10
Robust cycles from symbolic blender-horseshoes

In this chapter we obtain robust heterodimensional cycles of any co-index (Definition 1.1) using symbolic blender-horseshoes (Definition 1.10). Our goal is prove Theorem E.

10.1
Robust symbolic cycles

To define symbolic cu-blender-horseshoes, we introduce the inverse symbolic skew product maps. Given $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$, for a bounded open subset D of a compact manifold G, constants $1 < \lambda < \beta$, $\alpha > 0$, and $k > 1$ (recall Definition 1.7), the symbolic skew product map

$$\Phi^* = \tau \ltimes \phi_\xi^* \in \mathcal{S}^\alpha_{k,\beta^{-1},\lambda^{-1}}(D),$$

where $\phi_\xi^* : D \to D$ is given by $\phi_\xi^*(x) = \phi_{\xi^*}^{-1}(x)$, is called *associated inverse skew product map for* Φ. Here ξ and ξ^* are points of Σ_k of the form: $\xi = (\ldots \xi_{-1}; \xi_0, \xi_1, \ldots)$ and $\xi^* = (\ldots \xi_1; \xi_0, \xi_{-1}, \ldots)$ (the conjugate sequence of ξ). Since $\tau(\xi)^* = \tau^{-1}(\xi^*)$, iterates of Φ^* correspond to iterates of Φ^{-1}. A *symbolic cu-blender-horseshoes* for Φ in $\mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ is a symbolic cs-blender-horseshoe for Φ^*.

There is a similar result for symbolic cu-blender-horseshoes of Corollary D.

Corollary 10.1. *Let* $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k) \in \mathcal{S}^{0,\alpha}_{k,\lambda_u,\beta_u}(D_u)$ *with* $1 < \lambda_u < \beta_u < \nu^{-\alpha}$. *Assume that there exists an open set* B_u *in* D_u *satisfying the covering property for* $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$. *Then the maximal invariant set* Γ_Φ *of* Φ *in* $\Sigma_k \times \overline{D}_u$ *is a symbolic cu-blender-horseshoe for* Φ *whose superposition domain contains* B_u.

Remark 10.2 (Continuations of blenders)**.** *Let* Γ^s_Φ *be a symbolic cs-blender-horseshoe (resp. cu-blender-horseshoe) for a map* $\Phi \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$ *(resp.* $\mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u)$*). We consider* Γ^s_Ψ *(resp.* Γ^u_Ψ*) the continuation of* Γ^s_Φ *(resp.* Γ^u_Φ*) for* $\Psi \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$ *(resp.* $\mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u)$*).*

Let Γ_Φ be a symbolic blender-horseshoe of Φ. Recall that

$$W^s(\Gamma_\Phi; \Phi) = \{(\xi, x) \in \Sigma_k \times G : \lim_{n \to \infty} d(\Phi^n(\xi, x), \Gamma_\Phi) = 0\} \quad \text{and}$$

$$W^u(\Gamma_\Phi; \Phi) = \{(\xi, x) \in \Sigma_k \times G : \lim_{n \to \infty} d(\Phi^{-n}(\xi, x), \Gamma_\Phi) = 0\}.$$

Definition 10.3 (Cycles associated to symbolic blenders). *Let Φ be a skew-product map defined on $\Sigma_k \times G$. Assume that there are open bounded subsets D_s and D_u of G such that*

- *$\Phi_s = \Phi_{|D_s} \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$ and $\Phi_u = \Phi_{|D_u} \in \mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_s)$, here $\Phi_{|D_i}$ denotes the restriction of Φ to D_i, $i = s, u$.*

- *Φ_s has a symbolic cs blender-horseshoe Γ^s_Φ contained in $\Sigma_k \times D_s$ and Φ_u has a symbolic cu blender-horseshoe Γ^u_Φ contained in $\Sigma_k \times D_u$.*

We say that Φ has a symbolic cycle *associated to Γ^s_Φ and Γ^u_Φ if their stable and unstable sets meet cyclically:*

$$W^s(\Gamma^s_\Phi; \Phi) \cap W^u(\Gamma^u_\Phi; \Phi) \neq \emptyset \quad \text{and} \quad W^u(\Gamma^s_\Phi; \Phi) \cap W^s(\Gamma^u_\Phi; \Phi) \neq \emptyset.$$

We say that this symbolic cycle is $\mathcal{S}$-robust if there is a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s) \cap \mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u)$ such that for every $\Psi \in \mathcal{V}$ it holds $\Psi_{|D_s} \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$, $\Psi_{|D_u} \in \mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_s)$, and Ψ has a symbolic cycle associated to the cs blender Γ^s_Ψ and the cu blender Γ^u_Ψ, where Γ^s_Ψ and Γ^s_Ψ are the continuations of Γ^s_Φ and Γ^u_Φ, respectively.

The main technical step of the proof of Theorem E is the following:

Proposition 10.4. *Let $\phi_1, \ldots, \phi_k, \varphi$ be $(\gamma, \hat{\gamma})$-Lipschitz C^1-diffeomorphisms on a compact Riemannian manifold G, recall Equation (1.3). Consider a one-step skew product map $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$ defined on $\Sigma_k \times G$. Assume that there are disjoint open and bounded subsets D_s, D_u of G such that*

$$\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k) \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s) \cap \mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u),$$

where $\alpha \in (0, 1]$, $\nu \in (0, 1)$, and

$$0 < \nu^\alpha < \gamma \leq \lambda_s < \beta_s < 1 < \lambda_u < \beta_u \leq \hat{\gamma} < \nu^{-\alpha}.$$

Assume also that there are open subsets $B_s \subset D_s$ and $B_u \subset D_u$ such that the following two properties hold.

i) Covering properties: $\overline{B}_s \subset \bigcup_{i=1}^k \phi_i(B_s)$ and $\overline{B}_u \subset \bigcup_{i=1}^k (\phi_i)^{-1}(B_u)$.

ii) Cyclic intersections: *there are a subset $\tilde{B}_s$ of B_s, points $x \in \tilde{B}_s$ and $y \in B_u$ such that*

$$\varphi^n(x) \in B_u, \quad \phi_j^m(y) \in D_s, \quad and \quad (\phi_\ell)^{-1}(y) \in D_u$$

for some $n, m > 0$ and $j, \ell \in \{1, \ldots, k\}$.

Then the one-step map $\hat{\Phi} = \tau \ltimes (\phi_1, \ldots, \phi_k, \varphi)^1$ defined on $\Sigma_{k+1} \times G$ has a robust symbolic cycle associated to a cs-blender-horseshoe Γ_Φ^s and to a cu-blender-horseshoe Γ_Φ^u contained in $\Sigma_k \times D_s$ and $\Sigma_k \times D_u$, respectively.

To prove this proposition we need the following lemma:

Lemma 10.5. *Let $\phi_1, \ldots, \phi_k, \varphi$ be $(\gamma, \hat{\gamma})$-Lipschitz be C^1-diffeomorphisms on a compact Riemannian manifold G. Consider a one-step skew product $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$ defined on $\Sigma_k \times G$ with a symbolic cs-blender-horseshoe Γ_Φ contained in $\Sigma_k \times D$, $D \subset G$. Assume that superposition domain of the blender contains an open subset B of D satisfying the covering property:*

$$\overline{B} \subset \phi_1(B) \cup \cdots \cup \phi_k(B).$$

Let $\hat{\Phi} = \tau \ltimes (\phi_1, \ldots, \phi_k, \varphi)$ be a skew product map defined on $\Sigma_{k+1} \times G$ such that

$$\Phi = \hat{\Phi}|_{\Sigma_k \times G}.$$

There are a small $\delta > 0$ and a subset $\tilde{B}$ of B such that every small perturbation $\hat{\Psi}$ of $\hat{\Phi}$ satisfies

$$W^{uu}(\Gamma_\Psi^s; \hat{\Psi}) \cap \hat{H}^s \neq \emptyset,$$

for every almost δ-horizontal disk $\hat{H}^s$ in $\Sigma_{k+1} \times B$, recall Definition 1.9, where Γ_Ψ^s is the continuation of Γ_Φ^s for $\Psi = \hat{\Psi}|_{\Sigma_k \times G}$.

Proof. First observe that for each $(\xi, x) \in \Sigma_{k+1} \times G$, we have that

$$W_{loc}^{uu}((\xi, x); \hat{\Phi}) = W_{loc}^u(\xi; \tau) \times \{x\},$$

for the one-step map $\hat{\Phi}$. Then, by shrinking slightly the superposition domain B we can assume that there is an open set $\tilde{B} \subset B$ such that for every $\hat{\Psi}$ close enough to $\hat{\Phi}$, it holds

$$W_{loc}^{uu}((\xi, x); \hat{\Psi}) \subset \Sigma_{k+1} \times B \quad \text{for all } (\xi, x) \in \Sigma_{k+1} \times \tilde{B}. \qquad (10.1)$$

We can also assume that $\tilde{B}$ satisfies the covering property for IFS(Φ).

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[1]Note that the maps ϕ_i correspond to the symbol i, for $i \in \{1, \ldots, k\}$, and the map φ correspond to the symbol $k + 1$.

A slight modification on the proof of Theorem C (using that $\nu^\alpha < \gamma$, instead of $\nu^\alpha < \lambda$) shows that there is a small $\delta > 0$ such that for every small perturbation $\hat{\Psi}$ of $\hat{\Phi}$ we have

$$\Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B}) \cap \hat{H}^s \neq \emptyset \tag{10.2}$$

for every δ-horizontal disk $\hat{H}^s$ in $\Sigma_{k+1} \times \tilde{B}$, where

$$\Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B}) \overset{\text{def}}{=} \bigcap_{n \geq 0} \hat{\Psi}^n(\Sigma^-_{k,k+1} \times \tilde{B})$$

and

$$\Sigma^-_{k,k+1} \overset{\text{def}}{=} \{\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_{k+1} : \xi_i \in \{1, \dots, k\} \text{ for every } i < 0\},$$

this means that the symbol $k + 1$ corresponding to iterations by φ is not involved.

Consider a skew-product map $\hat{\Psi}$ close to $\hat{\Phi}$ satisfying (10.1) and (10.2), and let $\Gamma^s_\Psi \subset \Sigma_k \times G$ be the continuation of Γ^s_Φ for $\Psi = \hat{\Psi}|_{\Sigma_k \times G}$.

Claim 10.6. $\Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B}) \subset W^{uu}_{loc}(\Gamma^s_\Psi; \hat{\Psi})$.

Proof. For a given $(\xi, x) \in \Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B})$ we will show that

$$W^{uu}_{loc}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times B) \subset \Gamma^s_\Psi. \tag{10.3}$$

Note that this inclusion implies the claim: if $(\zeta, z) \in W^{uu}_{loc}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times B)$ then by equation above, $(\zeta, z) \in \Gamma^s_\Psi$. Thus

$$W^{uu}_{loc}((\zeta, z); \hat{\Psi}) \subset W^{uu}_{loc}(\Gamma^s_\Psi; \hat{\Psi}).$$

Observe that $(\xi, x) \in W^{uu}_{loc}((\zeta, z); \hat{\Psi})$, then inclusion above implies that $(\xi, x) \in W^{uu}_{loc}(\Gamma^s_\Psi; \hat{\Psi})$, ending the proof of the claim.

To prove inclusion (10.3), recall that Γ^s_Ψ is a symbolic blender-horseshoe and thus, by Equation (1.11), we have that

$$\Gamma^+_\Psi(\Sigma_k \times B) \overset{\text{def}}{=} \bigcap_{n \geq 0} \Psi^n(\Sigma_k \times B) \subset \Gamma^s_\Psi \overset{\text{def}}{=} \bigcap_{n \in \mathbb{Z}} \Psi^n(\Sigma_k \times \overline{D}).$$

Hence, given $(\zeta, z) \in W^{uu}_{loc}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times B)$ it is enough to show that

$$\hat{\Psi}^{-n}(\zeta, z) = \Psi^{-n}(\zeta, z) \in \Sigma_k \times B, \quad \text{for all } n \geq 0.$$

Observe that from the invariance of the local strong unstable set we get

$$\hat{\Psi}^{-n}(\zeta, z) \in \hat{\Psi}^{-n}\big(W^{uu}_{loc}((\xi, x); \hat{\Psi})\big) \subset W^{uu}_{loc}(\hat{\Psi}^{-n}(\xi, x); \hat{\Psi}) \tag{10.4}$$

for all $n \geq 0$. Since $(\xi, x) \in \Gamma_{\hat{\Psi}}^+(\Sigma_{k,k+1}^- \times \tilde{B})$ we have that $\hat{\Psi}^{-n}(\xi, x) \in \Sigma_{k+1} \times \tilde{B}$ for all $n \geq 0$, thus by (10.1) it follows that

$$W_{loc}^{uu}(\hat{\Psi}^{-n}(\xi, x); \hat{\Psi}) \subset \Sigma_{k+1} \times B, \quad \text{for all } n \geq 0. \qquad (10.5)$$

Therefore $\zeta \in \Sigma_k$, (10.4) and (10.5) imply that $\hat{\Psi}^{-n}(\zeta, z) \in \Sigma_k \times B$ for all $n \geq 0$. Thus $(\zeta, z) \in \Gamma_{\Psi}^+(\Sigma_k \times B) \subset \Gamma_{\Psi}^s$ proving (10.3), and thus the claim. $\square$

The lemma follows noting that Claim 10.6 and Equation (10.2) imply that $W_{loc}^{uu}(\Gamma_{\Psi}^s; \hat{\Psi}) \cap \hat{H}^s \neq \emptyset$ for every δ-horizontal disk $\hat{H}^s$ in $\Sigma_{k+1} \times \tilde{B}$. $\square$

Proof of Proposition 10.4. By Corollaries D and 10.1, the maximal invariant sets Γ_{Φ}^s and Γ_{Φ}^u of Φ in $\Sigma_k \times \overline{D}_s$ and $\Sigma_k \times \overline{D}_u$, respectively, are a cs and a cu-blenders horseshoes whose superpositions domains contain B_s and B_u, respectively.

We split the proof of the proposition into several steps. First, we prove that $\hat{\Phi}$ has a symbolic cycle associated to Γ_{Φ}^s and Γ_{Φ}^u (see Claims 10.7, 10.8, and 10.9).

Claim 10.7. *Let $\vartheta = (\vartheta_i)_{i \in \mathbb{Z}} \in \Sigma_{k+1}$ with $\vartheta_i = \ell \in \{1, \ldots, k\}$ for all $i \in \mathbb{Z}$. Then $W_{loc}^s(\vartheta; \tau) \times D_s \subset W^s(\Gamma_{\Phi}^s; \hat{\Phi})$ and $W_{loc}^u(\vartheta; \tau) \times D_u \subset W^u(\Gamma_{\Phi}^u; \hat{\Phi})$.*

Proof. We see that $W_{loc}^s(\vartheta; \tau) \times D_s \subset W^s(\Gamma_{\Phi}^s; \hat{\Phi})$, the other inclusion is analogous. Let p_ℓ be the attracting fixed point of ϕ_ℓ in D_s. Note that $(\vartheta, p_\ell) \in \Gamma_{\Phi}^s$ and that for every $(\xi, x) \in W_{loc}^s(\vartheta; \tau) \times D_s$ one has that $d(\hat{\Phi}^n(\xi, x), (\vartheta, p_\ell)) \to 0$ as $n \to \infty$. Thus $(\xi, x) \in W^s(\Gamma_{\Phi}^s; \hat{\Phi})$, ending the proof of the claim. $\square$

To get a symbolic cycle for $\hat{\Phi}$ associated to Γ_{Φ}^s and Γ_{Φ}^u we will see that the invariant sets meet cyclically.

Claim 10.8. $W^s(\Gamma_{\Phi}^s; \hat{\Phi}) \cap W^u(\Gamma_{\Phi}^u; \hat{\Phi}) \neq \emptyset$.

Proof. By hypothesis, there are $y \in D_u$ and $m > 0$ such that $\phi_j^m(y) \in D_s$ and $\phi_\ell^{-1}(y) \in D_u$ for some $j, \ell \in \{1, \ldots, k\}$. Let

$$\zeta = (\ldots, \ell, \ell; j, \overset{m}{\ldots}, j, \ell, \ell, \ldots).$$

Note that $\tau^{-1}(\zeta) \in W_{loc}^u(\vartheta; \tau)$, where ϑ is the constant sequence with $\vartheta_i = \ell \in \{1, \ldots, k\}$ for all i. By Claim 10.7,

$$\hat{\Phi}^{-1}(\zeta, y) = (\tau^{-1}(\zeta), \phi_\ell^{-1}(y)) \in W_{loc}^u(\vartheta; \tau) \times D_s \subset W^u(\Gamma_{\Phi}^u; \hat{\Phi}).$$

Thus, $(\zeta, y) \in W^u(\Gamma_{\hat{\Phi}}^u; \hat{\Phi})$. Analogously we have,

$$\hat{\Phi}^m(\zeta, y) = \Phi^m(\zeta, y) = (\tau^m(\zeta), \phi_j^m(y)) \in W_{loc}^s(\vartheta; \tau) \times D_s \subset W^s(\Gamma_{\hat{\Phi}}^s; \hat{\Phi}),$$

obtaining that $(\zeta, y) \in W^s(\Gamma_{\hat{\Phi}}^s; \hat{\Phi})$. Thus $(\zeta, y) \in W^s(\Gamma_{\hat{\Phi}}^s; \hat{\Phi}) \cap W^u(\Gamma_{\hat{\Phi}}^u; \hat{\Phi})$. $\square$

Claim 10.9. $W^u(\Gamma_{\hat{\Phi}}^s; \hat{\Phi}) \cap W^s(\Gamma_{\hat{\Phi}}^u; \hat{\Phi}) \neq \emptyset$.

Proof. By hypothesis, there are $x \in B_s$, $n > 0$ such that $\varphi^n(x) \in B_u$. Since $\Gamma_{\hat{\Phi}}^u$ is a *cu*-blender-horseshoe we have that $B_u \subset \mathscr{P}(\Gamma_{\hat{\Phi}}^u)$. Hence there is $\xi \in \Sigma_k$ such that $(\xi, \varphi^n(x)) \in \Gamma_{\hat{\Phi}}^u$. Consider the sequence

$$\zeta = (\dots, 1, 1; k+1, .\overset{n}{.}., k+1, \xi_0, \dots, \xi_n, \dots) \in \Sigma_{k+1}.$$

As the strong stable set of $\Gamma_{\hat{\Phi}}^u$ is contained in its stable set we get

$$\hat{\Phi}^n\big(W_{loc}^s(\zeta; \tau) \times \{x\}\big) \subset W_{loc}^s(\xi; \tau) \times \{\varphi^n(x)\} = W^{ss}\big((\xi, \varphi^n(x)); \hat{\Phi}\big)$$
$$\subset W^s(\Gamma_{\hat{\Phi}}^u; \hat{\Phi}).$$

On the one hand, inclusion above implies that the horizontal disk $\hat{H}^s = W_{loc}^s(\zeta; \tau) \times \{x\}$ in $\Sigma_{k+1} \times B_s$ satisfies $\hat{H}^s \subset W^s(\Gamma_{\hat{\Phi}}^u; \hat{\Phi})$. On the other hand, by Lemma 10.5 we have that $W^u(\Gamma_{\hat{\Phi}}^s; \hat{\Phi}) \cap \hat{H}^s \neq \emptyset$. Therefore, $W^u(\Gamma_{\hat{\Phi}}^s; \hat{\Phi}) \cap W^s(\Gamma_{\hat{\Phi}}^u; \hat{\Phi}) \neq \emptyset$, proving the claim. $\square$

Consider a neighborhood $\mathcal{V}$ of $\hat{\Phi}$ such that for every $\hat{\Psi} \in \mathcal{V}$ it holds Lemma 10.5, covering property and cyclic intersection, that is,

- there are neighborhoods $\mathcal{U}_i^s$ of ϕ_i and $\mathcal{U}_i^u$ of ϕ_i^{-1} such that for $i = 1, \dots, k$

$$B_{s,i} = \text{int}\big(\cap_{\psi \in \mathcal{U}_i^s} \psi(B_s)\big) \quad \text{and} \quad B_{u,i} = \text{int}\big(\cap_{\psi \in \mathcal{U}_i^u} \psi(B_u)\big),$$

  are open covering of $\overline{B}_s$ and $\overline{B}_u$, respectively;

- fix $x \in \tilde{B}_s$ and $y \in B_u$, $n, m > 0$, and $j, \ell \in \{1, \dots, k\}$ in cyclic intersection condition. Then for every $\hat{\Psi} = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds

$$\psi_\eta^n(x) \in B_u, \quad \psi_{\hat{\eta}}^m(y) \in D_s \quad \text{and} \quad \psi_{\tilde{\eta}}^{-1}(y) \in B_u$$

  for every $\eta, \hat{\eta}, \tilde{\eta} \in \Sigma_{k+1}$ such that $\eta_0 = \dots = \eta_{n-1} = k+1$ and $\eta_i \in \{1, \dots k\}$ for $i < 0$ and $i \geq n$, $\hat{\eta}_0 = \dots = \hat{\eta}_{m-1} = j$ and $\hat{\eta}_i \in \{1, \dots k\}$ for $i < 0$ and $i \geq m$, and $\tilde{\eta}_0 = \ell$.

We will see that for every map in the neighborhood $\mathcal{V}$ the cyclic conditions hold, obtaining a robust symbolic cycle (Definition 10.3).

Let $\hat{\Psi} \in \mathcal{V}$ and Γ_Ψ^s and Γ_Ψ^u be the continuation of Γ_Φ^s and Γ_Φ^u, respectively for $\Psi = \hat{\Psi}|_{\Sigma_k \times G}$. We have the assertions below which are analogous to Claims 10.7, 10.8 and 10.9.

Claim 10.10. *Let $\vartheta = (\vartheta_i)_{i \in \mathbb{Z}} \in \Sigma_{k+1}$ with $\vartheta_i = \ell \in \{1, \ldots, k\}$ for all $i \in \mathbb{Z}$. Then $W_{loc}^s(\vartheta; \tau) \times D_s \subset W^s(\Gamma_\Psi^s; \hat{\Psi})$ and $W_{loc}^u(\vartheta; \tau) \times D_u \subset W^u(\Gamma_\Psi^u; \hat{\Psi})$.*

The proof is analogous to Claim 10.7 and thus it is omitted.

Claim 10.11. $W^s(\Gamma_\Psi^s; \hat{\Psi}) \cap W^u(\Gamma_\Psi^u; \hat{\Psi}) \neq \emptyset$.

Proof. Consider $\zeta_0 = \cdots = \zeta_{m-1} = j$ and the sequence

$$\zeta = (\ldots, \ell, \ell; \zeta_0, \zeta_1, \ldots, \zeta_{m-1}, \ell, \ell, \ldots).$$

Note that $\tilde{\eta} \stackrel{\text{def}}{=} \tau^{-1}(\zeta) \in W_{loc}^u(\vartheta; \tau)$, where ϑ is the constant sequence with $\vartheta_i = \ell \in \{1, \ldots, k\}$ for all i and, by hypothesis, $\psi_{\tilde{\eta}}^{-1}(y) \in B_u$. By Claim 10.10,

$$\hat{\Psi}^{-1}(\zeta, y) = \Psi^{-1}(\zeta, y) = (\tau^{-1}(\zeta), \psi_{\tilde{\eta}}^{-1}(y)) \in W^u(\Gamma_\Psi^u; \hat{\Psi}).$$

Thus, $(\zeta, y) \in W^u(\Gamma_\Psi^u; \hat{\Psi})$. Analogously we have $\tau^m(\zeta) \in W_{loc}^s(\vartheta; \tau)$ and $\psi_\zeta^m(y) \in D_s$. Then by Claim 10.10,

$$\hat{\Psi}^m(\zeta, y) = \Psi^m(\zeta, y) = (\tau^m(\zeta), \psi_\zeta^m(y)) \in W_{loc}^s(\vartheta; \tau) \times D_s \subset W^s(\Gamma_\Psi^s; \hat{\Psi}),$$

getting that $(\zeta, y) \in W^s(\Gamma_\Psi^s; \hat{\Psi})$. Thus $(\zeta, y) \in W^s(\Gamma_\Psi^s; \hat{\Psi}) \cap W^u(\Gamma_\Psi^u; \hat{\Psi})$. $\square$

Lemma 10.12. $W^u(\Gamma_\Psi^s; \hat{\Psi}) \cap W^s(\Gamma_\Psi^u; \hat{\Psi}) \neq \emptyset$.

Proof. Let $\eta = (\eta_i)_i \in \Sigma_{k+1}$ such that $\eta_0 = \cdots = \eta_{n-1} = k + 1$. By hypothesis $\psi_\eta^n(x) \in B_u$. Since Γ_Ψ^u is the continuation of the *cu*-blender-horseshoe Γ_Φ^u we have that $B_u \subset \mathscr{P}(\Gamma_\Psi^u)$. Hence there is $\xi \in \Sigma_k$ such that $(\xi, \psi_\eta^n(x)) \stackrel{\text{def}}{=} (\xi, w) \in \Gamma_\Psi^u$.

Claim 10.13. *Consider small $\delta > 0$ and the almost δ-horizontal disk $\hat{H}^s$ associated to $W_{loc}^s(\zeta; \tau) \times \{x\}$ (recall Definition 1.9) where*

$$\zeta \stackrel{\text{def}}{=} (\ldots, \eta_{-1}; k + 1, \overset{n}{\ldots}, k + 1, \xi_0, \ldots, \xi_{n-1}, \xi_n \ldots) \in \Sigma_{k+1}.$$

Then

$$\hat{H}^s \subset W^s((\xi, w); \hat{\Psi}) \subset W^s(\Gamma_\Psi^u; \hat{\Psi}).$$

Proof of the claim. Note that if $(\xi, w) \in \Gamma_\Psi^u$, then the second inclusion of the claim holds. Thus it is enough to see that for any given $(\mu, z) \in \hat{H}^s$ we have that $\hat{\Psi}^n(\mu, z) \in W^s((\xi, w); \hat{\Psi})$ (where n is as in hypothesis (ii)), that is,

$$d\big(\hat{\Psi}^m(\hat{\Psi}^n(\mu, z)), \hat{\Psi}^m(\xi, w)\big) \to 0 \quad \text{as } m \to \infty. \tag{10.6}$$

First note that by Definition 1.9, one has that $\mu \in W_{loc}^s(\zeta; \tau)$, thus $\tau^n(\mu)^+ = \xi^+$. Then

$$\lim_{m\to\infty} d\big(\hat{\Psi}^m(\hat{\Psi}^n(\mu, z)), \hat{\Psi}^m(\xi, w)\big) = \lim_{m\to\infty} d\big(\hat{\Psi}^m(\tau^n(\mu), \psi_\mu^n(z)), \hat{\Psi}^m(\xi, \psi_\eta^n(x))\big)$$

$$= \lim_{m\to\infty} \|\psi_{\tau^n(\mu)}^m(\psi_\mu^n(z)) - \psi_\xi^m(\psi_\eta^n(x))\|.$$

Fact 10.14. $\|\psi_\mu^n(z) - \psi_\eta^n(x)\|$ *is bounded, say,* $\|\psi_\mu^n(z) - \psi_\eta^n(x)\| \le M$.

Let $A \stackrel{\text{def}}{=} \max\{\beta, \nu^\alpha\}$ and note that $A < 1$. Let us prove that

$$\|\psi_{\tau^n(\mu)}^m(\psi_\mu^n(z)) - \psi_\xi^m(\psi_\eta^n(x))\| \le \beta^m M + C_\Psi \, m \, A^m. \tag{10.7}$$

To prove inequality above, first recall that $\mu \in W_{loc}^s(\zeta; \tau)$, then by definition of ζ, we have that $\tau^n(\mu)^+ = \xi^+$, and thus $d_{\Sigma_k}(\tau^n(\mu), \xi) \le \nu$.

Using the triangle inequality, β-contraction of ψ_μ, the Hölder property, Fact 10.14 and observation above, we have the following estimates for $m = 1$:

$$\|\psi_{\tau^n(\mu)}\big(\psi_\mu^n(z)\big) - \psi_\xi\big(\psi_\eta^n(x)\big)\| \le \|\psi_{\tau^n(\mu)}\big(\psi_\mu^n(z)\big) - \psi_{\tau^n(\mu)}\big(\psi_\eta^n(x)\big)\| +$$

$$+ \|\psi_{\tau^n(\mu)}\big(\psi_\eta^n(x)\big) - \psi_\xi\big(\psi_\eta^n(x)\big)\|$$

$$\le \beta \|\psi_\mu^n(z) - \psi_\eta^n(x)\| + C_\Psi \, d_{\Sigma_k}(\tau^n(\mu), \xi)^\alpha$$

$$\le \beta M + C_\Psi \nu^\alpha.$$

Using the same arguments it follows the estimate for $m = 2$:

$$\|\psi_{\tau^n(\mu)}^2\big(\psi_\mu^n(z)\big) - \psi_\xi^2\big(\psi_\eta^n(x)\big)\| =$$

$$= \|\psi_{\tau^{n+1}(\mu)} \circ \psi_{\tau^n(\mu)}\big(\psi_\mu^n(z)\big) - \psi_{\tau(\xi)} \circ \psi_\xi\big(\psi_\eta^n(x)\big)\|$$

$$\le \|\psi_{\tau^{n+1}(\mu)} \circ \psi_{\tau^n(\mu)}\big(\psi_\mu^n(z)\big) - \psi_{\tau^{n+1}(\mu)} \circ \psi_\xi\big(\psi_\eta^n(x)\big)\| +$$

$$+ \|\psi_{\tau^{n+1}(\mu)} \circ \psi_\xi\big(\psi_\eta^n(x)\big) - \psi_{\tau(\xi)} \circ \psi_\xi\big(\psi_\eta^n(x)\big)\|$$

$$\le \beta \|\psi_{\tau^n(\mu)}\big(\psi_\mu^n(z)\big) - \psi_\xi\big(\psi_\eta^n(x)\big)\| + C_\Psi \, d_{\Sigma_k}\big(\tau^{n+1}(\mu), \tau(\xi)\big)^\alpha$$

$$\le \beta\,(\beta M + C_\Psi \nu^\alpha) + C_\Psi \nu^{2\alpha} = \beta^2 M + C_\Psi \beta \nu^\alpha + C_\Psi \nu^{2\alpha}.$$

Using analogous estimates, we get that $\|\psi_{\tau^n(\mu)}^m(\psi_\mu^n(z)) - \psi_\xi^m(\psi_\eta^n(x))\|$ is smaller than

$$\beta^m M + C_\Psi \beta^{m-1} \nu^\alpha + C_\Psi \beta^{m-2} \nu^{2\alpha} + \cdots + C_\Psi \beta \nu^{(m-1)\alpha} + C_\Psi \nu^{m\alpha}.$$

Since $A = \max\{\beta, \nu^\alpha\} < 1$, we have that

$$\|\psi_{\tau^n(\mu)}^m(\psi_\mu^n(z)) - \psi_\xi^m(\psi_\eta^n(x))\| \le \beta^m M + C_\Psi \, m \, A^m.$$

Then Equation (10.7) is proved. Taking $m \to \infty$ in the same equation, we get (10.6), and thus that the almost horizontal disk $\hat{H}^s \subset W^s(\Gamma^u_{\hat{\Psi}}; \hat{\Psi})$, proving the claim. $\qquad\square$

On the other hand, by Lemma 10.5, we have that $W^u(\Gamma^s_{\hat{\Psi}}; \hat{\Psi}) \cap \hat{H}^s \neq \emptyset$. Therefore, $W^u(\Gamma^s_{\hat{\Psi}}; \hat{\Psi}) \cap W^s(\Gamma^u_{\hat{\Psi}}; \hat{\Psi}) \neq \emptyset$, proving the lemma. $\qquad\square$

Claim 10.11 and Lemma 10.12 imply that the invariant sets $\Gamma^s_{\hat{\Psi}}$ and $\Gamma^u_{\hat{\Psi}}$ meet cyclically for $\hat{\Psi}$, proving the proposition. $\qquad\square$

10.2
Heterodimensional cycles

In this section, we prove Theorem E. First, we need the following lemma which is a reformulation of a result in [19] using our terminology.

Lemma 10.15 (Proposition 2.3 in [19])**.** *Let D be a bounded subset of G, $\phi : \overline{D} \to D$ be a (λ, β)-Lipschitz map with $\nu^\alpha < \lambda < \beta < 1$ and p_ϕ its attracting fixed point. Then there are $k \in \mathbb{N}$, an open neighborhood B of p_ϕ, and translations (in local coordinates) $\phi_1 \overset{\text{def}}{=} \phi, \phi_2, \ldots, \phi_k$ of ϕ such that the maps $\phi_1, \ldots, \phi_k$ satisfy the covering property for the set B:*

$$\overline{B} \subset \phi_1(B) \cup \ldots \cup \phi_k(B).$$

We observe that the number k of translations of ϕ depends on the dimension of G and the constant λ.

From Lemma 10.15 and Corollary D we immediately obtain the following consequence.

Corollary 10.16. *Let $\phi_1, \phi_2, \ldots, \phi_k$ be as in Lemma 10.15. Then the skew-product map $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k) \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ has a symbolic cs-blender-horseshoe whose superposition domain contains B.*

Proof of Lemma 10.15. Consider the open ball $B_\varepsilon(p_\phi) \subset D$ of radius $\varepsilon > 0$ centered at p_ϕ. Note that there are $k > 0$ and points $d_1 = p_\phi, d_2, \ldots, d_k \in B_\varepsilon(p_\phi)$, such that

$$\overline{B}_\varepsilon(p_\phi) \subset B_{\frac{\lambda}{2}\varepsilon}(d_1) \cup B_{\frac{\lambda}{2}\varepsilon}(d_2) \cup \ldots \cup B_{\frac{\lambda}{2}\varepsilon}(d_k).$$

Consider (in local coordinates) translations ϕ_i of ϕ, such that $B_{\frac{\lambda}{2}\varepsilon}(d_i) \subset \phi_i\big(B_\varepsilon(p_\phi)\big)$, for all $i = 1, \ldots, k$. The choice of the points d_i and the inclusion above imply that

$$\overline{B}_\varepsilon(p_\phi) \subset \phi_1\big(B_\varepsilon(p_\phi)\big) \cup \ldots \cup \phi_k\big(B_\varepsilon(p_\phi)\big),$$

proving the lemma. □

Proof of Theorem E. By hypothesis, the map F has a horseshoe Λ in N. Suppose that the contraction rate of F is $0 < \nu < 1$. Fix positive constants $\gamma, \hat{\gamma}$ with $\nu < \gamma < 1 < \hat{\gamma} < \nu^{-1}$.

Consider $\phi\colon G \to G$ a small perturbation of the identity map such that ϕ has two hyperbolic fixed points p and q satisfying the properties:

i) p is a sink,

ii) q is a source, and

iii) $W^s(p, \phi) \pitchfork W^u(q, \phi) \neq \emptyset$.

There are disjoint neighborhoods D_s of p and D_u of q such that $\phi_{|D_s}$ and $\phi^{-1}_{|D_u}$ are (λ, β)-Lipschitz maps for some $\lambda < \beta < 1$, recall Definition 1.3, satisfying $\gamma < \lambda < \beta < 1 < \beta^{-1} < \lambda^{-1} < \hat{\gamma}$.

By Lemma 10.15 there are $k > 0$, open sets $B_s \subset D_s$ and $B_u \subset D_u$ containing p and q, respectively, and maps $\phi_1^s = \phi, \ldots, \phi_k^s$ and $\phi_1^u = \phi, \ldots, \phi_k^u$ such that

$$\overline{B}_s \subset \phi_1^s(B_s) \cup \ldots \cup \phi_k^s(B_s) \quad \text{and} \quad \overline{B}_u \subset (\phi_1^u)^{-1}(B_u) \cup \ldots \cup (\phi_k^u)^{-1}(B_u). \quad (10.8)$$

Consider the maps $\phi_i\colon G \to G$, $i = 1, \ldots, k$, as follows:

$$\phi_i(x) = \begin{cases} \phi_i^s(x) & \text{if } x \in D_s \\ \phi_i^u(x) & \text{if } x \in D_u. \end{cases}$$

Note that these maps are $(\gamma, \hat{\gamma})$-Lipschitz C^1-diffeomorphisms. Then (10.8) is equivalent to

$$\overline{B}_s \subset \bigcup_{i=1}^{k} \phi_i(B_s) \quad \text{and} \quad \overline{B}_u \subset \bigcup_{i=1}^{k} \phi_i^{-1}(B_u).$$

By construction,

$$\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k) \in \mathcal{S}_{k,\lambda_s,\beta_s}^{1,\alpha}(D_s) \cap \mathcal{S}_{k,\lambda_u,\beta_u}^{1,\alpha}(D_u),$$

where $\lambda_s = \lambda$, $\beta_s = \beta$, $\lambda_u = \beta^{-1}$ and $\beta_u = \lambda^{-1}$.

Let $\varphi\colon G \to G$ be a $(\gamma, \hat{\gamma})$-Lipschitz C^1-diffeomorphism such that $\varphi^n(x) \in B_u$ for some $x \in B_s$ and some $n \in \mathbb{N}$.

Claim 10.17. *The map $\hat{\Phi} = \tau \ltimes (\phi_1, \ldots, \phi_k, \varphi)$ has a symbolic cycle associated to a cs-blender-horseshoe and cu-blender-horseshoe whose superposition domains contain B_s and B_u, respectively.*

Proof. By condition (iii) there is a point $y \in B_u \cap (W^s(p, \phi_1) \cap W^u(q, \phi_1))$ such that $\phi_1^m(y) \in B_s$, for some $m \in \mathbb{N}$. Since $y \in B_u \cap (W^s(p, \phi_1) \cap W^u(q, \phi_1))$ we have that $\phi_1^{-1}(y) \in D_u$. Then adding the choice of φ above, the hypothesis of Proposition 10.4 are satisfied. Ending the proof of the claim. $\square$

To continue the proof, we need the following Proposition[2]:

Proposition 10.18. *[15, 13] Let f be a skew product diffeomorphism given by*

$$f \colon N \times G \to N \times G, \quad f(z, x) = \big(F(z), \phi_z(x) \big),$$

where $F \colon N \to N$ is a C^1-diffeomorphism with a horseshoe $\Lambda \subset N$ of d "legs" and with contraction rate $\nu < 1$, and $\phi_z(\cdot) \colon G \to G$ is a C^1-diffeomorphism such that

$$\gamma < \|D\phi_z(x)\, v\| < \hat{\gamma}$$

for $v \in T_x G$ and $z \in \Lambda$, where $\nu < \gamma < \hat{\gamma} < \nu^{-1}$.

Then for every map g C^1-close to f has a locally maximal invariant set $\Delta \subset N \times G$ homoeomorphic to $\Lambda \times G$ such that $g|_\Delta$ is conjugated to a symbolic skew-product map $\Psi_g \in \mathcal{S}_{d,\gamma,\hat{\gamma}}^{1,\alpha}(G)$, for $\alpha > 0$, recall Definition 1.7.

Recall that F has a horseshoe Λ in N. There is $\ell \in \mathbb{N}$ such that $F^\ell|_\Lambda$ is conjugated to a full shift of d symbols, for some $d > k$. Consider "rectangles" $R_1, \ldots, R_d$ in N such that $\{R_1 \cap \Lambda, \ldots, R_d \cap \Lambda\}$ is a Markov partition for $F^\ell|_\Lambda$.

Let $f : N \times G \to N \times G$ be the C^1-diffeomorphism such that

$$f_{|(R_i \cap \Lambda) \times G} = F^\ell \times \phi_i \quad \text{for } i = 1, \ldots, k$$
$$f_{|(R_i \cap \Lambda) \times G} = F^\ell \times \varphi \quad \text{for } i = k+1, \ldots, d.$$

The map f restricted to $\Lambda \times G$ is conjugated to the symbolic one-step skew product map $\hat{\Phi} = \tau \ltimes (\phi_1, \ldots, \phi_k, \varphi, \ldots, \varphi) \in \mathcal{S}_{d,\gamma,\hat{\gamma}}^{1,\alpha}(G)$. To emphasize this conjugacy between f and $\hat{\Phi}$, we denote $f \stackrel{\text{def}}{=} f_{\hat{\Phi}}$.

Therefore, by Claim 10.17 and Proposition 10.18 we have that $f_{\hat{\Phi}}$ has a robust heterodimensional cycle associated with hyperbolic sets Γ_f^s and Γ_f^u, which came from the symbolic *cs*-blender-horseshoe and the *cu*-blender-horseshoe.

The proof of the theorem is now complete. $\square$

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[2]This proposition is detailed in [1].