7
Symbolic skew product maps

In this chapter we start to study the second main topic of this work: symbolic blender-horseshoes, in particular, we prove Theorem B.

Given a map $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$, with $0 < \lambda < \beta < 1$, we will study the maximal invariant set of Φ in $\Sigma_k \times \overline{D}$. For notational convenience, we denote $\mathcal{S}$ in the place of $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$.

Throughout this work we use the following notations.

Given a bi-sequence $\xi = (\dots, \xi_{-1}; \xi_0, \xi_1, \dots) \in \Sigma_k$ the symbol at the right of ";" is the "0 coordinate" of ξ. In what follows the number $k \geq 2$ of symbols, the contractivity ratio $0 < \nu < 1$ of $\tau : \Sigma_k \to \Sigma_k$ (recall (1.5)), the Hölder exponent $\alpha > 0$, the non-empty bounded open set D and the locally Lipschitz constants λ and β remain fixed.

Given a skew product map $\Phi = \tau \ltimes \phi_\xi$, for every $n > 0$ and every $(\xi, x) \in \Sigma_k \times \overline{D}$ we set

$$\phi^n_\xi(x) \overset{\text{def}}{=} \phi_{\tau^{n-1}(\xi)} \circ \cdots \circ \phi_\xi(x) \quad \text{and} \quad \phi^{-n}_\xi(x) \overset{\text{def}}{=} \phi^{-1}_{\tau^{-(n-1)}(\xi)} \circ \cdots \circ \phi^{-1}_\xi(x). \tag{7.1}$$

Note that $\Phi^n(\xi, x) = (\tau^n(\xi), \phi^n_\xi(x))$ and $\Phi^{-n}(\xi, x) = (\tau^{-n}(\xi), \phi^{-n}_{\tau^{-1}(\xi)}(x))$, for all $n \geq 0$.

7.1
Invariant graph

The invariant graph theorem is an important tool to understand symbolic blender-horseshoes. Next result claims the existence of a "unique invariant attracting graph" on $\Sigma_k \times \overline{D}$ for $\Phi \in \mathcal{S}$. Indeed these graphs depend continuously on Φ. The theorem below is a reformulation of the results in [15] (see also [24, Theorem 1.1]) (items (i-ii)) and [10, Section 6] (item (iii)).

Given a function $g : \Sigma_k \to \overline{D}$ its *graph map* is defined by

$$\mathrm{graph}[g] \colon \Sigma_k \to \Sigma_k \times G, \qquad \mathrm{graph}[g](\xi) = \big(\xi, g(\xi)\big)$$

and its *graph set* by

$$\Gamma_g \overset{\text{def}}{=} \mathrm{image}\Big(\mathrm{graph}[g]\Big) = \{(\xi, g(\xi)) \colon \xi \in \Sigma_k\} \subset \Sigma_k \times \overline{D}.$$

Theorem 7.1 ([15, 24, 10]). *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$ with $\beta < 1$. Then there exists a unique bounded continuous function $g_\Phi : \Sigma_k \to \overline{D}$ such that*

i) $\Phi\big(\xi, g_\Phi(\xi)\big) = \big(\tau(\xi), g_\Phi(\tau(\xi))\big)$, *for all $\xi \in \Sigma_k$, that is, $\Phi(\Gamma_{g_\Phi}) = \Gamma_{g_\Phi}$,*

ii) $\|\phi_\xi^n(x) - g_\Phi(\tau^n(\xi))\| \leq \beta^n \|g_\Phi(\xi) - x\|$, *for all $(\xi, x) \in \Sigma_k \times \overline{D}$ and $n \geq 0$, and*

iii) The set Γ_{g_Φ} depends continuously on Φ.

For notational simplicity in what follows we just write Γ_Φ in then place of Γ_{g_Φ} to denote the unique contracting invariant graph set. The next proposition shows that Γ_Φ is the locally maximal invariant set in $\Sigma_k \times \overline{D}$ of Φ.

Proposition 7.2. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$, with $\beta < 1$. Then*

i) the restriction $\Phi|_{\Gamma_\Phi}$ of Φ to the set Γ_Φ is conjugate to τ, and

ii) the invariant graph set is the (forward) maximal invariant set in $\Sigma_k \times \overline{D}$

$$\Gamma_\Phi = \bigcap_{n \in \mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D}) = \bigcap_{n \in \mathbb{N}} \Phi^n(\Sigma_k \times \overline{D}).$$

Proof. By (i) in Theorem 7.1, one has that $\Phi \circ \mathrm{graph}[g_\Phi] = \mathrm{graph}[g_\Phi] \circ \tau$. Hence $\mathrm{graph}[g_\Phi]$ conjugates the maps $\Phi|_{\Gamma_\Phi}$ and τ. To get the continuity just note that $\mathrm{graph}[g_\Phi]$ is continuous and that $\mathrm{graph}[g_\Phi]^{-1} \colon \Sigma_k \times G \to \Sigma_k$ is the projection on the first coordinate, thus it is also continuous. Then we conclude item (i) of the proposition.

Recall that periodic points of the shift map τ are dense in Σ_k, that is, $\Sigma_k = \overline{\mathrm{Per}(\tau)}$. Conjugation in the first part of this proposition implies that $\Gamma_\Phi = \overline{\mathrm{Per}(\Phi|_{\Gamma_\Phi})}$. Let Γ be the local maximal invariant set of Φ in $\Sigma_k \times D$, that is, $\Gamma = \cap_{n \in \mathbb{Z}} \Phi^n(\Sigma_k \times \overline{D})$, and then $\Gamma_\Phi \subset \Gamma$.

To prove that $\Gamma \subset \Gamma_\Phi$, consider $(\xi, x) \in \Gamma$ then it is enough to see that $x = g_\Phi(\xi)$. As the set Γ_Φ is bounded, we have that $K = \sup\{d(\gamma, \Gamma_\Phi), \gamma \in \Gamma\} \in [0, +\infty)$. Since the maps ϕ_ξ are contractions with contraction constant $0 < \beta < 1$ we deduce that

$$\begin{aligned}
\|x - g_\Phi(\xi)\| &= \|\phi_\xi^n \circ \phi_\xi^{-n}(x) - \phi_\xi^n \circ \phi_\xi^{-n}(g_\Phi(\xi))\| \\
&\leq \beta^n \|\phi_\xi^{-n}(x) - \phi_\xi^{-n}(g_\Phi(\xi))\| \\
&= \beta^n d\big(\Phi^{-n}(\xi, x), \Phi^{-n}(\xi, g_\Phi(\xi))\big) \leq K\beta^n.
\end{aligned}$$

Taking $n \to \infty$ we get $x = g_\Phi(\xi)$ and thus $(\xi, x) \in \Gamma_\Phi$. Thus $\Gamma \subset \Gamma_\Phi$, proving the proposition. □

7.2 Unstable sets

In this section we continue the study of the maximal invariant set Γ_Φ for $\Phi \in \mathcal{S}$. We analyse the relation between the set Γ_Φ and the unstable sets of points $(\xi, x) \in \Gamma_\Phi$. We first show the existence of a strong unstable lamination:

Proposition 7.3. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$, with $\beta < 1$, and let Γ_Φ be the maximal invariant set of $\Sigma_k \times \overline{D}$. Then, there exists a partition of Γ_Φ*

$$\mathcal{W}^u_{\Gamma_\Phi} = \{W^{uu}_{loc}((\xi, x); \Phi) : \ (\xi, x) \in \Gamma_\Phi\}$$

such that for every $(\xi, x) \in \Gamma_\Phi$ one has that

i) $W^{uu}_{loc}((\xi, x); \Phi)$ *is the graph of an α-Hölder function* $\gamma^u_\xi : W^u_{loc}(\xi; \tau) \to G$,

ii) $\gamma^u_\xi(\xi) = g_\Phi(\xi) = x$,

iii) $\gamma^u_\xi(\xi'') = \gamma^u_{\xi'}(\xi'')$ *for all* $\xi', \xi'' \in W^u_{loc}(\xi; \tau)$,

iv) $\phi^{-1}_{\tau^{-1}(\xi')} \circ \gamma^u_\xi(\xi') = \gamma^u_{\tau^{-1}(\xi)} \circ \tau^{-1}(\xi')$ *where* $\xi' \in W^u_{loc}(\xi; \tau)$, *and*

v) $W^{uu}_{loc}((\xi, x); \Phi) \subset W^u((\xi, x); \Phi)$.

Proof. Let $(\xi, x) \in \Gamma_\Phi$. Note that $x = g_\Phi(\xi)$ and that $(\phi^n_{\tau^{-n}(\xi)})^{-1} = \phi^{-n}_{\tau^{-1}(\xi)}$, indeed:

$$\underbrace{\phi^{-1}_{\tau^{-n}(\xi)} \circ \cdots \circ \phi^{-1}_{\tau^{-1}(\xi)}}_{\phi^{-n}_{\tau^{-1}(\xi)}} \circ \underbrace{\phi_{\tau^{-1}(\xi)} \circ \cdots \circ \phi_{\tau^{-n}(\xi)}}_{\phi^{n}_{\tau^{-n}(\xi)}} = \mathrm{Id}.$$

We define a sequence of maps $\gamma^{u,n}_\xi : W^u_{loc}(\xi; \tau) \to G$ by

$$\gamma^{u,n}_\xi(\xi') = \phi^n_{\tau^{-n}(\xi')} \circ (\phi^n_{\tau^{-n}(\xi)})^{-1}(x) = \phi^n_{\tau^{-n}(\xi')} \circ \phi^{-n}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi).$$

Note that $\{\gamma^{u,n}_\xi\}$ is a sequence in the complete metric space $C^0(W^u_{loc}(\xi; \tau), G)$.

Lemma 7.4. *The sequence $\{\gamma^{u,n}_\xi\}$ is Cauchy sequence and so it converges to some map γ^u_ξ.*

Proof. Since $\beta, \nu < 1$, to prove the first part in the lemma it is enough to see that

$$\|\gamma^{u,n+1}_\xi(\xi') - \gamma^{u,n}_\xi(\xi')\| \leq C_\Phi(\beta\nu^\alpha)^{n+1}\, d_{\Sigma_k}(\xi, \xi')^\alpha. \tag{7.2}$$

To prove this inequality first recall notation in (7.1) and observe that item (i) of Theorem 7.1 implies that for every $n > 0$ and $\xi \in \Sigma_k$

$$\phi_\xi^n \circ g_\Phi(\xi) = g_\Phi \circ \tau^n(\xi) \quad \text{and} \quad \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) = g_\Phi \circ \tau^{-n}(\xi). \tag{7.3}$$

Then, since $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$, we have that

$$\phi_{\tau^{-n}(\xi')}^{n-i} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi) = \phi_{\tau^{-n}(\xi')}^{n-i} \circ g_\Phi \circ \tau^{-n}(\xi) \in \overline{D} \quad \text{for every } 0 < i \leq n.$$

To prove the inequality in (7.2), we have $\|\gamma_\xi^{u,n+1}(\xi') - \gamma_\xi^{u,n}(\xi')\|$ equal to

$$\begin{aligned}
&\|\phi_{\tau^{-n-1}(\xi')}^{n+1} \circ \phi_{\tau^{-1}(\xi)}^{-n-1} \circ g_\Phi(\xi) - \phi_{\tau^{-n}(\xi')}^{n} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| = \\
&\quad = \|\phi_{\tau^{-n}(\xi')}^{n} \circ \phi_{\tau^{-n-1}(\xi')} \circ \phi_{\tau^{-1}(\xi)}^{-n-1} \circ g_\Phi(\xi) - \phi_{\tau^{-n}(\xi')}^{n} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| \\
&\quad \leq \beta^n \|\phi_{\tau^{-n-1}(\xi')} \circ \phi_{\tau^{-1}(\xi)}^{-n-1} \circ g_\Phi(\xi) - \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| \\
&\quad = \beta^n \|\phi_{\tau^{-n-1}(\xi')} \circ \phi_{\tau^{-1}(\xi)}^{-n-1} \circ g_\Phi(\xi) - \phi_{\tau^{-n-1}(\xi')} \circ \phi_{\tau^{-n-1}(\xi')}^{-1} \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| \\
&\quad \leq \beta^{n+1} \|\phi_{\tau^{-1}(\xi)}^{-n-1} \circ g_\Phi(\xi) - \phi_{\tau^{-n-1}(\xi')}^{-1} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| \\
&\quad = \beta^{n+1} \|\phi_{\tau^{-n-1}(\xi)}^{-1} \circ \underbrace{\phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)}_{*\in\overline{D}} - \phi_{\tau^{-n-1}(\xi')}^{-1} \circ \underbrace{\phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)}_{*\in\overline{D}}\| \\
&\quad \leq \beta^{n+1}\, C_\Phi\, d_{\Sigma_k}\big(\tau^{-n-1}(\xi), \tau^{-n-1}(\xi')\big)^\alpha \\
&\quad \leq \beta^{n+1}\, C_\Phi \nu^{\alpha(n+2)} = C_\Phi(\beta\nu^\alpha)^{n+1}\, d_{\Sigma_k}(\xi,\xi')^\alpha.
\end{aligned}$$

The proof of the lemma is now completed. □

To prove that $\mathcal{W}^u_{\Gamma_\Phi}$ is a partition of Γ_Φ we need to show that $W^{uu}_{loc}((\xi,x);\Phi)$ is contained in Γ_Φ for all $(\xi,x) \in \Gamma_\Phi$. To do this let $(\xi,x) \in \Gamma_\Phi$ and $(\xi',x') = (\xi', \gamma_\xi^u(\xi')) \in W^{uu}_{loc}((\xi,x);\Phi)$. We will prove that $x' = g_\Phi(\xi')$. From (7.3) and noting that $x = g_\Phi(\xi)$ we have the equalities:

$$\begin{aligned}
&\|g_\Phi(\xi') - \gamma_\xi^u(\xi')\| = \\
&\qquad = \lim_{n\to\infty} \|\phi_{\tau^{-n}(\xi')}^{n} \circ \phi_{\tau^{-1}(\xi')}^{-n} \circ g_\Phi(\xi') - \phi_{\tau^{-n}(\xi')}^{n} \circ \phi_{\tau^{-1}(\xi)}^{-n} \circ g_\Phi(\xi)\| \\
&\qquad = \lim_{n\to\infty} \|\phi_{\tau^{-n}(\xi')}^{n} \circ g_\Phi \circ \tau^{-n}(\xi') - \phi_{\tau^{-n}(\xi')}^{n} \circ g_\Phi \circ \tau^{-n}(\xi)\|.
\end{aligned}$$

Since the maps ϕ_ξ are β-contractions and for $g_\Phi : \Sigma_k \to \overline{D}$, if K is an upper bound of the diameter of $\overline{D}$ we get that

$$\|g_\Phi(\xi') - \gamma_\xi^u(\xi')\| \leq \lim_{n\to\infty} \beta^n \|g_\Phi \circ \tau^{-n}(\xi') - g_\Phi \circ \tau^{-n}(\xi)\| \leq \lim_{n\to\infty} K\beta^n = 0.$$

Thus $g_\Phi(\xi') = \gamma_\xi^u(\xi') = x'$ and so $(\xi',x') \in \Gamma_\Phi$. That is, $W^{uu}_{loc}((\xi,x);\Phi) \subset \Gamma_\Phi$ for all $(\xi,x) \in \Gamma_\Phi$.

Indeed, we have proved that $\gamma^u_\xi(\xi') = g_\Phi(\xi')$ for all $\xi' \in W^u_{loc}(\xi;\tau)$, proving (ii). In particular, $\gamma^u_\xi(\xi'') = g_\Phi(\xi'') = \gamma^u_{\xi'}(\xi'')$ for all $\xi', \xi'' \in W^u_{loc}(\xi;\tau)$, that proves (iii).

To prove that γ^u_ξ is an α-Hölder map (item (i)) first note that by the triangle inequality we get

$$\|\gamma^{u,n}_\xi(\xi') - x\| = \|\gamma^{u,n}_\xi(\xi') - \gamma^{u,n}_\xi(\xi)\| \le \sum_{i=1}^{n} s_i(\xi') \tag{7.4}$$

where $x = g_\Phi(\xi)$ and $s_i(\xi')$ is given by

$$\begin{aligned}&\|\phi^{n-i}_{\tau^{-n+i}(\xi')} \circ \phi_{\tau^{-n-1+i}(\xi')} \circ \phi^{i-1}_{\tau^{-n}(\xi)} \circ \phi^{-n}_{\tau^{-1}(\xi)}(x) - \\ &\qquad\qquad \phi^{n-i}_{\tau^{-n+i}(\xi')} \circ \phi_{\tau^{-n-1+i}(\xi)} \circ \phi^{i-1}_{\tau^{-n}(\xi)} \circ \phi^{-n}_{\tau^{-1}(\xi)}(x)\|.\end{aligned}$$

Claim 7.5. $s_i(\xi') \le C_\Phi(\beta\nu^\alpha)^{n-i} d_{\Sigma_k}(\xi,\xi')^\alpha$, *for every* $i \in \{1,\dots,n\}$.

We postpone the proof of this claim. Taking $n \to \infty$ in (7.4) we get

$$\|\gamma^u_\xi(\xi') - \gamma^u_\xi(\xi)\| \le C_\gamma d_{\Sigma_k}(\xi',\xi)^\alpha \quad \text{for all } \xi' \in W^s_{loc}(\xi;\tau)\ ,$$

where $C_\gamma = C_\Phi(1-\beta\nu^\alpha)^{-1}$. This shows that γ^u_ξ is α-Hölder. Indeed, for ξ', $\xi'' \in W^u_{loc}(\xi;\tau)$, observe that $\gamma^u_\xi(\xi') = \gamma^u_{\xi'}(\xi')$ and $\gamma^u_\xi(\xi'') = \gamma^u_{\xi'}(\xi'')$. Thus

$$\|\gamma^u_\xi(\xi') - \gamma^u_\xi(\xi'')\| = \|\gamma^u_{\xi'}(\xi') - \gamma^u_{\xi'}(\xi'')\| \le C_\gamma d_{\Sigma_k}(\xi',\xi'')^\alpha.$$

Note that the Hölder constant obtained is uniform on ξ and $x = g_\Phi(\xi)$.

Proof of Claim 7.5. For every $i \in \{1,\dots,n\}$, we have that

$$\begin{aligned} s_i(\xi') &= \|\phi^{n-i}_{\tau^{-n+i}(\xi')} \circ \phi_{\tau^{-n-1+i}(\xi')} \circ \phi^{i-1}_{\tau^{-n}(\xi)} \circ \phi^{-n}_{\tau^{-1}(\xi)}(x) - \\ &\qquad\qquad \phi^{n-i}_{\tau^{-n+i}(\xi')} \circ \phi_{\tau^{-n-1+i}(\xi)} \circ \phi^{i-1}_{\tau^{-n}(\xi)} \circ \phi^{-n}_{\tau^{-1}(\xi)}(x)\| \\ &\le \beta^{n-i}\|\phi_{\tau^{-n-1+i}(\xi')} \circ \phi^{i-1}_{\tau^{-n}(\xi)} \circ \phi^{-n}_{\tau^{-1}(\xi)}(x) - \\ &\qquad\qquad \phi_{\tau^{-n-1+i}(\xi)} \circ \phi^{i-1}_{\tau^{-n}(\xi)} \circ \phi^{-n}_{\tau^{-1}(\xi)}(x)\| \\ &\le \beta^{n-i}\, C_\Phi\, d_{\Sigma_k}(\tau^{-n-1+i}(\xi'), \tau^{-n-1+i}(\xi))^\alpha \\ &\le \beta^{n-i}\, C_\Phi\, \nu^{(n-i)\alpha}\, \nu^\alpha = (\beta\,\nu^\alpha)^{n-i}\, C_\Phi\, d_{\Sigma_k}(\xi',\xi)^\alpha, \end{aligned}$$

where first inequality follows from β-contraction of ϕ_ξ, and second one from α-Hölder continuity of ϕ_ξ. □

This completes the proof of item (i).

To prove item (iv) observe that

$$
\begin{aligned}
\phi^{-1}_{\tau^{-1}(\xi')} \circ \gamma^u_\xi(\xi') &= \lim_{n\to\infty} \phi^{-1}_{\tau^{-1}(\xi')} \circ \phi^{n}_{\tau^{-n}(\xi')} \circ \phi^{-n}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi) \\
&= \lim_{n\to\infty} \phi^{-1}_{\tau^{-1}(\xi')} \circ \phi_{\tau^{-1}(\xi')} \circ \phi^{n-1}_{\tau^{-n}(\xi')} \circ \phi^{-n}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi) \\
&= \lim_{n\to\infty} \phi^{n-1}_{\tau^{-n}(\xi')} \circ \phi^{-n}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi) \\
&= \lim_{n\to\infty} \phi^{n-1}_{\tau^{-n}(\xi')} \circ \phi^{-(n-1)}_{\tau^{-2}(\xi)} \circ \phi^{-1}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi) \\
&= \lim_{n\to\infty} \phi^{n-1}_{\tau^{-n}(\xi')} \circ \phi^{-(n-1)}_{\tau^{-2}(\xi)} \circ g_\Phi \circ \tau^{-1}(\xi) \qquad (7.5) \\
&= \gamma^u_{\tau^{-1}(\xi)} \circ \tau^{-1}(\xi'),
\end{aligned}
$$

where equality (7.5) is consequence of (7.3): $\phi^{-1}_{\tau^{-1}(\xi)} \circ g_\Phi(\xi) = g_\Phi \circ \tau^{-1}(\xi)$.

Now we prove item (v): $W^{uu}_{loc}((\xi,x);\Phi) \subset W^u((\xi,x);\Phi)$. Let $(\xi',x') \in W^{uu}_{loc}((\xi,x);\Phi)$, Then $\xi' \in W^u_{loc}(\xi;\tau)$ and $x' = \gamma^u_\xi(\xi')$. Thus,

$$
\begin{aligned}
d(\Phi^{-n}\big(\xi',x'),\Phi^{-n}(\xi,x)\big) &= d_{\Sigma_k}(\tau^{-n}(\xi'),\tau^{-n}(\xi)) + \\
&\quad + \|\phi^{-n}_{\tau^{-1}(\xi')}(x') - \phi^{-n}_{\tau^{-1}(\xi)}(x)\| \\
&\le \nu^n d_{\Sigma_k}(\xi',\xi) + \|\phi^{-n}_{\tau^{-1}(\xi')} \circ \gamma^u_\xi(\xi') - \phi^{-n}_{\tau^{-1}(\xi)}(x)\|.
\end{aligned} \qquad (7.6)
$$

Since $(\xi,x) \in \Gamma_\Phi$ we have that $x = g_\Phi(\xi)$, and using that $\gamma^u_{\tau^{-n}(\xi)} \circ \tau^{-n}(\xi) = g_\Phi \circ \tau^{-n}(\xi) = \phi^{-n}_{\tau^{-1}(\xi)}(x)$ and $\phi^{-n}_{\tau^{-1}(\xi')} \circ \gamma^u_\xi(\xi') = \gamma^u_{\tau^{-n}(\xi)} \circ \tau^{-n}(\xi')$ we get

$$
\begin{aligned}
\|\phi^{-n}_{\tau^{-1}(\xi')} \circ \gamma^u_\xi(\xi') - \phi^{-n}_{\tau^{-1}(\xi)}(x)\| &= \|\gamma^u_{\tau^{-n}(\xi)} \circ \tau^{-n}(\xi') - \gamma^u_{\tau^{-n}(\xi)} \circ \tau^{-n}(\xi)\| \\
&\le \nu^{\alpha n} d_{\Sigma_k}(\xi',\xi)^\alpha.
\end{aligned}
$$

This implies that (7.6) goes to zero as n goes to infinity and therefore (ξ',x') belongs to $W^u((\xi,x);\Phi)$. The proof of item (v), and of the proposition, is now complete. □

By construction, the sets of the partition $\mathcal{W}^u_{\Gamma_\Phi}$ are the *local strong unstable set* $W^{uu}_{loc}((\xi,x);\Phi)$ throughout the point (ξ,x) in Γ_Φ. We define the (global) *strong unstable set* of $(\xi,x) \in \Gamma_\Phi$ as

$$
W^{uu}\big((\xi,x);\Phi\big) \stackrel{\text{def}}{=} \bigcup_{n\ge 0} \Phi^n\big(W^{uu}_{loc}(\Phi^{-n}(\xi,x);\Phi)\big)
$$

Before stating the next proposition let us recall that for each $\Phi = \tau \ltimes \phi_\xi$ we denote by $\mathrm{Per}(\Phi)$ the set of periodic points of Φ and that $\mathscr{P} : \Sigma_k \times G \to G$ is the projection on the fiber space.

Proposition 7.6. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$, with $\beta < 1$. Then* *i)* $W^{uu}\big((\xi,x);\Phi\big) = W^{u}\big((\xi,x);\Phi\big) \subset \Gamma_\Phi$ *for all* $(\xi,x)\in\Gamma_\Phi$*, and*

ii) for all periodic point (ϑ,p) *of* Φ *in* $\Sigma_k\times\overline{D}$

$$K_\Phi \stackrel{\text{def}}{=} \overline{\mathscr{P}\big(\mathrm{Per}(\Phi)\big)\cap D} = \overline{\mathscr{P}\big(W^u((\vartheta,p);\Phi)\big)} = \mathscr{P}(\Gamma_\Phi) = g_\Phi(\Sigma_k).$$

Proof. To prove the inclusion $W^{uu}((\xi,x);\Phi)\subset W^{u}((\xi,x);\Phi)$ for all (ξ,x) in Γ_Φ, we take $(\xi',x')\in W^{uu}((\xi,x);\Phi)$ and show that

$$\lim_{n\to\infty} d(\Phi^{-n}(\xi',x'),\Phi^{-n}(\xi,x)) = 0. \tag{7.7}$$

Since $(\xi',x')\in W^{uu}((\xi,x);\Phi)$, by definition there are $m\in\mathbb{N}$ and $(\zeta,z)\in W^{uu}_{loc}(\Phi^{-m}(\xi,x);\Phi)$ such that $(\xi',x')=\Phi^m(\zeta,z)$. Let $(\eta,y)=\Phi^{-m}(\xi,x)$. Note that $(\eta,y)\in\Gamma_\Phi$, $(\zeta,z)\in W^{uu}_{loc}((\eta,y);\Phi)$, and

$$d\big(\Phi^{-n}(\xi',x'),\Phi^{-n}(\xi,x)\big) = d\big(\Phi^{-(n-m)}(\zeta,z),\Phi^{-(n-m)}(\eta,y)\big).$$

By item (v) of Proposition 7.3, we have that $W^{uu}_{loc}((\eta,y);\Phi)\subset W^{u}((\eta,y);\Phi)$ and thus it follows (7.7).

To prove the converse inclusion, that is, if $(\xi,x)\in\Gamma_\Phi$ then $W^{u}((\xi,x);\Phi)\subset W^{uu}((\xi,x);\Phi)$, take $(\xi',x')\in W^{u}((\xi,x);\Phi)$. We have to show that there is $m\in\mathbb{N}$ such that $\Phi^{-m}(\xi',x')\in W^{uu}_{loc}(\Phi^{-m}(\xi,x);\Phi)$. By definition of unstable set,

$$\lim_{n\to\infty} d_{\Sigma_k}\big(\tau^{-n}(\xi'),\tau^{-n}(\xi)\big)=0 \quad\text{and}\quad \lim_{n\to\infty}\|\phi^{-n}_{\tau^{-1}(\xi')}(x')-\phi^{-n}_{\tau^{-1}(\xi)}(x)\|=0. \tag{7.8}$$

Since $(\xi,x)\in\Gamma_\Phi$ then $\phi^{-n}_{\tau^{-1}(\xi)}(x)\in D$ for all $n\geq 0$. Thus, there exists $m\in\mathbb{N}$ such that

$$\tau^{-m}(\xi')\in W^{u}_{loc}(\tau^{-m}(\xi);\tau)\quad\text{and}\quad \phi^{-(n+m)}_{\tau^{-1}(\xi')}(x')\in D,\quad \text{for every } n\geq m.$$

Write $(\eta,y)=\Phi^{-m}(\xi,x)\in\Gamma_\Phi$ and $(\eta',y')=\Phi^{-m}(\xi',x')$. Hence, one has that $y=g_\Phi(\eta)$ and that $\phi^{n-i}_{\tau^{-n}(\eta)}\circ\phi^{-n}_{\tau^{-1}(\eta)}(y)$ and $\phi^{n-i}_{\tau^{-n}(\eta')}\circ\phi^{-n}_{\tau^{-1}(\eta')}(y')$ belong to D for all $0<i\leq n$. Recalling the definition of γ^u_η and $\gamma^{u,n}_\eta$ we have that

$$\begin{aligned}\|y'-\gamma^u_\eta(\eta')\| &= \lim_{n\to\infty}\|\phi^{n}_{\tau^{-n}(\eta')}\circ\phi^{-n}_{\tau^{-1}(\eta')}(y')-\phi^{n}_{\tau^{-n}(\eta')}\circ\phi^{-n}_{\tau^{-1}(\eta)}(y)\|\\ &\leq \lim_{n\to\infty}\beta^n\,\|\phi^{-n}_{\tau^{-1}(\eta')}(y')-\phi^{-n}_{\tau^{-1}(\eta)}(y)\|.\end{aligned}$$

From (7.8) and since $\beta<1$, we get this limit is zero and hence $y'=\gamma^u_\eta(\eta')$. That is, $\Phi^{-m}(\xi',x')\in W^{uu}_{loc}(\Phi^{-m}(\xi,x);\Phi)$ and therefore $(\xi',x')\in W^{uu}((\xi,x);\Phi)$, concluding our assertion.

Note that we proved that $W^{uu}((\xi,x);\Phi) = W^u(\xi,x);\Phi)$ for all $(\xi,x) \in \Gamma_\Phi$. Then by Proposition 7.3 we have that $W^u(\xi,x);\Phi) \subset \Gamma_\Phi$, ending the first item of the proposition.

To prove item (ii), consider a periodic point $\vartheta \in \Sigma_k$ of τ and note that $W^u(\vartheta;\tau)$ and $\mathrm{Per}(\tau)$ are both dense in Σ_k. Using the conjugation in Proposition 7.2 and item (i), we get

$$\overline{\mathrm{Per}(\Phi|_{\Gamma_\Phi})} = \Gamma_\Phi = \overline{W^u\big((\vartheta,g_\Phi(\vartheta));\Phi\big)} = \overline{W^{uu}\big((\vartheta,g_\Phi(\vartheta));\Phi\big)}. \tag{7.9}$$

Note that if $(\vartheta,p) \in \Sigma_k \times \overline{D}$ is a periodic point of Φ, from the assumption $\phi_\xi(\overline{D}) \subset D$, we have that $\Phi^n(\vartheta,p) \in \Sigma_k \times D$ for all $n \in \mathbb{Z}$. Moreover, since g_Φ is the unique invariant graph of Φ restricted to $\Sigma_k \times \overline{D}$, by item (ii) in Theorem 7.1 we get $p = g_\Phi(\vartheta)$. From this, we have

$$\mathrm{Per}(\Phi|_{\Gamma_\Phi}) = \mathrm{Per}(\Phi|_{\Sigma_k\times\overline{D}}) = \mathrm{Per}(\Phi) \cap (\Sigma_k \times D). \tag{7.10}$$

Recalling that K_Φ is the closure of projection by $\mathscr{P}$ of the periodic points of Φ in D and since the projection $\mathscr{P}$ is a closed map and Σ_k is a compact set, Equations (7.9) and (7.10) imply that

$$\mathscr{P}(\Gamma_\Phi) = \overline{\mathscr{P}\big(W^u((\vartheta,p);\Phi)\big)} = \overline{\mathscr{P}\big(\mathrm{Per}(\Phi|_{\Gamma_\Phi})\big)} = \overline{\mathscr{P}\big(\mathrm{Per}(\Phi)\big) \cap D} \stackrel{\text{def}}{=} K_\Phi.$$

Finally, note that by definition $\mathscr{P}(\Gamma_\Phi) = g_\Phi(\Sigma_k)$ and from the above equation, $K_\Phi = g_\Phi(\Sigma_k)$. Now, the proof of the proposition is now complete. □

7.3 Continuation of the reference cube

In this subsection we will conclude the proof of Theorem B. Note that Proposition 7.2 implies the first part of Theorem B and to conclude the proof of Theorem B, it remains to show item (v), that is, K_Φ depends continuously with respect to Φ.

Given two subsets A and B of $\overline{D}$ we define

$$d_H(A,B) \stackrel{\text{def}}{=} \sup\{d(a,B), d(b,A) : a \in A, b \in B\}, \quad A,\, B \in \mathcal{K}(\overline{D}).$$

We have the following properties of d_H.

(H1) $d_H(\overline{A},\overline{B}) = d_H(A,B)$,

(H2) $d_H(T(A),T(B)) \leq \mathrm{Lip}(T)\, d_H(A,B)$ where $T : \overline{D} \to \overline{D}$ is a Lipschitz map, (H3) if A_i and B_i are non-empty subsets for all i in a set of index I then

$$d_H\Big(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\Big) \le \sup_{i\in I} d_H(A_i, B_i).$$

We consider the set $\mathcal{K}(\overline{D})$ whose elements are the compact subsets of $\overline{D}$ endowed with the Hausdorff metric d_H obtaining a complete metric space. Define the map

$$\mathscr{L} : \mathcal{S} \to \mathcal{K}(\overline{D}), \qquad \mathscr{L}(\Phi) \stackrel{\text{def}}{=} K_\Phi.$$

Note that by Proposition 7.6 we have $K_\Phi \in \mathcal{K}(\overline{D})$ for all $\Phi \in \mathcal{S}$ and thus this map is well defined.

The next proposition claims that $\mathscr{L}$ is continuous, completing the proof of Theorem B.

Proposition 7.7. *Consider $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}$ with $\beta < 1$. Then, for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{S}$ with*

$$d_{C^0}(\phi_\xi, \psi_\xi) < \delta \quad \textit{it holds that} \quad d_H(K_\Phi, K_\Psi) = d_H(\mathscr{L}(\Phi), \mathscr{L}(\Psi)) < \varepsilon.$$

Proof. Fixed small $\varepsilon > \epsilon > 0$, let $\delta = \epsilon(1-\beta)/2 > 0$. Take a fixed point $\vartheta \in \Sigma_k$ of τ. As $\phi_\vartheta(\overline{D}) \subset D$, by Brouwer's fixed point theorem, there is $p_\Phi \in D$ such that $\phi_\vartheta(p_\Phi) = p_\Phi$. As the map ϕ_ϑ is a contraction this fixed point is hyperbolic. Thus, (ϑ, p_Φ) is a fixed point of Φ in $\Sigma_k \times D$. If δ is small enough then for every $\Psi = \tau \ltimes \psi_\xi$ with $d_{C^0}(\phi_\xi, \psi_\xi) < \delta$ there is $p_\Psi \in D$ close to p_Φ which is a fixed point of ψ_ϑ. Thus $(\vartheta, p_\Psi) \in \Gamma_\Psi$ is a fixed point of Ψ, called the continuation of $(\vartheta, p_\Phi) \in \Gamma_\Phi$ for Ψ. Take $\Theta = \tau \ltimes \theta_\xi \in \{\Phi, \Psi\}$. By Proposition 7.6 the strong unstable set and the unstable set of (ϑ, p_Θ) are equal. Thus

$$\mathscr{P}\big(W^u((\vartheta, p_\Theta); \Theta)\big) = \bigcup_{n\ge 0} \mathscr{P} \circ \Theta^n\big(W^{uu}_{loc}((\vartheta, p_\Theta); \Theta)\big).$$

By item (i) of Proposition 7.3, the graph set of $\gamma^u_{\vartheta, p_\Theta} : W^u_{loc}(\vartheta; \tau) \to G$ is the local strong unstable set of (ϑ, p_Θ) for Θ. Thus, for each $n \ge 0$,

$$\mathscr{P}\Big(\Theta^n(W^{uu}_{loc}((\vartheta, p_\Theta); \Theta))\Big) = \big\{\theta^n_\xi \circ \gamma^u_{\vartheta, p_\Theta}(\xi) \colon\ \xi \in W^u_{loc}(\vartheta; \tau)\big\} \stackrel{\text{def}}{=} E_n(\Theta).$$

Hence, by Proposition 7.6

$$K_\Theta = \overline{\mathscr{P}\big(W^u\big((\vartheta, p_\Theta); \Theta\big)\big)} = \overline{\bigcup_{n\ge 0} E_n(\Theta)}, \qquad \Theta = \Phi,\ \Psi.$$

By item (H3) of properties of d_H

$$d_H\big(K_\Phi, K_\Psi\big) \leq \sup_{n\geq 0} d_H\big(E_n(\Phi), E_n(\Psi)\big).$$

On the other hand, for each $n \geq 0$, we have that

$$d_H\big(E_n(\Phi), E_n(\Psi)\big) \leq \sup_{\xi \in W^u_{loc}(\vartheta;\tau)} \|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Phi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\|. \tag{7.11}$$

Fix $\xi \in W^u_{loc}(\vartheta;\tau)$. First we will estimate (7.11) for $n = 0$.

Claim 7.8. *For every $m \in \mathbb{N}$ it holds $\gamma^u_\vartheta(\xi) = \phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_\vartheta \circ \tau^{-m}(\xi)$, for $\xi \in W^u(\vartheta,\tau)$.*

Proof. Recall that $\big(\phi^m_{\tau^{-1}(\xi)}\big)^{-1} = \phi^{-m}_{\tau^{-1}(\xi)}$ and by notation in (7.1), and item (iv) in Proposition 7.3, we get:

$$\begin{aligned}
\phi^{-m}_{\tau^{-m}(\xi)} \circ \gamma^u_\vartheta(\xi) &= \phi^{-1}_{\tau^{-m}(\xi)} \circ \cdots \circ \underbrace{\phi^{-1}_{\tau^{-1}(\xi)} \circ \gamma^u_\vartheta(\xi)} \\
&= \phi^{-1}_{\tau^{-m}(\xi)} \circ \cdots \circ \underbrace{\phi^{-1}_{\tau^{-2}(\xi)} \circ \gamma^u_{\tau^{-1}(\vartheta)} \circ \tau^{-1}(\xi)} \\
&= \phi^{-1}_{\tau^{-m}(\xi)} \circ \cdots \circ \phi^{-1}_{\tau^{-3}(\xi)} \circ \gamma^u_{\tau^{-2}(\vartheta)} \circ \tau^{-2}(\xi) \\
&\;\;\vdots \\
&= \gamma^u_{\tau^{-m}(\vartheta)} \circ \tau^{-m}(\xi) = \gamma^u_\vartheta \circ \tau^{-m}(\xi).
\end{aligned}$$

□

Using the claim above, the triangle inequality and the β-contraction of ϕ_ξ we get

$$\begin{aligned}
&\|\gamma^u_{\vartheta,p_\Phi}(\xi) - \gamma^u_{\vartheta,p_\Psi}(\xi)\| = \\
&\quad = \|\phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Phi} \circ \tau^{-m}(\xi) - \psi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| \\
&\quad \leq \|\phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Phi} \circ \tau^{-m}(\xi) - \phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| + \\
&\qquad + \|\phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi) - \psi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| \\
&\quad \leq \beta^m \|\gamma^u_{\vartheta,p_\Phi} \circ \tau^{-m}(\xi) - \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| + \\
&\qquad + \|\phi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi) - \psi^m_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\|.
\end{aligned}$$

Write the last inequality in the form $\beta^m\,(A) + (B)$. By continuity and since ξ is in the local unstable manifold of the fixed point ϑ for τ we get that

$$(A) = \|\gamma^u_{\vartheta,p_\Phi} \circ \tau^{-m}(\xi) - \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| \overset{m\to\infty}{\longrightarrow} \|p_\Phi - p_\Psi\|.$$

Hence the first term in the sum goes to zero as $m \to \infty$. To estimate the term (B) recall that $d_{C^0}(\phi_\xi, \psi_\xi) < \delta$. Thus

$$\begin{aligned}
&\|\phi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi) - \psi_{\tau^{-1}(\xi)} \circ \psi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| \leq \\
&\leq \|\phi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi) - \psi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Phi} \circ \tau^{-m}(\xi)\| \\
&\quad + \|\psi_{\tau^{-1}(\xi)} \circ \phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Phi} \circ \tau^{-m}(\xi) - \psi_{\tau^{-1}(\xi)} \circ \psi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| \\
&\leq \delta + \beta\, \|\phi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Phi} \circ \tau^{-m}(\xi) \,-\, \psi_{\tau^{-1}(\xi)} \circ \psi^{m-1}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\|.
\end{aligned}$$

Arguing inductively we get

$$\|\phi^{m}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi) - \psi^{m}_{\tau^{-m}(\xi)} \circ \gamma^u_{\vartheta,p_\Psi} \circ \tau^{-m}(\xi)\| \leq \delta \sum_{k=0}^{m-1} \beta^k \leq \frac{\delta}{1-\beta}.$$

Putting together the estimates for (A) and (B) we get

$$\|\gamma^u_{\vartheta,p_\Phi}(\xi) - \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq \delta(1-\beta)^{-1} < \epsilon,$$

proving (7.11) for $n = 0$.

Now, for each $n \geq 1$, with a similar calculation we obtain that

$$\begin{aligned}
\|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Phi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq \beta^n \|\gamma^u_{\vartheta,p_\Phi}(\xi) - \gamma^u_{\vartheta,p_\Psi}(\xi)\| + \\
+ \|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\|.
\end{aligned}$$

Arguing analogously (using the triangle inequality and $d_{C^0}(\phi_\xi, \psi_\xi) < \delta$), we get that $\|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq \delta(1-\beta)^{-1}$. Putting together these estimates and using that $\beta^n < 1$, we have that

$$\|\phi^n_\xi \circ \gamma^u_{\vartheta,p_\Phi}(\xi) - \psi^n_\xi \circ \gamma^u_{\vartheta,p_\Psi}(\xi)\| \leq \beta^n \frac{\delta}{1-\beta} + \frac{\delta}{1-\beta} \leq \frac{2\,\delta}{1-\beta} = \epsilon.$$

By (7.11) this implies that

$$d_H(K_\Phi, K_\Psi) \leq \sup_{n\in\mathbb{N}} d_H\big(E_n(\Phi), E_n(\Psi)\big) \leq \epsilon < \varepsilon,$$

ending the proof of the proposition. □

The proof of Theorem B is now complete.