Pontifícia Universidade Católica do Rio de Janeiro

Edhin Franklin Mamani Castillo

About the measure of maximal entropy and horospherical foliations of geodesic flows of compact manifolds without conjugate points

Tese de doutorado

Thesis presented to the Programa de Pós–graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Rafael Oswaldo Ruggiero Rodriguez

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Abstract

Mamani Castillo, Edhin Franklin; Ruggiero Rodriguez, Rafael Oswaldo (Advisor). **About the measure of maximal entropy and horospherical foliations of geodesic flows of compact manifolds without conjugate points**. Rio de Janeiro, 2022. 125p. Tese de Doutorado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

In this thesis, we study some dynamical and geometrical properties of the geodesic flow of certain compact manifolds without conjugate points. The thesis has two main parts. We first extend Gelfert-Ruggiero's work about the existence of an expansive factor for the geodesic flow to the case of compact surfaces without conjugate points and genus greater than one. The main idea is to define an equivalence relation that collapses bi-asymptotic orbits of the geodesic flow. This induces a factor time-preserving semi-conjugate to the geodesic flow under the quotient map. Moreover, the factor is expansive, topologically mixing and has a local product structure. These properties imply that the factor has a unique measure of maximal entropy. We lift this measure to the unit tangent bundle and make sure that it is the unique measure of maximal entropy for the geodesic flow. This provides an alternative proof of Climenhaga-Knieper-War's theorem for the uniqueness result. In the last part of the thesis, we extend some results of Gelfert and Ruggiero from compact higher genus surfaces without conjugate points to compact n-manifolds without conjugate points and Gromov hyperbolic universal covering. Assuming that Green bundles are continuous and the existence of a hyperbolic closed geodesic, we show that Green bundles are tangent to the horospherical foliations. Moreover, the horospherical foliations are the only continuous foliations of the unit tangent bundle, invariant by the geodesic flow and satisfying a condition of local transversality. This fact was only known for compact surfaces without conjugate points by Barbosa-Ruggiero's work, and in higher dimensions assuming the stronger condition of bounded asymptote by Eschenburg's work.

Keywords

Geodesic flow; Measure of maximal entropy; Horospherical foliations; Green bundles; Visibility; Manifolds without conjugate points Entropy Gap; Gromov hyperbolic spaces.

Resumo

Mamani Castillo, Edhin Franklin; Ruggiero Rodriguez, Rafael Oswaldo. **Sobre a medida de máxima entropia e foliações horósfericas de fluxos geodésicos em variedades sem pontos conjugados**. Rio de Janeiro, 2022. 125p. Tese de Doutorado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

Nesta tese, estudamos algumas propriedades dinâmicas e geométricas do fluxo geodésico de certas variedades compactas sem pontos conjugados. A tese tem duas partes principais. Primeiro estendemos o trabalho de Gelfert-Ruggiero sobre a existência de um fator expansivo para o fluxo geodésico ao caso de superfícies compactas sem pontos conjugados e gênero maior que um. A idéia principal é definir uma relação de equivalência que colapsa as órbitas bi-asintóticas do fluxo geodésico. Isto induz um fator que preserva o tempo e é semi-conjugado ao fluxo geodésico sob o mapa do quociente. Além disso, o fator é expansivo, topologicamente misto e tem uma estrutura de produto local. Estas propriedades implicam que o fator tem uma única medida de máxima entropia. Levantamos esta medida para o fibrado tangente unitário e nos certificamos de que é a única medida de máxima entropia para o fluxo geodésico. Isto fornece uma prova alternativa do teorema de Climenhaga-Knieper-War para o resultado de unicidade. Na última parte da tese, estendemos alguns resultados de Gelfert e Ruggiero de superfícies compactas do gênero superior e sem pontos conjugados para n-variedades compactas sem pontos conjugados e recobrimento universal Gromov hiperbólico. Assumindo que os fibrados de Green são contínuos e a existência de uma geodésica fechada hiperbólica, mostramos que os fibrados de Green são tangentes às foliações horósfericas. Além disso, as foliações horósfericas são as únicas foliações contínuas do fibrado tangente unitário, invariantes pelo fluxo geodésico e que satisfazem uma condição de transversalidade local. Este fato só foi conhecido para superfícies compactas sem pontos conjugados pelo trabalho de Barbosa-Ruggiero, e em dimensões mais elevadas assumindo a condição mais forte de assíntota limitada pelo trabalho de Eschenburg.

Palavras-chave

Fluxo geodésico; Medida de máxima entropia; Folhações horosféricas; Fibrados de Green; Visibilidade; Variedade sem pontos conjugados; Gap de entropia; Espaços Gromov hiperbólicos.

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1
Introduction

The theory of the geodesic flow on compact Riemannian manifolds without conjugate points is a rich interplay between many areas in mathematics. The study of the topological, geometric, dynamical and ergodic properties of the geodesic flow, gives rise to a great variety of challenging problems.

The theory begins with Morse's works [1] in the 1920s, about globally minimizing geodesics in the universal covering of compact surfaces without conjugate points. Then, Koebe, Lobell, Hedlund, and Hopf in the 1930s and 1940s developed a theory to study the geodesic flows of compact surfaces of constant negative curvature [2, 3, 4, 5]. Shortly after, Hedlund, Morse and Grant extended this theory to the case of compact surfaces of variable negative curvature [6, 7, 8].

In the 1950s, Busemann [9] made a great contribution to the study of the global theory of geodesics by direct variational methods. Hopf [10] and Green [11] considered manifolds without conjugate points admitting some regions of zero or positive curvature. In this more general context, the divergence of geodesic rays and Morse's shadowing play an important role in the theory. From the classification of compact surfaces, the case of genus one, i.e., the torus was solved by Hopf [12]. Many interesting problems of the theory of geodesic flows on compact surfaces without conjugate points and genus greater than one, where the curvature has variable sign, remain open.

The general objective of this thesis is to investigate some ergodic, dynamical and geometrical properties of the geodesic flow on some families of compact manifolds without conjugate points.

As first specific goal, we study compact surfaces without conjugate points and genus grater than one. We intend to extend Gelfert-Ruggiero's approach [13] from compact surfaces without focal points to this setting. That is, build a factor flow time-preserving semi-conjugate to the geodesic flow of the surface. Following their strategy, we want to prove the relevant dynamical properties of the factor flow such as: expansivity, topological mixing, local product structure, specification property and uniqueness of the measure of maximal entropy.

Finally, we attempt to use the properties of this factor flow to show the uniqueness of the measure of maximal entropy of the geodesic flow of the surfaces.

As second specific goal, we work with compact n-manifolds without conjugate points and Gromov hyperbolic universal covering. Moreover, we assume that Green bundles are continuous and there exists a hyperbolic periodic geodesic. In this setting we intend to generalize to higher dimension some results that holds for surfaces. We first aim for an extension of Gelfert-Ruggiero's Theorem [14] that holds for compact higher genus surfaces without conjugate points and continuous Green bundles. This theorem says that Green bundles are tangent to horospherical foliations. In 1980's for compact n-manifolds without conjugate points and continuous Green bundles, Knieper [15] showed that Green bundles integrate to continuous foliations of T_1M invariant by the geodesic flow. Although he did not prove whether the Green bundles are tangent to the horospherical leaves or whether they are uniquely integrable. Thus, in our context we desire to show that Green bundles are uniquely integrable. On the other hand, we seek to extend Barbosa-Ruggiero's result [16] that holds for compact higher genus surfaces without conjugate points. This result asserts that the horospherical foliations are the only co-dimension one continuous foliations of T_1M invariant by the geodesic flow. We pursue the extension of this conclusion to our higher dimensional context under an additional local assumption given in Chapter 4. Finally, we want to show some dynamical consequences of Ruggiero's work [17] in our context. Specifically, about some set closely related to Pesin set and the hyperbolic periodic points of the geodesic flow.

The problem of the uniqueness of the measure of maximal entropy is a difficult problem for non-uniformly hyperbolic systems. The following theorem was proved by Climenhaga, Knieper and War in 2021 [18].

Theorem 1.1. *Let M be a compact surface without conjugate points and genus greater than one. Then the geodesic flow has a unique measure of maximal entropy.*

In fact, they proved the uniqueness of the measure of maximal entropy for a large family of compact n-manifolds without conjugate points. Their method uses the Climenhaga-Thompson [19] generalization of the classical Bowen-Franco criterion [20]. In this thesis we give another proof of this theorem that extends Gelfert-Ruggiero's approach [13]. This approach differs from Climenhaga-Knieper-War, giving a more direct proof of the uniqueness of the measure of maximal entropy. Moreover, Gelfert-Ruggiero's method could give more information because it has more control over the expansive factor.

Anosov's work [21] on uniformly hyperbolic flows extended a great deal of the theory of geodesic flows on the surface case. In particular, Anosov's theory applies to compact n-dimensional manifolds of variable negative curvature.

The theory of geodesic flows on compact manifolds of negative curvature evolved naturally to the theory of geodesic flows in more general categories of manifolds without conjugate points such as:

- Visibility manifolds.
- Rank-1 and higher rank manifolds.
- Manifolds of hyperbolic type.
- Manifolds with expansive geodesic flow.
- Manifolds of bounded asymptote.
- Manifolds satisfying the asymptoticity axiom.
- Gromov hyperbolic manifolds.

These categories of manifolds capture important aspects of the global geometry of manifolds of negative curvature. Notably, the visibility manifolds introduced by Eberlein and O'Neill [22].

Later, Thurston [23] and Gromov [24] extended the visibility condition to more general metric spaces, not necessarily manifolds, like graphs and simplicial complexes. Gromov introduced the notion of hyperbolic groups, and he extended Eberlein's work [25] about the global geometry of visibility manifolds.

Combining Gromov's ideas with Eberlein's work, in [26], Ruggiero showed that for the case of compact manifolds without conjugate points, Gromov hyperbolic manifolds are exactly the visibility manifolds where divergence of geodesic rays holds.

One of the main properties of hyperbolic dynamics not holding in the context of manifolds without conjugate points is the regularity of horospheres. Anosov [27] showed that horospheres form invariant submanifolds dynamically defined by the hyperbolic geodesic flow. Horospheres always exist in compact manifolds without conjugate points providing invariant sets for every point in the phase space of the geodesic flow. However, without restrictions on the curvature of the manifold, the regularity of horospheres might no longer hold.

In certain cases like geodesic flows of compact manifolds without focal points, horospheres induce invariant continuous foliations on the unit tangent bundle of the manifold called horospherical foliations. Some regularity of these foliations, even their continuity, proved to be useful to study global properties of the geodesic flow, extending some features of nonpositive curvature geodesics. Heintze-Imhof [28], Eschenburg [29] and Pesin [30] introduced categories of manifolds without conjugate points where horospheres define continuous invariant foliations, categories that are more general than the category of manifolds without focal points: manifolds of bounded asymptote [29] and manifolds satisfying the asymptoticity axiom [30].

The second main goal of this thesis is to show the following:

Theorem 1.2. *Let (M, g) be a compact, C^∞, n-dimensional Riemannian manifold without conjugate points, Gromov hyperbolic universal covering and continuous Green bundles. If there exists a hyperbolic periodic geodesic then*

1. *The set where the Lyapunov exponents of all vectors transverse to the geodesic vector field are non-zero agrees almost everywhere with an open dense set, with respect to Liouville measure.*
2. *Hyperbolic periodic points are dense on the unit tangent bundle T_1M.*
3. *The horospherical foliations are the only foliations of T_1M such that: they have C^1-leaves of dimension $n-1$, are continuous, invariant by the geodesic flow, and transverse to $\mathcal{F}^u$ (or $\mathcal{F}^s$) at some hyperbolic periodic point of T_1M.*
4. *Green bundles are uniquely integrable, and tangent to the horospherical foliations.*

Barbosa and Ruggiero [16] proved item 3 for compact surfaces without conjugates points and genus greater than one. Thus, item 3 is a partial extension of their result to higher dimension.

By Katok's work [31], the existence of a hyperbolic periodic geodesic is guaranteed for compact surfaces without conjugate points. However, in the higher dimensional setting of visibility manifolds, the existence of hyperbolic periodic geodesics remains an open problem.

The so-called Green bundles were introduced by Hopf [12] for compact surfaces without conjugate points, and later defined by Green [32] for any manifold without conjugate points whose sectional curvatures are bounded from below. Though Green bundles were used to get geometrical results, several authors showed their importance for the dynamics of the geodesic flow. Hopf [12] proved that a torus without conjugate points must be flat applying the theory of the Riccati equation associated with Green bundles. In the 1970s Eberlein [33] showed that Green bundles provide as well information about the hyperbolicity of the geodesic flow.

Eberlein [33] characterized Anosov flows by the linear independence of Green bundles. This result is considered a landmark in the theory, and gave rise to huge literature about the applications of Green bundles to the study of the regularity of horospheres.

This thesis is organized as follows. Chapter 2 gives all basic definitions and results of the theory. We divide the chapter into three major sections. In Section 2.1 we introduce the geodesic flows and related objects. Section 2.2 describes the dynamical and ergodic properties of the flows we work with. Finally, Section 2.3 states some tools used in the investigation of geodesic flows. In particular, we restrict ourselves to the tools most closely related to the hypotheses of the previous theorems. Chapter 3 is devoted to proving Theorem 1.1. Chapter 4 deals with the proof of Theorem 1.2. In Appendix A, we give a proof of the well-known result: for compact manifolds without conjugate points and visibility universal covering, the geodesic flow is topologically mixing.

2
Preliminaries

2.1
The dynamical system under study

This section is devoted to introduce the dynamical system and related objects, with which we shall work throughout the text. For the section, the references are [34, 35, 36].

The dynamical system under study is a smooth flow acting on a Riemannian manifold. First let us introduce the Riemannian manifold we will work with. Let (M, g) be a C^∞ compact connected Riemannian n-manifold, TM be its tangent bundle and T_1M be the associated unit tangent bundle. We know that T_1M is a compact differentiable $(2n-1)$-dimensional manifold if the Riemannian n-manifold M is also compact. We will endow T_1M with a Riemannian structure in a moment, but in the meantime we consider T_1M as our compact $2n-1$-dimensional Riemannian manifold.

Once we introduced the Riemannian manifold T_1M on which we will work, we define the smooth flow acting on T_1M using the geodesics of the manifold M. Recall that every Riemannian metric g induces in a unique way a connection ∇ called the Levi-Civita connection. This connection provides a way to differentiate covariantly vector fields on M. Covariant differentiation allows us to define geodesics: a smooth curve $\gamma \subset M$ is called a **geodesic** if

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

that is, the covariant derivative of its velocity vector field $\dot\gamma$ vanishes identically. Some properties of geodesics are given by the theory of differential equations. The existence and uniqueness of solutions guarantees that there exists a unique geodesic for each given initial conditions. Thus, given the initial conditions $\theta = (p, v) \in TM$, we denote by γ_θ to the unique geodesic satisfying

$$\gamma_\theta(0) = p \quad \text{and} \quad \dot\gamma_\theta(0) = v.$$

Moreover, since M is compact, Hopf-Rinow's theorem says that geodesics are defined for every time parameter. Therefore, we define the **geodesic flow** of (M, g) as the 1-parameter family of diffeomorphisms $(\phi_t)_{t\in\mathbb{R}}$ acting on TM:

$$\phi : \mathbb{R} \times TM \to TM$$
$$(t, \theta) \mapsto \phi(t, \theta) = \phi_t(\theta) = \dot{\gamma}_\theta(t).$$

Roughly speaking, the geodesic flow acting on θ is the transport of θ for a time t, through velocity vectors of the geodesic γ_θ. The existence and uniqueness of geodesics provides the composition flow property. The smooth dependence of the geodesics with respect to initial data yields the smooth structure of the geodesic flow. We know that the length of velocity vectors of a geodesic does not change along the geodesic. Thus requiring that all geodesics have velocity vectors of unit length, we can restrict the geodesic flow to T_1M. Therefore

$$\phi : \mathbb{R} \times T_1M \to T_1M \text{ or } \phi_t : T_1M \to T_1M$$

is the smooth flow we will work with and we will refer to it simply as the geodesic flow of (M, g).

In order to give a suitable Riemannian structure to T_1M, we shall introduce the decomposition of TTM(tangent bundle of TM) into horizontal and vertical subspaces. This decomposition will provide a natural setting for defining the Sasaki metric on TM. We will carry out this program first for TM and then we will move on to T_1M setting.

We first define the vertical and horizontal vector subspaces. Let us recall the canonical projection

$$P : TM \to M \qquad \theta = (p, v) \mapsto P(p, v) = p.$$

The derivative of P at θ

$$dP : TTM \to TM \qquad d_\theta P : T_\theta TM \to T_pM$$

allows us to define the vertical subspaces. The vertical subspace at θ is defined as

$$V(\theta) = \ker(d_\theta P) \subset T_\theta TM.$$

The horizontal subspace is defined in the following way. For every $\theta = (p, v) \in TM$, we use the Levi-Civita connection ∇ associated to g, to define the connection map

$$C_\theta : T_\theta TM \to T_pM.$$

For every $\xi \in T_\theta TM$ there exists a curve

$$z : (-\epsilon, \epsilon) \to TM \text{ such that } z(0) = \theta, \quad \dot{z}(0) = \xi.$$

In local coordinates, this means that there exist curves, the components of z,

$$\begin{aligned} \alpha : (-\epsilon, \epsilon) &\to M \\ t &\mapsto \alpha(t) \end{aligned} \qquad\qquad \begin{aligned} Z : (-\epsilon, \epsilon) &\to TM \\ t &\mapsto Z(t) \in T_{\alpha(t)}M, \end{aligned}$$

satisfying

$$z(t) = (\alpha(t), Z(t)), \quad z(0) = (\alpha(0), Z(0)) = (p, v) \text{ and } \quad \dot{z}(0) = (\dot{\alpha}(0), \dot{Z}(0)) = \xi.$$

From this α is a curve on M and Z is a vector field along α, hence Z can be differentiated along α. Thus we define the **connection map** as

$$\begin{aligned} C_\theta : T_\theta TM &\to T_pM \\ \xi &\mapsto C_\theta(\xi) = (\nabla_{\dot{\alpha}} Z)(0). \end{aligned}$$

The connection map C_θ is a well-defined linear map that helps to define the horizontal subspaces. The horizontal subspace at $\theta \in TM$ is defined as

$$H(\theta) = \ker(C_\theta) \subset T_\theta TM.$$

Concerning the dimensions of $V(\theta)$ and $H(\theta)$, we have the following property: for every $\theta \in TM$, the restricted maps

$$d_\theta P|_{H(\theta)} : H(\theta) \to T_pM, \qquad C_\theta|_{V(\theta)} : V(\theta) \to T_pM$$

are linear isomorphisms. So, if the manifold M is n-dimensional so are $V(\theta)$ and $H(\theta)$. This, together with the fact that $V(\theta)$ and $H(\theta)$ are transverse, gives the following decomposition for every $\theta \in TM$:

$$T_\theta TM = H(\theta) \oplus V(\theta).$$

Thus every vector $\xi \in T_\theta TM$ can be decomposed into unique horizontal and vertical components $\xi_1 \in H(\theta)$ and $\xi_2 \in V(\theta)$ as

$$\xi = \xi_1 \oplus \xi_2.$$

Thanks to $d_\theta P|_{H(\theta)}$ and $C_\theta|_{V(\theta)}$, for every $\theta = (p, v)$ the last decomposition can be simplified by the linear isomorphism

$$j_\theta : T_\theta TM = H(\theta) \oplus V(\theta) \to T_pM \times T_pM$$
$$\xi \mapsto j_\theta(\xi) = (d_\theta P(\xi), C_\theta(\xi)) = (\xi_h, \xi_v).$$

Taking into account this identification, we also call ξ_h and ξ_v the horizontal and vertical components of ξ. The linear isomorphism j_θ helps to define the Sasaki metric in such a way that

– Sasaki metric is defined in terms of the metric g of M and
– $H(\theta)$ and $V(\theta)$ are orthogonal.

The Sasaki metric at $\theta = (p, v) \in TM$ is defined by

$$\langle , \rangle_s : T_\theta TM \times T_\theta TM \to \mathbb{R}$$
$$(\xi, \eta) \mapsto \langle \xi, \eta \rangle_{s,\theta} = \langle \xi_h, \eta_h \rangle_p + \langle \xi_v, \eta_v \rangle_p.$$

Therefore TM with Sasaki metric, is a $2n$-dimensional Riemannian manifold.

Now, moving on to T_1M framework, we use the Riemannian metric of TM and the decomposition of TTM, to induce analog structures on T_1M. To define a Riemannian metric on T_1M, we first observe that T_1M is a closed submanifold of TM. This inclusion as submanifold automatically provides:

– For every $\theta \in T_1M$, we have the inclusion of vector subspaces $T_\theta T_1M \subset T_\theta TM$.
– We define the Riemannian metric on T_1M as the restriction of the Sasaki metric to the subspace $T_\theta T_1M \subset T_\theta TM$.

Thus T_1M is a $(2n - 1)$-dimensional Riemannian manifold. To study the dynamics of smooth flows on smooth Riemannian manifolds, a key tool is the distance induced by the Riemannian metric. In our setting, we call d_s the Sasaki distance induced by the Sasaki metric restricted to T_1M.

To carry out the decomposition of TT_1M, we will define an additional vector field and subspace: For every $\theta \in TM$,

– Let $G(\theta)$ be the vector field tangent to the geodesic flow ϕ_t at θ.
– The 1-dimensional subspace generated by $G(\theta)$ will be denoted by same symbol.

The vector field $G(\theta)$ helps to define a convenient one-form α of TM: for every $\theta = (p, v) \in TM$,

$$\alpha_\theta : T_\theta TM \to \mathbb{R}$$
$$\xi \mapsto \alpha_\theta(\xi) = \langle \xi, G(\theta) \rangle_{s,\theta} = \langle \xi_h, v \rangle_p.$$

Geometrically α_θ is the orthogonal projection onto $G(\theta)$ with respect to the Sasaki metric. Analogous to the definition of vertical and horizontal subspaces, for every $\theta \in T_1M$ we define the subspace

$$S(\theta) = \ker(\alpha_\theta) \subset T_\theta T_1 M.$$

From the geometric interpretation of α_θ, $S(\theta)$ is the orthogonal complement to $G(\theta)$ with respect to the Sasaki metric. From this and the dimensions of T_1M and $G(\theta)$, we conclude that $S(\theta)$ has dimension $2n-2$. Recalling that $T_\theta T_1 M \subset T_\theta TM$, we define the following subspaces: for every $\theta \in T_1M$

$$\mathcal{H}(\theta) = H(\theta) \cap S(\theta) \qquad \mathcal{V}(\theta) = V(\theta) \cap S(\theta).$$

It can be shown that these subspaces have dimension $n-1$, and therefore we have the orthogonal decomposition

$$S(\theta) = \mathcal{H}(\theta) \oplus \mathcal{V}(\theta).$$

Moreover, the decomposition of $T_\theta TM$ immediately gives the following orthogonal decomposition of $T_\theta T_1 M$:

$$T_\theta T_1 M = \mathcal{H}(\theta) \oplus \mathcal{V}(\theta) \oplus G(\theta).$$

Mimicking the same ideas for the case of $T_\theta TM$, any vector $\xi \in T_\theta T_1 M$ can be expressed as

$$\xi = (j_\theta(\xi), \xi_g) = (\xi_h, \xi_v, \xi_g) \text{ with } \xi_h, \xi_v \in T_pM, \xi_g \in G(\theta).$$

In this work we will almost always deal with vectors in $S(\theta)$, therefore for every $\theta \in T_1M$ and every $\xi \in T_\theta T_1 M$ we can express ξ as

$$\xi = (\xi_h, \xi_v),$$

where ξ_h and ξ_v are the horizontal and vertical components of ξ.

For the geodesic flow (T_1M, ϕ_t), we shall see that the Riemannian structure of T_1M provides a natural volume measure that turns the geodesic flow into a conservative smooth dynamical system. We first observe that the compactness of T_1M, implies that the Riemannian volume form $d\Omega$ of T_1M has finite integral over all T_1M. This property induces a finite volume measure μ on the Borel σ-algebra of T_1M. The measure μ assigns to every Borel set $U \subset T_1M$, the integral of $d\Omega$ over U. Normalizing the measure μ we get a probability Borel measure on T_1M (denoted by same symbol) called the **Liouville measure** of M. It can be shown that Liouville measure μ is invariant by the geodesic flow. Therefore the geodesic flow $\phi : \mathbb{R} \times T_1M \to T_1M$ is a smooth flow acting by diffeomorphisms that preserves measure: a conservative smooth dynamical system.

Furthermore, all the structures related to (T_1M, ϕ_t) are totally determined by the Riemannian manifold (M, g). Indeed the differential structure of M furnishes the tangent bundle TM, i.e., the tangent vectors to M. While to define T_1M we need the Riemannian metric g because g determines the length of vectors. We also observe that g through its Levi-Civita connection ∇ furnishes the equation governing the geodesics of M. Recalling that the geodesic flow is the transport of vectors along velocity vectors of geodesics, g totally determines the geodesic flow. In the last paragraph we saw that the g specifies the Liouville measure μ of M. Putting it all together, we can see that the system (T_1M, ϕ_t) and μ are totally defined by the Riemannian manifold (M, g).

In order to extend the families of geodesic flows determined by manifolds of nonpositive sectional curvature, we shall introduce Riemannian manifolds without focal points and without conjugate points. One way to define these manifolds is through the exponential map associated to the metric. Thus, let (M, g) be a compact Riemannian manifold with exponential map exp_p for every $p \in M$.

- (M, g) has no focal points if the restriction of exp_p to $v^\perp \subset T_pM$, is nonsingular for every $(p, v) \in TM$, $v \neq 0$ and $v^\perp$ the orthogonal subspace of v.
- (M, g) has no conjugate points if exp_p is nonsingular at every $p \in M$. Equivalently, the exponential map $\exp_p$ is a covering map at every $p \in M$.

It follows from the definitions that manifolds without focal points are a subfamily of manifolds without conjugate points. With respect to sectional curvature, these manifolds may have some regions of positive curvature.

Therefore, these manifolds are a certain type of generalization of the compact manifolds of non-positive curvature.

In this work we always work with geodesic flows (T_1M, ϕ_t) of compact Riemannian manifolds (M, g) without conjugate points.

2.1.1
The universal covering

Let ϕ_t be the geodesic flow of a compact manifold M without conjugate points. To prove relevant properties of ϕ_t, we rely on related dynamical systems. One such system is the geodesic flow of the associated universal covering space. Proofs in this system are often simpler than in M. This is useful because we can translate some results to M. Thus, we introduce the universal covering, its geodesic flow, and related objects.

Since we built geodesic flows from smooth manifolds, we review some results on covering spaces of these manifolds. A first basic result says that every connected compact smooth manifold M has a universal covering $\tilde{M}$ and a covering map

$$\pi : \tilde{M} \to M.$$

Moreover, the covering map π has an associated group of covering transformations Γ. We say that a homeomorphism $T : \tilde{M} \to \tilde{M}$ is a **covering transformation** if T preserves the covering map:

$$\pi \circ T = \pi. \tag{2.1}$$

The set Γ of all covering transformations forms a group with the composition of maps. The group Γ has an important connection with the topology of M. In the special case of the universal covering: Γ is isomorphic to the fundamental group $\pi_1(M)$ of M. Thus, we say that $\pi_1(M)$ acts on $\tilde{M}$ by homeomorphisms. Noting that we are in the smooth setting, we have:

- The universal covering $\tilde{M}$ is a simple connected smooth manifold.
- the covering map π is a smooth covering map.
- Every covering transformation is a diffeomorphism. Hence, $\pi_1(M)$ acts on M by diffeomorphisms.

If we add a metric to the smooth manifold M, we get more structure in the universal covering and related objects. So, consider a Riemannian metric g without conjugate points defined on M. Using π, we lift the metric g to

$\tilde{M}$ and thus get a Riemannian metric $\tilde{g} = \pi^* g$ on $\tilde{M}$, called the **pullback metric**. This definition of $\tilde{g}$ guarantees that π is a local isometry and hence a Riemannian covering map. Moreover, $(\tilde{M}, \tilde{g})$ becomes a complete Riemannian manifold without conjugate points. As for the covering transformations, taking derivatives in Equation (2.1) we see that T is a local isometry, hence a global isometry. So, from now on, every $T \in \Gamma$ is called a **covering isometry**. Updating some properties to the current setting we have:

- The universal covering $(\tilde{M}, \tilde{g})$ is a simple connected complete Riemannian manifold without conjugate points.
- the covering map π is a Riemannian covering map.
- Every covering transformation is an isometry. Hence, $\pi_1(M)$ acts on M by isometries called covering isometries.

There is another important relation between M and its universal covering $\tilde{M}$: we can obtain M as a quotient space of $\tilde{M}$ by Γ. Note that we can interpret Γ as a group action on $\tilde{M}$. Thus, we get a quotient space $\tilde{M}/\Gamma$ by the group action. The quotient $\tilde{M}/\Gamma$ is a topological space endowed with quotient topology. From the construction of Γ, we see that Γ is a discrete subgroup of isometries of $(\tilde{M}, \tilde{g})$. Moreover, the action of Γ is free and properly discontinuous. Under these conditions, the quotient $\tilde{M}/\Gamma$ is a Riemannian manifold with quotient metric $\pi_* \tilde{g}$. Furthermore, $(\tilde{M}/\Gamma, \pi_* \tilde{g})$ is isometric to (M, g). Thus, (M, g) is isometric to the quotient of the universal covering $(\tilde{M}, \tilde{g})$ by the group action $\Gamma \simeq \pi_1(M)$.

We now define the smooth dynamical system related to the geodesic flow of M. Recall that every Riemannian manifold induces a smooth dynamical system preserving volume: its geodesic flow. So, let us consider the universal covering $(\tilde{M}, \tilde{g})$ and denote by:

- $(T_1\tilde{M}, \tilde{\phi}_t)$ its geodesic flow.
- $\tilde{\mu}$ its Liouville measure.

Similar to the last section, in this case the Liouville measure $\tilde{\mu}$ is defined on a compact fundamental domain and then defined on $T_1\tilde{M}$ using covering isometries. Just as in the case of M and $\tilde{M}$, their corresponding objects are also related. For example: their geodesic flows, isometry groups, Liouville measures, and so on. We first extend the covering property of $\pi : \tilde{M} \to M$ to the setting of unit tangent bundles:

- $T_1\tilde{M}$ is a covering space of T_1M.
- The derivative $d\pi : T_1\tilde{M} \to T_1M$ is a Riemannian covering map.

The covering map immediately yields a time-preserving semi-conjugacy between the geodesic flows $\tilde{\phi}_t$ and ϕ_t. That is, for every $\theta \in T_1\tilde{M}$ and every $t \in \mathbb{R}$

$$\phi_t(d\pi(\theta)) = d\pi(\tilde{\phi}_t(\theta)), \qquad \begin{array}{ccc} T_1\tilde{M} & \xrightarrow{\tilde{\phi}_t} & T_1\tilde{M} \\ \big\downarrow{\scriptstyle d\pi} & & \big\downarrow{\scriptstyle d\pi} \\ T_1M & \xrightarrow{\phi_t} & T_1M \end{array} .$$

The canonical projections of $\tilde{M}$ and M give a relation between the covering maps. For every $\theta \in T_1\tilde{M}$,

$$\pi(\tilde{P}(\theta)) = P(d\pi(\theta)), \qquad \begin{array}{ccc} T_1\tilde{M} & \xrightarrow{\tilde{P}} & \tilde{M} \\ \big\downarrow{\scriptstyle d\pi} & & \big\downarrow{\scriptstyle \pi} \\ T_1M & \xrightarrow{P} & M \end{array} .$$

Furthermore, $\tilde{P}$ provides a relation between the covering isometries of π and $d\pi$. For every covering isometry T of π there is a unique covering isometry T' of $d\pi$ such that for each $\theta \in T_1\tilde{M}$

$$\tilde{P}(T'(\theta)) = T(\tilde{P}(\theta)), \qquad \begin{array}{ccc} T_1\tilde{M} & \xrightarrow{T'} & T_1\tilde{M} \\ \big\downarrow{\scriptstyle \tilde{P}} & & \big\downarrow{\scriptstyle \tilde{P}} \\ \tilde{M} & \xrightarrow{T} & \tilde{M} \end{array} .$$

On the other hand, the Liouville measure of (M, g) is the push-forward of the Liouville measure of $(\tilde{M}, \tilde{g})$:

$$\mu = (d\pi)_*\tilde{\mu}.$$

Since M is compact, there exist special isometries acting on $\tilde{M}$: the axial isometries. Let $T \in \Gamma$ be a covering isometry and

$$\eta : \mathbb{R} \to \tilde{M}$$

be a geodesic. We say that T **translates** η if there exists $\tau \neq 0$ such that for every $t \in \mathbb{R}$,

$$T(\eta(t)) = \eta(t + \tau).$$

In this case, we say that T has axis η and η is invariant by T. A covering isometry $T \in \Gamma$ is **axial** if T has no fixed points and T has an axis.

In the compact case we see that every covering isometry T different from the identity is axial, i.e., there is always a geodesic that is invariant by T [34]. Indeed, let $T \in \Gamma$ be a covering isometry. The isomorphism between Γ and $\pi_1(M)$ provides a nontrivial free homotopy class on M associated to T. In this homotopy class, Birkhoff's Theorem ensures the existence of a closed geodesic

$$\beta : [0, \tau] \to M$$

of minimum length τ. We choose any lift $\tilde{\beta} : [0, \tau] \to \tilde{M}$ of β under the covering map π. Since $\tilde{M}$ is complete, we extend $\tilde{\beta}$ to a geodesic $\beta' \subset \tilde{M}$ defined on all $\mathbb{R}$. This geodesic β' is an axis for T (see Chapter 12 of [34]). Therefore, for compact M, every covering isometry different from the identity has an invariant geodesic.

Besides the above, there is another reason why $\tilde{M}$ is important. The universal covering $(\tilde{M}, \tilde{g})$ is the framework of the so-called global geometry of M. This framework offers a variety of tools that have no analogy in M such as: asymptoticity of geodesics, ideal boundary, covering isometries, axial isometries, axis, strips, flats, etc.

The tools in $\tilde{M}$ allow a better understanding of ϕ_t through the dynamics of $\tilde{\phi}_t$. Roughly speaking, the geodesic flow $\tilde{\phi}_t$ on $\tilde{M}$ unwinds the geodesic flow ϕ_t on M. So, we can analyze $\tilde{\phi}_t$ and then transfer some results to ϕ_t under certain conditions. The transfer is through the relations mentioned above.

Therefore, the geodesic flow $\tilde{\phi}$ is an important tool to study the geodesic flow of compact manifolds without conjugate points. We discuss some concepts and results on global geometry in Subsection 2.3.9.

2.2
Some dynamical and ergodic properties of dynamical systems related to the geodesic flow of compact manifolds without conjugate points

Once we defined the dynamical system under study, we show that the system has relevant dynamical properties. We will also see that there are other dynamical systems related to the geodesic flows that are useful to show important results. Thus, we introduce some dynamical and ergodic properties related to the dynamical systems involved in our work. These properties are important in the study of geodesic flows of compact manifolds without conjugate points. We do not describe all the properties, but only the most relevant to our work. We use these books [37, 38] as main references for this section.

2.2.1
Dynamical properties

We start with properties of topological nature. More precisely, we deal with dynamical properties defined for continuous flows acting on compact metric spaces. For the remainder of this part we will assume a continuous flow $\phi_t : X \to X$ acting on the compact metric space X. We say that

- ϕ_t is **topologically transitive** if there exists a dense orbit.
- ϕ_t is **topologically mixing** if for every nonempty open sets $A, B \subset X$ there exists $t_0 > 0$ such that $\phi_t(A) \cap B \neq \emptyset$ for $|t| \geq t_0$.

We now define the strong sets associated to the flow. For every $x \in X$, the points $y \in Y$ such that the orbit of y tends to the orbit of x in the future or the past, form the strong sets.

Strong stable set of x: $W^{ss}(x) = \{y \in X : d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to \infty\}$.

Strong unstable set of x: $W^{uu}(x) = \{y \in X : d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to -\infty\}$.

Note that *a priori*, the above strong stable and unstable sets are only topological spaces with the subspace topology and have no further regularity. We also see that the elements of $W^{ss}(x)$ (or $W^{uu}(x)$) may be far from x. To consider points close to x, for every $\epsilon > 0$, we define the **ϵ-strong stable** and **ϵ-strong unstable set**.

$$W^{ss}_\epsilon(x) = \{y \in W^{ss}(x) : d(\phi_t(x), \phi_t(y)) \leq \epsilon, \text{ for every } t \geq 0\}$$
$$W^{uu}_\epsilon(x) = \{y \in W^{uu}(x) : d(\phi_t(x), \phi_t(y)) \leq \epsilon, \text{ for every } t \leq 0\}$$

These ϵ-strong sets allow us to define the so-called local product structure for a continuous flow $\phi_t : X \to X$ acting on a compact metric space. We say that ϕ_t has a **local product structure** if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $x, y \in X$ satisfies $d(x, y) \leq \delta$ then there exists a unique $\tau \in \mathbb{R}$ with $|\tau| \leq \epsilon$ and

$$W^{ss}_\epsilon(x) \cap W^{uu}_\epsilon(\phi_\tau(y)) \neq \emptyset.$$

We observe that the intersection point accompanies x in the future and y in the past.

In the context of smooth flows, Anosov's closing lemma is quite related to two important concepts: tracing and specification. We note that for discrete-time dynamical systems, these concepts have simple definitions. In contrast, for continuous flows, the rigorous analogous definitions are complicated. Since we do not deal directly with these definitions, we give sketches of these properties for continuous flows. For the technical definitions of these properties the reader is referred to section 6.1 of [13].

We begin with pseudo orbits. Let us consider a sequence of points $x_1, \ldots, x_n \in X$ and a sequence of positive numbers $\tau_1, \ldots, \tau_n \in \mathbb{R}_+$. These numbers τ_k help to mimic the discrete setting. Given $\delta, a > 0$, we say that the pair (x_k, τ_k) is a (δ, a)**-pseudo orbit** for ϕ_t if for $k = 1, \ldots n-1$ $\tau_k \geq a$ and

$$d(\phi_{\tau_k}(x_k), x_k + 1) \leq \delta.$$

Note that the numbers τ_k give time intervals that allow to mimic the iterates of a diffeomorphism on X. Moreover, the intervals τ_k may be nonuniform but are always greater than a. So, analogous to the discrete case, the image of x_k by ϕ during τ_k time is δ close to x_{k+1}.

Pseudo orbits are interesting because we can trace them by orbits under certain conditions. For simplicity, we first give a restricted definition of tracing by orbits and then we explain the extension. We refer to any orbit $(\phi_t(y))_{t\in\mathbb{R}}$ as y-orbit. For $k = 1, \ldots n$, let (x_k, τ_k) be a (δ, a)-pseudo orbit for ϕ_t. Given $\epsilon > 0$, we say that (x_k, τ_k) is ϵ**-traced** by a y-orbit if

$$d(\phi_t(y), \phi_t(x_1)) \leq \epsilon \text{ for every } t \in [0, \tau_1),$$

$$d(\phi_t(y), \phi_{t-\tau_1}(x_2)) \leq \epsilon \text{ for every } t \in [\tau_1, \tau_1 + \tau_2),$$

$$d(\phi_t(y), \phi_{t-\tau_1-\tau_2}(x_3)) \leq \epsilon \text{ for every } t \in [\tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3)$$

and so on for the remaining points. This means that y-orbit only accompanies x_k-orbit during a time interval τ_k. Note that x_k-orbits start at x_k and we only follow the orbits for one time interval τ_k. In contrast, the y-orbit start at y and we follow the orbit for a time interval $\tau_1 + \ldots + \tau_n$. The general definition of tracing allows a reparametrization of the y-orbit. A time reparametrization is an increasing homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$. Thus, we replace the time t of y by a new time $\alpha(t)$. For example the first formula says

$$d(\phi_{\alpha(t)}(y), \phi_t(x_1)) \leq \epsilon \text{ for every } t \in [0, \tau_1).$$

We now state the tracing property of a flow. Given $a > 0$, we say that a continuous flow ϕ_t have the **tracing property** with respect to a if for every $\epsilon > 0$ there exists $\delta > 0$ such that every (δ, a)-pseudo orbit is ϵ-traced by an orbit of ϕ_t.

A generalization of the tracing property deals with a collection of pieces of orbits instead of a collection of points: the specification property. Consider the points $x_1, \ldots, x_n \in X$, the positive numbers $\tau_1, \ldots, \tau_n \in \mathbb{R}_+$ and define for every $k = 1, \ldots, n$

$$J_k = \{\phi_t(x_k) : t \in [0, \tau_k]\} \subset X.$$

We see that each J_k is a piece of orbit starting at x_k for a time τ_k. Thus, we call $J_1, \ldots, J_n$ a sequence of orbit pieces. We use this sequence by over-understanding as implicit information x_k and τ_k. As above, we first give a restricted definition and then sketch the complete one. We say that a continuous flow ϕ_t has the **specification property** if for every $\epsilon > 0$ there exists $T > 0$ such that for every sequence of orbit pieces $J_1, \ldots, J_n$ there exist a sequence $t_1, \ldots, t_n \in \mathbb{R}$ and a periodic point $y \in X$ of period $b > 0$ such that setting $s_k = t_k + \tau_k$ for every k we have

- For every $k = 1, \ldots, n$,

$$d(\phi_t(y), \phi_{t-t_k}(x_k)) \leq \epsilon \quad \text{for } t \in [t_k, s_k].$$

- $|t_{k+1} - s_k| \geq T$ for $k = 1, \ldots, n-1$.

Roughly speaking, the y-orbit shadows all orbit pieces J_k at certain time intervals $[t_k, s_k]$. For every k, the time interval $[t_k, s_k]$ has a duration of τ_k. Thus, we can consider t_k and s_k as the start and end time, of the J_k-shadowing by the y-orbit. The second condition states that the time intervals between consecutive shadowings are always greater than T. The general definition allows small tolerances or errors in the shadowing times of the y-orbit. Thus, we can rephrase a part of the definition: for every sequence of orbit pieces $J_1, \ldots, J_n$, satisfying the above conditions, there exist a sequence of positive numbers $r_1, \ldots, r_n$ such that for every $k = 1, \ldots, n$,

$$d(\phi_{t+r_k}(y), \phi_{t-t_k}(x_k)) \leq \epsilon \quad \text{for } t \in [t_k, s_k].$$

The tolerances r_k satisfy additional technical conditions, we refer to section 6.1 of [13] for details.

We conclude the subsection by defining a characteristic property of the systems studied by Anosov. Let $\phi_t : X \to X$ be a continuous flow acting on a compact metric space. We say that ϕ_t is **expansive** if there exists $\epsilon > 0$ such that if $x, y \in X$ satisfy

$$d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \epsilon \text{ for every } t \in \mathbb{R} \text{ and some reparametrization } \rho,$$

then there exists $\tau \in [-\epsilon, \epsilon]$ with $y = \phi_\tau(x)$. We call ϵ the constant of expansivity of ϕ_t. Roughly speaking, if $x, y \in X$ have ϵ-close orbits then both orbits agree and x is ϵ-close to y in the same orbit. In the context of continuous flows without singularities acting on compact manifolds, the above definition is equivalent to Bowen-Walters expansivity definition (see [39]).

2.2.2
Ergodic properties

We now introduce properties of a more ergodic nature. We restrict ourselves to the case of continuous flows $\phi_t : X \to X$ acting on compact metric spaces. To deal with ergodic properties we add to the metric space X, a natural σ-algebra to X: the Borel σ-algebra $\mathcal{B}(X)$. For further analysis, we set up the following measurable system $(X, \mathcal{B}(X), \phi_t)$. We also consider a probability measure μ defined on $\mathcal{B}(X)$. In this case, μ is called a probability **Borel measure** on X. From now on, all measures considered will be Borel probability measures. Ergodic properties require that μ be invariant by the flow ϕ_t, i.e., for every measurable set $A \subset X$ and every $t \in \mathbb{R}$,

$$\mu(\phi_t(A)) = \mu(A).$$

We denote by $\mathcal{M}(\phi)$ the set of all invariant measures on X. We know that $\mathcal{M}(\phi)$ is a compact set in the weak* topology.

We mention the most relevant properties for our work. We start with the invariant sets. A measurable set $A \subset X$ is called **flow-invariant** if for every $t \in \mathbb{R}$

$$\phi_t(A) \subset A.$$

An invariant measure μ is ergodic with respect to ϕ_t if for every invariant set A, either

$$\mu(A) = 0 \quad \text{or} \quad \mu(A) = 1.$$

There is an equivalent characterization in terms of invariant functions. We say that a function $f : X \to \mathbb{R}$ is **invariant** by the flow ϕ_t if for every $t \in \mathbb{R}$

$$f \circ \phi_t = f.$$

This implies that f is constant on each orbit of the flow ϕ_t. We say that the measure μ is **ergodic** with respect to ϕ_t if every invariant integrable function f is constant μ-almost everywhere. Thus, we can interpret ergodicity as follows: μ is ergodic if every integrable function which is constant on orbits, implies that f is constant μ-almost everywhere. In this case, we say that the system $(X, \mathcal{B}(X), \phi_t, \mu)$ is **ergodic**.

2.2.2.1
Topological and metric entropies

Another important concept of ergodic theory is the so-called Kolmogorov metric entropy. The classical Kolmogorov entropy has an equivalent definition in the context of diffeomorphisms acting on compact Riemannian manifolds. This definition is due to Katok [31] and simplifies the definition by the use of the distances. We give Katok's definition of metric entropy. Let $f : M \to M$ be a C^1 diffeomorphism on a compact Riemannian manifold M and d be the Riemannian distance. There is a convenient concept that underlies the metric entropy and its topological analog: the dynamic balls. Given $\epsilon > 0$, $n \in \mathbb{N}$ and $p \in M$, we define the **dynamic ball** centered at p of radius ϵ and length n by

$$B(p, \epsilon, n) = \{q \in M : d(f_1^k(q), f_1^k(p)) < \epsilon, \text{ for every } k = 0, \dots, n-1\}.$$

Thus, $B(p, \epsilon, n)$ is the set of points q that accompanies p by a n-iterates. We can associate a metric entropy to every invariant set with an invariant measure. So, consider an invariant measurable set $Y \subset M$ and an invariant measure $\mu \in \mathcal{M}(f)$ supported on Y. Given $\epsilon > 0$, $n \in \mathbb{N}$ and $\delta \in (0,1)$, a finite set $K \subset M$ is called a (Y, n, ϵ, δ)**-covering set of** Y if

$$\mu\left(\bigcup_{p \in K} B(p, \epsilon, n)\right) \geq \delta.$$

Denote by $N(Y, n, \epsilon, \delta)$ the smallest cardinality of (n, ϵ, δ)-covering sets of Y. We define the **metric entropy** of μ on Y by

$$h_\mu(Y, f) = \lim_{\delta \to 1} \lim_{\epsilon \to 0} \lim_{n \to \infty} \inf \frac{1}{n} \log N(Y, n, \epsilon, \delta).$$

When $Y = X$ and μ is a f-invariant measure on X, we denote by $h_\mu(f)$ the metric entropy of μ.

Regarding a smooth flow $\phi_t : M \to M$ on a compact Riemannian manifold M. Let $\mu \in \mathcal{M}(\phi)$, i.e., μ is invariant by ϕ_t for every $t \in \mathbb{R}$. Choosing

$t = 1$, we obtain a smooth map $\phi_1 : M \to M$, which is called the **time-1 map of the flow** ϕ. In particular, from above we have $\mu \in \mathcal{M}(\phi_1)$. We define the **metric entropy** of μ with respect to the flow ϕ as the metric entropy of its time-1 map:

$$h_\mu(\phi) = h_\mu(\phi_1).$$

We now deal with the topological counterpart of the metric entropy: the topological entropy. Though the original definition is purely topological, we use Bowen definition of topological entropy [40]. This definition is equivalent to the general one in the context of homeomorphisms (continuous flows) acting on compact metric spaces. We first give the definition for the discrete case. Let $f : X \to X$ be a homeomorphism on a compact metric space. We define the topological entropy for every compact subset $Y \subset X$. Given $\epsilon, n > 0$, denote by $M(Y, n, \epsilon, f)$ the smallest cardinality of a cover of Y by dynamical balls $B(x, \epsilon, n)$ with $x \in X$. We define the **topological entropy** of Y by

$$h(Y, f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log M(Y, n, \epsilon, f).$$

When $Y = X$, we denote by $h(\phi)$ the topological entropy of X.

Let $\phi_t : X \to X$ be a continuous flow on a compact metric space. For every $\epsilon, T > 0$ and $x \in X$, the dynamic ball centered at x of radius ϵ and length T is defined by

$$B(x, \epsilon, T) = \{y \in X : d(\phi_s(x), \phi_s(y)) < \epsilon, \text{ for every } s \in [0, T]\}.$$

Let $Y \subset X$ be a compact subset and $M(Y, T, \epsilon)$ be the smallest cardinality of a cover of Y by dynamical balls $B(x, \epsilon, T)$ with $x \in X$. We define the **topological entropy** of Y with respect to the flow ϕ by

$$h(Y, \phi) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log M(Y, T, \epsilon, \phi).$$

When $Y = X$, we denote by $h(\phi)$ the topological entropy of X.

Both topological and metric entropy are related through the so-called **variational principle** for entropy. We first state this principle for the discrete case.

Theorem 2.1 ([41, 40])**.** *Let $f : X \to X$ be a homeomorphism acting on a compact metric space X. Then*

$$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f).$$

We now consider a continuous flow $\phi_t : X \to X$ on a compact metric space. It follows from the definitions that the topological entropy of the continuous flow and the time-1 map agree:

$$h(\phi) = h(\phi_1).$$

Furthermore, Dinaburg [41] shows that for every measure μ invariant by ϕ_1 there exists a measure ν invariant by the flow ϕ such that

$$h_\mu(\phi_1) < h_\nu(\phi_1).$$

From this, applying Theorem 2.1 to ϕ_1, we see that $h(\phi_1)$ is also the supremum of entropies $h_\mu(\phi_1)$ when μ varies over all measures invariant by the flow ϕ. From these considerations, we get the variational principle for entropy, for the continuous case.

Theorem 2.2. *Let $\phi_t : X \to X$ be a continuous flow acting on a compact metric space X. Then*

$$h(\phi_1) = h(\phi) = \sup_{\mu \in \mathcal{M}(\phi)} h_\mu(\phi) = \sup_{\mu \in \mathcal{M}(\phi)} h_\mu(\phi_1)$$

Therefore, it suffices to consider the time-1 map ϕ_1 when dealing with continuous flows. We only have to keep in mind that the measures considered must be invariant by the flow and not only invariant by time-1 map ϕ_1.

The variational principle is one of the greatest achievements of ergodic theory. This principle also holds for certain subsets of X. Recall that we can define metric entropy for all measures supported on a flow-invariant measurable subset of X. On the other hand, we can define the topological entropy for any compact subset of X. Joining these restrictions, we consider a compact flow-invariant subset $Y \subset X$. Thus, the variational principle for entropy for Y reads:

$$h(Y, \phi_1) = \sup_\mu h_\mu(Y, \phi_1),$$

where μ varies over all probability measures supported on Y and invariant by the flow ϕ.

We now introduce the beginnings of the so-called thermodynamic formalism. The variational principle for a continuous flow $\phi_t : X \to X$, raises the following natural questions:

- Does exist a measure $\mu \in \mathcal{M}(\phi)$ that attains the supremum?, that is, the metric entropy $h_\mu(\phi_1)$ is equal to the topological entropy $h(\phi_1)$.

– If so, is this measure unique?

The answers to these questions define one of the central concepts of the thermodynamic formalism. A measure $\mu \in \mathcal{M}(\phi)$ is called **measure of maximal entropy** if its metric entropy $h_\mu(\phi_1)$ achieves the supremum in the variational principle. If μ is also the only measure that satisfies this definition, we say that μ is the **unique measure of maximal entropy** for the flow ϕ_t. Given a continuous flow ϕ_t, an important problem in the thermodynamic formalism, is to show the existence and uniqueness of the measure of maximal entropy ϕ_t. In Chapter 3, we show this property for geodesic flows of compact surfaces without conjugate points and genus greater than one.

2.2.2.2
Lyapunov exponents

Lyapunov exponents are an important tool for understanding the dynamics of smooth systems. Let $\phi_t : M \to M$ be a smooth flow acting on a Riemannian manifold M and μ be a measure on M, invariant by the flow ϕ. In this context, we can study the flow ϕ_t through its linear approximation: the derivative $d\phi_t$. This is done through the Lyapunov exponents of the system (M, ϕ_t). For every $p \in M$ and every nonzero vector $v \in T_pM$, we define the **Lyapunov exponent** by

$$\chi(p, v) = \limsup_{t\to\infty} \frac{1}{t} \log \|d_p\phi_t(v)\|.$$

These exponents represent the exponential growth rate of $d\phi_t$ at point p in direction v. The exponents give an idea of the asymptotic behavior of orbits near to p in direction v. A natural question arise on these exponents: are Lyapunov exponents true limits for every p and every v?. The answer is given by Oseledets theorem. Among other things, this theorem says that Lyapunov exponents are true limits for μ-almost every p with respect to any invariant measure μ on M. We recall that a subset $A \subset M$ has total measure if $\mu(A) = 1$ for every invariant measure μ on M.

Theorem 2.3 ([42]). *Let $\phi_t : M \to M$ be a smooth flow acting on a compact Riemannian manifold M, and μ be a probability measure on M invariant by ϕ_t. Then, for μ-almost every $p \in M$, there exist*

1. *numbers $\chi_1(p) > \ldots > \chi_{k(p)}(p)$,*

2. and a filtration of vector subspaces

$$T_pM \supset F_1(p) \supset \dots F_{k(p)}(p) \supset \{0\},$$

such that for every $v \in F_i \setminus F_{i+1}, i = 1, \dots, k(p)$*,*

$$\lim_{t\to\infty} \frac{1}{t} \log |d_p\phi_t(v)| = \chi_i(p).$$

Moreover, k*,* χ_i *and* F_i *are measurable functions on* M*, invariant by* ϕ_t *and*

$$d_p\phi_t(F_i(p)) = F_i(\phi_t(p)).$$

Observe that Lyapunov exponents are μ-integrable functions, and also constant on the orbits of the flow. Thus, if the measure μ is ergodic then the Lyapunov exponents are constant μ-almost everywhere. Continuing the above interpretation, for q close to p in direction v,

– If $\chi(p, v) > 0$ then the orbits of p and q eventually diverge exponentially with rate $\chi(p, v) > 0$.

– If $\chi(p, v) < 0$ then the orbits of p and q eventually converge exponentially with rate $\chi(p, v) > 0$.

If $\chi(p, v) = 0$, the exponent does not give precise asymptotic behavior of the orbits of p and q. For every smooth flow $\phi_t : M \to M$, the Lyapunov exponent is always zero, for every $p \in M$ in the flow direction.

There is an important identity relating the Lyapunov exponents and the metric entropy of an invariant measure: Ruelle's inequality.

Theorem 2.4 ([43])**.** *Let* M *be a smooth compact manifold and* $f : M \to M$ *be a* C^1 *map. For every Borel probability measure* μ *on* M*, invariant by* f*,*

$$h_\mu(f) \le \int_M \chi^+(p)d\mu(p) \quad \textit{where} \quad \chi^+(p) = \sum_{r:\chi^r(p)>0} m_p^r \chi^r(p) > 0,$$

and m_p^r *being the multiplicity of* $\chi^r(p)$*.*

Note that $\chi^+(p)$ is the sum of positive Lyapunov exponents of the system at p.

Nonzero Lyapunov exponents are related to the so-called Pesin set [30]. Let us introduce a set closely related to Pesin set.

Definition 2.2.1. Let $\phi_t : M \to M$ be a smooth flow acting on a compact manifold M. The set Λ is the collection of all points $p \in M$ such that

- There exists a subspace $S_p \subset T_pM$ transverse to ϕ_t at p.
- The Lyapunov exponents are nonzero on S_p, i.e., $\chi(p, v) \neq 0$ for every $v \in S_p$.

Oseledets' Theorem ensures that the definition does not depend on the chosen transverse subspace S_p. Note that for each $p \in \Lambda$, for each close point $q \in M$ in any direction different to the flow direction, the orbit of q: either converges to the orbit of p eventually at an exponential rate or diverges from the orbit of p eventually at an exponential rate.

For geodesic flows, several of the above properties are satisfied under certain conditions, but not in general. Roughly speaking, there is a pattern in varying the Riemannian metric g: the more general g is, the weaker are the dynamical and ergodic properties of its geodesic flow. In fact, some concepts and results of the theory were motivated by trying to obtain some of these properties. We mention some of these motivations in the following sections.

2.3 Some results about geodesic flows of compact manifolds without conjugate points

This section introduces some concepts and results used in the study of geodesic flows of certain subfamilies of compact manifolds without conjugate points. These results relate the local and global geometry of the manifold with the dynamical and ergodic properties of the geodesic flow of the compact manifold without conjugate points.

2.3.1 Compact surfaces of constant negative curvature

To motivate some important concepts we give a brief historical account of some transitivity properties of geodesic flows of compact surfaces without conjugate points. In the 1920s, Morse, Hedlund and Hopf studied the transitivity properties of geodesic flows of surfaces of constant negative curvature. In the following we introduce some of these properties in the general context of compact manifolds without conjugate points.

- Topological transitivity: For any A, B open subsets of T_1M, there exists $t \in \mathbb{R}$ such that $\phi_t(A) \cap B \neq \emptyset$.

- Metric transitivity(Ergodicity): For any A, B measurable subsets of T_1M, with positive measure, there exists $t \in \mathbb{R}$ such that $\phi_t(A) \cap B \neq \emptyset$.
- Topological Mixing: For any A, B open subsets of T_1M, there exists $t_0 > 0$ such that $\phi_t(A) \cap B \neq \emptyset$ for $|t| \geq t_0$.
- Permanent metric transitivity: For any A, B measurable subsets of T_1M, with positive measure, there exists $t_0 > 0$ such that $\phi_t(A) \cap B \neq \emptyset$ for $|t| \geq t_0$.
- Mixing with respect to a measure m: For any A, B, C measurable subsets of T_1M, with positive measure,

$$\lim_{t \to \pm\infty} \frac{m(\phi_t(A) \cap B)}{m(\phi_t(A) \cap C)} = \frac{m(A)}{m(B)}$$

For locally compact Hausdorff spaces, we know that topological transitivity is equivalent to the existence of a dense orbit in T_1M. Metric Transitivity is equivalent to ergodicity of the geodesic flow ϕ_t with respect to the Liouville measure m. Clearly, topological mixing implies topological transitivity and permanent metric transitivity implies metric transitivity. Also, metric transitivity implies topological transitivity and the same happens with permanent metric transitivity and topological mixing. Moreover, mixing implies permanent metric transitivity and hence all the properties listed before.

The first family of surfaces where the above transitivity properties were obtained were surfaces of constant negative curvature. To prove these properties, the following conclusion was useful.

Theorem 2.5. *Let M be a compact surface of constant negative curvature. Then the set of periodic orbits is dense in T_1M.*

This result is better known as the density of periodic geodesics. It was proved in full generality for Koebe [2] and Lobell [3] in 1929. Using this property, these authors obtained topological transitivity.

Theorem 2.6. *Let M be a compact surface of constant negative curvature. Then the geodesic flow is topologically transitive.*

For topological mixing, Hedlund introduced horocycles and studied their transitivity properties. Thus, in 1936 Hedlund obtained the following interesting property of the horocycles. We give the precise definition of horocycles later in Subsection 2.3.2.

Theorem 2.7. *Let M be a compact surface of constant negative curvature. Then every stable and unstable horocycle is dense in T_1M.*

This property is also known as the minimality of the horocycle flow. The proof of this theorem depends on the density of periodic geodesics. Using this minimality, Hedlund [4] proved the topological mixing.

Theorem 2.8. *Let M be a compact surface of constant negative curvature. Then the geodesic flow is topologically mixing.*

In the same year, Hopf [5] proved ergodicity of the geodesic flow by different methods. Using Hopf's result and the minimality of the horocycle flow, in 1939 Hedlund [44] showed a stronger transitivity property.

Theorem 2.9. *Let M be a compact surface of constant negative curvature. Then the geodesic flow is mixing with respect to the Liouville measure m.*

2.3.2
Horospheres and Busemann functions

Horocycles can be defined not only for surfaces but, more generally, for any compact manifold without conjugate points. In this general framework, they are called horospheres in some references. The definition can be done through the so-called Busemann functions (see [29]). Let us note that these concepts are naturally defined in the universal covering of the manifold. Then, we can transfer these concepts to the manifold through the covering map. Let M be a compact manifold without conjugate points, and $\tilde{M}$ be its universal covering. Consider $\theta \in T_1\tilde{M}$ and the geodesic γ_θ induced by θ. For every θ, we define the **forward Busemann function**

$$\begin{aligned} b_\theta : \tilde{M} &\to \mathbb{R} \\ p &\mapsto b_\theta(p) = \lim_{t\to\infty} d(p, \gamma_\theta(t)) - t. \end{aligned}$$

The level sets of Busemann functions allows us to define the stable and unstable horospheres associated to θ by

$$H^+(\theta) = b_\theta^{-1}(0) \qquad H^-(\theta) = H^+(-\theta).$$

These horospheres can be lifted to corresponding sets associated to θ, in the unit tangent bundle $T_1\tilde{M}$,

$$\tilde{\mathcal{F}}^s(\theta) = \{(p, -\nabla_p b_\theta) : p \in H^+(\theta)\} \quad \tilde{\mathcal{F}}^u(\theta) = -\tilde{\mathcal{F}}^s(-\theta),$$

where ∇b_θ is the gradient vector field.

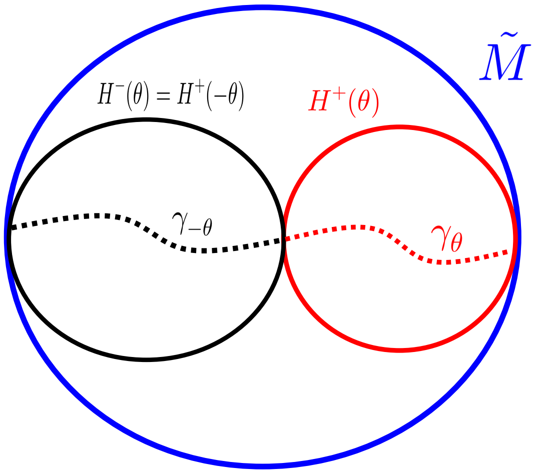

Figure 2.1: Stable and unstable horospheres

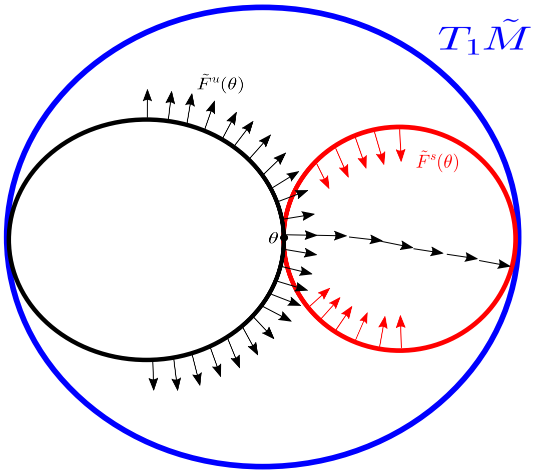

Figure 2.2: Lifts of stable and unstable horospheres

These sets have names related to their foliation property that is fulfilled in certain families of manifolds. We will return to this issue in Subsection 2.3.6. Until then we will only refer to these sets as lift of the horospheres associated to θ. Note that the horospheres are projections of the sets $\tilde{\mathcal{F}}^s(\theta)$ and $\tilde{\mathcal{F}}^u(\theta)$ under the canonical projection map $\tilde{P}$. This relation implies some analogous properties between horospheres and their lifts.

Theorem 2.10 ([29, 30])**.** *Let M be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. Then*

- *All Busemann functions b_θ are $C^{1,L}$ with L-Lipschitz unitary gradient vector fields ∇b_θ where $L > 0$ is an uniform constant depending on curvature bounds.*
- *All horospheres $H^+(\theta), H^-(\theta) \subset \tilde{M}$ and $\tilde{\mathcal{F}}^s(\theta), \tilde{\mathcal{F}}^u(\theta) \subset T_1\tilde{M}$ are embedded submanifolds of dimension $n-1$.*

– *Horospheres are equidistant: for every* $q \in H^+(\gamma_\theta(t))$,

$$d(q, H^+(\gamma_\theta(s))) = |t - s|.$$

The regularity of Busemann functions is related to the existence of tangent spaces to the sets $\tilde{\mathcal{F}}^s(\theta), \tilde{\mathcal{F}}^u(\theta)$. Busemann functions have the same type of regularity as horospheres. From the definitions of horospheres, we see that regularity of $\tilde{\mathcal{F}}^s(\theta), \tilde{\mathcal{F}}^u(\theta)$ decreases by one: they are only topological manifolds. Therefore, in general, we cannot guarantee the existence of tangent spaces to $\tilde{\mathcal{F}}^s(\theta)$ and $\tilde{\mathcal{F}}^u(\theta)$. However, for several special cases, Busemann functions are C^2 hence $\tilde{\mathcal{F}}^s(\theta), \tilde{\mathcal{F}}^u(\theta)$ are C^1, and they have tangent spaces. The regularity is an important problem in the theory of manifolds without conjugate points. We will come back to this problem in Chapter 4.

For every $\theta \in T_1M$, the integral flow σ_t^θ of the gradient vector field $\nabla_p b_\theta$, is called the **Busemann flow** associated to θ. The integral curves of σ_t^θ are called the **Busemann asymptotes** of γ_θ. Busemann asymptotes of γ_θ are geodesics orthogonal to the horospheres associated to θ. In particular, γ_θ is a Busemann asymptote. We highlight that Busemann asymptotes to γ_θ are related to the family of geodesics asymptotic to γ_θ. We will return to this topic in Subsection 2.3.9.

For $\theta \in T_1M$, consider geodesic flow orbits starting at $\tilde{\mathcal{F}}^s(\theta)$. Their projection through the projection $\tilde{P}$ are the Busemann asymptotes associated to θ, starting at $H^+(\theta)$ with initial velocity in $\tilde{\mathcal{F}}^s(\theta)$. Conversely, we can lift these Busemann asymptotes to corresponding geodesic flow orbits. This property provides some analog relations between, on the one hand horospheres and Busemann asymptotes, and on the other hand $\tilde{\mathcal{F}}^s(\theta), \tilde{\mathcal{F}}^u(\theta)$ and geodesic flow orbits.

Proposition 2.3.1 ([29, 30]). *Let* M *be a compact manifold without conjugate points and* $\theta \in T_1\tilde{M}$. *Then,*

1. *Horospheres are invariant by the integral flow* σ_t^θ*: for every* $t \in \mathbb{R}$,

$$\sigma_t^\theta(H^+(\theta)) = H^+(\phi_t(\theta)).$$

2. *The sets* $\tilde{\mathcal{F}}^s(\theta), \tilde{\mathcal{F}}^u(\theta)$ *are invariant by the geodesic flow:*

$$\phi_t(\tilde{\mathcal{F}}^s(\theta)) = \tilde{\mathcal{F}}^s(\phi_t(\theta)).$$

We can see these some ideas of the proposition in the following commuting diagram

$$\begin{array}{ccc} \tilde{\mathcal{F}}^s(\theta) & \xrightarrow{\phi_t} & \tilde{\mathcal{F}}^s(\phi_t(\theta)) \\ \Big\downarrow \tilde{P} & & \Big\downarrow \tilde{P} \\ H^+(\theta) & \xrightarrow{\sigma_t^\theta} & H^+(\phi_t(\theta)). \end{array}$$

For every $\theta \in T_1M$, there are several important collections of sets related to the sets $\tilde{\mathcal{F}}^s(\theta), \tilde{\mathcal{F}}^u(\theta)$. The collections of lifts of stable and unstable horospheres are denoted by

$$\tilde{\mathcal{F}}^s = \{\tilde{\mathcal{F}}^s(\theta)\}_{\theta \in T_1\tilde{M}} \qquad \tilde{\mathcal{F}}^u = \{\tilde{\mathcal{F}}^u(\theta)\}_{\theta \in T_1\tilde{M}}.$$

We define the **center stable and center unstable sets** associated to θ by

$$\tilde{\mathcal{F}}^{cs}(\theta) = \bigcup_{t\in\mathbb{R}} \tilde{\mathcal{F}}^s(\phi_t(\theta)) \quad \tilde{\mathcal{F}}^{cu}(\theta) = \bigcup_{t\in\mathbb{R}} \tilde{\mathcal{F}}^u(\phi_t(\theta)).$$

Note that we describe all ideas in the covering spaces $\tilde{M}$ and $T_1\tilde{M}$. However, we are interested in the properties and results in the manifolds M and T_1M. Thus, we transfer some relevant objects from $T_1\tilde{M}$ to T_1M. To do this, we use the derivative of the covering map

$$d\pi : T\tilde{M} \to TM.$$

Recall that $d\pi$ is a smooth covering map. For every $\theta \in T_1M$ and every lift $\tilde{\theta} \in T_1\tilde{M}$ of θ, we define the sets

$$\mathcal{F}^s(\theta) = d\pi(\tilde{\mathcal{F}}^s(\tilde{\theta})) \quad \mathcal{F}^u(\theta) = d\pi(\tilde{\mathcal{F}}^u(\tilde{\theta})).$$

We define the **center stable and center unstable sets** of θ, by

$$\mathcal{F}^{cs}(\theta) = d\pi(\tilde{\mathcal{F}}^{cs}(\tilde{\theta})) \quad \mathcal{F}^{cu}(\theta) = d\pi(\tilde{\mathcal{F}}^{cu}(\tilde{\theta})).$$

Similarly, we define the collections

$$\mathcal{F}^s = d\pi(\tilde{\mathcal{F}}^s) \qquad \mathcal{F}^u = d\pi(\tilde{\mathcal{F}}^u)$$

Finally, the notation 's' and 'u' of $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$ is inspired by the Anosov setting. Recall that when the manifold M is a compact manifold of negative curvature, the geodesic flow $\phi_t : T_1M \to T_1M$ is Anosov. A significant property of Anosov flows is the existence of stable and unstable invariant submanifolds. For Anosov geodesic flows, these invariant submanifolds locally agree with the sets $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$ respectively. Thus, all associated objects inherited the stable and unstable notation even though this property is only satisfied in particular cases. Although in the general case $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$ do not exhibit hyperbolic behavior, under certain hypotheses they have a weak hyperbolic behavior. We will review this observation in Subsection 2.3.6.

2.3.3
Compact surfaces of variable negative curvature and Morse's shadowing

Surfaces of variable negative curvature were the next case where transitivity properties were studied. One of the first approaches was made by Morse. In 1924, Morse [1] studied a fundamental class of geodesics in the universal covering of any closed surface of genus greater than one, i.e., surfaces that always admit a metric of negative curvature called **hyperbolic metric**. In this case, geodesics on the surface are called **hyperbolic geodesics**.

Theorem 2.11. *Let (M, g) be a compact surface without conjugate points of genus greater than one, $\tilde{M}$ be its universal covering and g' be a hyperbolic metric on M. Then, there exists $R(g, g') > 0$ satisfying: for every g-geodesic $\gamma \subset \tilde{M}$ there exists a hyperbolic geodesic $\gamma' \subset \tilde{M}$ such that the Hausdorff distance between γ and γ' is bounded above by R.*

This theorem is sometimes called **Morse's shadowing**. Thus, in the universal covering both types of geodesics are bi-asymptotic with respect to the hyperbolic distance. One of the implications is the existence of a uniform bound for the distance between bi-asymptotic geodesics.

Theorem 2.12. *Let (M, g) be a compact surface without conjugate points and genus greater than one. Then, there exists a universal constant $Q(M) > 0$ such that the Hausdorff distance between any two bi-asymptotic geodesics is bounded above by Q.*

Furthermore, for every hyperbolic geodesic γ, there exists a geodesic of (M, g) bi-asymptotic to γ. This leads to the following useful notion. Let (M, g) be a closed surface without conjugate points, genus greater than one and universal covering $\tilde{M}$. In Morse's terminology [7], the surface M satisfy unicity if for every hyperbolic geodesic $\beta \subset \tilde{M}$ there exists a unique g-geodesic bi-asymptotic to β. In this case there exists an injective correspondence between hyperbolic geodesics and g-geodesics of closed surfaces of genus greater than one. The unicity is satisfied by closed surfaces of variable negative curvature.

The importance of unicity lies in the fact that many dynamical properties are preserved by this correspondence. In this way, in 1935 Morse [7] proved the topological transitivity for compact surfaces of variable negative curvature.

On the other hand, in 1937 Grant extended Hedlund's horocycles and minimality of the horocycle flow to surfaces of variable negative curvature. Thus, Grant [8] proved the topological mixing for geodesic flows of compact surfaces of variable negative curvature.

2.3.4
Compact surfaces without conjugate points and divergence of geodesic rays

To go beyond the case of surfaces of negative curvature, i.e., where regions with positive or zero curvature may exist, some hypotheses of instability were considered. In 1935 Morse [7] proved for closed orientable surfaces of genus greater than one that the geodesic flow is topologically transitive assuming a condition called uniform instability. In particular, he showed that uniform instability implies unicity. In 1936 Hedlund [6] weakened uniform instability to a condition called ray instability. The uniform and ray instability conditions independently imply that the compact surface has no conjugate points. This means that the surfaces considered by Morse and Hedlund are subfamilies of compact surfaces without conjugate points. In 1942 Morse and Hedlund [45] proved the topological transitivity of geodesic flows of compact surfaces without conjugate points and genus greater than one.

The dynamics of the geodesic flow of a torus (closed surface of genus one) was better understood when Hopf [12] proved the following remarkable theorem.

Theorem 2.13. *Let M be a closed surface without conjugate points. If the genus of M is one then the sectional curvature vanishes everywhere.*

In 1954 Green [11] identified a purely geometric instability property that was sufficient to prove topological transitivity by the method of Morse and Hedlund. Green called this property, **divergence of geodesic rays**. We say that a geodesic $\beta \subset \tilde{M}$ has **base point** $p \in \tilde{M}$ if $\beta(0) = p$.

Definition 2.3.1. Let (M, g) be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. Geodesics rays diverge in $\tilde{M}$ if for every $p \in \tilde{M}$, every $\epsilon, A > 0$, there exists $T(p, \epsilon, A)$ such that for every geodesics $\gamma, \beta \subset \tilde{M}$ with same base point p and $\angle(\gamma'(0), \beta'(0)) \geq \epsilon$, then $d(\gamma(t), \beta(t)) \geq A$ for every $t \geq T(p, \epsilon, A)$. We say that geodesic rays diverge uniformly if $T(p, \epsilon, A)$ does not depend on p.

Although, Morse and Hedlund essentially proved this divergence for a large family of compact surfaces without conjugate points, the result is generally attributed to Green [11].

Theorem 2.14. *Let M be a compact surface without conjugate points, genus greater than one and universal covering $\tilde{M}$. Then, geodesic rays diverge uniformly in $\tilde{M}$.*

In 1973 Eberlein [33] extended the divergence of geodesic rays to higher dimension, i.e., compact manifolds without conjugate points. However, he pointed out that the divergence is not uniform in general. In the previous cases, a lower bound on the curvature is always guaranteed by compactness. In 1978 Goto [46] removed this restriction and proved the divergence of geodesic rays, topological transitivity and density of periodic orbits for non-compact complete manifolds without focal points with no lower bound on the curvature.

2.3.5
Transitivity properties in higher dimension and visibility manifolds

Although in 1940 Hopf [10] proved the ergodicity of the geodesic flow for compact manifolds of constant negative curvature, new and abundant results came with Anosov's theory. For this, we recall the definition of a hyperbolic set in the context of geodesic flows.

Definition 2.3.2. Let M be a compact manifold and ϕ_t be its geodesic flow. A compact ϕ_t-invariant subset $X \subset T_1M$ is called **hyperbolic** if there exist $C, \lambda > 0$ and $d\phi_t$-invariant subspaces $E^s(\theta), E^u(\theta) \subset T_\theta T_1M$ for every $\theta \in X$ such that

1. $T_\theta T_1 M = E^s(\theta) \oplus E^u(\theta) \oplus G(\theta)$.
2. $\|d_\theta\phi_t(\xi^s)\| \leq C \exp^{-\lambda t}$ for every $\xi^s \in E^s(\theta)$ and every $t \geq 0$.
3. $\|d_\theta\phi_t(\xi^u)\| \leq C \exp^{\lambda t}$ for every $\xi^u \in E^u(\theta)$ and every $t \leq 0$.

If $X = T_1M$ the geodesic flow ϕ_t is called **Anosov**. We call $E^s(\theta)$ and $E^u(\theta)$ the stable and unstable dynamical subspaces at θ respectively.

In 1967, for a compact n-manifold M of variable negative curvature, Anosov [21] showed that its geodesic flow is Anosov. Moreover, in this context

Anosov and Sinai [21, 27] proved, among other results, that periodic orbits are dense in T_1M and that the geodesic flow is ergodic.

For compact manifolds that admit regions of positive or zero curvature, there were several generalization directions. In 1971 Klingenberg [47] introduced the so-called **manifolds of hyperbolic type**: compact Riemannian manifolds that admit some Riemannian metric of negative curvature. For this family of manifolds, he [47] showed the density of periodic orbits and the topological transitivity of the geodesic flow. Furthermore, for compact manifolds with Anosov geodesic flow, in 1974 Klingenberg [48] extended the density of periodic orbits and proved the ergodicity of the geodesic flow.

On the other hand, for compact manifolds of nonpositive curvature, in 1972 Eberlein [49] proved the density of periodic orbits, the minimality of horospheres, and the topological mixing assuming a condition called visibility. Eberlein [25] later generalized this condition to manifolds without conjugate points. Visibility condition says that the farther you move a geodesic segment away from a point, the smaller the angle from the point to the ends of the geodesic.

Definition 2.3.3. A complete simply connected Riemannian manifold M is called a **visibility** manifold if it has no conjugate points and for every $p \in M$ and every $\epsilon > 0$ there exists $R(\epsilon, p) > 0$ such that for every $x, y \in M$, the angle at p formed by the geodesic segments $[p, x]$ and $[p, y]$ is less than ϵ if the distance from p to the geodesic segment $[x, y]$ is greater than $R(\epsilon, p)$. In addition M is called a uniform visibility manifold if $R(\epsilon, p)$ does not depend on p.

In this context, in 1973 Eberlein [25] extended some transitive properties previously proved in the case of manifolds of nonpositive curvature. Recall that a foliation is **minimal** if any of its leaves is dense.

Theorem 2.15 ([49, 25])**.** *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then*

1. *The families of sets $\mathcal{F}^s$ and $\mathcal{F}^u$ are minimal.*

2. *The geodesic flow ϕ_t is topologically mixing.*

Actually, the topological mixture was proved for the case of visibility manifolds of non-positive curvature. However, the argument can be extended to visibility manifolds without conjugate points, as we do in Appendix A.

We highlight that Eberlein extended Hedlund's work on surfaces to higher dimensions to obtain the transitivity properties. For this, the topology of the ideal boundary of the universal covering is important because the arguments are made in the compactification of the universal covering. This compactification is homeomorphic to the closed unit n-ball of $\mathbb{R}^n$. We review these concepts below.

We say that geodesics $\gamma, \beta \subset \tilde{M}$ are **asymptotic** if $d(\gamma(t), \beta(t)) \leq C$ for every $t \geq 0$ and some $C > 0$. Note that we do not require that the distance goes to zero as happens in manifolds of negative curvature. Asymptoticity is a equivalence relation on the set of geodesics of $\tilde{M}$. We denote by $\gamma(\infty)$ the equivalence class of any $\gamma \subset T_1\tilde{M}$. The set of equivalence classes is denoted by $\tilde{M}(\infty)$. This set is called the **set of points at infinity** or the **ideal boundary** of $\tilde{M}$. The visibility condition implies the existence and uniqueness of a geodesic asymptotic to any other given geodesic.

Proposition 2.3.2 ([25])**.** *Let M be a uniform visibility manifold. For every geodesic $\beta \subset \tilde{M}$ and every $p \in \tilde{M}$ there exists a unique geodesic γ asymptotic to β passing through p.*

In this case, we say that γ **joins** p to $\beta(\infty)$ because we can choose $\gamma(0) = p$ and $\gamma(\infty) = \beta(\infty)$. Moreover, we can find a geodesic joining two points in $\tilde{M}(\infty)$. Given a geodesic $\gamma \subset \tilde{M}$, we denote by $\gamma(-\infty)$ the equivalence class of the geodesic $t \mapsto \gamma(-t)$.

Proposition 2.3.3 ([25])**.** *Let M be a uniform visibility manifold. For every distinct $x, y \in \tilde{M}(\infty)$ there exists a geodesic γ such that $\gamma(-\infty) = x$ and $\gamma(\infty) = y$.*

In this setting, we say that γ **joins** x to y. Given distinct $p, q \in \tilde{M} \cup \tilde{M}(\infty)$, we denote by $V(p, q)$ the initial vector of the unique geodesic joining p to q. We now define the open sets of the cone topology on $\tilde{M} \cup \tilde{M}(\infty)$. Let $R > 0$, $\delta \in (0, \pi)$, $p \in \tilde{p}$ and $v \in T_p\tilde{M}$, the truncated cone $T_{p,v,R,\epsilon}$ with vertex p, axis v, angle δ and radius R is the set

$$T_{p,v,R,\epsilon} = \{q \in \tilde{M} \cup \tilde{M}(\infty) : \angle_p(v, V(p, q)) \leq \delta, d(p, q) \geq R\},$$

with the convention that $d(p, q) = \infty$ whenever $q \in \tilde{M}(\infty)$. This truncated cones are the neighborhoods of the points in the ideal boundary of the universal covering.

Proposition 2.3.4 ([25])**.** *Let M be a uniform visibility manifold. Then*

1. *The open sets of $\tilde{M}$ and truncated cones form a basis for a topology on $\tilde{M} \cup \tilde{M}(\infty)$, called* ***cone topology****. The space $\tilde{M} \cup \tilde{M}(\infty)$ is compact endowed with this topology.*

2. *For every $x \in \tilde{M}(\infty)$, the set of truncated cones with vertex $p \in \tilde{M}$ and containing x, is a local basis for the cone topology at x.*

3. *$\tilde{M} \cup \tilde{M}(\infty)$ with the cone topology is homeomorphic to the closed unit n-ball of $\mathbb{R}^n$.*

2.3.6
More properties of horospheres

In Section 2.3.1, we introduced horospheres and their lifts $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$, together with their basic properties. The present subsection deals with additional important properties of horospheres and $\tilde{\mathcal{F}}^s$, $\tilde{\mathcal{F}}^u$: foliation structure, intersections between $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ and their weak hyperbolic behavior. For the first part, we assume a compact surface without conjugate points and genus greater than one.

Let us start with the foliation structure of the sets $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$. For compact manifolds of negative curvature, Anosov [21] showed that $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ give rise to continuous foliations invariant by the geodesic flow. This is because $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ agree with the invariant stable and unstable submanifolds given the hyperbolic structure. Green's divergence of geodesic rays and Morse's shadowing allows to extend this property to compact surfaces without conjugate points and genus greater than one.

Theorem 2.16. *Let M be a compact surface without conjugate points and genus greater than one. Then, the collection of sets*

$$(\tilde{\mathcal{F}}^s(\theta))_{\theta \in T_1\tilde{M}} \quad \text{and} \quad (\tilde{\mathcal{F}}^u(\theta))_{\theta \in T_1\tilde{M}}$$

are continuous foliations invariant by the geodesic flow.

Moreover, this theorem can be generalized to higher dimension through manifolds satisfying the visibility condition.

Theorem 2.17 ([25])**.** *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then, the collection of sets*

$$(\tilde{\mathcal{F}}^s(\theta))_{\theta \in T_1\tilde{M}} \quad \text{and} \quad (\tilde{\mathcal{F}}^u(\theta))_{\theta \in T_1\tilde{M}}$$

are continuous foliations invariant by the geodesic flow.

We highlight that for general compact manifolds without conjugate points, it is not known whether horospheres provide invariant foliations.

We use this foliation structure to name the relevant sets involved in the above theorem. Recall that every compact surface without conjugate points and genus greater than one, has a visibility universal covering. Therefore, in the visibility context, we call $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ the stable and unstable horospherical foliations of $T_1\tilde{M}$. Also, for every $\theta \in T_1\tilde{M}$, $\tilde{\mathcal{F}}^s(\theta)$ and $\tilde{\mathcal{F}}^u(\theta)$ are called the **stable and unstable horospherical leaves** associated to θ. Similarly, we call $\mathcal{F}^s$ and $\mathcal{F}^u$ the stable and unstable horospherical foliations of T_1M and $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$ their corresponding leaves for every $\theta \in T_1M$. So, from now on we use these names to refer to these sets.

Let us clarify about the topology in which the continuity of the foliations is fulfilled. To do this, we first recall the Hausdorff distance. For every $A, B \subset T_1\tilde{M}$,

$$d_H(A, B) = \max\left(\sup_{a\in A} d_s(a, B), \sup_{b\in B} d_s(A, b)\right).$$

In addition we consider the horospherical foliations as the maps

$$\theta \in T_1\tilde{M} \mapsto \tilde{\mathcal{F}}^s(\theta) \quad \text{and} \quad \theta \in T_1\tilde{M} \mapsto \tilde{\mathcal{F}}^u(\theta).$$

The stable horospherical foliation is continuous with respect to the open-compact topology, if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any compact set $K \subset T_1\tilde{M}$,

$$d_H(K \cap \tilde{\mathcal{F}}^s(\theta), K \cap \tilde{\mathcal{F}}^s(\eta)) \leq \epsilon \quad \text{whenever} \quad d_s(\theta, \eta) \leq \delta.$$

An analogous statement holds for the unstable horospherical foliation.

We now deal with the question of intersections of stable and unstable horospherical leaves in $T_1\tilde{M}$. We first address the intersection of the stable and unstable horospherical leaves associated to the same point. To do this, consider a compact manifold M without conjugate points and universal covering $\tilde{M}$. We denote the intersections as follows: for every $\theta \in \tilde{M}$,

$$I(\theta) = H_+(\theta) \cap H_-(\theta) \qquad \mathcal{I}(\theta) = \tilde{\mathcal{F}}^s(\theta) \cap \tilde{\mathcal{F}}^u(\theta).$$

We call $\mathcal{I}(\theta)$ a class of $\theta \in T_1\tilde{M}$ or simply a class. Note that $\tilde{P}(\tilde{\mathcal{I}}) = I(\theta)$ hence $\mathcal{I}(\theta)$ is a lift of $I(\theta) \subset \tilde{M}$ to $T_1\tilde{M}$ where $\tilde{P} : T\tilde{M} \to \tilde{M}$ is the canonical projection.

The topological structure of the intersections is simple if we restrict ourselves to the case of surfaces. Recall that if M is a surface then horospheres and horospherical leaves are just curves in the surface $\tilde{M}$ and the 3-manifold $T_1\tilde{M}$ respectively. For surfaces, Morse's Theorem 2.12 about shadowing and Green's Theorem 2.14 about divergence of geodesic rays provide the following behavior of the intersections.

Proposition 2.3.5. *Let M be a compact surface without conjugate points and genus greater than one. For every $\theta \in T_1\tilde{M}$,*

1. *$I(\theta)$ and $\mathcal{I}(\theta)$ are compact connected curves of $\tilde{M}$ and $T_1\tilde{M}$ respectively.*
2. *Diam$(I(\theta)) \leq Q$ and Diam$(\mathcal{I}(\theta)) \leq \tilde{Q}$ for some universal constants $Q, \tilde{Q} > 0$ depending only on M.*

From above we see that given $\theta \in T_1\tilde{M}$, $\mathcal{I}(\theta)$ is either a single point or a curve with a fixed maximum length. Thus,

– If $\mathcal{I}(\theta)$ is a single point, we say that θ is an **expansive point** and $\mathcal{I}(\theta)$ is a **trivial class**.
– If $\mathcal{I}(\theta)$ is a curve of nonzero length, we say that θ is a **non expansive point** and $\mathcal{I}(\theta)$ is a **non trivial class**.

The set of expansive points is called **the expansive set** and is denoted by

$$\mathcal{R}_0 = \{\theta \in T_1 M : \mathcal{F}^s(\theta) \cap \mathcal{F}^u(\theta) = \{\theta\}\}.$$

The complement of $\mathcal{R}_0$ is called the **non expansive set**. In addition, observe that any non trivial class $\mathcal{I}(\theta)$ has two boundary points.

Non expansive points induce strips of bi-asymptotic orbits and geodesics. For any non expansive point $\theta \in T_1\tilde{M}$, the action of the geodetic flow produces the non-trivial strip of orbits

$$\bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{I}(\theta)) = \bigcup_{t\in\mathbb{R}} \mathcal{I}(\phi_t\theta).$$

The last equality follows from the invariance of $\tilde{\mathcal{F}}^s(\theta)$ and $\tilde{\mathcal{F}}^u(\theta)$ by ϕ_t. If θ was an expansive point then $\bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{I}(\theta)) = \bigcup_{t\in\mathbb{R}} \phi_t(\theta)$ would be a trivial strip, i.e., a single orbit. We project this non-trivial strip of orbits onto a non-trivial strip of bi-asymptotic geodesics

$$\tilde{P}(\bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{I}(\theta))) = \bigcup_{t\in\mathbb{R}} \sigma_t(I(\theta)) = \bigcup_{t\in\mathbb{R}} I(\sigma_t\theta) = \bigcup_{\eta\in\mathcal{I}(\theta)} \bigcup_{t\in\mathbb{R}} \gamma_\eta(t).$$

If θ was an expansive point then $\tilde{P}(\bigcup_{t\in\mathbb{R}}\phi_t(\mathcal{I}(\theta))) = \bigcup_{t\in\mathbb{R}}\gamma_\theta(t)$ would be a trivial strip, i.e., a single geodesic.

Strips of bi-asymptotic geodesics and orbits have special properties in certain cases. Consider a non-expansive point $\theta \in T_1\tilde{M}$. Eschenburg [29] showed that the non-trivial strip of bi-asymptotic geodesics

$$\bigcup_{\eta\in\mathcal{I}(\theta)}\bigcup_{t\in\mathbb{R}}\gamma_\theta(t)$$

K is a flat strip if M has no focal points. A flat strip is an isometric and totally geodesic embedded copy of an infinite strip of nonzero length of the Euclidean plane. This result is true in higher dimension and it is called the **flat strip theorem**.

Theorem 2.18 ([29]). *Let M be a simply connected manifold with no focal points. Then any two bi-asymptotic geodesics bound a flat strip. Furthermore, for every $\theta \in T_1M$, the set $I(\theta)$ is a convex set.*

If we lift this flat strip to a strip $S \subset T_1\tilde{M}$ of bi-asymptotic orbits, then S is just the place where the geodesic flow has no hyperbolic behavior. Though the flat strip theorem is false for general compact manifolds without conjugate points [50], the non-hyperbolic behavior of the geodesic flow happens in these strips of bi-asymptotic orbits. We will return to this intersection question in higher dimension in Subsection 2.3.9.

We now consider the intersection of stable and unstable horospherical leaves associated to different points. We first recall that in higher dimensions, visibility condition provides a special property for geodesics in the universal covering. Proposition 2.3.3 says that for any two different geodesics $\gamma, \beta \subset \tilde{M}$ satisfying the hypothesis, there exists a (non necessarily unique) geodesic with the same past as γ and the same future as β. We lift this property to $T_1\tilde{M}$, i.e., from geodesics to orbits of of the geodesic flow ϕ_t. Thus, we get a bit weaker type of intersections between the stable horospherical leaves and central unstable sets.

Proposition 2.3.6 ([25]). *Let M be a compact manifold without conjugate and visibility universal covering $\tilde{M}$. Then for every $\theta, \xi \in T_1\tilde{M}$ such that $\theta \notin \tilde{\mathcal{F}}^{cu}(\xi)$ and $\xi \notin \tilde{\mathcal{F}}^s(\theta)$, there exists $\eta_1, \eta_2 \in T_1\tilde{M}$ satisfying*

$$\tilde{\mathcal{F}}^s(\theta) \cap \tilde{\mathcal{F}}^{cu}(\xi) = \mathcal{I}(\eta_1) \qquad \tilde{\mathcal{F}}^s(\xi) \cap \tilde{\mathcal{F}}^{cu}(\theta) = \mathcal{I}(\eta_2).$$

It is straightforward to transform these intersections into intersections of unstable horospherical leaves and central stable sets. More precisely, there exist $t_1, t_2 \in \mathbb{R}$ such that

$$\tilde{\mathcal{F}}^{cs}(\theta) \cap \tilde{\mathcal{F}}^u(\xi) = \mathcal{I}(\phi_{t_1}(\eta_1)) \qquad \tilde{\mathcal{F}}^{cs}(\xi) \cap \tilde{\mathcal{F}}^u(\theta) = \mathcal{I}(\phi_{t_2}(\eta_2)).$$

From now on, we will refer to all the above intersections as heteroclinic connections. Although the heteroclinic connections looks very much like a local product, this is not generally true. Since the intersections are classes $\mathcal{I}(\eta)$. For a non-expansive $\eta \in T_1\tilde{M}$, $\mathcal{I}(\eta)$ is a non-trivial class hence the intersection is not unique. To obtain a true local product, in Chapter 3 we collapse non-trivial classes onto single points.

We finish the section by looking at the weak hyperbolic behavior of the horospherical leaves. Recall that for a compact manifold of negative curvature, its geodesic flow is uniformly hyperbolic. This provides invariant submanifolds with hyperbolic behavior. Moreover, these invariant submanifolds agree with the horospherical leaves. However, for a general compact manifold without conjugate points, its geodesic flow may not be uniformly hyperbolic because there may be regions with no hyperbolic behavior. Despite this, the horospherical leaves still have some weak hyperbolic properties: for points starting in the same stable horospherical leaf, the distance between their future orbits is bounded.

Proposition 2.3.7 ([25]). *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then, there exists $A, B > 0$ such that for every $\theta \in T_1\tilde{M}$ and every $\eta \in \tilde{\mathcal{F}}^s(\theta)$,*

$$d_s(\phi_t(\theta), \phi_t(\eta)) \leq A d_s(\theta, \eta) + B, \text{ for every } t \geq 0.$$

A similar statement holds for $\eta \in \tilde{\mathcal{F}}^u(\theta)$ considering $t \leq 0$.

2.3.7
Jacobi fields

Jacobi fields are one of the most important local objects for the study of geodesic flows. Their importance lies in the fact that these fields are directly related to the derivative of the geodesic flow. In this section we introduce Jacobi fields and their basic properties. The references of the section are [33] and [36].

Let (M, g) be a compact Riemannian n-manifold without conjugate points and ∇ be its Levi-Civita connection. Recall that ∇ provides a covariant derivative of vector fields along curves. Let $\beta \subset M$ be a curve and X be a vector field along β. We denote by

$$X' = \nabla_{\dot{\beta}} X$$

the covariant derivative of X along β. Given $\theta \in T_1M$, let γ_θ be the geodesic induced by θ. A vector field J along γ_θ is called a **Jacobi field** if J satisfies the Jacobi equation

$$J''(t) + R(\dot{\gamma}_\theta(t), J(t))\dot{\gamma}_\theta(t) = 0,$$

where R is the curvature tensor induced by g. The Jacobi fields have a close relationship with the tangent bundle of TM and T_1M. From Section 2.1, recall the decomposition of TM into horizontal and vertical subspaces. Thus, let $\theta = (p, v) \in TM$ and γ_θ be the geodesic induced by θ. For every $\xi \in T_\theta TM$, its decomposition into horizontal and vertical components reads

$$\xi = (\xi_h, \xi_v) \in T_pM \times T_pM.$$

Using this decomposition, we denote by J_ξ the unique Jacobi field along γ_θ with initial conditions

$$J_\xi(0) = \xi_h \quad \text{and} \quad J'_\xi(0) = \xi_v.$$

From linearity of the Jacobi equation, the assignation

$$\xi \mapsto J_\xi$$

is linear. Conversely, for every Jacobi field J along γ_θ, we assign linearly to J, the unique vector $\xi \in T_\theta TM$ such that

$$\xi = (J(0), J'(0)).$$

The above maps are inverses of each other. Moreover, the map $\xi \mapsto J_\xi$ is a linear isomorphism between $T_\theta TM$ and $\mathcal{J}_\theta$.

The linear isomorphism between $T_\theta TM$ and $\mathcal{J}_\theta$ can be restricted to T_1M. The image of T_1M under this isomorphism is the family of orthogonal Jacobi fields along γ_θ. We first observe that for every $\theta \in T_1M$, $\dot{\gamma}_\theta$ is a trivial solution of the Jacobi equation. Since we are not interested in this solution, we eliminate all Jacobi fields with parallel component along $\dot{\gamma}_\theta$. A Jacobi field J on γ_θ is called **orthogonal** if $J(t)$ is orthogonal to $\dot{\gamma}_\theta(t)$ for every $t \in \mathbb{R}$. A direct calculation shows that a Jacobi field J is orthogonal if and only if

$$\langle J(0), \theta \rangle = \langle J'(0), \theta \rangle = 0.$$

This means that both $J(0)$ and $J'(0)$ are orthogonal to $\dot{\gamma}_\theta(0) = \theta$. Thus the set of orthogonal Jacobi fields $\mathcal{J}_\theta^\perp$ is a vector subspace of dimension $2n-2$. This characterization is related with a vector subspace of $T_\theta TM$. If $\xi \in T_\theta TM$, then $\xi \in S(\theta) \subset T_\theta T_1 M$ if and only if

$$\langle J_\xi(0), \theta \rangle = \langle J'_\xi(0), \theta \rangle = 0,$$

where $S(\theta) \subset T_\theta T_1 M$ is the orthogonal complement of the vector subspace $G(\theta)$. We put all together in the following proposition.

Proposition 2.3.8 ([36]). *Let M be a compact manifold without conjugate points. Then*

1. *For every $\theta \in T_\theta TM$, the map $\xi \mapsto J_\xi$ is a linear isomorphism between the $2n$-dimensional vector spaces $T_\theta TM$ and $\mathcal{J}_\theta$.*
2. *For every $\theta \in T_\theta T_1 M$, the map $\xi \mapsto J_\xi$ is a linear isomorphism between the $(2n-2)$-dimensional vector spaces $S(\theta) \subset T_\theta TM$ and $\mathcal{J}_\theta^\perp$.*

From now on, we only consider orthogonal Jacobi fields.

The correspondence given by the above linear isomorphism can be extended to Jacobi fields evolving in time, with the help of the geodesic flow. Let $\theta \in T_1 M$ and $\xi \in S(\theta) \subset T_\theta T_1 M$. Note that for every $t \in \mathbb{R}$, the Jacobi field J_ξ defines a unique tangent vector

$$\xi(t) = (J_\xi(t), J'_\xi(t)) \in S(\phi_t(\theta)) \subset T_{\phi_t(\theta)} T_1 M.$$

This correspondence uses the above linear isomorphisms at each point $\phi_t(\theta)$ for every $t \in \mathbb{R}$. Thus, while Jacobi equation evolves $J_\xi(t)$ and $J'_\xi(t)$ over the time, the following questions arise:

- is there any mechanism that evolves $\xi(t)$ over the time through the tangent spaces $T_{\phi_t(\theta)} T_1 M$?
- In addition, is this mechanism compatible with the linear isomorphisms at each point $\phi_t(\theta)$?

The following proposition answers the questions: the counterpart mechanism to the Jacobi equation is the geodesic flow derivative.

Proposition 2.3.9 ([36]). *Let M be a compact manifold without conjugate points and $\theta \in T_1M$. For every $\xi \in S(\theta) \subset T_\theta T_1 M$ and every $t \in \mathbb{R}$,*

$$d_\theta \phi_t(\xi) = (J_\xi(t), J'_\xi(t)).$$

We see the correspondence in the diagram

$$\begin{array}{ccc} T_\theta T_1 \tilde{M} & \xrightarrow{d_\theta \phi_t} & T_{\phi_t(\theta)} T_1 \tilde{M} \\ \downarrow & & \downarrow \\ \mathcal{J}_\theta^\perp & \xrightarrow{\text{Jacobi equation}} & \mathcal{J}_{\phi_t(\theta)}^\perp \end{array} \qquad \begin{array}{ccc} \xi = (J_\xi(0), J'_\xi(0)) & \xmapsto{d_\theta \phi_t} & d_\theta \phi_t(\xi) = (J_\xi(t), J'_\xi(t)) \\ \downarrow & & \downarrow \\ J_\xi & \xmapsto{\text{shift by a time t}} & J_\xi^t \end{array},$$

where J_ξ^t is the Jacobi field J_ξ shifted by a time t: for every $s \in \mathbb{R}$,

$$J_\xi^t(s) = J_\xi(s+t).$$

Thus J_ξ starts at θ while J_ξ^t starts at $\phi_t(\theta)$.

An important consequence of the proposition arises from the use of Sasaki metric:

$$\|d_\theta \phi_t(\xi)\|_{s,v}^2 = \|J_\xi(t)\|_p^2 + \|J'_\xi(t)\|_p^2 \tag{2.2}$$

While Jacobi equation states that a Jacobi field is related to the infinitesimal variation of geodesics, this equation says that a Jacobi field and its derivative is related to the infinitesimal variation of orbits of the geodesic flow. This result allows relating the geometry of the manifold to the dynamical properties of the geodesic flow. One approach to the dynamics of the geodesic flow analyzes the asymptotic behavior of the derivative of the geodesic flow. Equation (2.2) says that this analysis can be done through Jacobi Fields. The asymptotic behavior of the derivative is related to dynamical properties such as hyperbolicity and Lyapunov exponents.

To end the section, we see that Jacobi fields provide an alternative characterization of manifolds without focal points and manifolds without conjugate points. We note that these manifolds admit regions of curvature positive. Let M be a compact manifold, we say that

– M **has no focal points** if for every Jacobi field J with $J(0) = 0$, $|J(t)|$ is increasing for $t \geq 0$.

– M **has no conjugate points** if for every Jacobi field J with $J(0) = 0$, $|J(t)|$ does not vanish for $t \neq 0$.

We see that the definitions are based on the asymptotic behavior of certain

Jacobi fields. In Subsection 2.3.10, we will review the asymptotic behavior of certain Jacobi fields, for different types of manifolds.

2.3.8
Green bundles

In this section we introduce the Green bundles, their properties and some relationships with other concepts.

Green bundles were first used by Hopf [12] in 1948 to show that a torus without conjugate points is flat. To extend this result to higher dimensions, in 1958 Green [32] defined the Green bundles in full generality. For compact manifolds without conjugate points, in 1973 Eberlein [33] showed that a geodesic flow is Anosov if and only if the Green bundles are transverse everywhere. In 1977 Eberlein, Heintze-Imhof and Eschenburg used Green bundles to prove some regularity of horospheres in several classes of manifolds. From these results, a new family of manifolds defined through Green bundles emerged: the manifolds of bounded asymptote. In 1982, Freire and Mañé [51] related the unstable Green bundle to the metric entropy of Liouville measure for compact manifolds without conjugate points. Based on this work, in 1986 Knieper [15] connected the positive metric entropy of the Liouville measure to the linear independence of Green bundles. In several papers, Ruggiero and his collaborators used the continuity of Green bundles to obtain stronger conclusions on the dynamics and geometry of geodesic flows. All these works make clear the importance of Green bundles in the study of the geodesic flows.

We first define the stable and unstable Jacobi fields. These objects arise as asymptotic limit fields of Jacobi fields with same initial condition. Let M be a compact n-manifold without conjugate points, $\theta = (p, v) \in T_1M$ and γ_θ be the geodesic induced by θ. For every $w \in v^\perp \subset T_pM$ and every $T \in \mathbb{R}$, consider the Jacobi field $J_T \in \mathcal{J}_\theta$ with boundary conditions

$$J_T(0) = w \quad \text{and} \quad J_T(T) = 0.$$

Hopf [12] and Green [32] showed that the limits when $T \to \pm\infty$ always exist. Thus, we call

$$J_s = \lim_{T\to\infty} J_T \quad \text{and} \quad J_u = \lim_{T\to-\infty} J_T,$$

the stable and unstable Jacobi fields with initial condition w, because $J_s(0) = J_u(0) = w$. These Jacobi fields never vanish. Moreover, every stable and unstable Jacobi field is always orthogonal to γ_θ.

We now introduce the Green bundles. Denote by $\mathcal{J}^s_\theta$ and $\mathcal{J}^u_\theta$ the vector subspaces of stable and unstable Jacobi fields along γ_θ. Since stable and unstable Jacobi fields are defined for every vector orthogonal to v, $\mathcal{J}^s_\theta$ and $\mathcal{J}^u_\theta$ are $(n-1)$-dimensional vector subspaces of $\mathcal{J}_\theta$. Proposition 2.3.8 allows us to lift $\mathcal{J}^s_\theta$ and $\mathcal{J}^u_\theta$ to $(n-1)$-dimensional vector subspaces $G^s(\theta)$ and $G^u(\theta)$ of $T_\theta T_1M$. The collection of these vector subspaces give rise to the stable and unstable Green bundles G^s and G^u:

$$\theta \mapsto G^s(\theta) \quad \text{and} \quad \theta \mapsto G^u(\theta).$$

We call $G^s(\theta)$ and $G^u(\theta)$ the stable and unstable Green subspaces at θ.

These notations are inspired by the Anosov geodesic flow of compact manifolds of negative curvature. More precisely, consider a hyperbolic set $X \subset T_1M$. For every $\theta \in X$, $G^s(\theta)$ and $G^u(\theta)$ agree with the stable and unstable dynamical subspaces $E^s(\theta)$ and $E^u(\theta)$ respectively. Furthermore, $G^s(\theta)$ and $G^u(\theta)$ are tangent to the stable and unstable horospherical leaves $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$. However, in general Green bundles may not be tangent to horospherical leaves as Ballmann-Brin-Burns showed [52].

We now discuss some properties of Green bundles. In 1972, Eberlein [33] formalized the basic properties of Green bundles.

Proposition 2.3.10. *Let M be a compact n-manifold without conjugate points. Then, the Green bundles are $(n-1)$-dimensional measurable bundles on TTM, invariant by the derivative of the geodesic flow.*

We say that **Green bundles are continuous** if $G^s(\theta)$ and $G^u(\theta)$ depend continuously on $\theta \in T_1M$ in the open-compact topology of T_1M. This property holds for manifolds of non-positive curvature and manifolds without focal points. However, Ballman-Brin-Eberlein [52] constructed examples of high genus compact surfaces without conjugate points whose Green bundles are not continuous.

Eberlein [33] gave a useful criteria to identify stable and unstable Jacobi fields. This criteria says that every nonzero Jacobi field that is bounded in the future(past) is actually a stable(unstable) Jacobi field.

Proposition 2.3.11. *Let M be a compact manifold without conjugate points. Then, every Jacobi field J with $\|J(t)\| \leq C$ for every $t \geq 0$ ($t \leq 0$) and some $C > 0$, is a stable (unstable) Jacobi field.*

In fact, for a large family of manifolds, Eberlein [33] characterized all stable and unstable Jacobi fields by this criteria. This family includes manifolds of non-positive curvature and manifolds without focal points.

Green bundles are related to the derivative of geodesic flow. Let $\theta \in T_1M$ and $\xi \in S(\theta) \subset T_\theta T_1 M$. From Equation (2.2) of Subsection 2.3.7, it follows that for every $t \in \mathbb{R}$,

$$\|J_\xi(t)\| \leq \|d_\theta \phi_t(\xi)\|.$$

Eberlein showed that if we restrict ourselves to vectors in Green bundles, the induced stable and unstable Jacobi fields are comparable to the derivative of the geodesic flow.

Proposition 2.3.12 ([33]). *Let M be a compact manifold without conjugate points. Then, there exists $K > 0$ such that for every vector $\xi \in G^s \cup G^u$ and every $t \in \mathbb{R}$,*

$$\|J_\xi(t)\| \leq \|d\phi_t(\xi)\| \leq K\|J_\xi(t)\|.$$

Thus, we can work with stable and unstable Jacobi fields instead of the derivative of the geodesic flow. Knowing the behavior of these fields, one could know the asymptotic behavior of the derivative of the geodesic flow. Thus, it is natural to wonder about the asymptotic behavior of stable and unstable Jacobi fields. These have different behavior depending on the manifolds considered. We will see this in the next section.

We now mention some relations of Green bundles to important objects defined above. In connection with the expansive set $\mathcal{R}_0$ of Subsection 2.3.6, we define

$$\mathcal{R}_1 = \{\theta \in T_1M : G^s(\theta) \cap G^u(\theta) = \{0\}\},$$

that is, the set where Green bundles are linearly independent or transverse. From the literature, it is known that

$$\mathcal{R}_1 \subset \mathcal{R}_0$$

in special cases: manifolds of negative curvature [27], manifolds of non-positive curvature [28] and manifolds without focal points [29]. This is because, in these cases, Green bundles are tangent to horospherical leaves everywhere. We explain this property below.

Let $X \subset T_1M$ be a set where the horospherical foliations have tangent spaces. We say that **Green bundles are tangent to horospherical foliations** on X if for every $\eta \in X$, $G^s(\eta)$ and $G^u(\eta)$ are tangent to $\mathcal{F}^s(\eta)$ and $\mathcal{F}^u(\eta)$ respectively.

If Green bundles are tangent to horospherical foliations on $\mathcal{R}_1$, then $\mathcal{R}_1 \subset \mathcal{R}_0$. Indeed, suppose that for every $\theta \in \mathcal{R}_1$, $G^s(\theta)$ and $G^u(\theta)$ are tangent to $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$. Since $G^s(\theta)$ and $G^u(\theta)$ are transverse at θ, so are $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$. Thus, $\mathcal{F}^s(\theta) \cap \mathcal{F}^u(\theta) = \{\theta\}$. By definition given in Subsection 2.3.6, $\theta \in \mathcal{R}_0$. We will return to this topic in Chapter 4.

2.3.9
Uniform global properties of the universal covering of compact manifolds without conjugate points

This subsection introduces some uniform global properties of the universal covering, i.e., the global geometry of compact manifolds without conjugate points. We also see some consequences of these properties.

The global geometry of compact manifolds without conjugate points deals with the geometric properties in the universal covering of the manifold. Poincaré and Hadamard were among the first to study the global properties of geodesics in the universal covering. Some examples of geometric properties in the universal covering are:

- Morse's shadowing.
- Divergence of geodesic rays.
- Horospheres and horospherical leaves.
- Visibility condition.
- The cone topology for the boundary of the universal covering.
- Gromov hyperbolic groups.
- Quasi convexity.

From previous subsections, we know that some of these properties allows to obtain dynamical properties of the geodesic flow.

Before mentioning some consequences of the divergence of geodesic rays, we will see that divergence is implied by the continuity of Green bundles. We first define radial Jacobi fields. Let M be a compact manifold without conjugate points and $\gamma \subset M$ be a geodesic. A Jacobi field J along γ is called **radial** if J vanishes at a single point. We say that **radial Jacobi fields diverge** if for every $\theta \in T_1\tilde{M}$, every $A > 0$, and every radial Jacobi field J on γ_θ there exists $T(\theta, A) > 0$ such that $\|J(t)\| \geq A$ for every $t \geq T$. If $T(\theta, A)$ does not depend on θ, we say that radial Jacobi fields diverge uniformly. We next restate Theorem 3.1 of Ruggiero's work [17].

Theorem 2.19. *Let M be a compact manifold without conjugate points whose Green bundles are continuous. Then, radial Jacobi fields diverge uniformly.*

Eberlein [33] pointed out that this property yields the divergence of geodesic rays. Thus, uniform divergence of radial Jacobi fields implies uniform divergence of geodesic rays. This is proposition 3.6 of [17].

Proposition 2.3.13. *Let M be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. If Green bundles are continuous then geodesics rays diverge uniformly in $\tilde{M}$.*

Since Green bundles are continuous for manifolds of non-positive curvature and manifolds without focal points, geodesic rays diverge uniformly in these manifolds. Recall that Green [11] proved the uniform divergence of geodesic rays for compact surfaces without conjugate points and genus greater than one. Eberlein [33] showed the divergence of geodesic rays for n-dimensional visibility manifolds, although this divergence is not uniform in general. In the family of compact manifolds without conjugate points, Ruggiero [53] found a useful characterization of uniform divergence of geodesic rays from the continuity of horospherical foliations.

Proposition 2.3.14. *Let M be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. Then geodesic rays diverge uniformly in $\tilde{M}$ if and only if $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ are continuous foliations by Lipschitz leaves, invariant by the geodesic flow.*

Thus $\tilde{\mathcal{F}}^s$ and $\tilde{\mathcal{F}}^u$ are continuous invariant foliations for manifolds of non-positive curvature, manifolds without focal points and compact surfaces without conjugate points and genus greater than one.

We now see a characterization of visibility manifolds in terms of uniform divergence of geodesic rays and Gromov hyperbolic manifolds. We first introduce Gromov hyperbolic spaces. Let (X,d) be a metric space. Let $p,q \in X$, a geodesic segment joining p to q is defined as an isometry $\gamma : [0, d(p,q)] \to X$ with $\gamma(0) = p$ and $\gamma(d(p,q)) = q$. We say that (X,d) is a **geodesic space** if for every two points in X there exists a geodesic segment joining them. A geodesic triangle with vertices $x,y,z \in X$ is a union of three geodesic segments joining respectively x to y, y to z and z to x. A complete geodesic space (X,d) is called **Gromov hyperbolic** if there exists $\delta > 0$ such that every geodesic triangle T satisfies: for every point x in a given side of T, the distance from x to the union of the other two sides is at most δ.

Clearly Gromov hyperbolic spaces are fairly general metric spaces. They characterize δ-uniformly the geodesic triangles of the space. We know that some Riemannian manifolds may be complete geodesic spaces when considered as metric spaces with the Riemannian distance. So, although it is not necessary that Gromov hyperbolic spaces have an underlying Riemannian metric, Riemannian manifolds may be Gromov hyperbolic spaces. In fact, Gromov hyperbolic spaces are inspired by the properties of geodesic triangles of hyperbolic geometry. Thus, the universal covering of compact manifolds of negative curvature are Gromov hyperbolic spaces. Another example is the universal covering of compact surfaces without conjugate points and genus greater than one. Another source of Gromov hyperbolic spaces that come from Riemannian manifolds is given by Ruggiero's characterization of visibility manifolds.

Theorem 2.20 ([54, 26]). *Let M be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. Then, $\tilde{M}$ is a visibility manifold if and only if $\tilde{M}$ is Gromov hyperbolic and geodesic rays diverge uniformly in $\tilde{M}$.*

So, every complete visibility manifold is automatically a Gromov hyperbolic space. This is the case of the universal covering of compact manifolds without conjugate points and expansive geodesic flow [54].

For a better understanding of the strips of bi-asymptotic geodesics defined in Subsection 2.3.6, we introduce the quasi-convexity property. This property says that the distance between geodesic segments is controlled by the distance between the ends of the segments. Let M be a compact manifold without conjugate points. We say that M is **quasi-convex** if there exist constants $A, B > 0$ such that for every two geodesics

$$\gamma : [t_1, t_2] \to \tilde{M}, \qquad \beta : [s_1, s_2] \to \tilde{M},$$

the Hausdorff distance is defined by

$$d_H(\gamma, \beta) \leq A \sup\{d(\gamma(t_1), \beta(s_1)), d(\gamma(t_2), \beta(s_2))\} + B.$$

This property holds for the universal covering of several manifolds: manifolds of non-positive curvature, manifolds without focal points and compact surfaces without conjugate points of genus greater than one. Moreover, it holds for Gromov hyperbolic spaces that come from Riemannian manifolds.

Proposition 2.3.15 ([25, 24]). *Let M be a compact manifold without conjugate points and Gromov hyperbolic universal covering $\tilde{M}$. Then $\tilde{M}$ is quasi-convex.*

The combination of quasi-convexity with uniform divergence of geodesic rays, provides a general framework for establishing some asymptotic properties of geodesics. The following proposition of Rifford and Ruggiero [55] says that strips of bi-asymptotic geodesics have a good topological decomposition as a product of sets.

Proposition 2.3.16. *Let M be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. If $\tilde{M}$ is quasi-convex and geodesic rays diverge uniformly, then for every $\theta \in T_1\tilde{M}$,*

1. *The set $I(\theta)$ is a compact, connected set.*
2. *There exists $L > 0$ such that the diameter of $I(\dot{\gamma_\theta}(t))$ is bounded above by L for every $t \in \mathbb{R}$.*
3. *$S(\gamma_\theta)$ is homeomorphic to the product $I(\theta) \times \mathbb{R}$.*

This proposition is a topological generalization of the flat strip theorem, that in the case of compact manifolds without focal points yields that $S(\gamma_\theta)$ is isometric to a flat strip of $\mathbb{R}^2$.

In particular, this result also implies that $\tilde{\mathcal{I}}(\theta)$ is a compact connected set for every $\theta \in T_1\tilde{M}$. So, this proposition tell us more about the intersection of horospheres and horospherical leaves. For general compact manifolds without conjugate points, these intersections in higher dimension may have a complicated behavior.

Finally, we highlight that these general hypothesis imply that asymptoticity and Busemann asymptoticity are equivalent. Indeed, if $\tilde{M}$ is quasi-convex then Busemann asymptoticity implies asymptoticity. The reverse implication holds if we assume divergence of geodesic rays.

Theorem 2.21 (Lemma 2.9 of [55]). *Let M be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. If $\tilde{M}$ is quasi-convex and geodesic rays diverge uniformly, then for every $\theta \in T_1\tilde{M}$, a geodesic β is asymptotic to γ_θ if and only if β is a Busemann asymptote of γ_θ.*

This equivalence has important consequences in the intersection of the stable and unstable horospherical leaves. This is because asymptoticity concepts help to characterize the intersection of horospherical leaves. Indeed, if a geodesic β starts at $H^+\theta$ and β is bi-asymptotic to γ_θ, then $\beta(0)$ must belong to $I(\theta)$. We can lift this relationship between asymptoticity and intersection, to $T_1\tilde{M}$.

Corollary 2.3.1 ([55]). *Let M be a compact manifold without conjugate points and $\tilde{M}$ be its universal covering. Assume that $\tilde{M}$ is quasi-convex and geodesic rays diverge uniformly. For every $\theta \in T_1\tilde{M}$, if $\eta = (q, w) \in \tilde{\mathcal{F}}^s(\theta)$ and γ_η is bi-asymptotic to γ_θ, then*

$$\eta \in \mathcal{I}(\theta) = \tilde{\mathcal{F}}^s(\theta) \cap \tilde{\mathcal{F}}^u(\theta) \quad \text{and} \quad q \in I(\theta) = H^+(\theta) \cap H^-(\theta).$$

This result simplifies several arguments because it allows to find points in the intersection of horospherical leaves using bi-asymptoticity of orbits.

2.3.10
The behavior of stable Jacobi fields for manifolds without conjugate points

In this subsection we mention the behavior of stable Jacobi fields for different subfamilies of compact manifolds without conjugate points.

We first recall the definition of manifolds without focal points and manifolds without conjugate points, in terms of Jacobi fields. Let M be a compact manifold, let $\gamma \subset M$ be a geodesic and J be a radial Jacobi field along γ with $J(0) = 0$.

- M has no focal points if and only if $|J|$ is a convex function and $|J|$ is non-decreasing for $t \geq 0$.
- M has no conjugate points if and only if $|J|$ does not vanish for $t \neq 0$.

Clearly, there is inclusion among these families of manifolds. Gulliver [56] constructed explicit examples showing that the inclusion is strict.

From Subsection 2.3.8, stable and unstable Jacobi fields always exist in compact manifold without conjugate points. Furthermore, for a large family of manifolds, the stable and unstable Jacobi fields are exactly the fields that are bounded in the future or in the past. But for certain subfamilies of manifolds we know more about the behavior of the stable and unstable Jacobi fields.

Let M be a compact manifold. Considering the Riemannian metric of M, we have the following chain of inclusions: M has

$$\text{Negative curvature} \subset \text{no focal points} \subset \text{no conjugate points.}$$

Let $\gamma \subset M$ be a geodesic and J be a stable Jacobi field along γ. The behavior of stable Jacobi fields for these manifolds is as follows.

- M has negative Curvature: $|J(t)|$ is exponentially decreasing as $t \to \infty$.

– M has no focal points: $|J(t)|$ is a convex non-increasing function as $t \to \infty$.

– M has bounded asymptote: there exists an universal constant $C > 0$ such that for any stable Jacobi field J, $|J(t)| \leq C|J(0)|$ for every $t \geq 0$, i.e., stable Jacobi fields are uniformly bounded in the future.

– M has no conjugate points: $|J(t)|$ never vanishes on $t \in \mathbb{R}$ (the most general behavior obtained by Eberlein [33]).

Analogous behaviors exist for unstable Jacobi fields when $t \leq 0$. The second-last item introduces an important family of manifolds. These manifolds are widely studied due to the good asymptotic behavior of their stable Jacobi fields. Recall that Eberlein [33] showed that the first three families are included in the family of manifolds with continuous Green bundles.

Note that the more general the family, the less controlled is the asymptotic behavior of the Jacobi stable fields. For general compact manifolds without conjugate points, the asymptotic behavior of Jacobi stable fields can be very complicated.

3
Existence of a unique measure of maximal entropy for compact surfaces

In this chapter we give another proof of the following result in the ergodic theory of geodesic flows on compact surfaces.

Theorem 3.1. *Let M be a compact surface without conjugate points of genus greater than one and ϕ_t be its geodesic flow. Then ϕ_t has a unique measure of maximal entropy.*

The existence and uniqueness of the measure of maximal entropy is a problem that appeared in the 1960s. In 1964, Parry [57] gave the formula for the unique measure of maximal measure entropy for irreducible subshifts of finite type. For compact manifolds of negative curvature, in 1969 Margulis [58] constructed an invariant measure using the stable and unstable invariant submanifolds provided by Anosov-Sinai's development of the theory of hyperbolic flows. Later, Bowen showed that this measure is actually the measure of maximal entropy for the geodesic flow. Moreover, using symbolic dynamics Bowen [59, 60] proved the existence and uniqueness of the measure of maximal entropy for hyperbolic flows which includes the case of Margulis. Later, in 1977 Bowen and Franco [20], proved the existence and uniqueness of the measure of maximal entropy for continuous flows satisfying expansivity and the specification property. In 1985, Katok [61] conjectured the existence and uniqueness of the measure of maximal entropy for compact rank-1 manifolds of non-positive curvature. This conjecture was proved by Knieper [62] in 1998, using Patterson-Sullivan measures. In 2015, Bosche [63] extended the conclusion to compact manifolds without conjugate points but assuming the expansivity of the geodesic flow. In 2016, Climenhaga and Thompson [19] extended Bowen-Franco's criteria to show the existence and uniqueness of the measure of maximal entropy for continuous flows. They proved the theorem assuming non-uniform generalized versions of expansivity and specification. Using this criterion, in 2018 Burns-Climenhaga-Fisher-Thompson [64] showed the existence and uniqueness of equilibrium states for compact rank-1 manifolds of non-positive curvature. In 2018, Gelfert and Ruggiero [13] proved the existence of a unique measure of maximal entropy for compact surfaces without focal points and genus greater than one. They used another approach based on the study of an expansive flow semi-conjugate to the geodesic flow. Shortly after, they extended the conclusion to compact surfaces without conjugate points and genus greater than one, by assuming a geometrical hypothesis: the continuity of Green bundles [14]. Following Climenhaga-Thompson's criterion, in 2020 Chen-Kao-Park [65] recovered Gelfert-Ruggiero theorem for closed surfaces without focal points. On the other hand, following Knieper's approach, in 2020 Liu-Wang [66] proved the existence of a unique measure of maximal entropy for compact rank-1 manifolds without focal points. In 2021, Climenhaga-Knieper-War [18] extended the result to a large family of compact manifolds without conjugate points, which includes the case of compact surfaces without conjugate points and genus greater than one. They also used Climenhaga-Thompson's criterion to show the theorem. We give another proof of Theorem 3.1 based on an extension of Gelfert-Ruggiero's strategy to the case of compact surfaces without conjugate points and genus greater than one. Gelfert-Ruggiero's approach differs from Climenhaga-Knieper-War's one, giving a more direct proof of the statement.

To prove Theorem 3.1, we define a quotient flow on a compact metric space, i.e., the quotient model. We prove some dynamical properties for this quotient model.

Theorem 3.2. *Let M be a compact surface without conjugate points and genus greater than one, and ϕ_t be its geodesic flow. Then, there exists a continuous flow ψ_t acting on a compact metric space X such that*

1. *ψ_t is time-preserving semi-conjugate to ϕ_t.*
2. *X has topological dimension at least two.*
3. *ψ_t is topologically mixing, expansive and has a local product. Moreover ϕ_t has the pseudo-orbit tracing and specification properties.*

This theorem extends analogous results obtained in Gelfert-Ruggiero's work [13, 14] for a compact higher genus surface S in two cases:

– S has no focal points.

– S has no conjugate points and the Green bundles are continuous.

We highlight that we deal with compact surfaces without conjugate points of genus greater than one and no further hypothesis. The proof of Theorem 3.2 follows Gelfert-Ruggiero's strategy.

We organize the chapter as follows. Section 3.1 discusses the problems that arise when we try to extend Gelfer-Ruggiero's approach to our setting. In Section 3.2, we introduce all the elements of the quotient model from an equivalence relation in T_1M. In addition we prove item 1 of Theorem 3.2. Section 3.3 builds a basis for the topology of X and shows that X is a compact metrizable space. Section 3.5 is devoted to estimate the topological dimension of X. Section 3.4 constructs a covering space for X. Section 3.6 is concerned with the dynamical properties of the quotient model. Furthermore, we prove item 3 of Theorem 3.2. Section 3.7 shows the uniqueness of the measure of maximal entropy for the geodesic flow.

3.1
Some problems that arise when Green bundles are not continuous

In this section we assume a compact surface without conjugate points and genus greater than one. We discuss some problems that appear when Green bundles are not continuous. That is, when trying to extend Gelfert-Ruggiero's work [13, 14] to our setting. We also mention how we deal with these problems. In this way we highlight the new contributions of this work regarding the previous articles by Gelfert and Ruggiero.

The continuity of Green bundles mainly provides the following consequences:

1. The set $\mathcal{R}_1$ is open and dense in T_1M.
2. Green bundles are tangent to the horospherical foliations hence $\mathcal{R}_1 \subset \mathcal{R}_0$.

We first mention the loss of global methods that use the covering space $\Pi : \tilde{X} \to X$. From items 1 and 2, it follows that there exists $\theta \in T_1M$ such that $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$ are included in $\mathcal{R}_0$. That is, $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$ are composed only of expansive points. This condition allows us to show that the quotient spaces X and $\tilde{X}$ are topological 3-manifolds. Moreover, in this setting X and $\tilde{X}$ are smooth 3-manifolds by [67]. Thus, we can endow X with a Riemannian metric g. Using the covering map $\Pi : \tilde{X} \to X$, the pullback metric Π^*g is a Riemannian metric on $\tilde{X}$. In particular, the map Π would be a local isometry hence a Riemannian covering map satisfying the following commutative diagram

$$\begin{array}{ccc} T_1\tilde{M} & \xrightarrow{d\pi} & T_1M \\ \big\downarrow{\scriptstyle \tilde{\chi}} & & \big\downarrow{\scriptstyle \chi} \\ \tilde{X} & \xrightarrow{\Pi} & X \end{array} . \tag{3.1}$$

We provide $\tilde{X}$ and X with Riemannian distances called quotient distances. So, we can state asymptotic relationships between Sasaki and quotient distance on the covering spaces $T_1\tilde{M}$ and $\tilde{X}$. Global methods refer to the use of relationships between covering spaces $T_1\tilde{M}$ and $\tilde{X}$ as tools to prove relevant results. In this way, we can show some properties of the quotient flow ψ_t such as:

- Expansivity.
- For every $\theta \in T_1M$, $\chi(\mathcal{F}^s(\theta))$ and $\chi(\mathcal{F}^u(\theta))$ are the strong stable and strong unstable sets of $[\theta]$ with respect to the quotient flow ψ_t.

Local methods deal with the relationships between the base spaces T_1M and X. The local approach lacks some important tools. For example, we cannot talk about the divergence of the distance between orbits in the future (or in the past).

Without assuming the continuity of Green bundles, the quotient spaces $\tilde{X}$ and X are not necessarily topological 3-manifolds. Thus, we cannot give Riemannian metrics to $\tilde{X}$ and X such that Π is a Riemannian covering map. Therefore we do not dispose of global methods in our setting.

To deal with this problem, we rely on local methods. In Section 3.3 we show that the quotient spaces $\tilde{X}$ and X are metric spaces and hence have lower regularity than in Gelfert-Ruggiero's work. Despite this, we show that the quotient spaces have topological dimension at least 2 in Section 3.5. Although we do not have global methods, we show that Π is still a covering map satisfying the above diagram 3.1 in Section 3.4. Section 3.6 deals with the dynamical properties of the quotient flow ψ_t. There, we use local arguments in X and T_1M to prove the expansivity and local product structure of ψ_t. In particular, we show that for every $\theta \in T_1M$, $\chi(\mathcal{F}^s(\theta))$ and the strong stable set of $[\theta] \in X$ agree when restricted to the closed ball $B([\theta], r_0)$. An analogous statement is true for the strong unstable set. The value r_0 comes from a local estimate. In Gelfert-Ruggiero's work [13], they used global methods to prove the full coincidence of the above sets.

Another important consequence of the above items 1 and 2 guarantees that $T_1M \setminus \mathcal{R}_1$ is a compact set containing the non-expansive set $T_1M \setminus \mathcal{R}_0$. In previous works, Gelfert and Ruggiero used this property to estimate the topological entropy of $T_1M \setminus \mathcal{R}_1$. The estimate was performed using the variational principle for entropy. Indeed, they used Ruelle's inequality to show that the metric entropy vanishes, for every flow-invariant measure supported on $T_1M \setminus \mathcal{R}_1$. Hence, applying the variational principle to the flow-invariant compact metric space $T_1M \setminus \mathcal{R}_1$, they obtained that $T_1M \setminus \mathcal{R}_1$ has zero topological entropy. This property is fundamental to apply Buzzi-Fisher-Sambarino-Vasquez's Theorem [68]. This result grants the uniqueness of the measure of maximal entropy of the geodesic flow of the surface.

In our case, to apply Buzzi-Fisher-Sambarino-Vasquez's Theorem we rely on a classical Katok's argument for an ergodic measure μ with positive metric entropy. Indeed, Ruelle's inequality implies that μ has non-zero Lyapunov exponents. Thus, we can apply Pesin's theory to build transverse local submanifolds with weak hyperbolic behavior. This fact entails that the expansive set $\mathcal{R}_0$ has positive μ-measure. This conclusion helps to fulfill Buzzi-Fisher-Sambarino-Vasquez's hypotheses and so guarantees the uniqueness of the measure of maximal entropy in our setting. This is done in more detail in Section 3.7.

3.2 The quotient model

In the remainder of this chapter we will assume that (M, g) is a compact surface without conjugate points and genus greater than one. This section introduces quotient models for the geodesic flows acting on $T_1\tilde{M}$ and T_1M respectively. We give some basic properties of the quotient models. Furthermore, we show how the quotient models are related to the corresponding geodesic flows. The relationships between the geodesic flows on $T_1\tilde{M}$ and T_1M will induce analogous relationships in the corresponding quotient models.

For general geodesic flows, a basic problem is to dealing with strips of bi-asymptotic orbits. These strips are regions where there is no hyperbolicity. Moreover, the geodesic flows are not expansive in these strips. This is clear for the case of compact manifolds without focal points. In these manifolds, the strips of bi-asymptotic geodesics are flat strips hence their lifts are strips with a rigid behavior. Therefore, the strips of bi-asymptotic orbits are a kind of obstructions for the flow to have better dynamical properties.

To deal with strips of bi-asymptotic orbits, we introduce a quotient model related to the geodesic flow. This model consists of a quotient space and a quotient flow: a continuous flow on a compact metric space. The quotient space is induced by a special equivalence relation on T_1M, while the quotient flow is induced by the geodesic flow. The equivalence relation collapses the strips of bi-asymptotic orbits into single orbits of the quotient flow. Thus, by collapsing the obstructions, we hope to obtain new properties such as expansivity. Inherit some properties and improve some other properties of the geodesic flow.

We introduce the quotient model for the geodesic flows on T_1M and $T_1\tilde{M}$. The constructions follow the same ideas of Section 4 of [13]. We start with the quotient model for the geodesic flow on T_1M. Let us define the equivalence relation on T_1M. Two points $\theta, \eta \in T_1M$ are equivalent $\theta \sim \eta$ if, and only if,

- $\eta \in \mathcal{F}^s(\theta)$.
- if $\tilde{\theta}, \tilde{\eta} \in T_1\tilde{M}$ are lifts of $\theta, \eta \in T_1M$ respectively with $\tilde{\eta} \in \tilde{\mathcal{F}}^s(\tilde{\theta})$, then $\gamma_{\tilde{\theta}}$ and $\gamma_{\tilde{\eta}}$ are bi-asymptotic.

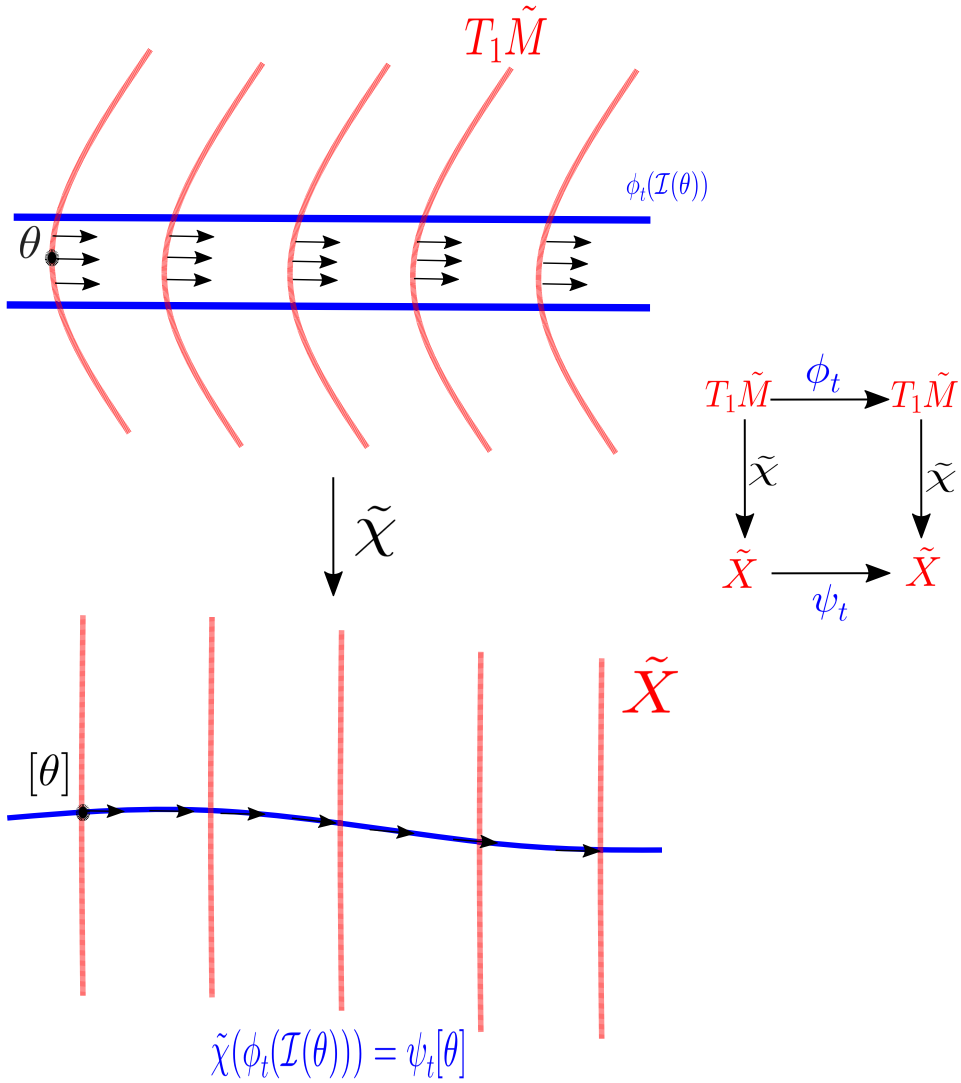

Figure 3.1: The equivalence relation

To deal with this relation, we rely on the properties of the horospherical leaves given in Section 2.3. In particular, the equivalence relation has a simpler characterization in terms of intersections of stable and unstable horospherical leaves. Indeed, Theorem 2.3.5 shows that these intersections are connected compact curves. Recall that for every $\eta \in T_1M$,

$$\mathcal{I}(\eta) = \mathcal{F}^s(\eta) \cap \mathcal{F}^u(\eta).$$

Lemma 3.2.1. *For every $\eta, \theta \in T_1M$, $\eta \sim \theta$ if and only if $\eta \in \mathcal{I}(\theta)$.*

Proof. If $\eta \sim \theta$ then there exist lifts $\tilde{\eta}, \tilde{\theta} \in T_1\tilde{M}$ of η and θ such that $\tilde{\eta} \in \tilde{\mathcal{F}}^s(\tilde{\theta})$, and $\gamma_{\tilde{\eta}}$ and $\gamma_{\tilde{\theta}}$ are bi-asymptotic. By Corollary 2.3.1, $\tilde{\eta} \in \tilde{\mathcal{I}}(\tilde{\theta})$. Projecting through $d\pi$ we get that $\eta \in \mathcal{I}(\theta)$. Conversely, if $\eta \in \mathcal{I}(\theta)$, for every lift $\tilde{\eta}$ of η, there exists a lift $\tilde{\theta}$ of θ so that $\tilde{\theta} \in \tilde{\mathcal{I}}(\tilde{\eta})$. Thus there exist lifts $\tilde{\eta}, \tilde{\theta} \in T_1\tilde{M}$ of η and θ respectively such that $\tilde{\eta} \in \tilde{\mathcal{F}}^s(\tilde{\theta})$, and the induced geodesics $\gamma_{\tilde{\eta}}$ and $\gamma_{\tilde{\theta}}$ are bi-asymptotic. ∎

With this characterization, we show that the relation $\sim$ on T_1M is actually an equivalence relation on T_1M.

Lemma 3.2.2. *The above relation $\sim$ on T_1M is an equivalence relation.*

Proof. We will use the alternative characterization of the relation given by Lemma 3.2.1. Clearly, the relation is reflexive, so we deal with the property of symmetry and transitivity. We see that $\eta \sim \theta$ if

$$\eta \in \mathcal{I}(\theta) = \mathcal{F}^s(\theta) \cap \mathcal{F}^u(\theta).$$

Recall that for any $\mathcal{F}^s(\theta_1)$ and $\mathcal{F}^u(\theta_2)$, if $\eta_1 \in \mathcal{F}^s(\theta_1)$ and $\eta_2 \in \mathcal{F}^u(\theta_2)$, we have

$$\mathcal{F}^s(\eta_1) = \mathcal{F}^s(\theta_1) \text{ and } \mathcal{F}^u(\eta_2) = \mathcal{F}^u(\theta_2).$$

For the symmetry, if $\eta \sim \theta$ then

$$\mathcal{F}^s(\eta) = \mathcal{F}^s(\theta) \text{ and } \mathcal{F}^u(\eta) = \mathcal{F}^u(\theta).$$

So, $\theta \in \mathcal{F}^s(\eta) \cap \mathcal{F}^u(\eta)$ hence $\theta \sim \eta$. Finally, for the transitivity, if $\eta \sim \theta$ and $\theta \sim \xi$ we see that

$$\mathcal{F}^s(\eta) = \mathcal{F}^s(\theta) = \mathcal{F}^s(\xi) \text{ and } \mathcal{F}^u(\eta) = \mathcal{F}^u(\theta) = \mathcal{F}^u(\xi).$$

Thus, $\eta \in \mathcal{F}^s(\xi) \cap \mathcal{F}^u(\xi)$ hence $\eta \sim \xi$. ∎

This equivalence relation induces a quotient space X and a quotient map

$$\begin{aligned} \chi : T_1M &\to X \\ \theta &\mapsto \chi(\theta) = [\theta], \end{aligned}$$

where $[\theta]$ is the equivalence class of θ. While the geodesic flow and the quotient map induce a quotient flow

$$\begin{aligned} \psi : \mathbb{R} \times X &\to X \\ (t, [\theta]) &\mapsto \psi(t, [\theta]) = \phi_t[\theta] = [\phi_t(\theta)]. \end{aligned}$$

Thus, we have a quotient flow ϕ_t acting on a quotient space X, and related to the geodesic flow through the quotient map χ. These elements form the quotient model of the geodesic flow ϕ_t.

We now verify some basic properties of the quotient model. We endow the quotient space X with the quotient topology. An immediate consequence follows: the quotient map χ is continuous. Seeing that T_1M is compact, the continuity of χ provides that X is also compact.

Lemma 3.2.3. *Let ψ_t be the quotient flow on the quotient space X, χ be the quotient map, and ϕ_t be the geodesic flow on T_1M. Then,*

1. *The quotient flow ψ_t is a well-defined continuous flow.*
2. *The quotient map χ is a semi-conjugacy between ϕ_t and ψ_t, which preserves time: for every $t \in \mathbb{R}$,*

$$\chi \circ \phi_t = \psi_t \circ \chi, \qquad \begin{array}{ccc} T_1M & \xrightarrow{\phi_t} & T_1M \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} \\ X & \xrightarrow{\psi_t} & X. \end{array}$$

Proof. Let $\eta, \xi \in T_1M$ with $\eta \sim \xi$, hence $[\eta] = [\xi]$ and $\eta \in \mathcal{I}(\xi)$. By the invariance of the horospherical foliations, for every $t \in \mathbb{R}$ we have

$$\phi_t(\mathcal{I}(\xi)) = \phi_t(\mathcal{F}^s(\xi) \cap \mathcal{F}^u(\xi)) = \mathcal{F}^s(\phi_t(\xi)) \cap \mathcal{F}^u(\phi_t(\xi)) = \mathcal{I}(\phi_t(\xi)).$$

Thus, $\phi_t(\eta) \in \mathcal{I}(\phi_t(\xi))$ and hence ψ_t is well defined:

$$\psi_t[\eta] = [\phi_t(\eta)] = [\phi_t(\xi)] = \psi_t[\xi].$$

To show that ψ_t is continuous, let U be an open set in X. For every $t \in \mathbb{R}$, we must show that $\psi_t^{-1}(U)$ is an open set in X. We know that $\psi_t^{-1}(U)$ is open if, and only if, $\chi^{-1}(\psi_t^{-1}(U))$ is open in T_1M. However, by definition of ψ_t we have

$$\chi^{-1} \circ \psi_t^{-1}(U) = \phi_t^{-1} \circ \chi^{-1}(U).$$

Since U is open and ϕ_t is a homeomorphism, $\chi^{-1} \circ \psi_t^{-1}(U)$ is open and hence $\psi_t^{-1}(U)$ is open. Thus ψ_t is continuous for every $t \in \mathbb{R}$. ■

An equivalent restatement of item 2 is to say that the quotient flow ψ_t is a time-preserving factor of the geodesic flow ϕ_t.

Regarding the quotient map $\chi : T_1M \to X$, the concept of saturation of sets is useful for the arguments. Let A be a subset of T_1M, we say that A is **saturated** with respect to χ, or simply saturated, if

$$\chi^{-1}\chi(A) = A.$$

This concept has an interpretation through the equivalence relation. Indeed, observe that for every $\eta \in T_1M$, the equivalence class of η, seen as subset of T_1M, satisfies

$$\{\xi \in T_1M : \xi \sim \eta\} = \mathcal{I}(\eta).$$

Thus, a set $A \subset T_1M$ is saturated if and only if A satisfies the condition:

$$\text{if } \eta \in A \quad \text{then} \quad \mathcal{I}(\eta) \subset A.$$

We mention two useful consequences of this characterization. If U is a saturated open set of T_1M then its image $\chi(U)$ is an open set of X. The other consequence deals with the action ϕ_t on saturated sets. If $A \subset T_1M$ is saturated then so is $\phi_t(A)$ for every $t \in \mathbb{R}$. This is because ϕ_t carries the classes $\mathcal{I}(\eta) \subset A$ onto $\mathcal{I}(\phi_t(\eta)) \subset \phi_t(A)$.

We introduce the corresponding quotient model for the geodesic flow $\tilde{\phi}_t$ on $T_1\tilde{M}$. Following the equivalent definition of the relation on T_1M, we say that $\eta, \theta \in T_1\tilde{M}$ are equivalent $\eta \sim \theta$ if, and only if, $\eta \in \tilde{\mathcal{I}}(\theta)$. As above the relation induces a quotient space $\tilde{X}$ and quotient map

$$\begin{aligned} \tilde{\chi} : T_1\tilde{M} &\to \tilde{X} \\ \theta &\mapsto \tilde{\chi}(\theta) = [\theta], \end{aligned}$$

where $[\theta]$ is the equivalence class of θ. While the geodesic flow $\tilde{\phi}_t$ induces a quotient flow

$$\begin{aligned} \tilde{\psi} : \mathbb{R} \times \tilde{X} &\to \tilde{X} \\ (t, [\theta]) &\mapsto \tilde{\psi}(t, [\theta]) = \tilde{\phi}_t[\theta] = [\tilde{\phi}_t(\theta)]. \end{aligned}$$

Lemma 3.2.4. *Let $\tilde{\psi}_t$ be the quotient flow on the quotient space $\tilde{X}$, $\tilde{\chi}$ be the* *quotient map, and $\tilde{\phi}_t$ be the geodesic flow on $T_1\tilde{M}$. Then,*

1. *The quotient flow $\tilde{\psi}_t$ is a well-defined continuous flow.*

2. *The quotient map $\tilde{\chi}$ is a semi-conjugacy between $\tilde{\phi}_t$ and $\tilde{\psi}_t$, which preserves time: for every $t \in \mathbb{R}$,*

$$\tilde{\chi} \circ \tilde{\phi}_t = \tilde{\psi}_t \circ \tilde{\chi}, \qquad \begin{array}{ccc} T_1\tilde{M} & \xrightarrow{\tilde{\phi}_t} & T_1\tilde{M} \\ \downarrow{\scriptstyle \tilde{\chi}} & & \downarrow{\scriptstyle \tilde{\chi}} \\ \tilde{X} & \xrightarrow{\tilde{\psi}_t} & \tilde{X}. \end{array}$$

As above we can say that the quotient flow $\tilde{\psi}_t$ is a time-preserving factor of the geodesic flow $\tilde{\phi}_t$. Thus, we have a quotient flow $\tilde{\psi}_t$ acting on a quotient space $\tilde{X}$. This flow is semi-conjugate to the geodesic flow $\tilde{\phi}_t$ through the quotient map $\tilde{\chi}$. Thus we have described the quotient model of the geodesic flow $\tilde{\phi}_t$.

In summary, we introduced the quotient models $\tilde{\psi}$ and ψ of the geodesic flows $\tilde{\phi}$ and ϕ. These models are time-preserving semi-conjugate to the geodesic flows through the quotient maps $\tilde{\chi}$ and χ. Furthermore, similar to the case of $\tilde{\phi}$ and ϕ, the quotient models $\tilde{\psi}$ and ψ are time-preserving semi-conjugate through the covering map Π. In the following sections we will study other properties of the quotient models. We will see how improved properties of these models can help in the study of geodesic flows.

3.3
Some basic topological properties of the quotient model

In this section we build a special basis for the quotient topology of $\tilde{X}$ and X. The construction is a natural extension of the work done by Gelfert and Ruggiero for higher genus compact surfaces without focal points (see Section 4 of [13]). As application, we show that X is a compact metrizable space.

We first build a topological basis for $T_1\tilde{M}$ and then for T_1M. Before we start with the construction of the topological basis, we give some preparatory remarks. Note that horospherical leaves $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$ can be identified with $\mathbb{R}$. Recall that $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$ are continuous connected 1-dimensional sub-manifolds of $T_1\tilde{M}$. So, given any arc-length parametrization of $\tilde{\mathcal{F}}^s(\eta)$, we see that $\tilde{\mathcal{F}}^s(\eta)$ is homeomorphic to $\mathbb{R}$. Thus, every connected subset of $\tilde{\mathcal{F}}^s(\eta)$ is homeomorphic to an interval of $\mathbb{R}$. To simplify notation, we always choose some arc-length parametrizations for $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$. This choice gives us homeomorphisms from $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$ onto $\mathbb{R}$. In this way, we identify connected subsets of $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$ with their images (intervals) through these homeomorphisms.

We next show another ingredient for the construction: expansive points accumulate around boundary points of nontrivial classes. In the context that identifies connected subsets of horospherical leaves with intervals of $\mathbb{R}$, the following lemma is useful.

Lemma 3.3.1. *Every interval $I \subset \mathbb{R}$ cannot be the union of disjoint closed intervals.*

The following result says that we can always find expansive points in $\tilde{\mathcal{F}}^s(\eta)$, as close as we want, to any nontrivial class in $\tilde{\mathcal{F}}^s(\eta)$. A similar conclusion is valid for $\tilde{\mathcal{F}}^u(\eta)$.

Lemma 3.3.2. *Let $\tilde{\mathcal{I}}(\theta)$ be a nontrivial class for some $\theta \in T_1\tilde{M}$. Then the boundary points of $\tilde{\mathcal{I}}(\theta)$ are accumulated by expansive points.*

Proof. Let $c : \mathbb{R} \to \tilde{\mathcal{F}}^s(\theta)$ be an arc-length parametrization of $\tilde{\mathcal{F}}^s(\theta)$. Since $\tilde{\mathcal{I}}(\theta)$ is non-trivial there exist $a, b \in \mathbb{R}$ with $a < b$ such that $\tilde{\mathcal{I}}(\theta) = c([a, b])$. We will identify connected compact subsets of $\tilde{\mathcal{F}}^s(\theta)$ with real intervals to simplify notation. By contradiction suppose that the boundary point b is not accumulated by expansive points. Thus, there exists $\delta > 0$ such that $(b, b + \delta)$ does not contain expansive points. From the equivalence relation properties, either $(b, b + \delta)$ is a single class or it is a disjoint union of distinct classes. By Lemma 3.3.1, $(b, b + \delta)$ must be a subset of a single class. Since classes are closed sets we also obtain that $[b, b + \delta]$ is a subset of a single class. Thus b is a common point of both classes hence $[a, b + \delta]$ must be a single class, a contradiction because b is a boundary point. ■

For compact higher genus surfaces without focal points, Gelfert-Ruggiero [13] built a basis for the quotient topology of X. Following the same procedure, we build a similar topological basis of $\tilde{X}$, for compact higher genus surfaces without conjugate points. For every $\theta \in T_1\tilde{M}$, the idea is to define special neighborhoods A_i of $\tilde{\mathcal{I}}(\theta)$ such that $\tilde{\chi}(A_i)$ are even neighborhoods of $[\theta] \in \tilde{X}$. These neighborhoods $\tilde{\chi}(A_i) \subset \tilde{X}$ will form the desired basis. We construct the neighborhoods A_i in several steps:

1. We define local cross sections to the geodesic flow, containing $\tilde{\mathcal{I}}(\theta)$. Thus the action of $\tilde{\phi}_t$ on the sections provides suitable neighborhoods of $\tilde{\mathcal{I}}(\theta)$.

2. In these cross sections, we define special intersections between stable and unstable horospherical leaves.

3. Using these intersections we build saturated smaller cross sections containing $\tilde{\mathcal{I}}(\theta)$. Again the action of $\tilde{\phi}_t$ on these sections gives saturated neighborhoods of $\tilde{\mathcal{I}}(\theta)$.

For step 1, we will see that a homeomorphism gives the desired cross section. To build the homeomorphism for every $\theta \in T_1\tilde{M}$, we rely on its vertical fiber V_θ. Roughly speaking, we join all the stable horospherical leaves of points in V_θ. More precisely, for every $\delta_0, \epsilon_0 > 0$, there exist $a, b \in \mathbb{R}$ and a map

$$R : (a - \epsilon, b + \epsilon) \times (-\delta, \delta) \to T_1\tilde{M}$$

satisfying the conditions:

1. For $s \in (-\delta, \delta)$, consider a δ_0-neighborhood of θ in $V(\theta)$. We denote by $R(0, s)$ the arc-length parametrization of this neighborhood with respect to Sasaki metric.

2. For fixed $s \in (-\delta, \delta)$, for every $r \in (a-\epsilon, b+\epsilon)$ consider a ϵ-neighborhood of $R(0, s)$ in $\tilde{\mathcal{F}}^s(\theta)$. We denote by $R(r, s)$ the arc-length parametrization of this neighborhood.

3. We require
$$R([a, b], 0) = \tilde{\mathcal{I}}(\theta) \text{ and } R(0, 0) = \theta.$$

We denote by

$$\Sigma = \Sigma(\theta, \epsilon, \delta) = R((a - \epsilon, b + \epsilon) \times (-\delta, \delta))$$

the image of R.

The continuity of the horospherical foliations ensures that R is a homeomorphism. Therefore, Σ is a 2-dimensional section containing $\tilde{\mathcal{I}}(\theta)$. Note that Σ is foliated by stable horospherical leaves of points in $V(\theta)$. Since these leaves are topologically transverse to the geodesic flow, Σ is a cross section. Finally, for $\tau > 0$, Brower's open mapping theorem provides that

$$B = B(\theta, \epsilon, \delta, \tau) = \bigcup_{|t|<\tau} \tilde{\phi}_t(\Sigma)$$

is an open 3-dimensional neighborhood of $\tilde{\mathcal{I}}(\theta)$.

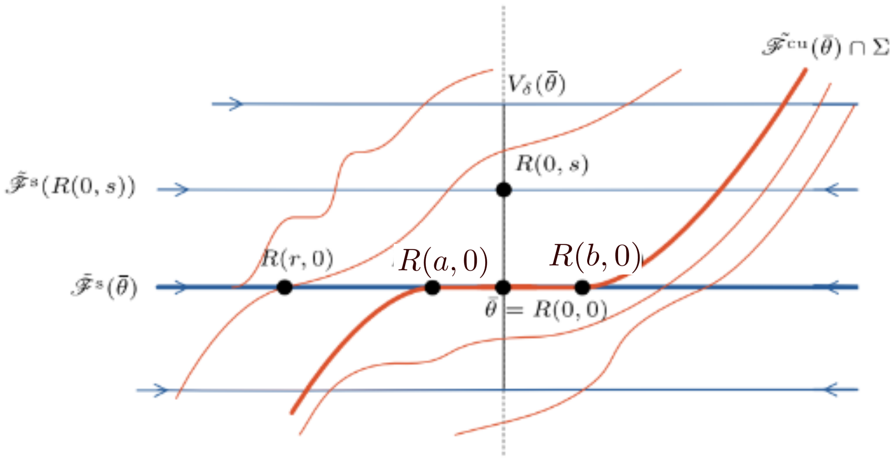

Figure 3.2: The parametrization through the homeomorphism R

To begin step 2, note that for every $\eta \in \Sigma$, Σ contains an interval of $\tilde{\mathcal{F}}^s(\eta)$ containing η. However, the unstable horospherical leaf $\tilde{\mathcal{F}}^u(\eta)$ may intersect Σ only at η. Since we need intervals related to $\tilde{\mathcal{F}}^u(\eta)$ included in Σ, we define a **projection map**

$$Pr : B \to \Sigma.$$

For every $\eta \in B$, $Pr(\eta)$ is the projection of η through the geodesic flow $\tilde{\phi}_t$. From the properties of $\tilde{\phi}_t$, we see that Pr is continuous and surjective. Using Pr, for every $\eta \in \Sigma$, we define the **stable and unstable intervals** in Σ,

$$W^s(\eta) = \tilde{\mathcal{F}}^s(\eta) \cap \Sigma \text{ and } W^u(\eta) = Pr(\tilde{\mathcal{F}}^u(\eta) \cap B).$$

We can think of $W^s(\eta)$ and $W^u(\eta)$ as the projections of $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$ to Σ. Using these intervals we define the special intersections mentioned above. For every $\xi, \eta \in \Sigma$, we define the **intersection**

$$[\xi, \eta] = W^s(\xi) \cap W^u(\eta).$$

Lemma 3.3.3. *For every $\xi, \eta \in \Sigma$, the intersection $[\xi, \eta] = Pr(\tilde{\mathcal{I}}(\zeta))$ is nonempty and belongs to Σ.*

Proof. Proposition 2.3.6 says that, in particular, for every $\xi, \eta \in \Sigma$,

$$\tilde{\mathcal{F}}^s(\xi) \cap \tilde{\mathcal{F}}^{cu}(\eta) = \tilde{\mathcal{I}}(\zeta) \quad \text{for some } \zeta \in T_1\tilde{M}.$$

If δ is small enough then $\tilde{\mathcal{I}}(\zeta)$ belongs to Σ. Thus, applying the projection map Pr to the above intersection we have

$$W^s(\xi) \cap W^u(\eta) = \tilde{\mathcal{I}}(\zeta) \in \Sigma.$$

■

For step 3, we will choose four expansive points $\theta_1, \theta_2, \eta_1, \eta_2 \in \Sigma$ that will serve as vertices of the new cross section. We choose the first two. For $\tilde{\mathcal{I}}(\theta) \in \Sigma$, Lemma 3.3.2 says that expansive points accumulate around the boundary points of $\tilde{\mathcal{I}}(\theta)$. This means that for $\epsilon > 0$ there exists $c \in (a-\epsilon, a), d \in (b, b+\epsilon)$ such that

$$\theta_1 = R(c, 0) \text{ and } \theta_2 = R(d, 0)$$

are expansive points in $W^s(\theta)$.

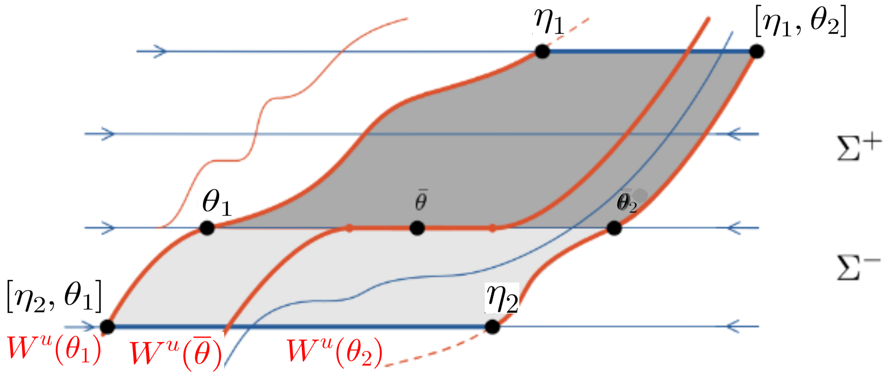

Figure 3.3: The new cross section

To choose the last two expansive points, we define the **upper and lower region** of Σ by

$$\Sigma_+ = \{R(r, s) : r \in (a-\epsilon, b+\epsilon), s > 0\} \text{ and } \Sigma_- = \{R(r, s) : r \in (a-\epsilon, b+\epsilon), s < 0\}.$$

Pick some expansive points

$$\eta_1 \in W^u(\theta_1) \cap \Sigma_+ \text{ and } \eta_2 \in W^u(\theta_2) \cap \Sigma_-.$$

Thus, the new cross section

$$U = U(\theta, \epsilon, \delta, \theta_1, \theta_2, \eta_1, \eta_2) \subset \Sigma$$

is the open 2-dimensional region in Σ, bounded by $W^u(\theta_1)$, $W^u(\theta_2)$, $W^s(\eta_1)$ and $W^s(\eta_2)$.

Clearly, since $\theta_1, \theta_2 \in W^s(\theta)$ are around $\tilde{\mathcal{I}}(\theta)$, we have that U contains $\tilde{\mathcal{I}}(\theta)$. Since U is bounded by stable and unstable intervals W^s and W^u, it follows that if $\eta \in U$ then $\tilde{\mathcal{I}}(\eta) \subset U$. This means that U is saturated. As above, for $\tau > 0$ Brouwer's open mapping theorem provides that

$$A = A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2) = \bigcup_{|t|<\tau} \tilde{\phi}_t(U)$$

is an open 3-dimensional neighborhood of $\tilde{\mathcal{I}}(\theta)$. Since U is saturated so is A.

Thus, for every $\theta \in T_1\tilde{M}$, we have built a family

$$\{A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2) : \epsilon, \delta, \tau > 0, \theta_1, \theta_2 \in W^s(\theta), \eta_1 \in W^u(\theta_1), \eta_2 \in W^u(\theta_2)\}$$

of saturated neighborhoods of $\tilde{\mathcal{I}}(\theta)$. Therefore we obtain a corresponding family

$$\{\tilde{\chi}(A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2)) : \epsilon, \delta, \tau > 0, \theta_1, \theta_2 \in W^s(\theta), \eta_1 \in W^u(\theta_1), \eta_2 \in W^u(\theta_2)\}$$

of neighborhoods of $[\theta] \in \tilde{X}$. We now show that this family induces a basis for the quotient topology of $\tilde{X}$.

Lemma 3.3.4. *For every $\theta \in T_1\tilde{M}$, the family*

$$\mathcal{A}_\theta = \{\tilde{\chi}(A(\theta, \epsilon_l, \delta_m, \tau_n)) : \epsilon_l = 1/l, \delta_m = 1/m, \tau_n = 1/n \text{ with } l, m, n \in \mathbb{N}\}$$

is a countable basis of neighborhoods of $[\theta] \in \tilde{X}$. Hence $\tilde{X}$ is first countable and

$$\{\mathcal{A}_\theta : \theta \in T_1\tilde{M}\}$$

is a basis for the quotient topology of $\tilde{X}$.

Proof. For every $\theta \in T_1\tilde{M}$, the above construction shows that

$$\{\tilde{\chi}(A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2))\}$$

is a family of neighborhoods of $[\theta] \in X$. Note that by choosing parameters $\epsilon, \delta, \tau > 0$ small enough, every neighborhood V of $\tilde{\mathcal{I}}(\theta)$ contains some $A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2)$. Thus, given an open set $U \subset X$ containing $[\theta]$, $\tilde{\chi}^{-1}(U)$ is an open neighborhood of $\tilde{\mathcal{I}}(\theta)$. So, there exists

$$A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2) \subset \tilde{\chi}^{-1}(U) \text{ hence } \tilde{\chi}(A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2)) \subset U.$$

Therefore the collection

$$\{\tilde{\chi}(A(\theta, \epsilon, \delta, \tau, \theta_1, \theta_2, \eta_1, \eta_2))\}$$

is a basis of neighborhoods of $[\theta] \in X$. This property is not affected by specific choices of parameters $\theta_1, \theta_2, \eta_1, \eta_2 \in \Sigma$, but by the parameters $\epsilon, \delta, \tau > 0$. By choosing $\epsilon_l = 1/l, \delta_m = 1/m, \tau_n = 1/n$ with $l, m.n \in \mathbb{N}$, we still have that

$$\mathcal{A}_\theta = \{\tilde{\chi}(A(\theta, \epsilon_l, \delta_m, \tau_n)) : l, m, n \in \mathbb{N}\}$$

is a basis of neighborhoods of $[\theta]$. Therefore $\mathcal{A}_\theta$ is a countable basis of neighborhoods of $[\theta]$. ■

This basis is important because it provides an explicit description of special basic sets for the quotient topology. Thus, we can prove some results about the quotient topology with simpler arguments. The metrization of the quotient space, below exemplifies this claim.

So, we have a family of neighborhoods of every $\tilde{\mathcal{I}}(\theta)$ in $T_1\tilde{M}$ and a basis of neighborhoods of $[\theta]$ in $\tilde{X}$. Projecting through the covering maps Π and $d\pi$ we get a family of neighborhoods of every $\mathcal{I}(\theta)$ in T_1M and a basis of neighborhoods of $[\theta]$ in X. Thus, X is first countable and

$$\{\Pi(\mathcal{A}_\theta) : \theta \in T_1\tilde{M}\}$$

is a basis for the quotient topology of X. This illustrates the use of the covering spaces $\tilde{M}$, $T_1\tilde{M}$ and $\tilde{X}$ to get results and then transfer them to M, T_1M and X.

We next show that the quotient space X is metrizable. Observe that many arguments are easier if there exists some distance compatible with the quotient topology. Moreover, there are strong theorems that hold for metric spaces but not for topological spaces. So, although we have suitable open basic sets for X, it is convenient to have a metric distance for X. We first recall a basic result about metrizability of topological spaces [69].

Proposition 3.3.1. *If $f : X \to Y$ is a continuous surjection from a compact metric space onto a Hausdorff space, then Y is metrizable.*

We now prove the metric structure of the quotient space.

Lemma 3.3.5. *Let M be a compact surface without conjugate points and genus greater than one. Then, the quotient space X is a compact metrizable space.*

Proof. Since χ is continuous and surjective, we see that X is compact. We next show that X is Hausdorff. Choose two different points $[\theta], [\eta] \in X$ and suppose that $\mathcal{F}^s(\theta) \cap \mathcal{F}^s(\eta) = \emptyset$. By choosing δ small enough in Lemma 3.3.4, we can build disjoint basic open sets because $\mathcal{F}^s$ is a foliation of T_1M. Now, consider the case $\mathcal{F}^s(\theta) \cap \mathcal{F}^s(\eta) \neq \emptyset$ hence $\mathcal{F}^s(\theta) = \mathcal{F}^s(\eta)$. Choosing ϵ small enough in Lemma 3.3.4, the basic open sets of θ and η are disjoint because $\mathcal{F}^u$ is a foliation of T_1M. Therefore, X is Hausdorff. Since $\chi : T_1M \to X$ is a continuous surjection from a compact metric space onto a Hausdorff space, Proposition 3.3.1 asserts that X is metrizable. ■

This lemma ensures the existence of a distance inducing the quotient topology, but the lemma does not give an explicit formula to do calculations.

3.4 The covering space of the quotient space

The goal of the section is to show that $\tilde{X}$ is a covering space of X.

Similar to the relationships between the geodesic flows $\tilde{\phi}_t$ and ϕ_t, there are relationships between the corresponding quotient models. Since $T_1\tilde{M}$ is a covering space of T_1M with covering map $d\pi$, we see that $\tilde{X}$ is a covering space of X through a covering map induced by $d\pi$. Moreover, the Deck transformations of both covering maps are related through the quotient map $\tilde{\chi}$. We define the induced covering map by

$$\begin{aligned} \Pi : \tilde{X} &\to X \\ [\theta] &\mapsto \Pi[\theta] = \chi \circ d\pi(\theta). \end{aligned}$$

While for the induced Deck transformations: for every covering isometry T of $d\pi$, we define the induced map by

$$\begin{aligned} T' : \tilde{X} &\to \tilde{X} \\ [\theta] &\mapsto T'[\theta] = \tilde{\chi} \circ T(\theta). \end{aligned}$$

Lemma 3.4.1. *Let Π and T' be the above defined maps. Then*

1. *The maps Π and T' are well-defined.*

2. *The set $\tilde{X}$ is a covering space of X and Π is a covering map satisfying*

$$\begin{array}{ccc} T_1\tilde{M} & \xrightarrow{\tilde{\chi}} & \tilde{X} \\ \big\downarrow{\scriptstyle d\pi} & & \big\downarrow{\scriptstyle \Pi} \\ T_1M & \xrightarrow{\chi} & X. \end{array}$$

3. *For every covering isometry T of $d\pi$, T' is a Deck transformation of* Π *satisfying*

$$\begin{array}{ccc} T_1\tilde{M} & \xrightarrow{T} & T_1\tilde{M} \\ \downarrow{\scriptstyle \tilde{\chi}} & & \downarrow{\scriptstyle \tilde{\chi}} \\ \tilde{X} & \xrightarrow{T'} & \tilde{X}. \end{array}$$

Proof. For 1, let $\eta, \theta \in T_1\tilde{M}$ such that $\eta \sim \theta$. Then $\eta \in \tilde{\mathcal{I}}(\theta)$ and projecting through $d\pi$, we get $d\pi(\eta) \in \mathcal{I}(d\pi(\theta))$: $d\pi(\eta) \sim d\pi(\theta)$. Therefore,

$$\Pi[\eta] = \chi \circ d\pi(\eta) = \chi \circ d\pi(\theta) = \Pi[\theta].$$

To verify the well definition of T', pick $\eta \sim \theta$ hence $\eta \in \tilde{\mathcal{I}}(\theta)$. Since T is a covering isometry, $T(\eta) \in T(\tilde{\mathcal{I}}(\theta)) = \tilde{\mathcal{I}}(T(\theta))$. This means that $T(\eta) \sim T(\theta)$ and so

$$T'[\eta] = \tilde{\chi} \circ T(\eta) = \tilde{\chi} \circ T(\theta) = T'[\theta].$$

For 2, the definition of Π provides that Π is a continuous surjection. We will verify that every $[\eta] \in X$ has a neighborhood evenly covered. Let $\tilde{\eta} \in T_1\tilde{M}$ be some lift of η. Since $\mathcal{F}^s$ and $\mathcal{F}^u$ are minimal foliations by Theorem 2.15, we know that $d\pi$ maps homeomorphically $\tilde{\mathcal{I}}(\tilde{\eta})$ onto $\mathcal{I}(\eta)$. Now, we choose a tubular neighborhood V of $\tilde{\mathcal{I}}(\tilde{\eta})$ such that

- $d\pi|_V$ is still a homeomorphism onto its image.
- If $\xi \in V$ then $\tilde{\mathcal{I}}(\xi) \subset V$.

We can generate all the preimages of $d\pi(V)$ with the action of the covering isometries:

$$(d\pi)^{-1}(d\pi(V)) = \bigsqcup_{S \in \pi_1(M)} S(V).$$

The union is disjoint because by above condition 3, every preimage $S(V)$ belongs to a different fundamental domain $S(K)$. By condition 2, $\chi(d\pi(V)) \subset X$ is an open set containing $[\eta]$. Thus, by the definition of Π we have

$$\Pi^{-1}(\chi(d\pi(V))) = \bigsqcup_{S \in \pi_1(M)} \tilde{\chi}(S(V)).$$

Since $d\pi$ maps homeomorphically every preimage $S(V)$ onto $d\pi(V)$, Π maps homeomorphically every preimage $\tilde{\chi}(S(V))$ onto $\chi(d\pi(V))$. Therefore, every $[\eta] \in X$ has a neighborhood evenly covered. This shows that Π is a covering map with covering space $\tilde{X}$. The diagram follows from the definition of Π.

For item 3, from the definition of T', we see that T' is continuous and surjective. For the injectivity, consider $[\eta], [\theta] \in \tilde{X}$ with $[\eta] \neq [\theta]$. This means that η and θ are not equivalent and hence $\mathcal{I}(\eta)$ and $\mathcal{I}(\theta)$ are disjoint classes. Since T is a homeomorphism, $T(\mathcal{I}(\eta)) = \mathcal{I}(T(\eta))$ and $T(\mathcal{I}(\theta)) = \mathcal{I}(T(\theta))$ are disjoint classes as well. Therefore, $T(\eta)$ and $T(\theta)$ are not equivalent and

$$T'([\eta]) = \tilde{\chi} \circ T(\eta) \neq \tilde{\chi} \circ T(\theta) = T'([\theta]).$$

This proves the injectivity of T' and thus T' is a continuous bijection. Applying the same argument to T^{-1} provides a continuous inverse to T', so T' is a homeomorphism. Finally, for every $[\eta] \in \tilde{X}$, T' preserves the covering map Π:

$$\Pi \circ T'[\eta] = \Pi \circ \tilde{\chi} \circ T(\eta) = \chi \circ d\pi \circ T(\eta) = \chi \circ d\pi(\eta) = \Pi \circ \tilde{\chi}(\eta) = \Pi[\eta].$$

Thus, T' is a Deck transformation of Π. Note that T and T' satisfy the above diagram by the definition of T'. ■

With this information, we can write the relationship between the quotient models of both geodesic flows. The covering map Π provides a time-preserving semi-conjugacy between $\tilde{\psi}$ and ψ: for every $t \in \mathbb{R}$,

$$\tilde{\Pi} \circ \tilde{\psi}_t = \psi_t \circ \tilde{\Pi}, \qquad \begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\psi}_t} & \tilde{X} \\ \downarrow{\scriptstyle \tilde{\Pi}} & & \downarrow{\scriptstyle \tilde{\Pi}} \\ X & \xrightarrow{\psi_t} & X. \end{array}$$

Therefore we transferred the relationships between geodesic flows to analog relationships between their corresponding quotient models.

3.5
Topological dimension of the quotient space

This section is devoted to show that the topological dimension of the quotient space is at least two.

We first define the topological dimension [35]. Let X be a topological space and U_i be any open cover of X. The order of U_i is the smallest integer n such that every $x \in X$ belongs to at most n sets of U_i. An open refinement of U_i is another open cover, each of whose sets is a subset of a set in U_i. The Lebesgue covering dimension or topological dimension of X is the minimum n such that any U_i has an open refinement of order $n+1$ or less. If no such minimal n exists, X has infinite topological dimension.

We have as standard examples the open sets of $\mathbb{R}^n$. For every open set $U \subset \mathbb{R}^n$, the topological dimension of U is n.

The following theorem says that topological dimension is a topological invariant. That is, the topological dimension is preserved by homeomorphisms.

Theorem 3.3 ([69]). *Let $f : X \to Y$ be a homeomorphism between topological spaces. Then, the topological dimension of X and Y are equal.*

Let X be a topological space and $f : X \to R^2$ be a continuous map and $U \subset X$ be an open set. We say that U is a topological surface if the restriction of f to U is a homeomorphism. The above theorem implies that every topological surface has topological dimension 2.

Let $\tilde{X}$ and X be the quotient spaces defined in Section 3.2. To show that $\tilde{X}$ has topological dimension at least two, for every $[\theta] \in \tilde{X}$, we will find a topological surface passing through $[\theta]$.

Lemma 3.5.1. *Let M be a compact surface without conjugate points and genus greater than one, and $\tilde{M}$ be its universal covering. If $\tilde{X}$ and X are the quotient spaces defined in Section 3.2 then*

1. *For every $[\theta] \in \tilde{X}$ there exists a topological surface $S_{[\theta]}$ containing $[\theta]$.*
2. *$\tilde{X}$ and X have topological dimension at least two.*

Proof. Let $\theta \in T_1\tilde{M}$ and V_θ be the vertical fiber of θ. Using the geodesic flow we define the set

$$W_\theta = \bigcup_{t\in\mathbb{R}} \phi_t(V_\theta).$$

Since V_θ is homeomorphic to the circle S^1, we conclude that W_θ is homeomorphic to a cylinder, i.e., $W_\theta \subset T_1\tilde{M}$ is a topological surface. We next show that for every $\theta \in T_1\tilde{M}$, $\tilde{\chi}$ maps W_θ bijectively. The divergence of geodesic rays guarantees that for every $\eta, \xi \in V_\theta$,

$$\eta \notin \tilde{\mathcal{I}}(\xi).$$

So, the restriction of $\tilde{\chi}$ to W_θ is injective, hence bijective onto its image. This implies that $\tilde{\chi}(W_\theta) \subset \tilde{X}$ is homeomorphic to a cylinder. Thus, for every $\theta \in T_1\tilde{M}$, there exists a topological surface $\tilde{\chi}(W_\theta)$ containing $[\theta]$. It follows that the topological dimension of $\tilde{X}$ is at least two. This conclusion extends to X because $\tilde{X}$ and X are locally homeomorphic. ■

In [13, 14], Gelfert-Ruggiero showed that X and $\tilde{X}$ are topological 3-manifolds for:

– Compact surfaces without focal points and genus greater than one.

– Compact surfaces without conjugate points, genus greater than one and continuous Green bundles.

These cases are included in our context: compact surfaces without conjugate points and genus greater than one. Thus, the above lemma is a weaker result than the one obtained by Gelfert and Ruggiero. It remains as an open problem to know exactly when the quotient space is a compact 3-manifold.

3.6 Topological dynamics of the quotient flow

We continue supposing a compact surface M without conjugate points and genus greater than one. In Section 2, we defined a quotient model: a continuous flow $\psi_t : X \to X$ time-preserving semi-conjugate to the geodesic flow. In this section, we want to show some dynamical properties for the quotient model. These properties are somehow induced by corresponding weak properties of ϕ_t through the semi-conjugacy χ. Indeed, ϕ_t is not expansive and does not have a local product in the general case. Instead, ϕ_t has geometrical properties related to expansiveness and the local product. These geometrical properties are exactly expansiveness and local product in particular cases: manifolds of negative curvature and manifolds with Anosov geodesic flow. The main impediment for this to happen in the general case is the existence of strips of bi-asymptotic orbits. Since the quotient collapses these strips into single orbits, we expect these properties for the quotient flow. We summarize the main dynamical properties that we prove in the section.

Theorem 3.4. *Let M be a compact surface without conjugate points of genus greater than one and $\psi_t : X \to X$ be the quotient flow. Then, ψ_t is topologically mixing, expansive and has a local product structure. Moreover, ψ_t has the pseudo-orbit tracing and specification properties.*

We prove the theorem in a series of auxiliary lemmas below. We show the dynamical properties in the order stated in the theorem. So, we begin with the topological mixing property. The continuity of χ allows us to project the topological mixing property onto the quotient space.

Lemma 3.6.1. *The quotient flow ψ_t is topologically mixing.*

Proof. Let $U, V \subset X$ be two open sets. Clearly, $\chi^{-1}(U)$ and $\chi^{-1}(V)$ are open sets in T_1M. By Theorem 2.15, the geodesic flow is topologically transitive. From the definition of the topological mixing property given in Subsection

2.3.1, there exists $T \geq 0$ such that

$$\phi_t(\chi^{-1}(U)) \cap \chi^{-1}(V) \neq \emptyset, \qquad \text{for every } |t| \geq T.$$

Applying the time-preserving semi-conjugacy χ we get that ψ_t is topologically mixing:

$$\psi_t(U) \cap (V) \neq \emptyset, \qquad \text{for every } |t| \geq T.$$

■

We now prove a lemma relating the Sasaki and quotient distance. The continuity of χ says that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\eta, \xi \in T_1M$ with

$$d_s(\eta, \xi) \leq \delta \quad \text{then} \quad d([\eta], [\xi]) \leq \epsilon.$$

Since χ has no inverse, we do not have a similar property in the reverse direction. However, for every $\eta \in T_1M$, Lemma 3.3.4 gives a relationship between basic open sets of $[\eta]$, and special neighborhoods of $\mathcal{I}(\eta)$. We use this basic open sets to get a relationship between Sasaki and quotient distance in a special case.

Lemma 3.6.2. *There exist $r_0, s_0 > 0$ such that for every $[\xi], [\eta] \in X$ with $d([\xi], [\eta]) \leq r_0$ then*

$$d_s(\tilde{\xi}, \tilde{\eta}) \leq Q + s_0,$$

for some lifts $\tilde{\xi}, \tilde{\eta} \in T_1\tilde{M}$ of $\xi, \eta \in T_1M$ respectively, where Q is Morse's constant.

Proof. We will consider the basic open sets $A(\eta, \epsilon, \delta, \tau)$ provided by Lemma 3.3.4. For every $\theta \in T_1M$, choose $\epsilon, \delta, \tau > 0$ small enough so that $A(\theta, \epsilon, \delta, \tau)$ is evenly covered by $d\pi$. Clearly, the family

$$\mathcal{A} = \{\chi(A(\theta, \epsilon, \delta, \tau)) : \theta \in T_1M\}$$

is an open cover of X. Since X is compact, let $r_0 > 0$ be the Lebesgue number of $\mathcal{A}$. Thus, for every $[\eta], [\xi] \in X$ with $d([\eta], [\xi]) \leq r_0$, there exists $\theta \in T_1M$ such that

$$[\xi] \in B([\eta], r_0) \subset \chi(A(\theta, \epsilon, \delta, \tau)) \in \mathcal{A},$$

where $B([\eta], r_0)$ is the closed ball of radius r_0 centered at $[\eta]$. Seeing that $A(\theta, \epsilon, \delta, \tau)$ is evenly covered by $d\pi$, for every lift $\tilde{\theta} \in T_1\tilde{M}$ of θ, there exist lifts

$$\tilde{A}(\tilde{\theta}, \epsilon, \delta, \tau) \quad \text{and} \quad \tilde{\eta}, \tilde{\xi} \in \tilde{A}(\tilde{\theta}, \epsilon, \delta, \tau)$$

of $A(\theta, \epsilon, \delta, \tau), \eta, \xi$ respectively. Since we chose ϵ, δ, τ small enough, there exists $s_0 > 0$ such that

$$Diam(\tilde{A}(\tilde{\theta}, \epsilon, \delta, \tau)) \leq Q + s_0 \quad \text{hence} \quad d_s(\tilde{\eta}, \tilde{\xi}) \leq Q + s_0.$$

■

The following lemma relates the time parameter of the quotient flow with the quotient distance. We use this property to bound the time parameter when proving expansivity. Thus, for two points in the same orbit, small distance implies small parameter time.

Lemma 3.6.3. *For every $\epsilon > 0$ there exists $\delta > 0$ such that if $[\eta] \in X, \tau \in \mathbb{R}$ with*

$$d([\xi], \psi_\tau[\xi]) \leq \delta \quad \textit{then} \quad |\tau| \leq \epsilon.$$

Proof. By contradiction, suppose there exist $\delta_0 > 0$ and sequences $[\xi_n] \in X, \tau_n \in \mathbb{R}$ such that for every $n \geq 1$

$$d([\xi_n], \psi_{\tau_n}[\xi_n]) \leq \frac{1}{n} \quad \text{and} \quad |\tau_n| \geq \epsilon_0. \tag{3.2}$$

By choosing subsequences, we can assume that $\tau_n \to T$ and $[\xi_n] \to [\xi]$. Since ψ_t is continuous we see that $\psi_{\tau_n}[\xi_n] \to \psi_T[\xi]$. While on the other hand Equation (3.2) says that $[\xi_n]$ and $\psi_{\tau_n}[\xi_n]$ converge to the same limit $[\xi] = \psi_T[\xi]$. This conclusion holds if and only if $T = 0$. Thus $\tau_n \to 0$, which contradicts Equation (3.2) and proves the lemma. ■

We now show the expansivity of the quotient flow. We first recall the definition given in Subsection 2.2.1. Let $\phi_t : X \to X$ be a continuous flow acting on a compact metric space. We say that ϕ_t is **expansive** if there exists $\epsilon > 0$ such that if $x, y \in X$ satisfy

$$d(\phi_t(x), \phi_{\rho(t)}(y)) \leq \epsilon \text{ for every } t \in \mathbb{R} \text{ and some reparametrization } \rho,$$

then there exists $\tau \in [-\epsilon, \epsilon]$ with $y = \phi_\tau(x)$.

We remark that ϕ might not be expansive. Indeed, for every non-expansive point $\eta \in T_1M$, and every $\xi \in \mathcal{I}(\eta) \setminus \{\eta\}$, for some $\epsilon > 0$

$$d_s(\phi_t(\eta), \phi_t(\xi)) \leq \epsilon \quad \text{for every } t \in \mathbb{R}.$$

Thus the strips of bi-asymptotic orbits

$$\bigcup_{t\in\mathbb{R}} \phi_t(\mathcal{I}(\eta))$$

obstructs expansivity. By collapsing these strips, we expect the expansivity in the quotient model.

Lemma 3.6.4. *The quotient flow ψ_t is expansive.*

Proof. Let $r_0 > 0$ be given by Lemma 3.6.2. We first show that if there are two quotient orbits having Hausdorff distance bounded by r_0, then the orbits are the same. More precisely, consider $[\eta], [\xi] \in X$ such that for some reparametrization ρ,

$$d(\psi_t[\eta], \psi_{\rho(t)}[\xi]) \leq r_0, \text{ for every } t \in \mathbb{R}.$$

By Lemma 3.6.2, there exist lifts $\tilde{\eta}, \tilde{\xi} \in T_1\tilde{M}$ of η, ξ such that

$$d_s(\phi_t(\tilde{\eta}), \phi_{\rho(t)}(\tilde{\xi})) \leq Q + s_0, \text{ for every } t \in \mathbb{R}.$$

Thus, the orbits of $\tilde{\eta}$ and $\tilde{\xi}$ have Hausdorff distance bounded by $Q + s_0$, hence the orbits are bi-asymptotic. This implies that there exists $\tau \in \mathbb{R}$ so that

$$\tilde{\xi} \in \tilde{\mathcal{I}}(\phi_\tau(\tilde{\eta})) \quad \text{hence} \quad [\xi] = \psi_\tau[\eta].$$

Now, given $\epsilon > 0$, Lemma 3.6.3 provides $\delta_1 > 0$ satisfying its statement. Choose $\delta = \min(\delta_1, r_0)$. If the orbits of $[\eta]$ and $[\xi]$ have Hausdorff distance bounded by $\delta \leq r_0$ then $[\xi] = \psi_\tau[\eta]$. Moreover $|\tau| \leq \epsilon$ because

$$d([\eta], \psi_\tau[\eta]) = d([\eta], [\xi]) \leq \delta \leq \delta_1.$$

■

3.6.1 Local product structure

We now turn to the local product structure problem. We first recall some definitions given in Subsection 2.2.1. Let $\phi_t : X \to X$ be a continuous flow acting on compact metric space. For each $\epsilon > 0$, the ϵ-strong stable set of $x \in X$ is defined by

$$W^{ss}_\epsilon(x) = \{y \in X : \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)) = 0, d(\phi_t(x), \phi_t(y)) \leq \epsilon, \text{ for each } t \geq 0\}.$$

The ϵ-strong unstable set $W_\epsilon^{uu}(x)$ is defined analogously considering negative times. We say that ϕ_t has a local product structure if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $x, y \in X$ satisfies $d(x,y) \leq \delta$ then there exists a unique $\tau \in \mathbb{R}$ with $|\tau| \leq \epsilon$ and

$$W_\epsilon^{ss}(x) \cap W_\epsilon^{uu}(\phi_\tau(y)) \neq \emptyset.$$

Even though ϕ_t has no local product in the general case, ϕ has a related property. In fact, the strong stable and strong unstable sets of ϕ_t are complicated: these sets may have low regularity or even be empty. Instead, we replace the strong sets with the horospherical leaves, which have weak hyperbolic properties. Accordingly, the intersection between strong sets is replaced with the heteroclinic connections of $\tilde{\phi}_t$: for every $\eta, \xi \in T_1\tilde{M}$ with $\xi \notin \tilde{\mathcal{F}}^{cu}(-\eta)$, there exists $\theta \in T_1\tilde{M}$ such that

$$\tilde{\mathcal{F}}^s(\eta) \cap \tilde{\mathcal{F}}^{cu}(\xi) = \tilde{\mathcal{I}}(\theta). \tag{3.3}$$

Though these intersections always exist, in general they are not unique. This is because $\tilde{\mathcal{I}}(\theta)$ may be nontrivial, and also generates a strip of bi-asymptotic orbits.

To speak properly of the local product of ψ_t, we need to identify of strong sets of ψ_t. We see natural candidates: the images of the horospherical leaves through χ. For every $\eta \in T_1M$, we define

$$V^s[\eta] = \chi(\mathcal{F}^s(\eta)) \quad \text{and} \quad V^u[\eta] = \chi(\mathcal{F}^u(\eta)),$$
$$V^{cs}[\eta] = \chi(\mathcal{F}^{cs}(\eta)) \quad \text{and} \quad V^{cu}[\eta] = \chi(\mathcal{F}^{cu}(\eta)).$$

Since χ is a time-preserving semi-conjugation, it follows that

$$V^{cs}[\eta] = \bigcup_{t\in\mathbb{R}} V^s(\psi_t[\eta]) \quad \text{and} \quad V^{cu}[\eta] = \bigcup_{t\in\mathbb{R}} V^u(\psi_t[\eta]).$$

Since χ is continuous and the horospherical foliations are minimal by Theorem 2.15, we see that for every $[\eta] \in X$, $V^s[\eta]$, $V^{cs}[\eta]$, $V^u[\eta]$ and $V^{cu}[\eta]$ are dense in X.

To deal with the local product, we consider connected components of $V^s[\eta]$ and $V^u[\eta]$. For every $[\eta] \in X$ and every open set $U \subset X$ containing $[\eta]$, we denote by $V^s[\eta] \cap U_c$ the connected component of $V^s[\eta] \cap U$ containing $[\eta]$. Similarly, we write $V^u[\eta] \cap U_c$, $V^{cs}[\eta] \cap U_c$ and $V^{cu}[\eta] \cap U_c$. Thus, let $[\eta], [\xi] \in X$ close enough. If there exists an open set $U \subset X$ with $[\eta], [\xi] \in U$ such that $V^s[\eta] \cap U_c$ and $V^{cu}[\eta] \cap U_c$ intersect then we define

$$V^s[\eta] \cap V^{cu}[\xi] = (V^s[\eta] \cap U_c) \cap (V^{cu}[\eta] \cap U_c). \tag{3.4}$$

By the heteroclinic connections given in Equation (3.3), there exist $\tilde{\eta}, \tilde{\xi} \in T_1\tilde{M}$ lifts of η and ξ so that $\tilde{\xi} \notin \tilde{\mathcal{F}}^{cu}(-\tilde{\eta})$. Moreover, there exist $\tilde{\theta} \in T_1\tilde{M}$ and $\tau \in \mathbb{R}$ such that

$$\tilde{\mathcal{F}}^s(\tilde{\eta}) \cap \tilde{\mathcal{F}}^{cu}(\tilde{\xi}) = \tilde{\mathcal{F}}^s(\tilde{\eta}) \cap \tilde{\mathcal{F}}^u(\tilde{\phi}_\tau(\tilde{\xi})) = \tilde{\mathcal{I}}(\tilde{\theta}).$$

Setting $\theta = d\pi(\tilde{\theta}) \in T_1M$, we have

$$V^s[\eta] \cap V^{cu}[\xi] = V^s[\eta] \cap V^u(\psi_\tau[\xi]) = \chi \circ d\pi(\tilde{\mathcal{I}}(\tilde{\theta})) = [\mathcal{I}(\theta)] = [\theta].$$

This equation shows that the intersection is always unique when it exists.

We always mean the definition given in Equation (3.4) when we speak about intersection $V^s[\eta] \cap V^{cu}[\xi]$. Roughly speaking, intersection of the connected components of $V^s[\eta]$ and $V^{cu}[\xi]$ containing $[\eta]$ and $[\xi]$. Thus, we will build the local product of ψ_t from the sets $V^s[\eta]$ and $V^u[\eta]$. The remaining lemmas help to work out the details of the local product.

The following lemma is a useful technical result that helps in contradiction arguments. For every $[\xi] \in X$ and $r > 0$, let $B([\xi], r)$ be the open ball of radius r centered at $[\xi]$.

Lemma 3.6.5. *Let $r_0 > 0$ be given by Lemma 3.6.2. There cannot exist $\epsilon_0 > 0$ and sequences $[\eta_n], [\xi_n] \in X$, $t_n \in \mathbb{R}$ such that $t_n \to \infty$ and for every $n \geq 1$, $[\eta_n] \in V^s[\xi_n]$, $[\eta_n]$ belongs to the connected component of $V^s[\xi_n] \cap B([\xi_n], r_0)$ containing $[\xi_n]$,*

$$d([\eta_n], [\xi_n]) \leq r_0 \quad \text{and} \quad d(\psi_{t_n}[\eta_n], \psi_{t_n}[\xi_n]) \geq \epsilon_0. \tag{3.5}$$

An analog statement holds for the unstable case.

Proof. By contradiction, suppose there exist the objects of the statement satisfying Equation (3.5). Lemma 3.6.2 says that there exist lifts $\tilde{\eta}_n, \tilde{\xi}_n \in T_1\tilde{M}$ of η_n, ξ_n such that for every $n \geq 1$,

$$d_s(\tilde{\eta}_n, \tilde{\xi}_n) \leq Q + s_0.$$

We claim that for every $n \geq 1$, $\tilde{\eta}_n \in \tilde{\mathcal{F}}^s(\tilde{\xi}_n)$. Otherwise, for some covering isometry T, $T(\tilde{\eta}_n) \in \tilde{\mathcal{F}}^s(\tilde{\xi}_n)$. Hence $[\eta_n] = [d\pi(T(\tilde{\eta}_n))]$ does not belong to the connected component of $V^s[\xi_n] \cap B([\xi_n], r_0)$ containing $[\xi_n]$ and the claim is proved.

By Proposition 2.3.7 there exist $A, B > 0$ such that for every $n \in \mathbb{N}$,

$$d_s(\phi_t(\tilde{\eta}_n), \phi_t(\tilde{\xi}_n)) \leq A d_s(\tilde{\eta}_n, \tilde{\xi}_n) + B \leq A(Q + s_0) + B = C, \text{ for every } t \geq 0.$$

This implies that for every $n \in \mathbb{N}$,

$$d_s(\phi_t(\phi_{t_n}\tilde{\eta}_n), \phi_t(\phi_{t_n}\tilde{\xi}_n)) \leq C, \text{ for every } t \geq -t_n. \tag{3.6}$$

By choosing subsequences and using covering isometries we can assume that

$$\phi_{t_n}(\tilde{\eta}_n) \to \tilde{\eta} \quad \text{and} \quad \phi_{t_n}(\tilde{\xi}_n) \to \tilde{\xi}. \tag{3.7}$$

By invariance of the horospherical foliations we see that $\phi_{t_n}(\tilde{\eta}_n) \in \tilde{\mathcal{F}}^s(\phi_{t_n}(\tilde{\xi}_n))$. Moreover, the continuity of the horospherical foliations yields that $\tilde{\eta} \in \tilde{\mathcal{F}}^s(\tilde{\xi})$. Now, choose any fixed $t \in \mathbb{R}$. Since $t_n \to \infty$, $t \geq -t_n$ for large enough n, and hence Equation (3.6) yields

$$d_s(\phi_t(\phi_{t_n}\tilde{\eta}_n), \phi_t(\phi_{t_n}\tilde{\xi}_n)) \leq C.$$

By continuity we obtain

$$d_s(\phi_t(\tilde{\eta}), \phi_t(\tilde{\xi})) \leq C, \quad \text{for every } t \in \mathbb{R}.$$

As $\tilde{\eta} \in \tilde{\mathcal{F}}^s(\tilde{\xi})$, Corollary 2.3.1 shows that $\tilde{\eta} \in \mathcal{I}(\tilde{\xi})$ hence $[\eta] = [\xi]$. Applying the map $\chi \circ d\pi$ to the sequences in Equation (3.7), we get

$$\chi \circ d\pi(\phi_{t_n}(\tilde{\eta}_n)) \to \chi \circ d\pi(\tilde{\eta}) \quad \text{and} \quad \chi \circ d\pi(\phi_{t_n}(\tilde{\xi}_n)) \to \chi \circ d\pi(\tilde{\xi}),$$

$$\psi_{t_n}[\eta_n] \to [\eta] \quad \text{and} \quad \psi_{t_n}[\xi_n] \to [\xi].$$

Therefore, $d(\psi_{t_n}[\eta_n], \psi_{t_n}[\xi_n]) \to 0$ as $n \to \infty$. This contradicts Equation (3.5) and proves the lemma. ∎

An intermediate result to show the relationship between $W^{ss}[\eta]$ and $W^{uu}[\eta]$, and $V^s[\eta]$ and $V^u[\eta]$, is the so-called **uniform contraction**. We prove this contraction for $V^s[\eta]$ and $V^u[\eta]$, but only for distances smaller than r_0.

Lemma 3.6.6. *Let $r_0 > 0$ be given by Lemma 3.6.2. For every $\epsilon > 0$ and $D \in (0, r_0]$ there exists $T > 0$ such that if $[\eta] \in V^s[\xi]$, $[\eta]$ belongs to the connected component of $V^s[\xi] \cap B([\xi], r_0)$ containing $[\xi]$ and $d([\eta], [\xi]) \leq D$, then*

$$d(\psi_t[\eta], \psi_t[\xi]) \leq \epsilon \quad \textit{for every } t \geq T.$$

An analog result holds for the unstable case.

Proof. By contradiction suppose there exist $\epsilon_0 > 0$, $D_0 \in (0, r_0]$ and sequences $[\eta_n], [\xi_n] \in X$, $t_n \in \mathbb{R}$, such that $t_n \to \infty$ and for every $n \geq 1$, $[\eta_n] \in V^s[\xi_n]$, $[\eta_n]$ belongs to the connected component of $V^s[\xi_n] \cap B([\xi_n], r_0)$ containing $[\xi_n]$,

$$d([\eta_n], [\xi_n]) \leq D_0 \leq r_0 \quad \text{and} \quad d(\psi_{t_n}[\eta_n], \psi_{t_n}[\xi_n]) \geq \epsilon_0.$$

This contradicts Lemma 3.6.5 and proves the statement. ∎

As an immediate consequence we see that $V^s[\eta]$ and $V^u[\eta]$ agree with the strong sets of ψ_t locally for distances smaller than r_0.

Lemma 3.6.7. *Let $r_0 > 0$ be given by Lemma 3.6.2. If $[\eta] \in V^s[\xi]$, $[\eta]$ belongs to the connected component of $V^s[\xi] \cap B([\xi], r_0)$ containing $[\xi]$ and $d([\eta], [\xi]) \leq r_0$ then*

$$d(\psi_t[\eta], \psi_t[\xi]) \to 0 \quad \text{as} \quad t \to \infty.$$

In particular, $[\eta] \in W^{ss}[\xi]$. An analog statement holds for the unstable case.

Proof. For every $n \geq 1$, set $\epsilon_n = 1/n$ and $D = r_0$ in the last lemma. So, there exists a sequence $T_n \to \infty$ such that

$$d(\psi_t[\eta], \psi_t[\xi]) \leq \frac{1}{n} \quad \text{for every } t \geq T_n.$$

This implies that $d(\psi_t[\eta], \psi_t[\xi]) \to 0$ as $t \to \infty$. ∎

The local product requires not only the intersection of strong sets $W^{ss}[\eta]$ and $W^{uu}[\eta]$ but the intersection of ϵ-strong sets $W^{ss}_\epsilon[\eta]$ and $W^{uu}_\epsilon[\eta]$ for $\epsilon > 0$. The following lemma sets a criteria to identify points of $W^{ss}_\epsilon[\eta]$ and $W^{uu}_\epsilon[\eta]$. Thus, close enough points in $V^s[\eta]$ have their future orbits in a ϵ-tubular neighborhood.

Lemma 3.6.8. *Let $r_0 > 0$ be given by Lemma 3.6.2. For every $\epsilon > 0$ there exists $\delta \in (0, r_0]$ such that if $[\eta] \in V^s[\xi]$, $[\eta]$ belongs to the connected component of $V^s[\xi] \cap B([\xi], r_0)$ containing $[\xi]$ and $d([\eta], [\xi]) \leq \delta$, then*

$$d(\psi_t[\eta], \psi_t[\xi]) \leq \epsilon \quad \text{for every } t \geq 0.$$

An analog result holds for the unstable case.

Proof. By contradiction suppose there exist $\epsilon_0 > 0$ and sequences $[\eta_n], [\xi_n] \in X$, $\delta_n \in (0, r_0]$ and $t_n \in \mathbb{R}$, such that $\delta_n \to 0$ and for every $n \geq 1$, $[\eta_n] \in V^s[\xi_n]$,

$[\eta_n]$ belongs to the connected component of $V^s[\xi_n] \cap B([\xi_n], r_0)$ containing $[\xi_n]$,

$$d([\eta_n], [\xi_n]) \leq \delta_n \leq r_0 \quad \text{and} \quad d(\psi_{t_n}[\eta_n], \psi_{t_n}[\xi_n]) \geq \epsilon_0. \tag{3.8}$$

We claim that $t_n \to \infty$. Otherwise, t_n is bounded and by choosing a subsequence we can assume that $t_n \to T \in \mathbb{R}$. By choosing subsequences, Equation (3.8) implies that $[\eta_n]$ and $[\xi_n]$ converge to the same limit $[\eta] \in X$. Since ψ_t is continuous, $\psi_{t_n}[\eta_n]$ and $\psi_{t_n}[\xi_n]$ converge to the same limit $\psi_T[\eta]$. This contradicts Equation (3.8) and proves the claim. Having $t_n \to \infty$, Equation (3.8) contradicts Lemma 3.6.5 and proves the lemma. ■

Combining this with Lemma 3.6.7, we get that $W^{ss}_\epsilon[\eta]$ and $W^{uu}_\epsilon[\eta]$ agree with $V^s[\eta]$ and $V^u[\eta]$ locally.

Regarding the geodesic flow, if $\tilde{\eta}, \tilde{\xi} \in T_1\tilde{M}$ are close enough then the intersection

$$\tilde{\mathcal{I}}(\tilde{\theta}) = \tilde{\mathcal{F}}^s(\tilde{\eta}) \cap \tilde{\mathcal{F}}^{cu}(\tilde{\xi})$$

is close to $\tilde{\eta}$ and $\tilde{\xi}$ [33]. The following lemma says that the same holds for $V^s[\eta]$ and $V^u[\eta]$, i.e., if $[\eta]$ and $[\xi]$ are close enough then the intersection $[\theta]$ is close to $[\eta]$ and $[\xi]$.

Lemma 3.6.9. *For every $\epsilon > 0$ there exists $\delta > 0$ such that if $[\eta], [\xi] \in X$, $[\theta] \in V^s[\eta] \cap V^u(\psi_\tau[\xi])$ and $d([\eta], [\xi]) \leq \delta$, then*

$$d([\theta], [\eta]) \leq \epsilon, \quad d([\theta], \psi_\tau[\xi]) \leq \epsilon \quad \textit{and} \quad |\tau| \leq \epsilon.$$

Proof. By contradiction suppose there exist $\epsilon_0 > 0$ and sequences $[\eta_n], [\xi_n], [\theta_n] \in X$, $\tau_n \in \mathbb{R}$ such that for every $n \geq 1$, $[\theta_n] \in V^s[\eta_n] \cap V^u(\psi_{\tau_n}[\xi_n])$, $|\tau_n| \geq \epsilon_0$,

$$d([\eta_n], [\xi_n]) \leq \frac{1}{n}, \quad d([\theta_n], [\eta_n]) \geq \epsilon_0 \quad \text{and} \quad d([\theta_n], \psi_{\tau_n}[\xi_n]) \geq \epsilon_0. \tag{3.9}$$

Given $r_0 > 0$ from Lemma 3.6.2, for every $n \geq 1$ large enough,

$$d([\eta_n], [\xi_n]) \leq \frac{1}{n} \leq r_0.$$

So, we can choose lifts $\tilde{\eta}_n, \tilde{\xi}_n, \tilde{\theta}_n$ of η_n, ξ_n, θ_n such that

$$d_s(\tilde{\eta}_n, \tilde{\xi}_n) \leq Q + s_0$$

and $\tilde{\theta}_n$ belongs to the fundamental domain containing $\tilde{\eta}_n$ and $\tilde{\xi}_n$. We claim that for every $n \geq 1$,

$$\tilde{\theta}_n \in \tilde{\mathcal{F}}^s(\tilde{\eta}_n) \cap \tilde{\mathcal{F}}^u(\phi_{\tau_n}(\tilde{\xi}_n)). \tag{3.10}$$

Otherwise there exist sequences of covering isometries T_n, T'_n such that $\tilde{\theta}_n \in \tilde{\mathcal{F}}^s(T_n(\tilde{\eta}_n)) \cap \tilde{\mathcal{F}}^u(\phi_{\tau_n}(T'_n(\tilde{\xi}_n)))$. Thus, there exists an open set U containing $[\eta_n]$ such that $[\theta_n]$ does not belong to the connected component of $V^s[\eta_n] \cap U$ containing $[\eta_n]$. Similarly, $[\theta_n]$ does not belong to the connected component of $V^{cu}[\xi_n] \cap U$ containing $[\xi_n]$. This contradicts the definition of intersection and proves the claim.

Thus, using the same covering isometries for all sequences and choosing suitable subsequences, we can assume that

$$\tilde{\eta}_n \to \tilde{\eta}, \quad \tilde{\xi}_n \to \xi, \quad \tilde{\theta}_n \to \tilde{\theta} \quad \text{and} \quad \tau_n \to T.$$

Since we used the same covering isometries for the sequences, the continuity of the horospherical foliations applied to Equation (3.10) yields

$$\tilde{\theta} \in \tilde{\mathcal{F}}^s(\tilde{\eta}) \cap \tilde{\mathcal{F}}^u(\phi_T(\tilde{\xi})). \tag{3.11}$$

We claim that $\tilde{\eta} \in \mathcal{I}(\tilde{\xi})$. Otherwise $d([\eta], [\xi]) > 0$. But applying the limit to Equation (3.9), we get $d([\eta], [\xi]) = 0$ and the claim is proved. The claim and Equation (3.11) provide that

$$\tilde{\theta} \in \tilde{\mathcal{F}}^s(\tilde{\eta}) \cap \tilde{\mathcal{F}}^u(\phi_T(\tilde{\eta})).$$

From this, Corollary 2.3.1 shows that $\tilde{\theta} \in \mathcal{I}(\tilde{\eta})$. Therefore $[\theta_n]$ and $[\eta_n]$ converge to the same limit $[\theta] = [\eta]$, which contradicts Equation (3.9). This provides $\delta_1 > 0$ such that $d([\theta], [\eta]) \leq \epsilon$. A similar reasoning provides $\delta_2 > 0$ such that $d([\theta], \psi_\tau[\xi]) \leq \epsilon$. Finally, we see that

$$\tilde{\theta} \in \mathcal{I}(\tilde{\eta}) \subset \tilde{\mathcal{F}}^u(\tilde{\eta}), \quad \text{hence} \quad \tilde{\theta} \in \tilde{\mathcal{F}}^u(\tilde{\eta}) \cap \tilde{\mathcal{F}}^u(\phi_T(\tilde{\eta})).$$

This holds if and only if $T = 0$. Therefore $\tau_n \to 0$, contradicting $|\tau_n| \geq \epsilon_0$. We thus get $\delta_3 > 0$ such that $|\tau| \leq \epsilon$. Choosing $\delta = \min(\delta_1, \delta_2, \delta_3)$ we obtain the result. ∎

This result is important because local product requires that close enough points have their intersection near them. Moreover, this lemma allows us to apply previous lemmas if we choose close enough points.

We now gather all the lemmas and show that ψ_t has local product.

Lemma 3.6.10. *The quotient flow ψ_t has a local product structure.*

Proof. Let $r_0 > 0$ be given by Lemma 3.6.2 and $\epsilon \in (0, r_0]$. Consider $[\eta], [\xi] \in X$ and $[\theta] \in V^s[\eta] \cap V^u(\psi_\tau[\xi])$. By Lemma 3.6.8 there exists $\delta_1 > 0$ such that if

$d([\theta],[\eta]) \leq \delta_1$ and $d([\theta],\psi_\tau[\xi]) \leq \delta_1$ then

$$d(\psi_t[\theta],\psi_t[\eta]) \leq \epsilon \quad \text{and} \quad d(\psi_{-t}[\theta],\psi_{-t}\psi_\tau[\xi]) \leq \epsilon, \quad \text{for} \quad t \geq 0. \tag{3.12}$$

For the same $\epsilon > 0$, Lemma 3.6.4 provides $\delta_2 > 0$ such that expansivity holds. Set $\delta_m = \min(\delta_1, \delta_2, \epsilon)$. By Lemma 3.6.9, for $\delta_m > 0$ there exists $\delta > 0$ such that if $[\eta],[\xi] \in X$, $[\theta] \in V^s[\eta] \cap V^u(\psi_\tau[\xi])$ and $d([\eta],[\xi]) \leq \delta$ then

$$d([\theta],[\eta]) \leq \delta_m, \quad d([\theta],\psi_\tau[\xi]) \leq \delta_m \quad \text{and} \quad |\tau| \leq \delta_m \leq \epsilon.$$

From this and $\delta_m \leq \epsilon \leq r_0$, Lemma 3.6.7 implies that

$$[\theta] \in W^{ss}[\eta] \cap W^{uu}(\psi_\tau[\xi]).$$

Furthermore, since $\delta_m \leq \delta_1$, we deduce that $[\theta],[\eta],[\xi]$ satisfy Equation (3.12) and hence

$$[\theta] \in W^{ss}_\epsilon[\eta] \cap W^{uu}_\epsilon(\psi_\tau[\xi]) \quad \text{and} \quad |\tau| \leq \epsilon.$$

Since $\delta_m \leq \delta_2$, expansivity guarantees the above intersection is unique. Finally, τ is unique because classes $\mathcal{I}(\theta)$ are transverse to the geodesic flow. Therefore ψ_t has a local product. ■

Finally, pseudo-orbit tracing and specification properties are consequences of previous dynamical properties. More precisely,

1. By Theorem 7.1 of [70], if ψ_t is expansive and has local product then ψ_t has the pseudo-orbit tracing property.
2. By Proposition 6.2 of [13], if ψ_t is expansive, topological mixing and has the pseudo-orbit tracing property then ψ_t has the specification property.

3.7 Existence and uniqueness of the measure of maximal entropy for the geodesic flow

This section deals with the problem of the existence and uniqueness of the measure of maximal entropy for the geodesic flow. We will see that existence is an easier problem than the uniqueness. Thus, we first show that the existence of the measure of maximal entropy through the h-expansiveness property. Next, we deal with uniqueness using the quotient model. This model allows to build the unique measure of maximal entropy for the geodesic flow. This approach is an adaptation of the procedure of Gelfert and Ruggiero [13].

In our setting, the geodesic flow always has a measure of maximal entropy. Indeed, the classical Newhouse's result [71] implies that every smooth flow acting on a compact smooth Riemannian manifold always has a measure of maximal entropy. Since T_1M and the geodesic flow are smooth, there exists at least one measure of maximal entropy for the geodesic flow. However, in our setting we can say more. To do this, recall that $\mathcal{M}(\phi)$ is the set of all Borel probability measures on T_1M invariant by the geodesic flow. The **entropy map** associated to the geodesic flow ϕ_t is given by

$$h : \phi \to \mathbb{R}$$
$$\mu \mapsto h_\mu \phi_1.$$

By the variational principle, a measure of maximal entropy is a global maximum for this entropy map. In this regard, we note that $\mathcal{M}(\phi_t)$ is compact in the weak* topology because T_1M is a compact metric space. Theorem 8.7 of [38] asserts that if $\mathcal{M}(\phi)$ is compact and the entropy map is upper semi-continuous then the entropy map always has a global maximum. The entropy map is upper semi-continuous for instance if the geodesic flow is smooth [71] or if its time-1 map ϕ_1 is h-expansive [72, 73]. Thus, the existence of the measure of maximal entropy also follows from the h-expansiveness of ϕ_1.

By the variational principle, the above discussion is vacuous if the geodesic flow has zero topological entropy. Thus, we verify that the geodesic flow has positive topological entropy. This ensures that the existence of the measure of maximal entropy is a nontrivial problem in our context.

Proposition 3.7.1. *Let M be a compact surface without conjugate points and genus greater than one. Then, the geodesic flow has positive topological entropy.*

Proof. We will give three different proofs. We first recall that every homeomorphism with an expansive factor, has positive topological entropy. Applying this conclusion to the time-1 map of the geodesic flow ϕ_t, Lemma 3.6.4 concludes the proof. Another approach is related the growth rate of the fundamental group $\pi_1(M)$. By Morse's theorem 2.11, for compact surfaces without conjugate points and genus greater than one, its universal covering is Gromov hyperbolic. From Gromov's work [24], it follows that $\pi_1(M)$ has an exponential growth rate hence the geodesic flow has positive topological entropy. For the last method, we know that every compact surface without conjugate points of genus greater than one admits a Riemannian metric g' of negative curvature. Denoting by $\phi_t^{g'}$ the geodesic flow of (M, g'), we see that $\phi_t^{g'}$ has positive topological entropy. By Freire-Mañe's Theorem [51], $h(\phi_1^{g'})$ agrees with the volume growth rate of closed balls in the universal covering of (M, g'). Since M is compact, this volume growth rate is the same for all Riemannian metrics without conjugate points on M. Using Freire-Mañe's Theorem again we conclude that the geodesic flow of (M, g) has positive topological entropy. ■

We now show that the time-1 map of the geodesic flow is h-expansive. We first recall the definition. Let $f : X \to X$ be a homeomorphism acting on a metric space. For every $\epsilon > 0$ and $x \in X$, we define

$$Z_\epsilon(x, f) = \{y \in X : d(f^n(x), f^n(y)) \leq \epsilon, \text{ for every } n \in \mathbb{Z}\}.$$

We say that f is **h-expansive** if there exists $\epsilon > 0$ such that the topological entropy

$$h(f, Z_\epsilon(x, f)) = 0 \quad \text{for every } x \in X.$$

In [74, 75, 14], the authors proved h-expansiveness for several contexts including our case. Note that in our case, for every $\eta \in T_1\tilde{M}$, the set $Z_\epsilon(\eta, \tilde{\phi}_1)$ is a subset of $\tilde{\mathcal{I}}(\eta)$. We remark that h-expansiveness holds because the classes $\tilde{\mathcal{I}}(\eta)$ are uniformly bounded. Thus, we next sketch Gelfert-Ruggiero's proof of zero topological entropy of the classes $\tilde{\mathcal{I}}(\eta)$ [14].

Lemma 3.7.1. *For every $\eta \in T_1M$, the topological entropy $h(\phi_1, \mathcal{I}(\eta)) = 0$. In particular, ϕ_1 is h-expansive and the entropy map of the geodesic flow is upper semi-continuous.*

Proof. Let $\epsilon > 0$, $n \geq 1$ and E be a (n, ϵ)-separated set of $\mathcal{I}(\eta)$. For each $k = 0, \dots, n-1$ we define the set $E_k \subset E$ such that if $\eta_1, \eta_2 \in E_k$ then

$$d_s(\phi_1^k(\eta_1), \phi_1^k(\eta_2)) \geq \epsilon.$$

Although sets E_k may have nonempty intersection between them, these sets cover E: $E = \cup_{k=0}^{n-1} E_k$. Recall that diameters of $\mathcal{I}(\xi)$ are uniformly bounded by Morse's constant Q. Since every $\mathcal{I}(\xi)$ is a connected curve, for every $k = 0, \dots, n-1$ we have

$$\epsilon.Card(E_k) \leq Q \quad \text{hence} \quad Card(E) \leq \sum_{k=0}^{n-1} Card(E_k) \leq \sum_{k=0}^{n-1} \frac{Q}{\epsilon} = \frac{nQ}{\epsilon}.$$

Thus, applying the entropy definition we get

$$h(\phi_1, \mathcal{I}(\eta)) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Card(E) \leq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \frac{nQ}{\epsilon} = 0.$$

■

The uniqueness of the measure of maximal entropy is a more complicated problem. To prove the uniqueness we use the quotient model semi-conjugate to the geodesic flow. This is one of the main motivations for the introduction of the model. The dynamical properties of the model will ensure the existence of a unique measure of maximal entropy. We lift this measure to T_1M and then verify that it is the unique measure of maximal entropy.

The quotient model has a unique measure of maximal entropy. Recall the quotient model, i.e., a quotient flow ψ_t time-preserving semi-conjugate to the geodesic flow ϕ_t. From the last section, Theorem 3.4 says that ψ_t is expansive and has the specification property. Assuming these properties for discrete systems, Bowen [40] showed the existence of a unique measure of maximal entropy. Franco [20] extended this result to the continuous setting.

Theorem 3.5. *Let $\phi_t : X \to X$ be a continuous flow acting on a compact metric space. If ϕ_t is expansive and has the specification property then ϕ_t has a unique measure of maximal entropy.*

Applying this theorem to our case implies that the quotient flow ψ has a unique measure of maximal entropy ν.

To lift ν to T_1M and verify the uniqueness property we rely on a abstract theorem proved by Buzzi-Fisher-Sambarino-Vasquez [68] for discrete systems and applied to certain derived from Anosov systems. They constructed a measure of maximal entropy using a classical argument due to Ledrappier and Walters [76]. We remark that Ures [77] also proved the existence of a unique measure of maximal entropy for some derived from Anosov discrete systems using geometrical methods. The following proposition establishes Buzzi-Fisher-Sambarino-Vasquez's theorem for continuous systems.

Proposition 3.7.2. *Let $\phi_t : Y \to Y$ and $\psi_t : X \to X$ be two continuous flows on compact metric spaces, $\chi : Y \to X$ be a time-preserving semi-conjugacy and ν be the measure of maximal entropy of ψ_t. Assume that ψ_t is expansive, has the specification property and*

1. $h(\phi_1, \chi^{-1}(x)) = 0$ *for every* $x \in X$.
2. $\nu\Big(\{\chi(y) : \chi^{-1} \circ \chi(y) = \{y\}\}\Big) = 1.$

Then, there exists a unique measure of maximal entropy μ of ϕ_t with $\chi_*\mu = \nu$.

Proof. We first build the measure $\mu \in \mathcal{M}(\phi)$. Let $\epsilon > 0$ be an expansivity constant for ψ_t. For each $T > 0$, we define the set

$$Per(T, \epsilon) = \{\chi^{-1}(\gamma) \subset Y : \gamma \text{ is a periodic orbit of } \psi_t \text{ with period in } [T-\epsilon, T+\epsilon]\}.$$

By expansivity this set has a finite number of elements. Note that every $\chi^{-1}(\gamma) \in Per(T, \epsilon)$ is compact and invariant by the flow ϕ_t. Thus, there exists a probability measure μ_γ supported on $\chi^{-1}(\gamma)$ and invariant by the flow ϕ_t. So, we can take the average

$$\mu_T = \frac{\sum_{\chi^{-1}(\gamma) \in Per(T,\epsilon)} \mu_\gamma}{\#Per(T, \epsilon)}.$$

Therefore, μ_T is a Borel probability measure on Y invariant by the flow ϕ_t. Let $\mu \in \mathcal{M}(\phi)$ be an accumulation point of the set $(\mu_T)_{T>0}$ in the weak* topology. So, there exists a sequence $T_n \to \infty$ such that $\mu_{T_n} \to \mu$ weakly.

We now show that $\chi_*\mu = \nu$. Note that for every $\chi^{-1}(\gamma) \in Per(T, \epsilon)$, $\chi_*\mu_\gamma$ is a probability measure supported on γ and invariant by the flow ψ_t. Since γ is a periodic orbit of ψ_t with period in $[T - \epsilon, T + \epsilon]$, $\chi_*(\mu_{T_n})$ is a probability measure supported on the union of periodic orbits of ψ_t with period in $[T - \epsilon, T + \epsilon]$. In this case, Bowen [59] showed that $\chi_*(\mu_{T_n}) \to \nu$ in the weak* topology. By continuity of χ_* and $\mu_{T_n} \to \mu$ we get

$$\chi_*\mu = \nu.$$

We verify that μ is a measure of maximal entropy. Since (Y, ϕ_t, μ) is an extension of (X, ψ_t, ν), we have $h_\nu(\psi_1) \le h_\mu(\phi_1)$. Applying Bowen's formula [40] and hypothesis 1, we deduce that

$$h(\phi_1) \le h(\psi_1) + \sup_{x \in X} h(\phi_1, \chi^{-1}(x)) = h(\psi_1) \quad \text{hence} \quad h(\phi_1) = h(\psi_1).$$

Since ν is a measure of maximal entropy for ψ_t, we obtain

$$h(\phi_1) = h(\psi_1) = h_\nu(\psi_1) \le h_\mu(\phi_1).$$

Therefore $h(\phi_1) = h_\mu(\phi_1)$ and μ is a measure of maximal entropy for ϕ_t.

So far we proved the following. Assuming hypothesis 1, every accumulation point $\mu \in \mathcal{M}(\phi)$ of the set $(\mu_T)_{T>0}$ satisfies:

$$\mu \text{ is a measure of maximal entropy for } \phi_t \text{ such that } \chi_*\mu = \nu. \tag{3.13}$$

We will use this remark in the proof of uniqueness for the case of geodesic flow.

From here we will start using hypothesis 2 to prove some properties of μ and then uniqueness. We first define the set

$$\mathcal{E} = \{y \in Y : \chi^{-1} \circ \chi(y) = \{y\}\}.$$

This notation translates assumption 2 into $\nu(\chi(\mathcal{E})) = 1$ which implies that

$$1 = \nu(\chi(\mathcal{E})) = \chi_*\mu(\chi(\mathcal{E})) = \mu(\chi^{-1} \circ \chi(\mathcal{E})) = \mu(\mathcal{E}).$$

Thus, for every Borel set $A \subset Y$ we see that

$$\mu(Sat(A)) = \mu(Sat(A) \cap \mathcal{E}) = \mu(A \cap \mathcal{E}) = \mu(A). \tag{3.14}$$

We finally prove the uniqueness property. Let μ' be a measure of maximal entropy for ϕ_t. Applying Ledrappier-Walter's formula [76] and hypothesis 1 we obtain

$$h_{\mu'}(\phi_1) \leq h_{\chi_*\mu'}(\psi_1) + \int_Y h(\phi_1, \chi^{-1}(x))\, d\mu'(x) = h_{\chi_*\mu'}(\psi_1)$$

hence $h_{\mu'}(\phi_1) = h_{\chi_*\mu'}(\psi_1)$. From this we deduce that $\chi_*\mu'$ is a measure of maximal entropy for ψ_t:

$$h(\psi_1) = h(\phi_1) = h_{\mu'}(\phi_1) = h_{\chi_*\mu'}(\psi_1).$$

By uniqueness we conclude that $\chi_*\mu' = \nu = \chi_*\mu$. Similarly, we can show that $\mu'(\mathcal{E}) = 1$ and $\mu'(Sat(A)) = \mu'(A)$ for every Borel set $A \subset Y$. Thus,

$$\mu'(Sat(A)) = \mu'(\chi^{-1}\chi(A)) = \chi_*\mu'(\chi(A)) = \chi_*\mu(\chi(A)) = \mu(\chi^{-1}\chi(A)) = \mu(Sat(A)).$$

Therefore, equation 3.14 implies that $\mu = \mu'$ and so the uniqueness of the measure of maximal entropy for ϕ_t. ■

We will apply this proposition to our setting to prove the uniqueness of the measure of maximal entropy. Thus, let $Y = T_1M$, ϕ_t be the geodesic flow, X be the quotient space, ψ_t be the quotient flow, χ be the quotient map and ν be the unique measure of maximal entropy of ψ_t. With these choices we satisfy proposition's assumptions except for hypothesis 1 and 2. Regarding hypothesis 1, we see that for every $[\eta] \in X$,

$$\chi^{-1}[\eta] = \chi^{-1} \circ \chi(\eta) = \mathcal{I}(\eta). \tag{3.15}$$

Therefore, hypothesis 1 follows from Lemma 3.7.1. Moreover, by the remark given in Equation (3.13), there exists a measure of maximal entropy μ for the geodesic flow ϕ_t such that $\chi_*\mu = \nu$. So, to show the uniqueness it only remains to prove hypothesis 2.

We will express hypothesis 2 of Proposition 3.7.2 in our context. By equation 3.15, this hypothesis has the following form

$$\{\chi(y) : \chi^{-1} \circ \chi(y) = \{y\}\} = \{\chi(\eta) \in X : \mathcal{I}(\eta) = \{\eta\}\} = \chi(\mathcal{R}_0).$$

In consequence, hypothesis 2 becomes

$$\nu(\chi(\mathcal{R}_0)) = 1. \tag{3.16}$$

To prove this equation, we use Proposition 3.3 of Climenhaga-Knieper-War's work [18]. This proposition states a classical Katok's result in the context of geodesic flows on surfaces. The result says that the non-expansive set $T_1M \setminus \mathcal{R}_0$ cannot support an ergodic measure of positive metric entropy. In particular, the non-expansive set cannot support a measure of maximal entropy.

Lemma 3.7.2. *Let M be a surface without conjugate points of genus greater than one and μ be an ergodic measure on T_1M invariant by the geodesic flow.*

$$\textit{If} \quad h_\mu(\phi_1) > 0 \quad \textit{then} \quad \mu(\mathcal{R}_0) = 1.$$

Proof. Since μ is ergodic, Ruelle's inequality reads

$$0 < h_\mu(\phi_1) \leq \int_{T_1M} \chi^+ d\mu = \sum_{\chi_i > 0} \chi_i.$$

Thus, at least one $\chi_i > 0$ μ-almost everywhere. Using Ruelle's inequality for ϕ_1^{-1},

$$0 < h_\mu(\phi_1) = h_\mu(\phi_1^{-1}) \leq \int_{T_1M} \chi^+(\phi_1^{-1}) d\mu = \sum_{\chi_j(\phi_1^{-1})>0} \chi_j(\phi_1^{-1}),$$

where $\chi_j(\phi_1^{-1})$ stands for the Lyapunov exponents with respect to ϕ_1^{-1}. Oseledets theorem ensures that at least one $-\chi_j(\phi_1) = \chi_j(\phi_1^{-1}) > 0$ and hence $\chi_j < 0$ μ-almost everywhere. Since we are in the surface case, except in the direction tangent to the geodesic flow, all Lyapunov exponents are nonzero μ-almost everywhere. Thus, for μ-almost every $\eta \in T_1M$ with nonzero Lyapunov exponents, Pesin theory provides local transverse submanifolds $\mathcal{W}^s(\eta)$ and $\mathcal{W}^u(\eta)$. Moreover $\mathcal{W}^s(\eta) \subset \mathcal{F}^s(\eta)$ and $\mathcal{W}^u(\eta) \subset \mathcal{F}^u(\eta)$, so $\mathcal{F}^s(\eta)$ and $\mathcal{F}^u(\eta)$ are also transverse. Therefore $\mathcal{F}^s(\eta) \cap \mathcal{F}^u(\eta) = \{\eta\}$ and $\eta \in \mathcal{R}_0$. Since $\mathcal{R}_0$ is invariant by the geodesic flow, we conclude that $\mu(\mathcal{R}_0) = 1$. ∎

Finally, we prove Equation (3.16) and so the uniqueness of the measure of maximal entropy for the geodesic flow.

Proof. As remarked above, by Equation (3.13), μ is a measure of maximal entropy and hence $h_\mu(\phi_1) = h(\phi_1) > 0$. Ergodic decomposition of μ provides an ergodic component τ with $h_\tau(\phi_1) > 0$. Lemma 3.7.2 implies that $\tau(\mathcal{R}_0) = 1$ hence $\mu(\mathcal{R}_0) > 0$. So, we have

$$\nu(\chi(\mathcal{R}_0)) = \chi_* \mu(\chi(\mathcal{R}_0)) = \mu(\chi^{-1}\chi\mathcal{R}_0) = \mu(\mathcal{R}_0) > 0.$$

Since ν is ergodic and $\chi(\mathcal{R}_0)$ is invariant by the flow ψ_t, we conclude that $\nu(\chi(\mathcal{R}_0)) = 1$. ■

4
Continuity of Green bundles in higher dimension and regularity of horospherical foliations

This chapter studies the relationship between regularity properties of the horospherical foliations and the continuity of Green bundles. This is done in the context of compact manifolds without conjugate points and Gromov hyperbolic universal covering. In [14], Gelfert and Ruggiero investigated among other things, similar questions for compact higher genus surfaces without conjugate points and continuous Green bundles. The results of the chapter extend to higher dimension part of the theory of Gelfert and Ruggiero. More precisely, we prove the following theorem.

Theorem 4.1. *Let M be a compact n-manifold without conjugate points, Gromov hyperbolic universal covering $\tilde{M}$ and continuous Green bundles. If there exists a hyperbolic periodic geodesic then*

1. *The set where the Lyapunov exponents of all vectors transverse to the geodesic vector field are non-zero agrees almost everywhere with an open dense set, with respect to Liouville measure.*
2. *Hyperbolic periodic points are dense on T_1M.*
3. *The horospherical foliations are the only foliations of T_1M such that: they have C^1-leaves of dimension $n-1$, are continuous, invariant by the geodesic flow, and transverse to $\mathcal{F}^u$ (or $\mathcal{F}^s$) at some hyperbolic periodic point of T_1M.*
4. *Green bundles are uniquely integrable, and tangent to the horospherical foliations.*

While horospheres were defined by Hedlund [4] in 1936, Green bundles were not formalized by Green [32] until 1958. For compact manifolds of negative curvature, Anosov [21] showed that horospherical foliations agree with the invariant foliations of the geodesic flow. Moreover, he pointed out that Green bundles are the invariant bundles of the geodesic flow. Thus, in this case Green bundles are tangent to horospherical foliations. Later in

1973, in the context of compact manifolds without conjugate points, Eberlein [33] showed this tangency for Anosov geodesic flows. Note that there are Anosov geodesic flows associated to manifolds that do not have strictly negative curvature. In 1977, Eberlein [22] and Heintze-Imhof [28] extended the tangency to compact manifolds of non-positive curvature. In the same year, Eschenburg [29] generalized this conclusion to a larger family of manifolds that includes manifolds without focal points, manifolds of bounded asymptote and manifolds with Anosov geodesic flow. In the above works, Eberlein, Heintze-Imhof and Eschenburg show that horospherical leaves are C^1. In 1977, Pesin [30] showed the tangency for closed manifolds without focal points. Note that in all of the above categories of manifolds, Green bundles are automatically continuous. This does not necessarily hold for general compact manifolds without conjugate points as showed by Ballmann-Brin-Burns [52] in 1987. For general compact manifolds without conjugate points, in 1986 Knieper [15] observed that if Green bundles are continuous, then they are integrable, but neither necessarily tangent to horospherical leaves, nor uniquely integrable. In 2020, Gelfert-Ruggiero [14] proved that Green bundles are tangent to smooth horospherical leaves for compact higher genus surfaces without conjugate points and continuous Green bundles. Theorem 4.1 extends this fact to higher dimension.

The chapter is organized as follows. Section 4.1 proves items 1 and 2 of Theorem 4.1. Moreover, we show basic consequences of the hypothesis of Theorem 4.1. In Section 4.2, we show item 3 and 4 of Theorem 4.1. While we devote most of the section to item 3, item 4 will be a consequence of item 3. Throughout the chapter, we will assume the hypothesis of Theorem 4.1.

4.1
Hyperbolic periodic points

This section establishes some basic consequences of the hypothesis of Theorem 4.1. Furthermore we prove items 1 and 2 of the same theorem.

In 1977, Pesin [30] established a theory that deals with nonzero Lyapunov exponents. He defined an important set in his theory, which has been called the Pesin set ever since. Pesin developed this theory to prove the ergodicity of geodesic flows of closed surfaces without focal points. The missing theorem in the theory was: the non-expansive set has zero Liouville measure. This is not known even in the case of closed surfaces of nonpositive curvature and remains as an open problem. An important ingredient of the theory is to verify that Pesin set has positive measure. Despite we do not prove this result, we study a set related to the Pesin set: the set where the Lyapunov exponents of all vectors transverse to the geodesic vector field are non-zero. We show that this set agrees almost everywhere with an open dense set with respect to Liouville measure. In 1985, Ballmann-Brin-Eberlein [78] proved this property for compact rank-1 manifolds of non-positive curvature. In 2003, Ruggiero and Rosas [79] extended this conclusion to compact manifolds without conjugate points, with bounded asymptote and expansive geodesic flow. Note that Theorem 4.1 holds for a family of manifolds that includes the previous cases.

We first give some basic consequences of the hypothesis of Theorem 4.1. Recall that for every $\theta \in T_1M$ we denoted by $G^s(\theta)$ and $G^u(\theta)$ the stable and unstable Green bundles at θ. Furthermore, $\mathcal{R}_1$ is the set of vectors $\theta \in T_1M$ where $G^s(\theta)$ and $G^u(\theta)$ are transverse.

Lemma 4.1.1. *Assume the hypothesis of Theorem 4.1. Then*

1. *The universal covering $\tilde{M}$ is a uniform visibility manifold.*
2. *If $\eta \in T_1M$ is a hyperbolic periodic point then $\eta \in \mathcal{R}_0 \cap \mathcal{R}_1$. In particular, $\mathcal{R}_1$ is nonempty.*
3. *$\mathcal{R}_1$ is open and dense.*
4. *If $\eta \in T_1M$ is a hyperbolic periodic point and $\xi \in \tilde{\mathcal{F}}^s(\eta)$ with $\xi \neq \eta$ then*

$$d_s(\phi_t(\eta), \phi_t(\xi)) \to \infty \quad \text{as} \quad t \to -\infty.$$

Proof. In item 1, since Green bundles are continuous, Proposition 2.3.13 says that geodesics rays diverge uniformly in the universal covering $\tilde{M}$. This property together with the Gromov hyperbolicity of $\tilde{M}$ implies that $\tilde{M}$ is a visibility manifold by Theorem 2.20. Since the fundamental domains of $\tilde{M}$ are compact it follows that $\tilde{M}$ is actually a uniform visibility manifold.

For item 2, suppose that $\eta \in T_1M$ is hyperbolic and periodic. By the hyperbolic structure of η, $G^s(\eta)$ and $G^u(\eta)$ agree with the invariant subspaces associated to η. Since the invariant subspaces are transverse at η, so are $G^s(\eta)$ and $G^u(\eta)$, hence $\eta \in \mathcal{R}_1$. Recall that $G^s(\eta)$ and $G^u(\eta)$ are tangent to $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$. From this we deduce that $\tilde{\mathcal{F}}^s(\eta)$ and $\tilde{\mathcal{F}}^u(\eta)$ are transverse hence $\eta \in \mathcal{R}_0$.

For item 3, the continuity of Green bundles implies that $\mathcal{R}_1$ is an open set. From the above item 1, we know that $\tilde{M}$ is a visibility manifold. From this, Theorem 2.15 says that ϕ_t is transitive. So, let U be an open set of T_1M. By transitivity there exists an orbit O of ϕ_t that is dense in T_1M. Thus, O

intersects U and $\mathcal{R}_1$. Since $\mathcal{R}_1$ is invariant by ϕ_t, we conclude that $O \subset \mathcal{R}_1$. Therefore $\mathcal{R}_1$ is dense.

For item 4, by contradiction we suppose that there exists $C > 0$ such that

$$d_s(\phi_t(\eta), \phi_t(\xi)) \leq C \quad \text{for every } t \leq 0.$$

This implies that $\xi \in \tilde{\mathcal{F}}^{cu}(\eta)$ and hence

$$\xi \in \tilde{\mathcal{F}}^s(\eta) \cap \tilde{\mathcal{F}}^{cu}(\eta).$$

By Corollary 2.3.1, we conclude that $\xi \in \mathcal{I}(\eta)$. Since η is expansive, $\mathcal{I}(\eta) = \{\eta\}$ and so $\xi = \eta$, a contradiction. ■

We now deal with the set closely related to the Pesin set, mentioned in the introduction. To do this, we recall Definition 2.2.1 given in Subsection 2.2.2. Let $\phi_t : M \to M$ be a smooth flow acting on a compact manifold M. We define the set

$$\Lambda_\phi = \{p \in M : \text{for every } v \in S_p \subset T_pM, \chi(p, v) \neq 0\},$$

where S_p is some subspace transverse to the flow ϕ_t at p.

As said in the introduction, we rely on a Ruggiero's Theorem (Theorem 4.1 of [17]) which we restate below.

Theorem 4.2. *Let M be a compact manifold without conjugate points and continuous Green bundles. If $\mathcal{R}_1$ is nonempty, then for m-almost every $\theta \in \mathcal{R}_1$, for every $\xi^s \in G^s(\theta), \xi^u \in G^u(\theta)$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \|d_\theta \phi_t(\xi^s)\| < 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{1}{t} \log \|d_\theta \phi_t(\xi^u)\| > 0.$$

From this we deduce that for m-almost every $\theta \in \mathcal{R}_1$, the Lyapunov exponents are nonzero on $G^s(\theta) \oplus G^u(\theta)$. Recall that $G(\theta)$ is the vector field tangent to the geodesic flow at θ. Since $G^s(\theta)$ and $G^u(\theta)$ are transverse on each $\theta \in \mathcal{R}_1$, we conclude that $G^s(\theta)$ and $G^u(\theta)$ span $S(\theta)$, the orthogonal complement of $G(\theta)$. Therefore for m-almost every $\theta \in \mathcal{R}_1$, the Lyapunov exponents are nonzero on a subspace $S(\theta)$ orthogonal to $G(\theta)$, i.e., $\theta \in \Lambda_\phi$. We thus get item 1 of Theorem 4.1.

Corollary 4.1.1. *The set Λ_ϕ agrees m-almost everywhere with the open dense set $\mathcal{R}_1$.*

Proof. By item 2 of Lemma 4.1.1, we know that $\mathcal{R}_1$ is nonempty. By hypothesis, Green bundles are continuous so we can apply Theorem 4.2 to our case. The result follows from the above discussion since $\mathcal{R}_1$ is open and dense by Lemma 4.1.1(3). ■

We now state a version of Katok's classical result [31] about measures with non-zero Lyapunov exponents.

Theorem 4.3. *Let M be a compact manifold, $f : M \to M$ be a C^2 diffeomorphism and μ be a f-invariant ergodic measure on M. If μ is not concentrated on a single periodic orbit and μ has non-zero Lyapunov exponents then for every $x \in Supp(\mu)$ and every neighborhood U of x there exists a hyperbolic periodic point in U.*

Note that this theorem is intended for discrete systems. However, there is a large consensus among specialists that the theorem extends via local cross sections to flows without singularities. Thus there are several works that use the result or deal with things related to it. Among them, in the context of geodesic flows we can cite some due to Paternain [80], Barbosa-Ruggiero [81], Ruggiero [17], Ledrappier-Lima-Sarig [82] and Araujo-Lima-Poletti [83].

Thus, to apply Theorem 4.3, let us look at Liouville measure restricted to $\mathcal{R}_1$. We denote by m' the Liouville measure m restricted and normalized to $\mathcal{R}_1$. Thus, m' is a Borel probability measure on $\mathcal{R}_1$ invariant by ϕ_t. Corollary 4.1.1 says that except for the direction tangent to the geodesic flow, m' has non-zero Lyapunov exponents on $\mathcal{R}_1$. Since $\mathcal{R}_1$ is open, m' is not concentrated on a single orbit. Meanwhile, Pesin [30] proved that the Liouville measure is ergodic when restricted to $\mathcal{R}_1$, so m' is ergodic. As a result, we can apply Katok's Theorem to m'. We observe that $Supp(m') = \mathcal{R}_1$ up to m'-null set. We thus get item 2 of Theorem 4.1 since $\mathcal{R}_1$ is open and dense by Lemma 4.1.1(3).

Corollary 4.1.2. *Hyperbolic periodic points are dense on T_1M.*

4.2 Horospherical foliations are dynamically defined and tangent to Green bundles

The goal of this section is to prove item 3 and 4 of Theorem 4.1. We first show the uniqueness-type result for the horospherical foliations on T_1M. This property has as direct consequence the tangency between horospherical foliations and Green bundles.

For closed manifolds of negative curvature, a consequence of the Anosov's work [21] is the uniqueness of continuous foliations of T_1M invariant by geodesic flow. In 1993, Paternain [84] showed the uniqueness for compact surfaces with expansive geodesic flow. In 1997, Ruggiero [85] extended this conclusion to compact manifolds without conjugate points and expansive geodesic flow. Note that Anosov geodesic flows are particular cases of expansive geodesic flows. In 2007, Barbosa and Ruggiero [16] showed the uniqueness for compact surfaces without conjugate points and genus greater than one, but this time without assuming expansivity. The present work generalizes the uniqueness to any dimension under the assumptions of Theorem 4.1.

We prove the uniqueness result in several steps. Let $\mathcal{D}$ be a ϕ_t-invariant continuous foliation of T_1M with C^1-leaves of dimension $n-1$. Moreover, without loss of generality we assume that $\mathcal{D}$ is transverse to $\mathcal{F}^u$ at the hyperbolic periodic point $\theta \in T_1M$. We first verify the coincidence of special leaves of $\mathcal{D}$ and $\mathcal{F}^s$ locally: if U is a small neighborhood of θ then

$$\mathcal{D}(\theta) \cap U = \mathcal{F}^s(\theta) \cap U. \tag{4.1}$$

Suppose that $\mathcal{D}(\theta) \cap U = \mathcal{F}^s(\theta) \cap U$. Using the fact that we can generate $\mathcal{F}^s(\theta)$ with discrete ϕ_t-iterates of $\mathcal{F}^s(\theta) \cap U$, we extend the coincidence to the whole leave: for the hyperbolic periodic point θ it holds

$$\mathcal{D}(\theta) = \mathcal{F}^s(\theta). \tag{4.2}$$

From this, using the density of $\mathcal{F}^s(\theta)$ on T_1M, we deduce that all the leaves of $\mathcal{D}$ and $\mathcal{F}^s$ agree and so the foliations $\mathcal{D}$ and $\mathcal{F}^s$ agree.

As first step, the following lemma proves Equation (4.1). This lemma says that $\mathcal{D}$ and $\mathcal{F}^s$ agree on a neighborhood of the hyperbolic periodic point θ. We establish a convention for the remainder of the section. Whenever we work on T_1M, intersection notations such as $\mathcal{D}(\theta) \cap U$, always refer to the connected component of $\mathcal{D}(\theta) \cap U$ containing θ.

Lemma 4.2.1. *Let $\mathcal{D}$ be a ϕ_t-invariant continuous foliation of T_1M with C^1-leaves of dimension $n-1$ and transverse to $\mathcal{F}^u$ (or $\mathcal{F}^s$) at the hyperbolic periodic point $\theta \in T_1M$ of period $P > 0$. If $U \subset T_1M$ is a small neighborhood of θ then*

$$\text{either} \quad \mathcal{D}(\theta) \cap U = \mathcal{F}^s(\theta) \cap U \quad \text{or} \quad \mathcal{D}(\theta) \cap U = \mathcal{F}^u(\theta) \cap U.$$

Proof. Without loss of generality we assume that $\mathcal{D}$ is transverse to $\mathcal{F}^u$ at θ. By contradiction, suppose that

$$\mathcal{D}(\theta) \cap U \neq \mathcal{F}^s(\theta) \cap U. \tag{4.3}$$

Thus, for every $n \geq 1$, we define

$$D_n = \phi_{nP}(\mathcal{D}(\theta)) \cap U.$$

Since $\mathcal{D}$ is invariant by the flow ϕ_t and P is the period of θ, we have

$$\phi_{nP}(\mathcal{D}(\theta)) = \mathcal{D}(\theta) \quad \text{hence} \quad D_n \subset \mathcal{D}(\theta) \cap U.$$

Denoting by d_r the C^r-distance for every $r \geq 1$, we see that for every $n \geq 1$,

$$d_r(D_n, \mathcal{F}^u(\theta) \cap U) > 0. \tag{4.4}$$

On the other hand, since θ is a hyperbolic fixed point of ϕ_P, $\mathcal{D}(\theta) \cap U$ is transverse to $\mathcal{F}^u(\theta) \cap U$ at θ and equation (4.3), the inclination lemma implies

$$d_r(D_n, \mathcal{F}^u(\theta) \cap U) \to 0 \quad \text{as } n \to \infty.$$

This contradicts Equation (4.4) and shows that $\mathcal{D}(\theta) \cap U = \mathcal{F}^s(\theta) \cap U$. The other case is analogous. ∎

To prove Equation (4.2), we consider the sets

$$V^s(\theta) = \mathcal{F}^s(\theta) \cap U \quad \text{and} \quad V^u(\theta) = \mathcal{F}^u(\theta) \cap U. \tag{4.5}$$

We now generate $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$ with discrete ϕ_t iterates of $V^s(\theta)$ and $V^u(\theta)$ respectively. To do this, we work in the covering space $T_1\tilde{M}$. Choosing some lift $\tilde{\theta} \in T_1\tilde{M}$ of θ, we obtain the following lifts.

– $\tilde{\mathcal{F}}^s(\tilde{\theta})$ and $\tilde{\mathcal{F}}^u(\tilde{\theta})$ are lifts of $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$.
– $\tilde{V}^s(\tilde{\theta})$ and $\tilde{V}^u(\tilde{\theta})$ are lifts of $V^s(\theta)$ and $V^u(\theta)$.
– The lifts satisfy

$$\tilde{V}^s(\tilde{\theta}) \subset \tilde{\mathcal{F}}^s(\tilde{\theta}) \quad \text{and} \quad \tilde{V}^u(\tilde{\theta}) \subset \tilde{\mathcal{F}}^u(\tilde{\theta}).$$

On the other hand, since θ is periodic of period $P > 0$, γ_θ is a closed geodesic with same period P. From Subsection 2.1.1, we know that γ_θ has an associated axial isometry $T : \tilde{M} \to \tilde{M}$, with axis $\gamma_{\tilde{\theta}}$. This means that T is a translation along $\gamma_{\tilde{\theta}}$: for every $t \in \mathbb{R}$,

$$\gamma_{\tilde{\theta}}(t + P) = T \circ \gamma_{\tilde{\theta}}(t).$$

Therefore the covering isometry $dT : T_1\tilde{M} \to T_1\tilde{M}$ is a translation along the orbit of $\tilde{\theta}$: for every $t \in \mathbb{R}$,

$$\phi_{t+P}(\tilde{\theta}) = dT \circ \phi_t(\tilde{\theta}). \tag{4.6}$$

The following lemma says that we can generate $\tilde{\mathcal{F}}^s(\tilde{\theta})$ and $\tilde{\mathcal{F}}^u(\tilde{\theta})$, with discrete ϕ_t iterates of $\tilde{V}^s(\tilde{\theta})$ and $\tilde{V}^u(\tilde{\theta})$ up to isometric images.

Lemma 4.2.2. *Let $\theta \in T_1M$ be the hyperbolic periodic point of period $P > 0$, $\tilde{\theta} \in T_1$ be any lift of θ and T be the axial isometry associated to γ_θ, having axis $\gamma_{\tilde{\theta}}$. If $\tilde{V}^s(\tilde{\theta})$ and $\tilde{V}^u(\tilde{\theta})$ are the lifts introduced in the above list then*

1. *For every $n \in \mathbb{Z}$,*
$$(dT)^{-n}\phi_{nP}(\tilde{\theta}) = \tilde{\theta}.$$

2.
$$\tilde{\mathcal{F}}^s(\tilde{\theta}) = \bigcup_{n\in\mathbb{N}} (dT)^n \circ \phi_{-nP}(\tilde{V}^s(\tilde{\theta})) \quad \textit{and} \quad \tilde{\mathcal{F}}^u(\tilde{\theta}) = \bigcup_{n\in\mathbb{N}} (dT)^{-n} \circ \phi_{nP}(\tilde{V}^u(\tilde{\theta})). \tag{4.7}$$

Proof. For item 1, from Equation (4.6), we see that for every $t \in \mathbb{R}$ and every $n \in \mathbb{Z}$

$$\phi_{t+nP}(\tilde{\theta}) = (dT)^n \phi_t(\tilde{\theta}) \quad \text{hence} \quad (dT)^{-n} \circ \phi_{t+nP}(\tilde{\theta}) = \phi_t(\tilde{\theta}).$$

The result follows setting $t = 0$. For item 2, since horospherical foliations are invariant by ϕ_t and by the covering isometries, for every $n \geq 1$ we have

$$(dT)^n \circ \phi_{-nP}(\tilde{\mathcal{F}}^s(\tilde{\theta})) = \tilde{\mathcal{F}}^s((dT)^n \circ \phi_{-nP}(\tilde{\theta})) = \tilde{\mathcal{F}}^s(\tilde{\theta}).$$

Since $\tilde{V}^s(\tilde{\theta}) \subset \tilde{\mathcal{F}}^s(\tilde{\theta})$, we deduce that for every $n \geq 1$

$$(dT)^n \circ \phi_{-nP}(\tilde{V}^s(\tilde{\theta})) \subset \tilde{\mathcal{F}}^s(\tilde{\theta}) \quad \text{hence} \quad \bigcup_{n\in\mathbb{N}} (dT)^n \circ \phi_{-nP}(\tilde{V}^s(\tilde{\theta})) \subset \tilde{\mathcal{F}}^s(\tilde{\theta}).$$

For the reverse inclusion, Lemma 4.1.1(4) states that for every $\tilde{\eta} \in \tilde{V}^s(\tilde{\theta}) \subset \tilde{\mathcal{F}}^s(\tilde{\theta})$,

$$d_s(\phi_t(\tilde{\theta}), \phi_t(\tilde{\eta})) \to \infty \quad \text{as} \quad t \to -\infty.$$

From this, setting $t = -nP$ we get

$$\begin{aligned} d_s(\tilde{\theta}, (dT)^n \circ \phi_{-nP}(\tilde{\eta})) &= d_s((dT)^n \circ \phi_{-nP}(\tilde{\theta}), (dT)^n \circ \phi_{-nP}(\tilde{\eta})) \\ &= d_s(\phi_{-nP}(\tilde{\theta}), \phi_{-nP}(\tilde{\eta})) \to \infty \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence,

$$Diam((dT)^n \circ \phi_{-nP}(\tilde{V}^s(\tilde{\theta}))) \to \infty \quad \text{as} \quad n \to \infty.$$

We conclude that $\tilde{\mathcal{F}}^s(\tilde{\theta})$ is covered by $\bigcup_{n\in\mathbb{N}}(dT)^n \circ \phi_{-nP}(\tilde{V}^s(\tilde{\theta}))$ as $n \to \infty$. The proof for the unstable case is similar. ■

As an immediate consequence, we generate $\mathcal{F}^s(\theta)$ and $\mathcal{F}^u(\theta)$, with discrete ϕ_t-iterates of $V^s(\theta)$ and $V^u(\theta)$. Furthermore, we prove Equation (4.2) and so the coincidence of the entire leaves $\mathcal{D}(\theta)$ and $\mathcal{F}^s(\theta)$.

Lemma 4.2.3. *Let $\mathcal{D}$ be a ϕ_t-invariant continuous foliation of T_1M with C^1-leaves of dimension $n-1$ and $\theta \in T_1M$ be the hyperbolic periodic point of period $P>0$ where $\mathcal{D}$ is transverse to $\mathcal{F}^u$ (or $\mathcal{F}^s$). If $V^s(\theta)$ and $V^u(\theta)$ are the sets defined in Equation* (4.5) *then*

1. $$\mathcal{F}^s(\theta) = \bigcup_{n\in\mathbb{N}} \phi_{-nP}(V^s(\theta)) \quad \text{and} \quad \mathcal{F}^u(\theta) = \bigcup_{n\in\mathbb{N}} \phi_{nP}(V^u(\theta)).$$

2. *Either*
$$\mathcal{D}(\theta) = \mathcal{F}^s(\theta) \quad \text{or} \quad \mathcal{D}(\theta) = \mathcal{F}^u(\theta).$$

Proof. Item 1 follows by applying the covering map $d\pi$ to Equation (4.7) since

$$d\pi(\bigcup_{n\in\mathbb{N}}(dT)^n\circ\phi_{-nP}(\tilde{V}^s(\tilde{\theta}))) = \bigcup_{n\in\mathbb{N}} d\pi\circ(dT)^n\circ\phi_{-nP}(\tilde{V}^s(\tilde{\theta})) = \bigcup_{n\in\mathbb{N}} \phi_{-nP}(\tilde{V}^s(\tilde{\theta})).$$

For item 2, let $\eta \in \mathcal{F}^s(\theta)$. By the last item, $\eta \in \phi_{-nP}(V^s(\theta))$ for some $n \geq 1$. Without loss of generality, we suppose that $\mathcal{D}(\theta) = \mathcal{F}^s(\theta)$ on $V^s(\theta)$. Since $\mathcal{D}$ and $\mathcal{F}^s$ are invariant by ϕ_{-nP}, it follows that $\mathcal{D}(\theta) = \mathcal{F}^s(\theta)$ on $\phi_{-nP}(V^s(\theta))$ and so the sets agree on η. The other case is similar. ■

Now, using the density of the horospherical leaves we show item 3 of Theorem 4.1.

Corollary 4.2.1. *The horospherical foliations are the only foliations of T_1M such that: they have C^1-leaves of dimension $n-1$, are continuous, invariant by the geodesic flow, and transverse to $\mathcal{F}^u$ (or $\mathcal{F}^s$) at some hyperbolic periodic point of T_1M.*

Proof. Let $\mathcal{D}$ be a ϕ_t-invariant continuous foliation of T_1M with C^1-leaves of dimension $n-1$, $\xi \in T_1M$ and B be a closed ball containing ξ. Without loss of generality, we assume that $\mathcal{D}$ is transverse to $\mathcal{F}^u$ at the hyperbolic periodic point $\theta \in T_1M$. Applying Lemma 4.2.3 we obtain

$$\mathcal{D}(\theta) = \mathcal{F}^s(\theta). \tag{4.8}$$

Furthermore, Theorem 2.15 provides that $\mathcal{F}^s(\theta)$ is dense on T_1M. Thus, there exists a sequence $\xi_n \in \mathcal{F}^s(\theta)$ with $\xi_n \to \xi$. For every $n \geq 1$, let F_n and D_n be the respective connected components of $\mathcal{F}^s(\xi_n) \cap B$ and $\mathcal{D}(\xi_n) \cap B$ containing ξ_n. By Equation (4.8), for every $n \geq 1$,

$$F_n = D_n.$$

We denote by F and D be the respective connected components of $\mathcal{F}^s(\xi) \cap B$ and $\mathcal{D}(\xi) \cap B$ containing ξ. Since $\mathcal{F}^s$ and $\mathcal{D}$ are continuous foliations in the compact-open topology, we deduce that F_n converges to F and $F_n = D_n$ converges to D as $n \to \infty$ in the compact-open topology. Thus, $F = D$ and hence for every $\xi \in T_1M$,

$$\mathcal{F}^s(\xi) \cap B = \mathcal{D}(\xi) \cap B.$$

Therefore the foliations $\mathcal{D}$ and $\mathcal{F}^s$ agree. A similar reasoning shows that $\mathcal{D}$ and $\mathcal{F}^u$ agree if we assume $\mathcal{D}(\theta) = \mathcal{F}^u(\theta)$. ■

We now turn to the problem of the tangency between horospherical foliations and Green bundles. A related problem to this, deals with the integrability of Green bundles. Knieper [15] studied this problem in the context of compact manifolds without conjugate points. In the special case of continuous Green bundles, Knieper showed that these bundles integrate to continuous foliations invariant by ϕ_t. Moreover, he conjectured the tangency between horospherical foliations and Green bundles in this context.

Theorem 4.4. *Let M be a compact n-manifold without conjugate points. If Green bundles are continuous then these bundles integrate to continuous foliations $\mathcal{G}^s$ and $\mathcal{G}^u$ of T_1M with C^1-leaves of dimension $n-1$. Moreover, $\mathcal{G}^s$ and $\mathcal{G}^u$ are invariant by the geodesic flow.*

Though the importance of this theorem, this does not ensure the unique integrability of Green bundles. The following corollary solves the problems of unique integrability and tangency. The result is a direct consequence of the uniqueness of horospherical foliations.

Corollary 4.2.2. *Green bundles are uniquely integrable and tangent to the horospherical foliations.*

Proof. By Theorem 4.4, Green bundles integrate to some ϕ_t-invariant continuous foliations $\mathcal{G}^s$ and $\mathcal{G}^u$ of T_1M with C^1-leaves of dimension $n-1$. By hypothesis there exists a hyperbolic periodic point $\theta \in T_1M$ hence

$$\mathcal{G}^s(\theta) = \mathcal{F}^s(\theta) \quad \text{and} \quad \mathcal{G}^u(\theta) = \mathcal{F}^u(\theta). \tag{4.9}$$

Thus, Lemma 4.1.1(2) implies that $\mathcal{G}^s$ and $\mathcal{G}^u$ are transverse to $\mathcal{F}^u$ and $\mathcal{F}^s$ respectively at θ. By Corollary 4.2.1, either $\mathcal{G}^s = \mathcal{F}^s$ or $\mathcal{G}^s = \mathcal{F}^u$. From Equation (4.9) we conclude that $\mathcal{G}^s = \mathcal{F}^s$ and $\mathcal{G}^u = \mathcal{F}^u$. Therefore, the results follow since $\mathcal{G}^s$ and $\mathcal{G}^u$ are arbitrary foliations integrated from Green bundles. ■

Bibliography

[1] MORSE, H. M.. **A fundamental class of geodesics on any closed surface of genus greater than one**. Transactions of the American Mathematical Society, 26(1):25–60, 1924.

[2] KOEBE, P.. **Allgemeine theorie der riemannschen mannigfaltigkeiten**. Acta mathematica, 50(1):27–157, 1927.

[3] LOBELL, F.. **Uber die geodatischen linien der cl ifford-kleinschen flachen**. Math. Zeit., 30:572–607, 1929.

[4] HEDLUND, G. A.. **Fuchsian groups and transitive horocycles**. Duke Mathematical Journal, 2(3):530–542, 1936.

[5] HOPF, E.. **Fuchsian groups and ergodic theory**. Transactions of the American Mathematical Society, 39(2):299–314, 1936.

[6] HEDLUND, G. A.. **Two-dimensional manifolds and transitivity**. Annals of Mathematics, p. 534–542, 1936.

[7] MORSE, M.. **Instability and transitivity**. J. math, pures et appl, 14:49–71, 1935.

[8] GRANT, A.. **Asymptotic transitivity on surfaces of variable negative curvature.** PhD thesis, Bryn Mawr University, 1937.

[9] BUSEMANN, H.. **The geometry of finsler spaces**. Bulletin of the American Mathematical Society, 56(1):5–16, 1950.

[10] HOPF, E.. **Statistik der lösungen geodätischer probleme vom unstabilen typus. ii**. Mathematische Annalen, 117(1):590–608, 1940.

[11] GREEN, L. W.. **Surfaces without conjugate points**. Transactions of the American Mathematical society, 76(3):529–546, 1954.

[12] HOPF, E.. **Closed surfaces without conjugate points**. Proceedings of the National Academy of Sciences, 34(2):47–51, 1948.

[13] GELFERT, K.; RUGGIERO, R. O.. **Geodesic flows modeled by expansive flows**. Proc. Edinb. Math. Soc. (2), 62(1):61–95, 2019., 1995.

[14] GELFERT, K.; RUGGIERO, R. O.. **Geodesic flows modeled by expansive flows: Compact surfaces without conjugate points and continuous green bundles**. arXiv preprint arXiv:2009.11365, 2020.

[15] KNIEPER, G.. **Mannigfaltigkeiten ohne konjugierte Punkte**. Número 168 em 1. Mathematisches Institut der Universität Bonn, 1986.

[16] GOMES, J. B.; RUGGIERO, R. O.. **Uniqueness of central foliations of geodesic flows for compact surfaces without conjugate points**. Nonlinearity, 20(2):497, 2007.

[17] RUGGIERO, R.. **Does the hyperbolicity of periodic orbits of a geodesic flow without conjugate points imply the anosov property?** Transactions of the American Mathematical Society, 2021.

[18] CLIMENHAGA, V.; KNIEPER, G. ; WAR, K.. **Uniqueness of the measure of maximal entropy for geodesic flows on certain manifolds without conjugate points**. Advances in Mathematics, 376:107452, 2021.

[19] CLIMENHAGA, V.; THOMPSON, D. J.. **Unique equilibrium states for flows and homeomorphisms with non-uniform structure**. Advances in Mathematics, 303:745–799, 2016.

[20] FRANCO, E.. **Flows with unique equilibrium states**. American Journal of Mathematics, 99(3):486–514, 1977.

[21] ANOSOV, D. V.. **Geodesic flows on closed riemannian manifolds with negative curvature**. Trudy Matematicheskogo instituta imeni V.A. Steklova, 90:3–210, 1969.

[22] EBERLEIN, P.; O'NEILL, B.. **Visibility manifolds**. Pacific Journal of Mathematics, 46(1):45–109, 1973.

[23] THURSTON, W. P.. **Hyperbolic structures on 3-manifolds i: Deformation of acylindrical manifolds**. Annals of Mathematics, 124(2):203–246, 1986.

[24] GROMOV, M.. **Hyperbolic groups**. In: ESSAYS IN GROUP THEORY, p. 75–263. Springer, 1987.

[25] EBERLEIN, P.. **Geodesic flow in certain manifolds without conjugate points**. Transactions of the American Mathematical Society, 167:151–170, 1972.

[26] RUGGIERO, R. O.. **Dynamics and global geometry of manifolds without conjugate points**. Ensaios Matemáticos, 12:1–181, 2007.

[27] ANOSOV, D. V.; SINAI, Y. G.. **Some smooth ergodic systems**. Russian Mathematical Surveys, 22(5):103, 1967.

[28] HEINTZE, E.; IM HOF, H.-C.. **Geometry of horospheres**. Journal of Differential geometry, 12(4):481–491, 1977.

[29] ESCHENBURG, J.-H.. **Horospheres and the stable part of the geodesic flow**. Mathematische Zeitschrift, 153(3):237–251, 1977.

[30] PESIN, J. B.. **Geodesic flows on closed riemannian manifolds without focal points**. Mathematics of the USSR-Izvestiya, 11(6):1195, 1977.

[31] KATOK, A.. **Lyapunov exponents, entropy and periodic orbits for diffeomorphisms**. Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 51(1):137–173, 1980.

[32] GREEN, L. W.. **A theorem of e. hopf.** Michigan Mathematical Journal, 5(1):31–34, 1958.

[33] EBERLEIN, P.. **When is a geodesic flow of anosov type? i**. Journal of Differential Geometry, 8(3):437–463, 1973.

[34] DO CARMO, M. P.; FLAHERTY FRANCIS, J.. **Riemannian geometry**, volumen 6. Springer, 1992.

[35] MUNKRES, J. R.. **Topology**, volumen 2. Prentice Hall Upper Saddle River, 2000.

[36] PATERNAIN, G. P.. **Geodesic flows**, volumen 180. Springer Science & Business Media, 2012.

[37] BRIN, M.; STUCK, G.. **Introduction to dynamical systems**. Cambridge university press, 2002.

[38] WALTERS, P.. **An introduction to ergodic theory**, volumen 79. Springer Science & Business Media, 2000.

[39] BOWEN, R.; WALTERS, P.. **Expansive one-parameter flows**. Journal of differential Equations, 12(1):180–193, 1972.

[40] BOWEN, R.. **Entropy for group endomorphisms and homogeneous spaces**. Transactions of the American Mathematical Society, 153:401–414, 1971.

[41] DINABURG, E. I.. **On the relations among various entropy characteristics of dynamical systems**. Mathematics of the USSR-Izvestiya, 5(2):337, 1971.

[42] OSELEDEC, V. I.. **A multiplicative ergodic theorem. liapunov characteristic number for dynamical systems**. Trans. Moscow Math. Soc., 19:197–231, 1968.

[43] RUELLE, D.. **An inequality for the entropy of differentiable maps**. Bol. Soc. Brasil. Mat, 9(1):83–87, 1978.

[44] HEDLUND, G. A.. **Fuchsian groups and mixtures**. Annals of Mathematics, p. 370–383, 1939.

[45] HEDLUND, G.; MORSE, M.. **Manifolds without conjugate points**. Trans. Amer. Math. Soc, 51:362–386, 1942.

[46] GOTO, M. S.. **Manifolds without focal points**. Journal of Differential Geometry, 13(3):341–359, 1978.

[47] KLINGENBERG, W.. **Geodätischer fluss auf mannigfaltigkeiten vom hyperbolischen typ**. Inventiones mathematicae, 14(1):63–82, 1971.

[48] KLINGENBERG, W.. **Riemannian manifolds with geodesic flow of anosov type**. Annals of Mathematics, 99(1):1–13, 1974.

[49] EBERLEIN, P.. **Geodesic flows on negatively curved manifolds. ii**. Transactions of the American Mathematical Society, 178:57–82, 1973.

[50] BURNS, K.. **The flat strip theorem fails for surfaces with no conjugate points**. Proceedings of the American Mathematical Society, p. 199–206, 1992.

[51] FREIRE, A.; MANÉ, R.. **On the entropy of the geodesic flow in manifolds without conjugate points**. Inventiones mathematicae, 69(3):375–392, 1982.

[52] BALLMANN, W.; BRIN, M. ; BURNS, K.. **On surfaces with no conjugate points**. Journal of Differential Geometry, 25(2):249–273, 1987.

[53] RUGGIERO, R. O.. **On the divergence of geodesic rays in manifolds without conjugate points, dynamics of the geodesic flow and global geometry**. ASTERISQUE-SOCIETE MATHEMATIQUE DE FRANCE, 287:231–247, 2003.

[54] RUGGIERO, R. O.. **Expansive dynamics and hyperbolic geometry**. Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society, 25(2):139–172, 1994.

[55] RIFFORD, L.; RUGGIERO, R.. **On the stability conjecture for geodesic flows of manifold without conjugate points**. Annales Henri Lebesgue, 4:759–784, 2021.

[56] GULLIVER, R.. **On the variety of manifolds without conjugate points**. Transactions of the American Mathematical Society, 210:185–201, 1975.

[57] PARRY, W.. **Intrinsic markov chains**. Transactions of the American Mathematical Society, 112(1):55–66, 1964.

[58] MARGULIS, G. A.. **Certain measures associated with u-flows on compact manifolds**. Functional Analysis and Its Applications, 4(1):55–67, 1970.

[59] BOWEN, R.. **Periodic orbits for hyperbolic flows**. American Journal of Mathematics, 94(1):1–30, 1972.

[60] BOWEN, R.. **Maximizing entropy for a hyperbolic flow**. Mathematical systems theory, 7(3):300–303, 1973.

[61] BURNS, K.; KATOK, A.; BALLMAN, W.; BRIN, M.; EBERLEIN, P. ; OSSERMAN, R.. **Manifolds with non-positive curvature**. Ergodic Theory and Dynamical Systems, 5(2):307–317, 1985.

[62] KNIEPER, G.. **The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds**. Annals of mathematics, p. 291–314, 1998.

[63] BOSCHÉ, A.. **Expansive geodesic flows on compact manifolds without conjugate points.** PhD thesis, Ruhr-Universität, 2015.

[64] BURNS, K.; CLIMENHAGA, V.; FISHER, T. ; THOMPSON, D. J.. **Unique equilibrium states for geodesic flows in nonpositive curvature**. Geometric and Functional Analysis, 28(5):1209–1259, 2018.

[65] CHEN, D.; KAO, L.-Y. ; PARK, K.. **Unique equilibrium states for geodesic flows over surfaces without focal points**. Nonlinearity, 33(3):1118, 2020.

[66] LIU, F.; WANG, F.. **The uniqueness of knieper measure on non-compact rank 1 manifolds of non-positive curvature**. Acta Mathematica Sinica, English Series, 37(8):1219–1228, 2021.

[67] BING, R.. **An alternative proof that 3-manifolds can be triangulated**. Annals of Mathematics, p. 37–65, 1959.

[68] BUZZI, J.; FISHER, T.; SAMBARINO, M. ; VÁSQUEZ, C.. **Maximal entropy measures for certain partially hyperbolic, derived from anosov systems**. Ergodic theory and dynamical systems, 32(1):63–79, 2012.

[69] WILLARD, S.. **General topology**. Courier Corporation, 2012.

[70] THOMAS, R. F.. **Canonical coordinates and the pseudo orbit tracing property**. Journal of differential equations, 90(2):316–343, 1991.

[71] NEWHOUSE, S. E.. **Continuity properties of entropy**. Annals of Mathematics, 129(1):215–235, 1989.

[72] BOWEN, R.. **Entropy-expansive maps**. Transactions of the American Mathematical Society, 164:323–331, 1972.

[73] MISIUREWICZ, M.. **Topological conditional entropy**. Studia Mathematica, 55(2):175–200, 1976.

[74] GLASMACHERS, E.; KNIEPER, G.; OGOUYANDJOU, C. ; SCHRÖDER, J. P.. **Topological entropy of minimal geodesics and volume growth on surfaces**. Journal of Modern Dynamics, 8(1):75, 2014.

[75] LIU, F.; WANG, F.. **Entropy-expansiveness of geodesic flows on closed manifolds without conjugate points**. Acta Mathematica Sinica, English Series, 32(4):507–520, 2016.

[76] LEDRAPPIER, F.; WALTERS, P.. **A relativised variational principle for continuous transformations**. Journal of the London Mathematical Society, 2(3):568–576, 1977.

[77] URES, R.. **Intrinsic ergodicity of partially hyperbolic diffeomorphisms with a hyperbolic linear part**. Proceedings of the American Mathematical Society, 140(6):1973–1985, 2012.

[78] BALLMANN, W.; BRIN, M. ; EBERLEIN, P.. **Structure of manifolds of nonpositive curvature. i**. Annals of Mathematics, 122(1):171–203, 1985.

[79] RUGGIERO, R. O.; MENESES, V. A. R.. **On the pesin set of expansive geodesic flows in manifolds with no conjugate points**. Bulletin of the Brazilian Mathematical Society, 34(2):263–274, 2003.

[80] PATERNAIN, G. P.. **Finsler structures on surfaces with negative euler characteristic**. Houston Journal of Mathematics, 23:421–426, 1997.

[81] GOMES, J. B.; RUGGIERO, R. O.. **On finsler surfaces without conjugate points**. Ergodic Theory and Dynamical Systems, 33(2):455–474, 2013.

[82] LEDRAPPIER, F.; LIMA, Y. ; SARIG, O. M.. **Ergodic properties of equilibrium measures for smooth three dimensional flows**. Commentarii Mathematici Helvetici, 91(1):65–106, 2016.

[83] ARAUJO, E.; LIMA, Y. ; POLETTI, M.. **Symbolic dynamics for nonuniformly hyperbolic maps with singularities in high dimension**. arXiv preprint arXiv:2010.11808, 2020.

[84] PATERNAIN, M.. **Expansive geodesic flows on surfaces**. Ergodic Theory and Dynamical Systems, 13(1):153–165, 1993.

[85] RUGGIERO, R. O.. **Expansive geodesic flows in manifolds with no conjugate points**. Ergodic Theory and Dynamical Systems, 17(1):211–225, 1997.

A

The topological mixing property for compact manifolds with visibility universal covering

In this appendix we will assume a compact manifold M without conjugate points and visibility universal covering $\tilde{M}$. Eberlein [49] showed the topological mixing for the geodesic flow of complete manifolds of non-positive curvature. His argument can be extended to the visibility case almost unchanged. For completeness, we give Eberlein's proof for the visibility case, without assuming the non-positive curvature hypothesis. Eberlein's works [49, 25] are the main references for the proof.

Lemma A.1. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then*

1. *If $v \in T_1M$ is periodic with period $a > 0$ and $w \in \overline{\mathcal{F}^{cs}(v)}$, then $\phi_c(w) \in \overline{\mathcal{F}^s(v)}$ for some $c \in [0, a]$.*
2. *If $w \in \overline{\mathcal{F}^s(v)}$ then $\overline{\mathcal{F}^s(w)} \subset \overline{\mathcal{F}^s(v)}$.*
3. *If $w \in \overline{\mathcal{F}^{cs}(v)}$ then $\overline{\mathcal{F}^{cs}(w)} \subset \overline{\mathcal{F}^{cs}(v)}$.*

Proof. For item 1, let $(w_n) \subset \mathcal{F}^{cs}(v)$ with $w_n \to w$. For every $n \geq 1$ there exists $c_n \in [0, a]$ such that $\phi_{c_n}(w_n) \in \mathcal{F}^s(v)$. Passing to a subsequence $c_n \to c$ and hence there exists $c \in [0, a]$ such that

$$\phi_{c_n}(w_n) \to \phi_c(w) \in \overline{\mathcal{F}^s(v)}.$$

For item 2, let $v \in T_1M$, $w \in \overline{\mathcal{F}^s(v)}$, $u \in \overline{\mathcal{F}^s(w)}$ and $U \subset T_1M$ be an open neighborhood of u. We see that $U \cap \mathcal{F}^s(w) \neq \emptyset$. Since $\tilde{M}$ is a visibility manifold, Proposition 2.3.14 shows that horospherical foliations are continuous. Hence we can choose an open neighborhood V of w such that for every $x \in V$, $\mathcal{F}^s(x) \cap U \neq \emptyset$. Since $w \in \overline{\mathcal{F}^s(v)}$ there exists $v' \in \mathcal{F}^s(v) \cap V$ hence

$$\mathcal{F}^s(v) \cap U = \mathcal{F}^s(v') \cap U \neq \emptyset.$$

Therefore $u \in \overline{\mathcal{F}^s(v)}$ and $\overline{\mathcal{F}^s(w)} \subset \overline{\mathcal{F}^s(v)}$. The reasoning for item 3 is similar. ■

For every $v \in T_1M$, the positive prolongational limit set of v, $P^+(v)$, is the set of points $w \in T_1M$ such that for any open neighborhoods U, V of v, w respectively, there exists a sequence $t_n \to \infty$ satisfying

$$\phi_{t_n}(U) \cap V \neq \emptyset \quad \text{for every } n \geq 1.$$

The negative prolongational limit set $P^-(v)$ is defined similarly considering sequences $t_n \to -\infty$. The following basic properties are straightforward from the definitions.

Lemma A.2. *Let M be a compact manifold without conjugate points and $v \in T_1M$.*

1. *The sets $P^+(v)$ and $P^-(v)$ are closed and invariant by the geodesic flow.*
2. *$P^+(-v) = -P^-(v)$ and $P^-(-v) = -P^+(v)$.*
3. *$w \in P^+(v)$ if and only if there exist sequences $(v_n) \subset T_1M$, $t_n \to \infty(-\infty)$ such that $v_n \to v$ and $\phi_{t_n}(v_n) \to w$.*
4. *$w \in P^-(v)$ if and only if there exist sequences $(v_n) \subset T_1M$, $t_n \to -\infty$ such that $v_n \to v$ and $\phi_{t_n}(v_n) \to w$.*

Another definition related to the prolongational sets and specially useful for topological transitivity follows. Let $(t_n) \subset \mathbb{R}$ be a sequence with $t_n \to \infty$ or $t_n \to -\infty$. For every $v, w \in T_1M$, we say that v is t_n **related to** w if for any open neighborhoods U, V of v, w respectively, there exists $N(U, V) \geq 1$ such that for every $n \geq N$,

$$\phi_{t_n}(U) \cap V \neq \emptyset.$$

We list some properties that follow from the definitions.

Lemma A.3. *Let M be a compact manifold without conjugate points and $v, w \in T_1M$.*

1. *$w \in P^+(v)$ if and only if v is t_n related to w for some sequence $t_n \to \infty$.*
2. *v is t_n related to w if and only if w is $-t_n$ related to v.*
3. *v is t_n related to w if and only if $-w$ is t_n related to $-v$.*

4. *For any $t_n \to \pm\infty$, the set $\{w \in T_1M : v$ is t_n related to $w\}$ is closed in T_1M.*

5. *For any $t_n \to \pm\infty$, the set $\{v \in T_1M : v$ is t_n related to $w\}$ is closed in T_1M.*

6. *v is t_n related to w if and only if there exists a sequence $(v_n) \subset T_1M$ such that $v_n \to v$ and $\phi_{t_n}v_n \to w$.*

Lemma A.4. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. For every $v, w \in T_1M$ such that v is t_n related to w for some sequence $t_n \to \infty$, it holds*

1. *If $v' \in \overline{\mathcal{F}^s(v)}$ then v' is t_n related to w.*

2. *If $w' \in \overline{\mathcal{F}^u(w)}$ then v is t_n related to w'.*

Proof. For item 1, by Lemma A.3(5), it suffices to show the claim for $u \in \mathcal{F}^s(v)$. By Lemma A.3(6), there exists a sequence $v_n \to v$ with $\phi_{t_n}v_n \to w$. Let $(p, u'), (q, v'), (q_n, v_n') \in T_1\tilde{M}$ be lifts of u, v, v_n respectively such that $u' \in \tilde{\mathcal{F}}^s(v')$ and $v_n' \to v'$. It follows that $\gamma_n \to \gamma_{v'}$ where $\gamma_n = \gamma_{v_n'}$. This implies that $\gamma_n(t_n) \to \gamma_{v'}(\infty)$. If $s_n = d(p, \gamma_n(t_n))$ then

$$\begin{aligned}|s_n - t_n| &= |d(p, \gamma_n(t_n)) - d(q_n, \gamma_n(t_n))| \\ &\leq |d(p, \gamma_n(t_n)) - d(q, \gamma_n(t_n))| + |d(q, \gamma_n(t_n)) - d(q_n, \gamma_n(t_n))| \to 0\end{aligned}$$

because $p, q \in H^+(v')$, $\gamma_n(t_n) \to \gamma_{v'}(\infty)$ and $q_n \to q$. For every $n \geq 1$, let $u_n' \in T_1\tilde{M}$ such that $\gamma_{u_n'}$ joins p to $\gamma_n(t_n)$. Note that vectors $\phi_{t_n}(v_n')$ and $\phi_{s_n}(u_n')$ are tangent to $\tilde{M}$ at $\gamma_n(t_n)$. By visibility condition, $\angle_{\gamma_n(t_n)}(\phi_{t_n}v_n', \phi_{s_n}u_n') \to 0$ because $d(p, q_n)$ is bounded.

For every $n \geq 1$, let $u_n = d\pi(u_n')$. Since $\gamma_n(t_n) \to \gamma_{v'}(\infty) = \gamma_{u'}(\infty)$, we conclude that $u_n' \to u'$ hence $u_n \to u$. Thus, the result follows from

$$\lim_{n\to\infty} \phi_{t_n}u_n = \lim_{n\to\infty} \phi_{s_n}u_n = \lim_{n\to\infty} \phi_{t_n}v_n = w.$$

For item 2, if $w' \in \overline{\mathcal{F}^u(w)} = \overline{-\mathcal{F}^s(-w)}$ then $-w' \in \overline{\mathcal{F}^s(-w)}$. By Lemma A.3(3) we see that $-w$ is t_n related to $-v_n$ hence $-w'$ is t_n related to $-v$ by (1). This means that v is t_n related to w'. ■

We next study the relationship between center stable and negative prolongational sets.

Lemma A.5. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Given $v, w \in T_1M$,*

1. *If $w \in P^+(v)$ then $\overline{\mathcal{F}^{cu}(w)} \subset P^+(v)$.*
2. *If $w \in P^-(v)$ then $\overline{\mathcal{F}^{cs}(w)} \subset P^-(v)$.*

Proof. For item 1, let $w' \in \mathcal{F}^{cu}(w)$. Then there exists $t \in \mathbb{R}$ with $\phi_t(w') \in \mathcal{F}^u(w)$. By hypothesis, v is t_n related to w for some sequence $t_n \to \infty$. By Lemma A.4 we see that v is t_n related to $\phi_t(w')$ hence v is $(t_n - t)$ related to w'. This implies that $w' \in P^+(v)$ and $\mathcal{F}^{cu}(w) \subset P^+(v)$. The result follows because $P^+(v)$ is closed. Item 2 is analogous. ■

Recall that a point $v \in T_1M$ is called **non-wandering** if $v \in P^+(v)$ (equivalently $v \in P^-(v)$). The set of non-wandering points is denoted by Ω. Note that Ω is closed and invariant by the geodesic flow. In particular, for a compact manifold M, $\Omega = T_1M$. We will complete the relationship between center stable and negative prolongational sets given in Lemma A.5, for the case of non-wandering points.

Lemma A.6. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. For every $v \in T_1M$,*

$$P^+(v) = \overline{\mathcal{F}^{cu}(v)} \quad \text{and} \quad P^-(v) = \overline{\mathcal{F}^{cs}(v)}.$$

Proof. Let $w \in P^-(v)$ and $(p, v'), (q, w') \in T_1\tilde{M}$ be lifts of v, w respectively. By Lemma 3.2 of [25], $\gamma_{v'}(-\infty)$ and $\gamma_{w'}(\infty)$ are dual relative to $\pi_1(M)$. Thus, there exists a sequence of covering isometries (T_n) such that $T_n^{-1}(p) \to \gamma_{v'}(-\infty)$ and $T_n(p) \to \gamma_{w'}(\infty)$. This and Proposition 2.2 of [25] imply that

$$\begin{aligned}\angle_p(T_n(\gamma_{v'}(\infty)), \gamma_{w'}(\infty)) &\leq \angle_p(T_n(\gamma_{v'}(\infty)), T_n(p)) + \angle_p(T_n(p), \gamma_{w'}(\infty)) \\ &= \angle_{T_n^{-1}(p)}(\gamma_{v'}(\infty), p) + \angle_p(T_n(p), \gamma_{w'}(\infty)) \to 0.\end{aligned}$$

Therefore $T_n(\gamma_{v'}(\infty)) \to \gamma_{w'}(\infty)$. For every $n \geq 1$ let $w'_n \in T_1\tilde{M}$ such that $\gamma_{w'_n}$ joins q to $T_n(\gamma_{v'}(\infty))$. Since $\gamma_{w'}$ joins q to $\gamma_{w'}(\infty)$, we conclude that $w'_n \to w'$. We also see that $\gamma_{dT_n^{-1}(w'_n)}$ joins $T_n^{-1}(q)$ to $\gamma_{v'}(\infty)$ and hence $dT_n^{-1}w'_n \in \tilde{\mathcal{F}}^{cs}(v')$. This implies that $w_n = d\pi(w'_n) \in \mathcal{F}^{cs}(v)$, $w_n \to w$ and so, $P^-(v) \subset \overline{\mathcal{F}^{cs}(v)}$. Since every $v \in T_1M$ is a non-wandering point, $v \in P^-(v)$ and Lemma A.5 together imply the reverse inclusion. For $P^+(v)$, note that

$$P^+(v) = -P^-(-v) = -\overline{\mathcal{F}^{cs}(-v)} = \overline{\mathcal{F}^{cu}(v)}.$$

■

We will show that the center stable and center unstable foliations are minimal.

Lemma A.7. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Assume that b_τ is any Busemann function of the geodesic $\tau \subset \tilde{M}$. If $\sigma \subset \tilde{M}$ is a geodesic with $\sigma(\infty) \neq \tau(\infty)$ then*

$$\lim_{t\to\infty} b_\tau(\sigma(t)) = \infty.$$

Proof. Proposition 2.3.3 provides a geodesic $\beta \subset \tilde{M}$ such that $\beta(-\infty) = \sigma(\infty)$ and $\beta(\infty) = \tau(\infty)$. Denote by β^{-1} the geodesic $t \mapsto \beta(-t)$. We see that $b_\tau(\beta(t)) = b_\tau(\beta(0)) - t$ and so

$$\lim_{t\to-\infty} b_\tau(\beta(t)) = \lim_{t\to\infty} b_\tau(\beta^{-1}(t)) = \infty.$$

Since β^{-1} and σ are asymptotic, $d(\beta^{-1}(s), \sigma(s))$ is bounded for $s \geq 0$ and

$$\begin{aligned} b_\tau(\sigma(s)) &= \lim_{t\to\infty} d(\sigma(s), \tau(t)) - t \geq \lim_{t\to\infty} d(\tau(t), \beta^{-1}(s)) - d(\beta^{-1}(s), \sigma(s)) - t \\ &\geq \lim_{t\to\infty} d(\tau(t), \beta^{-1}(s)) - t = b_\tau(\beta^{-1}(s)). \end{aligned}$$

The result follows letting $s \to \infty$. ■

In geometrical terms, τ is the only geodesic starting at $p \in \tilde{M}$, belonging to the horoball centered at $\tau(\infty)$ and bounded by the horosphere $H^+(\dot{\tau}(0))$.

Lemma A.8. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then, for every open sets $U, V \subset T_1M$ there exists $v \in U$ such that*

$$\mathcal{F}^s(v) \cap V \neq \emptyset.$$

Proof. Let $(p, v), (q, w) \in T_1\tilde{M}$ be lifts of any vectors of $U, V \subset T_1M$ respectively and $x = \gamma_v(\infty)$, $y = \gamma_w(\infty)$. By Proposition 2.8 of [25], x and y are dual respect to $\pi_1(M)$ hence there exists a sequence of covering isometries (T_n) such that $T_n(r) \to x$ and $T_n^{-1}(r) \to y$ for every $r \in \tilde{M}$. For every $n \geq 1$, let $x_n \in \tilde{M}(\infty)$ be the center of the horosphere $H^+(v_n)$ passing through $p, T_nq \in \tilde{M}$, for a sequence $(p, v_n) \subset T_1\tilde{M}$.

We will show that $x_n \to x$. Fix any $t \geq 0$, and let σ_n be the geodesic joining p to $T_n(q)$. For large enough n, we see that $t \leq d(p, T_n(q))$. Since $p, T_n(q) \in H^+(v_n)$, we conclude that $b_{v_n}(\sigma_n(t)) \leq 0$. Passing to a subsequence

$x_n \to z \in \tilde{M}(\infty)$. Since $T_n(q) \to x$, we deduce that $\sigma_n \to \gamma_v$. Let b_z be the Busemann function determined by $p \in \tilde{M}$ and $z \in \tilde{M}(\infty)$. By continuity of Busemann functions, $b_{v_n} \to b_z$ and $\sigma_n \to \gamma_v$. Thus, we see that $b_z(\gamma_v(t)) \leq 0$ hence

$$\lim_{t\to\infty} b_z(\gamma_v(t)) \to -\infty.$$

Lemma A.7 shows that $x = \gamma_v(\infty) = z$. Therefore $x_n \to x$ because every convergent subsequence of x_n converges to x.

We next show that $T_n^{-1}(x_n) \to y$. For $0 \leq t \leq d(p, T_n(q))/2$, the triangle inequality ensures that $d(T_n(q), \gamma_{v_n}(t)) \geq d(p, T_n(q))/2$. If $t \geq d(p, T_n(q))/2$ then

$$d(T_n(q), \gamma_{v_n}(t)) \geq |b_{v_n}(T_n(q)) - b_{v_n}(\gamma_n(t))| \geq |b_{v_n}(\gamma_{v_n}(t))| = t \geq d(p, T_n(q))/2.$$

Therefore, we obtain

$$d(q, T_n^{-1} \circ \gamma_{v_n}) = d(T_n(q), \gamma_{v_n}) \geq d(p, T_n(q))/2 \to \infty.$$

By visibility condition $\angle_q(T_n^{-1}(p), T_n^{-1}(x_n)) \to 0$. Putting all together, since $T_n^{-1}(p) \to y$, we have

$$\angle_q(T_n^{-1}(x_n), y) \leq \angle_q(T_n^{-1}(x_n), T_n^{-1}(p)) + \angle_q(T_n^{-1}(p), y) \to 0,$$

and so $T_n^{-1}(x_n) \to y$.

For each $n \geq 1$, let $w_n \in T_1\tilde{M}$ such that γ_{w_n} joins q to $T_n^{-1}(x_n)$. Since $x_n \to x$ and $T_n^{-1}(x_n) \to y$, we see that $v_n \to v$, $w_n \to w$ and

$$dT_n(w_n) \in \tilde{\mathcal{F}}^s(v_n).$$

For large enough n, we deduce that $d\pi(v_n) \in U$, $d\pi(w_n) \in V$ and $d\pi(w_n) \in \mathcal{F}^s(v_n)$. ■

Theorem A.1. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then, there exists $v \in T_1M$ such that $\mathcal{F}^s(v)$ is dense in T_1M.*

Proof. Let (U_n) be a countable basis for the topology of T_1M and U be any open set. By Lemma A.8, there exists $v_1 \in U$ with $\mathcal{F}^s(v_1) \cap U_1 \neq \emptyset$. Choose an open neighborhood A_1 of v_1 with compact closure $\overline{A}_1 \subset U$ satisfying for every $v \in A_1$,

$$\mathcal{F}^s(v) \cap U_1 \neq \emptyset.$$

Inductively, we construct a sequence of vectors $v_n \in U$, a sequence of open neighborhoods A_n of v_n with compact closures $\overline{A}_n \subset A_{n-1}$ such that

$$\mathcal{F}^s(v) \cap U_n \neq \emptyset \quad \text{for any } v \in A_n.$$

Thus, there exists $w \in \cap_n \overline{A}_n$ such that $\mathcal{F}^s(w) \cap U_n \neq \emptyset$ for every $n \geq 1$, and the result follows. ■

Since $\mathcal{F}^s(w) \subset \mathcal{F}^{cs}(w)$, this theorem implies that there exists $w \in T_1M$ such that $\mathcal{F}^{cs}(w)$ is dense in T_1M. In fact, we see that every center stable set is dense in T_1M.

Lemma A.9. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then, for every $v \in T_1M$, $\mathcal{F}^{cs}(v)$ is dense in T_1M.*

Proof. Let $v, w, v' \in T_1M$ with $\overline{\mathcal{F}^{cs}(v')} = T_1M$. Thus, there exists a sequence $(w_n) \subset \mathcal{F}^{cs}(v')$ such that $w_n \to w$ and $\mathcal{F}^{cs}(w_n) = \mathcal{F}^{cs}(v')$ for every $n \geq 1$. By Lemma A.6,

$$P^-(w_n) = \overline{\mathcal{F}^{cs}(w_n)} = \overline{\mathcal{F}^{cs}(v')} = T_1M.$$

Therefore, for every $n \geq 1$, $v \in P^-(w_n)$ and $w_n \in P^+(v)$. Thus $w \in P^+(v)$ because $P^+(v)$ is closed. So, for any $v \in T_1M$,

$$\overline{\mathcal{F}^{cu}(v)} = P^+(v) = T_1M.$$

Note that for every $v \in T_1M$,

$$T_1M = -\overline{\mathcal{F}^{cs}(v)} = \overline{\mathcal{F}^{cu}(-v)}.$$

■

Using the minimality of center foliations, we finally show the topological mixing of the geodesic flow.

Lemma A.10. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. If $v \in T_1M$ is periodic with period $a > 0$ then $\mathcal{F}^s(v)$ is dense in T_1M.*

Proof. Theorem A.1 gives $w \in T_1M$ such that $\overline{\mathcal{F}^s(w)} = T_1M$. Lemma A.9 guarantees that $w \in \overline{\mathcal{F}^{cs}(v)}$. Moreover, Lemma A.1(1) shows that $\phi_c(w) \in \overline{\mathcal{F}^s(v)}$ for some $c \in [0, a]$. By Lemma A.1(2), $\overline{\mathcal{F}^s(\phi_c(w))} \subset \overline{\mathcal{F}^s(v)}$ and thus

$$T_1M = \phi_c(T_1M) = \phi_c(\overline{\mathcal{F}^s(w)}) = \overline{\mathcal{F}^s(\phi_c v)}.$$

■

Theorem A.2. *Let M be a compact manifold without conjugate points and visibility universal covering $\tilde{M}$. Then the geodesic flow is topologically mixing.*

Proof. Let $(t_n) \subset \mathbb{R}$ be a sequence with $t_n \to \infty$, and $v' \in T_1M$ be a periodic vector of period $a > 0$. Choose a subsequence (s_n) of (t_n) such that $\phi_{s_n} v' \to \phi_c v'$ for some $c \in [0, a]$. Clearly v' is s_n related to $\phi_c v'$. Since $\phi_{-c} v'$ is periodic, we have

$$\overline{\mathcal{F}^u(\phi_c v')} = \overline{-\mathcal{F}^s(-\phi_c v')} = -\overline{\mathcal{F}^s(\phi_{-c}(-v'))} = T_1M.$$

Lemma A.10 says that $\overline{\mathcal{F}^s(v')} = T_1M$. Moreover, Lemma A.4 shows that v is s_n related to w for every $v, w \in T_1M$. Thus for every open sets $U, V \subset T_1M$ there exists $A_1 > 0$ such that $t \geq A$ implies that

$$\phi_t(U) \cap V \neq \emptyset.$$

Considering the open sets $-U$ and $-V$, there exists $A_2 > 0$ such that $t \geq A_2$ implies that

$$\emptyset \neq \phi_t(-U) \cap (-V) = -\phi_{-t}(U) \cap (-V) = -(\phi_{-t}(U) \cap V).$$

Choosing $A = \max(A_1, A_2)$ we get the topological mixing:

$$\phi_t(U) \cap V \neq \emptyset \quad \text{for } |t| \geq A.$$

■