, then H_A contains only the edge vw. Since vw is in $G(n,p)$ with probability p, we have $E_2(H) = p$. If $|A| = \{uv\}$ then H_A is 2-uniform with edges $\{uw, vw\}$ for $w \in V(G)$. Thus, H_A has $n-2$ hyper-edges and the probability of both elements of each hyper-edge be in $G(n,p)$ is p^2. Therefore, $E_1(H) = p^2(n-2)$. Finally, $E_0(H) = \mathbb{E}\left[Y_H\right] = p^3\binom{n}{3}$. Assuming that $p \gg n^{-1/2}$ we get $E^*(H) = E_0(H)$ and $E'(H) = E_1(H)$. Using Theorem 3.1 with $k = 3$ and $\lambda = \delta^{1/3}p^{1/6}n^{1/3}$ we get

$$\mathbb{P}\left(N_\triangle(G_p) > (1+\delta)p^3(n)_3\right) = O(\exp(-\delta^{1/3}p^{1/6}n^{1/3} + 2\log n)) \,.$$

This shows that

$$r(\delta, p, n) = \Omega(\delta^{1/3}p^{1/6}n^{1/3})$$

whenever $\delta > 3(\log n)^3 p^{-1/2} n$.

Later on, Kim and Vu [4] proved an extension of their original inequality. This result applied to triangle counts gives

$$r(\delta, p, n) \;=\; \Omega(p^2 n^2)$$

whenever $p \geqslant n^{-1} \log n$ and δ is constant.

4
Setup of the deviations problem

In this thesis, we study triangle deviations in $G(n,m)$ and then deduce results for the model $G(n,p)$. Given a graph G we let $N_H(G)$ denote the number of isomorphic copies of the graph H in the graph G. Given $n \in \mathbb{N}$, we let $N := \binom{n}{2}$. For $m \leqslant N$, the random graph $G(n,m)$ may be obtained by choosing a graph uniformly among all graphs with n vertices and m edges. Sometimes we may also consider the Erdős-Rényi random process $G_i, i = 0, 1, \ldots, m$. We recall that this process is generated by starting with the empty graph and add edges $e_1, e_2, \ldots, e_m$ one at a time uniformly among all possible edges except the ones that have already been chosen. Another way to generate the process G_i is to consider a random uniform permutation $\{e_1, \ldots, e_N\}$of the edges of K_n and then consider G_i as the graph with edge set $\{e_1, \ldots, e_i\}$. It is easily verified that $G_i \sim G(n,i)$ for all $i \leqslant m$.

In order to prove deviation results, we consider the random variable $D_H(G_m) := N_H(G_m) - \mathbb{E}\left[N_H(G_m)\right]$. In this work, we consider the cases where H is a cherry or a triangle. In the rest of this chapter, we will present the basic notations and a martingale representation for this deviation, which will allow us to use Freedman's inequality.

4.1
Basic notations

The notations used here are mostly the same as in [19]. Consider the Erdős-Rényi random process $G_i, i = 0, 1, \ldots, m$ defined above, where G_i has edge set $\{e_1, \ldots, e_i\}$. We write $t := m/N$ for the edge density of G_m and $s := i/N$ for the edge density of G_i.

Let H be a graph with $v(H)$ vertices and $e(H)$ edges. We claim that the expected value of $N_H(G_m)$ is

$$L_H(m) := \mathbb{E}\left[N_H(G_m)\right] = \frac{(n)_{v(H)}(m)_{e(H)}}{(N)_{e(H)}}.$$

In the expression above, the term $(n)_{v(H)}$ comes from the number of choices for the set of vertices of H among the n vertices of the random graph. The rest of the expression is due to the probability of each edge of H being present in the random graph. Thus, we can write the deviation on the number of copies of H in G_m as

$$D_H(G_m) := N_H(G_m) - L_H(m) .$$

For each graph F we also let

$$A_F(G_m) := N_F(G_m) - N_F(G_{m-1})$$

to be the number of copies of F created with the addition of the mth edge. We also define

$$X_F(G_m) := A_F(G_m) - \mathbb{E}\left[A_F(G_m)|G_{m-1}\right] = N_F(G_m) - \mathbb{E}\left[N_F(G_m)|G_{m-1}\right] .$$

Note that the sequence $X_F(G_i)$ works as a sequence of martingales increments, as $\mathbb{E}\left[X_F(G_i)|G_{i-1}\right] = 0$ for each $i = 1, \ldots, m$.

4.2 Martingale representation

The following theorem gives a martingale representation for the deviation $D_H(G_m)$, which was proved in [19].

Theorem 4.1. *Let H be a graph with v vertices and e edges. Then*

$$D_H(G_m) = \sum_{i=1}^{m} \sum_{F \subseteq E(H)} \frac{(N-m)_{e(F)}(m-i)_{e-e(F)}}{(N-i)_e} X_F(G_i) ,$$

where the inner sum is taken over all 2^e graphs F with $V(F) = V(H)$ and $E(F) \subseteq E(H)$.

In this thesis, we are only interested in cherry deviations, $D_\wedge(G_m)$, and triangle deviations, $D_\triangle(G_m)$. Note that $X_F(G_i)$ is 0 if F has 0 edges or 1 edge. Thus,

$$D_\wedge(G_m) = \sum_{i=1}^{m} \frac{(N-m)_2}{(N-i)_2} X_\wedge(G_i)$$

and

$$D_\triangle(G_m) = \sum_{i=1}^{m} \left[3\frac{(N-m)_2(m-i)}{(N-i)_3} X_\wedge(G_i) + \frac{(N-m)_3}{(N-i)_3} X_\triangle(G_i) \right] .$$

4.3 Description of our method

Let us describe briefly the method used to solve our deviation problem in $G(n, m)$. We focus on the triangle case, as the cherry case requires only a simplified version of the same method.

Recall that we think the problem in terms of the associated rate e^{-b}. To prove the upper bound of Theorem 1.3 we need to show that there exists a constant C such that

$$\mathbb{P}\left(D_{\triangle}(G_m) > CM(b,t)\right) \leqslant \exp(-b)$$

for all $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant tn^2\ell$ where $M(b,t) = \max\{\text{NORMAL}(b,t), \text{STAR}(b,t), \text{HUB}(b,t), \text{CLIQUE}(b,t)\}$.

As we have just seen a martingale representation for $D_{\triangle}(G_m)$ the best idea would be to apply Freedman's inequality to obtain the result above. However, we shall not go on with this idea directly because the martingale increments $X_{\wedge}(G_i)$ and $X_{\triangle}(G_i)$ could be very large. To fix this, we will introduce truncated versions of the increments, $X'_{\wedge}(G_i)$ and $X'_{\triangle}(G_i)$. Then we let

$$D'_{\triangle}(G_m) = \sum_{i=1}^{m}\left[3\frac{(N-m)_2(m-i)}{(N-i)_3}\,X'_{\wedge}(G_i) + \frac{(N-m)_3}{(N-i)_3}\,X'_{\triangle}(G_i)\right].$$

where $D_u(G_{i-1})$ represents the degree deviation of u in the graph G_{i-1} and $D_{uw}(G_{i-1})$ represents the codegree deviation of the pair uw in G_{i-1}. We define the value to truncate so that the increments of $D'_{\triangle}(G_m)$ are at most $M(b,t)/b$. Also, $D'_{\triangle}(G_m)$ is a supermartingale which allow us to apply Freedman's inequality. We also need to control the conditional variance of the increments. We shall observe that

$$\mathbb{E}\left[X^2_{\wedge}(G_i)|G_{i-1}\right] \leqslant \frac{32}{n}\sum_{u} D_u(G_{i-1})^2\,.$$

and

$$\mathbb{E}\left[X^2_{\triangle}(G_i)|G_{i-1}\right] \leqslant \frac{32}{n^2}\sum_{uw} D_{uw}(G_{i-1})^2$$

An extensive part of our work resides on studying these degree and codegree deviations. After doing that, we may prove that the conditional variance of the increments are also at most $M(b,t)/b$, except with probability at most $\exp(-b)$. We then apply Freedman's inequality to $D'_{\triangle}(G_m)$, obtaining the desired bound for the truncated part.

We let $Z_{\wedge}(G_i) = X_{\wedge}(G_i) - X'_{\wedge}(G_i)$ and $Z_{\triangle}(G_i) = X_{\triangle}(G_i) - X'_{\triangle}(G_i)$, i.e., the non-truncated part of the original martingale increments. We then need to control the quantity

$$N^*_{\triangle}(G_m) = \sum_{i=1}^{m}\left[3\frac{(N-m)_2(m-i)}{(N-i)_3}\,Z_{\wedge}(G_i) + \frac{(N-m)_3}{(N-i)_3}\,Z_{\triangle}(G_i)\right].$$

Each parcel of the sum above has two parts, the first one being at most $tZ_\wedge(G_i)$ and the second being at most $Z_\triangle(G_i)$. We shall prove that each of them are at most $M(b,t)$ except with probability at most $\exp(-b)$. For this, we will use again the results derived from our analysis of degrees and codegrees in $G(n,m)$. Indeed, bounding $\sum_i tZ_\wedge(G_i)$ is equivalent to bounding the number of triangles that are created by edges in which at least one vertex has large degree. We shall see that this sum may be bounded by the sum of squared degree deviations. Moreover, bounding $\sum_i Z_\triangle(G_i)$ is equivalent to bounding the number of triangles that are created by edges with large codegree. To bound this sum, we divide the edges into two categories: the first one contains edges for which its large codegree is caused by a large degree of one of its vertices; the second one contains the other edges. For the first category of edges, $\sum_i Z_\triangle(G_i)$ is bounded using the same tools that we use to bound $\sum_i tZ_\wedge(G_i)$. For the second category of edges, we use the results obtained in our analysis of codegrees.

5
Degrees in $G(n,m)$

In this chapter, we present the results that we need about degrees in $G(n,m)$.

Let $d_u(G)$ be the degree of a vertex u in a graph G and note that the expectation of $d_u(G_m)$ is $t(n-1) = 2m/n$. Then

$$D_u(G_m) \;:=\; d_u(G_m) - \frac{2m}{n}\,.$$

is the deviation of the degree of u in G_m from its mean.

Note that the expectation of $d_u(G_m)$ is $t(n-1)$, which is of order tn. We will prove bounds related to the number of vertices of large degree (at least $2^j tn$ for each $j \geqslant 5$). We will also bound the maximum degree $\Delta(G_m)$ of G_m. We then prove bounds related to the sum of squares of degree deviations $\sum_u D_u(G_m)^2$. Finally, we present a bound on the conditional variance of the increments $X_\wedge(G_i)$.

5.1
Maximum degree and the number of vertices of large degree

Let us introduce some notation that will be useful to state our bounds. We set $\ell_b := \log(b/etn)$. We also let V_j be the set of vertices in $G_m \sim G(n,m)$ with degree at least $2^j tn$.

Lemma 5.1. *Suppose* $t \geqslant 2n^{-1}\log n$ *and let* $b \geqslant 4tn$*. Then, except with probability at most* $\exp(-b)$*, we have*

(i) $\Delta(G_m) \leqslant 2b/\ell_b$*, and*

(ii) $|V_j| \leqslant b/tnj2^{j-6}$ *for all* $j \geqslant 5$.

Proof. Fix $t \geqslant 2n^{-1}\log n$ and $b \geqslant 4tn$. For the first item, let $a = 2b/\ell_b$. For each vertex u the probability that $d_u(G_m) \geqslant a$ is at most

$$\binom{n}{a} t^a \;\leqslant\; \exp\left(-a\log(a/etn)\right) \;\leqslant\; \exp\left(\frac{-2b}{\ell_b}(\ell_b - \log \ell_b)\right) \;\leqslant\; \exp(-3b/2)\,.$$

Since $b \geqslant 8\log n$ the quantity above is at most $\exp(-b)/2n$. Taking an union bound over all $u \in V(G)$ we get

$$\mathbb{P}\left(\Delta(G_m) \geqslant \frac{2b}{\ell_b}\right) \leqslant \exp(-b)\,.$$

Let us now prove the second part. Fix $j \geqslant 5$ and let $a := b/(tnj2^{j-6})$. We assume that a is an integer, increasing b if necessary. If the event $|V_j| \geqslant a$ occurs then there is a set A of a vertices whose degrees sum to at least $2^j atn$. As this sum is equal to $2e(G_m[A]) + e(G_m[A, V \setminus A])$ we have either

(i) $e(G_m[A]) \geqslant 2^{j-2}atn$, or

(ii) $e(G_m[A, V \setminus A]) \geqslant 2^{j-1}atn$

Note that both of the variables above are hypergeometric with mean $\mu \leqslant atn$. Using the bound (2-4) from Theorem 2.10 with $\nu = atn$ we have

$$\mathbb{P}\left(X \geqslant 2^{j-2}atn\right) \leqslant \exp(-(j-2)2^{j-4}atn) \leqslant \exp(-2b)\,,$$

where X is either of the hypergeometric variables above. Taking an union bound over all $j \geqslant 5$ completes the proof of (ii). □

5.2
Sum of squares of degrees and their deviations

We now study the sum of squares of degrees as well as the sum of squares of degree deviations. Let $\kappa(b,t)$ be the function defined by

$$\kappa(b,t) := \begin{cases} tn^2 & 1 \leqslant b < t^{1/2}n\ell \\ b^2/\ell^2 & t^{1/2}n\ell \leqslant b < n\ell \\ bn/\ell & n\ell \leqslant b \leqslant tn^2\ell \end{cases}$$

and let

$$\kappa^+(b,t) := \begin{cases} b^2/\ell_b^2 & 1 \leqslant b < n\ell \\ bn/\ell & n\ell \leqslant b \leqslant tn^2\ell\,. \end{cases}$$

Proposition 5.2. *There exists an absolute constant $C > 0$ such that the following holds. Suppose that $t \geqslant 2n^{-1}\log n$ and that $b \geqslant 32tn$. Except with probability at most* $\exp(-b)$ *we have*

$$\sum_u D_u(G_i)^2 \leqslant C\kappa(b,t)$$

and

$$\sum_{u\,:\,d_u(G_m)\geqslant 32tn} d_u(G_m)^2 \;\leqslant\; C\kappa^+(b,t)$$

for all steps $i \leqslant m$*.*

We observe that for $b = tn$ the function $\kappa(b,t)$ is of the same order of magnitude, tn^2, as the expected value of $\sum_u D_u(G_m)^2$, since degrees are typically of order $t^{1/2}n^{1/2}$. On the other hand, as b approaches $tn^2\ell$ the function $\kappa(b,t)$ approaches tn^3 which is a trivial upper bound for $\sum_u D_u(G_m)^2 \leqslant n\sum_u |D_u(G_m)| \leqslant nm \leqslant tn^3$.

In order to prove Proposition 5.2 we bound the contribution of vertices with not too large degree separately in the following Lemma.

Lemma 5.3. *There is an absolute constant* $C > 0$ *such that the following holds. Suppose that* $t \geqslant 2n^{-1}\log n$ *and that* $b \geqslant n$*. Except with probability at most* $\exp(-b)$*, we have*

$$\sum_{u:d_u(G_i)\leqslant 32tn} D_u^2(G_i) \;\leqslant\; Cbtn$$

for all steps $i \leqslant m$*.*

Proof. Given a vector $\sigma \in \mathbb{Z}^{V(G)}$ with entries which are all either 0 or $\pm 2^j$ for some $j = 0,\ldots,\lfloor \frac{1}{2}\log tn\rfloor$, we define E_σ to be the event that

$$\sum_{u\in V(G_i)} \sigma_u D_u(G_i) \;\geqslant\; 16\,\|\sigma\|^2\, t^{1/2}n^{1/2}\,. \tag{5-1}$$

Let S be the set of such sequences σ with $\|\sigma\|^2 \geqslant 16b$. We make two claims:

Claim 1: The event that $\sum_{u:d_u(G_i)\leqslant 32tn} D_u^2(G_i) > 2^{20}btn$ is contained in the union $\bigcup_{\sigma\in S} E_\sigma$.

Claim 2: $\mathbb{P}\left(E_\sigma\right) \leqslant \exp(-\|\sigma\|^2)$ for all $\sigma \in S$.

Using both claims above together with an union bound we have that

$$\begin{aligned}\mathbb{P}\left(\sum_{u:d_u(G_i)\leqslant 32tn} D_u^2(G_i) \;>\; 2^{20}btn \text{ for some } i \;\leqslant\; m\right) &\leqslant\; m\sum_{\sigma\in S}\mathbb{P}\left(E_\sigma\right)\\ &\leqslant\; m\sum_{\sigma\in S}\exp\left(-||\sigma||^2\right).\end{aligned}$$

A direct application of Proposition 2.18 with $\beta = 1$, $d = n$ and $r = 4b^{1/2}$ gives

$$\begin{aligned} m\sum_{\sigma\in S}\exp(-\|\sigma\|^2) &\leqslant\; n^2(8\pi)^{n/2}\exp\left(\frac{-(4b^{1/2}-n^{1/2})^2}{2}\right)\\ &\leqslant\; n^2\exp(2n - 9b/2)\\ &\leqslant\; \exp(-b)\,.\end{aligned}$$

Therefore, the desired result follows from Claim 1 and Claim 2.

Proof of Claim 1: Suppose that the event $\sum_{u:d_u(G_i)\leqslant 32tn} D_u^2(G_i) > 2^{20}btn$ occurs for some graph G_i. We first note that this sum over all vertices u so that $|D_u(G_i)| \leqslant 32tn$ is at most $2^{10}tn^2 \leqslant 2^{10}btn$. Thus, defining U as the set of vertices such that $|D_u(G_i)| \in [32t^{1/2}n^{1/2}, 32tn]$ we have

$$\sum_{u\in U} D_u^2(G_i) > 2^{19}btn\,. \tag{5-2}$$

Let us define a vector $\sigma \in \mathbb{Z}^{V(G)}$ as follows. If $u \notin U$, let $\sigma_u = 0$. Also, for $j \geqslant 0$, let

$$\sigma_u := \begin{cases} 2^j & 2^{j+5}t^{1/2}n^{1/2} \leqslant D_u(G_i) < \min\{2^{j+6}t^{1/2}n^{1/2}, 32tn\} \\ -2^j & 2^{j+5}t^{1/2}n^{1/2} \leqslant -D_u(G_i) < \min\{2^{j+6}t^{1/2}n^{1/2}, 32tn\} \end{cases}$$

In order to prove Claim 1, it suffices to show that $\|\sigma\|^2 \geqslant 16b$ and that E_σ occurs. It follows from the definition of σ that

$$\frac{D_u(G_i)^2}{2^{12}tn} \leqslant \sigma_u^2 \leqslant \frac{D_u(G_i)^2}{2^{10}tn}$$

for all $u \in U$. It follows that

$$16b \leqslant \|\sigma\|^2 \leqslant \frac{1}{2^{10}tn}\sum_{u\in U} D_u(G_i)^2\,.$$

Now, to show that E_σ occurs, we observe that $\sigma_u D_u(G_i) \geqslant D_u(G_i)^2/(2^6t^{1/2}n^{1/2}$ for all $u \in U$. Thus,

$$\begin{aligned}\sum_{u\in V(G_i)} \sigma_u D_u(G_i) &\geqslant \frac{1}{2^6t^{1/2}n^{1/2}}\sum_{u\in U} D_u(G_i)^2 \\ &\geqslant 16t^{1/2}n^{1/2}\|\sigma\|^2\,,\end{aligned}$$

as we desired.

Proof of Claim 2: Fix $\sigma \in S$ and define the function

$$f_\sigma(G_i) := \sum_{u\in V(G_i)} \sigma_u D_u(G_i)$$

Then, we may write E_σ as the event that $f_\sigma(G_i)$ is at least $16t^{1/2}n^{1/2}\|\sigma\|^2$. We are now in the context of Theorem 2.15. The function $f_\sigma(G_i)$ is ψ-Lipschitz for the function $\psi(uw) = \sum_{u\in V(G_i)} \sigma_u D_u(G_i)$. We note that $\|\psi\|^2 \leqslant 2n\,\|\sigma\|^2$ and $\psi_{\max} \leqslant 2\sigma_{\max} \leqslant 2t^{1/2}n^{1/2}$. Also, $\mathbb{E}\,[f_\sigma(G_i)] = 0$, as $\mathbb{E}\,[D_u(G_i)] = 0$ for all

$u \in U$. Applying Theorem 2.15 we obtain

$$\begin{aligned}\mathbb{P}(E_\sigma) &= \mathbb{P}\left(f_\sigma(G_i) \geqslant 16\,\|\sigma\|^2\, t^{1/2} n^{1/2}\right)\\ &\leqslant \exp\left(\frac{-256\,\|\sigma\|^4\, tn}{48tn\,\|\sigma\|^2 + 192tn\,\|\sigma\|^2}\right)\\ &\leqslant \exp(-\|\sigma\|^2)\,.\end{aligned}$$

which completes the proof. □

We now proceed to prove Proposition 5.2.

Proof of Proposition 5.2. Fix $t \geqslant 2n^{-1}\log n$, and $i \leqslant m$. Note that it suffices to prove that the bounds fail with probability at most $\exp(-2b)$, as we can take an union bound over $i \leqslant m$ to complete the proof. We begin with the second statement, about the restricted sum $\sum_{u\,:\,d_u(G_m)\geqslant 32tn} d_u(G_m)^2$. We use a dyadic argument, based on Lemma 5.1. Let us define $J = J(b)$ as follows: if $b \leqslant n\ell$ then J is maximual such that $2^j tn \leqslant 2b/\ell_b$; if $b > n\ell$, we let $J = \log_2(1/t)$. By Lemma 5.1, except with probability at most $\exp(-2b)$, we have

(i) $|V_j| \leqslant b/tnj2^{j-6}$ for all $j \geqslant 5$, and

(ii) $V_j = \emptyset$ for all $j > J$.

Assume that the event given by (i) and (ii) occurs. We now divide into two ranges of values of b.

For the range $32tn \leqslant b \leqslant n\ell$:

$$\begin{aligned}\sum_{u:d_u(G_m)\geqslant 32tn} d_u^2(G_m) &\leqslant \sum_{j=5}^{J} 2^{2j+2} t^2 n^2 |V_j|\\ &\leqslant 2^8 \sum_{j=5}^{J} \frac{2^j btn}{j}\\ &\leqslant \frac{2^{J+10} btn}{J}\\ &\leqslant \frac{2^{11} b^2}{\ell_b^2}\,.\end{aligned}$$

The last inequality above holds since, by definition,

$$J \geqslant \log\left(\frac{b}{etn\ell_b}\right) = \ell_b - \log(\ell_b) \geqslant \frac{\ell_b}{2}.$$

For the range $b \geqslant n\ell$: we observe that $J = \log_2(1/t)$, and so $\lfloor J \rfloor \geqslant \ell$ and $2^{\lfloor J \rfloor} \leqslant t^{-1}$. The same argument from above gives

$$\begin{aligned}\sum_{u : d_u(G_m) \geqslant 32tn} d_u(G_m)^2 &\leqslant \sum_{j=5}^{\lfloor J \rfloor} 2^{2j+2} t^2 n^2 |V_j| \\ &\leqslant \frac{2^{\lfloor J \rfloor + 10} btn}{\lfloor J \rfloor} \\ &\leqslant \frac{2^{10} bn}{\ell}\,.\end{aligned}$$

From the analysis of the two ranges above, we may choose any $C \geqslant 2^{11}$ to complete the proof of the bound for the second sum of this Proposition.

Now, we prove the first bound on the unrestricted sum $\sum_u D_u(G_i)^2$. We note that if $d_u(G_i) \geqslant 32tn$ then $d_u(G_m) \geqslant 32tn$ and we have $D_u(G_i)^2 \leqslant d_u(G_m)^2$. Thus, we may apply the above bound to control the sum over vertices u such that $d_u(G_i) \geqslant 32tn$.

For the range $t^{1/2} n\ell \leqslant b < n\ell$, the bound from above gives

$$\sum_{u : d_u(G_i) \geqslant 32tn} D_u^2(G_i) \leqslant \frac{2^{12} b^2}{\ell_b^2}$$

except with probability at most $\exp(-3b)$. Moreover, Lemma 5.3 gives a constant C_1 such that

$$\sum_{u : d_u(G_i) < 32tn} D_u^2(G_i) \leqslant C_1 \max\{tn^2, btn\}$$

except with probability at most $\exp(-3b)$.

We now observe that $\max\{tn^2, btn\} \leqslant b^2/\ell^2$ and $b^2/\ell_b^2 \leqslant 4b^2/\ell^2$ over the range considered. Thus,

$$\sum_u D_u^2(G_i) \leqslant (2^{14} + C_1)\kappa(b,t)\,,$$

which gives the result in this range.

For the range $n\ell \leqslant\!< tn^2\ell$, the same argument from above shows that, except with probability at most $\exp(-3b)$,

$$\sum_{u : d_u(G_i) \geqslant 32tn} D_u^2(G_i) \leqslant \frac{2^{12} b^2}{\ell_b^2}$$

and

$$\sum_{u : d_u(G_i) < 32tn} D_u^2(G_i) \leqslant C_1 btn\,.$$

Since $\kappa(b,t) = btn \geqslant b^2/\ell^2$, we have

$$\sum_u D_u^2(G_i) \leqslant (2^{14} + C_1)\kappa(b,t)\,,$$

which gives the result in this range.

Finally, we note that if $b \leqslant t^{1/2}n\ell$ then $b^2/\ell^2 \leqslant tn^2 = \kappa(b,t)$. As this value does not depend on b, the result in this range follows by applying the result that we already obtained for $b = t^{1/2}n\ell$. □

5.3 The conditional variance of the increments $X_\wedge(G_i)$

We now present a direct consequence of the results proved earlier in this chapter. This is a bound on the conditional variance of the increments $X_\wedge(G_i)$ given the graph G_{i-1}, which will be useful later to apply Freedman's inequality to the martingale expressions $D_\wedge(G_m)$ and $D_\triangle(G_m)$.

Lemma 5.4. *There exists an absolute constant* C *such that the following holds. Suppose that* $2n^{-1}\log n \leqslant t \leqslant 1/2$*, that* $b \geqslant 32tn$ *and that* $i \leqslant m$*. Then, except with probability at most* $\exp(-b)$*, we have*

$$\mathbb{E}\left[X_\wedge(G_i)^2 | G_{i-1}\right] \leqslant \frac{C\kappa(b,t)}{n}\,.$$

Proof. Recall that $X_\wedge(G_i)$ is defined by $X_\wedge(G_i) := A_\wedge(G_i) - \mathbb{E}\left[A_\wedge(G_i)|G_{i-1}\right]$, which is the difference between $A_\wedge(G_i)$, the number of (isomorphic copies of) paths of lengths 2 created with the addition of the ith edge, and its expected value given G_{i-1}. Since $\mathbb{E}\left[X_\wedge(G_i)|G_{i-1}\right] = 0$, it follows that

$$\mathbb{E}\left[X_\wedge(G_i)^2 | G_{i-1}\right] = \mathrm{Var}(X_\wedge(G_i)|G_{i-1}) = \mathrm{Var}(A_\wedge(G_i)|G_{i-1})\,.$$

The number of isomorphic copies of cherries created is $A_\wedge(G_i) = 2d_u(G_{i-1}) + 2d_w(G_{i-1})$ where $e_i = uw$ is the ith edge included in G_i. Note that e_i is uniformly selected among all pairs from $K_n \setminus G_{i-1}$ and that $\mathbb{E}\left[d_u(G_{i-1}\right] = 2(i-1)/n$. Let us also recall that for any random variable X and any constant c we have $\mathrm{Var}(X) \leqslant \mathbb{E}\left[(X-c)^2\right]$. Applying this with $X = A_\wedge(G_i)$ and $c = 8(i-1)/n$ gives

$$
\begin{aligned}
\mathbb{E}[X_\wedge(G_i)^2|G_{i-1}] &\leqslant \frac{1}{N-i+1} \sum_{uw \notin G_{i-1}} \left(2d_u(G_{i-1}) \,+\, 2d_w(G_{i-1}) \,-\, \frac{8(i-1)}{n}\right)^2 \\
&\leqslant \frac{8}{N-i+1} \sum_{uw \notin G_{i-1}} \left(d_u(G_{i-1}) \,+\, \frac{2(i-1)}{n}\right)^2 + \left(d_w(G_{i-1}) \,-\, \frac{2(i-1)}{n}\right)^2 \\
&\leqslant \frac{32}{n^2} \sum_{uw \notin G_{i-1}} D_u(G_{i-1})^2 \,+\, D_w(G_{i-1})^2 \\
&\leqslant \frac{32}{n} \sum_u D_u(G_{i-1})^2 \,.
\end{aligned}
$$

Let C_1 be the constant given by Proposition 5.2. The required inequality now follows by choosing $C = 32C_1$.

□

6
Codegrees in $G(n,m)$

In this chapter, we present the results that we need about codegrees in $G(n,m)$.

Let $d_{uw}(G_m)$ be the codegree of vertices u and w in the graph G_m, meaning the number of common neighbors of these vertices. We note that the expectation of $d_{uw}(G_m)$ is $(n-2)(m)_2/(N)_2$, as there are $n-2$ possible common neighbors for u and w. Then

$$D_{uw}(G_m) := d_{uw}(G_m) - \frac{m(m-1)(n-2)}{N(N-1)}.$$

is the deviation of the codegree of the pair uw in G_m from its mean.

Note that the expectation of $d_{uw}(G_m)$ is of order of order of magnitude tn^2. As we have done in the section about degrees, we will bound the number of vertices of large codegree deviation. Here, we will also include vertices with codegree deviation between the order of the codegree standard deviation and the codegree mean. We will also deduce a bound for the sum of square of codegree deviations $\sum_{uw} D_{uw}(G_m)^2$. Finally, we present a bound on the conditional variance of the increments $X_\triangle(G_i)$.

6.1
Number of vertices with large codegree deviations

As we mentioned above, we shall consider two ranges of codegree deviations

(i) From $\Theta(tn^{1/2})$ to $\Theta(t^2n)$

(ii) Larger than $\Theta(t^2n)$

Corresponding to (i) and (ii) above,

(i) For $k \in K_1 := \{10, \dots, \lceil \log_2(tn^{1/2}) + 10 \rceil\}$, we define

$$F_k(G) := \{uw : |D_{uw}(G)| \in [2^k tn^{1/2}, 2^{k+1} tn^{1/2}), |D_u(G)|, |D_w(G)| \leqslant 2^{k-5} n^{1/2}\}|$$

and $f_k(G) := |F_k(G)|$.

(ii) For $k \in K_2 := \{10, \dots, \lfloor 2\log_2(1/t) \rfloor\}$, we define

$$H_k(G) := \{uw : d_{uw}(G) \in [2^k t^2 n, 2^{k+1} t^2 n), \, d_u(G), d_w(G) \leqslant 2^{k-5} tn\}$$

and $h_k(G) := |H_k(G)|$.

We note that K_1 and K_2 have been chosen so that all dyadic intervalos from $2^{10} t n^{1/2}$ up to n are covered. In the definition of $H_k(G)$, we choose to use $d_{uw}(G_i)$ since it is easier to understand and we have $|D_{uw}(G_i)| \leqslant d_{uw}(G_i)$ once $d_{uw}(G_i) \geqslant 2t^2 n$.

The reason for the condition over the degree deviations in the definiton of $F_k(G)$ and the degrees in the definition of $H_k(G)$ is that we would like to count only pairs which have large codegree which are not explained by a possible large degree of one of its vertices.

We prove the following result.

Proposition 6.1. *There exists an absolute constant C such that the following holds. Suppose that $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and that $b \geqslant n$. Then, except with probability at most* $\exp(-b)$*, we have*

$$f_k(G_i) \leqslant \frac{Cb^2}{2^{4k}} \qquad \text{for all } k \in K_1 \text{ and } i \leqslant m\,.$$

For $b \geqslant 4tn$, except with probability at most $\exp(-b)$*, we have*

$$h_k(G_i) \leqslant \frac{Cb^2}{k^2 2^{2k} t^4 n^2} \qquad \text{for all } k \in K_2 \text{ and } i \leqslant m\,.$$

The proof of the bounds above is a bit technical and also different from each other.

6.2 Controlling $h_k(G_i)$

In this section, we prove the second part of Proposition 6.6 above.

First, we note that if $k2^k \leqslant b/(t^2n^2)$ then the bound follows trivially, as $h_k(G_i) \leqslant n^2 \leqslant b^2/(k^2 2^{2k} t^4 n^2)$. So we fix $i \leqslant m$ and $k \in K_2$ such that $k2^k > b/(t^2 n^2)$.

Let us make a dyadic partition on $H_k(G)$ depending on the degrees of u, w in the pair uw. We define

$$H_{k,0}(G) := \{uw : d_{uw}(G) \in [2^k t^2 n, 2^{k+1} t^2 n), \, \max\{d_u(G), d_w(G)\} \leqslant 2^{k/2} tn\}$$

and $h_{k,0}(G) := |H_{k,0}(G)|$. Also, for $k/2 < j \leqslant \min\{k-6, \log_2(1/t)\}$, we define

$$H_{k,j}(G) := \{uw : d_{uw}(G) \in [2^k t^2 n, 2^{k+1} t^2 n)\,,\ \max\{d_u(G), d_w(G)\} \in [2^j tn, 2^{j+1} tn)\}$$

and $h_{k,j}(G) := |H_{k,j}(G)|$.

Note that every pair in $H_k(G)$ is in exactly one $H_{k,j}(G)$ for some index $j \in J$ where $J = \{0\} \cup \{j : k/2 < j \leqslant \min\{k-6, \log_2(1/t)\}\}$. In particular,

$$h_k(G) = \sum_{j \in J} h_{k,j}(G)\,. \tag{6-1}$$

We shall think of $H_{k,j}(G)$ and $H_k(G)$ as auxiliary graphs of G. We now prove bounds on the size of stars and matchings on these auxiliary graphs. This result will be stated and proved in terms of $G \sim G(n,p)$ for $p \in (0,t)$ instead of $G(n,m)$ only because it is easier to prove this way. As we will see later, there is no loss in changing from $G(n,m)$ to $G(n,p)$ in this case.

Lemma 6.2. *There exists an absolute constant C such that the following holds. Suppose that $t \geqslant Cn^{-1/2}(\log n)^{1/2}$, that $p \in (0,t)$, that $b \geqslant 4tn$. Let $k/2 < j \leqslant \min\{k-6, \log_2(1/t)\}$ and let $G \sim G(n,p)$. With probability at least* $1 - \exp(-b)$*:*

(i) $H_{k,0}(G)$ contains no star with degree $Cb/k2^k t^2 n$,

(ii) $H_{k,j}(G)$ contains no star with degree $Cb/(k-j)2^k t^2 n$,

(iii) $H_{k,0}(G)$ contains no matching with $Cb/k2^k t^2 n$ edges.

Let us show how to deduce the second part of Proposition 6.6 from Lemma 6.2. We shall find a constant C so that, except with probability at most $\exp(-2b)$,

$$h_k(G_i) \leqslant \frac{Cb^2}{k^2 2^{2k} t^4 n^2}\,.$$

The proof then follows by taking an union bound over $k \in K_2$ and $i \leqslant m$.

We first bound $h_{k,0}(G_i)$. In order to apply Lemma 6.2 we need to choose an appropriate value for p so that we can go from $G(n,p)$ to G_i. Choosing $p = s = i/N$, Lemma 6.2 gives a constant C_1 such that there is probability at most $\exp(-3b)$ that $H_{k,0}(G)$ contains a star or matching with $C_1 b/(k2^k t^2 n)$ edges where $G \sim G(n,s)$. Now, the number of edges of G is given by a binomial variable $Bin(N,s)$. Since this variable has expectation sN, there is probability at least n^{-2} that G has $i = sN$ edges. Thus, there is at most $n^2 \exp(-3b)$ probability that $H_{k,0}(G_i)$ has a star or matching with $C_1 b/(k2^k t^2 n)$ edges. We claim that, except with probability at most $n^2 \exp(-3b)$ we have

$$h_{k,0}(G_i) \leqslant \frac{C_1^2 b^2}{k^2 2^{2k} t^4 n^2}\,.$$

To see this, we use a result called Vizing's Theorem that says that $\chi'(G) \leqslant \Delta(G)+1$. Take a proper edge colouring of G_i which uses $\chi'(G_i)$ colours. As each set of edges with the same colour is a matching, we deduce that the number of edges using the same colour is at most $C_1 b/(k2^k t^2 n)$. Also, the same quantity bounds the value of $\Delta(G)$, as this is the same as the maximum size of a star em G_i. Therefore, $h_{k,0}(G_i) \leqslant \chi'(G) C_1 b/(k2^k t^2 n)$ which gives the inequality above.

We now bound $h_{k,j}(G_i)$, for $k/2 < j \leqslant \min\{k-6, \log_2(1/t)\}$. The same application of Lemma 6.2 from above gives that, except with probability at most $n^2 \exp(-3b)$, the graph $H_{k,j}(G_i)$ has a star or matching with $C_1 b/(k2^k t^2 n)$ edges. Moreover, we defined $H_{k,j}(G_i)$ so that every edge is incident to the set V_j, defined in Lemma 5.1. This result shows that, except with probability at most $\exp(-3b)$, we have $|V_j| \leqslant 2^8 b/(tnj2^j)$. It follows that, except with probability at most $(n^2+1)\exp(-3b)$, we have

$$h_{k,j}(G_i) \leqslant |V_j|\,\Delta(H_{k,j}(G_i)) \leqslant \left(\frac{2^8 b}{tnj2^j}\right)\left(\frac{C_1 b}{(k-j)2^k t^2 n}\right) = \frac{2^8 C_1 b^2}{j(k-j)2^{k+j} t^3 n^2}\,.$$

Recall from (11-5) that $h_k(G_i) = \sum_{j\in J} h_{k,j}(G_i)$. Using the bounds from above and taking an union bound over $j \in J$ we get, except with probability at most $n^2 \log_2(1/t)\exp(-3b)$,

$$h_k(G_i) \leqslant \frac{C_1^2 b^2}{k^2 2^{2k} t^4 n^2} + \sum_{j=\lceil k/2\rceil}^{\min\{k-6,\log_2(1/t)\}} \frac{2^8 C_1 b^2}{j(k-j)2^{k+j} t^3 n^2}\,.$$

We note that the last sum above is dominated by the geometric sum with ratio 3/4. In particular, the sum is at most 4 times its first term. Also, $n^2 \log_2(1/t)\exp(-3b) \leqslant \exp(-2b)$. So, except with probability at most $\exp(-2b)$, we have

$$h_k(G_i) \leqslant \frac{C_1^2 b^2}{k^2 2^{2k} t^4 n^2} + \frac{2^{10} C_1 b^2}{\lfloor k/2\rfloor \lceil k/2\rceil 2^{3k/2} t^3 n^2}\,.$$

Finally, we are considering $k \leqslant 2\log_2(1/t)$, so that $t^{-1} \geqslant 2^{k/2}$. Thus, except with probability at most $\exp(-2b)$, we have

$$h_k(G_i) \leqslant \frac{2^{12} C_1^2 b^2}{k^2 2^{2k} t^4 n^2}\,,$$

The proof now follows by choosing $C = 2^{12} C_1^2$.

We now prove Lemma 6.2

Proof of Lemma 6.2. We first prove (i). We fix $j = k/2$ and let C be a constant that will be determined later.

For each vertex $u \in V$, each subset $W \subseteq V \setminus \{u\}$ with $|W| = Cb/(k2^k t^2 n)$ and each subset $\Gamma_0 \subseteq V \setminus \{u\}$ with $|\Gamma_0| \leqslant 2^j tn$, we define $F(u, W, \Gamma_0)$ as the event in which Γ_0 is the neighbourhood of u in $G \sim G(n,p)$ and

$$e(W, \Gamma_0) \geqslant 2^k t^2 n |W| .$$

Consider a star with the centre being u, Γ_0 being the neighbourhood of u and W being $Cb/(2^{k-1})$ neighbours of u in $H_k(G)$. Clearly, the event described in (i) fail only if $F(u, W, \Gamma_0)$ occurs for some trio u, W, Γ_0. Therefore, we only need to prove that

$$\mathbb{P}\left(\bigcup_{u,W,\Gamma_0} F(u, W, \Gamma_0) \right) \leqslant \exp(-b) .$$

Fix a choice of W and Γ_0 and define $\nu := 2^j t^2 n |W|$. We have $\nu \geqslant \mathbb{E}\left[e(W, \Gamma_0)\right]$. Moreover, the event $F(u, W, \Gamma_0)$ implies that $e(W, \Gamma_0) \geqslant 2^{k-j}\nu$. Thus, (2-4) implies that

$$\begin{aligned} \mathbb{P}\left(F(u, W, \Gamma_0) \mid \Gamma(u) = \Gamma_0\right) &\leqslant 2\exp\left(-(k-j)2^{k-j-3}\nu\right) \\ &= 2\exp(-(k-j)2^{k-3}t^2 n|W|) \\ &\leqslant 2\exp(-Cb/16) \end{aligned}$$

where the last line uses the definition of $|W|$ and the fact that $j = k/2$. We shall now take an union bound over choices of u and W. There are n choices for the vertex u and at most

$$n^{|W|} = \exp\left(\frac{Cb\log n}{k2^k t^2 n}\right) \leqslant \exp\left(\frac{b}{k2^k C}\right) \leqslant \exp(b/2)$$

choices of W. Therefore,

$$\begin{aligned} \mathbb{P}\left(\bigcup_{u,W,\Gamma_0} F(u, W, \Gamma_0) \right) &\leqslant \sum_{u,W,\Gamma_0} \mathbb{P}\left(\Gamma(u) = \Gamma_0\right) \mathbb{P}\left(F(u, W, \Gamma_0) | \Gamma(u) = \Gamma_0\right) \\ &\leqslant 2\exp(-Cb/16) \sum_{u,W} \sum_{\Gamma_0} \mathbb{P}\left(\Gamma(u) = \Gamma_0\right) \\ &\leqslant 2n\exp(-3b/2) \\ &\leqslant \exp(-b) , \end{aligned}$$

where we used $C \geqslant 32$ in the penultimate inequality.

The proof of (ii) follows the exact same argument. The only difference is that now $k/2 < j \leqslant \min\{k - 6, \log_2(1/t)\}$ and the $(k - j)$ term cancels with the equivalent term in $|W|$.

The proof of *(iii)* is a bit more complicated, as we need to consider a matching $u_1w_1,\dots,u_fw_f$ with $f := Cb/(k2^kt^2n)$ and a sequence of sets $\Gamma_1,\dots,\Gamma_f$. We now define $F(u_1,\dots,u_f,w_1,\dots,w_f,\Gamma_1,\dots,\Gamma_f)$ as the event in which $\Gamma(u_g)=\Gamma_g$ for all $g=1,\dots f$, and

$$\sum_{g=1}^{f}|\Gamma(w_g)\cap\Gamma_g| \,\geqslant\, 2^kt^2nf\,.$$

Now, the event described in *(iii)* fails only if $F(u_1,\dots,u_f,w_1,\dots,w_f,\Gamma_1,\dots,\Gamma_f)$ for some matching $u_1w_1,\dots,u_fw_f$ and some sets $\Gamma_1,\dots,\Gamma_f$ of cardinality at most $2^{k/2}tn$.

We write $\mathbf{u}$ to represent $u_1,\dots,u_f$, $\mathbf{w}$ to represent $w_1,\dots,w_f$ and $\boldsymbol{\Gamma}$ to represent $\Gamma_1,\dots,\Gamma_f$. We also write $F(\mathbf{u},\mathbf{w},\boldsymbol{\Gamma})$ to abbreviate the event $F(u_1,\dots,u_f,w_1,\dots,w_f,\Gamma_1,\dots,\Gamma_f)$ and $\Gamma(\mathbf{u})=\boldsymbol{\Gamma}$ for the event that $\Gamma(u_g)=\Gamma_g$ for all $g=1,\dots,f$.

For each such choice, $\sum_{g=1}^{f}|\Gamma(w_g)\cap\Gamma_g|$ is a sum of indicator functions representing whether certain edges are included in G. As edges may occur twice, it might not necessarily be binomial, but can be written as a sum of two binomials. And so, just as in the "star" case above, by (2-4) we have

$$\mathbb{P}\left(F(\mathbf{u},\mathbf{w},\boldsymbol{\Gamma})|\Gamma(\mathbf{u})=\boldsymbol{\Gamma}\right) \,\leqslant\, 2\exp(-Cb/16)\,.$$

We conclude the proof by taking an union bound over all possible matchings. There are at most $n^{2f}\leqslant\exp(b/2)$ choices for this matching, which implies

$$\begin{aligned}\mathbb{P}\left(\bigcup_{\mathbf{u},\mathbf{w},\boldsymbol{\Gamma}}F(\mathbf{u},\mathbf{w},\boldsymbol{\Gamma})\right) &\leqslant \sum_{\mathbf{u},\mathbf{w},\boldsymbol{\Gamma}}\bigcup_{\mathbf{u},\mathbf{w},\boldsymbol{\Gamma}}\mathbb{P}\left(\Gamma(\mathbf{u})=\boldsymbol{\Gamma}\right)\mathbb{P}\left(F(\mathbf{u},\mathbf{w},\boldsymbol{\Gamma})|\Gamma(\mathbf{u})=\boldsymbol{\Gamma}\right)\\ &\leqslant 2\exp(-Cb/16)\sum_{\mathbf{u},\mathbf{w}}\sum_{\boldsymbol{\Gamma}}\mathbb{P}\left(\Gamma(\mathbf{u})=\boldsymbol{\Gamma}\right)\\ &\leqslant 2\exp(-3b/2)\\ &\leqslant \exp(-b)\,,\end{aligned}$$

where we used $C\geqslant 32$ in the penultimate inequality. □

6.3 Controlling $f_k(G_i)$

In this section, we prove the second part of Proposition 6.6, relative to $f_k(G_i)$.

We may use the same argument from the previous section to control $f_k(G_i)$ for almost all $k \in K_1$. In this case, we control the size of stars and matchings in $F_k(G)$.

However, we need to use another approach for small values of k (more precisely, when $2^k \leqslant 2^8\sqrt{\log n}$). The problem appears when considering large matchings, as there are $\exp\left(()\,\Theta(f\log n)\right)$ ways to choose a matching with f edges and the probability bound for a fixed matching is $\exp(-\Theta(2^{2k}f))$.

We now state bounds on the size of stars and matchings on $F_k(G)$, which allow us to prove Proposition 6.6 for k such that $2^k \geqslant 2^8\sqrt{\log n}$. As in the previous section, we state the result in $G(n,p)$.

Lemma 6.3. *There exists an absolute constant* C *such that the following holds. Suppose that* $t \geqslant Cn^{-1/2}(\log n)^{1/2}$*, that* $p \in (0,t)$*, and that* $b \geqslant 4tn$*. Let* $k \in K_1$ *be such that* $2^k \geqslant 2^8\sqrt{\log n}$ *and let* $G \sim G(n,p)$*. With probability at least* $1-\exp(-b)$*:*

(i) $F_k(G)$ *contains no star with degree* $Cb/2^{2k}$*,*

(ii) $F_k(G)$ *contains no matching with* $Cb/2^{2k}$ *edges.*

We omit the proof of the desired bound for $f_k(G_i)$ for k such that $2^k \geqslant 2^8\sqrt{\log n}$, as it follows from Lemma 6.3 in the same way that the bound of $h_{k_0}(G_i)$ follows from Lemma 6.2. Before we present the proof for the other values of k, let us prove Lemma 6.3.

Proof of Lemma 6.3. We first prove (i). Also, we let C be a constant that will be determined later.

For each vertex $u \in V$, each subset $W \subseteq V \setminus \{u\}$ with $|W| = Cb/2^{2k+1}$ and each subset $\Gamma_0 \subseteq V\setminus\{u\}$ with $||\Gamma_0|-pn| \leqslant 2^{k-4}n^{1/2}$, we define $E(u,W,\Gamma_0)$ as the event in which Γ_0 is the neighbourhood of u in $G \sim G(n,p)$ and

$$|e(W,\Gamma_0) - p^2n|W|| \geqslant 2^k tn^{1/2}|W|\,.$$

Consider a star with the centre being u, Γ_0 being the neighbourhood of u and W being $Cb/(2^{2k+1})$ neighbours of u in $H_k(G)$ chosen so that $D_{uw}(G)$ has the same sign for all $w \in W$. Clearly, the event described in (i) fail only if $F(u,W,\Gamma_0)$ occurs for some trio u,W,Γ_0. Therefore, we only need to prove that

$$\mathbb{P}\left(\bigcup_{u,W,\Gamma_0} F(u,W,\Gamma_0)\right) \leqslant \exp\left(-b\right)\,.$$

Fix a choice of W and Γ_0 and define $\mu := \mathbb{E}\left[e(W,\Gamma_0)\right] = p|W||\Gamma_0|$. By the definition of Γ_0, we have

$$\left|\mu - p^2n|W|\right| \leqslant 2^{k-4}pn^{1/2}|W| \leqslant 2^{k-4}tn^{1/2}|W|\,.$$

Thus, the event $E(u,W,\Gamma_0)$ implies that

$$\left|e(W,\Gamma_0) - \mu\right| \geqslant 2^{k-1}tn^{1/2}|W|\,.$$

Let us note that $\mu \leqslant 2p^2n|W| \leqslant 2t^2n|W|$ and $2^ktn^{1/2} \leqslant 2^{10}t^2n$ for all $k \in K_1$. From this and Theorem 2.10 we deduce that

$$\begin{aligned}\mathbb{P}\left(E(u,W,\Gamma_0)|\Gamma(u)=\Gamma_0\right) &\leqslant \exp\left(\frac{-2^{2k-2}t^2n|W|^2}{4t^2n|W| + 2^ktn^{1/2}|W|}\right)\\ &\leqslant \exp\left(-2^{2k-13}|W|\right)\,.\end{aligned}$$

We now take an union bound over choices of u and W. The number of choices for this pair u,W is bounded by

$$n^{|W|+1} = \exp\left((|W|+1)\log n\right) \leqslant \exp\left(2|W|\log n\right)\,.$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{u,W,\Gamma_0} E(u,W,\Gamma_0)\right) &\leqslant \sum_{u,W,\Gamma_0}\mathbb{P}\left(\Gamma(u)=\Gamma_0\right)\mathbb{P}\left(E(u,W,\Gamma_0)\mid\Gamma(u)=\Gamma_0\right)\\ &\leqslant \exp\Big(-2^{2k-13}|W|\Big)\sum_{u,W}\sum_{\Gamma_0}\mathbb{P}\left(\Gamma(u)=\Gamma_0\right)\\ &\leqslant \exp\Big((2\log n - 2^{2k-13})|W|\Big)\\ &\leqslant \exp\Big(-2^{2k-14}|W|\Big)\\ &\leqslant \exp\Big(-Cb/2^{14}\Big)\,.\end{aligned}$$

Note that we used the bound, $2^k \geqslant 2^8\sqrt{\log n}$, for the penultimate inequality. This proves the required bound provided $C \geqslant 2^{14}$.

We omit the proof of the matching argument, as it is very similar to the matching argument in the proof of Lemma 6.2. □

We now focus on bounding $f_k(G_i)$ for $k \in K_1^-$ where $K_1^- := \{k \in K_1 : 2^k \leqslant 2^8\sqrt{\log n}\}$. In this case, we consider a union of disjoint stars. For $k \in K_1^-$, the auxiliary graph $F_k(G_i)$ may be very dense. The upper bound that we want to prove on $f_k(G_i)$ is $Cb^2/2^{4k}$ which is at least $b^2/\log n \geqslant n^2/\log n$, if we choose C large enough. The following lemma states that in reasonably dense graphs we may find a small number of stars which cover a large number of vertices.

Lemma 6.4. *Let G be a graph on n vertices with at least n^2/r^2 edges. Then there are some r vertices $v_1,\ldots,v_r$ of G such that*

$$\left|\bigcup_{i=1}^{r} N(v_i)\right| \geqslant \frac{n}{r}.$$

We also need the following bound on unions of stars in $F_k(G)$.

Lemma 6.5. *There exists an absolute constant C such that the following holds. Suppose that $t \geqslant Cn^{-1/2}(\log n)^{1/2}$, that $p \in (0,t)$, and that $b \geqslant n$. Let $k \in K_1$ be such that $2^k \leqslant 2^8\sqrt{\log n}$ and let $G \sim G(n,p)$. With probability at least $1-\exp(-b)$ we have that any union of $r := 2^{2k}$ stars in $F_k(G)$ contains less than $Cb/2^{2k}$ vertices.*

Let us show how the two lemmas above show that, except with probability at most $\exp(-2b)$,

$$f_k(G_i) \leqslant \frac{Cb^2}{2^{4k}}$$

for all $k \in K_1^-$. Then the first part of Proposition 6.6 follows by an union bound over $k \in K_1$.

We repeat the process used to bound $h_{k,0}(G_i)$ to go from $G(n,p)$ to G_i. We choose $p = s = i/N$ and note that Lemma 6.5 gives a constant C_1 such that there is probability at most $\exp(-3b)$ there is an union of $r = 2^{2k}$ stars in $F_k(G)$ with at least $Cb/2^{2k}$ vertices where $G \sim G(n,s)$. Since $n^2\exp(-3b) \leqslant \exp(-2b)$, we have that union of r stars in $F_k(G_i)$ contains less than $C_1b/2^{2k}$ vertices. Now, suppose for contradiction that

$$f_k(G_i) > \frac{C_1^2b^2}{2^{4k}} = \frac{C_1^2n^2(b/n)^2}{2^{4k}}.$$

It would then follow from Lemma 6.4 that $F_k(G_i)$ contains $r' := n2^{2k}/C_1b$ vertices whose neighbourhoods cover at least $C_1b/2^{2k}$ vertices. This is a contradiction, as $r' \leqslant 2^{2k}$.

We now prove Lemma 6.4.

Proof of Lemma 6.4. Clearly the result is trivial if some vertex has degree at least n/r so we may assume all degrees are less than n/r.

Consider the digraph obtained from G by replacing each edge by two oriented edges (one in each direction). It clearly suffices to find $v_1,\ldots,v_r$ such that the union of the out-neighbourhoods in D of these vertices has cardinality at least n/r. We note that all in and out degrees in D are at most n/r and that $e(D) = 2e(G) \geqslant 2n^2/r^2$.

We may find $v_1, \ldots, v_r$ greedily. Let v_1 be a vertex of maximum out-degree and let us write d_1 for this degree and S_1 for $N^+(v_1)$. We now remove from the digraph all edges into S_1. In the remaining digraph we find a vertex of maximum out-degree v_2 with out-degree d_2, set $S_2 = S_1 \cup N^+(v_2)$, and remove any other edges into S_2. We continue.

At step i, we may assume the current set S_i has cardinality at most n/r (else we are already done) and so the total number of removed edges so far is at most n^2/r^2. It follows that at least n^2/r^2 edges remain and so $d_i \geqslant n/r^2$.

It is clear this process terminates in at most r steps, as $|S_i| = d_1 + \cdots + d_i \geqslant in/r^2$. □

Let us now finish this section by proving Lemma 6.5.

Let us introduce some notation. Given two sequences of sets $\mathbf{A} = (A_1, \ldots, A_r)$ and $\mathbf{B} = (B_1, \ldots, B_r)$, we define

$$|\mathbf{A}| := \sum_{j=1}^{r} |A_j| \qquad \text{and} \qquad \mathbf{A} \cdot \mathbf{B} := \sum_{j=1}^{r} |A_j||B_j|\,.$$

We recall that for two sets of vertices U, W, we let $e(U,W)$ count the number of edges with multiplicity. We extend this definition to sequences of vertex subsets. We define $\mathbf{A} \cap \mathbf{B} := (A_1 \cap B_1, \ldots, A_r \cap B_r)$ and $e(\mathbf{A}, \mathbf{B}) := \sum_{j=1}^{r} e(A_j, B_j)$. For example, in the random graph $G \sim G(n,p)$, we have

$$\mathbb{E}\left[e(\mathbf{A}, \mathbf{B})\right] = p\mathbf{A} \cdot \mathbf{B} - p|\mathbf{A} \cap \mathbf{B}| = p\mathbf{A} \cdot \mathbf{B} + O(p|\mathbf{A}|)\,.$$

Proof of Lemma 6.5. Let C be a constant that will be determined later.

In a similar way to the proof of Lemma 6.2 and Lemma 6.3 we define an event $E(\mathbf{u}, \mathbf{W}, \mathbf{\Gamma})$ that depends on

- a sequence of vertices $\mathbf{u} = (u_1, \ldots, u_r)$,
- vertex subsets $\mathbf{W} = (W_1, \ldots, W_r)$ such that $|\mathbf{W}| = Cb/2^{2k+1}$
- vertex subsets $\mathbf{\Gamma} = (\Gamma_1, \ldots, \Gamma_r)$ with $\Big||\Gamma_j| - pn\Big| \leqslant 2^{k-4} n^{1/2}$ for all $j = 1, \ldots, r$.

For each choice of the above, we set $E(\mathbf{u}, \mathbf{W}, \mathbf{\Gamma})$ to be the event that Γ_j is the neighbourhood of u_j in $G \sim G(n,p)$ for all $j = 1, \ldots, r$, and that

$$\Big| e(\mathbf{W}, \mathbf{\Gamma}) - p^2 n |\mathbf{W}| \Big| \geqslant 2^k t n^{1/2} |\mathbf{W}|\,.$$

Suppose that the event of the lemma fails. Then there are $r = 2^{2k}$ stars in $F_k(G)$ which cover at least $Cb/(2^{2k})$ vertices. Then at least $Cb/(2^{2k+1})$ of these pairs have deviations of the same sign. We take $\mathbf{u}$ as the centres of these stars, $\mathbf{\Gamma}$ to be their neighbourhoods in G, and define $\mathbf{W} = (W_1, \ldots, W_r)$ by taking the sets W_j to be disjoint and with W_j chosen among the neighbours of u_j in $F_k(G)$ with the favoured sign. Thus, $E(\mathbf{u}, \mathbf{W}, \mathbf{\Gamma})$ must occur for such a trio.

We now bound the probability of the event $E(\mathbf{u}, \mathbf{W}, \mathbf{\Gamma})$. For a fixed choice of $\mathbf{W}$ and $\mathbf{\Gamma}$ let $\mu = \mathbb{E}\left[e(\mathbf{W}, \mathbf{\Gamma})\right] = p\mathbf{W}\cdot\mathbf{\Gamma} + O(p|\mathbf{W}|)$. By the triangle inequality, and the bounds on the $|\Gamma_j|$, we have

$$\Big|\mu \,-\, p^2 n|\mathbf{W}|\Big| \,\leqslant\, 2^{k-3} p n^{1/2}|\mathbf{W}| \,\leqslant\, 2^{k-3} t n^{1/2}|\mathbf{W}|\,.$$

Furthermore, the event $E(\mathbf{u}, \mathbf{W}, \mathbf{\Gamma})$ implies that

$$\Big|e(\mathbf{W}, \mathbf{\Gamma}) - \mu\Big| \,\geqslant\, 2^{k-1} t n^{1/2}|\mathbf{W}|\,.$$

We now bound the probability of $E(\mathbf{u}, \mathbf{W}, \mathbf{\Gamma})$ using Theorem 2.10. Let us note that $\mu \leqslant 2p^2 n|\mathbf{W}| \leqslant 2t^2 n|\mathbf{W}|$, and that we have $2^k t n^{1/2} \leqslant 2^{10} t^2 n$ for $k \in K_1$. Thus

$$\begin{aligned}\mathbb{P}\left(E(\mathbf{u}, \mathbf{W}, \mathbf{\Gamma}) \mid \Gamma(\mathbf{u}) = \mathbf{\Gamma}\right) \,&\leqslant\, \exp\left(\frac{-2^{2k-2} t^2 n|\mathbf{W}|^2}{4t^2 n|\mathbf{W}| \,+\, 2^k t n^{1/2}|\mathbf{W}|}\right)\\ &\leqslant\, \exp\Big(-\,2^{2k-13}|\mathbf{W}|\Big)\,.\end{aligned}$$

We now take an union bound over choices of $\mathbf{u}$ and $\mathbf{W}$. There are at most $n^r \leqslant \exp((\log n)^2)$ ways to choose $\mathbf{u}$, at most $\binom{n}{|\mathbf{W}|} \leqslant \exp\Big(|\mathbf{W}|\log(en/|\mathbf{W}|)\Big)$ ways to choose the elements of $\mathbf{W}$ and $r^{|\mathbf{W}|}$ ways to assign these elements to the sets $W_1, \ldots, W_r$. The total number of choices of the pair $\mathbf{u}, \mathbf{W}$ is therefore at most

$$\exp\left((\log n)^2 \,+\, |\mathbf{W}|\log\left(\frac{enr}{|\mathbf{W}|}\right)\right) \,\leqslant\, \exp\Big((\log n)^2 \,+\, 4k|\mathbf{W}|\Big)$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(\bigcup_{\mathbf{u},\mathbf{W},\mathbf{\Gamma}} E(\mathbf{u},\mathbf{W},\mathbf{\Gamma})\right) &\leqslant \sum_{\mathbf{u},\mathbf{W},\mathbf{\Gamma}} \mathbb{P}\left(\Gamma(\mathbf{u})=\mathbf{\Gamma}\right)\mathbb{P}\left(E(\mathbf{u},\mathbf{W},\mathbf{\Gamma})|\Gamma(\mathbf{u})=\mathbf{\Gamma}\right)\\ &\leqslant \exp\Big(-2^{2k-13}|\mathbf{W}|\Big)\sum_{u,W}\sum_{\mathbf{\Gamma}}\mathbb{P}\left(\Gamma(\mathbf{u})=\mathbf{\Gamma}\right)\\ &\leqslant \exp\Big((\log n)^2 + 4k|\mathbf{W}| - 2^{2k-13}|\mathbf{W}|\Big)\\ &\leqslant \exp\Big(-2^{2k-14}|\mathbf{W}|\Big)\\ &\leqslant \exp\Big(-Cb/2^{14}\Big).\end{aligned}$$

This proves the required bound provided we choose $C \geqslant 2^{14}$. □

6.4 Sum of squares of codegree deviations

We now apply the results from the previous sections to bound the sum of the squares of codegree deviations.

Proposition 6.6. *There exists an absolute constant C such that the following holds. Suppose that $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and that $b \geqslant n$. Then, except with probability at most* $\exp(-b)$*, we have*

$$\sum_{uw} D_{uw}(G_i)^2 \leqslant C\max\{bt^2n^2, b^2\}$$

for all $i \leqslant m$.

Proof. Let us fix $i \leqslant m$. We consider four types of pairs that contribute to the sum $\sum_{uw} D_{uw}(G_i)^2$.

First, the sum over pairs uw such that $|D_{uw}(G_i)| \leqslant 2^{10}tn^{1/2}$ is at most $2^{20}t^2n^3 \leqslant 2^{20}bt^2n^2$.

Second, we use our bound on the sum of squares of degree deviations to control the sum over pairs uw such that $|D_{uw}(G_i)| \leqslant 64t\max\{|D_u(G_i)|, |D_w(G_i)|\}$. Indeed, by Proposition 5.2, except with probability at most $\exp(-2b)$, the total contribution of these terms is at most

$$2^{12}t^2n\sum_{u} D_u(G_i)^2 \leqslant C_1t^2n\kappa(b,t) \leqslant \frac{C_1bt^2n^2}{\ell}$$

for some constant C_1.

The other types of pairs are in $\bigcup_{k\in K_1} F_k(G_i)$ or $\bigcup_{k\in K_2} H_k(G_i)$. From Proposition 6.6 we have a constant C_2 so that, except with probability at most $\exp(-2b)$, we have

$$f_k(G_i) \leqslant \frac{C_2 b^2}{2^{4k}} \qquad \text{for all } k \in K_1, \textit{ and}$$

$$h_k(G_i) \leqslant \frac{C_2 b^2}{k^2 2^{2k} t^4 n^2} \qquad \text{for all } k \in K_2\,,$$

Assume that we are on the event described by the bounds above.

If $uw \in F_k(G_i)$ for some $k \in K_1$ then $|D_{uw}(G_i)| \leqslant 2^{k+1} t n^{1/2}$ and so we may bound the total contribution of pairs in $F_k(G_i)$ by $2^{2k+2} t^2 n f_k(G_i)$.

We use two different bounds on $f_k(G_i)$ depending on the value of k. Let k_0 be the maximum value of k so that $2^{2k} \leqslant b/n$. If $k \leqslant k_0$, we use that $f_k(G_i) \leqslant n^2$ to bound the total contribution of pairs in $F_k(G_i)$ by $2^{2k+2} t^2 n^3$.

If $k > k_0$, the bound on $f_k(G_i)$ from above gives that the contribution of pairs in $F_k(G_i)$ is at most

$$2^{2k+2} t^2 n f_k(G_i) \leqslant \frac{4 C_2 b^2 t^2 n}{2^{2k}}\,.$$

We thus have

$$\sum_{k \in K_1} \sum_{uw \in F_k(G_i)} D_{uw}(G_i)^2 \leqslant \sum_{k=10}^{k_0} 2^{2k+2} t^2 n^3 + \sum_{k=k_0+1}^{\infty} \frac{4 C_2 b^2 t^2 n}{2^{2k}} \leqslant 8 b t^2 n^2 + 8 C_2 b t^2 n^2\,.$$

If $uw \in H_k(G_i)$ for some $k \in K_2$ then $|D_{uw}(G_i)| \leqslant d_{uw}(G_i) \leqslant 2^{k+1} t^2 n$. Thus, the total contribution of these pairs to the sum is at most $2^{2k+2} t^4 n^2 h_k(G_i) \leqslant 4 C_2 b^2 / k^2$, where we used the bound for $h_k(G_i)$ from above. We have

$$\sum_{k \in K_2} \sum_{uw \in H_k(G_i)} D_{uw}(G_i)^2 \leqslant \sum_{k \geqslant 10} \frac{4 C_2 b^2}{k^2} \leqslant C_2 b^2\,.$$

Putting together the bounds for the sums over the four types of edges considered above we have, except with probability at most $2\exp(-2b)$,

$$\sum_{uw} D_{uw}(G_i)^2 \leqslant (2^{20} + C_1 + 8 + 9C_2) \max\{b t^2 n^2, b\}\,.$$

To finish the proof, we take an union bound over $i \leqslant m$ and note that the failure probability is at most $2m \exp(-2b) \leqslant \exp(-b)$. □

6.5 The conditional variance of the increments $X_\triangle(G_i)$

We now present a bound on the conditional variance of the increments $X_\triangle(G_i)$ given the graph G_{i-1}, which will be useful later to apply Freedman's inequality to the martingale expression $D_\triangle(G_m)$.

Lemma 6.7. *There is an absolute constant C such that the following holds. Suppose that $Cn^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/2$, that $b \geqslant n$ and that $i \leqslant m$. Then, except with probability at most $\exp(-b)$, we have*

$$\mathbb{E}\left[X_\triangle(G_i)^2|G_{i-1}\right] \leqslant C\max\left\{bt^2, \frac{b^2}{n^2}\right\}.$$

The proof follows the same lines of Lemma 5.4, using the codegree results from this chapter.

Proof. Recall that $X_\triangle(G_i)$ is defined by $X_\triangle(G_i) := A_\triangle(G_i) - \mathbb{E}\left[A_\triangle(G_i)|G_{i-1}\right]$, which is the difference between $A_\triangle(G_i)$, the number of (isomorphic copies of) triangles created with the addition of the ith edge, and its expected value given G_{i-1}. Since $\mathbb{E}\left[X_\triangle(G_i)|G_{i-1}\right] = 0$, it follows that

$$\mathbb{E}\left[X_\triangle(G_i)^2|G_{i-1}\right] = \mathrm{Var}(X_\triangle(G_i)|G_{i-1}) = \mathrm{Var}(A_\triangle(G_i)|G_{i-1}).$$

The number of isomorphic copies of cherries created is $A_\triangle(G_i) = 6d_{uw}(G_{i-1})$ where $e_i = uw$ is the ith edge included in G_i. We recall that e_i is uniformly selected among all pairs from $K_n \setminus G_{i-1}$ and that $\mathbb{E}\left[d_{uw}(G_{i-1})\right] = (i-1)(i-2)(n-2)/(N(N-1))$. Therefore,

$$\begin{aligned}
\mathrm{Var}(A_\triangle(G_i)|G_{i-1}) &\leqslant \frac{1}{N-i+1}\sum_{uw\notin E(G_{i-1})}\left(6d_{uw}(G_{i-1}) - \frac{6(i-1)(i-2)(n-2)}{N(N-1)}\right)^2 \\
&\leqslant \frac{36}{N-i+1}\sum_{uw\notin E(G_{i-1})} D_{uw}(G_{i-1})^2 \\
&\leqslant \frac{144}{n^2}\sum_{uw\notin E(G_{i-1})} D_{uw}(G_{i-1})^2.
\end{aligned}$$

Let C_1 be the constant given by Proposition 6.6. The required inequality now follows by choosing $C = 144C_1$. □

7
Cherry counts in $G(n, m)$ - upper bounds on deviations

In this chapter we prove the upper bound of our main result for deviations on cherry counts. This result works as a gentle introduction to the proof of triangle counts, as both proofs share the many techniques.

Let us recall some definitions and our main result about cherry counts. We defined $M_\wedge(b,t) := \max\{\text{NORMAL}_\wedge(b,t), \text{STAR}_\wedge(b,t), \text{HUB}_\wedge(b,t)\}$ where $\text{NORMAL}_\wedge(b,t) := b^{1/2}tn^{3/2}, \text{STAR}_\wedge(b,t) := b^2 1_{b<n\ell}/\ell^2$ and $\text{HUB}(b,t) := bn1_{b\geqslant n\ell}/\ell$. In particular, we note that

$$M_\wedge(b,t) = \begin{cases} b^{1/2}tn^{3/2} & \text{if } b \leqslant t^{2/3}n\ell^{4/3} \\ b^2/\ell^2 & \text{if } t^{2/3}n\ell^{4/3} < b \leqslant n\ell \\ bn/\ell & \text{if } n\ell < b\,. \end{cases}$$

We shall use some lower bounds on $M_\wedge(b,t)$, which we present now. First, we claim that

$$M_\wedge(b,t) \geqslant btn \tag{7-1}$$

for all $b \geqslant 1$. To verify this, observe that $\text{NORMAL}_\wedge(b,t) \geqslant btn$ if $b \leqslant n$. Also, $\text{STAR}_\wedge(b,t) \geqslant btn$ if $n < b \leqslant n\ell$ and $\text{HUB}_\wedge(b,t) \geqslant btn$ if $b > n\ell$.

We also claim that

$$M_\wedge(b,t)^2 \geqslant btn\kappa(b,t) \tag{7-2}$$

for all $b \geqslant 1$, where $\kappa(b,t)$ is as defined in Section 5.2. To verify this, note that $\text{NORMAL}_\wedge(b,t)^2 = bt^2n^3 = btn\kappa(b,t)$ if $b \leqslant t^{1/2}n\ell$. Also, $M_\wedge(b,t) = \kappa(b,t)$ for all $b \geqslant t^{1/2}n\ell$ and so (7-1) implies (7-2) for this range.

Let us also note that

$$M_\wedge(b,t)^2 \geqslant \frac{\kappa^+(b,t)}{9} \tag{7-3}$$

for all $b \geqslant 4tn$, where $\kappa^+(b,t)$ is as defined in Section 5.2. To verify this, first note that $M_\wedge(b,t) = \kappa^+(b,t)$ whenever $b > n\ell$. Then observe that $\text{NORMAL}_\wedge(b,t) = b^{1/2}tn^{3/2} \geqslant b^2/\ell_b^2$ whenever $4tn \leqslant b \leqslant t^{2/3}n\ell_b^{4/3}$. Moreover, if $\max\{4tn, t^{2/3}n\ell_b^{4/3}\} < b \leqslant n\ell$ then $b/(etn) \geqslant 1/t^{1/3}$ which implies that $\ell_b \geqslant \ell/3$. Thus, $\text{STAR}_\wedge(b,t) = b^2/\ell^2 \geqslant b^2/(9\ell_b^2)$ over this range, completing the proof of (7-3).

Let us recall that $\mathrm{DEV}_\wedge(b,t)$ is the minimal value of a such that $\mathbb{P}\left(N_\wedge(b,t) > \mathbb{E}\left[N_\wedge(b,t)\right] + a\right) \leqslant e^{-b}$. We may now restate Theorem 1.4.

Theorem 1.4. *There exist absolute constants c, C such that the following holds. Suppose that $2n^{-1}\log n \leqslant t \leqslant 1/2$ and that $3\log n \leqslant b \leqslant tn^2\ell$. Then*

$$cM_\wedge(b,t) \leqslant DEV_\wedge(b,t) \leqslant CM_\wedge(b,t)\,.$$

As we mentioned, the proof of the upper bound follows the same lines of the triangle counts deviations. In particular, we follow most of the steps described in Chapter 4. Note that we need to find a constant C so that

$$\mathbb{P}\left(D_\wedge(G_m) \geqslant CM_\wedge(b,t)\right) \leqslant \exp(-b)\,.$$

Let us recall from Theorem 4.1 that the martingale expression for $D_\wedge(G_m)$ is given by

$$D_\wedge(G_m) = \sum_{i=1}^{m} \frac{(N-m)_2}{(N-i)_2} X_\wedge(G_i)$$

We introduce now a truncated version of $X_\wedge(G_i)$, as Freedman's inequality fails if an increment is too large. Let

$$X'_\wedge(G_i) := X_\wedge(G_i) 1_{X_\wedge(G_i) \leqslant 128tn}$$

and

$$Z_\wedge(G_i) := X_\wedge(G_i) 1_{X_\wedge(G_i) > 128tn}\,.$$

It follows that

$$D_\wedge(G_m) = D'_\wedge(G_m) + N^*_\wedge(G_m)$$

where

$$D'_\wedge(G_m) := \sum_{i=1}^{m} \frac{(N-m)_2}{(N-i)_2} X'_\wedge(G_i) \tag{7-4}$$

and

$$N^*_\wedge(G_m) := \sum_{i=1}^{m} \frac{(N-m)_2}{(N-i)_2} Z_\wedge(G_i)\,. \tag{7-5}$$

Thus, to prove the upper bound of Theorem 1.4 it suffices to prove that there exist absolute constants C_1 and C_2 such that each of the events $D'_\wedge(G_m) \geqslant C_1 M_\wedge(b,t)$ and $N^*_\wedge(G_m) \geqslant C_2 M_\wedge(b,t)$ has probability at most $\exp(-1.1b)$. We may then take $C = C_1 + C_2$ and the required bound follows by the triangle inequality.

7.1 Bounding $D'_\wedge(G_m)$

Note that $X'_\wedge(G_i) \leqslant X_\wedge(G_i)$ and $X_\wedge(G_i)$ are martingale increments. Thus, the sequence $D'_\wedge(G_i), i = 1, \ldots, m$ is a supermartingale. We shall use Freedman's inequality, Theorem 2.12, to bound the probability that $D'_\wedge(G_m) \geqslant C_1 M_\wedge(b,t)$. Let us write

$$Y'_i := \frac{(N-m)(N-m-1)}{(N-i)(N-i-1)} X'_\wedge(G_i)$$

for the increments of this supermartingale. Clearly, each coefficient above is at most 1 so $|Y'_i| \leqslant 128tn$, by the definition of the truncation.

In order to apply Freedman's inequality, we also need to bound the quadratic variation of the process. Note that $(Y'_i)^2 \leqslant (X'_\wedge(G_i))^2 \leqslant X_\wedge(G_i)^2$. We now apply Lemma 5.4 to find a constant C_3 such that, for each $i \leqslant m$,

$$\mathbb{E}\left[(Y'_i)^2 | G_{i-1}\right] \leqslant C_3 n^{-1} \kappa(b,t)$$

except with probability at most $\exp(-2b)$ (Note that if $b < 16tn$ we may deduce the same result with failure probability equals to $\exp(-32tn) \leqslant \exp(-2b)$). Taking an union bound over $i \leqslant m = tN$ gives

$$\sum_{i=1}^{m} \mathbb{E}\left[(Y'_i)^2 | G_{i-1}\right] \leqslant C_3 \kappa(b,t) tn\,.$$

except with probability at most $\exp(-1.5b)$.

We may now apply Freedman's inequality, Theorem 2.12, substituting the value of "b" given there being $C_3\kappa(b,t)$ and $R = 128tn$. We thus deduce that

$$\begin{aligned}\mathbb{P}\left(D'_\wedge(G_m) \geqslant C_1 M_\wedge(b,t)\right) &\leqslant \exp\left(\frac{-C_1^2 M_\wedge(b,t)^2}{2C_3\kappa(b,t) + 256 C_1 M_\wedge(b,t)tn}\right) + \exp(-1.5b) \\ &\leqslant \exp\left(\frac{-C_1^2 b}{2C_3 + 256 C_1}\right) + \exp(-1.5b)\end{aligned}$$

where the last inequality follows from (7-1) and (7-2). If we choose $C_1 \geqslant \max\{3C_3, 2^9\}$ the last probability above is at most $2\exp(-1.5b) \leqslant \exp(-1.1b)$, as we desired.

7.2 Bounding $N_\wedge^*(G_m)$

Let us recall that

$$X_\wedge(G_i) \leqslant A_\wedge(G_i) = 2\sum_{v\in e_i} d_v(G_{i-1})\,.$$

In particular, if $X_\wedge(G_i) > 128tn$ then the vertex of larger degree has degree at least $32tn$. Thus,

$$Z_\wedge(G_i) \;=\; X_\wedge(G_i)1_{X_\wedge(G_i)>128tn} \;\leqslant\; 4\sum_{v\in e_i} d_v(G_m)1_{d_v(G_m)>32tn}$$

where the last inequality also uses that $d_v(G_{i-1}) \leqslant d_v(G_m)$. By summing the expression above over $i \leqslant m$ and using a double counting argument, we obtain

$$\begin{aligned} N_\wedge^*(G_m) \;&\leqslant\; 4\sum_{v\in V} d_v(G_m)1_{d_v(G_m)>32tn} \sum_{e\in E(G_m)} 1_{v\in e} \\ &= 4\sum_{v\in V} d_v(G_m)^2 1_{d_v(G_m)>32tn}\,. \end{aligned}$$

If $b < 32tn$ then, except with probability at most $\exp(-1.1b)$ we have $\Delta(G_m) \leqslant 32tn$ by Lemma 5.1. In particular, the sum above is equal to 0, except with probability at most $\exp(-1.1b)$.

If $b \geqslant 32tn$, we apply Proposition 5.2 to obtain a constant C_4 such that, except with probability at most $\exp(-1.1b)$, we have

$$N_\wedge^*(G_m) \;\leqslant\; C_4\kappa^+(b,t) \;\leqslant\; 9C_4M_\wedge(b,t)$$

where the final inequality uses (7-3). Choosing $C_2 \geqslant 9C_4$, we finish the proof.

8
Triangle counts in $G(n, m)$ - upper bounds on deviations

In this chapter we prove the upper bound of our main result for deviations on triangle counts.

We first recall that

$$M(b,t) = \max\{\text{NORMAL}(b,t), \text{STAR}(b,t), \text{HUB}(b,t), \text{CLIQUE}(b,t)\}$$

where $\text{NORMAL}(b,t) := b^{1/2}t^{3/2}n^{3/2}$, $\text{STAR}(b,t) := b^2t1_{b<n\ell}/\ell^2$, $\text{HUB}(b,t) := btn1_{b\geqslant n\ell}/\ell$ and $\text{CLIQUE}(b,t) = b^{3/2}/\ell^{3/2}$.

We shall present some lower bounds on $M(b,t)$ which will be useful later on this chapter. First, we claim that

$$M(b,t) \geqslant bt^{3/2}n\,. \tag{8-1}$$

for all $b \geqslant 1$. This is easily verified, as we have $\text{NORMAL}(b,t) \geqslant bt^{3/2}n$ for $b \leqslant n$, $\text{STAR}(b,t) \geqslant bt^{3/2}n$ for $n < b \leqslant n\ell$ and $\text{HUB}(b,t) \geqslant bt^{3/2}n$ for $n\ell < b \leqslant tn^2\ell$. In particular, (8-1) implies that

$$M(b,t) \geqslant bt^2n \tag{8-2}$$

for all $t \leqslant 1$.

We also claim that

$$M(b,t) \geqslant t\kappa(b,t) \tag{8-3}$$

for all $b \geqslant 32tn$, where $\kappa(b,t)$ is as defined in Section 5.2. To verify this, note that $\text{NORMAL}(b,t) = b^{1/2}t^{3/2}n^{3/2} \geqslant t^2n^2 = t\kappa(b,t)$ whenever $32tn \leqslant b < t^{1/2}n\ell$. Also, $\text{STAR}(b,t) = t\kappa(b,t)$ whenever $t^{1/2}n\ell \leqslant b < n\ell$ and $\text{HUB}(b,t) = t\kappa(b,t)$ if $n\ell < b \leqslant tn^2\ell$.

Let us recall that $\text{DEV}_\triangle(b,t)$ is the minimal value of a such that $\mathbb{P}\left(N_\triangle(b,t) > \mathbb{E}\left[N_\triangle(b,t)\right] + a\right) \leqslant e^{-b}$. We may now restate Theorem 1.3.

Theorem 1.3. *There exist absolute constants c, C such that the following holds. For all $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant tn^2\ell$ we have*

$$cM(b,t) \leqslant DEV_\triangle(b,t) \leqslant CM(b,t)\,.$$

In order to prove the upper bound from above, we need to find a constant

C so that

$$\mathbb{P}\left(D_\triangle(G_m) \geqslant CM(b,t)\right) \leqslant \exp(-b)\,.$$

Let us recall from Theorem 4.1 that the martingale expression for $D_\triangle(G_m)$ is given by

$$D_\triangle(G_m) \;=\; \sum_{i=1}^{m}\left[3\frac{(N-m)_2(m-i)}{(N-i)_3}\,X_\wedge(G_i) \;+\; \frac{(N-m)_3}{(N-i)_3}\,X_\triangle(G_i)\right].$$

As we explained in Section 4.3,we shall truncate the increments $X_\wedge(G_i)$ and $X_\triangle(G_i)$. The values at which we truncate these increments are

$$K_\wedge \;=\; K_\wedge(b,t) \;:=\; \frac{2^8M(b,t)}{bt} \qquad \text{and} \qquad K_\triangle \;=\; K_\triangle(b,t) \;:=\; \frac{2^{16}M(b,t)}{b}$$

respectively. We set

$$X'_\wedge(G_i) \;=\; X_\wedge(G_i)1_{X_\wedge(G_i)\leqslant K_\wedge} \qquad \text{and} \qquad X'_\triangle(G_i) \;=\; X_\triangle(G_i)1_{X_\triangle(G_i)\leqslant K_\triangle}\,.$$

We also set

$$Z_\wedge(G_i) \;=\; X_\wedge(G_i)1_{X_\wedge(G_i)> K_\wedge} \qquad \text{and} \qquad Z_\triangle(G_i) \;=\; X_\triangle(G_i)1_{X_\triangle(G_i)> K_\triangle}\,.$$

Let us remark that both $X'_\wedge(G_i)$ and $Z_\wedge(G_i)$ are defined in a different way than Chapter 7, as the truncation needed for the triangle deviation process is different than the truncation needed for the cherry deviation process.

It follows that

$$D_\triangle(G_m) \;=\; D'_\triangle(G_m) \;+\; N^*_\triangle(G_m)$$

where

$$D'_\triangle(G_m) \;=\; \sum_{i=1}^{m}\left[\frac{3(N-m)_2(m-i)}{(N-i)_3}\,X'_\wedge(G_i) \;+\; \frac{(N-m)_3}{(N-i)_3}\,X'_\triangle(G_i)\right] \tag{8-4}$$

and

$$N^*_\triangle(G_m) \;=\; \sum_{i=1}^{m}\left[\frac{3(N-m)_2(m-i)}{(N-i)_3}\,Z_\wedge(G_i) \;+\; \frac{(N-m)_3}{(N-i)_3}\,Z_\triangle(G_i)\right].$$

Let us observe that the two coefficients that multiplies $X'_\wedge(G_i)$ and $X_\triangle(G_i)$ are at most $3t$ and 1, respectively. In particular, we have

$$N^*_\triangle(G_m) \;\leqslant\; \sum_{i=1}^{m}\left[3t\,Z_\wedge(G_i) \;+\; Z_\triangle(G_i)\right]. \tag{8-5}$$

A main difference from the proof for cherry counts is that there are two types of non-truncated increments, $Z_\wedge(G_i)$ and $Z_\triangle(G_i)$ and they are not bounded in the same way, as we shall see later.

Thus, to prove the upper bound of Theorem 1.3 it suffices to prove that there exist absolute constants C_1, C_2 and C_3 such that each of the events $D'_\triangle(G_m) \geqslant C_1 M_\triangle(b,t)$, $\sum_{i=1}^m tZ_\wedge(G_i) \geqslant C_2 M_\triangle(b,t)$ and $\sum_{i=1}^m Z_\triangle(G_i) \geqslant C_3 M_\triangle(b,t)$ has probability at most $\exp(-1.1b)$. We may then take $C = C_1 + C_2 + C_3$ and the required bound follows by the triangle inequality.

8.1 Bounding $D'_\triangle(G_m)$

We note that $X'_\wedge(G_i)$ and $X'_\triangle(G_i)$ are truncations of the martingale increments, $X_\wedge(G_i)$ and $X_\triangle(G_i)$ respectively, and only positive values are truncated. Thus, the sequence $D'_\triangle(G_i), i = 1, \ldots, m$ is a supermartingale. We shall use Freedman's inequality, Theorem 2.12, to bound the probability that $D'_\triangle(G_m) \geqslant C_1 M(b,t)$ Let us write

$$\mathbb{X}'_i := \frac{3(N-m)_2(m-i)}{(N-i)_3} X'_\wedge(G_i) + \frac{(N-m)_3}{(N-i)_3} X'_\triangle(G_i) \,,$$

for the increments of this supermartingale. Recall that the coefficients from above are at most $3t$ and 1, respectively. By the truncation which occurs in the definition of the X' variables we have immediately that $|\mathbb{X}'_i| \leqslant 2^{17} M(b,t)/b$ deterministically. Moreover,

$$|\mathbb{X}'_i|^2 \leqslant 18t^2 X_\wedge(G_i)^2 + 2X_\triangle(G_i)^2 \,. \tag{8-6}$$

We now control the quadratic variation of the process. Combining Lemma 5.4 and Lemma 6.7 together with the last inequality, we may find a constant C_4 so that, except with probability at most $\exp(-1.3b)$ and for all $i \leqslant m$,

$$\mathbb{E}\left[(\mathbb{X}'_i)^2 | G_{i-1}\right] \leqslant \frac{C_4 t^2 \kappa(b,t)}{n} + C_4 \max\{bt^2, b^2/n^2\}$$

if $b \geqslant n$ and

$$\mathbb{E}\left[(\mathbb{X}'_i)^2 | G_{i-1}\right] \leqslant \frac{C_4 t^2 \kappa(b,t)}{n} + C_4 t^2 n \,.$$

if $32tn \leqslant b < n$.

Let us note that $t\kappa(b,t)/n \leqslant \max\{bt^2, b^2/n^2\}$ if $b \geqslant n$ and $t\kappa(b,t)/n \leqslant t^2 n$ if $32tn \leqslant b < n$. Thus, by taking an union bound over $i \leqslant m$, we have, except with probability at most $\exp(-1.2b)$

$$\sum_{i=1}^m \mathbb{E}\left[(\mathbb{X}'_i)^2 | G_{i-1}\right] \leqslant 2C_4 \max\{bt^3 n^2, b^2 t\}$$

if $b \geqslant n$ and

$$\sum_{i=1}^{m} \mathbb{E}\left[(\mathbb{X}_i')^2 | G_{i-1}\right] \leqslant 2C_4 t^3 n^3 .$$

if $32tn \leqslant b < n$. We now observe that $\text{NORMAL}(b,t)^2/b = t^3n^3$, $\text{CLIQUE}(b,t)^2/b = b^2/\ell^3 \geqslant b^2 t$ and $M(b,t)^2/b \geqslant bt^3n^2$ by (8-1). Therefore, except with probability at most $\exp(-1.2b)$, we have

$$\sum_{i=1}^{m} \mathbb{E}\left[(\mathbb{X}_i')^2 | G_{i-1}\right] \leqslant \frac{2C_4 M(b,t)^2}{b}$$

for all $b \geqslant 32tn$.

We may now apply Freedman's inequality, Theorem 2.12, substituting the value of "b" in its statement by $2C_4M(b,t)^2/b$ and letting $R = 2^{17}M(b,t)/b$. We thus deduce thats

$$\mathbb{P}\left(D_\triangle'(G_m) \geqslant C_1 M(b,t)\right) \leqslant \exp\left(\frac{-C_1^2 M(b,t)^2 b}{4C_4 M(b,t)^2 + 2^{18} C_1 M(b,t)^2}\right) + \exp(-1.2b)$$

If we choose $C_1 \geqslant \max\{5C_4, 2^{19}\}$ the last probability above is at most $2\exp(-1.2b) \leqslant \exp(-1.1b)$, as we desired.

8.2
Bounding $\sum_{i=1}^{m} tZ_\wedge(G_i)$

In this section, we need to find a constant C_2 such that, except with probability at most $\exp(-1.1b)$ we have

$$\sum_{i=1}^{m} tZ_\wedge(G_i) \leqslant C_2 M_\triangle(b,t) .$$

The proof here is essentially the same presented in Section 7.2. Let $i \leqslant m$. We recall that

$$X_\wedge(G_i) \leqslant A_\wedge(G_i) = 2\sum_{v \in e_i} d_v(G_{i-1}) .$$

Thus, if $X_\wedge(G_i) > K_\wedge$ then there must be a vertex $v \in e_i$ such that $d_v(G_{i-1}) > K_\wedge/4$. So

$$Z_\wedge(G_i) = X_\wedge(G_i) 1_{X_\wedge(G_i) > K_\wedge} \leqslant 4\sum_{v \in e_i} d_v(G_m) 1_{d_v(G_m) > K_\wedge/4} . \quad (8\text{-}7)$$

By summing the expression above over $i \leqslant m$ and using a double counting argument, we obtain

$$\sum_{i=1}^{m} Z_{\wedge}(G_i) \leqslant 4\sum_{v\in V} d_v(G_m) 1_{d_v(G_m)>K_\wedge/4} \sum_{e\in E(G_m)} 1_{v\in e}$$
$$= 4\sum_{v\in V} d_v(G_m)^2 1_{d_v(G_m)>K_\wedge/4}\,.$$

We note that (8-2) implies that $K_\wedge = 2^8 M(b,t)/(bt) \geqslant 2^8 tn$. In particular, the last summation above is 0 if $\Delta(G_m) \leqslant 64tn$. If $b < 32tn$, Lemma 5.1 guarantees that this event occurs except with probability at most $\exp(-1.1b)$

If $b \geqslant 32tn$, we note that $d_v(G_m) \leqslant 2D_v(G_m)$ for all v such that $d_v(G_m) \geqslant 32tn$ to deduce that

$$\sum_{i=1}^{m} tZ_{\wedge}(G_i) \leqslant 16t \sum_{v\in V(G_m)} D_v(G_m)^2\,.$$

We now use Proposition 5.2 to find a constant C_2 so that, except with probability at most $\exp(-1.1b)$, we have

$$\sum_{i=1}^{m} tZ_{\wedge}(G_i) \leqslant C_2 t\kappa(b,t) \leqslant C_2 M(b,t)$$

where the last inequality uses (8-3). This is the desired result.

8.3 Bounding $\sum_{i=1}^m Z_\triangle(G_i)$

In this section, we need to find a constant C_3 such that, except with probability at most $\exp(-1.1b)$ we have

$$\sum_{i=1}^{m} Z_{\triangle}(G_i) \leqslant C_3 M_\triangle(b,t)\,.$$

Let $i \leqslant m$. We recall that

$$X_\triangle(G_i) \leqslant A_\triangle(G_i) = 6d_{e_i}(G_{i-1})\,.$$

Thus, if $X_\triangle(G_i) > K_\triangle$ then $d_{e_i}(G_{i-1}) > K_\triangle/6$. Summing over $i \leqslant m$, we get

$$\sum_{i=1}^{m} Z_{\triangle}(G_i) \leqslant 6\sum_{i=1}^{m} d_{e_i}(G_{i-1}) 1_{\{d_{e_i}(G_{i-1})>K_\triangle/6\}}\,. \tag{8-8}$$

We now have to deal with the sum of codegrees of edges which have large codegree. As we mentioned in Chapter 6, we may divide these edges into two categories: those who have its large codegree explained by a large degree of one of its vertices and those who have this large codegree explained by the behaviour of the codegree itself. We call the first type good, as we may bound their contribution to the sum by the sum of their degrees, just as we have seen in the previous section. For each $i \leqslant m$, we say that $e_i = uv$ is **good** if $d_{e_i}(G_{i-1}) > K_\triangle/6$ and there exists a time $i \leqslant j \leqslant m$ such that

$$d_{e_i}(G_{j-1}) \leqslant 64t(d_u(G_{j-1}) + d_v(G_{j-1}))\,.$$

We also say that an edge e_i is **bad** if e_i is not good and $d_{e_i}(G_{i-1}) > K_\triangle/6$.

Clearly,

$$\sum_{i=1}^{m} Z_\triangle(G_i) \leqslant 6\sum_{i=1}^{m} d_{e_i}(G_{i-1})1_{e_i \text{ is good}} + 6\sum_{i=1}^{m} d_{e_i}(G_{i-1})1_{e_i \text{ is bad}}\,. \tag{8-9}$$

The next two Lemmas provide bounds for the sum of codegrees of good and bad edges.

Lemma 8.1. *There exists an absolute constant C such that the following holds. Suppose that $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant tn^2\ell$. Then, except with probability at most* $\exp(-1.2b)$*,*

$$\sum_{i=1}^{m} d_{e_i}(G_{i-1})1_{\{e_i \textit{ is good}\}} \leqslant CM(b,t)\,.$$

Lemma 8.2. *There exists an absolute constant C such that the following holds. Suppose that $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant tn^2\ell$. Then, except with probability at most* $\exp(-1.2b)$*,*

$$\sum_{i=1}^{m} d_{e_i}(G_{i-1})1_{\{e_i \textit{ is bad}\}} \leqslant CM(b,t)\,.$$

Let us see how the rest of the proof goes from the two Lemmas above. Let C' and C'' be the constants obtained in Lemmas 8.1 and 8.2, respectively. It follows from (8-9) that we have $\sum_{i=1}^m Z_\triangle(G_i) \leqslant 6(C'+C'')M(b,t)$, except with probability at most $2\exp(-1.3b) \leqslant \exp(-1.2b)$. The proof follows by choosing $C_3 = 6(C'+C'')$.

We shall now bound the sum of codegrees of good edges.

Proof of Lemma 8.1. Let $i \leqslant m$ and suppose that e_i is good. Then $d_{e_i}(G_{i-1}) \leqslant 64t(d_u(G_m)+d_v(G_m))$, as the degree and codegree are non-decreasing over time. In particular, the largest degree in G_m of the vertices in e_i must be at least $K_\triangle/2^9 t$. Therefore,

$$d_{e_i}(G_{i-1})1_{\{e_i \text{ is good}\}} \leqslant 128t\sum_{v\in e_i} d_v(G_m)1_{d_v(G_m)>K_\triangle/2^9 t}\,.$$

By summing over $i \leqslant m$ and repeating the double counting argument used to bound $\sum_{i=1}^m tZ_\wedge(G_i)$ we get

$$\sum_{i=1}^m d_{e_i}(G_{i-1})1_{\{e_i \text{ is good}\}} \leqslant 128t \sum_{v\in V(G_m)} d_v(G_m)1_{d_v(G_m)>K_\triangle/2^9t} \sum_{e\in E(G_m)} 1_{v\in e}$$
$$= 128t \sum_{v\in V(G_m)} d_v(G_m)^2 1_{d_v(G_m)>K_\triangle/2^9t}.$$

We note that (8-2) implies that $K_\triangle/2^9t = 2^7M(b,t)/bt \geqslant 2^7tn$. In particular, the last summation above is automatically 0 if $\Delta(G_m) \leqslant 128tn$. If $b < 32tn$, Lemma 5.1 guarantees that this event occurs except with probability at most $\exp(-1.3b)$.

If $b \geqslant 32tn$, we note that $d_v(G_m) \leqslant 2D_v(G_m)$ for all v such that $d_v(G_m) \geqslant 2^7tn$. Therefore,

$$128t \sum_{v\in V(G_m)} d_v(G_m)^2 1_{d_v(G_m)>K_\triangle/2^9t} \leqslant 256t \sum_{v\in V(G_m)} D_v(G_m)^2 .$$

We now use Proposition 5.2 to find a constant C so that, except with probability at most $\exp(-1.3b)$, we have

$$\sum_{i=1}^m d_{e_i}(G_{i-1})1_{\{e_i \text{ is good}\}} \leqslant Ct\kappa(b,t) \leqslant CM(b,t)$$

where the last inequality uses (8-3). This completes the proof. □

We shall now bound the sum of codegrees of bad edges.

Proof of Lemma 8.2. Recall that we want to find a constant C so that for all $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and all $3\log n \leqslant b \leqslant tn^2\ell$ we have

$$\sum_{i=1}^m d_{e_i}(G_{i-1})1_{\{e_i \text{ is bad}\}} \leqslant CM(b,t)\,. \tag{8-10}$$

except with probability at most $\exp(-1.3b)$.

We shall use a dyadic decomposition on the sum given above. Let $J = \lceil \log_2(6n/K_\triangle)\rceil$ and set j_0 to be the minimum integer j so that $2^j \geqslant \ell$. We shall find constants C_5 and C_6 so that we have

$$\sum_{i=1}^m d_{e_i}(G_{i-1})1_{e_i \text{ is bad}} \sum_{j=0}^{j_0-1} 1_{d_{e_i}(G_{i-1})\in[2^jK_\triangle/6,2^{j+1}K_\triangle/6)} \leqslant C_5M(b,t) \tag{8-11}$$

and

$$\sum_{i=1}^m d_{e_i}(G_{i-1})1_{e_i \text{ is bad}} \sum_{j=j_0}^{J} 1_{d_{e_i}(G_{i-1})\in[2^jK_\triangle/6,2^{j+1}K_\triangle/6)} \leqslant C_6M(b,t)\,, \tag{8-12}$$

each of them occurring except with probability at most $\exp(-1.4b)$.

Letting $C = C_5 + C_6$ we obtain from the two inequalities above together that (8-10) occurs except with probability at most $2\exp(-1.4b) \leqslant \exp(-1.3b)$, as we desired.

Let us prove that (8-12) occurs except with probability at most $\exp(-1.4b)$. Since the degree is non-decreasing over time, we have

$$\begin{aligned}&\sum_{i=1}^{m} d_{e_i}(G_{i-1}) 1_{e_i \text{ is bad}} \sum_{j=j_0}^{J} 1_{d_{e_i}(G_{i-1}) \in [2^j K_\triangle/6, 2^{j+1} K_\triangle/6)} \\ &\leqslant \sum_{j=j_0}^{J} \frac{2^{j+1} K_\triangle}{6} \sum_{i=1}^{m} 1_{e_i \text{ is bad}} 1_{d_{e_i}(G_m) > 2^j K_\triangle/6} \\ &\leqslant \frac{K_\triangle}{3} \sum_{j=j_0}^{J} 2^j \sum_{i=1}^{m} 1_{e_i \text{ is bad}} 1_{d_{e_i}(G_m) > 2^j K_\triangle/6} \,.\end{aligned}$$

Recall from Section 6.1 that we defined the sets

$$H_k(G) := \{uw : d_{uw}(G) \in [2^k t^2 n, 2^{k+1} t^2 n) \,,\, d_u(G), d_w(G) \leqslant 2^{k-5} tn\}$$

for $k \in K_2 := \{10, \ldots, \lfloor 2\log_2(1/t) \rfloor\}$.

Suppose $e_i = uw$ is a bad edge with $d_{e_i}(G_m) > 2^j K_\triangle/6$. By definition, $64t(d_u(G_m) + d_w(G_m)) \leqslant d_{e_i}(G_m)$. In particular, if $d_{e_i}(G_m) \leqslant 2^{k+1}t^2 n$ for some $k \in K_2$ then both $d_u(G_m)$ and $d_w(G_m)$ are at most $2^{k-5}tn$. Moreover, $d_{e_i}(G_m) > 2^k t^2 n$ for some $k \geqslant \lfloor j + \log_2(K_\triangle/6t^2 n) \rfloor$.

Therefore, the set of bad edges with $d_{e_i}(G_m) > 2^j K_\triangle/6$ is contained in the union of $H_k(G_m)$ over $k \in \{\lfloor j + \log_2(K_\triangle/6t^2 n)\rfloor, \ldots, \lfloor 2\log_2(1/t)\rfloor\}$. Thus,

$$\frac{K_\triangle}{3} \sum_{j=j_0}^{J} 2^j \sum_{i=1}^{m} 1_{e_i \text{ is bad}} 1_{d_{e_i}(G_m) > 2^j K_\triangle/6} \leqslant \frac{K_\triangle}{3} \sum_{j=j_0}^{J} 2^j \sum_{k=\lfloor j + \log_2(K_\triangle/6t^2 n)\rfloor}^{\lfloor 2\log_2(1/t)\rfloor} h_k(G_m)$$

where $h_k(G_m) = |H_k(G_m)|$.

We note that Proposition 6.6 gives a constant C_7 so that $h_k(G_m) \leqslant C_7 b^2/(k^2 2^{2k} t^4 n^2)$ except with probability at most $\exp(-1.5b)$. Since $2\log_2(1/t) \leqslant 4\ell$ and $\log(\ell) \leqslant b$ we have, except with probability at most $\exp(-1.4b)$,

$$\frac{K_\triangle}{3} \sum_{j=j_0}^{J} 2^j \sum_{k=\lfloor j + \log_2(K_\triangle/6t^2 n)\rfloor}^{\lfloor 2\log_2(1/t)\rfloor} h_k(G_m) \leqslant \frac{C_8 b^2 K_\triangle}{3t^4 n^2} \sum_{j=j_0}^{J} 2^j \sum_{k=\lfloor j + \log_2(K_\triangle/6t^2 n)\rfloor}^{\lfloor 2\log_2(1/t)\rfloor} \frac{1}{k^2 2^{2k}} \,.$$

Observe that sum of $1/(k^2 2^{2k})$ starting from some k_0 is bounded by $4k_0/3$. Therefore, the expression on the right-hand-side of the inequality above is bounded by

$$\frac{C_7 b^2 K_\triangle}{3t^4n^2} \frac{4t^4n^2}{3K_\triangle^2 \log_2^2(K_\triangle/6t^2n)} \sum_{j=j_0}^{J} \frac{1}{2^j} \leqslant \frac{2C_7b^2}{2^{j_0} K_\triangle \ell^2}$$

where we used (8-1) to deduce that $\log_2(K_\triangle/6t^2n) \geqslant \ell$ in the last inequality.

Recall that j_0 was defined so that $2^{j_0} \geqslant \ell$. Therefore, except with probability at most $\exp(-1.4b)$,

$$\sum_{i=1}^{m} d_{e_i}(G_{i-1}) 1_{e_i \text{ is bad}} \sum_{j=j_0}^{J} 1_{d_{e_i}(G_{i-1}) \in [2^j K_\triangle/6, 2^{j+1}K_\triangle/6)} \leqslant \frac{2C_7b^2}{K_\triangle \ell^3} = \frac{C_7 M(b,t)}{2^{15}}$$

where we used that $M(b,t) \geqslant \text{CLIQUE}(b,t) = b^{3/2}/\ell^{3/2}$ in the last inequality. The proof of (8-12) follows by choosing $C_6 = C_7/2^{15}$.

Finally, we may prove that (8-11) occurs except with probability at most $\exp(-1.4b)$.

We start bounding each codegree by its maximum value in the dyadic interval, i.e., $2^{j+1}K_\triangle/6$. After rearranging the terms, we get

$$\begin{aligned} &\sum_{i=1}^{m} d_{e_i}(G_{i-1}) 1_{e_i \text{ is bad}} \sum_{j=0}^{j_0-1} 1_{d_{e_i}(G_{i-1}) \in [2^j K_\triangle/6, 2^{j+1}K_\triangle/6)} \\ &\leqslant \frac{K_\triangle}{3} \sum_{j=0}^{j_0-1} 2^j \sum_{i=1}^{m} 1_{e_i \text{ is bad}} 1_{d_{e_i}(G_{i-1}) \in [2^j K_\triangle/6, 2^{j+1}K_\triangle/6)} \, . \end{aligned} \tag{8-13}$$

Let $j \in \{0, 1, \ldots, j_0 - 1\}$ and suppose $e_i = uw$ is a bad edge with $d_{e_i}(G_{i-1}) \in [2^j K_\triangle/6, 2^{j+1}K_\triangle/6)$. By definition, $64t(d_u(G_m) + d_w(G_m)) \leqslant d_{e_i}(G_m)$. In particular, if $d_{e_i}(G_{i-1}) \leqslant 2^{k+1}t^2n$ for some $k \in K_2$ then both $d_u(G_{i-1})$ and $d_w(G_{i-1})$ are at most $2^{k-5}tn$. Moreover, $2^k t^2 n \leqslant d_{e_i}(G_{i-1}) < 2^{k+2}t^2n$ where $k = \lfloor j + \log_2(K_\triangle/6t^2n) \rfloor$.

Let $\mathcal{H}_j(G_{i-1}) = H_k(G_{i-1}) \cup H_{k+1}(G_{i-1})$ where $k = \lfloor j + \log_2(K_\triangle/6t^2n) \rfloor$. Then each bad edge e_i with $d_{e_i}(G_{i-1}) > 2^j K_\triangle/6$ must be in $\mathcal{H}_j(G_{i-1})$. In the spirit of this observation, we define the random sets

$$A_{j,m} := \{e_i \in E(G_m) : e_i \in \mathcal{H}_j(G_{i-1})\} \, .$$

We also set $a_{j,m} := |A_{j,m}|$. For each $j \in \{0, 1, \ldots, j_0 - 1\}$, we have

$$\sum_{i=1}^{m} 1_{e_i \text{ is bad}} 1_{d_{e_i}(G_{i-1}) > 2^j K_\triangle/6} \leqslant a_{j,m} \, .$$

Thus,

$$\frac{K_\triangle}{3}\sum_{j=0}^{j_0-1} 2^j \sum_{i=1}^{m} 1_{e_i \text{ is bad}} 1_{d_{e_i}(G_{i-1})>2^j K_\triangle/6} \leqslant \frac{K_\triangle}{3}\sum_{j=0}^{j_0-1} 2^j a_{j,m}\,. \tag{8-14}$$

We recall that Proposition 6.6 gives a constant C_8 so that $h_k(G_{i-1}) \leqslant C_8 b^2/(k^2 2^{2k} t^4 n^2)$ for all $i \leqslant m$ and all $k \in K_2 = \{10,\dots,2\log_2(1/t)\}$ except with probability at most $\exp(-1.5b)$. Now, observe that $\log_2(K_\triangle/6t^2n) - 1 \geqslant \ell/2$ according to (8-1). Therefore, except with probability at most $\exp(-1.5b)$, we have

$$h_k(G_{i-1}) \leqslant 36 C_8 b^2/(\ell^2 K_\triangle^2)$$

for all $i \leqslant m$ and all $k \geqslant \lfloor \log_2(K_\triangle/6t^2n)\rfloor$. Let us define E as the event that

$$|\mathcal{H}_j(G_{i-1})| \leqslant 72 C_8 b^2/(\ell^2 K_\triangle^2) \tag{8-15}$$

for all $j \in \{0,1,\dots,j_0-1\}$ and all $i \leqslant m$. Since $|\mathcal{H}_j(G_{i-1})| \leqslant h_k(G_{i-1}) + h_{k+1}(G_{i-1})$ with $k = \lfloor j + \log_2(K_\triangle/6t^2n)\rfloor$, we have $\mathbb{P}(E^c) \leqslant \exp(-1.5b)$. From this together with (8-13) and (8-14), we deduce the following: in order to prove that (8-11) happens except with probability at most $\exp(-1.4b)$, it suffices to prove that

$$\mathbb{P}\left(\frac{K_\triangle}{3}\sum_{j=0}^{j_0-1} 2^j a_{j,m} 1_E > C_5 M(b,t)\right) \leqslant \exp(-1.5b)\,.$$

Moreover, as $K_\triangle = 2^{16} M(b,t)/b$, we only need to prove that

$$\mathbb{P}\left(\sum_{j=0}^{j_0-1} 2^j a_{j,m} 1_E > \frac{C_5 b}{2^{14}}\right) \leqslant \exp(-1.5b)\,.$$

Fix $j \in \{0,1,\dots,j_0-1\}$ and suppose we are on the event E. We note that $a_{j,m}$ may be seen as a sum of Bernoulli random variables $X_i = 1_{e_i \in \mathcal{H}_j(G_{i-1})}$ with $i = 1,2,\dots,m$. We also observe that, by definition, the sets $\mathcal{H}_j(G_{i-1})$ are completely determined by the first $i-1$ edges. Then

$$\mathbb{P}\left(1_{e_i \in \mathcal{H}_j(G_{i-1})} | e_1,\dots,e_{i-1}\right) \leqslant \frac{|\mathcal{H}_j(G_{i-1})|}{N} \leqslant \frac{72 C_9 b^2}{N \ell^2 K_\triangle^2}$$

where the last inequality comes from the definition of the event E given on (8-15).

We let $p := (72 C_8 b^2)/(N\ell^2 K_\triangle^2)$. Also, consider a random variable Y which is the sum of m Bernoulli independent random variables Y_i such that $\mathbb{P}(Y_i = 1) = p$. Then

$$\mathbb{E}[Y] = pm = \frac{72 C_8 b^2 m}{N\ell^2 K_\triangle^2} \leqslant \frac{72 C_8 t b^2}{\ell^2 K_\triangle^2}\,.$$

An application of Theorem 2.10 ((2-3)) with $\theta = C_5 b/2^{15}\mu\ell$ gives

$$\mathbb{P}\left(Y > \frac{C_5 b}{2^{15}\ell}\right) \leqslant \exp\left(-\frac{C_5 b}{2^{15}\ell}\log\left(\frac{C_5 \ell K_\triangle^2}{2^{23}C_8 bt}\right)\right)$$
$$\leqslant \exp\left(-\frac{C_5 b}{2^{15}\ell}\log\left(\frac{C_5}{2^{23}C_8 t^{1/2}}\right)\right)$$

where the last inequality above uses that $\ell K_\triangle^2/(bt) \geqslant 1/t^{1/2}$. Then, choosing $C_5 = \max\{2^{16}, 2^{23}C_8\}$ we obtain

$$\mathbb{P}\left(Y > \frac{C_5 b}{2^{15}\ell}\right) \leqslant \exp(-1.6b)\,.$$

We now observe that Lemma 2.17 gives

$$\mathbb{P}\left(a_{j,m}1_E > \frac{C_5 b}{2^{15}\ell}\right) \leqslant \mathbb{P}\left(Y > \frac{C_5 b}{2^{15}\ell}\right).$$

Therefore,

$$\mathbb{P}\left(a_{j,m}1_E > \frac{C_5 b}{2^{15}\ell}\right) \leqslant \exp(-1.6b)\,.$$

Recall that j_0 was chosen so that $2^{j_0} \leqslant 2\ell$. Taking an union bound over $j = \{0, 1, \dots, j_0 - 1\}$, we obtain

$$\mathbb{P}\left(\sum_{j=0}^{j_0-1} 2^j a_{j,m}1_E > \frac{C_5 b}{2^{14}}\right) = \mathbb{P}\left(\sum_{j=0}^{j_0-1} 2^j a_{j,m}1_E > \frac{C_5 b}{2^{15}\ell}\sum_{j=0}^{j_0-1} 2^j\right) \leqslant \exp(-1.5b)\,.$$

This concludes the proof. □

9
Lower bounds on deviations in $G(n,m)$

In this chapter we prove the lower bound of our main result for deviations on triangle counts. We also provide a lower bound for cherry counts.

Let us restate Theorem 1.3.

Theorem 1.3. *There exist absolute constants c, C such that the following holds. For all $t \geqslant Cn^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant tn^2\ell$ we have*

$$cM(b,t) \leqslant DEV_{\triangle}(b,t) \leqslant CM(b,t)\,.$$

We shall divide the proof of the lower bound into two parts. For the normal regime, we may use the lower bound of Freedman's inequality, Theorem 2.14. For the other three regimes (star, hub and clique) we provide an explicit structure (graph) which is present in $G(n,m)$ with probability at least $\exp(-b)$ and is directly responsible for the triangle count deviation.

9.1
Normal regime

In this section we prove that there exists a constant $c > 0$ such that

$$\mathbb{P}\left(D_{\triangle}(G_m) \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) \geqslant \exp(-b) \tag{9-1}$$

for all pairs (b,t) in the normal regime. This gives the lower bound of Theorem 1.3.

The first idea would be to use $D'_{\triangle}(G_m)$, our truncated version of the original martingale. However, we cannot apply Theorem 2.14 to this sequence, as it is a supermartingale rather than a martingale. In order to have a martingale again, let us define $D''_{\triangle}(G_m) := \sum_{i=1}^{m} \mathbb{X}''_i$ as the sum of the rebalanced increments

$$\mathbb{X}''_i := \frac{3(N-m)_2(m-i)}{(N-i)_3}\,X''_{\wedge}(G_i) + \frac{(N-m)_3}{(N-i)_3}\,X''_{\triangle}(G_i)\,,$$

where $X''_{\wedge}(G_i)$ and $X''_{\triangle}(G_i)$, are defined by

$$X''_{\wedge}(G_i) = X'_{\wedge}(G_i) + \mathbb{E}\left[Z_{\wedge}(G_i)|G_{i-1}\right] \text{ and } X''_{\triangle}(G_i) = X'_{\triangle}(G_i) + \mathbb{E}\left[Z_{\triangle}(G_i)|G_{i-1}\right]\,.$$

Note that

$$\begin{aligned}\mathbb{E}\left[X''_{\wedge}(G_i)|G_{i-1}\right] &= \mathbb{E}\left[X'_{\wedge}(G_i)|G_{i-1}\right] + \mathbb{E}\left[Z_{\wedge}(G_i)|G_{i-1}\right] \\ &= \mathbb{E}\left[X_{\wedge}(G_i)|G_{i-1}\right] = 0\,.\end{aligned}$$

The same reasoning shows that $\mathbb{E}\left[X''_{\triangle}(G_i)|G_{i-1}\right] = 0$. Therefore, $D''_{\triangle}(G_m)$ is the final value of a martingale with increments $\mathbb{X}''_i$.

The next result gives the desired lower bound for the martingale $D''_{\triangle}(G_m)$. We shall prove this using Freedman's lower bound inequality, Theorem 2.14.

Proposition 9.1. *There is an absolute constant $c > 0$ such that the following holds. Let $n^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/2$, and suppose that $b \geqslant 3\log n$ is such that we are in the normal regime (i.e., $M(b,t) = N(b,t)$). Then*

$$\mathbb{P}\left(D''_{\triangle}(G_m) \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) \geqslant \exp(-b)\,.$$

Before we step into the proof of this proposition, we state and prove some bounds on the quadratic variation of $Z_{\wedge}(G_i)$ and $Z_{\triangle}(G_i)$.

Lemma 9.2. *There is an absolute constant $c > 0$ such that the following holds. Let $n^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/2$ and $3\log n \leqslant b \leqslant n$. Except with probability at most $\exp(-b)$, we have*

$$\mathbb{E}\left[(Z_{\wedge}(G_i))^2|G_{i-1}\right] \leqslant \frac{cb^2}{n\ell_b^2}$$

for all $1 \leqslant i \leqslant m$. Moreover, except with probability at most $\exp(-b)$,

$$\mathbb{E}\left[(Z_{\triangle}(G_i))^2|G_{i-1}\right] \leqslant \frac{ct^2b^2}{n\ell_b^2}$$

for all $1 \leqslant i \leqslant m$.

Proof. Fix $1 \leqslant i \leqslant m$. Using the inequality (8-7), we have

$$\mathbb{E}\left[Z_{\wedge}(G_i)^2|G_{i-1}\right] \leqslant \frac{32}{N-i+1}\sum_{e\notin E(G_{i-1})}\sum_{v\in e} d_v(G_m)^2 1_{d_v(G_m)>K_{\wedge}/4}$$

where $K_{\wedge}(b,t) = 2^8 M(b,t)/(bt) \geqslant 2^8 tn$. Now, each vertex appear in at most $n-1$ edges and $N-i+1 \geqslant n(n-1)/4$. The last upper bound gives

$$\mathbb{E}\left[Z_{\wedge}(G_i)^2|G_{i-1}\right] \leqslant \frac{2^7}{n}\sum_{v\in V} d_v(G_m)^2 1_{d_v(G_m)>2^6 tn}\,. \tag{9-2}$$

If $b \leqslant 32tn$ the sum above is 0 except with probability $\exp(-1.3b)$ by Lemma 5.1. If $b > 32tn$ then Proposition 5.2 gives a constant $c' > 0$ such that, except with probability at most $\exp(-2b)$

$$\mathbb{E}\left[Z_{\wedge}(G_i)^2|G_{i-1}\right] \leqslant \frac{2^7c'b^2}{n\ell_b^2}\,. \tag{9-3}$$

The proof of the first identity of the statement follows by an union bound over $i \leqslant m \leqslant \exp(b)$.

Let us prove the second identity. Using the inequality (8-8), we have

$$\mathbb{E}\left[Z_{\triangle}(G_i)^2|G_{i-1}\right] \leqslant \frac{36}{N-i+1}\sum_{e\notin E(G_{i-1})} d_e(G_m)^2 1_{\{d_e(G_m)>K_\triangle/6\}} \tag{9-4}$$

where $K_\triangle = 2^{16}M(b,t)/b$.

The proof now uses a very similar reasoning to the one that we used to bound $\sum_{i=1}^m Z_\triangle(G_i)$ in Section 8.3. In particular, we split the sum above into two parts, depending on whether or not the codegree of an edge is bounded by t times the degree of one of its vertices. More precisely, suppose that $d_e(G_m) \leqslant 64t(d_u(G_m)+d_v(G_m)$ for $e=uv$. Then at least one of the vertices has degree at least $K_\triangle/(2^9t) \geqslant 2^6tn$. Thus,

$$d_e(G_m)^2 1_{\{K_\triangle/6<d_e(G_m)\leqslant 64t(d_u(G_m)+d_v(G_m)\}} \leqslant 2^{13}t^2\sum_{v\in e} d_v(G_m)^2 1_{d_v(G_m)>2^6tn}\,.$$

Note that the last sum above is the same sum from (9-2) multiplied by 2^6t^2. We may use the same argument from the proof of (9-3) to deduce that, except with probability at most $\exp(-2b)$,

$$\frac{36}{N-i+1}\sum_{e\notin E(G_{i-1})} d_e(G_m)^2 1_{\{K_\triangle/6<d_e(G_m)\leqslant 64t(d_u(G_m)+d_v(G_m))\}} \leqslant \frac{2^{13}c't^2b^2}{n\ell_b^2}\,. \tag{9-5}$$

Recall from Section 6.1 that we defined the sets

$$H_k(G) := \{uw : d_{uw}(G)\in[2^kt^2n, 2^{k+1}t^2n)\,,\, d_u(G), d_w(G)\leqslant 2^{k-5}tn\}$$

for $k \in K_2 := \{10,\ldots,\lfloor 2\log_2(1/t)\rfloor\}$. Now, we use a dyadic argument to bound the sum over edges e such that $d_e(G_m) > 64t(d_u(G_m)+d_v(G_m))$. We note that such an edge must be in H_k for some $k_0 \leqslant k \leqslant 2\log_2(1/t)$ where $k_0 = \log_2(K_\triangle/t^2n) \geqslant \ell$. Let us also recall that Proposition 6.6 gives a constant c_0 so that $h_k(G_m) \leqslant c_0b^2/(k^2 2^{2k}t^4n^2)$ except with probability at most $\exp(-2b)$. Therefore,

$$\frac{36}{N-i+1}\sum_{e\notin E(G_{i-1})} d_e(G_m)^2 1_{\{K_\triangle/6<d_e(G_m)\}}1_{\{64t(d_u(G_m)+d_v(G_m))\leqslant d_e(G_m)\}} \leqslant$$

$$\leqslant \frac{72}{n^2}\sum_{k=k_0}^{2\log_2(1/t)} 2^{2k+2}t^4n^2h_k(G_m) \leqslant 2^9c_0\sum_{k=k_0}^{2\log_2(1/t)}\frac{1}{k^2} \tag{9-6}$$

except with probability at most $\exp(-2b)$. Since $k_0 \geqslant \ell$ and $2\log_2(1/t) \leqslant 4\ell$, the last expression above is at most $2^{11}c_0/\ell$. From this together with (9-4), (9-5) and (9-6) we deduce that, except with probability at most $\exp(-1.1b)$,

$$\mathbb{E}\left[Z_\triangle(G_i)^2|G_{i-1}\right] \leqslant (2^{13}c' + 2^{11}c_0)\frac{t^2b^2}{n\ell_b^2}. \tag{9-7}$$

The proof of the second identity of the statement follows by an union bound over $i \leqslant m \leqslant \exp(b)$. □

The main prerequisite to prove this proposition is a lower bound on the quadratic variation of the process. For this, we need to give lower bounds on the quadratic variation of both $X''_\wedge(G_i)$ and $X''_\triangle(G_i)$. We now prove a series of lemmas towards this goal.

Lemma 9.3. *Let $p \in (0,1/2)$. For all $\delta > 0$ there exist a constant $c > 0$ and $n \in \mathbb{N}$ such that the following holds. For all $n' \geqslant n$ and all intervals $I \subseteq \mathbb{R}$ of length $2cp^{1/2}n^{1/2}$, if $X \sim Bin(n',p)$ then $\mathbb{P}(X \in I) < \delta$.*

Proof. Note that $2cp^{1/2}n^{1/2}$ is smaller than the standard deviation of $X \sim Bin(n',p)$. As n increases, the central limit theorem implies that the maximum over $I \subseteq \mathbb{R}$ of $\mathbb{P}(X \in I)$ converges to a limit that is at most $\mathbb{P}(Z \in [-c,c]) \leqslant 2c$ where $Z \sim N(0,1)$. □

Lemma 9.4. *There is a constant $c > 0$ such that the following holds. Let $p \in (0,1/2)$ and $G \sim G(n,p)$. Then, except with probability at most* $\exp(-3n)$,

$$|\{u \in V(G) : p(n-1) - cp^{1/2}n^{1/2} \leqslant d_u(G) \leqslant p(n-1) + cp^{1/2}n^{1/2}\}| \leqslant \frac{n}{2}.$$

Proof. Let c be the constant given by Lemma 9.3 for $\delta = e^{-6}$ and let

$$I := [p(n-1) - cp^{1/2}n^{1/2}, p(n-1) + cp^{1/2}n^{1/2}].$$

Fix an enumeration of the vertices $v_1, v_2, \ldots, v_n$ and define, for all $1 \leqslant i \leqslant n/2$, the events

$$E_i := \{d_{v_1}(G) \in I, \ldots, d_{v_i}(G) \in I\}.$$

We claim that $\mathbb{P}(E_1) < e^{-8}$ and

$$\mathbb{P}(d_{v_i}(G) \in I|E_{i-1}) < e^{-8} \tag{9-8}$$

for all $2 \leqslant i \leqslant n/2$. Clearly, the claim implies that

$$\mathbb{P}(d_{v_i}(G) \in I \text{ for all } 1 \leqslant i \leqslant n/2) = \mathbb{P}(E_1)\prod_{i=2}^{n/2}\mathbb{P}(d_{v_i}(G) \in I|E_{i-1}) < e^{-4n}.$$

Taking an union bound over all choices for $v_1, v_2, \ldots, v_{n/2}$ we obtain $\mathbb{P}(|I| \leqslant n/2) \leqslant 2^n e^{-4n} \leqslant e^{-3n}$, as desired.

Let us now prove the claim. We begin with $i = 1$. We know that $d_{v_1}(G)$ has distribution $X \sim Bin(n-1, p)$. Then, using our choice of c and Lemma 9.3, we obtain that $\mathbb{P}(E_1) = \mathbb{P}(X \in I) < e^{-8}$. Now, fix $2 \leqslant i \leqslant n/2$. Given a vertex v, we write $N(v)$ for the neighbourhood of v in G and $N_i(v) := N(v) \cap \{v_1, \ldots, v_i\}$. We then have

$$\begin{aligned}&\mathbb{P}\left(d_{v_i}(G) \in I | E_{i-1}\right) \\ &= \sum_{N \subseteq \{v_1, \ldots, v_{i-1}\}} \mathbb{P}\left(d_{v_i}(G) \in I | N_{i-1}(v_i) = N\right) \mathbb{P}\left(N_{i-1}(v_i) = N | E_{i-1}\right).\end{aligned}$$

Now, suppose that $N_{i-1}(v_i) = N$. Then $d_{v_i}(G)$ is the sum of the number of neighbours already observed ($|N|$) and a variable X such that $X \sim Bin(n-i, p)$. Therefore,

$$\mathbb{P}\left(d_{v_i}(G) \in I | N_{i-1}(v_i) = N\right) = \mathbb{P}\left(X \in I - |N|\right)$$

where $I - |N|$ represents the interval I translated N units to the left. Then, by our choice of c, Lemma 9.3 implies that the probability above is at most e^{-6}. Thus,

$$\mathbb{P}\left(d_{v_i}(G) \in I | E_{i-1}\right) < e^{-8} \sum_{N \subseteq \{v_1, \ldots, v_{i-1}\}} \mathbb{P}\left(N_{i-1}(v_i) = N | E_{i-1}\right) = e^{-8}.$$

□

Lemma 9.5. *For all $c > 0$ there exists $\eta > 0$ such that the following holds. Let $\eta^{-2} n^{-1} \leqslant p \leqslant 1/2$ and $G \sim G(n,p)$. We have, except with probability at most* $\exp(-3n)$,

$$\left| \sum_{u \in A} d_u(G) - |A| p(n-1) \right| \leqslant c p^{1/2} n^{3/2} \tag{9-9}$$

for every set $A \subseteq V(G)$ with $|A| \leqslant \eta n$ where $\eta = c^2/(8+8c)$.

Proof. We note that $X = \sum_{u \in A} d_u(G)$ is a sum of two binomial random variables, $e(A)$ and $e(A, A^c) + e(A)$ where $e(A)$ counts the number of edges with both vertices in A and $e(A, A^c)$ counts the number of edges with one vertex in A and other in A^c. In particular, $\mathbb{E}[X] = |A| p(n-1)$. It follows from Theorem 2.10 that the event given by (9-9) may fail with probability at most

$$2 \exp\left(\frac{-c^2 p n^3}{2(|A| p n + c p^{1/2} n^{3/2})}\right) \leqslant 2 \exp\left(\frac{-c^2 n}{2\eta(1+c)}\right).$$

Choosing $\eta \leqslant c^2/(8+8c)$, the probability above is at most $\exp(-4n)$. Taking an union bound over all subsets of size at most ηn completes the proof. □

Corollary 9.6. *Let* $100n^{-1} \leqslant p \leqslant 1/2$ *and* $G \sim G(n,p)$. *Except with probability at most* $\exp(-2n)$, *there are at most* $n/100$ *vertices* u *so that* $d_u(G) < pn/2$ *and there are at most* $n/100$ *vertices so that* $d_u(G) > 2pn$.

Proof. Suppose that there exist some sets $A, B \subseteq V(G)$ such that $|A| = |B| = n/100$ and $d_u(G) < pn/2$ for all $u \in A$ and $d_w(G) > 2pn$ for all $w \in B$. Then

$$\sum_{u \in A} d_u(G) \leqslant |A|pn/2 < |A|p(n-1) - p^{1/2}n^{3/2}$$

and

$$\sum_{w \in B} d_u(G) \geqslant |B|pn/2 > |B|p(n-1) + p^{1/2}n^{3/2}$$

Applying Lemma 9.5 with $c = 1$, we deduce that the probability of both events above occur is at most $2\exp(-3n) \leqslant \exp(-2n)$. □

Lemma 9.7. *Let* $p \in (0, 1/4]$ *and* $G \sim G(n,p)$. *There exists* $c > 0$ *such that the following holds. For each* $u \in V(G)$, *let* F_u *be the event that* $d_u(G) \in [pn/2, 2pn]$ *and*

$$X_u^- := |w \notin N(u) : d_{uw}(G) \leqslant pd_u(G) - cpn^{1/2}| \leqslant n/20 \tag{9-10}$$

or

$$X_u^+ := |w \notin N(u) : d_{uw}(G) \geqslant pd_u(G) + cpn^{1/2}| \leqslant n/20\,. \tag{9-11}$$

Then

$$\mathbb{P}\left(\cup_{u \in V(G)} F_u\right) \leqslant \exp(-0.03n)\,.$$

Proof. Fix $u \in V(G)$. Let U be a set of vertices so that $pn/2 \leqslant |U| \leqslant 2pn$ and suppose that $N(u) = U$ where $N(u)$ is the neighbourhood of u in G_p. For each $w \notin U$, the codegree $d_{uw}(G_p)$ has a binomial distribution $Bin(|U|, p)$. Let $c' > 0$ be the constant given by Lemma 9.3 for $\delta = 0.2$. We note that $p^{1/2}|U|^{1/2} \geqslant pn^{1/2}/2$. So, Lemma 9.3 gives

$$\mathbb{P}\left(d_{uw}(G_p) \leqslant p|U| - cpn^{1/2} | N(u) = U\right) \geqslant 0.4$$

and

$$\mathbb{P}\left(d_{uw}(G_p) \geqslant p|U| + cpn^{1/2} | N(u) = U\right) \geqslant 0.4$$

where $c = c'/2$. We now observe that for a fixed vertex u and assuming that $N(u) = U$ as above, the family of variables $d_{uw}(G_p)$ with $w \notin U$ is independent. Therefore, given that $N(u) = U$, both X_u^- and X_u^+ are binomial random variables, $Bin(n - |U|, p_0)$, with $n - |U| \geqslant 0.5n$ and $0.4 \leqslant p_0 < 0.5$. Thus, an application of Theorem 2.10 to each of the variables X_u^- and X_u^+ shows that

$$\mathbb{P}\left(X_u^- \leqslant \frac{n}{20}\right) \leqslant \exp\left(-0.045n\right)$$

and

$$\mathbb{P}\left(X_u^+ \leqslant \frac{n}{20}\right) \leqslant \exp\left(-0.045n\right) .$$

Thus, $\mathbb{P}\left(F_u | N(u) = U\right) \leqslant 2\exp(-0.045n) \leqslant \exp(-0.04n)$ for all choices of $U \subseteq V$ such that $pn/2 \leqslant |U| \leqslant 2pn$. Therefore,

$$\mathbb{P}\left(F_u\right) \leqslant \max_{U \subset V : pn/2 \leqslant |U| \leqslant 2pn} \mathbb{P}\left(F_u | N(u) = U\right) \leqslant \exp(-0.04n) .$$

Taking an union bound over all $u \in V(G)$ we obtain

$$\mathbb{P}\left(\cup_{u \in V(G)} F_u\right) \leqslant n\exp(-0.04n) \leqslant \exp(-0.03n) .$$

□

Lemma 9.8. *Let* $100n^{-1} \leqslant p \leqslant 1/4$ *and* $G \sim G(n,p)$. *There exists* $c > 0$ *such that the following holds. Except with probability at most* $\exp(-0.02n)$, *there are two sets* $A, B \subseteq E(K_n) \setminus E(G)$ *so that* $|A| \geqslant 0.024n^2$, $|B| \geqslant 0.024n^2$ *and*

$$d_{u'w'}(G_p) - d_{uw}(G_p) \geqslant 2cpn^{1/2}$$

for all pairs $uw \in A$ *and* $u'w' \in B$.

Proof. Consider the random set of vertices

$$V' := \{u \in V(G) : d_u(G) \in [pn/2, 2pn]\} .$$

Let $n' = |V'|$. We order the vertices of V' according to their degree in G, i.e., we write $V' = \{u_1, u_2, \ldots, u_{n'}\}$ with $d_{u_i}(G) \leqslant d_{u_{i+1}}(G)$ for all $i < n'$. Let $u^* = u_{\lfloor n'/2 \rfloor}$. Also, let $c > 0$ be the constant given by Lemma 9.7. For each $i \leqslant \lfloor n'/2 \rfloor$, we define the following set:

$$A_i := \{w \notin N(u_i) : d_{u_i w}(G) \leqslant pd_{u^*}(G) - cpn^{1/2}\}$$

Also, for each $\lfloor n'/2 \rfloor < i \leqslant n'$, we define

$$B_i := \{w \notin N(u_i) : d_{u_i w}(G) \geqslant pd_{u^*}(G) + cpn^{1/2}\}.$$

Let $i \leqslant \lfloor n'/2 \rfloor$. Observing that $d_{u_i}(G) \leqslant d_{u^*}(G)$ and recalling the definition of $X^-_{u_i}$ given in (9-10), we have $X^-_{u_i} \leqslant |A_i|$. Analogously, if $\lfloor n'/2 \rfloor < i \leqslant n'$, we shall observe that $d_{u^*}(G) \leqslant d_{u_i}(G) \leqslant 2pn$ and recall the definition of $X^+_{u_i}$ given in (9-11) to deduce that $X^+_{u_i} \leqslant |B_i|$.

We finally define the sets

$$A := \{(u_i, w) : i \leqslant \lfloor n'/2 \rfloor \text{ and } w \in A_i\}$$

and

$$B := \{(u_i, w) : i > \lfloor n'/2 \rfloor \text{ and } w \in B_i\}\,.$$

Note that if $(u, w) \in A$ and $(u', w') \in B$, we have

$$d_{u'w'}(G) - d_{uw}(G) \geqslant pd_{u^*}(G) + cpn^{1/2} - (pd_{u^*}(G) - cpn^{1/2}) = 2cpn^{1/2}\,.$$

Now, Corollary 9.6 and Lemma 9.7 together show that, except with probability at most $2\exp(-0.03n) \leqslant \exp(-0.02n)$, we have $n' \geqslant 0.98n$ vertices and both X^-_u and X^+_u are larger than $n/20$ for all $u \in V'$. Therefore, except with probability at most $\exp(-0.01n)$, we have

$$|A| = \left\lfloor \frac{n'}{2} \right\rfloor |A_i| \geqslant 0.48n \left(\frac{n}{20}\right) = 0.024n^2 \text{ and } |B| = \left\lceil \frac{n'}{2} \right\rceil |B_i| \geqslant 0.024n^2$$

where we used that $|A_i| \geqslant X^-_{u_i}$ and $|B_i| \geqslant X^+_{u_i}$. □

We are now ready to prove a lower bound for the conditional variance of the increments $X_\triangle(G_i)$.

Lemma 9.9. *There is an absolute constant $c > 0$ such that the following holds. Let $n^{-1/2} \leqslant t \leqslant 1/32$. Then, except with probability at most* $\exp(-0.01n)$*,*

$$\mathbb{E}\left[X_\triangle(G_i)^2 | G_{i-1}\right] \geqslant ct^2 n\,.$$

for all $m/2 \leqslant i \leqslant m$.

Proof. Let $m/2 \leqslant i \leqslant m$ and $s = (i-1)/N$. We recall from the proof of Lemma 6.7 that

$$\mathbb{E}\left[X_\triangle(G_i)^2 | G_{i-1}\right] = Var(A_\triangle(G_i)|G_{i-1})\,.$$

Now, let G_i and G'_i be random graphs containing G_{i-1} but with the i-th edge chosen independently. We may rewrite the expression above as

$$\begin{aligned}\mathbb{E}\left[X_\triangle(G_i)^2|G_{i-1}\right] &= \frac{1}{2}\mathbb{E}\left[(A_\triangle(G_i)-A_\triangle(G_i'))^2|G_{i-1}\right] \\ &= \frac{1}{2}\left(\frac{1}{N-i+1}\right)^2 \sum_{e,e'\notin E(G_{i-1})} 36\left(d_e(G_{i-1})-d_{e'}(G_{i-1})\right)^2 .\end{aligned}$$

We observe that Lemma 9.8 gives a constant $c>0$ such that the following holds: if $G\sim G(n,p)$ then, except with probability at most $\exp(-0.01n)$ there are two sets $A,B\subseteq E(G)^c$ such that $|A|\geqslant 0.024n^2$, $|B|\geqslant 0.024n^2$ and $d_e(G_p)-d_{e'}(G_p)\geqslant 2cpn^{1/2}$ for all $e'\in A$ and $e\in B$. Once again, we take $p=s$ and note that there is probability at least n^{-2} that G has $i-1=sN$ edges. Thus, except with probability at most $n^2\exp(-0.02n)\leqslant\exp(-0.015n)$ there are two sets $A,B\subseteq E(G_{i-1})^c$ as described above. In particular,

$$\left(d_e(G_{i-1})-d_{e'}(G_{i-1})\right)^2 \geqslant 4c^2s^2n$$

for all $e\in B$ and $e'\in A$. Therefore, except with probability at most $\exp(-0.015n)$,

$$\begin{aligned}\mathbb{E}\left[X_\triangle(G_i)^2|G_{i-1}\right] &\geqslant \frac{1}{2}\left(\frac{1}{N-i+1}\right)^2 \sum_{e\in B,e'\in A} 36\left(d_e(G_{i-1})-d_{e'}(G_{i-1})\right)^2 \\ &\geqslant \frac{36(0.024n^2)^2}{N^2}(4c^2s^2n)\geqslant 0.08c^2t^2n\end{aligned}$$

where we used that $s\geqslant t/4$ in the last inequality. Taking an union bound over $m/2\leqslant i\leqslant m$ gives the desired result. □

Let us now give a lower bound for the conditional variance of the increments $X''_\triangle(G_i)$.

Lemma 9.10. *There is an absolute constant $c>0$ such that the following holds. Let $n^{-1/2}\leqslant t\leqslant 1/32$ and $3\log n\leqslant b$. Then, except with probability at most* $\exp(-0.001n)$,

$$\mathbb{E}\left[X''_\triangle(G_i)^2|G_{i-1}\right] \geqslant ct^2n\,.$$

for all $m/2\leqslant i\leqslant m$.

Proof. Let $m/2\leqslant i\leqslant m$. We recall that $X_\triangle(G_i)=X'_\triangle(G_i)+Z_\triangle(G_i)$ and, by definition, $X'_\triangle(G_i)Z_\triangle(G_i)=0$. So,

$$\mathbb{E}\left[X_\triangle(G_i)^2|G_{i-1}\right] = \mathbb{E}\left[X'_\triangle(G_i)^2|G_{i-1}\right]+\mathbb{E}\left[Z_\triangle(G_i)^2|G_{i-1}\right]. \qquad (9\text{-}12)$$

Moreover, $X'_\triangle(G_i)=X''_\triangle(G_i)-\mathbb{E}\left[Z_\triangle(G_i)|G_{i-1}\right]$ and $\mathbb{E}\left[X''_\triangle(G_i)|G_{i-1}\right]=0$. Thus,

$$\mathbb{E}\left[X'_{\triangle}(G_i)^2|G_{i-1}\right] = \mathbb{E}\left[X''_{\triangle}(G_i)^2|G_{i-1}\right] + \mathbb{E}\left[Z_{\triangle}(G_i)|G_{i-1}\right]^2 . \qquad (9\text{-}13)$$

From the two inequalities above, we obtain

$$\mathbb{E}\left[X''_{\triangle}(G_i)^2|G_{i-1}\right] \geqslant \mathbb{E}\left[X_{\triangle}(G_i)^2|G_{i-1}\right] - 2\mathbb{E}\left[Z_{\triangle}(G_i)^2|G_{i-1}\right] .$$

Note that, by Lemma 9.2 there is a constant $c' > 0$ such that

$$\mathbb{E}\left[Z_{\triangle}(G_i)^2|G_{i-1}\right] \leqslant \frac{c't^2n}{\ell} \ll t^2n .$$

except with probability at most $\exp(-0.01n)$. Furthermore, Lemma 9.9 gives a constant $c > 0$ so that

$$\mathbb{E}\left[X_{\triangle}(G_i)^2|G_{i-1}\right] \geqslant 2ct^2n$$

except with probability at most $\exp(-0.01n)$. From the three last inequalities, we obtain

$$\mathbb{E}\left[X''_{\triangle}(G_i)^2|G_{i-1}\right] \geqslant ct^2n .$$

except with probability at most $\exp(-0.01n)$. The proof follows by taking an union bound over $m/2 \leqslant i \leqslant m$. □

Our bound on the quadratic variation of the process $D''_{\triangle}(G_m)$ comes from the next result.

Lemma 9.11. *There is an absolute constant $c > 0$ such that the following holds. Let $n^{-1/2} \leqslant t \leqslant 1/2$ and $3\log n \leqslant b$. Except with probability at most $\exp(-cn)$, we have*

$$\mathbb{E}\left[(\mathbb{X}''_i)^2|G_{i-1}\right] \geqslant ct^2n$$

for all $m/2 \leqslant i \leqslant m$.

Proof. Let us define

$$\alpha_{\wedge}(i) := \frac{3(N-m)_2(m-i)}{(N-i)_3} \text{ and } \alpha_{\triangle}(i) := \frac{(N-m)_3}{(N-i)_3} .$$

Note that $\alpha_{\wedge}(i) \leqslant 6t$ and $1/32 \leqslant \alpha_{\triangle}(i) \leqslant 1$. Moreover,

$$\begin{aligned}(\mathbb{X}''_i)^2 &= \alpha_{\wedge}(i)^2X''_{\wedge}(G_i)^2 + 2\alpha_{\wedge}(i)\alpha_{\triangle}(i)X''_{\wedge}(G_i)X''_{\triangle}(G_i) + \alpha_{\triangle}(i)^2X''_{\triangle}(G_i)^2 \\ &\geqslant 2^{-10}X''_{\triangle}(G_i)^2 - 12t|X''_{\wedge}(G_i)X''_{\triangle}(G_i)| . \qquad (9\text{-}14)\end{aligned}$$

We shall prove that $\mathbb{E}\left[X''_{\triangle}(G_i)^2|G_{i-1}\right]$ is the dominating term on the expansion of $\mathbb{E}\left[(\mathbb{X}''_i)^2|G_{i-1}\right]$.

Let us recall that Lemma 5.4 gives a constant C such that, except with probability at most $\exp(-n)$,

$$t^2\mathbb{E}\left[X_\wedge(G_i)^2|G_{i-1}\right] \leqslant \frac{Ct^2n}{\ell} \ll t^2n\,.$$

We also observe that the same argument used to deduce (9-12) and (9-13) now applied to $X_\wedge(G_i)$ shows that

$$\mathbb{E}\left[X''_\wedge(G_i)^2|G_{i-1}\right] = \mathbb{E}\left[X_\wedge(G_i)^2|G_{i-1}\right]-\mathbb{E}\left[Z_\wedge(G_i)^2|G_{i-1}\right]-\mathbb{E}\left[Z_\wedge(G_i)|G_{i-1}\right]^2\,.$$

In particular,

$$t^2\mathbb{E}\left[X''_\wedge(G_i)^2|G_{i-1}\right] \leqslant t^2\mathbb{E}\left[X_\wedge(G_i)^2|G_{i-1}\right] \ll t^2n$$

except with probability at most $\exp(-n)$. It follows from the inequality above together with Holder's inequality that

$$12t\mathbb{E}\left[|X''_\wedge(G_i)X''_\triangle(G_i)||G_{i-1}\right] \ll 12tn^{1/2}(\mathbb{E}\left[X''_\triangle(G_i)^2|G_{i-1}\right])^{1/2} \qquad (9\text{-}15)$$

except with probability at most $\exp(-n)$.

Let c_0 be the constant given by Lemma 9.10 so that $\mathbb{E}\left[(X''_\triangle)(G_i)^2|G_{i-1}\right] \geqslant c_0t^2n$, except with probability at most $\exp(-0.001n)$. It follows from (9-14) and (9-15) that

$$\mathbb{E}\left[(\mathbb{X}''_i)^2|G_{i-1}\right] \geqslant (2^{-10}c_0 - 12c_0^{1/2})t^2n$$

except with probability at most $\exp(-0.001n)$. The proof now follows by taking an union bound over $m/2 \leqslant i \leqslant m$ and taking $c < \min\{c_0 - 12c_0^{1/2}, 10^{-5}\}$. □

After proving a lower bound on the conditional variance of the increments of $D''_\triangle(G_m)$ we are now ready to prove Proposition 9.1.

Proof of Proposition 9.1. We fix t such that $n^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/2$ and fix b such that we are in the normal regime, i.e., $M(b,t) = \text{NORMAL}(b,t)$ (in particular, $3\log n \leqslant b \leqslant n$). We shall prove that there is a constant $c > 0$ such that

$$\mathbb{P}\left(D''_\triangle(G_m) \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) \geqslant \exp(-b)\,.$$

As we mentioned, we intend to use the converse Freedman's inequality, given in Theorem 2.14.

We recall that $D''_\triangle(G_m) = \sum_{i=1}^m \mathbb{X}''_i$ is a martingale. We observe that Theorem 2.14 gives a lower bound on the probability that $\sum_{i=1}^{m'} \mathbb{X}''_i$ for some $m' \leqslant m$. In order to get around this issue, we write $\alpha' = cb^{1/2}t^{3/2}n^{3/2}$ (with c

to be determined later) and note that it suffices to prove that

$$\mathbb{P}\left(\sum_{i=1}^{m} \mathbb{X}_i'' \geqslant \alpha' \mid \sum_{i=1}^{m'} \mathbb{X}_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) \geqslant 1/2 \tag{9-16}$$

and

$$\mathbb{P}\left(\sum_{i=1}^{m'} \mathbb{X}_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) \geqslant \exp(-b/2)\,. \tag{9-17}$$

We shall use Theorem 2.12 to prove (9-16) and Theorem 2.14 to prove (9-17).

In order to use each of the theorems above, we need to bound the maximum of the increments as well as their conditional variance. We note that

$$|\mathbb{X}_i''| \leqslant 6t(|X_\wedge'(G_i)| + \mathbb{E}\left[Z_\wedge(G_i)|G_{i-1}\right]) + |X_\triangle'(G_i)| + \mathbb{E}\left[Z_\triangle(G_i)|G_{i-1}\right]\,.$$

We claim that, except with probability at most $\exp(-0.01n)$,

$$|\mathbb{X}_i''| \leqslant \frac{8M(b,t)}{b} = \frac{8t^{3/2}n^{3/2}}{b^{1/2}} \tag{9-18}$$

for all $i \leqslant m$. To see this, we note that $6t|X_\wedge'(G_i)| + |X_\triangle'(G_i)| \leqslant 7M(b,t)/b$ by the choice of the truncation used to define $X_\wedge'(G_i)$ and $X_\triangle'(G_i)$. Now, applying Lemma 9.2 we obtain, except with probability at most $\exp(-0.01n)$,

$$\mathbb{E}\left[Z_\wedge(G_i)|G_{i-1}\right] \leqslant \mathbb{E}\left[Z_\triangle(G_i)^2|G_{i-1}\right]^{1/2} \leqslant \frac{ctn^{1/2}}{\ell} \ll \frac{t^{3/2}n^{3/2}}{b^{1/2}}$$

and

$$t\mathbb{E}\left[Z_\triangle(G_i)|G_{i-1}\right] \leqslant t\mathbb{E}\left[Z_\triangle(G_i)^2|G_{i-1}\right]^{1/2} \leqslant \frac{ctn^{1/2}}{\ell} \ll \frac{t^{3/2}n^{3/2}}{b^{1/2}}$$

Thus, (9-18) holds, except with probability at most $\exp(-0.01n)$.

We now claim that there are two constants $c_0, c_1 > 0$ such that, except with probability at most $\exp(-c_0 n)$,

$$c_0 t^3 n^3 \leqslant \sum_{i=1}^{m} \mathbb{E}\left[(\mathbb{X}_i'')^2|G_{i-1}\right] \leqslant c_1 t^3 n^3\,. \tag{9-19}$$

The lower bound is a direct consequence of Lemma 9.11. For the upper bound we note that

$$\mathbb{E}\left[(\mathbb{X}_i'')^2|G_{i-1}\right] \leqslant 72t^2\mathbb{E}\left[X_\wedge(G_i)^2|G_{i-1}\right] + 2\mathbb{E}\left[X_\triangle(G_i)^2|G_{i-1}\right]$$

for all $i \leqslant m$. Applying Lemma 5.4 and Lemma 6.7, we may find a constant

c_2 such that, except with probability at most $\exp(-c_0 n)$,

$$\mathbb{E}\left[(\mathbb{X}_i'')^2|G_{i-1}\right] \leqslant \frac{c_2 t^2 n}{\ell^2} + c_2 t^2 n\,.$$

The upper bound of (9-19) follows by summing the last expression over $i \leqslant m$ and taking $c_1 = 2c_2$.

Let us now prove (9-16). Suppose that $\sum_{i=1}^{m} \mathbb{X}_i'' \geqslant 2\alpha'$ for some $m' \leqslant m$. We shall prove that after conditioning on this event, the probability that $\sum_{i=1}^{m} \mathbb{X}_i'' \leqslant \alpha'$ is very small. Consider the process starting at $m'+1$ with increments $-\mathbb{X}_i''$. Clearly, $\sum_{i=1}^{m} \mathbb{X}_i'' \leqslant \alpha'$ if and only if $\sum_{i=m'+1}^{m} -\mathbb{X}_i'' \geqslant \alpha'$. To bound this probability we shall use Theorem 2.12 with "a" $= \alpha' = cb^{1/2}t^{3/2}n^{3/2}$, "$b$" $= c_1 t^3 n^3$ and $R = 8t^{3/2}n^{3/2}/b^{1/2}$. Then

$$\begin{aligned}
&\mathbb{P}\left(\sum_{i=m'+1}^{m} -\mathbb{X}_i'' \geqslant \alpha' \mid \sum_{i=1}^{m'} \mathbb{X}_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) \leqslant \\
&\leqslant \exp\left(\frac{-(\alpha')^2}{2c_1 t^3 n^3 + 8\alpha' t^{3/2} n^{3/2}/b^{1/2}}\right) + \exp\left(-c_0 n\right) + \exp\left(-0.01 n\right) \\
&\leqslant \exp\left(\frac{-c^2 b t^3 n^3}{(2c_1 + 8c) t^3 n^3}\right) + 2\exp\left(-c_0 n\right) \\
&\leqslant \exp\left(\frac{-c^2 n}{2c_1 + 8c}\right) + 2\exp\left(-c_0 n\right) \leqslant 1/2\,.
\end{aligned}$$

We now prove (9-17). We may apply Theorem 2.14 with "a" $= 2\alpha' = 2cb^{1/2}t^{3/2}n^{3/2}$, "$b$" $= c_0 t^3 n^3$ and $R = 8t^{3/2}n^{3/2}/b^{1/2}$. We obtain

$$\mathbb{P}\left(\sum_{i=1}^{m'} \mathbb{X}_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) \geqslant \frac{1}{2}\exp\left(\frac{-(1+4\delta)\alpha^2}{2\beta}\right) - 2\exp\left(-c_0 n\right)$$

where $\delta > 0$ satisfy $c_0/(2c) \geqslant \delta^{-2}$ and $4c^2 b/c_0 > 16\delta^{-2}\log(64\delta^{-2})$. We may choose $c < c_0/2$ so that $\delta = 1$. Then

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{m'} \mathbb{X}_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) &\geqslant \exp\left(\frac{-5\alpha^2}{2\beta} - 1\right) - 2\exp\left(-c_0 n\right) \\
&\geqslant \exp\left(-c_0 b/50\right) - \exp\left(-c_0 n/2\right) \geqslant \exp\left(-b\right)
\end{aligned}$$

where we have assumed that $c < c_0/100$ in the last inequality. □

We complete this section by showing how Proposition 9.1 implies that there exists a constant $c > 0$ such that

$$\mathbb{P}\left(D_\triangle(G_m) \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) \geqslant \exp\left(-b\right)$$

for all pairs (b,t) in the normal regime.

Fix $n^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/32$ and let $3\log n \leqslant b \leqslant n$. Let c be the constant given by Proposition 9.1 such that $\mathbb{P}\left(D''_{\triangle}(G_m) \geqslant 2cb^{1/2}t^{3/2}n^{3/2}\right) \geqslant \exp(-b/2)$. Then

$$\mathbb{P}\left(D_{\triangle}(G_m) \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) + \mathbb{P}\left(D''_{\triangle}(G_m) - D_{\triangle}(G_m) \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) \geqslant \exp(-b/2).$$

Therefore, it suffices to prove that

$$\mathbb{P}\left(D''_{\triangle}(G_m) - D_{\triangle}(G_m) \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) \leqslant \exp(-b) .$$

Let us note that

$$\begin{aligned} D''_{\triangle}(G_m) - D_{\triangle}(G_m) &\leqslant \sum_{i=1}^{m} 6t(X''_{\wedge}(G_i) - X_{\wedge}(G_i)) + X''_{\triangle}(G_i) - X_{\triangle}(G_i) \\ &\leqslant \sum_{i=1}^{m} 6t\mathbb{E}\left[Z_{\wedge}(G_i)|G_{i-1}\right] + \mathbb{E}\left[Z_{\triangle}(G_i)|G_{i-1}\right] . \end{aligned}$$

Now, Lemma 9.2 implies that, except with probability at most $\exp(-2b)$,

$$\mathbb{E}\left[Z_{\triangle}(G_i)|G_{i-1}\right] \leqslant \mathbb{E}\left[Z^2_{\triangle}(G_i)|G_{i-1}\right]^{1/2} \leqslant \frac{ctb}{n^{1/2}\ell_b} \ll b^{1/2}t^{3/2}n^{3/2}$$

for all $i \leqslant m$. Moreover, the same holds for $t\mathbb{E}\left[Z_{\wedge}(G_i)|G_{i-1}\right]$. Taking an union bound over $i \leqslant m$ we deduce that

$$\mathbb{P}\left(\sum_{i=1}^{m} 6t\mathbb{E}\left[Z_{\wedge}(G_i)|G_{i-1}\right] + \mathbb{E}\left[Z_{\triangle}(G_i)|G_{i-1}\right] \geqslant cb^{1/2}t^{3/2}n^{3/2}\right) \leqslant \exp(-b) .$$

This completes the proof of (9-1) in the normal regime.

9.2 The other regimes

In this section we complete the proof of the lower bound of Theorem 1.3 by proving that there exists a constant $c > 0$ such that

$$\mathbb{P}\left(D_{\triangle}(G_m) \geqslant M(b,t)\right) \geqslant \exp(-b)$$

for all pairs (b,t) such that $t \geqslant n^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant tn^2\ell$.

We may assume that the pair (b,t) is not in the normal regime, as the result has already been proved for such pairs in the previous section. For each of the three remaining regimes (star, hub and clique) we shall provide a graph

G_* containing at most $b/(10\ell)$ edges and such that

$$\mathbb{E}\left[N_\triangle(G_{m-b/10\ell}\cup G_*)\right] > \mathbb{E}\left[N_\triangle(G_m)\right] + 2cM(b,t)$$

with c to be determined later. Let us see that this is sufficient to prove the lower bound. We first observe that the last inequality is equivalent to

$$\mathbb{E}\left[N_\triangle(G_m)|E_m\right] > \mathbb{E}\left[N_\triangle(G_m)\right] + 2cM(b,t)\,.$$

where E_m is the event that $G_*\subseteq G_m$. We also observe that, since G_* contains at most $b/(10\ell)$ edges, we have

$$\mathbb{P}\left(E_m\right) \geqslant \frac{(m)_{b/10\ell}}{(N)_{b/10\ell}} \geqslant \left(\frac{t}{2}\right)^{b/10\ell} \geqslant \exp\left(-\frac{b}{2}\right)\,.$$

Moreover,

$$\mathbb{E}\left[N_\triangle(G_m)|E_m\right] \leqslant \mathbb{E}\left[N_\triangle(G_m)\right] + cM(b,t) + n^3\mathbb{P}\left(D_\triangle(G_m) > cM(b,t)|E_m\right)\,.$$

Therefore,

$$\mathbb{P}\left(D_\triangle(G_m) > cM(b,t)|E_m\right) \geqslant cM(b,t)n^{-3} \geqslant n^{-3}$$

and so

$$\mathbb{P}\left(D_\triangle(G_m) > cM(b,t)\right) \geqslant n^{-3}\exp\left(-b/2\right) \geqslant \exp\left(-b\right)\,.$$

Let us now give the three structures that give the required lower bound in each of the three remaining regimes.

The star regime: We may take G_* to be a star of degree $b/(10\ell)$. This adds at least $tb^2/(100\ell^2)$ to the expected number of triangles in G_m.

The hub regime: We may take G_* to be a hub which consists of $b/(10n\ell)$ vertices of degree $n-1$. This adds at least $btn/(80\ell)$ to the expected number of triangles in G_m.

The clique regime: We may take G_* to be a clique with $b^{1/2}/(4\ell^{1/2})$ vertices. This adds at least $(b/4\ell)^{3/2}$ to the expected number of triangles in G_m.

9.3 Lower bounds on cherry deviations

We now prove the lower bound of our main theorem about cherry deviations. We shall restate this theorem now.

Theorem 1.4. *There exist absolute constants c, C such that the following holds. Suppose that $2n^{-1}\log n \leqslant t \leqslant 1/2$ and that $3\log n \leqslant b \leqslant tn^2\ell$. Then*

$$cM_\wedge(b,t) \leqslant DEV_\wedge(b,t) \leqslant CM_\wedge(b,t)\,.$$

The proof is analogue to the proof for triangle deviations. We start with the normal regime for cherry counts. We shall prove that there exists a constant $c > 0$ such that

$$\mathbb{P}\left(D_\wedge(G_m) \geqslant cb^{1/2}tn^{3/2}\right) \geqslant \exp(-b) \tag{9-20}$$

for all pairs (b,t) in the normal regime of cherry deviation, i.e., for $t > n^{-1/2}(\log n)^{1/2}$ and $3\log n \leqslant b \leqslant t^{2/3}n\ell^{4/3}$.

Repeating the ideas from triangle deviations, we define the martingale $D''_\wedge(G_m) = \sum_{i=1}^m Y''_i$ where

$$Y''_i = \frac{(N-m)_2}{(N-i)_2} X''_\wedge(G_i)$$

and

$$X''_\wedge(G_i) = X'_\wedge(G_i) + \mathbb{E}\left[Z_\wedge(G_i)|G_{i-1}\right]\,.$$

The following result gives the lower bound for $D''_\wedge(G_m)$.

Proposition 9.12. *There is an absolute constant $c > 0$ such that the following holds. Let $n^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/2$, and suppose that $3\log n \leqslant b \leqslant t^{2/3}n\ell^{4/3}$. Then*

$$\mathbb{P}\left(D''_\wedge(G_m) \geqslant cb^{1/2}tn^{3/2}\right) \geqslant \exp(-b)\,.$$

Again we shall use the converse Freedman's inequality, Theorem 2.14 to prove this proposition. In order to use this method, we need to give a lower bound for the conditional variance of the increments Y''_i. We start proving a lower bound for $\mathbb{E}\left[X^2_\wedge(G_i)|G_{i-1}\right]$.

Lemma 9.13. *There is an absolute constant $c > 0$ such that the following holds. Let $n^{-1/2} \leqslant t \leqslant 1/32$. Then, except with probability at most $\exp(-2n)$,*

$$\mathbb{E}\left[X_\wedge(G_i)^2|G_{i-1}\right] \geqslant ctn$$

for all $m/2 \leqslant i \leqslant m$.

Proof. Let $m/2 \leqslant i \leqslant m$ and let $p = s = (i-1)/N$. Let us recall that $A_\wedge(G_i) = 2d_u(G_{i-1}) + 2d_v(G_{i-1})$ where $e_i = uv$ is the i-th edge added in the process G_i. We then have

$$\begin{aligned}\mathbb{E}\left[A_\wedge(G_i)|G_{i-1}\right] &= \frac{2}{N-i+1}\sum_{uw\notin E(G_{i-1})} d_u(G_{i-1})+d_w(G_{i-1})\\ &= \frac{2}{N-i+1}\sum_{u\in V(G)} d_u(G_{i-1})(n-1-d_u(G_{i-1}))\\ &= \frac{4(i-1)(n-1)}{N-i+1}-\frac{2}{N-i+1}\sum_{u\in V(G)} d_u(G_{i-1})^2\end{aligned}$$

where the last equality uses the fact that the sum of degrees over all vertices of a graph is twice the number of its edges.

Recall that we defined the degree deviation as $D_u(G_{i-1}) = d_u(G_{i-1}) - s(n-1)$. We also note that $N-i+1 = N(1-s)$. Thus,

$$\begin{aligned}\mathbb{E}\left[A_\wedge(G_i)|G_{i-1}\right] &= \frac{2sn(n-1)^2}{N(1-s)}-\frac{2s^2n(n-1)^2}{N(1-s)}-\frac{2}{N(1-s)}\sum_{u\in V(G)} D_u(G_{i-1})^2\\ &= 4s(n-1)-\frac{2}{N(1-s)}\sum_{u\in V(G)} D_u(G_{i-1})^2\,.\end{aligned}$$

We also recall that $X_\wedge(G_i) = A_\wedge(G_i) - \mathbb{E}\left[A_\wedge(G_i)|G_{i-1}\right]$ and so

$$X_\wedge(G_i) = 2\sum_{u\in e_i} d_u(G_{i-1}) - 4s(n-1) + \frac{2}{N(1-s)}\sum_{u\in V(G)} D_u(G_{i-1})^2 \qquad (9\text{-}21)$$

We now observe that Lemma 9.4 gives a constant $c>0$ such that the following holds: if $G\sim G(n,p)$ then, except with probability at most $\exp(-3n)$, either there is a set $F\subseteq V$ of size at least $n/4$ with $d_u(G) > p(n-1)+cp^{1/2}n^{1/2}$ for all $u\in F$ or there is a set $F\subseteq V$ of size at least $n/4$ with $d_u(G) < p(n-1)-cp^{1/2}n^{1/2}$ for all $u\in F$. We take $p=s$ and assume that the first case holds, as the proof if only the latter holds is completely analogous. Now, since the number of edges of $G(n,s)$ is given by a binomial variable $Bin(N,s)$, there is probability at least n^{-2} that G has $i-1=sN$ edges. Thus, except with probability at most $n^2\exp(-3n)\leqslant\exp(-2.5n)$, the graph G_{i-1} has a set F as described above.

Suppose that the edge $e_i = uw$ is contained in the set F. From (9-21) and the definition of F we have

$$X_\wedge(G_i)1_{e_i\in E(F)} \geqslant 4cs^{1/2}n^{1/2} + \frac{2}{N(1-s)}\sum_{u\in V(G)} D_u(G_{i-1})^2\,.$$

Moreover, since F has at least $n/4$ vertices, the number of edges $e\in E(K_n)\setminus$

$E(G_{i-1})$ that are completely contained in F is at least

$$\binom{n/4}{2} - i + 1 \geqslant \frac{n^2}{64}\,.$$

where we used that $s \leqslant 1/32$. Therefore, except with probability at most $\exp(-2n)$,

$$\begin{aligned}
\mathbb{E}\left[X_\wedge(G_i)^2 | G_{i-1}\right] &= \frac{1}{N-i+1} \sum_{uw \notin E(G_{i-1})} X_\wedge(G_i)^2 1_{\{e_i = uw\}} \\
&\geqslant \frac{1}{N-i+1} \sum_{uw \in E(F) \cap E(G_{i-1})^c} X_\wedge(G_i)^2 1_{\{e_i = uw\}} \\
&\geqslant \frac{n^2}{64N(1-s)} \left(4cs^{1/2}n^{1/2} + \frac{2}{N(1-s)} \sum_{u \in V(G)} D_u(G_{i-1})^2 \right)^2 \\
&\geqslant 8c^2 sn\,.
\end{aligned}$$

We note that since $i \geqslant m/2$ we have $s \geqslant t/4$ and thus $\mathbb{E}\left[X_\wedge(G_i)^2 | G_{i-1}\right] \geqslant 2c^2 tn$ except with probability at most $\exp(-2.5n)$. Taking an union bound over $m/2 \leqslant i \leqslant m$ gives the desired result. □

We are now ready to bound the quadratic variation of the process $D''_\wedge(G_m)$.

Lemma 9.14. *There is an absolute constant $c > 0$ such that the following holds. Let $n^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/2$ and $3\log n \leqslant b \leqslant t^{2/3} n\ell^{4/3}$. Except with probability at most $\exp(-2b)$, we have*

$$\mathbb{E}\left[(Y''_i)^2 | G_{i-1}\right] \geqslant ctn$$

for all $m/2 \leqslant i \leqslant m$.

Proof. We note that $Y''_i \geqslant 2^{-3} X''_\wedge(G_i)$ and thus

$$\mathbb{E}\left[(Y''_i)^2 | G_{i-1}\right] \geqslant 2^{-6}\mathbb{E}\left[X''_\wedge(G_i)^2 | G_{i-1}\right]$$

We recall from the proof of Lemma 9.11 that

$$\mathbb{E}\left[X''_\wedge(G_i)^2 | G_{i-1}\right] \geqslant \mathbb{E}\left[X_\wedge(G_i)^2 | G_{i-1}\right] - \mathbb{E}\left[Z_\wedge(G_i)^2 | G_{i-1}\right] - \mathbb{E}\left[Z_\wedge(G_i) | G_{i-1}\right]^2\,.$$

Now, Lemma 9.2 shows that, except with probability at most $\exp(-3b)$ we have

$$\mathbb{E}\left[(Z_\wedge(G_i))^2 | G_{i-1}\right] \leqslant \frac{cb^2}{n\ell_b^2} \ll tn\,.$$

Let c' be the constant given by Lemma 9.13 such that, except with probability at most $\exp(-2n)$, we have

$$\mathbb{E}\left[X_\wedge(G_i)^2|G_{i-1}\right] \geqslant c'tn\,.$$

Therefore, we have

$$\mathbb{E}\left[X''_\wedge(G_i)^2|G_{i-1}\right] \geqslant c'tn/2$$

except with probability at most $\exp(-2n)+\exp(-3b) \leqslant \exp(-b)$. The proof follows by choosing $c = c'/2$. □

Let us now prove Proposition 9.12, which is analogue to the proof of Proposition 9.1.

Proof of Proposition 9.12. We fix t such that $n^{-1/2}(\log n)^{1/2} \leqslant t \leqslant 1/2$ and fix b such that $3\log n \leqslant b \leqslant t^{2/3}n\ell^{4/3}$. We shall find a constant $c>0$ such that

$$\mathbb{P}\left(D''_\wedge(G_m) \geqslant cb^{1/2}tn^{3/2}\right) \geqslant \exp(-b)\,.$$

We recall that $D''_\wedge(G_m) = \sum_{i=1}^m Y''_i$ is a martingale. Repeating the proof of Proposition 9.1, we set $\alpha' = cb^{1/2}tn^{3/2}$ and note that it suffices to prove that

$$\mathbb{P}\left(\sum_{i=1}^{m} Y''_i \geqslant \alpha' \mid \sum_{i=1}^{m'} Y''_i \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) \geqslant 1/2 \qquad (9\text{-}22)$$

and

$$\mathbb{P}\left(\sum_{i=1}^{m'} Y''_i \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) \geqslant \exp(-b/2)\,. \qquad (9\text{-}23)$$

In order to bound the maximum of the increments, we note that $|Y''_i| \leqslant |X'_\wedge(G_i)| + \mathbb{E}\left[Z_\wedge(G_i)|G_{i-1}\right]$. By the definition of $X'_\wedge(G_i)$ we have $|X'_\wedge(G_i)| \leqslant 2^7tn$. Moreover, Lemma 9.2 shows that $\mathbb{E}\left[(Z_\wedge(G_i))|G_{i-1}\right] \ll tn$, except with probability at most $\exp(-2b)$. Thus,

$$|Y''_i| \leqslant 2^8tn$$

except with probability at most $\exp(-2b)$.

Moreover, there are constants $c_0, c_1 > 0$ such that, except with probability at most $\exp(-2b)$,

$$c_0t^2n^3 \leqslant \sum_{i=1}^{m} \mathbb{E}\left[(Y''_i)^2|G_{i-1}\right] \leqslant c_1t^2n^3\,.$$

The lower bound is given by Lemma 9.14 and the upper bound follows from Lemma 5.4 together with the fact that $\mathbb{E}\left[(Y_i'')^2|G_{i-1}\right] \leqslant 2\mathbb{E}\left[X_\wedge(G_i)^2|G_{i-1}\right]$ for all $i \leqslant m$.

Let us now prove (9-22). As in the proof of (9-16) we assume that $\sum_{i=1}^{m'} Y_i'' \geqslant 2\alpha'$ for some $m' \leqslant m$ and consider the process starting at $m'+1$. We then apply Theorem 2.12 with "a" $= \alpha' = cb^{1/2}tn^{3/2}$, "$b$" $= c_1t^2n^3$ and $R = 2^8tn$ to deduce that

$$\begin{aligned}\mathbb{P}\left(\sum_{i=m'+1}^{m} Y_i'' \geqslant \alpha' \mid \sum_{i=1}^{m'} Y_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) &\leqslant \\ \leqslant \exp\left(\frac{-(\alpha')^2}{2c_1t^2n^3 + 2^8\alpha' tn}\right) + \exp\left(-2b\right) + \exp\left(-2b\right) \\ \leqslant \exp\left(\frac{-c^2b}{2c_1+2^8}\right) + 2\exp\left(-c_0n\right) \leqslant 1/2\,.\end{aligned}$$

We now prove (9-23). We may apply Theorem 2.14 with "a" $= 2\alpha' = 2cb^{1/2}tn^{3/2}$, "$b$" $= c_0t^2n^3$ and $R = 2^8tn$. We obtain

$$\mathbb{P}\left(\sum_{i=1}^{m'} Y_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) \geqslant \frac{1}{2}\exp\left(\frac{-(1+4\delta)\alpha^2}{2\beta}\right) - 2\exp\left(-2b\right)$$

where $\delta > 0$ satisfy $c_0/(2c) \geqslant \delta^{-2}$ and $4c^2b/c_0 > 16\delta^{-2}\log(64\delta^{-2})$. We may choose $c < c_0/2$ so that $\delta = 1$. Then

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{m'} Y_i'' \geqslant 2\alpha' \text{ for some } m' \leqslant m\right) &\geqslant \exp\left(\frac{-5\alpha^2}{2\beta} - 1\right) - 2\exp\left(-2b\right) \\ &\geqslant \exp\left(-c_0b/50\right) - \exp\left(-2b\right) \geqslant \exp\left(-b\right)\end{aligned}$$

where we have assumed that $c < c_0/100$ in the last inequality. □

For the other regimes, we mimic the proof given for triangles in Section 9.2. Recall that we only need to give a graph G_* containing at most $b/(10\ell)$ edges and such that

$$\mathbb{E}\left[N_\wedge(G_m)|E_m\right] > \mathbb{E}\left[N_\wedge(G_m)\right] + 2cM(b,t)\,.$$

where E_m is the event that $G_* \subseteq G_m$. Note that for cherry deviations, there are only two remaining regimes (star and hub). Let us give the structures that give the lower bound in each of them.

The star regime: We may take G_* to be a star of degree $b/(10\ell)$, which adds at least $b^2/(100\ell^2)$ to the expected number of cherries in G_m.

The hub regime: We may take G_* to be a hub with $b/(10n\ell)$ vertices of degree $n-1$. This adds at least $bn/(80\ell)$ to the expected number of cherries in G_m.

10
Triangle deviations in $G(n,p)$

In this chapter we prove our results for triangle deviations in $G(n,p)$, Theorem 1.1 and Theorem 1.2.

10.1
Some binomial estimates

As we shall use our results from $G(n,m)$ to prove our results in $G(n,p)$ it will be useful to have estimates for

$$b_n(m) \;:=\; \mathbb{P}\left(Bin(n,p) \;=\; m\right)$$

and

$$B_n(m) \;:=\; \mathbb{P}\left(Bin(n,p) \;\geqslant\; m\right) .$$

We state the results in terms of

$$x_N(m) \;:=\; \frac{m - pN}{\sqrt{Npq}}$$

where $q = 1 - p$. They are valid for $p \in (0,1)$ a constant or $p = p_N$ a function. We also define the expression

$$E(x,N) \;:=\; \sum_{i=1}^{\infty} \frac{(p^{i+1} + (-1)^i q^{i+1}) x^{i+2}}{(i+1)(i+2) p^{i/2} q^{i/2} N^{i/2}} ,$$

We write $E(x,N,J)$ for the partial sum up to $i = J$. The next result was stated in [19] and it follows from Bahadur [30], Theorem 2.

Theorem 10.1. *Suppose that (x_N) is a sequence such that $1 \ll x_N \ll \sqrt{Npq}$. Then*

$$b_N(\lfloor pN + x_N\sqrt{Npq}\rfloor) \;=\; (1+o(1))\frac{1}{\sqrt{2\pi Npq}} \exp\left(-\frac{x_N^2}{2} - E(x_N,N)\right)$$

and

$$B_N(pN + x_N\sqrt{Npq}) \;=\; (1+o(1))\frac{1}{x_N\sqrt{2\pi}} \exp\left(-\frac{x_N^2}{2} - E(x_N,N)\right) .$$

Furthermore, if $1 \ll x_N \ll (pqN)^{1/2}(pqN)^{-1/(J+3)}$ then the infinite sum $E(x_N, N)$ may be replaced by the finite sum $E(x_N, N, J)$ in both expressions.

10.2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 which gives the asymptotic value of $r(\delta_n, p, n)$ for a certain range of the parameters δ, p. Let us recall that

$$r(\delta_n, p, n) := -\log \mathbb{P}\left(N_\triangle(G_p) > (1+\delta_n)p^3(n)_3\right).$$

We now restate Theorem 1.1.

Theorem 1.1. *Let $n^{-1/2}\log n \ll p \ll 1$ and let δ_n be a sequence satisfying*

$$p^{-1/2}n^{-1} \ll \delta_n \ll p^{3/4}(\log n)^{3/4},\ n^{-1/3}(\log n)^{2/3} + p\log(1/p).$$

Then

$$r(\delta_n, p, n) = (1+o(1))\frac{\delta_n^2 pn^2}{36}.$$

We shall use the following identity to go from $G(n,m)$ to $G(n,p)$:

$$\mathbb{P}\left(N_\triangle(G_p) > (1+\delta_n)p^3(n)_3\right) = \sum_{m=0}^{N} b_N(m)\mathbb{P}\left(N_\triangle(G_m) > (1+\delta_n)p^3(n)_3\right) \tag{10-1}$$

Let us define

$$m_* := pN(1+\delta_n)^{1/3}.$$

This is approximately the value of m so that the expected number of triangles in G_m is $(1+\delta_n)p^3(n)_3$. In particular, we do not require deviation in G_{m_*} in order to have $N_\triangle(G_m) > (1+\delta_n)p^3(n)_3$. We also let

$$x_* := x_N(m_*) = \frac{m_* - pN}{\sqrt{Npq}}.$$

We also define

$$\tilde{M}(\delta, p) := CM(\delta^2 pn^2, 2p)$$

where $M(b,t)$ is as defined in Chapter 1, before the statement of Theorem 1.3 and C is the constant obtained from the upper bound of the same theorem. In particular, it follows from Theorem 1.3 that

$$\mathbb{P}\left(N_\triangle(G_m) > \mathbb{E}\left[N_\triangle(G_m)\right] + \tilde{M}(\delta_n, p)\right) \leqslant \exp(-\delta^2 pn^2) \tag{10-2}$$

for all $m \leqslant 2pN$.

We now state a more precise version of Theorem 1.1

Proposition 10.2. *Let $n^{-1/2}\log n \ll p \ll 1$ and let δ_n be a sequence satisfying*

$$p^{-1/2}n^{-1} \;\ll\; \delta_n \;\ll\; p^{3/4}(\log n)^{3/4}\,,\; n^{-1/3}(\log n)^{2/3} \;+\; p\log(1/p)\,.$$

Then

$$r(\delta_n,p,n) \;=\; \frac{x_*^2}{2} \;+\; E(x_*,N) \;+\; \log x_* \;+\; O\left(\frac{\delta_n\tilde{M}(\delta_n,p)}{p^2n}\right) \;+\; O(1)\,.$$

Let us see how this proposition implies Theorem 1.1.

Proof of Theorem 1.1. We first observe that

$$x_* \;=\; [(1+\delta_n)^{1/3}-1]\sqrt{\frac{pN}{q}}\,.$$

From the expansion of $(1+\delta_n)^{1/3}$ we get

$$\begin{aligned} x_* \;&=\; \left(\frac{\delta_n}{3}+\frac{\delta_n^2}{9}+O(\delta_n^3)\right)\sqrt{\frac{pN}{q}} \\ &\leqslant\; \frac{\delta_n p^{1/2}N^{1/2}}{3q^{1/2}}\left(1+\frac{\delta_n}{3}\right)+O(\delta_n^3 n)\,. \end{aligned}$$

Since $N=n(n-1)/2$ we obtain from the last expression that

$$x_* \;=\; \frac{\delta_n p^{1/2}n}{3\sqrt{2}}(1+o(1))\,.$$

Thus,

$$\frac{x_*}{2} \;=\; \frac{\delta_n^2 pn^2}{36}(1+o(1))\,.$$

Moreover, $x_* \gg 1$ for this range of values of δ_n and thus $\log x_* \ll x_*^2$. Furthermore, we have $E(x_*,N) \ll x_*^2$ and $\tilde{M}(\delta_n,p) \ll \delta_n p^3 n^3$ over the specified region of values of δ_n. Indeed, using (10-2) we may note that $\tilde{M}(\delta_n,p)$ was chosen so that the contribution to the deviation made by $G(n,m)$ is small, as the probability of having a deviation of at least $\tilde{M}(\delta_n,p)$ is bounded by $\exp(-\delta_n^2 pn^2) \leqslant \exp(-x_*^2)$. □

Let us now prove Proposition 10.2. We recall that $m_* = pN(1+\delta_n)^{1/3}$ and define

$$m_- \;:=\; m_* - 2p^{-2}n^{-1}\tilde{M}(\delta_n,p)\,.$$

We also define

$$x_- \;:=\; x_N(m_-) \;=\; \frac{m_- - pN}{\sqrt{Npq}}\,.$$

Proof of Proposition 10.2. We start with the proof of the lower bound on $\mathbb{P}\left(N_\triangle(G_p) > (1+\delta_n)p^3(n)_3\right)$, which gives the upper bound on $r(\delta_n, p, n)$. Note that (10-1) and monotonicity gives

$$\mathbb{P}\left(N_\triangle(G_p) > (1+\delta_n)p^3(n)_3\right) = B_N(m_*)\mathbb{P}\left(N_\triangle(G_{m_*}) > (1+\delta_n)p^3(n)_3\right)$$

Now, the expected number of triangles in G_{m_*} is

$$\mathbb{E}\left[N_\triangle(G_{m_*})\right] = \frac{(m_*)_3(n)_3}{(N)_3}$$

which is of order $(1+\delta_n)p^3(n)_3$ by the definition of m_*. Thus, $\mathbb{P}\left(N_\triangle(G_{m_*}) > (1+\delta_n)p^3(n)_3\right)$ converges to $1/2 + o(1)$ by the central limit theorem. Then it follows from Theorem 10.1 that

$$\mathbb{P}\left(N_\triangle(G_p) > (1+\delta_n)p^3(n)_3\right) \geqslant \exp\left(\frac{-x_*^2}{2} - E(x_*, N) - \log x_* + O(1)\right).$$

We now prove an upper bound on $\mathbb{P}\left(N_\triangle(G_p) > (1+\delta_n)p^3(n)_3\right)$, which gives a lower bound on $r(\delta_n, p, n)$. We may again use (10-1), splitting the sum into two parts: considering only the contribution above m_- and only the contribution below m_-. Using Theorem 10.1 we note that the contribution from above m_- is at most

$$B_N(m_-) \leqslant \exp\left(\frac{-x_-^2}{2} - E(x_-, N) - \log x_- + O(1)\right).$$

Let us note that

$$\begin{aligned} x_- &= x_* - \frac{2p^{-2}n^{-1}\tilde{M}(\delta_n, p)}{\sqrt{Npq}} \\ &= x_* - \Theta\left(\frac{\tilde{M}(\delta_n, p)}{p^{5/2}n^2}\right). \end{aligned}$$

Thus,

$$\begin{aligned} B_N(m_-) &\leqslant \exp\left(\frac{-x_*^2}{2} + O\left(\frac{x_*\tilde{M}(\delta_n, p)}{p^{5/2}n^2}\right) - E(x_*, N) - \log x_* + O(1)\right) \\ &\leqslant \exp\left(\frac{-x_*^2}{2} + O\left(\frac{\delta_n\tilde{M}(\delta_n, p)}{p^2 n}\right) - E(x_*, N) - \log x_* + O(1)\right) \end{aligned}$$

where we used that $x_* = \Theta(\delta_n^2 pn^2)$ in the last inequality.

We now bound the contribution from $m < m_-$. We claim that

$$\mathbb{P}\left(N_\triangle(G_{m_-}) \geqslant (1+\delta_n)p^3\binom{n}{3}\right) \leqslant \exp(-\delta_n^2 pN). \qquad (10\text{-}3)$$

In order to prove this, let us bound $\mathbb{E}\left[N_\triangle(G_{m_-})\right]$. We have

$$\begin{aligned}\mathbb{E}\left[N_\triangle(G_{m_-})\right] &\leqslant \left(\frac{m_-}{N}\right)^3 (n)_3 \\ &= \left(\frac{m_* \,-\, 2p^{-2}n^{-1}\tilde{M}(\delta_n,p)}{N}\right)^3 (n)_3 \\ &\leqslant \left(\frac{m_*}{N}\right)^3 (n)_3 \,-\, \tilde{M}(\delta_n,p) \\ &= (1+\delta_n)p^3(n)_3 \,-\, \tilde{M}(\delta_n,p)\,.\end{aligned}$$

Therefore,

$$\mathbb{P}\left(N_\triangle(G_{m_-}) \geqslant (1+\delta_n)p^3\binom{n}{3}\right) \leqslant \mathbb{P}\left(D_\triangle(G_{m_-}) \geqslant \tilde{M}(\delta_n,p)\right) \leqslant \exp\left(-\delta_n^2 pN\right).$$

By monotonocity, we have

$$\sum_{m=0}^{N} b_N(m)\mathbb{P}\left(N_\triangle(G_m) \,>\, (1+\delta_n)p^3(n)_3\right) \leqslant m_-\exp\left(-\delta_n^2pN\right) + B_N(m_-)\,.$$

Since $B_N(m_-)$ has the exact desired upper bound and δ_n^2pN is much smaller than the desired bound, the proof is complete. □

10.3
Proof of Theorem 1.2

In this section, we prove Theorem 1.2 which gives the asymptotic value of $r(\delta_n,p,n)$ for the localised region.

Let us restate Theorem 1.2.

Theorem 1.2. *Let* $n^{-1/2}\log n \ll p \ll 1$ *and let* δ_n *be a sequence satisfying*

$$p^{3/4}(\log n)^{3/4}\,,\; n^{-1/3}(\log n)^{2/3} \,+\, p\log(1/p) \leqslant \delta_n \leqslant 1\,.$$

Then

$$r(\delta_n,p,n) \,=\, \Theta(1)\min\{\delta_n^{2/3}p^2n^2\log n, \delta_n^{1/2}pn^{3/2}\log n \,+\, \delta_np^2n^2\log(1/p)\}\,.$$

As this is the localised region, the proof of the lower bound on $\mathbb{P}\left(N_\triangle(G_p) > (1+\delta)p^3(n)_3\right)$ goes by showing that G_p has a certain structure (hub, star or clique) with probability $r(\delta_n,p,n)$ and that this structure causes the required deviation.

For the upper bound, we show that the probability of having deviation without any structure is small. We let $m_0 = pN + \delta_npn^2/10$ and note that by

Theorem 2.10 we have

$$B_N(m_0) \leqslant \exp\left(\frac{-\delta_n^2 pn^2}{200}\right). \tag{10-4}$$

Proof of Theorem 1.2. We have three regimes for δ_n: hub, star and clique. The hub region corresponds to

$$p\log(1/p)\,,\ \frac{1}{p^2n} \leqslant \delta_n \leqslant 1\,.$$

In this region, the minimum value in the required bound is $\Theta(\delta_n p^2 n^2 \log(1/p))$. Moreover, as $\delta_n \geqslant p\log(1/p)$, we have from (10-4) that

$$\mathbb{P}\left(e(G_p) > m_0\right) = B_N(m_0) \leqslant \exp\Big(-\Omega(\delta_n p^2 n^2 \log(1/p))\Big)\,.$$

We now give a lower bound on $\mathbb{P}\left(N_\triangle(G_p) > (1+\delta)p^3(n)_3\right)$ in this region. We note that the probability of having a hub of size $\Theta(\delta_n p^2 n)$ vertices, each of them with degree $n-1$, is $\exp(-O(\delta_n p^2 n^2 \log(1/p)))$. This structure adds an extra $\delta_n p^3 n^3$ triangles to the expectation, as required. For the upper bound we have

$$\begin{aligned}
&\mathbb{P}\Big(N_\triangle(G_p) \geqslant (1+\delta_n)p^3(n)_3\Big)\\
&\leqslant \mathbb{P}\left(e(G_p) > m_0\right) + \mathbb{P}\Big(N_\triangle(G_p) \geqslant (1+\delta_n)p^3(n)_3 \mid e(G_p) \leqslant m_0\Big)\\
&\leqslant \exp(-\Omega(\delta_n p^2 n^2 \log(1/p)) + \mathbb{P}\Big(N_\triangle(G_{m_0}) \geqslant (1+\delta_n)p^3(n)_3\Big)\,.
\end{aligned}$$

We now observe that as $\delta_n \leqslant 1$, we have

$$\mathbb{E}\left[N_\triangle(G_{m_0})\right] = \frac{(m_0)_3(n)_3}{(N)_3} \leqslant \left(1+\frac{9\delta_n}{10}\right)p^3(n)_3\,.$$

Thus,

$$\begin{aligned}
\mathbb{P}\Big(N_\triangle(G_{m_0}) \geqslant (1+\delta_n)p^3(n)_3\Big) &\leqslant \mathbb{P}\Big(D_\triangle(G_{m_0}) \geqslant \Theta(\delta_n p^3 n^3)\Big)\\
&\leqslant \exp\Big(-\Omega(\delta_n p^2 n^2 \log(1/p))\Big)
\end{aligned}$$

where we used Theorem 1.3 to deduce the last inequality. This completes the proof for the hub region.

Let us now prove the result for the star region, which corresponds to

$$\frac{1}{p^3(\log n)^3}\,,\ n^{-1/3}(\log n)^{2/3} \leqslant \delta_n \leqslant \frac{1}{p^2n}\,.$$

The proof follows in a very similar way to the hub region. In the star region, the minimum value in the required bound is $\Theta(\delta_n^{1/2} p n^{3/2} \log n)$. Moreover, as $\delta_n \geqslant n^{-1/3}(\log n)^{2/3}$, we have from (10-4) that

$$\mathbb{P}\left(e(G_p) > m_0\right) \;=\; B_N(m_0) \;\leqslant\; \exp\left(-\Omega(\delta_n^{1/2} p n^{3/2} \log n)\right) .$$

The lower bound on $\mathbb{P}\left(N_\triangle(G_p) > (1+\delta)p^3(n)_3\right)$ in this region follows by asking that there is a vertex of degree at least $\delta_n^{1/2} p n^{3/2}$. This increases the expectation of the number of triangles in $\delta_n p^3 n^3$, as required. Furthermore, the probability that such a structure happens is $\exp(-O(\delta_n^{1/2} p n^{3/2} \log n))$, as $\log n$ and $\log(1/p)$ are equivalent in this regime. For the upper bound, we proceed exactly as in the hub regime, with the difference being that Theorem 1.3 gives

$$\begin{aligned}\mathbb{P}\left(N_\triangle(G_{m_0}) \geqslant (1+\delta_n)p^3(n)_3\right) \;&\leqslant\; \mathbb{P}\left(D_\triangle(G_{m_0}) \geqslant \Theta(\delta_n p^3 n^3)\right) \\ &\leqslant\; \exp\left(-\Omega(\delta_n^{1/2} p n^{3/2} \log n)\right)\end{aligned}$$

as required.

We finally move to the clique region, which corresponds to

$$p^{3/4}(\log n)^{3/4} \;\leqslant \delta_n \;\leqslant\; \frac{1}{p^3(\log n)^3} .$$

In this region, the minimum value in the required bound is $\Theta(\delta_n^{2/3} p^2 n^2 \log n)$. Moreover, as $\delta_n \geqslant p^{3/4}(\log n)^{3/4}$ we have from (10-4) that

$$\mathbb{P}\left(e(G_p) > m_0\right) \;=\; B_N(m_0) \;\leqslant\; \exp\left(-\Omega(\delta_n^{2/3} p^2 n^2 \log n)\right) .$$

The lower bound on $\mathbb{P}\left(N_\triangle(G_p) > (1+\delta)p^3(n)_3\right)$ in this region follows by asking for a clique on $\Theta(\delta_n^{1/3} pn)$ vertices. This adds an extra $\delta_n p^3 n^3$ triangles to the expectation and has probability $\exp(-O(\delta_n^{2/3} p^2 n^2 \log n))$. For the upper bound, we repeat the same proof from the previous regimes, with the difference being that Theorem 1.3 gives

$$\begin{aligned}\mathbb{P}\left(N_\triangle(G_{m_0}) \geqslant (1+\delta_n)p^3(n)_3\right) \;&\leqslant\; \mathbb{P}\left(D_\triangle(G_{m_0}) \geqslant \Theta(\delta_n p^3 n^3)\right) \\ &\leqslant\; \exp\left(-\Omega(\delta_n^{2/3} p^2 n^2 \log n)\right)\end{aligned}$$

as required. □

11
A version of Freedman's inequality

In this chapter we prove new versions of Freedman's inequalities - both the upper and lower bound. Our version includes some additional symmetric conditions on the martingale increments.

11.1
Upper bound

In this section, we shall prove Theorem 1.5. This is related to the upper bound of Freedman's inequality, Theorem 2.12.

Recall that, given a martingale S_i with increments X_i with respect to the filtration $\mathcal{F}_i$, we defined

$$T_n := \sum_{i=1}^{n} \mathbb{E}\left[X_i^2|\mathcal{F}_{i-1}\right] .$$

with $n \in \mathbb{N} \cup \{+\infty\}$.

We now restate Theorem 1.5.

Theorem 1.5. *Let $m \in \mathbb{N}$. Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. Suppose that there exists $R \in \mathbb{R}$ so that $|X_i| \leqslant R$ a.s. for all i. Assume that, for each i, there are real numbers ε_i so that $|\mathbb{E}\left[X_i^3|\mathcal{F}_{i-1}\right]| \leqslant \varepsilon_i$. If $0 < a \leqslant 2b$ then*

$$\mathbb{P}\left(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n \leqslant m\right) \leqslant \exp\left(\frac{-a^2}{2b} + \frac{\xi a^3}{6R^3b^3} + \frac{a^4}{12R^2b^3}\right)$$

where $\xi = \sum_{i=1}^{m} \varepsilon_i$.

The proof of this result is very similar to the proof of Theorem 2.12, given in [22].

The main idea of this proof goes through the definition of a certain supermartingale. Before we go through this, let us fix $\lambda > 0$ and define the number $e(\lambda) = e^{\lambda} - 1 - \lambda - (\lambda^3/6)$. The proof of the next Lemma follows the same lines of the proof of Freedman's inequality given in [31].

Lemma 11.1. *Let $\lambda > 0$, $\varepsilon \geqslant 0$ and X be a random variable such that $|X| \leqslant 1$, $\mathbb{E}[X] = 0$ and $|\mathbb{E}[X^3]| \leqslant \varepsilon$. Then,*

$$\mathbb{E}[\exp(\lambda X)] \leqslant \exp\left(e(\lambda)\mathbb{E}\left[X^2\right] + \frac{\varepsilon\lambda^3}{6}\right)$$

for any $\lambda > 0$.

Proof. This follows from

$$\begin{aligned}
\mathbb{E}[\exp(\lambda X)] &= \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{\lambda^k X^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{\lambda^k \mathbb{E}\left[X^k\right]}{k!} \\
&\leqslant 1 + \left(\frac{\lambda^2}{2} + \sum_{k=4}^{\infty} \frac{\lambda^k}{k!}\right)\mathbb{E}\left[X^2\right] + \frac{\lambda^3}{6}\mathbb{E}\left[X^3\right] \\
&= 1 + e(\lambda)\mathbb{E}\left[X^2\right] + \frac{\varepsilon\lambda^3}{6} \\
&\leqslant \exp\left(e(\lambda)\mathbb{E}\left[X^2\right] + \frac{\varepsilon\lambda^3}{6}\right).
\end{aligned}$$

□

We observe that Lemma 11.1 also applies to conditional expectations. For $\lambda > 0$, let us define the sequence

$$Y_n(\lambda) := \exp\left(\lambda S_n - e(\lambda)T_n - \frac{\lambda^3}{6}\sum_{i=1}^{n}\varepsilon_i\right).$$

Corollary 11.2. *Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. For each i, suppose that there are ε_i so that $|\mathbb{E}[X_i^3|\mathcal{F}_{i-1}]| \leqslant \varepsilon_i$ and that $|X_i| \leqslant 1$. For any $\lambda > 0$, the sequence $Y_i(\lambda)$ is a supermartingale with respect to the filtration $\mathcal{F}_i$.*

Proof. Fix $\lambda > 0$ and $i \in \mathbb{N}$. Then,

$$\begin{aligned}
\mathbb{E}[Y_i \mid \mathcal{F}_{i-1}] &= \mathbb{E}\left[\exp\left(\lambda S_i - e(\lambda)T_i - \frac{\lambda^3}{6}\sum_{j=1}^{i}\varepsilon_j\right) \mid \mathcal{F}_{i-1}\right] \\
&= Y_{i-1}\mathbb{E}\left[\exp\left(\lambda X_i - e(\lambda)V_i - \frac{\varepsilon_i\lambda^3}{6}\right) \mid \mathcal{F}_{i-1}\right]
\end{aligned}$$

where $V_i = \mathbb{E}[X_i^2|\mathcal{F}_{i-1}]$.

Now, apply Lemma 11.1 to the variable X_i and its conditional expectation on the sigma-algebra $\mathcal{F}_{i-1}$. This shows that

$$\mathbb{E}[\exp(\lambda X_i) \mid \mathcal{F}_{i-1}] \leqslant \exp\left(e(\lambda)V_i - \frac{\varepsilon_i\lambda^3}{6}\right)$$

and the result follows immediately. □

Given any martingale (or supermartingale) S_i and a number $a \geqslant 0$, we define the stopping time τ_a as the first time i such that $S_i - S_0 \geqslant a$ with $\tau_a = \infty$ if this never occurs. We then define the sequence of stopping times $\sigma_n^a := \min\{n, \tau_a\}$ for $n \in \mathbb{N}$. We write $\sigma_n = \sigma_n^a$ if a is clearly given by the context.

We now have all the tools to prove Theorem 1.5.

Proof of Theorem 1.5. We assume, without loss of generality, that $|X_i| \leqslant 1$ for all i. The proof for the general case given in the statement goes by considering $X_i' = X_i/R$. We also note that $S_i' = S_i - S_0$ is a martingale with $S_i' = 0$ and the same increments as S_i. Thus, we may assume that $S_0 = 0$

Note that Y_i is a supermartingale and each σ_n is a bounded stopping time, so $1 = \mathbb{E}[Y_0] \geqslant \mathbb{E}[Y_{\sigma_n}]$. In addition, $Y_{\sigma_n} 1_{\{\tau_a<\infty\}}$ converges to $Y_{\tau_a} 1_{\{\tau_a<\infty\}}$ as n goes to infinity. Now, Fatou's Lemma, Lemma 2.16, implies that $\mathbb{E}\left[Y_{\tau_a} 1_{\{\tau_a<\infty\}}\right] \leqslant 1$.

Fix $m \in \mathbb{N}$ and define the event $A_m = \{S_n \geqslant a$ and $T_n \leqslant b$ for some $n \leqslant m\}$. On this event, we have $\tau_a \leqslant m$, $S_{\tau_a} \geqslant a$ and $T_{\tau_a} \leqslant b$. Therefore,

$$1 \geqslant \mathbb{E}\left[Y_{\tau_a} 1_{A_m}\right] \geqslant \mathbb{P}(A_m) \exp\left(\lambda a - e(\lambda) b - \frac{\lambda^3}{6} \sum_{i=1}^{m} \varepsilon_i\right)$$

for any $\lambda > 0$. Also, $e(\lambda) \leqslant \lambda^2/2 + \lambda^4/12$ for $\lambda < 1/2$. Choosing $\lambda = a/b$, we get

$$\begin{aligned} \mathbb{P}(A_m) &\leqslant \exp\left(-\lambda a + b\left(\frac{\lambda^2}{2} + \frac{\lambda^4}{12}\right) + \frac{\lambda^3}{6} \sum_{i=1}^{m} \varepsilon_i\right) \\ &\leqslant \exp\left(-\frac{a^2}{2b} + \frac{\xi a^3}{6b^3} + \frac{a^4}{12b^3}\right). \end{aligned}$$

□

11.2 Lower bound

In this section, we shall prove Theorem 1.6. This is related to the lower bound of Freedman's inequality, Theorem 2.14.

Let us restate Theorem 1.6.

Theorem 1.6. *Let $m \in \mathbb{N}$. Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. Suppose that $|X_i| \leqslant 1$ a.s. for all i and that there are*

ε_i so that $|\mathbb{E}[X_i^3|\mathcal{F}_{i-1}]| \leqslant \varepsilon_i$. If $2 < a \leqslant b/8$, $a/b < \min\{\varepsilon_i/3 : 1 \leqslant i \leqslant 3m\}$ and $8b < a^2$ then

$$\mathbb{P}(S_n - S_0 \geqslant a \text{ and } T_n \leqslant b \text{ for some } n \leqslant m) \geqslant \frac{1}{2}\exp\left(\frac{-(1+\eta)a^2}{2b}\right) - \mathbb{P}(T_m < b) .$$

where $\gamma = \sum_{i=1}^{3m} \varepsilon_i$ and $0 < \eta < 1/16$ is minimal such that $b^2/a > 36\gamma\eta^{-1}$, $b^2/a^2 > 108\eta^{-2}$ and $a^2/b > 180\eta^{-2}\log(90\eta^{-2})$.

The proof of this result depends on a series of results that we state and prove in this section. We first fix $m \in \mathbb{N}$ and define a process S_i^* with increments X_i^* as follows: let $X_i^* = X_i$ if $i \leqslant m$ and $\mathbb{P}(X_i^* = 1) = \mathbb{P}(X_i^* = -1) = 1/2$ if $i > m$. Clearly, S_i^* is an "extended" version of our original process S_i after time m since $S_i^* = S_i$ for all $i \leqslant m$. We note that $|X_i^*| = 1 = \mathbb{E}[(X_i^*)^2|\mathcal{F}_{i-1}]$ and $\mathbb{E}[(X_i^*)^3|\mathcal{F}_{i-1}] = 0$ for all $i > m$. Thus, the process S_i^* satisfies all the assumptions from Theorem 1.6.

We also define, for $a > 0$, the stopping time $\tau_a^* := \min\{i : S_i^* - S_0 \geqslant a\}$. Also, let $W_a^* = T^*_{\tau_a^*}$. This variable is essential for the proof of Theorem 1.6. Clearly, if $W_a^* < b$ then either $T_m < b$ or S_i reaches a before time m and $T_i < b$. Thus, our result follows if we prove that

$$\mathbb{P}(W_a^* < b) \geqslant \frac{1}{2}\exp\left(-\frac{a^2}{2b}(1+\eta)\right) . \tag{11-1}$$

As we shall see later, we need to also prove an upper bound on $\mathbb{P}(W_a^* < x)$ for some values of x. This part of the proof is essentially the same presented in the previous section. We first define, for $\lambda > 0$, the sequence

$$Y_n^*(\lambda) := \exp\left(\lambda S_n^* - e(\lambda)T_n^* - \lambda^3 \sum_{i=1}^{n\wedge m} \varepsilon_i\right) .$$

We note that all processes defined in this section depend on the value of m. For convenience, we omit this dependence. We also observe that $Y_n^* = Y_n$ for $n \leqslant m$. Indeed, Y_n^* is also a supermartingale.

Corollary 11.3. *Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. For each i, suppose that there are ε_i so that $|\mathbb{E}[X_i^3|\mathcal{F}_{i-1}]| \leqslant \varepsilon_i$ and that $|X_i| \leqslant 1$. Let $m \in \mathbb{N}$ and S_i^* be a process with increments X_i^* with respect to the filtration $\mathcal{F}_i$, where $X_i^* = X_i$ for $1 \leqslant i \leqslant m$ and $X_i^* \sim Unif\{-1,1\}$ for $i > m$. For any $\lambda > 0$, the sequence $Y_i^*(\lambda)$ is a supermartingale with respect to the filtration $\mathcal{F}_i$.*

Proof. For $i \leqslant m$, we just observe that $Y_i^* = Y_i$ and Y_i is a supermartingale by Corollary 11.2. For $i \geqslant m$ we note that Lemma 11.1 gives

$$\mathbb{E}\left[\exp\left(\lambda X_i^*\right) \mid \mathcal{F}_{i-1}\right] \leqslant \exp\left(e(\lambda) V_i^*\right)$$

for $i > m$, completing the proof. □

We now present an upper bound on $\mathbb{P}\left(W_a^* < x\right)$ which is the same bound given in Theorem 1.5.

Theorem 11.4. *Let $m \in \mathbb{N}$, $a > 0$ and $x > 2a$. Then*

$$\mathbb{P}\left(W_a^* < x\right) \leqslant \exp\left(\frac{-a^2}{2x} + \frac{\xi a^3}{6x^3} + \frac{a^4}{12x^3}\right)$$

where $\xi = \sum_{i=1}^m \varepsilon_i$.

Proof. Let $\lambda > 0$. Since $Y_i^*(\lambda)$ is a supermartingale, we have $1 = \mathbb{E}\left[Y_0^*\right] \geqslant ExY_{\tau_a^*}^* 1_{\{\tau_a^* < \infty\}}$. Also, $S_{\tau_a^*}^* 1_{\{\tau_a^* < \infty\}} \geqslant a$. Thus,

$$1 \geqslant \exp\left(\lambda a - \frac{\lambda^3 \sum_{i=1}^m \varepsilon_i}{6}\right) \mathbb{E}\left[\exp\left(-e(\lambda) W_a^*\right) 1_{\{\tau_a^* < \infty\}}\right].$$

Moreover, $T_\infty^* = \infty$ which implies that $\exp\left(-e(\lambda) W_a^*\right) 1_{\{\tau_a^* = \infty\}} = 0$ almost surely. Therefore,

$$\mathbb{E}\left[\exp\left(-e(\lambda) W_a^*\right)\right] \leqslant \exp\left(-\lambda a + \frac{\xi \lambda^3}{6}\right).$$

Now, an application of Markov inequality gives

$$\begin{aligned} \mathbb{P}\left(W_a^* < x\right) &= \mathbb{P}\left(\exp\left(-e(\lambda) W_a^*\right) > \exp\left(-e(\lambda) x\right)\right) \\ &\leqslant \exp\left(e(\lambda) x\right) \mathbb{E}\left[\exp\left(-e(\lambda) W_a^*\right)\right] \\ &\leqslant \exp\left(-\lambda a + e(\lambda) x + \frac{\xi \lambda^3}{6}\right). \end{aligned}$$

The proof follows by choosing $\lambda = a/x$ and recalling that $e(\lambda) \leqslant \lambda^2/2 + \lambda^4/12$ for $\lambda < 1/2$. □

We now define, for any $\lambda > 0$, $f(\lambda) = e^{-\lambda} - 1 + \lambda + \lambda^3/6$. Also, for $\lambda \leqslant 3m$, define

$$Z_n(\lambda)^* = \exp\left(\lambda S_n^* - f(\lambda) T_n^* + \varepsilon_n \lambda^3 n\right)$$

We shall see next that $Z_n(\lambda)^*$ is a finite submartingale.

The proof of the next lemma also follows the lines of the proof of Freedman's inequality given in [31].

Lemma 11.5. *Let $\varepsilon \in (0,1)$ and let X be a random variable such that $|X| \leqslant 1$, $E(X) = 0$ and $|E(X^3)| \leqslant \varepsilon$. Then,*

$$\mathbb{E}\left[\exp(\lambda X)\right] \geqslant \exp\left(f(\lambda)\mathbb{E}\left[X^2\right] - \varepsilon\lambda^3\right)$$

for any $0 < \lambda < \varepsilon/3$.

Proof. We recall that $f(\lambda) = e^{-\lambda} - 1 + \lambda + \lambda^3/6$ and so $f'(\lambda) = -e^{-\lambda} + 1 + \lambda^2/2$ and $f''(\lambda) = e^{-\lambda} + \lambda$. We note that $f(0) = 0 = f'(0)$. From the expansion of $e^{-\lambda}$ we have

$$\frac{\lambda^2}{2} \leqslant f(\lambda) \leqslant \frac{\lambda^2}{2} + \frac{\lambda^4}{24}. \tag{11-2}$$

Moreover, we have

$$\lambda \leqslant f'(\lambda) \leqslant \lambda + \frac{\lambda^3}{6} \tag{11-3}$$

and

$$1 \leqslant f''(\lambda) \leqslant 1 + \frac{\lambda^2}{2}. \tag{11-4}$$

Let $\sigma^2 = \mathbb{E}\left[X^2\right]$ and define $g(\lambda) = \mathbb{E}\left[\exp(\lambda X)\right] - \exp\left(f(\lambda)\sigma^2 - \varepsilon\lambda^3\right)$. We shall prove that $g(\lambda) \geqslant 0$ if $0 < \lambda < \varepsilon/3$. Differentiating $g(\lambda)$ twice with respect to λ, we get

$$g'(\lambda) = \mathbb{E}\left[X\exp(\lambda X)\right] - \left(f'(\lambda)\sigma^2 - 3\varepsilon\lambda^2\right)\exp\left(f(\lambda)\sigma^2 - \varepsilon\lambda^3\right)$$

and

$$g''(\lambda) = \mathbb{E}\left[X^2\exp(\lambda X)\right] + h(\lambda)\exp\left(f(\lambda)\sigma^2 - \varepsilon\lambda^3\right)$$

where

$$h(\lambda) = 6\varepsilon\lambda - f''(\lambda)\sigma^2 - \left(f'(\lambda)\sigma^2 - 3\varepsilon\lambda^2\right)^2.$$

Since $f(0) = 0 = f'(0)$ we also have $g(0) = 0 = g'(0)$. Thus, it suffices to prove that $g''(\lambda) \geqslant 0$ for $0 < \lambda < \varepsilon/3$.

We split the rest of the proof in two cases, depending on the sign of $h(\lambda)$. First, we assume that $h(\lambda) \geqslant 0$. We observe that the inequality $e^{\lambda x} \geqslant 1 + \lambda x$ implies that

$$\mathbb{E}\left[X^2\exp(\lambda X)\right] \geqslant \sigma^2 - \lambda\mathbb{E}\left[X^3\right]$$

and so

$$g''(\lambda) \geqslant \sigma^2 - \lambda\mathbb{E}\left[X^3\right] + h(\lambda)\exp\left(f(\lambda)\sigma^2 - \varepsilon\lambda^3\right).$$

If $h(\lambda) \geqslant 0$ then

$$g''(\lambda) \geqslant \sigma^2 - \lambda\mathbb{E}\left[X^3\right] \geqslant \frac{2\sigma^2}{3}$$

where we used that $|X| \leqslant 1$ and $\lambda < 1/3$ in the last inequality.

We may now assume that $h(\lambda) \leqslant 0$. Using (11-3) and (11-4), we obtain

$$\begin{aligned} h(\lambda) &= 6\varepsilon\lambda - f''(\lambda)\sigma^2 - \left(f'(\lambda)\sigma^2 - 3\varepsilon\lambda^2\right)^2 \\ &\geqslant 6\varepsilon\lambda - \left(1 + \frac{\lambda^2}{2}\right)\sigma^2 - 4\lambda^2\sigma^2 + 6\varepsilon\lambda^3\sigma^2 - 9\varepsilon^2\lambda^4 \\ &\geqslant 4\varepsilon\lambda - \left(1 + \frac{\lambda^2}{2}\right)\sigma^2 \end{aligned} \tag{11-5}$$

where we used that $\lambda < \varepsilon/3$ in both inequalities above. In particular, we have $\lambda < \sigma^2/(2\varepsilon)$ which, together with (11-2), imply that

$$f(\lambda)\sigma^2 \geqslant \frac{\lambda^2}{2} \geqslant \varepsilon\lambda^3 .$$

Using (11-5) and the fact $e^x \leqslant 1 + 2x$ if $0 < x < 1$, we deduce that

$$h(\lambda)\exp(f(\lambda)\sigma^2 - \varepsilon\lambda^3) \geqslant (4\varepsilon\lambda - 2\sigma^2)(1 + 2f(\lambda)\sigma^2 - 2\varepsilon\lambda^3).$$

Now, observing that $\mathbb{E}[X^3] \geqslant -\varepsilon$ and using (11-2) we get

$$g''(\lambda) \geqslant \sigma^2 - \varepsilon\lambda + (4\varepsilon\lambda - 2\sigma^2)(1 + 2\lambda^2\sigma^2 - 2\varepsilon\lambda^3)$$

Expanding the term on the right of the last inequality and using that $\lambda < \varepsilon/3$ and $\sigma^2 < 1$, we obtain

$$g''(\lambda) \geqslant 2\varepsilon\lambda - 3\lambda^2\sigma^2 \geqslant 0 .$$

This concludes the proof. □

We observe that Lemma 11.5 also applies to conditional expectations. Thus we can mimic the proof of 11.3 to obtain the following result.

Corollary 11.6. *Let S_i be a martingale with increments X_i with respect to a filtration $\mathcal{F}_i$. For each i, suppose that there are $\varepsilon_i \in (0,1)$ so that $|\mathbb{E}[X_i^3|\mathcal{F}_{i-1}]| \leqslant \varepsilon_i$ and that $|X_i| \leqslant 1$. Let $m \in \mathbb{N}$ and S_i^* be a process with increments X_i^* with respect to the filtration $\mathcal{F}_i$, where $X_i^* = X_i$ for $1 \leqslant i \leqslant m$ and $X_i^* \sim Unif\{-1,1\}$ for $i > m$. For any $0 < \lambda \leqslant \min\{\varepsilon_i/3 : 1 \leqslant i \leqslant 3m\}$, the sequence $Z_i(\lambda)^*$ is a supermartingale with respect to the filtration $\mathcal{F}_i$.*

Proof. Fix $\lambda \leqslant \min\{1/2, \varepsilon_i : 1 \leqslant i \leqslant 3m\}$ and $i \leqslant 3m$. Then

$$\begin{aligned} \mathbb{E}[Z_n \mid \mathcal{F}_{n-1}] &= \mathbb{E}\left[\exp\left(\lambda S_n - f(\lambda)T_n + \frac{\lambda^3}{6}\sum_{i=1}^{n}\varepsilon_i\right) \mid \mathcal{F}_{n-1}\right] \\ &= Z_{n-1}\mathbb{E}\left[\exp\left(\lambda X_n - f(\lambda)V_n - \varepsilon_n\lambda^3\right) \mid \mathcal{F}_{n-1}\right] . \end{aligned}$$

Now, an application of Lemma 11.5 to X_i and its conditional expectation on $\mathcal{F}_{i-1}$ gives

$$\mathbb{E}\left[\exp\left(\lambda X_n\right) \mid \mathcal{F}_{n-1}\right] \geqslant \exp\left(f(\lambda)V_n + \varepsilon_n\lambda^3\right).$$

and the result follows immediately. □

Let us now define, for $a > 0$, the variable $R_a^* = W_a^* 1_{\{\tau_a \leqslant 3m\}} + T_{3m}^* 1_{\{\tau_a > 3m\}}$. In words, R_a^* measures the total quadratic variation necessary for the process to reach the value a, if this happens before $3m$ steps; otherwise, R_a^* is just the quadratic variation up to this point. We have the following result.

Lemma 11.7. *Let $m \in \mathbb{N}$, $a > 0$ and the processes S_i and S_i^* defined as before. Also, define $\gamma = \sum_{i=1}^{3m} \varepsilon_i$. Then*

$$\mathbb{E}\left[\exp\left(-f(\lambda)R_a^*\right)\right]\exp\left(\gamma\lambda^3\right) \geqslant \exp\left(-\lambda(a+1)\right).$$

for all $0 < \lambda < \min\{\varepsilon_i/3 : 1 \leqslant i \leqslant 3m\}$.

Proof. Recall that we defined $\sigma_n^* = \min\{n, \tau_a^*\}$. Note that $S_{\sigma_{3m}^*} \leqslant a+1$ and $\sigma_{3m}^* \leqslant 3m$. Since $Z_n(\lambda)^*$ is a submartingale, we have

$$1 = \mathbb{E}\left[Z_0^*\right] \leqslant \mathbb{E}\left[Z_{\sigma_{3m}^*}\right] \leqslant \exp\left(\lambda(a+1) + \lambda^3\sum_{i=1}^{3m}\varepsilon_i\right)\mathbb{E}\left[\exp\left(-f(\lambda)T_{\sigma_{3m}}^*\right)\right].$$

The proof follows by observing that $T_{\sigma_{3m}}^* \geqslant R_a^*$. □

We now establish a relation between the variables R_a^* and W_a^*.

Lemma 11.8. *Let $m \in \mathbb{N}$ and positive real numbers a, b, x such that $b \leqslant m$ and $x \leqslant 2b$. Then, $\mathbb{P}\left(R_a^* < x\right) = \mathbb{P}\left(W_a^* < x\right)$.*

Proof. We first note that $T_{3m}^* \geqslant 2m \geqslant 2b \geqslant x$. This implies that $R_a^* < x$ if and only if $W_a^* < x$ and $\tau_a^* \leqslant 3m$. Also, in case that $\tau_a^* > 3m$, we have $W_a^* \geqslant T_{3m}^* \geqslant x$. Therefore,

$$\mathbb{P}\left(R_a^* < x\right) == \mathbb{P}\left(W_a^* < x, \tau_a^* \leqslant 3m\right) = \mathbb{P}\left(W_a^* < x\right).$$

□

Our last step before proving Theorem 1.6 is the following technical lemma.

Lemma 11.9. *If $c > 0$ and $x > 2\log c$, then $e^x > cx$.*

Proof. Note that $g(c) = c - \log c$ has a minimum at $c = 2$ and is positive there, so it is positive everywhere. Thus, $h(x) = e^x - cx$ is positive at $x = 2\log c$. Since $h'(x) = e^x - c$ is positive for all $x > 2\log c$, the result follows. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. We assume, without loss of generality, that $|X_i| \leqslant 1$ for all i. Now, since $|T_m| \leqslant m$, the result holds trivially if $b > m$. We thus assume that $b \leqslant m$. Let us recall that it suffices to prove (11-1) to complete the proof. From Lemma 11.8 we see that it suffices to prove

$$\mathbb{P}\left(R_a^* < b\right) \geqslant \frac{1}{2}\exp\left(-\frac{a^2}{2b}(1+\eta)\right). \tag{11-6}$$

where η is fixed as in the statement of this theorem.

Let us note that $f(\lambda) \leqslant \lambda^2/2$, and so Lemma 11.7 implies that $\exp(-\lambda(a+1)) \leqslant \mathbb{E}\left[\exp(-\lambda^2 R_a^*/2)\right]\exp(\gamma\lambda^3)$. Integrating by parts, we get

$$\exp(\gamma\lambda^3)\frac{\lambda^2}{2}\int_0^\infty \mathbb{P}\left(R_a^* < x\right)\exp\left(-\frac{\lambda^2}{2}x\right)dx \geqslant \exp(-\lambda(a+1)). \tag{11-7}$$

We let $\rho = \eta/3$ and fix $\lambda = (1+\rho)a/b$. Note that $\lambda < 1/2$ and $\lambda < \varepsilon_i$ for all $i \leqslant 3m$, so the inequality above holds for this choice of λ. Now, we divide the interval $(0, +\infty)$ in five subintervals, as follows: $I_1 = [0, b/5)$, $I_2 = [b/5, (1-2\rho)b)$, $I_3 = [b, 2b)$, $I_4 = [2b, +\infty)$ and $I_5 = [(1-2\rho)b, b)$. For each of these intervals, let

$$\delta_j = \exp(\gamma\lambda^3)\frac{\lambda^2}{2}\int_{I_j}\mathbb{P}\left(R_a^* < x\right)\exp\left(-\frac{\lambda^2}{2}x\right).$$

We claim that $\delta_j \leqslant \exp(-\lambda(a+1)-4)$ for $j = 1,2,3,4$. Let us now show how (11-6) follows from this claim. First, note that

$$\begin{aligned}\delta_5 &\geqslant \exp(-\lambda(a+1)) - \delta_1 - \delta_2 - \delta_3 - \delta_4 \\ &\geqslant \frac{9}{10}\exp(-\lambda(a+1)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\delta_5 &\leqslant \mathbb{P}\left(R_a^* < b\right)\exp(\gamma\lambda^3)\int_{(1-2\rho)b}^\infty \frac{\lambda^2}{2}\exp\left(-\frac{\lambda^2}{2}x\right)dx \\ &= \mathbb{P}\left(R_a^* < b\right)\exp\left(\gamma\lambda^3 - \frac{\lambda^2(1-2\rho)b}{2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{P}\left(R_a^* < b\right) &\geqslant \frac{9}{10}\exp\left(-\lambda(a+1) - \gamma\lambda^3 + \frac{\lambda^2(1-2\rho)b}{2}\right)\\ &\geqslant \frac{9}{10}\exp\left(-\frac{(1+\rho)a^2}{b} - \frac{(1+\rho)a}{b} - \frac{\gamma(1+\rho)^3a^3}{b^3} + \frac{(1+\rho)^2(1-2\rho)a^2}{2b}\right)\\ &\geqslant \frac{9}{10}\exp\left(-\frac{a^2}{2b}\left(2+2\rho-1+3\rho^2+2\rho^3+\frac{3\gamma a}{b^2}\right) - \frac{(1+\rho)a}{b}\right).\end{aligned}$$

Since $a/b < 1/5$, $\rho < 1/40$ and $\gamma a/b^2 < \rho^2/4$, we deduce that

$$\mathbb{P}\left(R_a^* < b\right) \geqslant \frac{1}{2}\exp\left(-\frac{a^2}{2b}(1+3\rho)\right) = \frac{1}{2}\exp\left(-\frac{a^2}{2b}(1+\eta)\right).$$

It is now left for us to prove that $\delta_j \leqslant \exp(-\lambda(a+1))$ for $1 \leqslant j \leqslant 4$. We start with δ_1. Recall that $I_1 = [0, b/5)$ and $\rho < 1/16$, so

$$\begin{aligned}\delta_1 &\leqslant \exp(\gamma\lambda^3)\mathbb{P}\left(R_a^* < \frac{b}{5}\right)\int_0^\infty \frac{\lambda^2}{2}\exp\left(-\frac{\lambda^2}{2}x\right)dx\\ &\leqslant \exp(\gamma\lambda^3)\mathbb{P}\left(R_a^* < \frac{b}{4(1+\rho)}\right).\end{aligned}$$

Since $a \leqslant b/8$, we can apply Lemmas 11.1 and 11.8 to deduce from the inequality above that

$$\begin{aligned}\delta_1 &\leqslant \exp\left(\frac{\gamma(1+\rho)^3a^3}{b^3} - \frac{4(1+\rho)a^2}{2b} + \frac{64\gamma(1+\rho)^3a^3}{6b^3} + \frac{64(1+\rho)^3a^4}{16b^3}\right)\\ &= \exp\left(-\frac{(1+\rho)a^2}{b} - \frac{(1+\rho)a^2}{2b} - \frac{(1+\rho)a^2}{2b} + \frac{70\gamma(1+\rho)^3a^3}{6b^3} + \frac{4(1+\rho)^3a^4}{b^3}\right).\end{aligned}$$

Note that, for simplicity, we changed ξ in the bound given by Lemma 11.1 by γ. Now, observe that the first term inside the exponential above is exactly λa. Also, $a^2/2b > a/b$ implies that the second term is bounded by $-\lambda$. Using that $\gamma a/b^2 < \rho^2/12$ and $a^2/b^2 < \rho^2/12$ we deduce that

$$\delta_1 \leqslant \exp\left(-\lambda a - \lambda + \frac{a^2}{b}\left(-\frac{1}{2} - \frac{\rho}{2} + \frac{5\rho^2(1+\rho)^3}{2}\right)\right)$$

Since $\rho < 1/40$ and $a^2/2b > 4$, the inequality above shows that $\delta_1 \leqslant \exp\left(-\lambda(a+1) - 4\right)$.

Let us now prove that the same upper bound holds for δ_2. Recalling that $I_2 = [b/5, (1-2\rho)b]$, we shall apply Lemmas 11.1 and 11.8 to bound $\mathbb{P}\left(R_a^* < x\right)$ for every $x \in I_2$. We get

$$\delta_2 \leqslant \exp(\gamma\lambda^3)\frac{\lambda^2}{2}\int_{b/5}^{(1-2\rho)b)} \exp(-\theta(x))dx \qquad (11\text{-}8)$$

where

$$\theta(x) = \frac{\lambda^2 x}{2} + \frac{a^2}{2x} - \frac{\gamma a^3}{6x^3} - \frac{a^4}{16x^3}. \qquad (11\text{-}9)$$

We shall prove that $-\theta(x)$ is increasing over I_2. Note that, for every $x > 0$,

$$\theta''(x) = \frac{a^2}{x^3}\left(1 - \frac{2\gamma a}{x^2} - \frac{3a^2}{4x^2}\right) > 0$$

So, $\theta'(x)$ is increasing over $[0, +\infty)$ where

$$\theta'(x) = \frac{\lambda^2}{2} - \frac{a^2}{2x^2} + \frac{\gamma a^3}{2x^4} + \frac{3a^4}{16x^4}.$$

In particular, if $x \in I_2$, we have

$$\begin{aligned}
\theta'(x) \leqslant \theta'((1-2\rho)b) &= \frac{(1+\rho)^2 a^2}{2b^2} - \frac{a^2}{2(1-2\rho)^2 b^2} + \frac{\gamma a^3}{2(1-2\rho)^4 b^4} + \frac{3a^4}{16(1-2\rho)^4 b^4} \\
&\leqslant \frac{a^2}{b^2}\left(\frac{(1+\rho)^2}{2} - \frac{(1+2\rho)^2}{2} + \frac{\rho^2}{8} + \frac{\rho^2}{64}\right) \\
&\leqslant \frac{a^2}{b^2}(-2\rho - 2\rho^2) < 0
\end{aligned}$$

which proves that θ is decreasing over I_2. Thus, $-\theta(x) \leqslant -\theta((1-2\rho)b)$ for every $x \in I_2$. This implies that

$$\begin{aligned}
-\theta(x) &\leqslant -\frac{(1+\rho)^2(1-2\rho)a^2}{2b} - \frac{a^2}{2(1-2\rho)b} + \frac{\gamma a^3}{6(1-2\rho)^3 b^3} + \frac{a^4}{16(1-2\rho)^2 b^3} \\
&\leqslant -\frac{a^2}{b}\left(\frac{2 + 2\rho + \rho^2 + 6\rho^3}{2} - \frac{\rho^2}{24} - \frac{\rho^2}{192}\right) \\
&\leqslant -\frac{(1+\rho)a^2}{b} - \frac{a^2}{b}\left(\frac{\rho^2}{2} - \frac{\rho^2}{24} - \frac{\rho^2}{192}\right) \qquad (11\text{-}10)
\end{aligned}$$

where we used that $1/(1-2\rho) \geqslant 1 + 2\rho + 4\rho^2 + 8\rho^3$. Observe that the first term on the right-hand side of the last inequality is equal to λa. We also note that

$$\gamma\lambda^3 = \frac{\gamma(1+\rho)^3 a^3}{b^3} \leqslant \frac{3\rho^2 a^2}{10b}. \qquad (11\text{-}11)$$

Then, it follows from (11-8), (11-10) and (11-11) together that

$$\begin{aligned}
\delta_2 &\leqslant \frac{b\lambda^2}{2}\exp\left(-\frac{(1+\rho)a^2}{b} - \frac{a^2}{b}\left(\frac{\rho^2}{2} - \frac{\rho^2}{72} - \frac{\rho^2}{192} - \frac{3\rho^2}{10}\right)\right) \\
&\leqslant \exp\left(\log\left(\frac{a^2}{b}\right) - \lambda a - \frac{\rho^2 a^2}{6b}\right)
\end{aligned}$$

where we also used that $b\lambda^2/2 \leqslant a^2/b$. Now, by assumption, $a^2/b >$

$(20/\rho^2)\log(10/\rho^2)$. Thus, a direct application of Lemma 11.9 with $x = \rho^2 a^2/10b$ and $c = 10/\rho^2$ shows that $\log(a^2/b) \leqslant \eta^2 a^2/10b$. This observation, together with the last inequality, proves that $\delta_2 \leqslant \exp(-\lambda a - (\rho^2 a^2/15b))$. A direct computation shows that $\rho^2 a^2/15b > 5 > (4 + ((1+\rho)a/b))$ and thus $\delta_2 \leqslant \exp(-\lambda a - \lambda - 4)$.

We may now bound δ_3 in a similar way. We begin again by applying Lemmas 11.1 and 11.8 to deduce that

$$\delta_3 \leqslant \exp(\gamma\lambda^3)\frac{\lambda^2}{2}\int_b^{2b} \exp(-\theta(x))dx \tag{11-12}$$

with $\theta(x)$ as defined in (11-9). Note that if $x \in I_3$, then $b \leqslant x < 2b$, so

$$\theta'(x) \geqslant \frac{(1+\rho)^2 a^2}{2b^2} - \frac{a^2}{2b^2} + \frac{\gamma a^3}{2b^4} + \frac{3a^4}{16b^4} \geqslant 0.$$

Thus, $\theta(x)$ is increasing over I_3 which implies that

$$\begin{aligned}
-\theta(x) \leqslant -\theta(b) &\leqslant -\frac{(1+\rho)^2 a^2}{2b} - \frac{a^2}{2b} + \frac{\gamma a^3}{6b^3} - \frac{a^4}{16b^3} \\
&\leqslant -\frac{a^2}{b}\left(\frac{2+2\rho+\rho^2}{2} - \frac{\rho^2}{24} - \frac{\rho^2}{192}\right) \\
&\leqslant -\frac{(1+\rho)a^2}{b} - \frac{a^2}{b}\left(\frac{\rho^2}{2} - \frac{\rho^2}{24} - \frac{\rho^2}{192}\right).
\end{aligned} \tag{11-13}$$

Note that this is the exact same bound that we obtained in equation (11-10) for the function $\theta(x)$ with $x \in I_2$. Now, using (11-12),(11-11) and (11-13) and applying the same argument used to bound δ_2, we conclude that $\delta_3 \leqslant \exp(-\lambda a - \lambda - 4)$.

Finally, we bound δ_4. Note that

$$\delta_4 = \exp(\gamma\lambda^3)\int_{2b}^{\infty} \frac{\lambda^2}{2}\exp\left(-\frac{\lambda^2 x}{2}\right)dx = \exp\left(\gamma\lambda^3 - \lambda^2 b\right).$$

Now, it follows from equation(11-11) and the equality above that

$$\begin{aligned}
\delta_4 &\leqslant \exp\left(-\frac{(1+\rho)^2 a^2}{b} + \frac{\rho^2 a^2}{4b}\right) \\
&\leqslant \exp\left(-\frac{(1+\rho)a^2}{b} - \frac{a^2}{b}\left(\rho + \frac{3\rho^2}{4}\right)\right) \\
&= \exp\left(-\lambda a - \frac{\rho a^2}{b^2}\right).
\end{aligned}$$

We recall from the proof of the upper bound for δ_2 that a direct computation shows that $\rho a^2/b > \rho^2 a^2/15b > (4 + ((1+\rho)a/b))$. Together with the last inequality above, this shows that $\delta_4 \leqslant \exp(-\lambda a - \lambda - 4)$. □

Bibliography

[1] SHAMIR, E.; SPENCER, J.. **Sharp concentration of the chromatic number on random graphs $g(n, p)$**. Combnatorica, 7:121–129, 1987.

[2] ALON, N.; SPENCER, J.. **The Probabilistic Method**. Wiley, Tel Aviv and New York, 4th edition, 2015.

[3] VU, V. H.. **A large deviation result on the number of small subgraphs of a random graph**. Combinatorics Probability and Computing, 10(1):79–94, 2001.

[4] KIM, J. H.; VU, V. H.. **Divide and conquer martingales and the number of triangles in a random graph**. Random Structures & Algorithms, 24(2):166–174, 2004.

[5] JANSON, S.; OLESZKIEWIVZ, K.; RUCIŃSKI, A.. **Upper tails for subgraph counts in random graphs**. Israel Journal of Mathematics, 142:61–92, 2004.

[6] CHATTERJEE, S.. **The missing log in large deviations for triangle counts**. Random Structures & Algorithms, 40(4):437–451, 2012.

[7] DEMARCO, B.; KAHN J.. **Upper tails for triangles**. Random Structures & Algorithms, 40(4):452–459, 2012.

[8] CHATTERJEE, S.. **An introduction to large deviations for random graphs**. Bulletin of the American Mathematical Society, 53(4):617–642, 2016.

[9] CHATTERJEE, S.; DEMBO, A.. **Nonlinear large deviations**. Advances in Mathematics, 299:396–450, 2016.

[10] LUBETZKY, E.; ZHAO, Y.. **On the variational problem for upper tails in sparse random graphs**. Random Structures & Algorithms, 50(3):420–436, 2017.

[11] ELDAN, R.. **Gaussian-width gradient complexity, reverse log-sobolev inequalities and nonlinear large deviations**. Geometric and Functional Analysis, 28:1548–1596, 2018.

[12] COOK, N.; DEMBO, A.. **Large deviations of subgraph counts for sparse Erdős-Rényi graphs**. Advances in Mathematics, 373:107289, 2020.

[13] AUGERI, F.. **Nonlinear large deviation bounds with applications to traces of wigner matrics and cycles counts in Erdős-Rényi graphs**. arXiv preprint arXiv:1810.01558v3, 2020.

[14] HAREL, M.; MOUSSET, F.; SAMOTIJ, W.. **Upper tails via high moments and entropic stability**. Duke Mathematical Journal, 171(10):2089–2192, 2022.

[15] RUCIŃSKI, A.. **When are small subgraphs of a random graph normally distributed?** Probability Theory and Related Fields, 78:1–10, 1988.

[16] JANSON, S.. **A functional limit theorem for random graphs with applications to subgraph count statistics**. Random Structures & Algorithms, 1(1):15–37, 1990.

[17] FÉRAY, V.; MÉLIOT, P.-L.; NIKEGHBALI, A.. **Mod-ϕ Convergence: Normality Zones and Precise Deviations**. Springer Briefs in Probability and Statistics, 2016.

[18] DÖRING, H.; EICHELSBACHER, P.. **Moderate deviations in a random graph and for the spectrum of bernoulli random matrices**. Eletronic Journal of Probability, 14:2636–2656, 2009.

[19] GOLDSCHMIDT, C.; GRIFFITHS, S.; SCOTT, A. D.. **Moderate deviations of subgraph counts in the Erdős-Rényi random graphs $g(n, m)$ and $g(n, p)$**. Transactions of the American Mathematical Society, 373(2020):5517–5585, 2020.

[20] HOEFFDING, W.. **Probability inequalities for sums of bounded random variables**. Journal of the American Statistical Association, 58(301):13–30, 1963.

[21] AZUMA, K.. **Weighted sums of certain dependent random variables**. Tohoku Mathematical Journal, 19(3):357–367, 1967.

[22] FREEDMAN, D. A.. **On tail probabilities for martingales**. The Annals of Probability, 3(1):100–118, 1963.

[23] DURRETT, R.. **Probability: Theory and Examples**. Duxbury Press, Belmont, CA, 6th edition, 2019.

[24] JANSON, S.. **Large deviation inequalities for sums of indicator variables**. arXiv:1609.00533 [math.PR], 2016.

[25] PIGA, S.; SANHUEZA-MATALA, N.. **Cycle decompositions in 3-uniform hypergraphs**. arXiv preprint arXiv:2101.12205v2, 2021.

[26] JANSON, S.; RUCIŃSKI, A. . **The infamous upper tail**. Random Structures & Algorithms, 20(3):317–342, 2001.

[27] TALAGRAND, M.. **Concentration of measure and isoperimetric inequalities in product spaces**. Publications Mathématiques de l'Institut des Hautes Scientifiques, 81:73–205, 1995.

[28] KIM, J. H.; VU, V. H.. **Concentration of multivariate polynomials and its applications**. Combinatorica, 20:417–434, 2000.

[30] BAHADUR, R. R.. **Some approximations to the binomial distribution function**. The Annals of Mathematical Statistics, 31(1):43–54, 1960.

[31] LIN, Z.; BAI, Z.. **Probability Inequalities**. Springer, Berlin, 1st edition, 2010.