

## 5 Blenders

In this chapter, following [9, Section 6.2.1], we first introduce a simplified model of a blender with central direction  $c > 1$ . The main result in Section 5.1 is the intersection property in Proposition 5.1 implying that an invariant manifold of a saddle of  $s$ -index  $s$  topologically behaves as a saddle of bigger  $s$ -index. We see in Section 5.2 that these blenders are robust.

### 5.1 Geometric model of blender

Fix  $c > 1$  and  $n \geq 2$  and consider a diffeomorphism  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that the maximal invariant set of  $f$  in the rectangle  $R = [0, 1]^n$  is an affine Smale horseshoe contained in the interior of  $R$ .

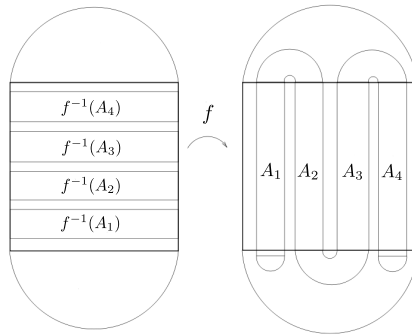
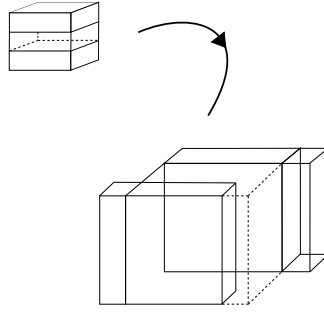


Figure 5.1: Rectangles

For notational simplicity, let us consider the case  $n = 2$  and  $c = 2$ . Suppose that the affine horseshoe has four legs. Let  $I_1, \dots, I_4$  be pairwise disjoint closed intervals in  $(0, 1)$  such that  $f(R) \cap R$  is the union of the vertical rectangles  $A_i = I_i \times [0, 1]$ ,  $i = 1, \dots, 4$ . Analogously, let  $J_1, \dots, J_4$  be pairwise disjoint closed intervals in  $(0, 1)$  such that  $f^{-1}(R) \cap R$  is the union of four horizontal rectangles  $f^{-1}(A_i) = [0, 1] \times J_i$ ,  $i = 1, \dots, 4$ . Denote by  $(x_q, y_q)$  the fixed point of  $f$  in the rectangle  $A_1$ .

Let  $\sigma < 1$  be the rate of contraction of  $f$  in  $R$  and take  $1 < \lambda < \beta < \max\{2, \sigma^{-1}\}$ . Consider  $0 < \lambda_0, \beta_0 < 1$  with  $\beta - \beta_0, \lambda - \lambda_0 \in (0, 1)$ .

Figure 5.2: The map  $F$ 

We define a diffeomorphism  $F: \mathbb{R}^4 \rightarrow \mathbb{R}^4$  whose restriction to the cube  $\mathbf{C} = [-1, 1]^2 \times R$  is defined as follows:

$$F(t, s, x, y) = \begin{cases} \left( \lambda t, \beta s, f(x, y) \right) & \text{if } (x, y) \in f^{-1}(A_1), \\ \left( \lambda t, \beta s - \beta_0, f(x, y) \right) & \text{if } (x, y) \in f^{-1}(A_2), \\ \left( \lambda t - \lambda_0, \beta s, f(x, y) \right) & \text{if } (x, y) \in f^{-1}(A_3), \\ \left( \lambda t - \lambda_0, \beta s - \beta_0, f(x, y) \right) & \text{if } (x, y) \in f^{-1}(A_4), \end{cases} \quad (5.1)$$

where  $t, s \in [-1, 1]$ . Let  $\Gamma$  be the maximal invariant (hyperbolic) set of  $F$  in the cube  $\mathbf{C}$ . Under these assumptions the set  $\mathbf{C} \cap F(\mathbf{C})$  is the union of the following four sets

$$\begin{aligned} \mathbb{A}_1 &= [-1, 1] \times [-1, 1] \times A_1, \\ \mathbb{A}_2 &= [-1, 1] \times [-1, \beta - \beta_0] \times A_2, \\ \mathbb{A}_3 &= [-1, \lambda - \lambda_0] \times [-1, 1] \times A_3, \\ \mathbb{A}_4 &= [-1, \lambda - \lambda_0] \times [-1, \beta - \beta_0] \times A_4. \end{aligned}$$

Consider the hyperbolic fixed point  $Q = (0, 0, x_q, y_q) \in \mathbb{A}_1$  of  $F$  of  $s$ -index 1 and its local stable manifold

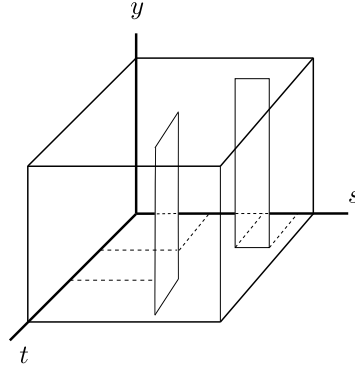
$$W_{loc}^s(Q; F) \stackrel{\text{def}}{=} \{(0, 0)\} \times [0, 1] \times \{y_q\},$$

which is the connected component of  $W^s(Q; F) \cap \mathbf{C}$  that contains  $Q$ .

We define *vertical segments*, *strips*, and *blocks* to the right of  $W_{loc}^s(Q; F)$  as follows:

- A *vertical segment to the right of  $W_{loc}^s(Q; F)$*  is a segment of the form  $\varsigma = \{(t, s, x)\} \times [0, 1]$ , where  $x \in [0, 1]$  and  $0 < t, s \leq 1$ .
- A *vertical block to the right of  $W_{loc}^s(Q; F)$*  is a set of the form  $B = [t_1, t_2] \times [s_1, s_2] \times \{x\} \times [0, 1]$ , where  $x \in [0, 1]$ ,  $0 < t_1 < t_2 \leq 1$  and  $0 < s_1 < s_2 \leq 1$ . Consider  $a(B) = (t_2 - t_1) \cdot (s_2 - s_1)$  the area of the base of this vertical block.

- A  $t$ -vertical strip to the right of  $W_{loc}^s(Q; F)$  is a set of the form  $\Delta^t = [t_1, t_2] \times \{s\} \times \{x\} \times [0, 1]$ , where  $x \in [0, 1]$ ,  $s \in (0, 1]$  and  $0 < t_1 < t_2 \leq 1$ . We denote by  $w_t(\Delta^t) = (t_2 - t_1)$  the  $t$ -width of  $\Delta^t$ .
- A  $s$ -vertical strip to the right of  $W_{loc}^s(Q; F)$  is a set of the form  $\Delta^s = \{t\} \times [s_1, s_2] \times \{x\} \times [0, 1]$ , where  $x \in [0, 1]$ ,  $t \in (0, 1]$  and  $0 < s_1 < s_2 \leq 1$ . We denote by  $w_s(\Delta^s) = (s_2 - s_1)$  the  $s$ -width of  $\Delta^s$ .

Figure 5.3:  $t$ -vertical strips and  $s$ -vertical strips

The following intersection result is the main step of this section.

**Proposition 5.1.** *Consider the map  $F$  and the cube  $\mathbf{C}$  in (5.1) and the hyperbolic fixed point  $Q$  of  $F$ . Then the stable manifold of  $Q$  intersects every vertical block  $B$  to the right of  $W_{loc}^s(Q; F)$  in  $\mathbf{C}$ .*

### 5.1.1

#### Proof of Proposition 5.1

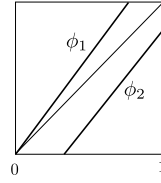
To prove the proposition, we will rewrite it using iterated function systems (IFS). See Proposition 5.2 which is just a reformulation of Proposition 5.1 in terms of the IFS's obtained considering the quotient dynamics by the hyperbolic part.

More precisely, consider the maps

- $\alpha_1(t, s) = (\phi_1(t), \psi_1(s))$  for  $t \in [-\frac{1}{\lambda}, \frac{1}{\lambda}]$  and  $s \in [-\frac{1}{\beta}, \frac{1}{\beta}]$ ;
- $\alpha_2(t, s) = (\phi_1(t), \psi_2(s))$  for  $t \in [-\frac{1}{\lambda}, \frac{1}{\lambda}]$  and  $s \in [\frac{-1+\beta_0}{\beta}, 1]$ ;
- $\alpha_3(t, s) = (\phi_2(t), \psi_1(s))$  for  $t \in [\frac{-1+\lambda_0}{\lambda}, 1]$  and  $s \in [-\frac{1}{\beta}, \frac{1}{\beta}]$ ;
- $\alpha_4(t, s) = (\phi_2(t), \psi_2(s))$  for  $t \in [\frac{-1+\lambda_0}{\lambda}, 1]$  and  $s \in [\frac{-1+\beta_0}{\beta}, 1]$ ,

where  $\phi_1, \phi_2, \psi_1, \psi_2: \mathbb{R} \rightarrow \mathbb{R}$  are the expanding maps given by

- $\phi_1(t) = \lambda t$  and  $\phi_2(t) = \lambda t - \lambda_0$ ,

Figure 5.4:  $\phi_1, \phi_2: \mathbb{R} \rightarrow \mathbb{R}$ 

- $\psi_1(s) = \beta s$  and  $\psi_2(s) = \beta s - \beta_0$ .

Indeed, considering the quotient of the dynamics of  $F$  by the sum of the strong stable and strong unstable directions, one gets the bidimensional central dynamics given by the compositions of maps  $\alpha_i$ ,  $i = 1, \dots, 4$ , above. More precisely, observe that

- $F^{-1}(\mathbb{A}_1) = [-\frac{1}{\lambda}, \frac{1}{\lambda}] \times [-\frac{1}{\beta}, \frac{1}{\beta}] \times f^{-1}(A_1)$ ;
- $F^{-1}(\mathbb{A}_2) = [-\frac{1}{\lambda}, \frac{1}{\lambda}] \times [\frac{-1+\beta_0}{\beta}, 1] \times f^{-1}(A_2)$ ;
- $F^{-1}(\mathbb{A}_3) = [\frac{-1+\lambda_0}{\lambda}, 1] \times [-\frac{1}{\beta}, \frac{1}{\beta}] \times f^{-1}(A_3)$ ;
- $F^{-1}(\mathbb{A}_4) = [\frac{-1+\lambda_0}{\lambda}, 1] \times [\frac{-1+\beta_0}{\beta}, 1] \times f^{-1}(A_4)$

and that the map  $\alpha_i$  is the “central part” of the restriction of  $F$  to  $F^{-1}(\mathbb{A}_i)$ .

**Proposition 5.2.** *Let  $\mathcal{G} = \mathcal{G}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  be the iterated function system generated by the compositions of the maps  $\alpha_i$ ,  $i = 1, \dots, 4$ . Then given any open set  $O \subset [0, 1]^2$  there is a map  $\alpha \in \mathcal{G}$  such that  $(0, 0) \in \alpha(O)$ .*

We need the following one-dimensional version of the proposition above.

**Lemma 5.3** (Lemma 6.7 in [9]). *Denote by  $\mathcal{G}(\phi_1, \phi_2)$  the iterated function system generated by the maps*

$$\phi_1(t) = \lambda t \quad \text{and} \quad \phi_2(t) = \lambda t - \lambda_0,$$

where  $0 < \lambda_0 < 1 < \lambda < 2$ . Then given any open interval  $I \subset [0, 1]$  there is  $\phi \in \mathcal{G}(\phi_1, \phi_2)$  such that  $0 \in \phi(I)$ .

*Proof.* Note that  $\phi_2(\frac{\lambda_0}{\lambda}) = 0$ , then it is enough to prove that there is a map  $\varphi \in \mathcal{G}(\phi_1, \phi_2)$  such that  $\frac{\lambda_0}{\lambda} \in \varphi(I)$ . Then  $\phi = \phi_2 \circ \varphi$  satisfies the conclusion in the lemma. Let  $I = I_0$  be an open interval in  $[0, 1]$  that does not contain  $\frac{\lambda_0}{\lambda}$ . We will define inductively intervals  $I_n \subset [0, 1]$  as follows: if  $I_n$  does not contain  $\frac{\lambda_0}{\lambda}$  we let

$$I_{n+1} = \begin{cases} \phi_1(I_n) & \text{if } I_n \subset (0, \frac{1}{\lambda}], \\ \phi_2(I_n) & \text{if } I_n \subset (\frac{\lambda_0}{\lambda}, 1]. \end{cases}$$

Note that the length of  $|I_{n+1}|$  of  $I_{n+1}$  is  $\lambda|I_n|$ . Thus if the intervals  $I_n$  are inductively defined and do not contain  $\frac{\lambda_0}{\lambda}$  then  $|I_{n+1}| = \lambda^n |I_0|$ . Since  $\lambda > 1$ , this implies that there is a first  $n_0 > 0$  such that  $I_{n_0}$  contains  $\frac{\lambda_0}{\lambda}$ . Taking  $\varphi$  the composition of the corresponding  $\phi_i$  we prove the claim and thus the lemma.  $\square$

*Proof of Proposition 5.2.* Given an open set  $O$  in  $[0, 1]^2$ , there are intervals  $I_t, I_s \subset [0, 1]$  such that  $O \supset I_t \times I_s$ . Applying Lemma 5.3 to the intervals  $I_t$  and  $I_s$ , and to the maps  $(\phi_1, \phi_2)$  and  $(\psi_1, \psi_2)$ , respectively, we get maps

$$\phi = \phi_{i_n} \circ \cdots \circ \phi_{i_1} \in \mathcal{G}(\phi_1, \phi_2) \quad \text{and} \quad \psi = \psi_{j_m} \circ \cdots \circ \psi_{j_1} \in \mathcal{G}(\psi_1, \psi_2)$$

such that  $0 \in \phi(I_t)$  and  $0 \in \psi(I_s)$ , respectively. Suppose that  $n \leq m$ . Then, since  $\phi_1(0) = 0$ , to prove the proposition it is enough to consider the maps

$$\alpha_k = \begin{cases} (\phi_{i_k}, \psi_{j_k}) & \text{for } k = 1, \dots, n \\ (\phi_1, \psi_{j_k}) & \text{for } k = n+1, \dots, m, \end{cases}$$

and their composition.  $\square$

*Proof of Proposition 5.1.* Let  $B = B_0 = [t_1, t_2] \times [s_1, s_2] \times \{x\} \times [0, 1]$  be a vertical block to the right of  $W_{loc}^s(Q; F)$ . It is enough to prove that

$$F^N(B_0) \cap W_{loc}^s(Q; F) \neq \emptyset \quad \text{for some } N \geq 0. \quad (5.2)$$

Recall that  $a(B_0) = (t_2 - t_1) \cdot (s_2 - s_1) > 0$  is the area of the base of the block  $B_0$ . Write  $I_t = [t_1, t_2]$  and  $I_s = [s_1, s_2]$  and consider  $w_t(B_0) = t_2 - t_1$  and  $w_s(B_0) = s_2 - s_1$  the  $t$  and  $s$ -widths of the block.

First recall that  $W_{loc}^s(Q; F) = \{(0, 0)\} \times [0, 1] \times \{y_q\}$ . In the two last coordinates the action of the map  $F$  is given by the map  $f$ , more precisely, we have a contraction and an expansion in each coordinate, respectively. To get the intersection in (5.2), we need to prove that there is  $N > 0$  such that  $F^N(B_0) \supset \{(0, 0, x(N))\} \times [0, 1]$  for some  $x(N) \in [0, 1]$ . Indeed, this inclusion is given by Proposition 5.2. Let us go to the details. We call  $t$ -interval the first interval of a block and  $s$ -interval the second one.

The intervals  $I_t = I_t(0)$  and  $I_s = I_s(0)$ , that form the base of the block  $B_0$ , might satisfy one of the following cases:

- i)  $I_t \subseteq (0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$  and  $I_s \subseteq (0, \frac{\beta_0}{\beta}) \cup (\frac{\beta_0}{\beta}, \frac{1}{\beta}]$ ;
- ii)  $I_t \subseteq (0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$  and  $I_s \subseteq (\frac{\beta_0}{\beta}, 1]$ ;
- iii)  $I_t \subseteq (\frac{\lambda_0}{\lambda}, 1]$  and  $I_s \subseteq (0, \frac{\beta_0}{\beta}) \cup (\frac{\beta_0}{\beta}, \frac{1}{\beta}]$ ;

- iv)  $I_t \subseteq (\frac{\lambda_0}{\lambda}, 1]$  and  $I_s \subseteq (\frac{\beta_0}{\beta}, 1]$ ;
- v)  $I_t \subseteq (0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$  and  $I_s \ni \frac{\beta_0}{\beta}$ ;
- vi)  $I_t \subseteq (\frac{\lambda_0}{\lambda}, 1]$  and  $I_s \ni \frac{\beta_0}{\beta}$ ;
- vii)  $I_t \ni \frac{\lambda_0}{\lambda}$  and  $I_s \subseteq (0, \frac{\beta_0}{\beta}) \cup (\frac{\beta_0}{\beta}, \frac{1}{\beta}]$ ;
- viii)  $I_t \ni \frac{\lambda_0}{\lambda}$  and  $I_s \subseteq (\frac{\beta_0}{\beta}, 1]$ ;
- ix)  $I_t \ni \frac{\lambda_0}{\lambda}$  and  $I_s \ni \frac{\beta_0}{\beta}$ .

We claim that there are essentially three cases to consider (cases (ix), (v), and (i)) and that the other cases follow analogously.

**Case (ix):** If the intervals  $I_t$  and  $I_s$  satisfy case (ix), then  $F(B_0) \cap \mathbb{A}_4$  intersects  $W_{loc}^s(Q; F)$ , thus we are done and  $N = 1$ .

**Cases (v), (vi), (vii), and (viii):** First, if the intervals  $I_t$  and  $I_s$  satisfy case (v), then the  $s$ -interval of  $F(B_0) \cap \mathbb{A}_2$  contains 0, and  $F(B_0) \cap \mathbb{A}_2$  contains a  $t$ -vertical strip

$$\Delta^t(1) = I_t(1) \times \{s(1)\} \times \{x(1)\} \times [0, 1]$$

to the right of  $W_{loc}^s(Q; F)$  with  $w_t(\Delta^t(1)) \geq \lambda w_t(B_0)$ . There are three possibilities for  $\Delta^t(1)$ :

- $I_t(1) \subset [0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$ . Then  $F(F(B_0) \cap \mathbb{A}_2) \cap \mathbb{A}_2$  contains a  $t$ -vertical strip  $\Delta^t(2) = I_t(2) \times \{s(2)\} \times \{x(2)\} \times [0, 1]$  to the right of  $W_{loc}^s(Q; F)$  with  $w_t(\Delta^t(2)) \geq \lambda^2 w_t(B_0)$ .
- $I_t(1) \subset (\frac{\lambda_0}{\lambda}, 1]$ . Then  $F(F(B_0) \cap \mathbb{A}_2) \cap \mathbb{A}_4$  contains a  $t$ -vertical strip  $\Delta^t(2) = I_t(2) \times \{s(2)\} \times \{x(2)\} \times [0, 1]$  to the right of  $W_{loc}^s(Q; F)$  with  $w_t(\Delta^t(2)) \geq \lambda^2 w_t(B_0)$ .
- $I_t(1) \ni \frac{\lambda_0}{\lambda}$ . Then  $F(F(B_0) \cap \mathbb{A}_2) \cap \mathbb{A}_4$  intersects  $W_{loc}^s(Q; F)$ .

Arguing as in Lemma 5.3 and considering the  $t$ -vertical strip  $\Delta^t(2)$  there are three possibilities as above and we repeat the process until getting a first  $N$  such that the interval  $I_t(N)$  contains  $\frac{\lambda_0}{\lambda}$ , (this natural number  $N$  exists because  $\lambda > 1$ , then the width of the strips are increasing).

The three cases below follow similarly:

- for case (vi),  $F(B_0) \cap \mathbb{A}_4$  contains a  $t$ -vertical strip  $\Delta^t(1)$  to the right of  $W_{loc}^s(Q; F)$  with  $w_t(\Delta^t(1)) \geq \lambda w_t(B_0)$ .
- for case (vii),  $F(B_0) \cap \mathbb{A}_3$  contains a  $s$ -vertical strip  $\Delta^s(1)$  to the right of  $W_{loc}^s(Q; F)$  with  $w_s(\Delta^s(1)) \geq \beta w_s(B_0)$ .

- for case (viii),  $F(B_0) \cap \mathbb{A}_4$  contains a  $s$ -vertical strip  $\Delta^s(1)$  to the right of  $W_{loc}^s(Q; F)$  with  $w_s(\Delta^s(1)) \geq \beta w_s(B_0)$ .

**Cases (i), (ii), (iii), and (iv):** First, suppose that the intervals  $I_t$  and  $I_s$  of the block  $B_0$  satisfy case (i). Then  $F(B_0) \cap \mathbb{A}_1$  contains a vertical block  $B_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0, 1]$  to the right of  $W_{loc}^s(Q; F)$  with  $a(B_1) \geq \lambda \beta a(B_0)$ . There are nine possibilities analogous to (i)  $\sim$  (ix) for the intervals  $I_t(1)$  and  $I_s(1)$ . Arguing again as in Proposition 5.2 (this is possible since  $\lambda > 1$  and  $\beta > 1$ ) we repeat the process until getting a first  $N$  such that intervals  $I_t(N)$  and  $I_s(N)$  satisfy  $I_t(N) \ni \frac{\lambda_0}{\lambda}$  and  $I_s(N) \ni \frac{\beta_0}{\beta}$ . The number  $N$  exists because  $\lambda > 1$  and  $\beta > 1$  and thus then the area/width of the base/strips are increasing.

The three cases bellow are similar to case (i):

- for case (ii),  $F(B_0) \cap \mathbb{A}_2$  contains a vertical block  $B_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0, 1]$  to the right of  $W_{loc}^s(Q; F)$  with  $a(B_1) \geq \lambda \beta a(B_0)$ .
- for case (iii),  $F(B_0) \cap \mathbb{A}_3$  contains a vertical block  $B_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0, 1]$  to the right of  $W_{loc}^s(Q; F)$  with  $a(B_1) \geq \lambda \beta a(B_0)$ .
- for case (iv),  $F(B_0) \cap \mathbb{A}_4$  contains a vertical block  $B_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0, 1]$  to the right of  $W_{loc}^s(Q; F)$  with  $a(B_1) \geq \lambda \beta a(B_0)$ .

The proof of the proposition is now complete.  $\square$

## 5.2

### Cone fields and robustness

#### 5.2.1

##### Cone fields

Consider the map  $F$  defined in Equation 5.1,  $F: \mathbf{C} \rightarrow \mathbb{R}^{n+2}$ , where  $\mathbf{C} = [-1, 1]^2 \times [0, 1]^n$ , and the maximal invariant set of the map  $F$  in the cube  $\mathbf{C}$  is  $\Gamma$ . We call the set  $\Gamma$  a *model of blender*. Note that  $\Gamma$  has a dominated splitting (recall Section 2.1) of the form  $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$ , where  $\mathbb{X}$  and  $\mathbb{Y}$  are the strong stable and strong unstable bundles (corresponding to the stable and unstable bundles of the quotient horseshoe), respectively, and  $\mathbb{T}$  and  $\mathbb{S}$  the weak unstable bundles (corresponding to the weak expansions  $\lambda$  and  $\beta$ ). This dominated splitting can be extended (and we do) to the whole cube  $\mathbf{C}$ . To emphasize this domination, from now on, we will write the strong stable direction in the first coordinate and the strong unstable in the last one. That is if  $[0, 1]^n = [0, 1]^s + [0, 1]^u$ , where  $s$  is the dimension of the stable direction of

the horseshoe and  $u$  is the dimension of the unstable one, we write

$$\mathbf{C} = [0, 1]^s \times [-1, 1]^2 \times [0, 1]^u.$$

Consider cone fields  $\mathfrak{C}^s, \mathfrak{C}^{uu}, \mathfrak{C}^u, \mathfrak{C}^1$  and  $\mathfrak{C}^2$  defined as follows. Given  $x \in \mathbf{C}$  and the decomposition of the tangent bundle  $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$  over  $x$ , we consider (for simplicity the dependence on  $x$  is omitted)

$$\begin{aligned} \mathfrak{C}^s &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^1, v^2, v^u)\| < \|v^s\|\}, \\ \mathfrak{C}^{uu} &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^s, v^1, v^2)\| < \|v^u\|\}, \\ \mathfrak{C}^u &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|v^s\| < \|(v^1, v^2, v^u)\|\}, \\ \mathfrak{C}^1 &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^2, v^u)\| < \|(v^s, v^1)\|\}, \\ \mathfrak{C}^2 &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^s, v^1)\| < \|(v^2, v^u)\|\}. \end{aligned}$$

Due to the domination<sup>1</sup> of  $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$  we have that the cones  $\mathfrak{C}^s, \mathfrak{C}^{uu}, \mathfrak{C}^u, \mathfrak{C}^1$  and  $\mathfrak{C}^2$  are invariant under the derivative  $DF^{\pm 1}$ . More precisely, the following inclusions hold for all  $x \in \mathbf{C}$  with  $F^{\pm 1}(x) \in \mathbf{C}$ :

$$\begin{aligned} DF_{F(x)}^{-1} \mathfrak{C}^s(F(x)) &\subset \mathfrak{C}^s(x), \\ DF_x \mathfrak{C}^u(x) &\subset \mathfrak{C}^u(F(x)), \\ DF_x \mathfrak{C}^{uu}(x) &\subset \mathfrak{C}^{uu}(F(x)), \\ DF_{F(x)}^{-1} \mathfrak{C}^1(F(x)) &\subset \mathfrak{C}^1(x), \\ DF_x \mathfrak{C}^2(x) &\subset \mathfrak{C}^2(F(x)). \end{aligned}$$

Similarly as in the previous section, using these cones we now define *almost vertical disks, strips, and blocks* to the right of  $W^s(Q; F)$ . For simplicity we use the same notation as in the previous section for them. First, we define *top* and the *base* of the cube by:

$$\mathbf{C}_{\text{top}} \stackrel{\text{def}}{=} [0, 1]^s \times [-1, 1]^2 \times \{1^u\} \quad \text{and} \quad \mathbf{C}_{\text{base}} \stackrel{\text{def}}{=} [0, 1]^s \times [-1, 1]^2 \times \{0^u\}.$$

Analogously, the  $t$  and  $s$ -*top*, and the  $t$  and  $s$ -*base* of the cube are defined as follows:

$$\begin{aligned} \mathbf{C}_{\text{top}}^t &\stackrel{\text{def}}{=} [0, 1]^s \times \{1\} \times [-1, 1] \times [0, 1]^u, \\ \mathbf{C}_{\text{base}}^t &\stackrel{\text{def}}{=} [0, 1]^s \times \{-1\} \times [-1, 1] \times [0, 1]^u, \\ \mathbf{C}_{\text{top}}^s &\stackrel{\text{def}}{=} [0, 1]^s \times [-1, 1] \times \{1\} \times [0, 1]^u, \\ \mathbf{C}_{\text{base}}^s &\stackrel{\text{def}}{=} [0, 1]^s \times [-1, 1] \times \{-1\} \times [0, 1]^u. \end{aligned}$$

<sup>1</sup> As in Section 2.1, the splitting with four bundles  $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$  is dominated if the bundles  $\mathbb{X} \oplus (\mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y})$ ,  $(\mathbb{X} \oplus \mathbb{T}) \oplus (\mathbb{S} \oplus \mathbb{Y})$  and  $(\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S}) \oplus \mathbb{Y}$  are all dominated.



In what follows we consider disks and strips contained in  $\mathbf{C}$ .

- An *almost vertical disk*  $\varsigma$  to the right of  $W_{loc}^s(Q; F)$  is a disk (of dimension  $u$ ) tangent to  $\mathfrak{C}^{uu}$ , that is,  $T_x \varsigma \subset \mathfrak{C}^{uu}(x)$  for all  $x \in \varsigma$ , and such that  $\varsigma$  intersects the top and the base of  $\mathbf{C}$  (note that by “transversality” these intersections consist of just a point).
- An *almost  $t$ -vertical strip* to the right of  $W_{loc}^s(Q; F)$  is a  $(u + 1)$ -dimensional disk tangent to  $\mathfrak{C}^1$  and foliated by almost vertical segments.
- An *almost  $s$ -vertical strip* to the right of  $W_{loc}^s(Q; F)$  is a  $(u + 1)$ -dimensional disk tangent to  $\mathfrak{C}^2$  and foliated by almost vertical segments.
- An *almost vertical block*  $B$  to the right of  $W_{loc}^s(Q; F)$  is a  $(u + 2)$ -dimensional disk tangent to unstable cone field  $\mathfrak{C}^u$ , foliated by almost  $t$ -vertical strips and by almost  $s$ -vertical strips.

Given an almost  $t$ -vertical (resp.  $s$ -vertical) strip  $\Delta^t$  (resp.  $\Delta^s$ ), we define the  $t$ -width of  $\Delta^t$  (resp.  $s$ -width of  $\Delta^s$ ) as the infimum of the lengths of the curves  $\alpha$  contained in  $\Delta^t$  (resp.  $\Delta^s$ ) joining  $\mathbf{C}_{top}^t$  and  $\mathbf{C}_{base}^t$  (resp.  $\mathbf{C}_{top}^s$  and  $\mathbf{C}_{base}^s$ ) and tangent to  $\mathfrak{C}^1$  (resp.  $\mathfrak{C}^2$ ).

Given an almost vertical block  $B$  (which is foliated by  $t$ -vertical and  $s$ -vertical strips), we define the *area of the base of a block* as the product of the infimum of the  $t$ -width of  $t$ -vertical strips in  $B$  and the infimum of the  $s$ -width of  $s$ -vertical strips in  $B$ .

Next result is the version of Proposition 5.1 for almost vertical blocks:

**Lemma 5.4.** *The stable manifold of  $Q$  intersects every almost vertical block  $B$  to the right of  $W_{loc}^s(Q; F)$ .*

*Proof.* To show this result we argue as in the proof of Proposition 5.1. Let  $B$  be an almost vertical block to the right of  $W_{loc}^s(Q; F)$ . Due to the domination and the forward invariance of the unstable cones  $\mathfrak{C}^u$ , we have that the area of the base of  $B$  exponentially increases under forward iterations of  $F$  (in fact, this area is increasing in the rate  $\lambda\beta > 1$ ). Then arguing as in Proposition 5.1, we have that  $W^s(Q; F)$  intersects  $B$ . This ends the sketch of proof of the lemma.  $\square$

### 5.2.2

#### Robustness

In previous section we got invariant cone fields under the derivative  $DF^{\pm 1}$  associated to the sum of the bundles of the dominated splitting

$$\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}.$$

We note that this invariance is an open property, thus the cones are also  $DG^{\pm 1}$  invariant for every map  $G$  which is  $C^1$ -close to  $F$ .

Observe also that the maximal invariant set  $\Gamma$  of  $F$  is contained in the interior of the cube  $\mathbf{C}$ . Thus the same holds for the maximal invariant set of  $G$  in  $\mathbf{C}$  that we denote by  $\Gamma_G$  and call it the continuation of  $\Gamma$ . Note also that the continuation  $Q_G$  of the point  $Q$  for  $G$  is well defined. Thus we have analogous definitions for almost vertical disks,  $t$ -strips,  $s$ -strips and blocks to the right of  $W_{loc}^s(Q_G; G)$ . We also have that every map  $G$  close to  $F$  are expansions in the central coordinate, then the expansion properties (for  $G$ ) are satisfied for these sets. Therefore Lemma 5.4 holds for every map  $G$  sufficiently close to  $F$ . Thus we have proved the following:

**Proposition 5.5.** *For every map  $G$   $C^1$ -close enough to  $F$  there are defined the continuation  $Q_G$  of  $Q$  and almost vertical disks,  $t$ -strips,  $s$ -strips and blocks to the right of  $W_{loc}^s(Q_G; G)$ . Then the stable manifold of  $Q_G$  intersects every almost vertical block  $B$  to the right of  $W_{loc}^s(Q_G; G)$ .*