Pontifícia Universidade Católica
do Rio de Janeiro

**Fiorella María Rendón García**

# Global boundary weak Harnack inequality for general uniformly elliptic equations in divergence form and applications.

**Tese de doutorado**

Thesis presented to the Programa de Pós–graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Boyan Slavchev Sirakov

Rio de Janeiro
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# Abstract

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This thesis focuses on global extension of the interior weak Harnack inequality for a general class of divergence-type elliptic equations, under very weak regularity assumptions on the differential operator. In this way we generalize and unify all previous results of this type.

As an application, we prove a priori estimates for a class of quasilinear elliptic problems with quadratic growth on the gradient and we investigate, under various assumptions, the multiplicity of the solutions obtained for this problem.



## Keywords

Harnack inequality; Global estimates; Regularity theory; Existence theory; Natural growth;

# Resumo

Rendón García, Fiorella María; Sirakov, Boyan. **Desigualdade de Harnack global para operadores ellípticos gerais na forma divergente com aplicações.**. Rio de Janeiro, 2022. 84p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta tese estudamos a extensão da desigualdade fraca de Harnack até o bordo para uma equação de segunda ordem elíptica geral na forma divergência, assumindo pouca regularidade sobre o operador diferencial. Assim, generalizamos e unificamos todos os resultados precedentes deste tipo.

Como aplicação, mostramos estimativas a priori para uma classe de problemas elípticos quasilineares com crescimento quadratico no gradiente e investigamos, sob várias hipóteses, a multiplicidade das soluções obtidas para este problema.



## Palavras-chave

Desigualdade de Harnack; Estimativas globais; Teoria de regularidade; Existência; Crescimento natural;

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# 1
# Introduction

This thesis proves global extensions of the interior Weak Harnack Inequality and the Zaremba-Hopf-Oleinik boundary point principle for a general divergence-type uniformly elliptic operator. We begin this introduction by recalling these two results, which have been fundamental in the development of the theory of elliptic PDE.

Let us have a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, in which is defined a second-order uniformly elliptic operator in either a non-divergence form

$$\mathcal{L}[u] = -\operatorname{tr}(A(x)D^2u) + b(x).Du + c(x)u, \ \ x \in \Omega, \tag{1.1}$$

or divergence form

$$L[u] = -\operatorname{div}(A(x)Du) + b(x).Du + c(x)u, \ \ x \in \Omega, \tag{1.2}$$

where $A$ is a uniformly positive bounded measurable matrix, i.e. $\vartheta I_n \leq (a_{ij}(x)) \leq \vartheta^{-1} I_n$, $\vartheta$ is a positive constant, and $I_n$ is the identity matrix; and the coefficients $b, c \in L^p(\Omega)$ for some $p > n$ (in particular $b, c$ can be bounded measurable), $\|b\|_{L^p}, \|c\|_{L^p} \leq \vartheta^{-1}$. Fix also $f \in L^p(\Omega)$.

The following basic result goes back to De Giorgi and Moser in the divergence case, and to Krylov and Safonov for non-divergence form operators.

**Theorem 1.1 (Interior Weak Harnack Inequality, IWHI)** *There exist constants $\varepsilon > 0$ depending only on $n, p, \vartheta$ and $C > 0$ depending only on $n, p, \vartheta, R > 0$ such that if $B_{2R} \subset \Omega$ then for each nonnegative solution of $\mathcal{L}[u] \geq f$ or $L[u] \geq f$ in $\Omega$ we have*

$$\left( \int_{B_R} u^\varepsilon \right)^{1/\varepsilon} \leq C \left( \inf_{B_R} u + \|f\|_{L^p(B_{2R})} \right). \tag{1.3}$$

*Hence, by a scaling and covering argument, for each compact $K \subset \Omega$*

$$\inf_K u \geq C \left( \int_K u^\varepsilon \right)^{1/\varepsilon} - C\|f\|_{L^p(\Omega)}, \tag{1.4}$$

*where $C$ depends also on $K$ and $dist(K, \partial\Omega)$.*

When $f = 0$ and $\inf u = 0$ this result reduces to the classical *strong maximum principle* (SMP), which says a nonnegative supersolution cannot vanish inside the domain unless it is trivial. Actually (1.3)-(1.4) for $f = 0$ can be seen as a quantitative extension of the SMP in the following sense: if we know that $u$ is positive somewhere, then $u$ is positive everywhere with a quantified lower bound; specifically, if $u \geq a > 0$ in some (unknown) $\omega \subset B_R$ then $u \geq aC^{-1}|\omega|^{1/\varepsilon}$ in $B_R$ ("a growth lemma"). If $u$ is not $L$-superharmonic, there is a correction in this inequality with the $L^p$-norm of the right hand side.

The discovery of the IWHI (in somewhat modified form for divergence form operators with $b = c = 0$) by E. De Giorgi in the 1960's was the final and decisive step in the resolution of the 19th Hilbert problem on the regularity of minimizers of variational integrals. In its full form Theorem 1.1 was proved by Moser and Trudinger for general divergence form operators a few years later. Furthermore, the corresponding result in the non-divergence case was reached in the early 1980s by Krylov and Safonov, and essentially opened up the theory of non-divergence form operators. The importance of the IWHI lies in particular in that it implies a Hölder bound for the solutions of $\mathcal{L}[u] = f$ or $L[u] = f$: there exists $\alpha > 0$ (depending on $n, p, \vartheta$) such that

$$\|u\|_{C^\alpha(B_R)} \leq C\left(\|u\|_{L^\infty(B_{2R})} + \|f\|_{L^p(B_{2R})}\right). \tag{1.5}$$

The latter is at the base of the *regularity theory* in Hölder spaces for solutions. Classical results of this theory, more references, and a proof of Theorem 1.1 and (1.5) can be found in Chapters 6, 8 and 9 of [GT01].

Another fundamental result in the elliptic theory is the so-called "Hopf lemma"[1], to which we will refer as the boundary point principle (BPP). In its classical form it says that if a nonnegative nontrivial supersolution vanishes at a point of the (sufficiently smooth) boundary of $\Omega$, then its gradient does not vanish at that point; specifically, if $\mathcal{L}[u] \leq 0$, $u > 0$ in $\Omega$, and $u(x_0) = 0$ for some $x_0 \in \partial\Omega$ such that there is an interior tangent ball to $\partial\Omega$ at $x_0$, then the interior normal derivative $\frac{\partial u}{\partial \nu}(x_0) > 0$. The optimal regularity of the boundary for this result to hold is interior $C^{1,Dini}$ (we write $C^{1,D}$, see below and the next section), and it is known that it may fail even for the Laplacian for a domain with a $C^1$ boundary.

Set $B'_R = B_R(x_0) \cap \Omega$ and $d(x) = \text{dist}(x, \partial\Omega)$. Another way to write the BPP is the following.

[1]This is the most often encountered name of the result, even though various particular cases were known before the classical work of Hopf from 1954, starting with a paper by Zaremba in 1910; Oleinik proved the same result simultaneously with Hopf, so in some sources it is called Zaremba-Hopf-Oleinik lemma.

**Theorem 1.2 (BPP, Zaremba-Hopf-Oleinik lemma)** *For each nonnegative solution $u$ of $\mathcal{L}[u] \geq 0$ in $B'_{2R}$ we have*

$$\inf_{B'_R} \frac{u}{d} > 0. \tag{1.6}$$

The BPP has immediate consequences for the uniqueness of solutions of Neumann and Robin (mixed-type) boundary value problems. Its quantitative forms are at the base of the up-to-the-boundary (global) Hölder regularity theory for solutions of the Dirichlet problem for uniformly elliptic operators.

In relation to the BPP, a rather distinction appears between divergence and non-divergence form operators. While no regularity assumptions on the coefficients are needed in the non-divergence case, for general divergence form with only bounded measurable coefficients the BPP fails. For its validity it is necessary that leading coefficients $a_{ij}$ be at least Dini continuous (simple continuity is not enough). We refer to [ADN16], [AN19], and the references there for details.

At a first glance the IWHI and the BPP have little in common. The main theoretical contribution of this thesis is a quantitative inequality which extends both results, and bridges them into a single statement, for divergence-form operators.



**Theorem 1.3 (Boundary Weak Harnack Inequality, BWHI)** *In addition to the above hypotheses, assume that the boundary of $\Omega$ is $C^{1,D}$-smooth, and that $a_{ij} \in C^{0,D}(\Omega)$, $\|a_{ij}\|_{C^{0,D}(\Omega)} \leq \vartheta^{-1}$, $i, j = 1, \ldots, n$. There exist constants $\varepsilon > 0$ depending only on $n, p, \vartheta$ and $C > 0$ depending only on $n, p, \vartheta, R > 0$ and the $C^{1,D}$-representation of the boundary such that for each nonnegative solution of $L[u] \geq f$ in $\Omega$ we have*

$$\inf_{B'_R} \frac{u}{d} \geq C \left( \int_{B'_R} \left(\frac{u}{d}\right)^{\varepsilon} \right)^{1/\varepsilon} - C\|f\|_{L^p(B'_{2R})}, \tag{1.7}$$

*and*

$$\inf_{\Omega} \frac{u}{d} \geq C \left( \int_{\Omega} \left(\frac{u}{d}\right)^{\varepsilon} \right)^{1/\varepsilon} - C\|f\|_{L^p(\Omega)}. \tag{1.8}$$

The possibility of proving such a result was only recently noticed by B. Sirakov in [S17] where he proved the above theorem in the non-divergence case (also for more general fully nonlinear operators). Here we prove the same result for operators in divergence form. The hypotheses we make on the coefficients and the domain are optimal for the result to hold. We shall observe that, as usual in the theory of elliptic PDE, the results for divergence and non-divergence form operators is very similar, however, essential points and techniques in the proofs are very different. In particular, in the non-divergence case it is quite straightforward to find an "approximate barrier function" (in terms of the distance to the boundary), and use it to deduce an up-to-the-boundary growth lemma, which in turn leads to the BWHI. In our case such a barrier function is not available, and we use an implicit construction, based on solving a sequence of approximating boundary value problems in an annulus around each point on the boundary, together with $C^1$-regularity estimates.

The second part of this thesis is devoted to an application of the BWHI. We will use this inequality as an important tool to prove an uniform a priori bound for solutions of a class of quasilinear elliptic equations with quadratic dependence in the gradient of the unknown function. Specifically, we study the equation

$$\begin{cases} -\operatorname{div}(A(x)Du) &= c_\lambda(x)u + (M(x)Du, Du) + h(x) \quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega; \end{cases} \tag{$P_\lambda$}$$



where $c^+(x)$ and $c^-(x)$ are nonnegative functions such that $c_\lambda(x) := \lambda c^+ - c^-$, for a parameter $\lambda \in \mathbb{R}$; $c^+(x), c^-(x), h(x) \in L^p(\Omega)$ with $p > n$. As before $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class $C^{1,D}$. The matrix $M(x)$ is a positive definite matrix such that

$$0 < \mu_1 I_n \leq M(x) \leq \mu_2 I_n \text{ in } \Omega$$

for some positive constants $\mu_1$ and $\mu_2$.

The main difference between this class of equations and linear equations is that the second-order and the gradient terms in the equation ($P_\lambda$) have the same scaling with respect to dilations (i.e. "zooms", changes of variable $x \to x/r$). Thus the first-order term does not disappear when a zoom is performed around a fixed point, and this leads to quite different properties.

The point in writing the zero order coefficient $c = c_\lambda$ with dependence on a parameter is that the solvability of the Dirichlet problem for ($P_\lambda$) changes dramatically according to the sign of that coefficient. In a few words, solutions are expected to be unique when that coefficient is negative, while phenomena of multiplicity appear for nonnegative coefficients. Thus $\lambda$ serves as a "measure" of the positivity of $c$, and we study, the solutions as functions of $\lambda$.

The study of elliptic equations with gradient dependence up to the critical growth $|Du|^2$ was essentially initiated by Boccardo, Murat and Puel in the 80's, and has been a very active field of research ever since. Up to ten years ago all results concerned the "coercive" case, i.e. $c \leq 0$, when uniqueness holds. On the other hand, from 2014 many works started to uncover the much more complex nature of noncoercive equations. We refer to [BMP1], [ACJT15], [CJ17], [CFJ19], [N18], [NS18], and the large lists of references in these works.

In simple situations, such as $M = const$, the problem $(P_\lambda)$ can be studied by variational methods, after an exponential change of the unknown function. More general equations have been studied by topological (fixed-point) techniques. In the latter, it is essential to prove various a priori bounds in $L^\infty$ for the solutions, with bounds that take into account the dependence in $\lambda$. We will use here some insights from [NS18] in order to generalize the results from [ACJT15] to operators in divergence form with maximal generality in the coefficients as well as the domain.

To illustrate the type of results that we obtain, we include in this introduction two theorems from Chapter 5, and visualize them on a chart describing a value of the solutions at a fixed point in $\Omega$, as a function of the parameter $\lambda$.



**Theorem 1.4** *Suppose that $(P_0)$ has a solution $u_0$ with $c^+(x)u_0 \gneqq 0$. Then*

*(i) For all $\lambda \leq 0$, the problem $(P_\lambda)$ has a unique solution $u_\lambda$ and this solution satisfies $u_0 - \|u_0\|_\infty \leq u_\lambda \leq u_0$.*

*(ii) There exists a continuum $\mathcal{C} \subset \Sigma$ such that the projection of $\mathcal{C}$ on the $\lambda$-axis is an unbounded interval $(-\infty, \overline{\lambda}]$ for some $\overline{\lambda} \in (0, +\infty)$ and $\mathcal{C}$ bifurcates from infinity to the right of the axis $\lambda = 0$.*

*(iii) There exists $\lambda_0 \in (0, \overline{\lambda}]$ such that, for all $\lambda \in (0, \lambda_0)$, the problem $(P_\lambda)$ has at least two solutions with $u_i \geq u_0$ for $i = 1, 2$.*

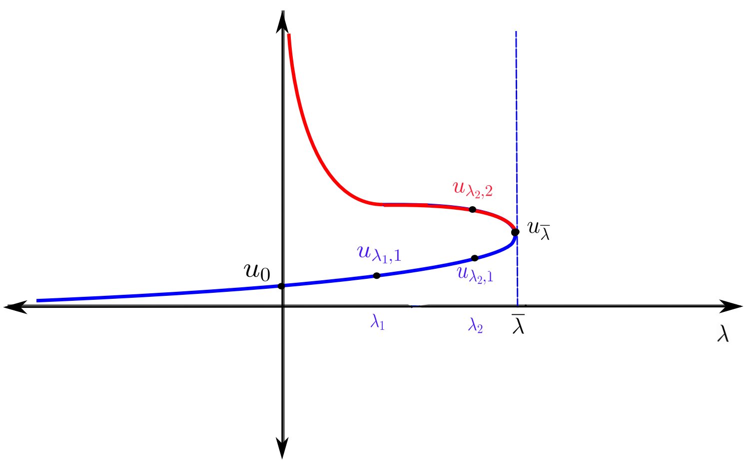


Figure 1.1: Illustration of Theorem 1.4

**Theorem 1.5** *Suppose that $(P_0)$ has a solution $u_0 \leq 0$ with $c^+(x)u_0 \lneqq 0$. Then* *(i)* *For* $\lambda \leq 0$, *the problem* ($P_\lambda$) *has a unique nonpositive solution* $u_\lambda$ *and this solution satisfies* $u_\lambda \geq u_0$;

*(ii)* *There exists a continuum* $\mathcal{C} \subset \Sigma$ *such that its projection of* $\mathcal{C}^+$ *on the* $\lambda$*-axis is* $[0, +\infty)$;

*(iii)* *For* $\lambda > 0$, *every non-positive solution of* ($P_\lambda$) *satisfies* $u_\lambda < u_0$. *Furthermore* ($P_\lambda$) *has at least two non-trivial solutions* $u_{\lambda,i}$ *for* $i = 1, 2$ *with*

$$u_{\lambda,1} < u_0 \leq u_{\lambda,2}, \quad u_{\lambda,1} < u_{\lambda,2}, \text{ and } \max_{\overline{\Omega}} u_{\lambda,2} > 0.$$

*Moreover we have* $u_{\lambda_2,1} \leq u_{\lambda_1,1} \leq u_0$ *if* $0 < \lambda_1 < \lambda_2$.

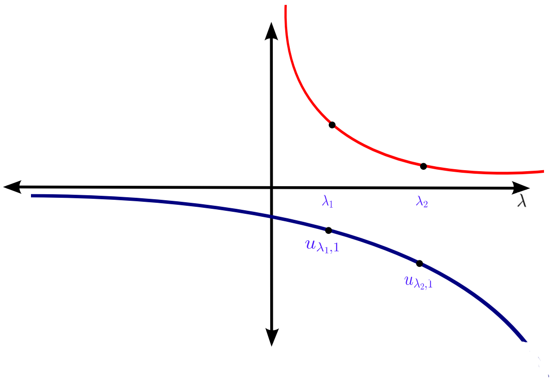


Figure 1.2: Illustration of Theorem 1.5

For the above a priori estimates and multiplicity results for ($P_\lambda$), we need to assume that the problem ($P_0$) has a solution $u_0$. Conditions on the coefficients of the operator which guarantees this are available, see [ACJT15], [CJ17], [CFJ19]. In addition, having a sign information on the solution $u_0$ of ($P_0$) allows us to give rather precise informations on the set of solutions of ($P_\lambda$).

More results on the existence and multiplicity for ($P_\lambda$) are given in Chapter 5, also with weaker assumptions on the solution for $\lambda = 0$. Their proofs are based on a construction of a completely continuous operator and the study of an auxiliary fixed point problem for that operator. Essential compactness properties are inferred from a set of uniform a priori bounds for the solutions, which in turn rely heavily on the BWHI.

This thesis is organized as follows. In Chapter 2 we present some known results and introduce some preliminaries. In Chapter 3 we provide a version of the Boundary Point Hopf Lemma for a divergence form operator, under our general assumptions on the coefficients of the equation and the boundary of the domain. In Chapter 4 we prove the full Theorem 1.3. Then in Chapter 5 we state our existence and multiplicity results on the boundary value problem ($P_\lambda$), and prove the essential a priori bounds for that problem. In the final Chapter 6 we give the proofs of the existence and multiplicity results .

The contents of this thesis are the object of two articles in preparation. The first, devoted to the boundary weak Harnack inequality (Chapters 3-4) is a collaboration with B. Sirakov and M. Soares, while the second, on the problem ($P_\lambda$) (Chapters 5-6) is a joint work with M. Soares.

# 2
# Preliminary Results

In this chapter we recall some notation, definitions and known results (at least for specialists) that will be used throughout the thesis. We begin with some definitions on Dini continuity.

**Definition 2.1** *Let $\sigma : [0,1] \to R_+$ be a function. We say that $\sigma$ belongs to the Dini class $\mathcal{D}$ if*

- *$\sigma$ is increasing and $\sigma(0) = 0$;*
- *$\frac{\sigma(\tau)}{\tau}$ is decreasing and summable at zero.*

*It should be noted that the assumption about the decay of $\sigma(\tau)/\tau$ is not restrictive (see Remark 1.2 in [ADN16] for more details).*

**Definition 2.2** *Let a function $\sigma \in \mathcal{D}$. We define the function $J_\sigma$ as*

$$J_\sigma(s) := \int_0^s \frac{\sigma(\tau)}{\tau} d\tau.$$

**Definition 2.3** *We say that a function $\psi : \Omega \to R$ belongs to the class $C^{0,D}(\Omega)$, if there exists some $\sigma \in \mathcal{D}$ such that*

*(i)* $\psi \in C(\overline{\Omega})$;

*(ii)* $|\psi(x) - \psi(y)| \leq \sigma(|x-y|)$, $\forall x, y \in \Omega$, *and $\sigma$ belongs to the class $\mathcal{D}$.*

We suppose that $\partial\Omega \in C^{1,D}$, which means that $\partial\Omega$ is locally the graph of a $C^1$-function $F$ satisfying $DF \in C^{0,D}$, where $DF$ is the gradient of $F$. Actually, for the boundary point principle it is sufficient that $\partial\Omega \in C^{1,D}$ only from the inside, as in the following definition.

**Definition 2.4** *We say that $\partial\Omega$ satisfies the interior $C^{1,D}$-paraboloid condition if in a local coordinate system $\partial\Omega$ is given by the equation $x_n = F(x')$, where*

*(i)* *$F$ is a $C^1$ function such that $F(0) = 0$;*

*(ii)* *The equation $F(x') \leq |x'|\sigma(|x'|)$ holds true in some neighborhood of the origin, for a $C^1$-function $\sigma \in D$.*

**Remark 2.5** *We observe that every Hölder continuous function* $\psi \in C^{\alpha}(\overline{\Omega})$ *belongs to the class* $C^{0,D}(\Omega)$ *for all* $\alpha \in (0,1)$. *Indeed, it is sufficient to choose* $\sigma(\tau) = C|\tau|^{\alpha}$ *for a suitable positive constante* $C > 0$.

**Remark 2.6** *Without loss of generality* $\sigma \in \mathcal{D}$ *can be assumed continuously differentiable on* $(0,1]$ *(see [AN19]).*

*Furthermore, when* $\partial\Omega$ *satisfies the interior* $C^{1,D}$*-paraboloid condition, in order to prove the BPP we may assume that locally* $\partial\Omega$ *is a paraboloid* $x_n = |x'|\sigma(|x'|)$ *for a smooth* $\sigma \in \mathcal{D}_1$.

**Remark 2.7** *Note that all the assumptions on the coefficients of* $L$ *are invariant under* $C^{1,D}$*-regular changes of variables. Hence, without loss of generality, we may consider* $\partial\Omega$ *locally as the flat boundary* $\{x_n = 0\}$. *We may assume without restriction that* $B_R \cap \mathbb{R}^n_+ \subset \Omega$ *for some* $R > 0$. *For details check Remark 1 and 2 in [AN19], and the end of Chapter 4 below.*

We next recall the definition of a weak Sobolev solution of the equations we study in this thesis, namely

$$L[u] = -\operatorname{div}(A(x)Du) + b(x)\cdot Du + c(x)u = f(x), \ x \in \Omega. \tag{2.1}$$

**Definition 2.8** *We say that* $u$ *is a weak (super,sub) solution of* (2.1)*, if* $u$ *satisfies:*

$$\int_\Omega A(x)DuD\varphi + \int_\Omega b(x)\varphi|Du| + \int_\Omega c(x)\varphi u = (\geq, \leq) \int_\Omega f\varphi$$

*for each* $\varphi \in C_0^{\infty}(\Omega)$, $\varphi \geq 0$.

Now we present a rescaled version of the generalized Maximum Principle (Stampacchia inequality) for our problem which is going to be useful later.

**Lemma 2.9** **[Rescaled Version of the Weak Maximum Principle]** *Let* $u$ *be a weak subsolution (supersolution) of problem* (2.1) *where* $L$ *is the uniformly elliptic operator given in* (2.1) *with* $a_{ij} \in L^{\infty}(\Omega)$, $|b| \in L^p(\Omega)$, $c \in L^{p/2}(\Omega)$, $f \in L^p(\Omega)$*, for some* $p > n$ *and* $c \geq 0$ *in* $\Omega$. *If* $\Omega$ *is a domain with width* $\delta > 0$, *there exists a constant* $C > 0$ *independent of* $\delta$ *such that*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \delta^{2-\frac{n}{p}} C\|f\|_{L^{p/2}(\Omega)}$$
$$\Big(\sup_{\Omega}(-u) \leq \sup_{\partial\Omega} u^- + \delta^{2-\frac{n}{p}} C\|f\|_{L^{p/2}(\Omega)}\Big).$$

*Proof.* We use Theorem 8.16 in [GT01], together with the remark on page 193 of [GT01]. We assume initially that $|\Omega| = 1$, and we apply Theorem 8.16 in

[GT01]. The general case comes from a coordinate transformation: given $\delta > 0$, if $\Omega$ has width $\delta$ we have $|\Omega| \leq \delta^n$, and can change variables to transform $\Omega$ into a domain $\Omega_\delta$ with $|\Omega_\delta| = 1$. In fact, we translate so that $0 \in \Omega$, and for all $x \in \Omega$ we set $y(x) = x/|\Omega|^{1/n}$ and $y(\Omega) = \Omega_\delta$.

Moreover, since $u$ is a subsolution of problem (2.1), setting $\tilde{u}(y) := u(x)$ we have that $\tilde{u}$ satisfies

$$-\frac{1}{|\Omega|^{2/n}} \operatorname{div}(\tilde{A}(y)D\tilde{u}(y)) + \frac{\tilde{b}(y)}{|\Omega|^{1/n}} D\tilde{u}(y) + \tilde{c}(y)\tilde{u}(y) \leq \tilde{f}(y)$$
$$-\operatorname{div}(\tilde{A}(y)D\tilde{u}) + |\Omega|^{1/n}\tilde{b}(y)D\tilde{u} + |\Omega|^{2/n}\tilde{c}(y)u \leq |\Omega|^{2/n}\tilde{f} \text{ in } \Omega$$

where $\tilde{\varphi}(y) = \varphi(x)$, for $\varphi = A, b, c, f$. Hence applying Theorem 8.16 [GT01] and using that $|\Omega| \leq \delta^n$, we obtain

$$\begin{aligned}
\sup_{\Omega} u = \sup_{\Omega_\delta} \tilde{u} &\leq \sup_{\partial\Omega_\delta} \tilde{u}^+ + C|\Omega|^{\frac{2}{n}} \|\tilde{f}\|_{L^p(\Omega_\delta)} \\
&= \sup_{\partial\Omega} u^+ + C|\Omega|^{\frac{2}{n}-\frac{1}{p}} \|f\|_{L^p(\Omega)} \\
&\leq \sup_{\partial\Omega} u^+ + C\delta^{2-\frac{n}{p}} \|f\|_{L^p(\Omega)}
\end{aligned}$$

as desired. ■

The maximun principle for small domains or small $c^+$ is a consequence of Lemma 2.9.

**Lemma 2.10** *Under the assumptions of Lemma 2.9, there exists $\delta_0 > 0$, such that if $|\Omega| \leq \delta^n$ or $\|c^+\|_{L^p(\Omega)} \leq \delta$, $\delta \leq \delta_0$, then any weak subsolution $u$ of*

$$\begin{cases} -\operatorname{div}(A(x)Du) + b(x)Du + c(x)u & \leq 0 \quad \textit{in } \Omega \\ u & \leq 0 \quad \textit{on } \partial\Omega \end{cases}$$

*satisfies $u \leq 0$ in $\Omega$. Analogously, if $v$ is a weak supersolution of*

$$\begin{cases} -\operatorname{div}(A(x)Dv) + b(x)Dv + c(x)v & \geq 0 \quad \textit{in } \Omega \\ v & \geq 0 \quad \textit{on } \partial\Omega \end{cases}$$

*then we have that $v \geq 0$ in $\Omega$.*

*Proof.* We apply the previous lemma with $c$ replaced by $-c^-$ and $f$ replaced by $-c^+u$, where $c^+, c^-$ are the positive and negative part of $c(x)$. ■

For completeness, we also state the Comparison Principle.

**Lemma 2.11 (Comparison Principle)** *Under the assumptions of Lemma 2.9, there exists $\delta_0 > 0$, such that if $|\Omega| \leq \delta^n$ or $\|c^+\|_{L^p(\Omega)} \leq \delta$, $\delta \leq \delta_0$, and*

*$u, v$ satisfy (in a weak sense)*

$$\begin{cases} L[u] & \leq L[v] \quad \text{in } \Omega \\ u & \leq v \quad \text{in } \Omega \end{cases}$$

*then $u \leq v$ in $\Omega$.*

The next theorem, the Strong Maximum Principle (SMP) is extremely important, it says that a nonnegative supersolution of an elliptic equation in a domain cannot vanish inside the domain, unless it vanishes identically.

**Theorem 2.12 (SMP)** *Let $\Omega \subset \mathbb{R}^n$ be a domain. If $u$ satisfies*

$$\begin{cases} L[u] & \geq 0 \quad \text{in } \Omega \\ u & \geq 0 \quad \text{in } \Omega \end{cases}$$

*then either $u > 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.*

The SMP is an immediate consequence of the interior weak Harnack inequality, Theorem 1.1.

**Lemma 2.13 (Exponential change)** *Let $u$ be a weak solution of problem (2.1). For $m > 0$ we define*

$$v := \frac{e^{mu} - 1}{m}, \qquad w := \frac{1 - e^{-mu}}{m}$$

*Then we have $Dv = (1 + mv)Du$, $Dw = (1 - mw)Du$,*

$$-\operatorname{div}(A(x)Du) - \vartheta^{-1}m|Du|^2 \leq \frac{-\operatorname{div}(A(x)Dv)}{1 + mv} \leq -\operatorname{div}(A(x)Du) - \vartheta m|Du|^2,$$
$$-\operatorname{div}(A(x)Du) + \vartheta m|Du|^2 \leq \frac{-\operatorname{div}(A(x)Dw)}{1 - mw} \leq -\operatorname{div}(A(x)Du) - \vartheta^{-1}m|Du|^2,$$

*and $\{u = 0\} = \{v = 0\}$ and $\{u > 0\} = \{v > 0\}$.*
*Therefore if $u$ is a weak supersolution of*

$$-\operatorname{div}(A(x)Du) \geq \mu_1|Du|^2 + c_\lambda(x)u + h(x) \tag{2.2}$$

*then $v = \frac{1}{m}(e^{mu} - 1)$ for $m = \mu_1\vartheta$, is a weak supersolution of*

$$-\operatorname{div}(A(x)Dv) \geq h(x)(1 + mv) + \frac{c_\lambda(x)}{m}(1 + mv)\ln(1 + mv).$$

*Proof.* This follows from a computation, since $div(fu) = f div(u) + uDf$. ∎

**Lemma 2.14 (Lipschitz Bound)** *Under the assumptions of Lemma 2.11, if $\Omega$ is a bounded domain such that $\partial\Omega \in C^{1,D}$ and $\Omega$ has width less than $\delta$, then*

$$u(x) \leq \frac{C}{\delta}\left(\sup_{\overline{\Omega}} u^+ + \delta^{2-\frac{n}{p}}\|f\|_{L^p(\Omega)}\right) d(x) \ \textit{in}\ \overline{\Omega}.$$

*Proof.* First we consider $\Omega$ with width $\delta = 1$, and let $\phi$ be the weak solution of the problem

$$\begin{cases} L\phi &= f \quad \text{in } \Omega \\ \phi &= h \quad \text{on } \partial\Omega \end{cases}$$

where $h \equiv \sup_{\partial\Omega} u^+$. By the Comparison Principle we have $u \leq \phi$ in $\overline{\Omega}$, since $u$ is a subsolution of

$$\begin{cases} Lu &= f \quad \text{in } \Omega \\ u &\leq h \quad \text{on } \partial\Omega. \end{cases}$$

Then,

$$\frac{u(x)}{d(x)} \leq \frac{\phi(x)}{d(x)} \leq C\|D\phi\|_{L^\infty(\overline{\Omega})} \leq C\|\phi\|_{C^1(\overline{\Omega})} \text{ in } \overline{\Omega}.$$

Furthermore, global $C^1$-estimates are valid for the Dirichlet problem satisfied by $\phi$ (see Chapter 8 in [GT01]), so

$$\begin{aligned} \|\phi\|_{C^1(\overline{\Omega})} &\leq C\left(\|\phi\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}\right) \\ &\leq C\left(\sup_{\overline{\Omega}} u^+ + \|f\|_{L^p(\Omega)}\right), \end{aligned}$$

where in the last inequality we used Lemma 2.9 with $\delta = 1$. Combining these two equations, it follows that

$$u(x) \leq C\left(\sup_{\overline{\Omega}} u^+ + \|f\|_{L^p(\Omega)}\right) d(x) \text{ in } \overline{\Omega}.$$

Now if $\Omega$ is a domain with width $\delta > 0$, repeating the argument above for the rescaled problem we obtain

$$\frac{u(x)}{d(x)} \leq \frac{\phi(x)}{d(x)} \leq \frac{C}{\delta}\|D\phi\|_{L^\infty(\overline{\Omega})} \leq \frac{C}{\delta}\left(\sup_{\overline{\Omega}} u^+ + \delta^{2-\frac{n}{p}}\|f\|_{L^p(\Omega)}\right) \text{ in } \overline{\Omega}.$$

■

# 3
# Boundary Point Hopf Lemma

## 3.1
## Hopf Lemma

This section is devoted to the proof of the Boundary Point Principle for the operators we consider. As we noted in the introduction, this result is somewhat delicate for operators in divergence form, as it requires some smoothness of the leading coefficients, and since barriers close to the boundary are not readily exhibited.

The first result for equations with divergence structure was proved by R. Finn and D. Gilbarg [FG57]. They considered a two-dimensional bounded domain with $C^{1,\alpha}$-regular boundary, Hölder continuous entries of the matrix $A(x)$ and continuous lower order coefficients. After various generalizations, as of today the most general result is due to Apushkinskaya and Nazarov [AN19], whom prove the BPP for a divergence form operators with Dini continuous leading coefficients, and integrable first-order coefficients. That paper contains a historical review of results on the BPP for divergence form operators. However, the result in [AN19] is valid for operators without zero order terms.

The goal of this section is to generalize the result from [AN19] for the full divergence-type operator $L$, with a $L^p$-integrable zero order term. The general idea of the proof is similar to that of [AN19], however, to deal with the zero-order term we use here one more tool, a fixed-point theorem. The proof we give and the result itself are also instrumental in the following sections, in particular for the proof of the boundary weak Harnack inequality.

Thus, this section will be devoted to the proof of Theorem 1.2 under our assumptions on $L$.

The proof below is based on the classical $C^1$ bounds of Gruter and Winter [GW82], and we will also need the following Fixed Point Theorem of Schäefer.

**Theorem 3.1** *[Corollary 1.19, [CQ04]] Let $F : X \to X$ be compact, where $X$ is a Banach space. Then the following alternative holds:*

*(i) $x - tF(x) = 0$ has a solution for every $t \in [0, 1]$*
*or*

*(ii)* $S = \{x : \exists t \in [0,1] : x - tF(x) = 0\}$ *is unbounded in* $X$*.*

We will follow the argument from [AN19], by sketching the parts of the proof that are taken from that paper, and giving more details when differences appear and in order to consider $c \not\equiv 0$ it is necessary to work a little deeper.

As we explained above, we can assume that $0 \in \partial\Omega$ and $\partial\Omega$ is flat around the origin. Consider for $0 < \rho < R/2$ the point $x^\rho = (0, \cdots, 0, \rho)$ and the annulus

$$A_\rho = \{x : \rho/2 < |x - x^\rho| < \rho\} \subset \Omega.$$

Let $x^*$ be an arbitrary point in $\overline{A}_\rho$, and define the auxiliary functions $z$ and $\varphi_{x^*}$ as the weak solutions for the Dirichlet problems

$$\begin{cases} \mathcal{L}_0 z & = 0 \quad \text{in } A_\rho \\ \quad z & = 1 \quad \text{on } \partial B_{\rho/2}(x^\rho) \\ \quad z & = 0 \quad \text{on } \partial B_\rho(x^\rho) \end{cases}, \quad \begin{cases} \mathcal{L}_0^{x^*} \varphi_{x^*} & = 0 \quad \text{in } A_\rho \\ \quad \varphi_{x^*} & = 1 \quad \text{on } \partial B_{\rho/2}(x^\rho) \\ \quad \varphi_{x^*} & = 0 \quad \text{on } \partial B_\rho(x^\rho), \end{cases} \tag{3.1}$$

where the operators $\mathcal{L}_0$ and $\mathcal{L}_0^{x^*}$ are given by

$$\begin{aligned} \mathcal{L}_0 z &:= -D_i(a_{ij}(x) D_j z) \\ \mathcal{L}_0^{x^*} \varphi_{x^*} &:= -D_i(a_{ij}(x^*) D_j \varphi_{x^*}) \\ \mathcal{L} v &:= \mathcal{L}_0 v + b.Dv. \end{aligned}$$

By repeating the proof of Lemma 2.2 in [AN19] we get the following $C^1$-estimate for the function $w^{(1)} = z - \varphi_{x^*}$

$$|Dz(x^*) - D\varphi_{x^*}(x^*)| \leq C_1 \frac{\mathcal{J}_\sigma(2\rho)}{\rho} \tag{3.2}$$

for all $\rho \leq R/2$, where $z \in C^1(\overline{A}_\rho)$ and $\varphi_{x^*} \in C^\infty(\overline{A}_\rho)$ are the unique weak solutions for (3.1). In the same way, according to Lemma 3.2 [GW82], we get

$$|Dz(y)| \leq \frac{N_1}{\rho} \tag{3.3}$$

for any $y \in A_\rho$ and some $N_1 > 0$, where $z$ is the solution of the Dirichlet problem in (3.1).

We observe that it is well known, from the general elliptic theory (see for instance Chapter 8 in [GT01]), that given $f \in L^p(A_\rho)$ there exists a unique weak solution $z_f \in C^1(\overline{A}_\rho)$ for the Dirichlet problem

$$\begin{cases} \mathcal{L}_0 z_f &= f \quad \text{in } A_\rho \\ z_f &= 1 \quad \text{on } \partial B_{\rho/2}(x^\rho) \\ z_f &= 0 \quad \text{on } \partial B_\rho(x^\rho). \end{cases} \tag{3.4}$$

Further, we introduce the barrier function $v$ defined as the weak solution of the Dirichlet problem

$$\begin{cases} \mathcal{L} v_f &= f \quad \text{in } A_\rho \\ v_f &= 1 \quad \text{on } \partial B_{\rho/2}(x^\rho) \\ v_f &= 0 \quad \text{on } \partial B_\rho(x^\rho). \end{cases} \tag{3.5}$$

Following the argument in Theorem 2.3 [AN19] we get the existence of a unique weak solution $v_f \in C^1(\overline{A}_\rho)$ to problem (3.5) for each $f \in L^p(A_\rho)$, provided $\rho$ is sufficiently small. Let us briefly recall this proof, for completeness.

Consider in $A_\rho$ the auxiliary function $w_f^{(2)} = v_f - z_f$. We observe that it vanishes on $\partial A_\rho$, and

$$\mathcal{L}_0 w_f^{(2)} = \mathcal{L} v_f - f(x) - b(x).Dv_f = -b(x)Dv_f = -b(x)(Dw_f^{(2)} + Dz_f) \text{ in } A_\rho.$$

Hence, $w_f^{(2)}$ can be represented in $A_\rho$ via the corresponding Green function $G_{0,\rho}(x,y)$ of the operator $\mathcal{L}_0$ as

$$w_f^{(2)}(x) = \int_{A_\rho} G_{0,\rho}(x,y)\mathcal{L}_0 w_f^{(2)}(y) = -\int_{A_\rho} G_{0,\rho}(x,y) b_i(y)(D_i w_f^{(2)}(y) + D_i z_f(y)).$$

Differentiating this equality we obtain a fixed-point problem for the gradient of $w^{(2)}$, and it can be shown that this problem is governed by a contractible operator, for sufficiently (but uniformly) small $\rho$.

We now turn to the treatment of operators with zero order terms. By the existence result we just proved for $\mathcal{L}$, since $c(x) \in L^p(\Omega)$, the following operator is well defined and compact: $F : C(\overline{A}_\rho) \to C(\overline{A}_\rho)$ given by

$$F(g) := \overline{v}_g, \text{ where } \begin{cases} \mathcal{L}\overline{v}_g &= -cg \quad \text{in } A_\rho \\ \overline{v}_g &= 1 \quad \text{on } \partial B_{\rho/2}(x_\rho) \\ \overline{v}_g &= 0 \quad \text{on } \partial B_\rho(x_\rho). \end{cases}$$

Indeed, given $g \in C(\overline{A}_\rho)$ we obviously have $-cg \in L^p(\overline{A}_\rho)$, hence the solution $\overline{v}_g \in C(\overline{A}_\rho)$, exists and is unique. Thus $F(g) = \mathcal{L}^{-1}(-cg) \in C^1(\overline{A}_\rho)$ is well defined and due to the compact embedding $C^1(\overline{A}_\rho) \hookrightarrow C(\overline{A}_\rho)$, $F$ is also compact.

We are going to show that the operator $F$ has a fixed point, that is, there exists $\overline{v} \in C(\overline{A}_\rho)$ such that $F\overline{v} = \overline{v}$, which means that $\overline{v} \in C^1(\overline{A}_\rho)$ and solves

$$\begin{cases} L\overline{v} = \mathcal{L}\overline{v} + c\overline{v} & = 0 \quad \text{in } A_\rho \\ \overline{v} & = 1 \quad \text{on } \partial B_{\rho/2}(x^\rho) \\ \overline{v} & = 0 \quad \text{on } \partial B_\rho(x^\rho). \end{cases} \tag{3.6}$$

To that end we will show that the set $S$ defined in Theorem 3.1 is bounded. We apply the Weak Maximum Principle (Theorem 2.9), to the problem

$$\begin{cases} \mathcal{L}\overline{v} & = -tc\overline{v} \quad \text{in } A_\rho \\ \overline{v} & = 1 \qquad \text{on } \partial B_{\rho/2}(x^\rho) \\ \overline{v} & = 0 \qquad \text{on } \partial B_\rho(x^\rho) \end{cases}$$

obtaining

$$\|\overline{v}\|_{C(\overline{A}_\rho)} \leq 1 + tC\rho^{2-\frac{n}{p}}\|c\|_{L^p(\overline{A}_\rho)}\|\overline{v}\|_{C(\overline{A}_\rho)}$$

which implies that given $\overline{v} \in S$ we have $\|\overline{v}\|_{C(\overline{A}_\rho)} \leq C_0$ for some $C_0 > 0$ and sufficiently small $\rho$, namely, for $\rho^{2-\frac{n}{p}} < (1/2)C\|c\|_{L^p(\overline{A}_\rho)}$. Therefore, by Theorem 3.1 we ensure the existence of $\overline{v}$ when $t = 1$. From the Weak Maximum Principle we conclude that $0 \leq \overline{v} \leq 1$, in particular, $\overline{v}$ is nonnegative.

At this point we consider the function $z = z_0$, which is the unique solution of the problem (3.4) when $f = 0$.

**Theorem 3.2** *There exists $\rho_0 > 0$ such that for all $\rho \leq \rho_0$ the problem (3.6) admits a unique solution $\overline{v} \in C^1(\overline{A}_\rho)$. Moreover, the inequality*

$$|(D\overline{v} - Dz)(x)| \leq C_2\rho^{-\frac{n}{p}} \tag{3.7}$$

*holds true for any $x \in A_\rho$. Here $z \in C^1(\overline{A}_\rho)$ is defined in (3.4).*

*Proof.* The existence of $\overline{v}$ was already proved. We define $w^{(3)} := \overline{v} - z$, so $w^{(3)}$ vanishes on $\partial A_\rho$ and

$$\begin{aligned} \mathcal{L}_0 w_f^{(3)} &= \mathcal{L}\overline{v} - b(x).D\overline{v} \\ &= -c(x)(w^{(3)} + z) - b(x) \cdot (Dw^{(3)} + Dz) \text{ in } A_\rho. \end{aligned}$$

Hence, $w_f^{(3)}$ can be represented in $A_\rho$ via the corresponding Green function

$G_{0,\rho}(x,y)$ as

$$\begin{aligned} w^{(3)}(x) &= \int_{A_\rho} G_{0,\rho}(x,y)\mathcal{L}_0 w^{(3)}(y) \\ &= -\int_{A_\rho} G_{0,\rho}(x,y)b_i(y)D_i[w^{(3)}(y)+z(y)]dy \\ &\quad -\int_{A_\rho} G_{0,\rho}(x,y)c_i(y)[w^{(3)}(y)+z(y)]dy \end{aligned}$$

Differentiating with respect the $x_k$ gives

$$\begin{aligned} D_k w^{(3)}(x) = &-\int_{A_\rho} D_{x_k}G_{0,\rho}(x,y)b_i(y)[D_i w^{(3)}(y)+D_i z(y)]dy \\ &-\int_{A_\rho} D_{x_k}G_{0,\rho}(x,y)c_i(y)[w^{(3)}(y)+z(y)]dy. \end{aligned}$$

Theorem 3.3 [GW82] provides the estimate

$$|D_x G_{0,\rho}(x,y)| \le C \min\{|x-y|^{1-n}, d(y,\partial A_\rho)|x-y|^{-n}\} \tag{3.8}$$

for any $x, y \in A_\rho$ where $C > 0$ does not depend on $\rho$. Hence, we get

$$\begin{aligned} \int_{A_\rho} |D_{x_k}G_{0,\rho}(x,y)b_i(y)|dy &\le C\|b\|_{L^p(A_\rho)}\left(\int_{A_\rho}|x-y|^{(1-n)p/(p-1)}dy\right)^{\frac{p-1}{p}} \\ &\le C\rho^{1-\frac{n}{p}}\|b\|_{L^p(A_\rho)} \end{aligned}$$

for sufficiently small $\rho = \left(\int_{A_\rho}|x-y|^{(1-n)p/(p-1)}dy\right)^{\frac{n-1}{p}} > 0$ and obtain

$$\begin{aligned} \|Dw^3\|_{L^\infty} &\le C\rho^{1-\frac{n}{p}}\left[\|b\|_{L^p(A_\rho)}\left(\|Dw^3\|_{L^\infty}+\|Dz\|_{L^\infty}\right)+C_0\|c\|_{L^p(A_\rho)}\right] \\ \frac{1}{2}\|Dw^3\|_{L^\infty} &\le [1-C\rho^{1-\frac{n}{p}}\|b\|_{L^p(A_\rho)}]\|Dw^3\|_{L^\infty} \\ &\le C\rho^{1-\frac{n}{p}}\left[\|b\|_{L^p(A_\rho)}\|Dz\|_{L^\infty}+C_0\|c\|_{L^p(A_\rho)}\right] \\ &\le C\rho^{1-\frac{n}{p}}\|Dz\|_{L^\infty}\left[\|b\|_{L^p(A_\rho)}+\|c\|_{L^p(A_\rho)}\right]. \end{aligned}$$

Using the estimate (3.3) for all $x \in A_\rho$, we get

$$|D\overline{v}(x) - Dz(x)| \le \|Dw^3\|_{L^\infty} \le C_2\rho^{-\frac{n}{p}}.$$

■

**Proof of Boundary Point Hopf Lemma**

It is well known that the Boundary Point Hopf Lemma holds true for the operator $\mathcal{L}_0^{x^*}$ with $x^* = 0$ in the annulus $A_1$, Theorem 1.2. Thus, rescaling $A_1$

into $A_\rho$ we get the estimate

$$D_n\varphi_0(0) \geq \frac{C_3}{\rho} > 0.$$

Furthermore, the inequalities (3.2) and (3.7) imply for sufficiently small $\rho$

$$\begin{aligned} D_n\overline{v}(0) &\geq D_n\varphi_0(0) - |Dz(0) - D\varphi_0(0)| - |D\overline{v}(0) - Dz(0)| \\ &\geq \frac{C_3}{\rho} - C_1\frac{\mathcal{J}_\sigma(2\rho)}{\rho} - C_2\frac{\rho^{1-\frac{n}{p}}}{\rho} \geq \frac{C_3}{2\rho}. \end{aligned}$$

Since $u$ satisfies $Lu \geq 0$ we observe that

$$L(u - u(0)) \geq -cu(x_0) \geq 0 \text{ in } \Omega.$$

There exists a ball $B$ such that $u - u(0) > 0$ in $B \cap \Omega$ (by the strong maximum principle). Hence given $\rho > 0$, we have for sufficiently small $\varepsilon > 0$

$$\begin{cases} \mathcal{L}(u - u(0) - \varepsilon\overline{v}) & \geq 0 \quad \text{in } A_\rho \\ \quad u - u(0) - \varepsilon\overline{v} & \geq 0 \quad \text{on } \partial A_\rho. \end{cases}$$

By Lemma 2.9 the estimate $u - u(0) \geq \varepsilon\overline{v}$ holds true in $A_\rho$. This gives

$$\frac{\partial u}{\partial n}(0) = -D_n u(0) \leq -\epsilon D_n\overline{v}(0) \leq -\epsilon\frac{C_3}{2\rho} < 0,$$

which completes the proof.

# 4
# Boundary Weak Harnack Inequality

In this chapter we prove our main theoretical result, the boundary weak Harnack inequality -BWHI (Theorem 1.3).

## 4.1
## The Growth Lemma and auxiliary results

The core of our argument is the following growth lemma. We define by $Q_\rho(y)$ the cube of center $y$ and side of length $\rho$, i.e.

$$Q_\rho(y) = \{x \in \mathbb{R}^n : |x_i - y_i| < \rho/2 \text{ for } i = 1, \cdots, n\}.$$

In case the center of the cube is $\rho e$ with $e = (0, 0, \cdots, 1/2)$, we use the notation $Q_\rho = Q_\rho(\rho e)$.

**Lemma 4.1 (The Growth Lemma)** *Let $u$ be a nonnegative weak supersolution of $L[u] \geq f$, in $\Omega$ under the assumptions of the BWHI (Theorem 1.3) and $f \in L^p(\Omega)$ is non-positive in $\Omega$. Given $\nu > 0$ there exist $k, a > 0$ depending on $\nu, n, p, \vartheta, \Omega$ such that if*

$$\|f\|_{L^p(\Omega)} \leq a$$

*and the following inequality holds*

$$|\{x \in \Omega : u(x) > d\} \cap \Omega| \geq \nu|\Omega|$$

*then $u > kd$ in $\Omega$.*

*Proof.* Take $d_1 = d_1(\nu, n, \Omega)$ for which the set $\Omega_\delta := \{x \in \Omega : d(x) < \delta\}$ be smooth and has measure such that $|\Omega_\delta| \leq \nu/2|\Omega|$, for all $0 < \delta \leq d_1$. Then, for $S_\delta = \Omega \setminus \Omega_\delta$ we have

$$|\{u \geq \delta\} \cap S_\delta| \geq |\{u \geq d\} \cap S_\delta| \geq \frac{\nu}{2}|S_\delta|.$$

Since $u \geq 0$ it follows that

$$\int_{S_\delta} u^s dx \geq \int_{\{u \geq \delta\} \cap S_\delta} u^s dx \geq |\{u \geq \delta\} \cap S_\delta| \delta^s \geq \delta^s \frac{\nu}{2}|S_\delta|.$$

By the interior weak Harnack inequality (Theorem 1.1), there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}\inf_{S_\delta} u &\geq C_1 \left( \int_{S_\delta} u^s dx \right)^{1/s} - C_2 \|f\|_{L^p(\Omega)} \\ &\geq C_1 \delta \left( \frac{\nu}{2} \right)^{1/s} |S_\delta|^{1/s} - C_2 a.\end{aligned}$$

We define $k'_\delta := C_1 \delta \left( \frac{\nu}{2} \right)^{1/s} |S_\delta|^{1/s} - C_2 a$. Thus $k'_\delta > 0$ if $a$ is chosen sufficiently small, and

$$u \geq \frac{k'_\delta}{diam(\Omega)} d \text{ in } S_\delta, \text{ for all } \delta \in (0, d_1). \tag{4.1}$$

It remains to prove that $u \geq k_\delta d$ in $\Omega_\delta$, for some $k_\delta > 0$ and some fixed $\delta > 0$, to be determined.

We are going to use an argument that proves this in a neighborhood of each boundary point separately. Given a point $x_0 \in \partial\Omega$ and the exterior normal unitary vector $\eta$ to $\partial\Omega$ passing by $x_0$, we can assume, after a change of variables, that $x_0 = 0$ and $\eta = -e_n$. We are going to prove that $u \geq k_\delta d$ at each point $\overline{x} = \overline{t} e_n$, $0 \leq \overline{t} < \delta$. Then the same inequality for all $x \in \Omega_\delta$ follows from repeating this argument for each $x_0 \in \partial\Omega$.

Consider the annulus $A_\rho$ (with $\overline{x} \in A_\rho$) given by $A_\rho = B_\rho \setminus B_{\rho/2}$ where $B_\rho = B_\rho(\rho e_n)$ and such that $B_{\rho/2} \cap \Omega_\delta = \emptyset$. For the latter, we can choose $\rho = 4\delta$ for instance. Since for $\overline{x} = \overline{t} e_n$ we have $\overline{x}$ closer to $\partial B_\rho$ than to $\partial B_{\rho/2}$, we have $\widetilde{d}(\overline{x}) := dist(\overline{x}, \partial A_\rho) = d(\overline{x})$, where $d(\overline{x}) := dist(\overline{x}, \partial\Omega) = \overline{t}$.

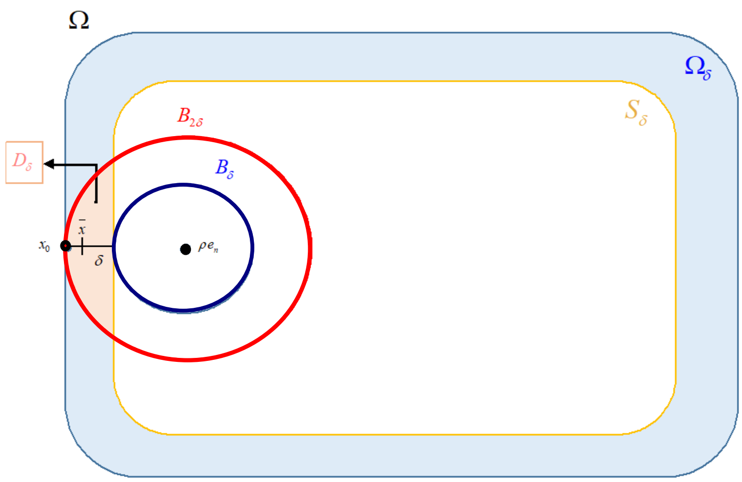


Figure 4.1: Annulus $A_\rho$

We introduce the auxiliary function $w_0 = \overline{v} - \varphi_0$, where $\widetilde{v}$ is the unique weak solution of (3.6), and $\varphi_0$ is defined in the previous section (or in Claim

4.2 below). We obtain from (3.2) and (3.7) that

$$\left|\frac{w_0(\overline{x})}{d(\overline{x})}\right| \leq C\|Dw_0\|_{L^\infty(A_\rho)} \leq \frac{C_3}{\rho}\left(\rho^{1-\frac{n}{p}} + \mathcal{J}_\sigma(2\rho)\right),$$

where $C_3 > 0$ does not depend on $\rho$ neither on $\overline{x}$. Then Claim 4.2 below yields

$$\frac{\overline{v}(\overline{x})}{d(\overline{x})} \geq \frac{\varphi_0(\overline{x})}{d(\overline{x})} - \frac{C_3}{\rho}\left(\rho^{1-\frac{n}{p}} + \mathcal{J}_\sigma(2\rho)\right) \geq \frac{C_1}{\rho} - \frac{C_3}{\rho}\left(\rho^{1-\frac{n}{p}} + \mathcal{J}_\sigma(2\rho)\right) \geq \frac{C_1}{2\rho} \tag{4.2}$$

for sufficiently small $\delta$ such that $\rho = 4\delta$ is small enough. Consider $D_\delta = \Omega_\delta \cap A_\rho$, and set $k_\delta = \dfrac{4\delta C_1}{\rho k'_\delta} > 0$, we have

$$\overline{v}(x) \leq \max \overline{v} \leq k_\delta u(x)$$

for all $x \in \partial\Omega_\delta \cap \partial D_\delta$, which is possible since by (4.1) we have $u \geq C_\delta d \geq C_\delta \delta$ in $S_\delta$. Moreover, $u \geq 0 = \overline{v}$ on $\partial A_\rho \cap \partial D_\delta$, since $x \in \partial B_\rho$. Then

$$\begin{cases} L(\overline{v} - k_\delta u) & \leq -k_\delta f \quad \text{in } D_\delta(\rho e_n) \\ \overline{v} - k_\delta u & \leq 0 \quad \text{on } \partial D_\delta(\rho e_n). \end{cases} \tag{4.3}$$

We apply the rescaled version of the Weak Maximum Principle, Lemma 2.9 for a domain with width $\delta$, and we get

$$\overline{v} - k_\delta u \leq C\delta^{2-\frac{n}{p}} k_\delta \|f\|_{L^p(D_\delta)}.$$

By the Lipschitz Bound at $\overline{\Omega}_\delta$ , Lemma 2.14, applied to (4.3) in $\overline{\Omega}_\delta$.

$$\begin{aligned} \overline{v} - k_\delta u &\leq \frac{C}{\delta}\left(\sup_{\overline{\Omega}_\delta}(\widetilde{v} - k_\delta u) + \delta^{2-\frac{n}{p}} k_\delta \|f\|_{L^p(D_\delta)}\right) d \\ &\leq C\delta^{1-\frac{n}{p}} k_\delta d \|f\|_{L^p(D_\delta)} \text{ in } D_{\delta/2} \end{aligned}$$

Thus, by using (4.2), we see that

$$\begin{aligned} k_\delta u(\overline{x}) &\geq \overline{v} - C\delta^{1-\frac{n}{p}} k_\delta d(\overline{x}) \|f\|_{L^p(D_\delta)} \\ &\geq \left(\frac{C_1}{2\rho} - C\delta^{1-\frac{n}{p}} k_\delta \|f\|_{L^p(D_\delta)}\right) d(\overline{x}) \text{ in } D_{\delta/2}. \end{aligned}$$

We now fix $\delta_0$ small enough so that $C\delta_0^{1-\frac{n}{p}} k_{\delta_0} a < \frac{C_1}{4\rho} < \frac{C_1}{4}$. Thus,

$$u(\overline{x}) \geq \frac{C_1}{4k_{\delta_0}} d(\overline{x}) \text{ in } D_{\delta_0/2}.$$

Repeating this argument for all $x_0 \in \partial\Omega$, we obtain the same inequality for all $x \in \Omega_\delta$.

On the other hand, $u(\overline{x}) \geq \dfrac{k'_{\delta_0/2}}{diam(\Omega)} d(\overline{x})$ in $S_{\delta_0/2}$ by (4.1). Setting

$$k = \min\left\{\frac{k'_{\delta_0/2}}{diam(\Omega)}, \frac{C_1}{4k_{\delta_0}}\right\} = \frac{k'_{\delta_0/2}}{diam(\Omega)},$$

where $\delta_0$ is set to be smaller than $diam(\Omega)/4$, since

$$\frac{k'_{\delta_0/2}}{diam(\Omega)} \leq \frac{2C_1\delta_0}{k_{\delta_0/2}diam(\Omega)} \leq \frac{C_1}{2k_{\delta_0/2}} \leq \frac{C_1}{4k_{\delta_0}}.$$

This ends the proof of the growth lemma, pending a proof of the following claim. ■

We need the following estimate for $\varphi_0$.

**Claim 4.2** *Let $\varphi_0$ be the unique solution of the problem*

$$\begin{cases} \mathcal{L}_0^0\varphi_0 = -\operatorname{div}(A(0)D\varphi_0(x)) = 0 & \text{in } A_\rho(\rho e_n) \\ \varphi_0 = 1 & \text{on } \partial B_{\rho/2}(\rho e_n) \\ \varphi_0 = 0 & \text{on } \partial B_\rho(\rho e_n). \end{cases} \tag{4.4}$$

*Then,*

$$\varphi_0(x) \geq \frac{C_1}{\rho}\tilde{d}(x) \text{ for all } x \in A_\rho$$

*where $C_1$ does not depend on $\rho$ neither on $\overline{x}$. In particular, $\varphi_0(\bar{x}) \geq \dfrac{C_1}{\rho} d(\bar{x})$.*

*Proof.* We define the new variable $y = \dfrac{1}{\rho}(x - \rho e_n)$, and the rescaled function $\tilde{\varphi}_0(y) = \varphi_0(x)$. Then $\tilde{\varphi}$ is a weak solution of

$$0 = -\operatorname{div}(A(0)D\varphi_0(x)) = -\frac{1}{\rho^2}\operatorname{div}(A(0)D\tilde{\varphi}_0(y))$$

and

$$d(y) = dist(y, \partial B_1) = \frac{1}{\rho}dist(x, \partial B_\rho) = \frac{1}{\rho}d(x).$$

Moreover, observe that

$$x \in A_\rho(\rho e_n) \text{ if and only if } y \in B_1(0)/B_{1/2}(0)$$

and so, that (4.4) is equivalent to

$$\begin{cases} -\operatorname{div}(A(0)D\widetilde{\varphi}_0(y)) &= 0 \quad \text{in } B_1(0)/B_{1/2}(0) \\ \widetilde{\varphi}_0 &= 1 \quad \text{on } \partial B_{1/2}(0) \\ \widetilde{\varphi}_0 &= 0 \quad \text{on } \partial B_1(0). \end{cases} \tag{4.5}$$

Applying the BPP, we get $\frac{\partial \widetilde{\varphi}_0}{\partial \eta}(y_0) < 0$, since $\widetilde{\varphi}_0$ attains its minimum at $y_0 \in \partial B_1(0)$, and $\frac{\partial \widetilde{\varphi}_0}{\partial \eta}(y_1) > 0$, since $\widetilde{\varphi}_0$ attains its maximum at $y_1 \in \partial B_{1/2}(0)$. Now, let us define

$$\widehat{\varphi}_0(y) = \begin{cases} \widetilde{\varphi}_0 & \text{in } B_1(0)/B_{1/2}(0) \\ 1 & \text{in } B_{1/2}(0). \end{cases}$$

Then $\widehat{\varphi}_0$ solves in the viscosity sense

$$-\operatorname{div}(A(0)D\widehat{\varphi}_0(y)) = -\sum_{i,j=1}^{n} a_{ij}(0)D_{ij}\widehat{\varphi}_0(y) \geq 0 \text{ in } B_1$$

since the normal derivative of $\widetilde{\varphi}_0$ at $\partial B_{1/2}$ is different from zero, and then no smooth function can touch $\widehat{\varphi}_0$ from below at a point on $\partial B_{1/2}$. That is, $\widehat{\varphi}_0$ is a weak supersolution in $B_1$, with

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} a_{ij}(0)D_{ij}\widehat{\varphi}_0(y) &\geq 0 \text{ in } B_1(0) \\ \widehat{\varphi}_0 &= 1 \text{ in } B_{1/2}(0). \end{cases}$$

The latter equation is in non-divergence form, so we can apply the growth lemma for such equations, already proved in [S17] (Theorem 4.2 in that paper, with $f = 0$ , $\Omega = B_1$). Since

$$|\{y \in B_1(0) : \widehat{\varphi}_0 \geq d(y)\}| \geq |\{y \in B_{1/2}(0) : \widehat{\varphi}_0 \geq d(y)\}| \geq |B_{1/2}| \geq C|B_1|.$$

we can find $k > 0$ such that $\widehat{\varphi}_0(y) \geq kd(y)$ for all $y \in B_1/B_{1/2}$. Equivalently, $\varphi_0(x) \geq \frac{k}{\rho}d(x)$ for all $x \in B_\rho/B_{\rho/2}$. The Claim is proved. ■

As a second ingredient, we introduce a technical result from measure theory which is an equivalent version of Krylov's famous propagating ink spot Lemma (see Lemma 4.2, [S17]).

**Lemma 4.3** *Let $A \subset B \subset Q_1$ be two open sets. Assume there exists $\alpha \in (0, 1)$ such that:*

*(i)* $|A| \leq (1 - \alpha)|Q_1|$.

*(ii) For any cube $Q \subset Q_1$, $|Q \cap A| \geq (1 - \alpha)|Q|$ implies $Q \subset B$.*

*Then, it follows that* $|A| \leq (1 - c_0\alpha)|B|$ *for some constant* $c_0 = c_0(n) \in (0, 1)$.

The next lemma is the basis for the Calderon-Zigmund-Caffarelli type iteration, which we will use to prove the boundary weak Harnack inequality. Although our proof is based in the correspondent theorem present in [S17], in [S17] such type of inequalities is established for a uniformly elliptic operator in non-divergence form. We exhibit a self-contained proof under our assumptions, simplifying the arguments employed there.

**Lemma 4.4** *Let* $u$ *be a nonnegative weak supersolution of* $L[u] \geq f$ *in* $Q_2$*, under the assumptions of the BWHI (Theorem 1.3), and* $f \in L^p(Q_2)$ *is non-positive in* $Q_2$*. Assume*

$$\inf_{Q_1} \frac{u(x)}{x_n} \leq 1.$$

*Then, there exist* $M > 1$*,* $\mu \in (0, 1)$ *and* $\delta_0 > 0$ *depending on* $n, p, \vartheta$ *such that if* $\|f\|_{L^p(Q_2)} \leq \delta_0$ *then*

$$|\{u/x_n > M^j\} \cap Q_1| \leq (1 - \mu)^j, \quad \forall j \in \mathbb{N}. \tag{4.6}$$

*Proof.* We choose $\nu = \dfrac{1}{2}\left(\dfrac{1}{4}\right)^n$ and denote by $M = \max\left\{\dfrac{1}{k}, \dfrac{4}{C_1}2^{1/p}\right\} > 1$, where $C_1$ the constant given by the interior Harnack inequality (see (4.13) below), and $k \in (0, 1)$ and $a > 0$ the constants given by the Growth Lemma 4.1 applied to our weak supersolution $u$, which is such that

$$L(ku) \geq kf \geq f \text{ in } Q_{3/2}. \tag{4.7}$$

If $u$ is a non-negative weak supersolution of $L[u] \geq f$, then $u$ is a non-negative weak supersolution of (4.7), since $f \leq 0$. We are going to show that (4.6) holds.

First of all, observe that

$$\{x \in Q_1 : u(x)/x_n > M\} \subset \{x \in Q_1 : ku(x) > x_n\}.$$

Hence, since $\inf\limits_{Q_1} \dfrac{ku(x)}{x_n} \leq k$ and $\|f\|_{L^p(Q_2)} \leq a$, Lemma 4.1 implies that

$$|\{x \in Q_1 : u(x)/x_n > M\}| \leq |\{x \in Q_1 : ku(x) > x_n\}| < \nu < \frac{1}{2} \tag{4.8}$$

and, in particular, (4.6) holds for $j = 1$ and $\mu < 1/2$.

Now, for $j > 1$ we fix $\mu = c_0/2$, where $c_0 < 1$ is given by the Lemma 4.3.

We introduce the sets

$$A = \{x \in Q_1 : u(x)/x_n > M^j\} \text{ and } B = \{x \in Q_1 : u(x)/x_n > M^{j-1}\}.$$

Since $M > 1$ and $j > 1$, observe that (4.8) implies that

$$|A| = |\{x \in Q_1 : ku(x) > x_n\}| < \frac{1}{2}, \tag{4.9}$$

and the first assumption of Lemma 4.3 is satisfied for $(1-\alpha) = \frac{1}{2}$. Thanks to (4.9) and the Claim 4.5 below we can apply Lemma 4.3, and we obtain that

$$|A| \leq \left(1 - \frac{c_0}{2}\right)|B|$$
$$\text{i.e., } |\{x \in Q_1 : u(x)/x_n > M^j\}| \leq (1-\mu)|\{x \in Q_1 : u(x)/x_n > M^{j-1}\}|.$$

Iterating in $j$ and using (4.8), the result follow with $\mu \in (0,1)$ depending only on $n$, once the following Claim is proved. ■

**Claim 4.5** *For every cube $Q_\rho(x_0) \subset Q_1$ such that*

$$|A \cap Q_\rho(x_0)| \geq \frac{1}{2}|Q_\rho(x_0)| = \frac{1}{2}\rho^n, \tag{4.10}$$

*we have $Q_\rho(x_0) \subset B$.*

*Proof.* Let us denote $x_0 = (x_0', x_{0_n})$ with $x_0' \in \mathbb{R}^{n-1}$. We define the new variable

$$y = (y', y_n) = \left(\frac{x' - x_0'}{\rho'}, \frac{x_n}{\rho'}\right) \text{ where } \rho' = 2x_{0_n}$$

and the rescaled function

$$v(y) = \frac{u(x)}{\rho'} = \frac{1}{\rho'}u(\rho' y' + x_0', \rho' y_n).$$

Then $v$ is a non-negative supersolution of

$$-\operatorname{div}(\widetilde{A}(y)Dv) + \rho'\widetilde{b}(y)Dv + (\rho')^2\widetilde{c}(y)v = \rho'\widetilde{f}(y), \text{ in } Q_{4/\rho'}\left(-\frac{x_0'}{\rho'}, \frac{2}{\rho'}\right) \tag{4.11}$$

where $\widetilde{\varphi}(y) := \varphi(x)$, for $\varphi = A, b, c, f$. In fact, $x \in Q_{\frac{3}{2}}$, then

$$2 > 3/4 > |x'| = |x' - x_0' + x_0'| = |y'\rho' + x_0'|$$
$$\text{and, } |y_n\rho' - 2| = |x_n - 2| \leq |x_n - \frac{3}{4}| + |\frac{3}{4} - 2| < \frac{3}{4} + \frac{5}{4} = 2.$$

Moreover, observe that

$$x \in A \cap Q_\rho(x_0) \text{ if and only if } y \in \{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_n\}$$

and so, (4.10) is equivalent to

$$|\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_n\}| \geq \frac{1}{2}|Q_{\rho/\rho'}(e)| = \frac{1}{2}\left(\frac{\rho}{\rho'}\right)^n. \tag{4.12}$$

Observe also that the embedding $Q_\rho(x_0) \subset Q_1$ implies that $\rho \leq \rho' \leq 2-\rho$ and $|x_{0,i}| \leq \frac{1-\rho}{2}$ $i \in \{1, \cdots, n-1\}$. In fact,

$$|x_i| \leq |x_i - x_{0,i}| + |x_{0,i}| < \frac{\rho}{2} + |x_{0,i}|$$

then $|x_{0,i}| \leq \frac{1-\rho}{2}$, similar with $x_n$, we obtain

$$|x_n - \frac{1}{2}| \leq |x_n - x_{0,n}| + |x_{0,n} - \frac{1}{2}| < \frac{\rho}{2} + |x_{0,n} - \frac{1}{2}|$$

with $\rho \leq 2x_{0,n} \leq 2-\rho$. In particular, we have $Q_{\frac{3}{4}} \subset Q_{\frac{4}{\rho'}}\left(-\frac{x_0'}{\rho'}, \frac{2}{\rho'}\right)$. In fact,

$$\begin{aligned} |x'\rho' + x_0'| \leq \rho'|x'| + \frac{1-\rho}{2} < 2 - \frac{5\rho}{4} < 2, \\ \text{and } |x_n\rho' - 2| \leq \frac{3\rho'}{4} + \frac{|3\rho' - 8|}{4} = 2. \end{aligned}$$

Now, we consider three cases:

**Case 1**: $\rho < \rho'/4$ . Then $v/M^j$ is also a non-negative supersolution of (4.11) and we apply the interior weak Harnack inequality, Theorem 1.1 (also see Theorem 8.18 in [GT01] ) to obtain

$$\inf_{Q_{\rho/\rho'}(e)} v \geq C_1 \left[\left(\frac{\rho}{\rho'}\right)^{-n} \int_{Q_{\rho/\rho'}(e)} v^p\right]^{\frac{1}{p}} - C\frac{\rho}{\rho'}\|\rho'\widetilde{f}\|_{L^p(Q_{2\rho/\rho'}(e))}. \tag{4.13}$$

Since $Q_\rho(x_0) \subset Q_1$ we have $Q_{2\rho}(x_0) \subset Q_2$ and

$$\|\rho'\widetilde{f}\|_{L^p(Q_{2\rho/\rho'}(e))} \leq C\|f\|_{L^p(Q_{2\rho})} \leq C_2\delta_0.$$

Now, let us introduce

$$G = \{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > 1/4\},$$

and, as $y_n > 1/4$ for all $y \in Q_{\rho/\rho'}(e)$, observe that (4.10) implies that

$$|G| \geq |\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_n\}| \geq \frac{1}{2}\left(\frac{\rho}{\rho'}\right)^n.$$

Hence, we deduce that

$$\begin{aligned}\inf_{Q_{\rho/\rho'}(e)} v &\geq C_1 \left[\left(\frac{\rho}{\rho'}\right)^{-n} \int_G v^p\right]^{\frac{1}{p}} - C_2\delta_0 \\ &\geq C_1 \left(\frac{\rho}{\rho'}\right)^{-\frac{n}{p}} \frac{M^j}{4} |G|^{\frac{1}{p}} - C_2\delta_0 \\ &\geq \frac{C_1}{2^{\frac{1}{p}}} \frac{M^j}{4} - C_2\delta_0.\end{aligned}$$

Using $M \geq \dfrac{4}{C_1} 2^{1/p}$, $y_n \leq 1$ in $Q_{\rho/\rho'}(e)$, increasing $M$ and diminishing $\delta_0$ we obtain

$$v \geq M^{j-1/2} - 1 \geq M^{j-1} y_n.$$

Thus, we conclude that $u(x)/x_n > M^{j-1}$ in $Q_\rho(x_0)$.
**Case 2:** $\rho \leq \rho'/4$ and $\rho' < 1$. Then $Q_2 \subset Q_{2/\rho'}$ and

$$\|\rho' \widetilde{f}\|_{L^p(Q_2)} \leq \|\rho' \widetilde{f}\|_{L^p(Q_{2/\rho'})} = (\rho')^{1-\frac{n}{p}} \|f\|_{L^p(Q_2)} \leq \delta_0 \leq a.$$

Observe that $v/M^j$ is a non-negative supersolution of

$$-\operatorname{div}(\widetilde{A}(y) D(v/M^j)) + \rho' \widetilde{b}(y).D(v/M^j) + \rho'^2 \widetilde{c}(y) v/M^j \geq \frac{\rho'}{M^j} \widetilde{f}(y) \geq \rho' \widetilde{f}$$

since $M^{-1} < 1$ and $\widetilde{f} < 0$. Moreover, as $\rho \leq \rho'$, (4.12) implies that

$$|\{y \in Q_1 : v(y)/M^j > y_n\}| \geq |\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_n\}| \geq \frac{1}{2}\left(\frac{\rho}{\rho'}\right)^n = \nu.$$

Hence by Lemma 4.1, we obtain $v(y)/M^j > k y_n$ in $Q_1$, and by the definition of $k$, $v(y)/y_n > M^{j-1}$ in $Q_{\rho/\rho'}(e)$. This implies that $u(x)/x_n > M^{j-1}$ in $Q_\rho(x_0)$.
**Case 3:** $\rho \geq \rho'/4$ and $\rho' \geq 1$. Then $\rho \geq 1/4$ and

$$|A \cap Q_1| \geq |A \cap Q_\rho| \geq \frac{1}{2}\rho^n \geq \frac{1}{2}\left(\frac{1}{4}\right)^n = \nu.$$

Hence by Lemma 4.1, we obtain directly $u(x)/M^j > k x_n$ in $Q_1$. This means that $Q_\rho(x_0) \subset B$ and so, the claim is proved. ■

## 4.2
## Boundary Weak Harnack Inequality for cubes

We state and prove the version for cubes of our main result in this section.

**Theorem 4.6 (Boundary Weak Harnack Inequality for cubes)** *Assume that $u$ is a nonnegative weak supersolution of problem $L[u] \geq f$ in $Q_2$, where $L$ is under the assumptions of Lemma 4.1, and $f \in L^p(Q_2)$ is nonpositive function in $Q_2$. Then, there exist constants $\epsilon > 0$ and $C > 0$ such that*

$$\inf_{Q_1} \frac{u}{x_n} \geq C \left[ \int_{Q_1} \left( \frac{u}{x_n} \right)^{\epsilon} \right]^{1/\epsilon} - C \|f\|_{L^p(Q_2)}.$$

*Proof.* Let us split the proof into three steps.

**Step 1:** Assume that $\inf_{Q_1} \dfrac{u(x)}{x_n} \leq 1$ and $\|f\|_{L^p(Q_2)} \leq \delta_0$. Then there exist $\epsilon > 0$ and $C > 0$ such that for all $t \geq 0$

$$|\{x \in Q_1 : u(x)/x_n > t\}| \leq C \min\{1, t^{-2\epsilon}\}.$$

To prove this, let us define the real valued function

$$g(t) = |\{x \in Q_1 : u(x)/x_n > t\}|$$

and let $M$ and $\mu$ be the constants obtained in Lemma 4.4. We define

$$C := \max\{(1-\mu)^{-1}, M^{2\epsilon}\} > 1 \quad \text{and} \quad \epsilon := -\frac{1}{2} \frac{\ln(1-\mu)}{\ln(M)} > 0.$$

If $t \in [0, M]$ we get

$$|\{x \in Q_1 : u(x)/x_n > t\}| \leq 1 \leq C M^{-2\epsilon} \leq C \min\{1, t^{-2\epsilon}\}.$$

Now, let us assume $t > M > 1$. Without loss of generality, we assume $t \in [M^j, M^{j+1}]$ for some $j \in \mathbb{N}$, and it follows that

$$\frac{\ln t}{\ln M} - 1 \leq j \leq \frac{\ln t}{\ln M}.$$

Since $g$ is non-increasing and $1 - \mu \in (0, 1)$, the above inequality and Lemma 4.4 imply

$$g(t) \leq f(M^j) \leq (1-\mu)^j \leq (1-\mu)^{\frac{\ln t}{\ln M} - 1}.$$

Observe that,

$$\ln\left((1-\mu)^{\frac{\ln t}{\ln M}-1}\right) = \left(\frac{\ln t}{\ln M}-1\right)\ln(1-\mu) = \frac{\ln(1-\mu)}{\ln M}\ln t - \ln(1-\mu)$$
$$\leq -2\epsilon \ln t + \ln C = \ln(Ct^{-2\epsilon}).$$

Finally, the conclusion of Step 1 follows from the two last equations and the fact $\min\{1, t^{-2\epsilon}\} = t^{-2\epsilon}$ for $t \geq 1$.

**Step 2:** Assume that $\inf_{Q_1} \dfrac{u(x)}{x_n} \leq 1$ and $\|f\|_{L^p(Q_2)} \leq \delta_0$. Then there exist $C > 0$ such that

$$\int_{Q_1} \left(\frac{u(x)}{x_n}\right)^{\epsilon} dx \leq C.$$

Hence, applying Lemma 9.7 [GT01] we obtain that

$$\int_{Q_1} \left(\frac{u(x)}{x_n}\right)^{\epsilon} dx \leq \epsilon \int_0^{\infty} t^{\epsilon-1} |\{x \in Q_1 : u(x)/x_n > t\}| dt$$
$$\leq C\epsilon \int_0^{\infty} t^{\epsilon-1} \min\{1, t^{-2\epsilon}\} dt = C.$$

**Step 3:** Conclusion. Let us introduce the functions

$$v = \frac{u}{\inf\limits_{y\in Q_1} \dfrac{u(y)}{y_n} + \beta + \delta^{-1}\|f\|_{L^p(Q_2)}} \qquad \text{and} \qquad \widetilde{f} = \frac{f}{\inf\limits_{y\in Q_1} \dfrac{u(y)}{y_n} + \beta + \delta^{-1}\|f\|_{L^p(Q_2)}}$$

where $\beta > 0$ is an arbitrary constant. Hence $v$ satisfies $\inf_{Q_1} \dfrac{v(x)}{x_n} \leq 1$ and $\|\widetilde{f}\| \leq \delta_0$ then applying Step 2, we obtain that

$$\int_{Q_1} \left(\frac{v(x)}{x_n}\right)^{\epsilon} dx,$$

so

$$\left(\int_{Q_1} \left(\frac{u(x)}{x_n}\right)^{\epsilon} dx\right)^{1/\epsilon} \leq C \inf_{y\in Q_1} \frac{u(y)}{y_n} + \beta + \delta_0^{-1}\|f\|_{L^p(Q_2)}.$$

Therefore, we get the result by letting $\beta \to 0$. ■

## 4.3 Boundary Weak Harnack Inequality

Finally, we enunciate the general version of boundary weak Harnack inequality. Namely, the statement for an arbitrary $\Omega \subset \mathbb{R}^n$ with $C^{1,D}$-regular boundary.

**Theorem 4.7 (Boundary Weak Harnack Inequality)** *Let $\Omega \subset R^n$, for $n \geq 2$, be a bounded domain. Assume that $u$ is a nonnegative weak supersolution of $L[u] \geq f$ in $\Omega$, where $\Omega$ and $L$ are under our hypotheses and $f \in L^p(\Omega)$ is non-positive in $\Omega$. Then for any $x_0 \in \partial\Omega$ there exist constants $\overline{R} > 0$, $\epsilon > 0$ and $C > 0$ such that for all $R \in (0, \overline{R}]$,*

$$\inf_{B_R(x_0)\cap\Omega} \frac{u(x)}{d(x)} \geq C \left( \int_{B_R(x_0)\cap\Omega} \left( \frac{u(x)}{d(x)} \right)^{\epsilon} dx \right)^{1/\epsilon} - C\|f\|_{L^p(\Omega)}. \qquad (4.14)$$

*Proof.* By the definition of a $C^{1,D}$-domain, at each point $x_0 \in \partial\Omega$ there is a neighborhood $N$ of $x_0$ and a $C^{1,D}$-diffeomorphism $\varphi$ that straightens the boundary in $N$, such that $D\varphi(0) = I_n$. Let $B_R(x_0) \subset\subset N$ and set $B' = B_R(x_0) \cap \Omega$, $\varphi(B') \subset Q_1$, $T = B_R(x_0) \cap \partial\Omega \subset \partial B'$, and $\varphi(T) \subset \{x \in Q_1; x_n = 0\}$ ($\varphi(T)$ is a hiperplane portion of $\partial Q_1$). Under the mapping $y = \varphi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$, let $\tilde{u}(y) = u(x)$, and $\tilde{L}\tilde{u}(y) = Lu(x)$, where

$$\tilde{L}\tilde{u} \equiv -\operatorname{div}(\tilde{A}(y)D\tilde{u}) + \tilde{b}(y)D\tilde{u} + \tilde{c}(y)\tilde{u} = \tilde{f}(y),$$

and $\tilde{A}(\varphi(x)) = A(x)D\varphi(x)$, $\tilde{b}(\varphi(x)) = D\varphi(x)b(x)$, $\tilde{c}(y) = c(x)$, $\tilde{f}(y) = f(x)$.

Since $D\varphi(0) = I_n$, by choosing $\overline{R}$ small enough we can ensure that the new equation has the same properties as the original one (by replacing, say, $\lambda$ by $\lambda/2$, and $\vartheta$ by $2\vartheta$). The assumptions of Theorem 4.6 are satisfied for the equation $\tilde{L}\tilde{u} = \tilde{f}$ in $\varphi(B')$ with the hyperplane portion $\varphi(T)$. We can therefore assert

$$\inf_{Q_1} \frac{\tilde{u}(y)}{y_n} \geq C \left( \int_{Q_1} \left( \frac{\tilde{u}(y)}{y_n} \right)^{\epsilon} dy \right)^{1/\epsilon} - C\|\tilde{f}\|_{L^p(Q_2)}$$

for any $Q_2 \subset \varphi(B_1)$. Returning to the original variable $x$, the latter inequality implies (4.14), by diminishing $\overline{R}$, if necessary.

■

## 4.4 Regularity Estimates

This section has expository character. For the reader's convenience we sketch how, using the original method of Safonov, Hölder estimates can be inferred from the weak Harnack inequality. We will use these estimates in later sections.

**Theorem 4.8 ( $C^\alpha$ Regularity Estimates)** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume $L$ is under the assumptions of Lemma 5.1, $f \in L^p(\Omega)$, where*

*$p > n$. Let $u$ be a weak solution of*

$$L[u] = f(x) \text{ in } \Omega$$

*with $\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} \leq C_0$. Then, there exists $\alpha \in (0,1)$ depending on $n, p, \vartheta, \|b\|_{L^p(\Omega)}$, such that $u \in C^\alpha_{loc}(\Omega)$ and for any subdomain $\Omega' \subset\subset \Omega$ we have*

$$\|u\|_{C^\alpha(\overline{\Omega'})} \leq C\left\{\|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}\right\}$$

*where $C$ depends only $n, p, \vartheta, \|b\|_{L^p(\Omega)}, \|c\|_{L^p(\Omega)}, dist(\Omega', \partial\Omega)$ and $C_0$. If, in addition $\Omega$ is $C^1$-smooth, then there exist some $\alpha_0, \rho_0 > 0$ depending only $n, p, \vartheta, \vartheta^{-1}, \|b\|_{L^p(\Omega)}$, such that for each ball $B_\rho$ with radius $\rho \leq \rho_0$ and center in $\overline{\Omega}$*

$$\operatorname*{osc}_{B_\rho \cap \Omega} u \leq C_1(\rho^{\alpha_0} + \operatorname*{osc}_{B_{\sqrt{\rho}} \cap \partial\Omega} u)$$

*where $C_1$ depends on $n, p, \vartheta, \vartheta^{-1}, \|b\|_{L^p(\Omega)}, \|c\|_{L^p(\Omega)}, \Omega$ and $C_0$. Hence if $u|_{\partial\Omega} \in C^\beta(\partial B)$ then $u \in C^\alpha(\overline{\Omega})$ with $\alpha = \min\{\alpha_0, \beta/2\}$.*

**Proof of Theorem 4.8**
Note we can assume $c = 0$ by replacing $f$ by $f - cu$.
First, we give the proof of the interior estimate in the case $f = 0$. Recall we have a solution $u \in C(\Omega)$ of $-\operatorname{div}(A(x)Du) + b(x).Du = f(x)$. Then for any $\rho$ such that $B_{2\rho} \subset \Omega$ the functions

$$u_1 := u - \inf_{B_{2\rho}} u \qquad u_2 := \sup_{B_{2\rho}} u - u$$

satisfy the hypotheses of Growth Lemma (Lemma 4.1) . In addition, we define

$$w(2\rho) := \operatorname*{osc}_{B_{2\rho}} u = u_1 + u_2.$$

Note the following equivalences are satisfied,

(i) $u \geq \frac{1}{2}(\sup_{B_{2\rho}} u + \inf_{B_{2\rho}} u) \iff u_1 \geq \frac{1}{2}w(2\rho)$

(ii) $u \leq \frac{1}{2}(\sup_{B_{2\rho}} u + \inf_{B_{2\rho}} u) \iff u_2 \geq \frac{1}{2}w(2\rho)$

so at each point of $B_{2\rho}$, either $u_1$ or $u_2$ is greater or equal than $\frac{1}{2}w(2\rho)$.
**Case 1:** Suppose that

$$\left|\left\{x \in B_\rho;\ u_1 \geq \frac{1}{2}w(2\rho)\right\}\right| \geq \frac{1}{2}|B_\rho|$$

Applying Lemma 4.1 to $u_1 \geq 0$ in $B_\rho$, we have that for some $k > 0$

$$u_1 \geq kw(2\rho) \text{ in } B_\rho,$$

which yields the following estimate:

$$\inf_{B_\rho} u \geq k \sup_{B_{2\rho}} + (1-k) \inf_{B_{2\rho}} u.$$

**Case 2:** Suppose that

$$\left|\left\{x \in B_\rho;\ u_2 \geq \frac{1}{2} w(2\rho)\right\}\right| \geq \frac{1}{2}|B_\rho|.$$

Similarly we obtain

$$u_2 \geq kw(2\rho) \text{ in } B_\rho,$$

which yield the following estimate:

$$\sup_{B_\rho} u \leq k \sup_{B_{2\rho}} + (1-k) \inf_{B_{2\rho}} u.$$

In both cases, we have $w(\rho) \leq (1-k)w(2\rho)$, for all $\rho \in (0, \rho_0)$. With the claim 4.9 below we have $u \in C^\alpha(B_\rho)$. Next, we deal with $f \neq 0$. We argue in the same way as in the case $f = 0$, but in the end we get

$$w(\rho) \leq (1-k)w(2\rho) + C\rho\|f\|_{L^p(B_{2\rho})}.$$

Applying Lemma 8.23 of [GT01], for any $\gamma \in (0,1)$ there exists $\alpha$ depending on $\gamma, n, \vartheta, \vartheta^{-1}, \|b\|_{L^p}$ such that

$$w(\rho) \leq C \sup_{B_{\rho_0}} |u| \rho^\alpha w(2\rho) + C\rho^\gamma \|f\|_{L^p(B_{2\rho})}.$$

The last case deals with the extension of the result to the boundary. We use the idea of extending the function $u$ as a constant outside the domain. Let $x_0 \in \partial\Omega$ we want to show that for some $k_0 < 1$,

$$\operatorname*{osc}_{B_\rho(x_0)\cap\Omega} u \leq k_0 \operatorname*{osc}_{B_{2\rho}(x_0)\cap\Omega} u + C\rho\|f\|_{L^p(\Omega)} + 2 \operatorname*{osc}_{B_{2\rho}(x_0)\cap\partial\Omega} u$$

which would imply, by Claim 4.9

$$w(\rho) \leq C\rho^\alpha + \omega(\rho^\nu), \forall \nu < 1, \forall \rho \leq \rho_0$$

where $\sigma(\rho) = C\rho\|f\|_{L^p(\Omega)} + 2\operatorname{osc}_{B_{2\rho}(x_0)\cap\partial\Omega} u$. Note that if

$$\operatorname*{osc}_{B_\rho(x_0)\cap\Omega} u \le 2 \operatorname*{osc}_{B_{2\rho}(x_0)\cap\partial\Omega} u$$

we are done. Suppose then

$$\operatorname*{osc}_{B_\rho(x_0)\cap\Omega} u > 2 \operatorname*{osc}_{B_{2\rho}(x_0)\cap\partial\Omega} u$$

and we have two possibilities:

$$1) \quad \sup_{B_{2\rho}(x_0)\cap\Omega} u - \sup_{B_{2\rho}(x_0)\cap\partial\Omega} u \le \frac{1}{4} \operatorname*{osc}_{B_{2\rho}(x_0)\cap\Omega} u,$$

and

$$2) \quad \inf_{B_{2\rho}(x_0)\cap\partial\Omega} u - \inf_{B_{2\rho}(x_0)\cap\Omega} u \le \frac{1}{4} \operatorname*{osc}_{B_{2\rho}(x_0)\cap\Omega} u.$$

For the case 2 we define $u_2 := u - \inf\limits_{B_{2\rho}(x_0)\cap\Omega} u$ so that

$$\begin{cases} L[u_2] = f & \text{in } B_{2\rho}(x_0)\cap\Omega \\ u_2 \ge 0 & \text{in } B_{2\rho}(x_0)\cap\Omega \\ u_2 \ge \dfrac{1}{4}w(2\rho) & \text{on } B_{2\rho}(x_0)\cap\partial\Omega. \end{cases}$$

In $B_{2\rho}(x_0)$ we define the function

$$\overline{u}_2 := \begin{cases} \min\{u_2, \dfrac{1}{8}w(2\rho)\} \text{ in } & B_{2\rho}(x_0)\cap\Omega \\ \dfrac{1}{8}w(2\rho) \text{ in } & B_{2\rho}(x_0)\setminus\Omega. \end{cases}$$

By Lemma 4.10 below we obtain that $\overline{u}_2$ is a supersolution, and we allow to use Lemma 4.1 to conclude that

$$\overline{u}_2 \ge k_0 a - C\rho\|f\|_{L^p} \text{ in } B_\rho$$

where $d = m = \frac{1}{8}w(2\rho)$ and by the definition of $\overline{u}_2$, we obtain that

$$\inf_{B_\rho} u_2 \ge \inf_{B_\rho} \overline{u}_2 \ge \frac{k_0}{8}w(2\rho) - C\rho\|f\|_{L^p(\Omega)}$$

which implies that

$$w(\rho) \le (1 - \frac{k_0}{8})w(2\rho) + C\rho\|f\|_{L^p(\Omega)}.$$

**Claim 4.9** *Suppose the hypotheses of Theorem 4.8 are satisfied and for all $\rho \in (0, \rho_0]$, the inequality*

$$w(\rho) \leq (1-k)w(2\rho)$$

*holds, $k > 0$. Then for any $\rho \in (0, \rho_0)$, we have $u$ is locally Hölder continuous in $\Omega$ and for any ball $B_{\rho_0}$ we have*

$$\sup_{x,y \in B_\rho} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq C \sup_{B_{\rho_0}} |u|$$

*where $C = C(a, n, p, \rho_0), \alpha = \alpha(a, n, p, \rho_0) > 0$ are positive constants.*

*Proof.* Let us fix initialy some number $\rho_1 \leq \rho_0$. Then for any $\rho \leq \rho_1$ we have

$$w(\rho) \leq (1-k)w(2\rho).$$

We now iterate this inequality to get, for any positive integer $m$,

$$w(2^{-m}\rho_1) \leq (1-k)^m w(\rho_1) \leq (1-k)^m w(\rho_0). \tag{4.15}$$

For any $\rho \leq \rho_1$, we can choose $m$ such that

$$2^{-m}\rho_1 < \rho \leq 2^{-m+1}\rho_1.$$

The inequality (4.15) can be written in the form of the Hölder condition. Hence,

$$\begin{aligned} |u(x)-u(y)| &\leq w(\rho) \leq w(2^{-m+1}\rho_1) \leq (1-k)^{m-1} w(\rho_0) \\ &\leq \frac{1}{(1-k)} \left(\frac{\rho}{\rho_1}\right)^{\log_{1/2}(1-k)} w(\rho_0). \end{aligned}$$

Now let $\rho_1 = \rho_0^{1-\alpha}\rho^\alpha$ so that we have

$$|u(x)-u(y)| \leq \frac{1}{(1-k)} \left(\frac{\rho}{\rho_0}\right)^{(1-\alpha)\log_{1/2}(1-k)} w(\rho_0).$$

Indeed, let $\alpha > 0$ be the number such that $(1-\alpha)\log_{1/2}(1-k) < \alpha$, then

$$|u(x)-u(y)| \leq \frac{C}{(1-k)} \left(\frac{\rho}{\rho_0}\right)^\alpha \sup_{B_{\rho_0}} |u| \leq C\rho^\alpha \sup_{B_{\rho_0}} |u|.$$

■

**Lemma 4.10** *Suppose for some* $B \subset \mathbf{R}^n$ *and for* $f \in L^p(B), f \leq 0, u \in C(\overline{\Omega}), m > 0$, *we have*

$$\begin{cases} \mathcal{L}[u] & \geq f(x) \quad \text{in } B_{2\rho} \cap \Omega \\ u & \geq 0 \quad \text{in } B_{2\rho} \cap \Omega \\ u & \geq 2m \quad \text{on } B_{2\rho} \cap \partial\Omega. \end{cases}$$

*Then for all* $B_{2\rho} \subset \Omega \cup B$, $\rho \leq \rho_0$ *and for any* $\nu, a > 0$ *it followa that*

$$\inf_{B_\rho} \overline{u} \geq ka - C\rho \|f\|_{L^p(\Omega)}$$

*where* $k, C$ *depends on* $\nu, n, \vartheta, \vartheta^{-1}, \|b\|_{L^p}, p > n$ *and* $\overline{u} \in C(B)$ *is defined by*

$$\overline{u} = \begin{cases} \min\{u, m\} & \text{in } B_{2\rho} \cap \Omega \\ m & \text{in } B_{2\rho} \setminus \Omega. \end{cases}$$

*Proof.* We note that $\overline{u}$ satisfies the hypotheses of Growth Lemma 4.1 in the ball $B$, since the minimum of two weak supersolution is a weak supersolution and $\mathcal{L}[\overline{u}] = \mathcal{L}[m] \equiv 0 \geq f(x)$. ■

*Proof.*[Proof of Theorem 4.8] Let us prove the interior estimate for

$$-\operatorname{div}(A(x)Du) - \mu|Du|^2 - b(x)|Du| \geq |f(x)| \tag{4.16}$$

where $u$ and $-u$ are solutions of (4.16). Hence by Lemma 2.13, the functions

$$w_1 = \frac{1 - e^{-mu_1}}{m}, \qquad w_2 = \frac{1 - e^{-mu_2}}{m}$$

where $u_1 = u - \inf_{B_{2\rho}} u$ and $u_2 = \sup_{B_{2\rho}} u - u$ satisfy

$$\begin{aligned} -\operatorname{div}(A(x)Dw_j) - \mu|Du|^2 - b(x)|Du| &\geq (1 - mw_j)[|f(x)| + \vartheta m|Du|^2] \\ &\geq (1 - mw_j)|f(x)| := \tilde{f}. \end{aligned}$$

Since at each point $x \in B_{2\rho}$

$$w_j \geq \frac{1 - e^{-m\frac{w(2\rho)}{m}}}{m}$$

for some $j$, say $j = 1$, reasoning as before and applying the Growth Lemma we get

$$w_1 \geq k\frac{1-e^{-m\frac{w(2\rho)}{m}}}{m} - C\rho\|f\|_{L^p(B_{2\rho})}$$

for $\rho \leq \rho_0$. Notice that for each $t_0$ there exits $\epsilon = \epsilon(t_0, m)$ such that

$$t \geq \frac{1-e^{-mt}}{m} \geq \epsilon t \text{ for } t \in [0, t_0].$$

We apply this with $t_0 = w(2\rho_0)/2$ and we get

$$u_1 \geq \frac{1-e^{-mu_1}}{m} = w_1 \geq k\frac{1-e^{-m\frac{w(2\rho)}{2}}}{m} - C\rho\|f\|_{L^p(B_{2\rho})} \geq \frac{\epsilon}{2}w(2\rho_0)k - C\rho\|f\|_{L^p(B_{2\rho})}$$

in $B_\rho$, so again

$$w(\rho) \leq C\rho^\alpha \sup_{B_{\rho_0}}|u| + C\rho^\gamma\|f\|_{L^p(B_{2\rho})}$$

for $\rho \in (0, \rho_0)$. ■

# 5
# A priori Bounds and Multiplicity results

In this chapter we obtain a priori bounds for solutions of a class of indefinite quasilinear elliptic equations, assuming lower regularity on their coefficients and on the boundary of $\Omega$ than in the previous works on the subject. These a priori bounds are going to be used to establish existence and multiplicity of solutions for these problems.

We consider the following class of boundary value problem

$$\begin{cases} -\operatorname{div}(A(x)Du) & = c_\lambda(x)u + (M(x)Du, Du) + h(x) \\ \qquad u & \in H_0^1(\Omega) \cup L^\infty(\Omega) \end{cases} \tag{$P_\lambda$}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded domain with boundary $\partial\Omega$ of class $C^{1,D}$. It is assumed that $c, h \in L^p(\Omega)$ for some $p > n$, where $c^+$ and $c^-$ are nonnegative functions such that $c_\lambda(x) := \lambda c^+(x) - c^-(x)$ for a parameter $\lambda \in \mathbb{R}$. Also $A(x)$ is a uniformly positive bounded measurable matrix, i.e. $\vartheta I_n \leq (a_{ij}(x)) \leq \vartheta^{-1} I_n$, $\vartheta$ is a positive constant, and $I_n$ is the identity matrix; and that $M(x)$ is an positive matrix such that

$$0 < \mu_1 I_n \leq M(x) \leq \mu_2 I_n \text{ in } \Omega \tag{5.1}$$

for some positive constants $\mu_1$ and $\mu_2$.

The specificity of these problems, and what makes them delicate to study, is the quadratic dependence in the gradient, which makes the gradient term of the same order as the Laplacian with respect to dilations. We refer to [CFJ19], [NS18] for a review of the large literature on this topic.

The study of the coercive case, i.e. $c \leq 0$, was initiated by Boccardo, Murat and Puel in the 80's, and we refer to [ACJT14] for the uniqueness. On the other hand, the noncoercive case remained unexplored until very recently. We refer a particular case of Jeanjean and Sirakov, study a problem directly connected to ($P_\lambda$).

As in [CFJ19] we assume the additional assumption

$$\begin{cases} |\Omega_{c^+}| > 0, & \text{where } \Omega_{c^+} := supp(c^+), \\ \text{There exists } \epsilon > 0 \text{ such that } c^- = 0 & \text{in } \{x \in \Omega : d(x, \Omega_{c^+}) < \epsilon\}. \end{cases} \tag{A}$$

This hypothesis means we are in the "hard" noncoercive case, when the zero order coefficient is not negative, and uniqueness of solutions is expected to fail. For a definition of $supp(f)$ with $f \in L^p(\Omega)$, for some $p \geq 1$, we refer to Proposition 4.17 [B11].

**Definition 5.1** *Let $f \in L^p(\Omega)$. Consider the family $(\omega_i)_{i\in I}$ of all open sets on $\Omega$ such that for each $i \in I$, $f = 0$ a.e. on $\omega_i$. Set $\omega = \cup_{i\in I}\omega_i$.*
*Then $f = 0$ a.e on $\omega$.*
*By definition, $supp(f)$ is the complement of $\omega$ in $\Omega$.*

We also observe that, under the above regularity assumptions, any solution of $(P_\lambda)$ belongs to $C^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. This can be deduced from ([LU68], Theorem IX-2,2).

As in [ACJT14], [ACJT15], [CFJ19], [NS18], we will obtain our results by using a topological approach, which relies heavily on the derivation of a priori bounds, and combines sub and supersolutions arguments together with degree theory.

We now recall a few definitions. We will denote with $\gamma_1 > 0$ the “first eigenvalue" of the linear problem, which in our case means that the problem

$$\begin{cases} -\operatorname{div}(A(x)D\varphi_1) &= c_{\gamma_1}(x)\varphi_1 & \text{in } \Omega \\ \varphi_1 &> 0 & \text{in } \Omega \\ \varphi_1 &= 0 & \text{on } \partial\Omega \end{cases} \qquad (P_{\gamma_1})$$

has a solution. In that case when $h(x) \gneqq 0$ the problem $(P_\lambda)$ has no solution $u$ with $c^+(x)u \gneqq 0$ when $\lambda = \gamma_1$ and no nonnegative solutions when $\lambda \geq \gamma_1$. See Lemma 6.1,[ACJT15] for details.

Further, we define strict comparison between functions in the following way.

**Definition 5.2** *Let $u, v \in C(\overline{\Omega})$. We say that $u \ll v$ in case there exists $\varepsilon > 0$ such that, for all $x \in \overline{\Omega}$, $v(x) - u(x) \geq \varepsilon\varphi_1(x)$, where $\varphi_1$ is the first eigenfunction of $(P_{\gamma_1})$. Recall that, for all $x \in \Omega$, $\varphi_1(x) > 0$ and, for $x \in \partial\Omega$, $\frac{\partial\varphi_1}{\partial\nu}(x) < 0$ where $\nu$ denotes the exterior unit normal.*

We make the convention, mainly when dealing with multiplicity results, that $\alpha$ and $\beta$ will always denote a pair of sub and supersolutions, in a sense to be specified.

Many of our results are valid without assuming that $h$ has a sign. However, when we require $h$ to have a sign, we will see that the set of solutions differs completely for $h \lneqq 0$ and $h \gneqq 0$.

## 5.1
## Main results

We now state our main multiplicity results which is going to be proved next chapter. In what follows continuum means a closed and connected set and the above assumptions on the coefficients of the equation are assumed to hold.

More precisely, defining

$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : u \text{ solves } (P_\lambda)\},$$

we will show that it is possible to obtain a description of the set $\Sigma$. In the next two theorems, following the strategy of [CFJ19], we show the existence of a continuum of solutions of $(P_\lambda)$ when the coercive problem $(P_0)$ with $\lambda = 0$ has a solution (conditions on the coefficients which ensure this can be found for instance in [ACJT15], [CFJ19]).

**Theorem 5.1** *Suppose that $(P_0)$ has a solution $u_0$ with $c^+(x)u_0 \gneqq 0$. Then*

*(i) For all $\lambda \leq 0$, the problem $(P_\lambda)$ has a unique solution $u_\lambda$ and this solution satisfies $u_0 - \|u_0\|_\infty \leq u_\lambda \leq u_0$.*

*(ii) There exists a continuum $\mathcal{C} \subset \Sigma$ such that the projection of $\mathcal{C}$ on the $\lambda$-axis is an unbounded interval $(-\infty, \overline{\lambda}]$ for some $\overline{\lambda} \in (0, +\infty)$ and $\mathcal{C}$ bifurcates from infinity to the right of the axis $\lambda = 0$.*

*(iii) There exists $\lambda_0 \in (0, \overline{\lambda}]$ such that, for all $\lambda \in (0, \lambda_0)$, the problem $(P_\lambda)$ has at least two solutions with $u_i \geq u_0$ for $i = 1, 2$.*

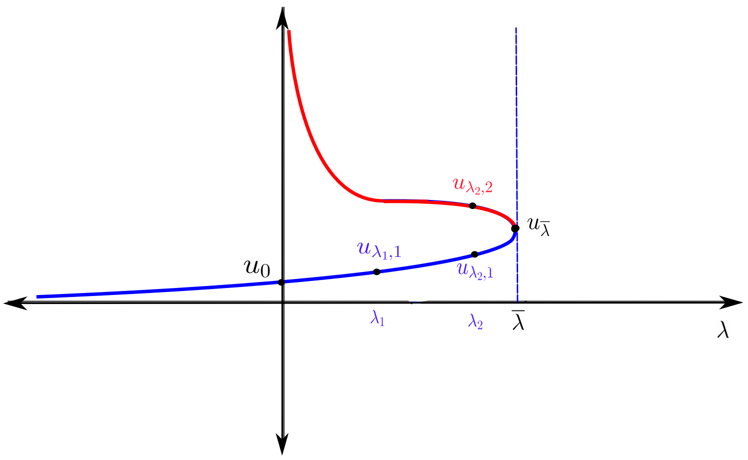


Figure 5.1: Illustration of Theorem 5.1

**Theorem 5.2** *Suppose that $(P_0)$ has a solution $u_0 \leq 0$ with $c^+(x)u_0 \lneqq 0$. Then* *(i) For $\lambda \leq 0$, the problem ($P_\lambda$) has a unique nonpositive solution $u_\lambda$ and this solution satisfies $u_0 + \|u_0\|_\infty \geq u_\lambda \geq u_0$;*

*(ii) There exists a continuum $\mathcal{C} \subset \Sigma$ such that its projection of $\mathcal{C}^+$ on the $\lambda$-axis is $[0, +\infty)$;*

*(iii) For $\lambda > 0$, every non-positive solution of ($P_\lambda$) satisfies $u_\lambda \ll u_0$. Furthermore ($P_\lambda$) has at least two non-trivial solutions $u_{\lambda,i}$ for $i = 1, 2$ with*

$$u_{\lambda,1} \ll u_0 \leq u_{\lambda,2}, \quad u_{\lambda,1} \ll u_{\lambda,2}, \ \textit{and} \ \max_{\overline{\Omega}} u_{\lambda,2} > 0.$$

*Moreover we have $u_{\lambda_2,1} \leq u_{\lambda_1,1} \leq u_0$ if $0 < \lambda_1 < \lambda_2$.*

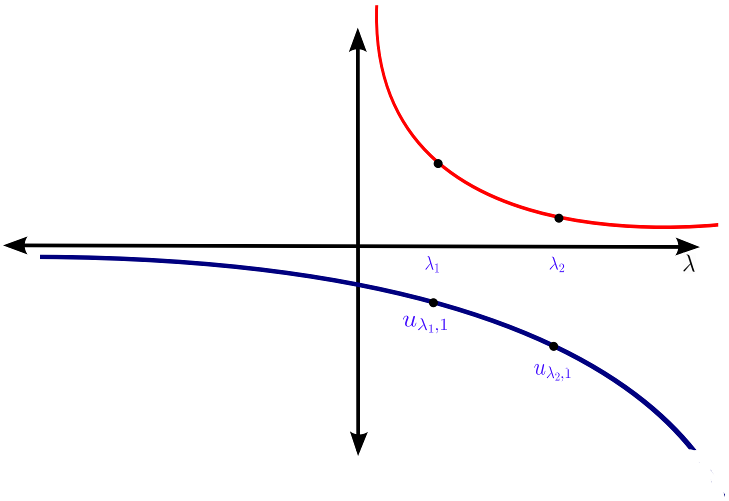


Figure 5.2: Illustration of Theorem 5.2



Note that our Theorems 5.1 and 5.2 require ($P_0$) to have a solution and thus we are in a situation where a branch of solutions starts from $(0, u_0)$. In our next results we consider the situation when a (super)solution of ($P_\lambda$) exists for some $\lambda_0 > 0$.

**Theorem 5.3** *Assume that*

*(a) ($P_0$) does not have a solution $u_0 \leq 0$;*

*(b) there exist $\lambda_0 > 0$ and $\beta_0$ a supersolution of ($P_{\lambda_0}$) with $\beta_0 \leq 0$.*

*Then there exists $0 < \underline{\lambda} \leq \lambda_0$ such that*

*(i) for every $\lambda \in (\underline{\lambda}, \infty)$, the problem ($P_\lambda$) has at least two solutions with $u_{\lambda,1} \leq 0$ and $u_{\lambda,1} \leq u_{\lambda,2}$. Moreover, if $\lambda_1 < \lambda_2$, we have $u_{\lambda_1,1} \gg u_{\lambda_2,1}$;*

(ii) *the problem* $(P_{\underline{\lambda}})$ *has a unique solution* $u_{\underline{\lambda}} \leq 0$;

(iii) *for* $\lambda < \underline{\lambda}$, *the problem* $(P_\lambda)$ *has no solution* $u \leq 0$.

*For every* $\lambda < 0$, *the problem* $(P_\lambda)$ *has at most one nonpositive solution* $u_\lambda$; *There exists an unbounded contiuum* $\mathcal{C}$ , $u_\lambda$ *and* $\lambda = 0$ *is a birfucation point from infinity.*

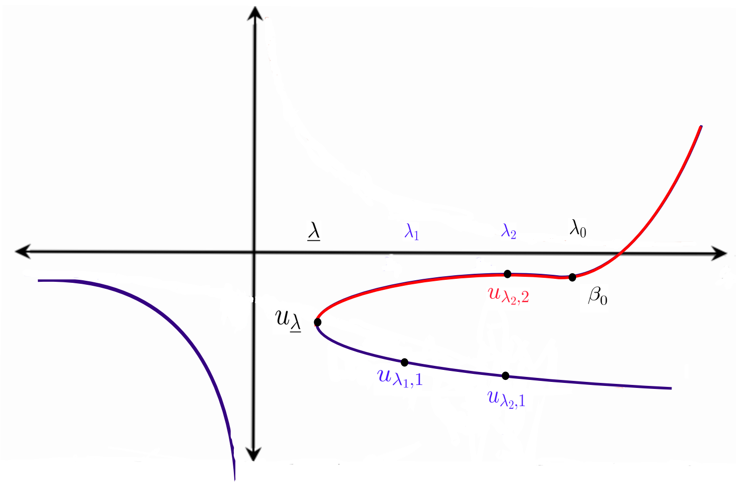


Figure 5.3: Illustration of Theorem 5.3

**Open Problem** Can we prove in Theorem 5.3 that the second solution changes sign?

In the proof of Theorem 5.2 below (see page 68) we define the auxiliary problem $(P_{\lambda,k})$, whose solutions are supersolutions of $(P_\lambda)$. In particular, from Theorem 5.1 and Lemma 6.7 below we can deduce the following corollary which concerns the case $h \lneqq 0$, and in which we see the simultaneous realization of two of the above theorems.

**Corollary 5.4** *Assume that* $h \lneqq 0$. *For all* $\widetilde{\lambda} > \gamma_1$ *where* $\gamma_1 > 0$ *is the first eigenvalue* $(P_{\gamma_1})$, *there exists* $\widetilde{k} > 0$ *such that, for all* $k \in (0, \widetilde{k}]$,

(i) *there exists* $\lambda_1 \in (0, \gamma_1)$ *such that*

- (a) *for all* $\lambda \in (0, \lambda_1)$, *the problem* $(P_{\lambda,k})$ *has at least two positive solutions;*
- (b) *for* $\lambda = \lambda_1$, *the problem* $(P_{\lambda,k})$ *has exactly one positive solution;*
- (c) *for* $\lambda > \lambda_1$, *the problem* $(P_{\lambda,k})$ *has no non-negative solution;*

(ii) *for* $\lambda = \gamma_1$ *the problem* $(P_{\lambda,k})$ *has no solution;*

(iii) *there exists* $\lambda_2 \in (\gamma_1, \widetilde{\lambda}]$ *such that*

(a) *for* $\lambda > \lambda_2$*, the problem* $(P_{\lambda,k})$ *has at least two solutions with* $u_{\lambda,1} \ll 0$ *and* $\min u_{\lambda,2} < 0$*;*

(b) *for* $\lambda = \lambda_2$*, the problem* $(P_{\lambda,k})$ *has a unique non-positive solution;*

(c) $\lambda < \lambda_2$*, the problem* $(P_{\lambda,k})$ *has no non-positive solution.*

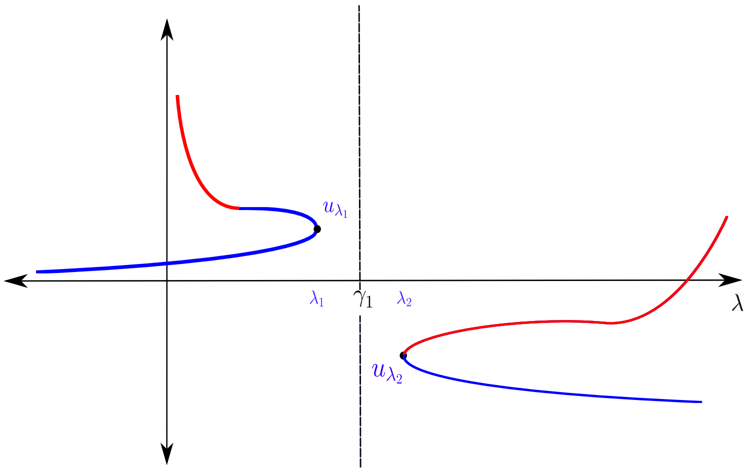


Figure 5.4: Illustration of Corollary 5.4



We conclude this section with a result on the particular but important case $h(x) \equiv 0$. Further considerations in case $h(x)$ has a sign are given in Remark 6.10 below.

**Theorem 5.5** *Assume* $h(x) \equiv 0$ *and recall that* $\gamma_1 > 0$ *denotes the first eigenvalue* $(P_{\gamma_1})$*. Then*

(i) *for all* $\lambda \in (0, \gamma_1)$*, the problem*

$$-\operatorname{div}(A(x)Du) = c_\lambda(x)u + (M(x)Du, Du) \qquad (P_{h\equiv 0})$$

*has at least two solutions* $u_{\lambda,1} \equiv 0$ *and* $u_{\lambda,2} \gneqq 0$*;*

(ii) *for* $\lambda = \gamma_1$ *the problem* $(P_{h\equiv 0})$ *has only the trivial solution;*

(iii) *for* $\lambda > \gamma_1$*, the problem* $(P_{h\equiv 0})$ *has at least two solutions* $u_{\lambda,1} \equiv 0$ *and* $u_{\lambda,2} \leq 0$*;*

(iv) *for all* $\lambda \leq 0$ *the problem* $(P_{h\equiv 0})$ *has a unique solution* $u_\lambda \equiv 0$*.*

(v) *There exists a continuum* $\mathcal{C} \subset \Sigma$ *such that the projection of* $\mathcal{C}$ *on the* $\lambda$*-axis is an unbounded interval* $(0, +\infty)$ *and* $\mathcal{C}$ *bifurcates from infinity to the right of the axis* $\lambda = 0$*.*

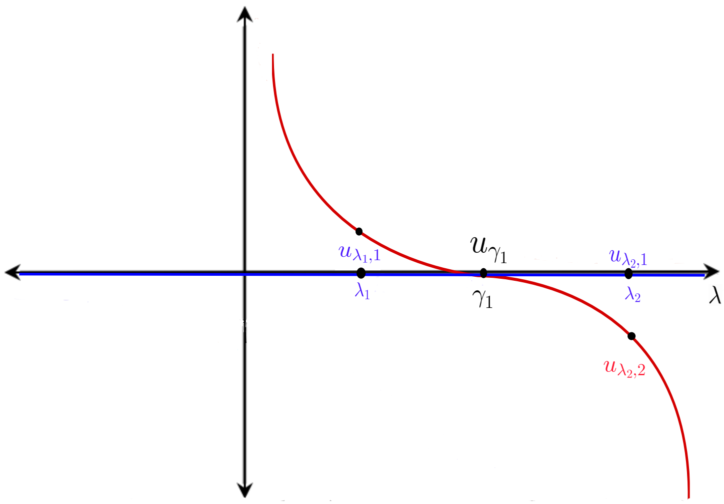


Figure 5.5: Illustration of Theorem 5.5

## 5.2 A priori Bound

The following essential upper bound shows that any unbounded continuum of solutions of $(P_\lambda)$ for $\lambda > 0$ in a bounded interval can only bifurcate to the right of $\lambda = 0$.

**Theorem 5.6 (A priori Upper Bound)** *Under the stated assumptions of problem $(P_\lambda)$, including hypothesis (A), for any $\Lambda_2 > \Lambda_1 > 0$, there exists a constant $\widetilde{M} > 0$ such that, for each $\lambda \in [\Lambda_1, \Lambda_2]$, any solution of $(P_\lambda)$ satisfies $\sup_\Omega u \leq \widetilde{M}$.*

To prove this theorem we will first show, in Lemma 5.8, that it is sufficient to control the behavior of the solutions on $\overline{\Omega}_{c^+}$. By compactness, it is equivalent to study what happens around any fixed point $\tilde{x} \in \overline{\Omega}_{c^+}$. We shall consider separately the alternative cases $\tilde{x} \in \overline{\Omega}_{c^+} \cap \Omega$ and $\tilde{x} \in \overline{\Omega}_{c^+} \cap \partial\Omega$.



**Remark 5.7** *Let us point out that if $\lambda = 0$ or $c^+ \equiv 0$ i,e. $|\Omega_{c^+}| = 0$ the problem $(P_\lambda)$ reduces to $(P_0)$ which is independent of $\lambda$, and has a solution, by [ACJT15],[CF18], where the authors give sufficient conditions to ensure the existence of a solution of $(P_0)$. Such a solution is unique and so, automatically we have an a priori bound.*

For the general case, an a priori bound to solution of $(P_\lambda)$ depends only on controlling the solution on $\Omega_{c^+}$.

**Lemma 5.8** *Assume the hypotheses of* $(P_\lambda)$*, there exists a constant* $M > 0$ *such that, for any* $\lambda \in \mathbb{R}$*, any solution* $u$ *of the problem* $(P_\lambda)$ *satisfies*

$$-\sup_{\Omega_{c^+}} u^- - M \leq u \leq \sup_{\Omega_{c^+}} u^+ + M.$$

*Proof.* In case problem $(P_\lambda)$ has no solution for any $\lambda \in \mathbb{R}$, there is nothing to prove. Hence, we assume the existence of $\widetilde{\lambda} \in \mathbb{R}$ such that $(P_{\widetilde{\lambda}})$ has a solution $\widetilde{u}$. We shall prove the result with $M := 2\|\widetilde{u}\|_\infty$. Let $u$ be an arbitrary solution of $(P_\lambda)$. Setting $\mathcal{D} := \Omega \setminus \overline{\Omega}_{c^+}$ and $v = u - \sup_{\partial D} u^+$, we have

$$\begin{aligned} -\operatorname{div}(A(x)Dv) &= -c^-(x)v + (M(x)Dv, Dv) + h(x) - c^-(x)\sup_{\partial \mathcal{D}} u^+ \\ &\leq -c^-(x)v + (M(x)Dv, Dv) + h(x) \text{ in } \mathcal{D}. \end{aligned}$$

Since $v \leq 0$ on $\partial\mathcal{D}$, the function $v$ is a subsolution of $(P_0)$. On the other hand, setting $\widetilde{v} = \widetilde{u} + \|\widetilde{u}\|_\infty$ we obtain

$$\begin{aligned} -\operatorname{div}(A(x)D\widetilde{v}) &= -c^-(x)\widetilde{v} + (M(x)D\widetilde{v}, D\widetilde{v}) + h(x) + c^-(x)\|\widetilde{u}\|_\infty \\ &\geq -c^-(x)\widetilde{v} + (M(x)D\widetilde{v}, D\widetilde{v}) + h(x) \text{ in } \mathcal{D} \end{aligned}$$

and thus, as $\widetilde{v} \geq 0$ on $\partial\mathcal{D}$, the function $\widetilde{v}$ is a supersolution of $(P_0)$. By standard regularity results (see for instance Lemma 2.1, [ACJT14], which can be applied under our hypotheses), we get $u, \widetilde{u} \in H^1(\Omega) \cap W^{1,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ and hence, $v, \widetilde{v} \in H^1_0(\mathcal{D}) \cap W^{1,n}_{loc}(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ and the right-hand sides of the above inequalities are $L^n$ functions. Therefore we are able to apply the Lemma 2.11 (Comparison Principle), and conclude that $v \leq \widetilde{v}$ in $\mathcal{D}$, namely that

$$u - \sup_{\partial D} u^+ \leq \widetilde{u} + \|\widetilde{u}\|_\infty \text{ in } \mathcal{D}$$

$$u \leq \widetilde{u} + \|\widetilde{u}\|_\infty + \sup_{\partial D} u^+ \text{ in } \mathcal{D}.$$

Hence $u \leq M + \sup_{\Omega_{c^+}} u^+$ in $\Omega$. For the other inequality, we now define $v := u + \sup_{\partial\mathcal{D}} u^-$ and obtain $v \geq 0$ on $\partial\mathcal{D}$, as well as,

$$\begin{aligned} -\operatorname{div}(A(x)Dv) &= -c^-(x)v + (M(x)Dv, Dv) + h(x) + c^-(x)\sup_{\partial \mathcal{D}} u^- \\ &\geq -c^-(x)v + (M(x)Dv, Dv) + h(x) \text{ in } \mathcal{D}. \end{aligned}$$

Thus $v$ is a supersolution of $(P_0)$. Now defining $\widetilde{v} = \widetilde{u} - \|\widetilde{u}\|_\infty$ again we have

$\widetilde{v} \leq 0$ on $\partial\mathcal{D}$ as well as

$$\begin{aligned} -\operatorname{div}(A(x)D\widetilde{v}) &= -c^-(x)\widetilde{v} + (M(x)D\widetilde{v}, D\widetilde{v}) + h(x) - c^-(x)\|\widetilde{u}\|_\infty \\ &\leq -c^-(x)\widetilde{v} + (M(x)D\widetilde{v}, D\widetilde{v}) + h(x) \text{ in } \mathcal{D}. \end{aligned}$$

Thus $\widetilde{v}$ is a subsolution of $(P_0)$. As previously we have that $v, \widetilde{v} \in H_0^1(\mathcal{D}) \cap W^{1,n}_{loc}(\mathcal{D}) \cap C(\overline{\mathcal{D}})$, and applying again the Comparison Principle (Lemma 2.11) we get $\widetilde{v} \leq v$ in $\mathcal{D}$. Namely

$$\widetilde{u} - \|\widetilde{u}\|_\infty \leq u + \sup_{\partial\mathcal{D}} u^- \text{ in } \mathcal{D}$$
$$u \geq \widetilde{u} - \|\widetilde{u}\|_\infty u - \sup_{\partial\mathcal{D}} u^- \text{ in } \mathcal{D}.$$

Therefore, it yields $u \geq -\sup_{\Omega_{c^+}} u^- - M$ in $\Omega$, ending the proof. ∎

Now, let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of $(P_\lambda)$. We introduce the exponential change of variable

$$w_i(x) := \frac{1}{\nu_i}(e^{\nu_i u(x)} - 1) \quad \text{and} \quad g_i(x) := \frac{1}{\nu_i}\ln(1 + \nu_i s),\ i = 1, 2 \tag{5.2}$$

where

$$\nu_1 := \mu_1\vartheta, \qquad \text{and} \qquad \nu_2 := \mu_2\vartheta^{-1}$$

for $\mu_1$, $\mu_2$ given in (5.1) and $\vartheta$ given in the definition of the matrix $A(x)$.

By Lemma 2.13 we have,

$$\begin{aligned} -\operatorname{div}(A(x)Dw_i) &= -\operatorname{div}(A(x)(1 + \nu_i w_i)Du) \\ &= -(1 + \nu_i w_i) div(A(x)Du) - (A(x)Du, D(1 + \nu_i w_i)) \\ &= (1 + \nu_i w_i)\Big[c_\lambda u(x) + (A(x)Du, Du) + h(x)\Big] \\ &\quad - (1 + \nu_i w_i)(\nu_i A(x)Du, Du). \end{aligned}$$

Then,

$$-\operatorname{div}(A(x)Dw_i) = (1 + \nu_i w_i)\left[c_\lambda(x)g_i(w_i) + h(x) + \Big([M(x) - \nu_i A(x)]Du, Du\Big)\right]. \tag{5.3}$$

Note that the last term is negative for $i = 1$ and positive for $i = 2$.

Using (5.3) we shall obtain a uniform a priori upper bound on $u$ in a neighborhood of any fixed point $\widetilde{x} \in \overline{\Omega}_{c^+}$. We consider the two cases $\widetilde{x} \in \overline{\Omega}_{c^+} \cap \Omega$ and $\widetilde{x} \in \overline{\Omega}_{c^+} \cap \partial\Omega$ separately.

**Lemma 5.9** *Assume that* (A) *holds and that* $\overline{x} \in \overline{\Omega}_{c^+} \cap \Omega$. *For each* $\Lambda_2 > \Lambda_1 > 0$, *there exist* $M_1 > 0$ *and* $R > 0$ *such that, for any* $\lambda \in [\Lambda_1, \Lambda_2]$, *any solution* $u$ *of* $(P_\lambda)$ *satisfies* $\sup\limits_{B_R(\widetilde{x})} u \leq M_1$.

*Proof.* Under the assumption (A) we can find a $R > 0$ such that $M(x) \geq \mu_1 I_n > 0$, $\quad c^- \equiv 0$ in $B_{4R}(\widetilde{x})$ and $c^+ \gneqq 0$ in $B_R(\widetilde{x})$. Observe that (5.3), for $i = 1$ turns into

$$\begin{aligned} -\operatorname{div}(A(x)Dw_1) &= (1+\nu_1 w_1)\Big[c_\lambda(x)g_1(w_1) + h(x) + \Big([M(x) - \nu_1 A(x)]Du, Du\Big)\Big] \\ &\geq (1+\nu_1 w_1)[\lambda c^+(x)g_1(w_1) + h^+(x)] - h^-(x) - \nu_1 h^-(x)w_1 \\ &\quad + (1+\nu_1 w_1)(\mu_1 - \vartheta^{-1}\nu_1)|Du|^2. \end{aligned}$$

Therefore in $B_{4R}(\widetilde{x})$,

$$-\operatorname{div}(A(x)Dw_1) + \nu_1 h^-(x)w_1 \geq (1+\nu_1 w_1)[\lambda c^+(x)g_1(w_1) + h^+(x)] - h^-(x). \tag{5.4}$$

Define $z_0$ to be the solution of

$$-\operatorname{div}(A(x)Dz_0) + \nu_1 h^-(x)z_0 = -\Lambda_2 c^+(x)\frac{e^{-1}}{\nu_1}, \;\; z_0 \in H_0^1(B_{4R}(\widetilde{x})). \tag{5.5}$$

By classical regularity (Theorem III-14.1 [LU68]), $z_0 \in C(\overline{B_{4R}(\widetilde{x})})$ and there exists a constant $\overline{C} > 0$ depending on $\widetilde{x}, \nu_1, \Lambda_2, p, R, \|h^-\|_{L^p(B_{4R})}, \|c^+\|_{L^p(B_{4R})}$ such that $z_0 \geq -\overline{C}$ in $B_{4R}$ (see Lemma 2.9). Further, by the Weak Maximum Principle (see Lemma 2.9) we know that $z_0 \leq 0$.

Observe that

$$\min_{(-\frac{1}{\nu_i}, \infty)} (1+\nu_i s)g_i(s) = -\frac{e^{-1}}{\nu_i},$$

and define $v_1 = w_1 - z_0 + \frac{1}{\nu_1}$. Thus $v_1$ satisfies

$$\begin{aligned} -\operatorname{div}(A(x)Dv_1) + \nu_1 h^-(x)v_1 &\geq (1+\nu_1 w_1)[\lambda c^+(x)g_1(w_1) + h^+(x)] + \Lambda_2 c^+(x)\frac{e^{-1}}{\nu_1} \\ &\geq (1+\nu_1 w_1)(\Lambda_2 - \Lambda_2)[c^+(x)g_1^-(w_1)] \\ &\quad +(1+\nu_1 w_1)\Lambda_1 c^+(x)g_1^+(w_1) \\ &\geq \frac{\Lambda_1 c^+(x)}{\nu_1}(1+\nu_1 w_1)\ln(1+\nu_1 w_1) \\ &= \Lambda_1 c^+(x)(v_1 + z_0)\ln(\nu_1(v_1 + z_0)) := f(x, v_1) \end{aligned} \tag{5.6}$$

in $B_{4R}(\widetilde{x})$, where

$$\begin{aligned} f:\Omega\times\mathbb{R} &\to \mathbb{R} \\ (x,s) &\to f(x,s) := \Lambda_1 c^+(x)\Big([s+z_0][\ln(\nu_1)+\ln(s+z_0(x))]\Big) \end{aligned} \tag{5.7}$$

is a superlinear function in the variable $s$. Since $w_1 > -1/\nu_1$ we have $v_1 > 0$ in $\overline{B_{4R}(\tilde{x})}$. On the other hand, for $i = 2$, in view of (5.1) in $\Omega$ and $w_2 > -1/\nu_2$, by (5.3) in a similar way we conclude that $w_2$ satisfies

$$\begin{aligned} -\operatorname{div}(A(x)Dw_2) &\leq [1+\nu_2 w_2](\lambda c^+(x)g_2(w_2)+h^+(x)) \\ &+ (\nu_1-\nu_2)h^-(x)w_2 - h^-(x) - \nu_1 h^-(x)w_2 \\ -\operatorname{div}(A(x)Dw_2)+\nu_1 h^-(x)w_2 &\leq [1+\nu_2 w_2]\left(\frac{\Lambda_2 c^+(x)}{\nu_2}\ln(1+\nu_2 w_2)+h^+(x)\right) \\ &=: g(x,w_2) \end{aligned} \tag{5.8}$$

in $B_{4R}(\tilde{x})$, where $g:\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

$$g(x,s) \leq a_0[1+s^{\alpha+1}], \quad \text{for each} \quad \alpha > 0. \tag{5.9}$$

In order to prove that (5.8) implies (5.9), let $c_\alpha > 0$ be a constant such that

$$\ln(1+x) \leq (1+x)^\alpha + c_\alpha, \text{ for all } x \geq 0.$$

Hence,

$$\begin{aligned} g(x,w_2) &= [1+\nu_2 w_2]\left(\frac{\Lambda_2}{\nu_2}c^+(x)\ln(1+\nu_2 w_2)+h^+(x)\right) \\ &\leq [1+\nu_2 w_2]\left(\frac{\Lambda_2}{\nu_2}c^+(x)(1+\nu_2 w_2)^\alpha + c_\alpha\frac{\Lambda_2}{\nu_2}c^+(x)+h^+(x)\right) \\ &\leq [1+\nu_2 w_2]^{\alpha+1}\left(\frac{\Lambda_2}{\nu_2}c^+(x)(1+c_\alpha)+h^+(x)\right) \\ &\leq [1+(\nu_2 w_2)^{\alpha+1}]a_0(x) \end{aligned}$$

where $a_0(x) \in L^p(\Omega)$, $\alpha > 0$. In addition, we note that

$$[1+\nu_2 w_2]^{\frac{\nu_1}{\nu_2}} = (e^{\nu_2 u})^{\frac{\nu_1}{\nu_2}} = (e^{\nu_1 u}) = 1+\nu_1 w_1 = \nu_1[v_1+z_0].$$

This means that $w_2 = \xi(v_1+z_0)$, where $\xi(s) := [(\nu_1 s)^{\frac{\nu_2}{\nu_1}}-1]\nu_2^{-1}$ is an increasing function satisfying

$$\lim_{s\to\infty}\frac{\xi(s)}{s^\beta} = \lim_{s\to\infty}\frac{(\nu_1 s)^{\nu_2/\nu_1}-1}{\nu_2 s^{\nu_2/\nu_1}} = \lim_{s\to\infty}\frac{\nu_1^{\nu_2/\nu_1}-\frac{1}{s^{\nu_2/\nu_1}}}{\nu_2} = \frac{\nu_1^{\nu_2/\nu_1}}{\nu_2} < \infty, \tag{5.10}$$

for $\beta = \nu_2/\nu_1$.

Thus we are in position to apply the following theorem, which under our assumptions is a rather straightforward generalization of Theorem 2 in [S20].

**Theorem 5.10** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary $\partial\Omega$ satisfying the interior $C^{1,D}$-paraboloid condition and $L$ be a uniformly elliptic operator under our standing assumptions. Assume that $z_0$ is a bounded function and $v \geq 0$, and $\xi(v+z_0)$ where $\xi$ satisfies (5.10) are functions in $H^1(\Omega)$ satisfying the following inequalities in the weak sense*

$$-\operatorname{div}(A(x)Dv) + \nu_1 h^{-1}(x)v \geq f(x,v)$$
$$-\operatorname{div}(A(x)D\xi(v+z_0)) + \nu_1 h^{-1}(x)\xi(v+z_0) \leq g(x,\xi(v+z_0)),$$

*where $f$ satisfies (5.7) and $g$ satisfies (5.9) for some $r = \alpha + 1$ with*

$$r < \frac{n+1}{n-1} + \left(\frac{1}{\beta} - 1\right)\frac{2}{n-1}.$$

*Then for some $C$ depending on the concerned quantities we have*

$$\xi(v(x)+z_0) \leq Cd(x) \text{ in } \Omega \quad \text{and hence} \quad v(x) \leq C.$$

*Proof.* In view of Theorem 1.8 all Theorems 3, 4, 5 and 6 stated in [S20] are valid under our assumptions on the domain and on the coefficients of L, hence, it remains to observe that the other generalizations on the hypotheses of Theorem 5.3 in comparison with Theorem 2, [S20] are natural, in view of Remark 4, [S20] and due to the injective property of $\xi$ and the boundedness of $z_0$. Thus, taking into account these observations, the proof follows almost verbatim the proof of Theorem 2, [S20], with minor changes. ■

In view of (5.6) and (5.8) we are able to apply Theorem 5.10 for $v = v_1$ and $w_2 = \xi(v_1+z_0)$ and conclude that $v_1$ and $w_2$ have upper bounds in $B_{4R}(\widetilde{x})$. As a consequence of this, the same holds for $w_1$ and also for $u$ as desired. ■

It is important to observe that Theorem 1.8 is fundamental to prove Theorem 5.10.

**Lemma 5.11** *Assume that (A) holds and that $\widetilde{x} \in \widetilde{\Omega}_{c^+} \cap \partial\Omega$. For each $\Lambda_2 > \Lambda_1 > 0$, there exist $R > 0$ and $M_2 > 0$ such that, for any $\lambda \in [\Lambda_1, \Lambda_2]$, any solution $u$ of ($P_\lambda$) satisfies $\sup\limits_{B_R(\widetilde{x})\cap\Omega} u \leq M_2$.*

*Proof.* The proof is very similar to the previous case, we only need to observe that our assumptions permit us to find $\Omega_1 \subset \Omega$ with $\partial\Omega_1$ of class $C^{1,D}$ such that $B_{2R}(\widetilde{x}) \cap \Omega \subset \Omega_1$ and $M(x) \geq \mu_1 I_n > 0$, $c^-(x) \equiv 0$ and $c^+(x) \gneqq 0$ in $\Omega_1$. Hence, for $i = 1$ note that (5.3) turn into (5.4) in $\Omega_1$ instead of $B_{4R}(\widetilde{x})$. Then, if $z_0$ is the solution of (5.5) in $H_0^1(\Omega_1)$ instead of $H_0^1(B_{4R}(\widetilde{x}))$, as in Lemma

5.9, we get $z_0 \in C(\overline{\Omega}_1)$ and $\overline{C} > 0$ depending on the usual quantities such that $-\overline{C} \leq z_0 \leq 0$ in $\Omega_1$. In addition, defining $v_1$ as in Lemma 5.9, we observe that $v_1$ satisfies equation (5.6) in $\Omega_1$ and $v_1 > 0$ on $\overline{\Omega}_1$. Arguing exactly as Lemma 5.9 we deduce (5.6),(5.8) and then we are able to apply Theorem 5.10 getting an upper bound to $u$ in $\Omega_1$. ■

**Proof of Theorem 5.6**

Once the previous two lemmas are available, that is, we have the existence of a uniform a priori upper bound on $u$ in a neighborhood of any fixed point $\tilde{x} \in \overline{\Omega}_{c^+}$ (see Lemma 5.9 and 5.11), then the proof of Theorem 5.6 follows exactly as the proof of Theorem 1.1 in [CFJ19].

We will now see that solutions are bounded from below, even when $\lambda \to 0$, $\lambda > 0$.

**Theorem 5.12 (A priori lower bound)** *Under the standing assumptions on problem* ($P_\lambda$)*, including hypothesis* (A)*, let* $\Lambda_2 > 0$*. Then every super-solution* $u$ *of* ($P_\lambda$) *satisfies*

$$\|u^-\|_{L^\infty} \leq C \textit{ for all } \lambda \in [0, \Lambda_2]$$

*where* $C$ *depends only on* $n, p, \nu_1, \Omega, \Lambda_2, \|c\|_{L^p(\Omega)}, \|h^-\|_{L^p(\Omega)}$.

*Proof.* First observe that both $U_1 = -u$ and $U_2 = 0$ are subsolutions of

$$-\operatorname{div}(A(x)DU) \leq c_\lambda U - (M(x)DU, DU) + h^-(x) \text{ in } \Omega.$$

Then these functions are also subsolutions of

$$\begin{cases} -\operatorname{div}(A(x)DU) + \mu_1|Du|^2 & \leq c_\lambda U + h^-(x) \quad \text{in } \Omega \\ U & \leq 0 \quad\quad\quad\quad\quad\; \text{on } \partial\Omega \end{cases}$$

and so is $U := u^- = max\{U_1, U_2\}$, as the maximum of subsolutions. Moreover $U \geq 0$ in $\Omega$ and $U = 0$ on $\partial\Omega$. We make the following exponential change of variables

$$w := \frac{1 - e^{-\nu_1 U}}{\nu_1}.$$

From Lemma 2.13,

$$-\operatorname{div}(A(x)Dw) \leq (1 - \nu_1 w)\left[c_\lambda(x)U + h^-(x)\right]$$

we know that $w$ is a weak solution of

$$\begin{cases} -\operatorname{div}(A(x)Dw)+\nu_1 h^-(x)w & \leq h^-(x)+\frac{c_\lambda(x)}{\nu_1}\ln(1-\nu_1 w)(1-\nu_1 w) \quad \text{in } \Omega \\ w & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{$Q_\lambda$}$$

Note that the logarithm above is well defined, since

$$0 \leq w = \frac{1-e^{-\nu_1 U}}{\nu_1} \leq \frac{1}{\nu_1} \text{ in } \Omega.$$

Now set $w_1 := \frac{1-e^{-\nu_1 u_1^-}}{\nu_1}$, where $u_1$ is some fixed supersolution of ($P_\lambda$), $\lambda \geq 0$ (if there was not such supersolution, we have nothing to prove). Then, by the above, $w_1 \in [0, 1/\nu_1)$ is a solution of ($Q_\lambda$). Define

$$\overline{w} := \sup \mathcal{A}, \text{ where } \mathcal{A} := \{w : w \text{ is a solution of } (Q_\lambda);\ 0 \leq w < 1/\nu_1 \text{ in } \Omega\}.$$

First, observe that $\mathcal{A} \neq \emptyset$ since $w_1 \in \mathcal{A}$, and $w_1 \leq \overline{w} \leq 1/\nu_1$ in $\Omega$. Also, as a supremum of subsolutions, $\overline{w}$ is a weak solution of ($Q_\lambda$), with $\overline{w} = 0$ on $\partial\Omega$. Then, the function

$$f(x) := h^-(x) + \frac{c_\lambda(x)}{\nu_1}|\ln(1-\nu_1\overline{w})|(1-\nu_1\overline{w}) \in L^p_+(\Omega)$$
$$\text{with } \|f^+\|_{L^p(\Omega)} \leq \|h^-\|_{L^p(\Omega)} + \frac{1}{\nu_1}\left(\Lambda_2\|c^+\|_{L^p(\Omega)} + \|c^-\|_{L^p(\Omega)}\right)C_0,$$

since $A(\overline{w}) := |\ln(1-\nu_1\overline{w})|(1-\nu_1\overline{w}) \leq C_0$. Therefore, by the Boundary Lipschitz bound, Lemma 2.14,

$$\overline{w} \leq C\delta^{1-n/p}\|f^+\|_{L^p(\Omega)}d(x) \to 0 \text{ as } x \to \partial\Omega$$

and so $\overline{w} \not\equiv 1/\nu_1$. Observe that the function $\overline{w}$ can be equal to $1/\nu_1$ at some interior points. In order to obtain a contradiction, assume that there is a sequence of supersolutions $u_k$ of ($P_\lambda$) in $\Omega$ with unbounded negative parts, then there would exist a subsequence such that

$$u_k^-(x_k) = \|u_k^-\|_{L^\infty} \to +\infty, x_k \in \overline{\Omega}, x_k \to x_0 \in \overline{\Omega}, k \to \infty$$

with $x_k \in \Omega$ for large $k$, since $u_k \geq 0$ on $\partial\Omega$. Then the respective sequence

$$w_k(x_k) = \frac{1-e^{-\nu_1 u_k^-(x_k)}}{\nu_1} \to \frac{1}{\nu_1}, \qquad w_k \in \mathcal{A}$$

i.e. for every $\epsilon > 0$, there exists some $k_0 \in \mathbb{N}$ such that

$$\frac{1}{\nu_1} - \epsilon \leq w_k(x_k) \leq \overline{w}(x_k) \leq \frac{1}{\nu_1}, \text{ for all } k \geq k_0.$$

Thus, there exists $\lim_{k\to\infty} \overline{w}(x_k) = \dfrac{1}{\nu_1}$ and also

$$\overline{w}(x_0) \geq \lim_{x_k \to x_0} \overline{w}(x_k) = \lim_{k\to\infty} \overline{w}(x_k) = \frac{1}{\nu_1}.$$

Hence, $x_0 \in \Omega$, since $\overline{w} = 0$ on $\partial\Omega$, and $\overline{w}(x_0) = \frac{1}{\nu_1}$. Finally, define $z := 1 - \nu_1 \overline{w}$, and observe that

$$\begin{aligned} div(A(x)Dz) &= -\nu_1 div(A(x)D\overline{w}) \\ &\leq \nu_1(1-\nu_1\overline{w}) \left[ \frac{c_\lambda(x)}{\nu_1} |\ln(1-\nu_1\overline{w})| + h^-(x) \right] \\ &= c_\lambda(x)|\ln z| z + \nu_1 h^-(x) z. \end{aligned}$$

Then $z$ is a supersolution of

$$\begin{cases} -\operatorname{div}(A(x)Dz) + \nu_1 h^-(x) z & \geq -c_\lambda(x)|\ln z| z \quad \text{in } \Omega \\ z & \gneqq 0 \quad \text{in } \Omega \\ z(x_0) & = 0. \end{cases}$$

But this contradicts the nonlinear version of the SMP, (Lemma 5.3, [NS18], and its extension in [SS21]) which says that $z \equiv 0$ or $z > 0$ in $\Omega$. ■

# 6
# Proof of Multiplicity results

## 6.1
## Preliminary observations

We first define strict sub and supersolutions. We observe that for the purposes of this section, where degree arguments will be employed, it will be sufficient to consider only supersolutions (resp. subsolutions) that are finite minima (resp. maxima) of regular (in $W^{2,p}$) supersolutions (resp. subsolutions).

**Definition 6.1** *A subsolution of* $(P_\lambda)$ *is said to be strict if every solution* $u$ *of* $(P_\lambda)$ *such that* $\alpha \leq u$ *on* $\Omega$ *satisfies* $\alpha \ll u$*. In the same way a strict supersolution of* $(P_\lambda)$ *is a supersolution such that every solution* $u$ *with* $u \leq \beta$ *is such that* $u \ll \beta$*.*

The next result is important in degree arguments.

**Lemma 6.2** *Under assumption* (A) *for every* $\lambda > 0$*, there exists a strict subsolution* $v_\lambda$ *of* $(P_\lambda)$ *such that, every supersolution* $\beta$ *of* $(P_\lambda)$ *satisfies* $v_\lambda \leq \beta$*.*

*Proof.* Let $C > 0$ be given by Theorem 5.12 and $\overline{M}$ be given by Theorem 5.6 such that, for every supersolution $\beta$ of

$$\begin{cases} -\operatorname{div}(A(x)Du) &= c_\lambda(x)u + (M(x)Du, Du) - h^-(x) - 1 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

we have $\beta \geq -C$. Let $k > C$ and consider $\alpha_k$ the solution of

$$\begin{cases} -\operatorname{div}(A(x)Dv) + c^-(x)v &= -\lambda k c^+(x) - h^-(x) - 1 \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

As $-\lambda k c^+(x) - h^-(x) - 1 < 0$ we have $\alpha_k \ll 0$ by the strong maximum principle and the Hopf lemma.

**Claim 1:** Every supersolution $\beta$ of $(P_\lambda)$ satisfies $\beta \geq \alpha_k$.

In fact, $\beta = \min\{\beta_j; 1 \leq j \leq l\}$ where $\beta_1, \cdots, \beta_l$ are regular supersolutions of

$(P_\lambda)$. Setting $w = \beta_j - \alpha_k$ for some $1 \le k \le l$ we have

$$\begin{cases} -\operatorname{div}(A(x)Dw) + c^-(x)w & \ge \lambda c^+(x)(\beta_j + k) + \mu_1 |D\beta_j|^2 \ge 0 \quad \text{in } \Omega \\ w & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By the maximum principle $w \ge 0$ i.e. $\beta_j \ge \alpha_k$. This proves the claim.

Consider the problem

$$-\operatorname{div}(A(x)Dv) = c_\lambda(x)T_k(v) + (M(x)Dv, Dv) - h^-(x) - 1 \tag{6.1}$$

where

$$T_k(v) = \begin{cases} -k, & \text{if } v \le -k, \\ v, & \text{if } v > -k. \end{cases}$$

We observe that $\beta$ is a supersolution of (6.1) with $\beta = T_k(\beta)$ and $\alpha_k$ is a subsolution of (6.1) (note that $-\lambda k c^+(x) = \lambda c^+(x)T_k(\alpha_k)$, $c^-(x)k = -c^-(x)T_k(\alpha_k)$); and hence by the standard method of sub- and super-solutions (6.1) has a minimal solution $v_k$ with $\alpha_k \le v_k \le \beta$.

**Claim 2:** Every supersolution $\beta$ of $(P_\lambda)$ satisfies $\beta \ge v_k$.

Observe that, by the construction of (6.1), every supersolution $\beta$ of $(P_\lambda)$ is also a supersolution of (6.1). As, by the Claim 1, we have $\beta \ge \alpha_k$, the minimality of $v_k$ implies that $v_k \le \beta$.

**Claim 3:** $v_k$ is a subsolution of $(P_\lambda)$.

Observe that $v_k \ge -C > -k$ and $v_k$ satisfies

$$\begin{aligned} -\operatorname{div}(A(x)Dv_k) &= c_\lambda(x)T_k(v_k) + (M(x)Dv_k, Dv_k) - h^-(x) - 1 \\ &\le c_\lambda(x)v_k + (M(x)Dv_k, Dv_k) + h(x). \end{aligned}$$

This implies that $v_k$ is a subsolution of $(P_\lambda)$.

**Claim 4:** $v_k$ is strict subsolution of $(P_\lambda)$.

Let $u$ be a solution of $(P_\lambda)$ with $u \ge v_k$. Then $w = u - v_k$ satisfies

$$\begin{aligned} -\operatorname{div}(A(x)Dw) &\ge c_\lambda(x)u + (M(x)Du, Du) + h(x) \\ &\quad - c_\lambda(x)v_k - (M(x)Dv_k, Dv_k) + h^-(x) + 1 \\ &= c_\lambda(x)w + (M(x)[Du + Dv_k], Dw) + h^+(x) + 1, \end{aligned}$$

which means that

$$\begin{cases} -\operatorname{div}(A(x)Dw) - (M(x)[Du + Dv_k], Dw) & \ge c_\lambda(x)w + h^+(x) + 1 \quad \text{in } \Omega \\ w & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, we deduce that $w \gg 0$ i.e. $u \gg v_k$. ■

**Remark 6.3** *Lemma 6.2 shows that, for $(P_0)$, having a supersolution is equivalent to have a solution.*

By adapting Lemma 5.1 from [CFJ19] to our setting we obtain the following auxiliary result for proving Theorem 5.1.

**Lemma 6.4** *Under the assumptions of Theorem 5.1, assume that $(P_0)$ has a solution $u_0$ such that $c^+(x)u_0 \gneqq 0$, $c^-(x) \equiv 0$. Then there exists $\overline{\Lambda} \in (0, \infty)$ such that, for $\lambda \geq \overline{\Lambda}$, the problem $(P_\lambda)$ has no solution $u$ with $u \geq u_0$ in $\Omega$.*

*Proof.* Let $\varphi_1 > 0$ the first eigenfunction of $(P_{\gamma_1})$. If $(P_\lambda)$ has a solution $u$ with $u \geq u_0$, multiplying $(P_\lambda)$ by $\varphi_1$ and integrating we obtain

$$\begin{aligned}\int_\Omega c_{\gamma_1}(x)u\varphi_1 &= \int_\Omega A(x)D\varphi_1 Du \\ &= \int_\Omega c_\lambda(x)u\varphi_1 dx + \int_\Omega (M(x)Du, \varphi_1 Du) + \int_\Omega h(x)\varphi_1 dx\end{aligned}$$

and hence $\lambda > \overline{\Lambda} > \gamma_1$, as $u \geq u_0$, we have

$$\begin{aligned}0 &\geq (\lambda - \gamma_1)\int_\Omega c^+(x)u\varphi dx + \mu_1 \int_\Omega \varphi_1 |Du|^2 dx + \int_\Omega h(x)\varphi_1 dx \\ &\geq (\lambda - \gamma_1)\int_\Omega c^+(x)u_0\varphi dx + \mu_1 \int_\Omega \varphi_1 |Du|^2 dx + \int_\Omega h(x)\varphi_1 dx\end{aligned}$$

which gives a contradiction for $\lambda$ large enough. ■

We also need a continuation theorem. Let $C(\overline{\Omega})$ be a real Banach space and $T : \mathbb{R} \times C(\overline{\Omega}) \to C(\overline{\Omega})$ a completely continous map, i.e. it is a continuous and maps bounded sets to relatively compact sets. For $\lambda \in \mathbb{R}$, we consider the problem of finding the zeroes of $\Phi(\lambda, u) := u - T(\lambda, u)$, i.e.

$$u \in C(\overline{\Omega}); \quad \Phi(\lambda, u) := u - T(\lambda, u) = 0, \qquad (Q_\lambda)$$

Let $\lambda_0 \in \mathbb{R}$ arbitrary but fixed and we assume that $u_{\lambda_0}$ is an isolated solution of $\Phi(\lambda_0, u)$, then the degree $\deg(\Phi(\lambda_0, .), B(u_{\lambda_0}, r), 0)$ is well defined and is constant for $r > 0$ small enough. Thus it is possible to define the index

$$i(\Phi(\lambda_0, .), u_{\lambda_0}) := \lim_{r \to 0} \deg(\Phi(\lambda_0, .), B(u_{\lambda_0}, r), 0).$$

**Theorem 6.5 (Theorem 2.2 of [ACJT15])** *If $(Q_\lambda)$ has a unique solution $u_{\lambda_0}$, and $i(\Phi(\lambda_0, .), u_{\lambda_0}) \neq 0$ then $\Sigma$ possesses two unbounded components $\mathcal{C}^+, \mathcal{C}^-$ in $[\lambda_0, +\infty] \times C(\overline{\Omega})$ and $[-\infty, \lambda_0] \times C(\overline{\Omega})$ respectively which meet at $(\lambda_0, u_{\lambda_0})$.*

## 6.2
## Proof of Theorem 5.1

Applying all previous results and adopting strategies presented in [ACJT14],[ACJT15],[CFJ19] we give the proof of Theorem 5.1. We treat separately the case $\lambda \leq 0$ and $\lambda > 0$.

**(i):** $\lambda \leq 0$.

This has been studied in previous works. We briefly recall the following argument. If $(P_0)$ has a solution $u_0$, then $u_0$ is a supersolution of $(P_\lambda)$ and by using Lemma 5.8 and [ACJT15] we obtain the existence of a solution $u_\lambda$ of $(P_\lambda)$ for any $\lambda < 0$, and by Proposition 4.1 [ACJT15] we have the uniqueness of solutions for $\lambda \leq 0$. Observe that for $\lambda \leq 0$, we have $c_\lambda(x) = \lambda c^+(x) - c^-(x) \leq -c^-(x)$ so by applying the comparison principle (Lemma 2.11), we get $u_\lambda \leq u_0$. Also by Lemma 5.8, setting $v = u_0 - \|u_0\|_\infty$ we see that $v_0$ a subsolution of $(P_\lambda)$ for $\lambda < 0$, so again by the Comparison Principle we get $u_0 - \|u_0\|_\infty \leq u_\lambda$.

**(ii):** $\lambda > 0$.

With the aim of showing the existence of a continuum of solution of $(P_\lambda)$, for $\lambda \geq 0$ we introduce the auxiliary problem

$$-\operatorname{div}(A(x)Du)+u = [c_\lambda(x)+1][(u-u_0)^+ + u_0] + \Big(M(x)Du, Du\Big) + h(x). \quad (\overline{P}_\lambda)$$

As in the case of $(P_\lambda)$, any solution of $(\overline{P}_\lambda)$ belongs to $C^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. Moreover observe that $u$ is a solution of $(\overline{P}_\lambda)$ if and only if it is a fixed point of the operator $\overline{T}_\lambda$ defined by $\overline{T}_\lambda : C(\overline{\Omega}) \to C(\overline{\Omega}) : v \to u$ with $u$ the solution of

$$-\operatorname{div}(A(x)Du) + u - \Big(M(x)Du, Du\Big) = [c_\lambda(x) + 1][(v - u_0)^+ + u_0] + h(x).$$

Applying Lemma 5.2 of [ACJT15], we see that $\overline{T}_\lambda$ is completely continuous. Now, we denote

$$\overline{\Sigma} := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}), u \text{ solves } (\overline{P}_\lambda)\}$$

and we split the rest of the proof into three steps.

**Step 1:** If $u$ is a solution of $(\overline{P}_\lambda)$ then $u \geq u_0$ and hence it is a solution of $(P_\lambda)$.

Observe that $(u-u_0)^+ + u_0 - u \geq 0$ and $\lambda c^+(x)[(u-u_0)^+ + u_0] \geq \lambda c^+(x)u_0 \geq 0$. Hence, we deduce that a solution $u$ of $(\overline{P}_\lambda)$ is a supersolution of

$$-\operatorname{div}(A(x)Du) = [c_\lambda(x) + 1][(u - u_0)^+ + u_0] + \Big(M(x)Du, Du\Big) + h(x). \quad (6.2)$$

Since $u_0$ is a solution of $(P_\lambda)$, it implies that $u_0$ solves (6.2). Then applying again the comparison principle we get $u \geq u_0$.

**Step 2:** $u_0$ is the unique solution to $(\overline{P}_0)$ as well as to the problem $(P_0)$. Furthermore $i(I - \overline{T}_0, u_0) = 1$.

For $\lambda = 0$, if $u$ is a solution of (6.2), then by Step 1, $u \geq u_0$ and $u$ solves $(P_\lambda)$. From case 1 we conclude that $u = u_0$. In order to prove that $i(I - \overline{T}_0, u_0) = 1$, we consider the operator $S_t$ defined by

$$S_t : C(\overline{\Omega}) \to C(\overline{\Omega})$$
$$v \to S_t(v) = t\overline{T}_0 v = u$$

with $u$ is the solution of

$$-\operatorname{div}(A(x)Du) + u = (M(x)Du, Du) + th(x) + t\Big([-c^-(x) + 1][u_0 + (v - u_0)^+ - (v - u_0 - 1)^+]\Big).$$

First, note that the complete continuity of $\overline{T}_\lambda$ follows from the fact that every solution $u$ of $(\overline{P}_\lambda)$ is $C^\alpha$ up to the boundary, then there exists $R > 0$ such that for all $t \in [0,1]$ and all $v \in C(\overline{\Omega})$,

$$\|S_t v\|_{L^\infty} < R.$$

Then $I - S_t$ does not vanish on $\partial B_R(0)$ and

$$\deg(I - \overline{T}_0, B_R(0)) = \deg(I - S_1, B_R(0)) = \deg(I - S_0, B_R(0)) = \deg(I, B_R(0)) = 1.$$

Therefore, $\overline{T}_0$ has a fixed point $u_0$ which is a solution of $(\overline{P}_0)$. Hence, by the property of the degree, for all $\varepsilon > 0$ small enough, it follows that

$$\deg(I - \overline{T}_0, B_\varepsilon(0)) = \deg(I - \overline{T}_0, B_R(0)) = 1.$$

Thus, for $\varepsilon < 1$, we conclude that

$$i(I - \overline{T}_0, u_0) = \lim_{\varepsilon \to 0} \deg(I - \overline{T}_0, B_\epsilon(0)) = 1.$$

**Step 3:** Existence and behavior of the continuum.

We are able to apply Theorem 6.5 (see also Theorem 2.2 [ACJT15] and Theorem 3.2 [R71]) to ensure the existence of a continuum $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^- \subset \overline{\Sigma}$ such that

$$\mathcal{C}^+ = \mathcal{C} \cap ([0, \infty) \times C(\overline{\Omega})) \text{ and } \mathcal{C}^- = \mathcal{C} \cap ((-\infty, 0] \times C(\overline{\Omega}))$$

are unbounded in $\mathbb{R}^{\pm} \times C(\overline{\Omega})$. By Step 1, we get that if $u \in \mathcal{C}^+$, then $u \geq u_0$ and is a solution of $(P_\lambda)$. Thus applying Lemma 6.4 we infer that the projection of $\mathcal{C}^+$ on $\lambda$-axis is $[0, \overline{\Lambda}]$, a bounded interval. A consequence of (i) is that none of $\lambda \in (-\infty, 0]$ is a bifurcation point from infinity of $(P_\lambda)$, and then deduce that the projection of $\mathcal{C}^-$ on $\lambda$-axis is $(-\infty, 0]$. Hence,

$$\mathrm{Proj}_{\mathbb{R}}\mathcal{C} = \mathrm{Proj}_{\mathbb{R}}\mathcal{C}^- \cup \mathrm{Proj}_{\mathbb{R}}\mathcal{C}^+ = (-\infty, \overline{\Lambda}]$$

for some $\overline{\Lambda} > 0$.

Finally, by Theorem 5.6 for any $0 < \Lambda_1 < \Lambda_2$ there is a priori bound for the solution of $(P_\lambda)$, for all $\lambda \in [\Lambda_1, \Lambda_2]$. Then by the $C^\alpha$ global estimates (Theorem 4.8), we have also a $C^\alpha$ a priori bound for these solutions i.e. the projection of $\mathcal{C} \cap ([\Lambda_1, \Lambda_2] \times C(\overline{\Omega}))$ on $C(\overline{\Omega})$ is bounded. Since the component $\mathcal{C}^+$ is unbounded in $\mathbb{R}^+ \times C(\overline{\Omega})$, its projection on the $C(\overline{\Omega})$ axis must be unbounded. By (i), the projection $\mathcal{C}^-$ on the $C(\overline{\Omega})$ is bounded. Hence,

$$\mathrm{Proj}_{C(\overline{\Omega})}\mathcal{C} = \mathrm{Proj}_{C(\overline{\Omega})}\mathcal{C}^- \cup \mathrm{Proj}_{C(\overline{\Omega})}\mathcal{C}^+ = [0, +\infty).$$

Therefore, we deduce that $\mathcal{C}$ must emanate from infinity on the right of axis $\lambda = 0$.

**(iii):** Multiplicity results.

Since $\mathcal{C}$ contains $(0, u_0)$, with $u_0$ being the unique solution of $(P_0)$, from (ii) we deduce that $\mathcal{C}$ also emanates from infinity on the right of axis $\lambda = 0$. We conclude that there exists $\lambda_0 \in (0, \overline{\Lambda})$ such that problems $(\overline{P}_\lambda)$ and $(P_\lambda)$ have at least two solutions satisfying $u \geq u_0$ for $\lambda \in (0, \lambda_0)$.

Next, the quantity

$$\overline{\lambda} := \sup\{\mu, \forall\lambda \in (0, \mu), (P_\lambda) \text{ has at least two solutions}\}$$

is well defined.

We now prove that, for all $\lambda \in (0, \overline{\lambda})$, the problem $(P_\lambda)$ has at least two solutions with $u_{\lambda,1} \ll u_{\lambda,2}$.

Let us consider the strict subsolution $\alpha_\lambda$ given by Lemma 6.2. As $\alpha_\lambda \leq u$ for all $u$ solution of $(P_\lambda)$, we can choose $u_{\lambda,1}$ as the minimal solution with $u_{\lambda,1} \geq \alpha$. Hence we have $u_{\lambda,1} \lneqq u_{\lambda,2}$, otherwise there exists a solution $u$ with $\alpha \leq u \leq \min\{u_{\lambda,1}, u_{\lambda,2}\}$, which contradicts the minimality of $u_{\lambda,1}$. Observe that, the function $\beta = \frac{1}{2}(u_{\lambda,1} + u_{\lambda,2})$ is a supersolution of $(P_\lambda)$ which is not a solution. As in the proof of the Lemma 6.8 below, we use the convexity of

$\varphi(\xi) = (M(x)\xi, \xi)$ for each $\xi \in \mathbf{R}^n$ in order to obtain

$$\begin{aligned}
-\operatorname{div}(A(x)D\beta) &= -\frac{1}{2}div(A(x)Du_{\lambda,1}) - \frac{1}{2}div(A(x)Du_{\lambda,2}) \\
&= c_\lambda(x)\beta + \frac{1}{2}(M(x)Du_{\lambda,1}, Du_{\lambda,1}) + \frac{1}{2}(M(x)Du_{\lambda,2}, Du_{\lambda,2}) + h(x) \\
&= c_\lambda(x)\beta + \frac{1}{2}\varphi(Du_{\lambda,1}) + \frac{1}{2}\varphi(Du_{\lambda,2}) + h(x) \\
&\gneqq c_\lambda(x)\beta + \varphi\Big(\frac{Du_{\lambda,1}}{2} + \frac{Du_{\lambda,2}}{2}\Big) + h(x) \\
&= c_\lambda(x)\beta + (M(x)D\beta, D\beta) + h(x).
\end{aligned}$$

Let us prove that $\beta$ is a strict supersolution of ($P_\lambda$). Consider a solution $u$ of ($P_\lambda$) with $u \leq \beta$. Then $v := \beta - u$ satisfies

$$\begin{aligned}
-\operatorname{div}(A(x)Dv) &\gneqq c_\lambda(x)\beta + (M(x)D\beta, D\beta) + h(x) - (M(x)Du, Du) - c_\lambda u - h(x) \\
&= (M(x)[D\beta + Du], Dv) + c_\lambda v,
\end{aligned}$$

and hence

$$-\operatorname{div}(A(x)Dv) - (M(x)D\beta + Du, Dv) + c^-(x)v \gneqq \lambda c^+(x)v \geq 0.$$

By Theorem 2.12 we deduce that either $v \gg 0$ or $v \equiv 0$. If $v \equiv 0$, then $\beta = u$ is solution, which contradicts the construction of $\beta$. Then we have $\beta \gg u$. As $u_{\lambda,1} \lneqq \beta \lneqq u_{\lambda,2}$ we deduce that, $u_{\lambda,1} \ll \beta \lneqq u_{\lambda,2}$ and hence we have $u_{\lambda,1} \ll u_{\lambda,2}$. We finish the proof with the following claim.

**Claim 6.6** *If $\overline{\lambda} < \infty$, the solution $u_{\overline{\lambda}}$ of ($P_{\overline{\lambda}}$) is unique.*

*Proof.* To prove that ($P_{\overline{\lambda}}$) has at least one solution, let $\{\lambda_n\} \subset (0, \overline{\lambda})$ such that $\lambda_n \to \overline{\lambda}$ and by the regularity result (Lemma 2.1 [ACJT14]) $\{u_n\} \subset H^1(\omega) \cap W^{1,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ be a sequence of corresponding solutions. By Theorem 5.6, there exists $M > 0$ such that $\|u_n\|_{L^\infty} < M$ for all $n \in \mathbb{N}$, and hence by the $C^{1,\alpha}$ global estimates we get $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$. Hence, up to a subsequence, $u_n \to u$ in $C^1_0(\Omega)$. From this strong convergence we easily observe that $u$ is a solution of ($P_{\overline{\lambda}}$). Now we proof the uniqueness of the solution of ($P_{\overline{\lambda}}$).
Let us assume by contradiction that we have two distinct solutions, $u_1$ and $u_2$ of ($P_{\overline{\lambda}}$), we prove that $\beta = \frac{1}{2}(u_1 + u_2)$ is a strict supersolution of ($P_{\overline{\lambda}}$). Let us consider the strict subsolution $\alpha_{\overline{\lambda}} \ll \beta$ of ($P_{\overline{\lambda}}$) given by Lemma 6.2, and look at the set,

$$\overline{\mathcal{S}} = \{u \in C^1_0(\overline{\Omega});\ \alpha \ll u \ll \beta, \|u\|_{C^1_0} < R\}$$

for some $R > C > 0$. Again, by the $C^{1,\alpha}$ estimates,

$$\|u\|_{C^{1,\alpha}} \leq C \text{ for all } u \text{ solution of } (P_\lambda),\ \lambda \in [\overline{\lambda}, \overline{\lambda}+1] \tag{6.3}$$

such that $deg(I - T_{\overline{\lambda}}, \overline{\mathcal{S}}) = 1$.

Now we prove the existence of $\varepsilon > 0$ such that

$$deg(I - T_\lambda, \overline{\lambda}) = 1, \text{ for all } \lambda \in [\overline{\lambda}, \overline{\lambda}+\varepsilon]. \tag{6.4}$$

We will verify that there exists some $\varepsilon \in (0,1)$ such that there is no fixed points of $T_\lambda$ on the boundary of $\overline{\mathcal{S}}$ for all $\lambda$ in the preceding interval. Indeed, if this was not the case, there would exists a sequence $\lambda_k \to \overline{\lambda}$ with the respective solutions $u_k$ of $(P_{\lambda_k})$ belonging to $\overline{\mathcal{S}}$. Say $\lambda_k \in [\overline{\lambda}, \overline{\lambda}+1]$ for $k \geq k_0$. Then, since $\alpha \ll u_k \ll \beta$ in $\Omega$, by (6.3) we must have $u_k \in \partial\overline{\mathcal{S}}$ for $k \geq k_0$, which means that for each such $k$,

$$\max_{\overline{\Omega}}(\alpha - u_k) = 0 \text{ or } \min_{\overline{\Omega}}(u_k - \beta) = 0. \tag{6.5}$$

By (6.3) and the compact inclusion $C^\alpha(\overline{\Omega}) \subset C(\Omega)$, $u_k \to u$ in $\Omega$ for some $u \in C(\Omega)$, up to subsequence. From this, we observe that $u$ is a solution of $(P_{\overline{\lambda}})$ and $\alpha \leq \beta$ in $\Omega$, by taking the limit as $k \to +\infty$ in the corresponding inequalities for $u_k$. Thus $\alpha \ll u \ll \beta$ in $\Omega$, since $\alpha$ and $\beta$ are strict. Passing (6.5) to the limits, we obtain that $u(x) = \alpha(x)$ or $u(x) = \beta(x)$ at a point $x \in \overline{\Omega}$, which contradicts the definition of $\alpha \ll u \ll \beta$. Hence for obtaining (6.4) it is just necessary to apply the homotopy invariance in $\lambda$ in the interval $[\overline{\lambda}, \overline{\lambda}+\varepsilon]$. Next, with (6.4) at hand, we repeat exactly the same argument done in (iii) to obtain the existence of a second solution $u_{\lambda,2}$ of $(P_\lambda)$, for all $\lambda \in [\overline{\lambda}, \overline{\lambda}+\varepsilon]$. But this, finally, contradicts the definition of $\overline{\lambda}$. ■

## 6.3
## Proof of Theorem 5.2

We start by constructing an auxiliary problem $(P_{\lambda,k})$, for which we can assume that there are no solutions for large $k$. This is a typical but essential argument that allows us to find a second solution via degree theory, by homotopy invariance in $k$. Fix $\Lambda_2 > 0$. Recall that Theorem 5.12 gives us an a priori lower uniform bound $C_0$ such that

$$u \geq -C_0, \text{ for every weak supersolution } u \text{ of } (P_\lambda), \text{ for all } \lambda \in [0, \Lambda_2].$$

Consider, the problem

$$\begin{cases} -\operatorname{div}(A(x)Du) = c_\lambda(x)u + (M(x)Du, Du) + h(x) + k\widetilde{c}(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_{\lambda,k})$$

for $k \geq 0$, $\lambda \in [0, \Lambda_2]$ and $\widetilde{c}$ being defined as

$$\widetilde{c}(x) := \widetilde{c}_{\Lambda_2}(x) = h^-(x) + \Lambda_2 C_0 c^+(x) + \widetilde{M} c^-(x) + Bc^+(x) \qquad (6.6)$$

with $B = \gamma_1/\nu_1$, where $\gamma_1 = \gamma_1^+ > 0$ is the first eigenvalue with weight $c$, associated to the eigenfunction $\varphi_1 \in W^{2,p}(\Omega)$, given by $(P_{\gamma_1})$. Note that every solution of $(P_{\lambda,k})$ is also a supersolution of $(P_\lambda)$ since $k\widetilde{c}(x) \geq 0$. From this and (6.6) we have for all $k \geq 1$ that

$$c_\lambda(x)u + h(x) + k\widetilde{c}(x) \geq -\Lambda_2 C_0 c^+(x) - \widetilde{M} c^-(x) - h^-(x) + \widetilde{c}(x) = Bc^+(x) \gneqq 0.$$

**Lemma 6.7** *Under assumption (A), assume that $(P_0)$ has a solution $u_0 \leq 0$ with $c^+(x)u_0 \lneqq 0$. Then for each fixed $\Lambda_2 > 0$ and $\lambda \in [0, \Lambda_2]$, there exists $k \geq 0$ such that*

*(i) For all $k > 1$, the problem $(P_{\lambda,k})$ has no solutions;*

*(ii) For all $k \in (0,1)$, $(P_{\lambda,k})$ has at least two solutions $u_{\lambda,1} \ll u_{\lambda,2}$;*

*(iii) For $k = 1$, and $h \leq 0$ the problem $(P_{\lambda,k})$ has exactly one solution.*

*Proof.* We proceed in several steps.
**Step 1:** For $k > 0$ small, $(P_{\lambda,k})$ admits a solution.
Let $\lambda > \gamma_1$ and $\varepsilon_0 > 0$ be given by Lemma 6.11 corresponding to $\bar{c} = c(x)$, $\bar{d} = \nu_2 h^-(x)$, $\bar{h} = \nu_2 \widetilde{c}(x) + \frac{1}{k}\nu_2 h^+(x)$, and choose
$\lambda_0 \in \left(\gamma_1, \min\left\{\gamma_1 + \varepsilon_0, \gamma_1 + \dfrac{\lambda - \gamma_1}{2}\right\}\right]$. Then the problem

$$-\operatorname{div}(A(x)Du) + \nu_2 h^-(x)u = c_{\lambda_0} u + \nu_2 \widetilde{c}(x) + \frac{1}{k}\nu_2 h^+(x)$$

has a solution $u \ll 0$. Also taking $\delta > 0$ small enough we have that

$$\lambda_0 s \geq (1 + \lambda s)\ln(1 + \lambda s)$$

for all $s \in [-\delta, 0]$. Thus defining $\widetilde{\beta}_k = \frac{k}{\lambda} u$ for $k > 0$ small enough, it follows that $\widetilde{\beta}_k \in [-\delta, 0]$ and satisfies

$$-\operatorname{div}(A(x)D\widetilde{\beta}_k) = c_{\lambda_0}\widetilde{\beta}_k + \nu_2 \frac{k}{\lambda}\widetilde{c}(x) - \nu_2 \frac{k}{\lambda} h^-(x)u + \frac{1}{\lambda}\nu_2 h^+(x), \text{ and hence}$$
$$-\operatorname{div}(A(x)D\widetilde{\beta}_k) + \nu_2 h^-(x)\widetilde{\beta}_k = c_{\lambda_0}(x)\widetilde{\beta}_k + \nu_2 \frac{k}{\lambda}\widetilde{c}(x) + \frac{1}{\lambda}\nu_2 h^+(x).$$

Hence for $\beta_k$ being defined by $\beta_k = \frac{1}{\nu_2}\ln(1+\lambda\widetilde{\beta}_k)$, we have

$$\begin{aligned}
-\operatorname{div}(A(x)D\beta_k) &= -\frac{\lambda}{\nu_2}\frac{\operatorname{div}(A(x)D\widetilde{\beta}_k)}{(1+\lambda\widetilde{\beta}_k)} - \frac{\lambda}{\nu_2}\left(A(x)D\widetilde{\beta}_k, D\left[\frac{1}{(1+\lambda\widetilde{\beta}_k)}\right]\right)\\
&\gneqq c_\lambda(x)\beta_k + \frac{k\widetilde{c}(x) + h^+(x) - \lambda h^-(x)\widetilde{\beta}_k}{1+\lambda\widetilde{\beta}_k}\\
&\quad + \frac{\lambda^2}{\nu_2(1+\lambda\widetilde{\beta}_k)^2}(A(x)D\widetilde{\beta}_k, D\widetilde{\beta}_k)\\
&\geq c_\lambda(x)\beta_k + k\widetilde{c}(x) + h^+(x) - h^-(x) + \nu_2\vartheta\frac{|D\widetilde{\beta}_k|^2}{(1+\nu_2\widetilde{\beta})^2}\\
&= c_\lambda(x)\beta_k + k\widetilde{c}(x) + h(x) + \mu_2|D\beta_k|^2\\
&\geq c_\lambda(x)\beta_k + k\widetilde{c}(x) + h(x) + (M(x)D\beta_k, D\beta_k).
\end{aligned}$$

We see that

$$\begin{cases}
-\operatorname{div}(A(x)D\beta_k) & \geq c_\lambda(x)\beta_k + k\widetilde{c}(x) + h(x) + (M(x)D\beta_k, D\beta_k) \quad \text{in } \Omega\\
\beta_k & = 0 \quad \text{on } \partial\Omega
\end{cases}$$

has a supersolution $\beta_k$ with $\beta_k \ll 0$. Hence we conclude that $(P_{\lambda,k})$ has at least one solution, by following the proof of Theorem 5.1.

**Step 2:** For $k > 1$ the problem $(P_{\lambda,k})$ has no solution.

First we observe that every solution of $(P_{\lambda,k})$, for $\lambda \in [0, \Lambda_2]$, is positive in $\Omega$. In fact, we observe that

$$\begin{cases}
-\operatorname{div}(A(x)Du) & \geq (M(x)Du, Du) + Bc^+(x) \geq 0 \quad \text{in } \Omega\\
u & = 0 \quad \text{on } \partial\Omega
\end{cases}$$

and this implies that $u \geq 0$ in $\Omega$ by Lemma 2.10. Then $u > 0$ in $\Omega$ by SMP. In order to obtain a contradiction, assume that $u$ is a solution of $(P_{\lambda,k})$ in $\Omega$. Let $\varphi \in C_0^\infty(\Omega)$ such that $\varphi^2 \gg 0$. Then using $\varphi^2$ as test function, by Theorem 5.12 we obtain

$$\begin{aligned}
\int \frac{1}{\mu_1}|D\varphi|^2 &\geq 2\int(\varphi Du, D\varphi) - \mu_1\int |Du|^2\varphi^2\\
&\geq 2\int(\varphi Du, D\varphi) - \int(M(x)Du, \varphi^2 Du)\\
&= \int c_\lambda(x)u\varphi^2 + h(x)\varphi^2 + k\tilde{c}(x)\varphi^2\\
&\geq -\Lambda_2 C_0\int c^+(x)\varphi^2 - M\int c^-(x)\varphi^2 - \int h^-(x)\varphi^2 + \int k\tilde{c}(x)\varphi^2
\end{aligned}$$

which is a contradiction for $k > 1$ large enough.

**Step 3:** For $k = 1$ $(P_{\lambda,k})$ has a unique solution, and $k \in (0,1)$ the problem

$(P_{\lambda,k})$ has a strict supersolution.
By Step 1 and 2 we have

$$1 = \sup\{k > 0; (P_{\lambda,k}) \text{ has at least one solution}\}.$$

Let $k \in (0,1)$ and $\widetilde{k} \in (k,1)$ be such that $(P_{\lambda,\widetilde{k}})$ has a solution $\widetilde{\beta}$. Then $\beta = \frac{k}{\widetilde{k}}\widetilde{\beta}$ is a supersolution of $(P_{\lambda,k})$. In fact note that,

$$\begin{aligned} -\operatorname{div}(A(x)D\beta) &= c_\lambda\beta + \Big(M(x)\frac{\widetilde{k}}{k}D\beta, D\beta\Big) + \frac{k}{\widetilde{k}}h(x) + k\widetilde{c}(x) \\ &\geq c_\lambda\beta + (M(x)D\beta, D\beta) + h(x) + k\widetilde{c}(x) \end{aligned}$$

i.e. $\beta$ is a supersolution of $(P_{\lambda,k})$. Now, as in (iii) of the proof of Theorem 5.1 we can prove that $\beta$ is a strict supersolution of $(P_{\lambda,k})$.
**Step 4:** Conclusion.
The proof of the existence of the second solution $u_{\lambda,2}$ with $u_{\lambda,1} \ll u_{\lambda,2}$ is derived exactly as (iii) of the proof of Theorem 5.2. ∎

**Lemma 6.8** *Under assumption* (A)*, assume that* $(P_0)$ *has a solution* $u_0 \leq 0$ *with* $c^+(x)u \lneqq 0$*. Then, for all* $\lambda \geq 0$*, problem* $(P_\lambda)$ *has at most one solution* $u \leq 0$*.*

*Proof.* The proof is divided in several steps.
**Step 1:** If $u$ is a subsolution of $(P_\lambda)$ with $u \leq 0$, then $u \ll 0$.
In fact, $u$ is a subsolution of $(P_0)$ and by Lemma 2.11, we have $u \leq u_0$. In addition for $w = u_0 - u$ we have

$$\begin{aligned} -\operatorname{div}(A(x)Dw) &\geq -c^-(x)u_0 + (M(x)Du_0, Du_0) - c_\lambda(x)u - (M(x)Du, Du) \\ &= (M(x)Du + Dw, Dw) - c^-(x)w - \lambda c^+(x)u, \end{aligned}$$

and hence, we get

$$\begin{cases} -\operatorname{div}(A(x)Dw) - (M(x)Du + Dw, Dw) - c^-(x)w & \gneqq 0 \quad \text{in } \Omega \\ w & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This implies that $w \gg 0$ i.e. $u \ll u_0 \leq 0$.
**Step 2:** If we have two solutions $u_1, u_2 \leq 0$ of $(P_\lambda)$ then we have two ordered solutions $\tilde{u}_1 \lneqq \tilde{u}_2 \leq u_0$.
By Step 1, we have $u_1, u_2 \ll u_0$ . In case $u_1$ and $u_2$ are not ordered, as $u_0$ is a supersolution of $(P_\lambda)$, applying Theorem 2.1 of [CJ17], there exists a solution $u_3$ of $(P_\lambda)$ with $\max\{u_1, u_2\} \leq u_3 \leq u_0$. This proves Step 2 by choosing $\tilde{u}_1 = u_1$ and $\tilde{u}_2 = u_3$.

**Step 3:** We prove the uniqueness of the nonpositive soluion of $(P_\lambda)$.
Let us assume by contradiction that we have two ordered solutions, we can suppose $u_1 \ll u_2 \ll 0$. As $|u_2| \gg 0$ the set $\{\epsilon > 0, u_2 - u_1 \leq \epsilon|u_2|\}$ is not empty. Then defining

$$\tilde{\epsilon} := \min\{\epsilon > 0, u_2 - u_1 \leq \epsilon|u_2|\}$$

and setting

$$w_{\tilde{\epsilon}} := \frac{(1+\tilde{\epsilon})u_2 - u_1}{\tilde{\epsilon}},$$

we can define for each $\xi \in \mathbf{R}^n$ the function $\varphi(\xi) := (M(x)\xi, \xi)$, and by assumption (A), we have $D^2(\varphi) > 0$, therefore $\varphi$ is convex. We can write

$$\begin{aligned} u_2 &= \frac{\tilde{\epsilon}}{1+\tilde{\epsilon}} w_{\tilde{\epsilon}} + \frac{1}{1+\tilde{\epsilon}} u_1, \\ (M(x)Du_2, Du_2) = \varphi(Du_2) &= \varphi\left(\frac{\tilde{\epsilon}}{1+\tilde{\epsilon}} Dw_{\tilde{\epsilon}} + \frac{1}{1+\tilde{\epsilon}} Du_1\right) \\ &\leq \frac{\tilde{\epsilon}}{1+\tilde{\epsilon}} \varphi(Dw_{\tilde{\epsilon}}) + \frac{1}{1+\tilde{\epsilon}} \varphi(Du_1) \\ &= \frac{1}{1+\tilde{\epsilon}} \Big[\tilde{\epsilon}(M(x)Dw_{\tilde{\epsilon}}, Dw_{\tilde{\epsilon}}) + (M(x)Du_1, Du_1)\Big], \end{aligned}$$

and hence

$$\frac{1+\tilde{\epsilon}}{\tilde{\epsilon}}(M(x)Du_2, Du_2) \leq (M(x)Dw_{\tilde{\epsilon}}, Dw_{\tilde{\epsilon}}) + \frac{1}{\tilde{\epsilon}}(M(x)Du_1, Du_1).$$

Thus, we obtain

$$\begin{aligned} -\operatorname{div}(A(x)Dw_{\tilde{\epsilon}}) &= -\frac{1+\tilde{\epsilon}}{\tilde{\epsilon}} div(A(x)Du_2) + \frac{1}{\tilde{\epsilon}} div(A(x)Du_1) \\ &\leq \frac{1+\tilde{\epsilon}}{\tilde{\epsilon}}\Big(c_\lambda(x)u_2 + (M(x)Du_2, Du_2) + h(x)\Big) \\ &\quad - \frac{1}{\tilde{\epsilon}}\Big(c_\lambda(x)u_1 + (M(x)Du_1, Du_1) + h(x)\Big) \\ &\leq c_\lambda(x)w_{\tilde{\epsilon}} + (M(x)Dw_{\tilde{\epsilon}}, Dw_{\tilde{\epsilon}}) + h(x). \end{aligned}$$

Then, applying again the Comparison Principle Lemma 2.11, $w_{\tilde{\epsilon}} \lneqq u_2 \leq 0$. Hence, we have a contradiction with the definition of $\tilde{\epsilon}$. ■

*Proof.*[Proof of Theorem 5.2] We treat separately the case $\lambda \leq 0$ and $\lambda > 0$.
**(i):** $\lambda \leq 0$.
As in the proof of Theorem 5.1 we can use Theorem 1.2 [CFJ19] and its proof. Moreover, observe that $u_0$ is a subsolution of $(P_\lambda)$. Hence we conclude that $u_\lambda \geq u_0$ applying the comparison principle. By Proposition 4.1 in [ACJT15] the problem $(P_\lambda)$ for $\lambda \leq 0$ has at most one solution.

By Lemma 5.8 the functions $v = u_0 + \|u_0\|_\infty$ is a supersolution of $(P_\lambda)$ for $\lambda < 0$, and by the comparison principle, we get $u_0 + \|u_0\|_\infty \geq u_\lambda$ .
**(ii):** $\lambda > 0$.
With the aim of showing the existence of a continuum of solution of $(P_\lambda)$, for $\lambda \geq 0$ we introduce the auxiliary problem

$$-\operatorname{div}(A(x)Du)+u = [c_\lambda(x)+1][u_0-(u-u_0)^-]+(M(x)Du, Du)+h(x). \quad (\underline{P_\lambda})$$

As in the case of $(P_\lambda)$, any solution of $(\underline{P_\lambda})$ belong to $\mathcal{C}^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. Moreover observe that $u$ is a solution of $(\underline{P_\lambda})$ if and only if it is a fixed point of the operator $\widehat{T}_\lambda$ defined by $\widehat{T}_\lambda : C(\overline{\Omega}) \to C(\overline{\Omega}) : v \to u$, where $u$ is the solution of

$$-\operatorname{div}(A(x)Du) + u - \big(M(x)Du, Du\big) = [c_\lambda(x) + 1][u_0 - (v - u_0)^-] + h(x).$$

Applying the same argument to $\overline{T}_\lambda$ as the one used in the proof of Theorem 5.1, we see that $\widehat{T}_\lambda$ is completely continuous, and we split the rest of the proof into three steps.
**Step 1:** If $u$ is a solution of $(\underline{P_\lambda})$ then $u \leq u_0$ and hence it is a solution of $(P_\lambda)$.
Observe that $u_0 - u - (u - u_0)^- \leq 0$. Moreover, we also have

$$\lambda c^+(x)[u_0 - u - (u - u_0)^-] \leq \lambda c^+(x)u_0 \leq 0.$$

Hence we deduce that a solution $u$ of $(\underline{P_\lambda})$ is a subsolution of

$$-\operatorname{div}(A(x)Du) = -c^-(x)[u_0 - (u - u_0)^-] + (M(x)Du, Du) + h(x). \qquad (6.7)$$

Since $u_0$ is a solution of $(P_\lambda)$, it implies that $u_0$ solves (6.7). Then applying again the comparison principle we get $u \leq u_0$.
**Step 2:** $u_0$ is the unique solution to $(\underline{P_0})$ as well as to the problem $(P_0)$ and $i(I - \widehat{T}_0, u_0) = 1$.
For $\lambda = 0$, if $u$ is a solution to (6.7), then by Step 1, $u \leq u_0$ and $u$ solves $(P_\lambda)$. From (i) we conclude that $u = u_0$. In order to prove that $i(I - \widehat{T}_0, u_0) = 1$, we consider the operator $S_t$ defined by

$$\begin{aligned} S_t : C(\overline{\Omega}) &\to C(\overline{\Omega}) \\ v &\to S_t(v) = t\widehat{T}_0 v = u \end{aligned}$$

where $u$ is the solution of

$$-\operatorname{div}(A(x)Du) + u = (M(x)Du, Du) + th(x) \\ + t[-c^{-}(x)+1][u_0 - (v-u_0)^{-} - (v-u_0+1)^{-}].$$

First, note that by the complete continuity of $\widehat{T}$ (recall also that every solution $u$ of $(\overline{P}_\lambda)$ is $C^\alpha$ up to the boundary), there exists $R > 0$ such that for all $t \in [0,1]$ and all $v \in C(\overline{\Omega})$,

$$\|S_t v\|_{C^\alpha} < R.$$

Then $I - S_t$ does not vanish on $\partial B_R(0)$ and

$$\begin{aligned}\deg(I - \widehat{T}_0, B_R(0)) &= \deg(I - S_1, B_R(0)) \\ &= \deg(I - S_0, B_R(0)) \\ &= \deg(I, B_R(0)) = 1.\end{aligned}$$

Therefore, $\widehat{T}_0$ has only a fixed point $u_0$ which is a solution of $(\underline{P_0})$. Hence, by the property of the degree, for all $\varepsilon > 0$ small enough, it follows that

$$\deg(I - \widehat{T}_0, B_\varepsilon(0)) = \deg(I - \widehat{T}_0, B_R(0)) = 1.$$

Thus, for $\varepsilon < 1$, we conclude that

$$i(I - \widehat{T}_0, u_0) = \lim_{\varepsilon \to 0} \deg(I - \widehat{T}_0, B_\epsilon(0)) = 1.$$

**Step 3:** Existence and behavior of the continuum.
Proceeding as the proof of Theorem 1.2 of [CJ17], we are able to apply Theorem 6.5 (see also Theorem 2.2 [ACJT15]) to ensure the existence of a continuum $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^- \subset \overline{\Sigma}$ such that

$$\mathcal{C}^+ = \mathcal{C} \cap ([0,\infty) \times C(\overline{\Omega})) \text{ and } \mathcal{C}^- = \mathcal{C} \cap ((-\infty, 0] \times C(\overline{\Omega}))$$

are unbounded in $\mathbb{R}^\pm \times C(\overline{\Omega})$. Since the component $\mathcal{C}^+$ is unbounded in $\mathbb{R}^+ \times C(\overline{\Omega})$, its projection on the $C(\overline{\Omega})$ axis must be unbounded and a consequence of (i) is that none of $\lambda \in (-\infty, 0]$ is a bifurcation point from infinity of $(P_\lambda)$, thus we deduce that the projection of $\mathcal{C}^-$ on $\lambda$-axis is $(-\infty, 0]$.
**(iii):** Multiplicity results.
We now prove that for $\lambda > 0$, $(P_\lambda)$ has at least two solutions, $u_{\lambda,1}$ and $u_{\lambda,2}$ with $u_{\lambda,1} \le u_{\lambda,2}$. By Step 1, we get the existence of a first solution $u_{\lambda,1} \le u_0$. To prove that $u_0$ is a strict supersolution of $(P_\lambda)$, we argue as in Step 2 of

the proof of Theorem 5.1, and by Lemma 6.2 $(P_\lambda)$ has a strict subsolution $\alpha$ with $\alpha \leq u_0$. Then, by Theorem 2.1 of [CJ17], there exists $R > 0$ such that $u_{\lambda,1} \in \mathcal{S}$, where $\mathcal{S} = \{u \in C_0^1(\overline{\Omega}); \alpha \ll u \ll u_0 \text{ in } \Omega, \|u\|_{C_0^1} < R\}$.

Fix $\lambda > 0$ and set $\Lambda_2 = 2\lambda$. Replace $h$ by $h + k\widetilde{c}$ in the problem $(P_{\lambda,k})$, then Theorem 5.6 gives us an $L^\infty$ a priori bound for solutions of $(P_{\lambda,k})$ for every $k \in [0,1]$. This provides, by the $C^{1,\alpha}$ global estimatives, an a priori bound for solutions in $C_0^1(\overline{\Omega})$, i.e. $\|u\|_{C_0^1(\overline{\Omega})} < R_0$ for every solution $u$ of $(P_{\lambda,k})$, for all $k \in [0,1]$ where $R_0 > R$ also depends on $\lambda$. Hence, by the homotopy invariance of the degree, and the fact that, for $k > 1$, $(P_{\lambda,k})$ has no solution we have

$$\deg(I - \widehat{T}_\lambda, B_{R_0}(0)) = \deg(I - \widehat{T}_{\lambda,0}, B_{R_0}(0)) = \deg(I - \widehat{T}_{\lambda,k}, B_{R_0}(0)) = 0$$

where $\widehat{T}_{\lambda,k}$ is the operator $\widehat{T}_\lambda$ in which we replace $h(x)$ by $h(x) + k\widetilde{c}$ (of course $\widehat{T}_{\lambda,k}$ is still completely continuous). But then, by the excision property of the degree,

$$\deg(I - \mathcal{T}_\lambda, B_{R_0} \setminus \mathcal{S}(0)) = \deg(I - \mathcal{T}_\lambda, B_{R_0}(0)) - \deg(I - \mathcal{T}_\lambda, S(0)) = -1$$

and the existence of a second solution $u_{\lambda,2} \in B_{R_0} \setminus \mathcal{S}$ is derived. By Lemma 6.8 we have $u_{\lambda,2} > 0$.



**Claim 6.9** *For $\lambda_1 < \lambda_2$, we have $u_{\lambda_2,1} \ll u_{\lambda_1,1}$.*

*Proof.* For fixed $\lambda_1 < \lambda_2$ note that

$$c_{\lambda_1}(x)u_{\lambda_1,1} = \lambda_1 c^+(x)u_{\lambda_1,1} - c^-(x)u_{\lambda_1,1} \gneqq \lambda_2 c^+(x)u_{\lambda_1,1} - c^-(x)u_{\lambda_1,1} = c_{\lambda_2}(x)u_{\lambda_1,1}$$

since $u_{\lambda_1,1} < 0$. Then $u_{\lambda_1,1}$ is a strict supersolution of $(P_{\lambda_2})$, which is not a solution and, in particular $u_{\lambda_1,1} \neq u_{\lambda_2,1}$. As in the proof of Claim 6.16 [NS18], observe that $u_{\lambda_2,1}$ is the minimal solution of $(P_{\lambda_2})$. In fact, recall that $\xi = \xi_{\lambda_2}$, given by Lemma 6.2, is such that $\xi \leq u$ for every strict supersolution of $(P_{\lambda_2})$, and in particular $\xi \leq u_{\lambda_1,1}$. Remember also that $u_{\lambda_2,1}$ is the minimal strict solution such that $u_{\lambda_2,1} \geq \xi$ in $\Omega$. Now, if there was a $x_0 \in \Omega$ such that $u_{\lambda_2,1}(x_0) > u_{\lambda_1,1}(x_0)$, by defining $\eta := \min\{u_{\lambda_1,1}, u_{\lambda_2,1}\}$, as the minimum of strict supersolutions of $(P_{\lambda_2})$ not less than $\xi$, we have $\xi \leq \eta$ in $\Omega$. Thus, Theorem 2.1 of [CJ17] provides a solution $u$ of $(P_{\lambda_2})$ such that $\xi \leq u \leq \eta \lneqq u_{\lambda_2,1}$ in $\Omega$, which contradicts the minimality of $u_{\lambda_2,1}$. ■

This ends the proof of Theorem 5.2. ■

## 6.4
## Proof of Theorem 5.3

**(i):** Multiplicity results.
First observe that if $(P_\lambda)$ has a supersolution $\beta_\lambda \leq 0$, then $\beta_\lambda$ satisfies also $c^+(x)\beta_\lambda \lneqq 0$, otherwise, it is also an supersolution of $(P_0)$, which contradicts the assumption (a). (See Remark 6.3). Let us define

$$\underline{\lambda} = \inf\{\lambda \geq 0; (P_\lambda) \text{ has a supersolution } \beta_\lambda \leq 0 \text{ with } c^+(x)\beta_\lambda \lneqq 0\}.$$

Let $\lambda > \underline{\lambda}$. By the definition of $\underline{\lambda}$ there exists $\widetilde{\lambda} \in [\underline{\lambda}, \lambda)$, such that $(P_{\widetilde{\lambda}})$ has a supersolution $\beta_{\widetilde{\lambda}} \leq 0$ with $c^+(x)\beta_{\widetilde{\lambda}} \lneqq 0$. Note that

$$c_{\widetilde{\lambda}}(x)\beta_{\widetilde{\lambda}} = \widetilde{\lambda}c^+(x)\beta_{\widetilde{\lambda}} - c^-(x)\beta_{\widetilde{\lambda}} \gneqq \lambda c^+(x)\beta_{\widetilde{\lambda}} - c^-(x)\beta_{\widetilde{\lambda}} = c_\lambda(x)\beta_{\widetilde{\lambda}}.$$

Then, $\beta_{\widetilde{\lambda}}$ is a supersolution of $(P_\lambda)$, which is not a solution and hence, as in (iii) of the proof of Theorem 5.2, it is a strict supersolution of $(P_\lambda)$. By Lemma 6.2, $(P_\lambda)$ has a strict subsolution $\alpha \leq \beta_{\widetilde{\lambda}}$ and $\alpha \leq u$ for all solutions $u$ of $(P_\lambda)$. As in Step 2 of the proof of Theorem 5.2, there exists $R > 0$ such that $\deg(I - \widehat{T}_\lambda, S) = 1$ with

$$S = \{u \in C_0^1(\overline{\Omega}),\ \alpha \ll u \ll \beta_{\widetilde{\lambda}},\ \|u\|_{C^1} \leq R\},$$

and by the property of the degree, the existence of the first solution $u_{\lambda,1} \ll 0$ is derived. To obtain a second solution $u_{\lambda,2}$ satisfying $u_{\lambda,1} \ll u_{\lambda,2}$ and $u_{\lambda,2} > \beta_{\widetilde{\lambda}}$ we now repeat the argument of (iii) of the proof of the Theorem 5.2. By Lemma 6.8 in this case we have $u_{\lambda,2} > u_{\underline{\lambda}}$. Again, Claim 6.9, we prove that if $\lambda_1 < \lambda_2$ we have $u_{\lambda_1,1} \gg u_{\lambda_2,1}$.
**(ii):** Uniqueness of the solution of $(P_{\underline{\lambda}})$.
To prove that $(P_{\underline{\lambda}})$ has at least one solution with $u \leq 0$, let $\{\lambda_n\} \subset (\underline{\lambda}, \infty)$ be a decreasing sequence such that $\lambda_n \to \underline{\lambda}$. By the regularity result (Lemma 2.1 [ACJT14]) $\{u_n\} \subset H^1(\omega) \cap W^{1,n}_{loc}(\Omega) \cap C(\overline{\Omega})$ be a sequence of corresponding solutions with $u_n \leq u_{n+1} \leq 0$. As $\{u_n\}$ is increasing and bounded above, by Theorem 5.6, there exists $M > 0$ such that $\|u_n\|_{L^\infty} < M$ for all $n \in \mathbb{N}$, and hence by the $C^{1,\alpha}$ global estimates, Theorem 4.8 we get $\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C$. Hence, up to a subsequence, $u_n \to u$ in $C_0^1(\Omega)$. From this strong convergence we easily observe that $u$ is a solution of $(P_{\underline{\lambda}})$ with $u \leq 0$.

Now we prove the uniqueness of the nonpositive solution of $(P_{\underline{\lambda}})$. Let us assume by contradiction that we have two distincts solutions, $u_1$ and $u_2$ of $(P_{\underline{\lambda}})$, then as in the Step 3 of the proof of Theorem 5.2, we prove that $\beta = \frac{1}{2}(u_1 + u_2)$ is a strict super solution of $(P_{\underline{\lambda}})$. Let us consider the strict subsolution $\alpha \ll \beta$

of $(P_{\underline{\lambda}})$ given by Lemma 6.2, and define the set,

$$\overline{\mathcal{S}} = \{u \in C_0^1(\overline{\Omega});\ \alpha \ll u \ll \beta, \|u\|_{C_0^1} < R\}$$

for some $R > C > 0$. Again, by the $C^{1,\alpha}$ estimates,

$$\|u\|_{C^{1,\alpha}} \leq C \text{ for all } u \text{ sol. of } (P_\lambda),\ \lambda \in [\underline{\lambda} - 1, \underline{\lambda}] \tag{6.8}$$

such that $deg(I - \widehat{T}_{\underline{\lambda}}, \overline{\mathcal{S}}) = 1$.

Now we prove the existence of $\varepsilon > 0$ such that

$$deg(I - \widehat{T}_\lambda, \overline{\lambda}) = 1, \text{ for all } \lambda \in [\underline{\lambda} - \varepsilon, \underline{\lambda}]. \tag{6.9}$$

We will verify that there exists some $\varepsilon \in (0, 1)$ such that there is no fixed points of $T_\lambda$ on the boundary of $\overline{\mathcal{S}}$ for all $\lambda$ in the preceding interval. Indeed, if this was not the case, there would exists a sequence $\lambda_k \to \underline{\lambda}$ with the respective solutions $u_k$ of $(P_{\lambda_k})$ belonging to $\overline{\mathcal{S}}$. Say $\lambda_k \in [\underline{\lambda} - 1, \underline{\lambda}]$ for $k \geq k_0$. Then, since $\alpha \ll u_k \ll \beta$ in $\Omega$, by (6.8) we must have $u_k \in \partial\overline{\mathcal{S}}$ for $k \geq k_0$, which means that for each such $k$,

$$\max_{\overline{\Omega}}(\alpha - u_k) = 0 \text{ or } \min_{\overline{\Omega}}(u_k - \beta) = 0. \tag{6.10}$$

By (6.8) and the compact inclusion $C^\alpha(\overline{\Omega}) \subset C(\Omega)$, $u_k \to u$ in $\Omega$ for some $u \in \Omega$, up to subsequence. From this, we observe that $u$ is a solution of $(P_{\underline{\lambda}})$; and $\alpha \leq \beta$ in $\Omega$, by taking the limit as $k \to +\infty$ in the corresponding inequalities for $u_k$. Thus $\alpha \ll u \ll \beta$ in $\Omega$, since $\alpha$ and $\beta$ are strict. Passing (6.10) to the limit and we obtain that $u(x) = \alpha(x)$ or $u(x) = \beta(x)$ at a point $x \in \overline{\Omega}$, which contradicts the definition of $\alpha \ll u \ll \beta$. Hence for obtaining (6.9) it is sufficient to apply the homotopy invariance in $\lambda$ in the interval $[\underline{\lambda} - \varepsilon, \underline{\lambda}]$. Next, with (6.9) at hand, we repeat exactly the same argument done in (i) to obtain the existence of a second solution $u_{\lambda,2}$ of $(P_\lambda)$, for all $\lambda \in [\underline{\lambda} - \varepsilon, \underline{\lambda}]$. But this, finally, contradicts the definition of $\underline{\lambda}$.

**(iii):** By the definition of $\underline{\lambda}$ and since $\beta$ is a strict supersolution of $(P_\lambda)$ we infer that the problem $(P_\lambda)$ has no solution $u \leq 0$.

**(iv):** Behaviour of the solutions for $\lambda \to 0^-$.

In Theorem 5.12 we proved that $\|u_\lambda\|_\infty \geq -2\|u_{\widehat{\lambda}}\|_\infty$ for all $\lambda \leq \widehat{\lambda} < 0$. In particular, if $C_0 := \liminf_{\lambda \to 0^-} -\|u_\lambda\|_\infty > -\infty$, then there exists a sequence $\widehat{\lambda}_n \to 0^-$ such that $C_0 = \lim_{n\to\infty} -\|u_{\widehat{\lambda}}\|_\infty > -\infty$. Hence, for every sequence $\lambda_n \to 0^-$ we deduce by the above inequality that $\liminf_{n\to\infty} \|u_{\lambda_n}\|_\infty \geq -2C_0$, which implies that $\liminf_{\lambda \to 0^-} -\|u_{\lambda_n}\|_\infty > -\infty$. Therefore, we have either

$\lim_{\lambda\to 0^-} -\|u_\lambda\|_\infty = -\infty$ or $\lim_{\lambda\to 0^-} -\|u_\lambda\|_\infty > -\infty$. By hypothesis we have that $(P_0)$ does not have a solution $u_0$, then we have the first case.

## 6.5 Proof of Corollary 5.4

First observe that $(P_{\gamma_1,k})$ has no solution. If we assume by contradiction that $u$ is a solution of $(P_{\lambda,k})$ and using $\varphi_1 > 0$ the first eigenfunction of $(P_{\gamma_1})$ as test function in $(P_{\lambda,k})$, we have

$$\begin{aligned}\int c_{\gamma_1}(x)u\varphi_1 &= \int A(x)DuD\varphi_1 \\ &= \int c_\lambda(x)u\varphi_1 + \int \varphi_1(M(x)Du, Du) + \int (h(x) + k\tilde{c}(x))\varphi_1\end{aligned}$$

and

$$(\gamma_1 - \lambda)\int c^+(x)u\varphi_1 \le -\int |h(x)|\varphi_1 < 0.$$

which is a contradiction for $\lambda = \gamma_1$. Hence also, for all $\lambda > 0$ $(P_\lambda)$ has no solution with $c^+(x)u \equiv 0$ as otherwise $u$ is a solution of $(P_\lambda)$ for every $\lambda \in \mathbb{R}$ which contradicts the nonexistence of a solution for $\lambda = \gamma_1$. By Step 3 of the proof of Lemma 6.7 there exists $\tilde{k} > 0$ such that, for all $k \in (0, \tilde{k}]$, the problem $(P_{\lambda,k})$ has a strict super solution $\beta_0$ with $\beta \ll 0$. The existence of $\lambda_2 > \gamma_1$ as in (iii) can then be deduced from Theorem 5.3. By Theorem 1.1 [ACJT15], decreasing $\tilde{k}$ if necessary, we know that for all $k \in (0, \tilde{k}]$, the problem $(P_{0,k})$ has a solution $u_0 \gg 0$. Hence the existence of $\lambda_1$ as in (i) can be deduced from Theorem 5.1.

## 6.6 Proof of Theorem 5.5

Let us begin with a preliminary remark.

**Remark 6.10** *Particular cases of Theorem 5.1 and 5.2 are given when* $h(x) \gneqq 0$ *and* $h(x) \lneqq 0$. *Indeed, if* $h \gneqq 0$ *holds, then* $u_0$ *is a supersolution of*

$$\begin{cases} -\operatorname{div}(A(x)Du_0) & \ge c_\lambda(x)u_0 + (M(x)Du_0, Du_0) + h(x) \gneqq 0 \quad \text{in } \Omega \\ \qquad u_0 & = 0 \quad \text{on } \partial\Omega \end{cases}$$

*and this implies that* $u_0 \ge 0$ *in* $\Omega$ *by Lemma 2.10. Then applying the SMP SMP gives us* $u_0 > 0$ *in* $\Omega$. *Furthermore, by Hopf,* $u_0 \gg 0$ *in* $\Omega$. *On the other hand , if* $h \lneqq 0$, *then* $u_0$ *is a subsolution of*

$$-\operatorname{div}(A(x)Du_0) \le c_\lambda(x)u_0 + (M(x)Du_0, Du_0) + h(x) \lneqq (M(x)Du_0, Du_0)$$

*and so* $v_0 = \frac{1}{\nu_2}(e^{\nu_2 u_0} - 1)$ *is a subsolution of*

$$\begin{aligned} -\operatorname{div}(A(x)Dv_0) &\leq [1+\nu_2 v][-\operatorname{div}(A(x)Du_0) - \mu_2|Du_0|^2] \\ &\lneqq [1+\nu_2 v][(M(x)Du_0, Du_0) - \mu_2|Du_0|^2] \\ &\lneqq 0 \text{ in } \Omega \end{aligned}$$

*by Lemma 2.10 with* $v_0 = 0$ *on* $\partial\Omega$*. Again by SMP we get* $v_0 < 0$ *in* $\Omega$ *(then* $v_0 \ll 0$ *in* $\Omega$ *by Hopf) and so does* $u_0 < 0$ *(with* $u_0 \ll 0$*).*

In order to consider the situation where $(P_{\lambda_0})$ has a supersolution, we need the following formulation of the anti-maximum principle. Under slightly more smooth data this result was established in [H81] but the proof given in [H81] directly extend under our regularity assumptions.

**Lemma 6.11** *Let* $\overline{c}, \overline{h}, \overline{d} \in L^p(\Omega)$ *with* $p > n$ *and assume* $\overline{h} \gneqq 0$*. We denote by* $\overline{\gamma}_1 > 0$ *the first eigenvalue of*

$$-\operatorname{div}(A(x)Du) + \overline{d}(x)u = \overline{c}_{\overline{\gamma}_1}(x)u, \;\; u \in H_0^1(\Omega).$$

*Then there exists* $\varepsilon_0 > 0$ *such that, for all* $\lambda \in (\overline{\gamma}_1, \overline{\gamma}_1 + \varepsilon_0)$*, the solution* $v$ *of*

$$-\operatorname{div}(A(x)Dv) + \overline{d}(x)v = \overline{c}_\lambda(x)v + \overline{h}(x), \;\; v \in H_0^1(\Omega).$$

*satisfies* $v \ll 0$.

*Proof.*[Proof of Theorem 5.5] Note that, for all $\lambda \in \mathbb{R}$, $u \equiv 0$ is a solution of $(P_{h\equiv 0})$.

**(i):** We proceed in several steps.

**Step 1:** We prove that for all $\lambda \in (0, \gamma_1)$ the problem $(P_{h\equiv 0})$ has a second solution $u_{\lambda,2} \gneqq 0$.

Let us prove that the problem $(P_{h\equiv 0})$ has a supersolution $\beta \gg 0$. Define $\lambda < \gamma_1$ and $\varepsilon > 0$ such that, for all $v \in [0, \varepsilon]$,

$$\lambda \frac{(1+\nu_2 v)\ln(1+\nu_2 v)}{\nu_2} \leq \gamma_1 v.$$

Consider then the function $\widetilde{\beta} = \varepsilon\varphi_1$ where $\varphi_1$ denotes the first eigenfunction of $(P_{\gamma_1})$ with $\|\varphi_1\|_{L^\infty} = 1$ and

$$\begin{cases} -\operatorname{div}(A(x)D\widetilde{\beta}) &= c_\gamma(x)\widetilde{\beta} \gneqq c_\lambda(x)\dfrac{(1+\nu_2\widetilde{\beta})\ln(1+\nu_2\widetilde{\beta})}{\nu_2}, \quad \text{in } \Omega \\ \widetilde{\beta} &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Hence for $\beta$ being defined by $\beta = \frac{\ln(1+\nu_2\widetilde{\beta})}{\nu_2}$, we have

$$\begin{aligned}
-\operatorname{div}(A(x)D\beta) &= -\frac{div(A(x)D\widetilde{\beta})}{(1+\nu_2\widetilde{\beta})} - \left(A(x)D\widetilde{\beta}, D\left[\frac{1}{(1+\nu_2\widetilde{\beta})}\right]\right) \\
&\gneqq c_\lambda(x)\beta + \frac{\nu_2}{(1+\nu_2\widetilde{\beta})^2}(A(x)D\widetilde{\beta}, D\widetilde{\beta}) \\
&\geq c_\lambda(x)\beta + \nu_2\vartheta\frac{|D\widetilde{\beta}|^2}{(1+\nu_2\widetilde{\beta})^2} \\
&= c_\lambda(x)\beta + \mu_2|D\beta|^2 \\
&\geq c_\lambda(x)\beta + (M(x)D\beta, D\beta)
\end{aligned}$$

and hence

$$\begin{cases} -\operatorname{div}(A(x)D\beta) & \gneqq c_\lambda(x)\beta + (M(x)D\beta, D\beta) \quad \text{in } \Omega \\ \qquad\qquad \beta & = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This implies, by Lemma 2.11 that $\beta \geq 0$ is a strict supersolution of $(P_{h\equiv 0})$. Then by Remark 6.10 we know that, every solution $u$ of the problem $(P_{h\equiv 0})$ satisfies $u \geq 0$, and by Lemma 6.2, $(P_{h\equiv 0})$ has a strict subsolution $\alpha \lneqq 0$. Hence we conclude that $(P_{h\equiv 0})$ has at least two solutions following the proof of Theorem 5.1 with the solution $u_{\lambda,1}$ being $u \equiv 0$.

**(ii):** Uniqueness of the solution.

Let $u \not\equiv 0$ be another solution of $(P_{h\equiv 0})$, and using $\varphi_1 > 0$ the first eigenfunction of $(P_{\gamma_1})$, as test function in $(P_{h\equiv 0})$, we have

$$\begin{aligned}
\int c_{\gamma_1}(x)u\varphi_1 &= \int A(x)DuD\varphi_1 = \int c_\lambda(x)u\varphi_1 + \int (M(x)Du, Du)\varphi_1 \\
(\gamma_1 - \lambda)\int c^+(x)u\varphi_1 &= \int (M(x)Du, Du)\varphi_1 \geq \mu_1 \int |Du|^2\varphi_1 > 0,
\end{aligned}$$

which is a contradiction for $\lambda = \gamma_1$. Hence $(P_{h\equiv 0})$ has only the trivial solution.

**(iii):** Multiplicity results.

For $\lambda > \gamma_1$, the problem $(P_{h\equiv 0})$ has a second solution $u_{\lambda,2} \ll 0$. Let $\lambda > \gamma_1$ and $\lambda_0 \in (\gamma_1, \lambda]$ such that, by Lemma 6.11, the problem

$$-\operatorname{div}(A(x)Du) = c_{\lambda_0}(x)u + 1,$$

has a solution $u \ll 0$. This implies that for $\varepsilon > 0$ small enough, the function

$\beta_0 = \varepsilon u$ satisfies

$$\begin{aligned} -\operatorname{div}(A(x)D\beta_0) &= c_{\lambda_0}(x)\varepsilon u + \varepsilon \\ &\geq c_{\lambda_0}(x)\beta_0 + \varepsilon^2 \mu_2 |Du|^2 \\ &\geq c_{\lambda_0}(x)\beta_0 + (M(x)D\beta_0, D\beta_0) \end{aligned}$$

and the problem $(P_{h\equiv 0})$ has a supersolution $\beta_0$ with $\beta_0 \leq 0$ and $c^+(x)\beta_0 \lneqq 0$. The result follows by Theorem 5.3 with $u_{\lambda,2} \equiv 0$.

**(v):** Continuum of solution of $(P_{h\equiv 0})$.
With the aim of showing the existence of a continuum of solution of $(P_{h\equiv 0})$, we use the operator

$$T_\lambda = \begin{cases} \overline{T}_\lambda, & \text{if } \lambda \leq \gamma_1, \\ \widehat{T}_\lambda, & \text{if } \lambda > \gamma_1. \end{cases}$$

where $\overline{T}_\lambda$ for $\lambda \leq \gamma_1$ is defined in (ii) of the proof of Theorem 5.1 and the operator $\widehat{T}_\lambda$ for $\lambda \geq \gamma_1$ is defined in (ii) of the proof of Theorem 5.2 in both cases with $h \equiv 0$. We proceed in several steps.
**Step 1:** For $\lambda \in (-\infty, \gamma_1]$.
This can be proved as in (ii) of the proof of Theorem 5.1. Then, if $u$ is a solution of $(\overline{P}_\lambda)$ then $u \geq u_{\gamma_1}$ and hence it is a solution of $(P_{h\equiv 0})$.
**Step 2:** For $\lambda \in [\gamma_1, +\infty)$.
The proof follows the lines of (ii) of the proof of Theorem 5.2. Then, if $u$ is a solution of $(\underline{P}_\lambda)$ then $u \leq u_{\gamma_1}$ and hence it is a solution of $(P_{h\equiv 0})$.
**Step 3:** We have $u_{\gamma_1} \equiv 0$ is the unique solution of the problem $(P_{h\equiv 0})$ for $\lambda = \gamma_1$ and $i(I - T_{\gamma_1}, u_{\gamma_1}) = 1$.
**Step 4:** Existence and behavior of the continuum.
To establish the existence of a continuum of solutions of $(P_{h\equiv 0})$ we use Theorem 6.5 (see also Theorem 2.2 [ACJT15]) with $\gamma_1 > 0$, to ensure the existence of a continuum $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^- \subset \overline{\Sigma}$ such that

$$\mathcal{C}^+ = \mathcal{C} \cap ([\gamma_1, +\infty) \times C(\overline{\Omega})) \text{ and } \mathcal{C}^- = \mathcal{C} \cap ((-\infty, \gamma_1] \times C(\overline{\Omega}))$$

are unbounded in $\mathbb{R}^\pm \times C(\overline{\Omega})$. By Step 1, we get that if $u \in \mathcal{C}^-$, then $u \geq u_{\gamma_1}$ and is a solution of $(P_{h\equiv 0})$. Thus by (iv) we infer that the projection of $\mathcal{C}^-$ on $\lambda$-axis is $(0, \gamma_1]$, a bounded interval, and then deduce that the projection of $\mathcal{C}^+$ on $\lambda$-axis is $[\gamma_1, +\infty)$. Hence,

$$\operatorname{Proj}_{\mathbb{R}} \mathcal{C} = \operatorname{Proj}_{\mathbb{R}} \mathcal{C}^- \cup \operatorname{Proj}_{\mathbb{R}} \mathcal{C}^+ = (0, +\infty).$$

Finally, by Theorem 5.6 for any $0 < \Lambda_1 < \Lambda_2 < \gamma_1$ there is a priori bound for the solution of $(P_{h\equiv 0})$, for all $\lambda \in [\Lambda_1, \Lambda_2]$. Then by the $C^\alpha$ global estimates (Theorem 4.8), we have also a $C^\alpha$ a priori bound for these solutions i.e. the projection of $\mathcal{C} \cap ([\Lambda_1, \Lambda_2] \times C(\overline{\Omega}))$ on $C(\overline{\Omega})$ is bounded. Since the component $\mathcal{C}^-$ is unbounded in $\mathbb{R}^- \times C(\overline{\Omega})$, its projection on the $C(\overline{\Omega})$ axis must be unbounded. Therefore, we deduce that $\mathcal{C}$ must emanate from infinity on the right of axis $\lambda = 0$. ■

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