9
Symbolic blenders in the Hölder setting

In this section we prove Theorem C. First, we introduce some notation and preliminary results.

Given a finite word $\bar{\omega} = \omega_{-m} \dots \omega_{-1}\, \omega_0\, \omega_1 \dots \omega_n$, where $m, n \geq 0$ and $\omega_i \in \{1, \dots, k\}$, we define the *bi-lateral cylinder* by

$$\mathcal{C}_{\bar{\omega}} \stackrel{\text{def}}{=} \{\xi \in \Sigma_k \colon \xi_j = \omega_j,\ -n \leq j \leq n\}.$$

Given $\zeta \in \Sigma_k$ and a word $\bar{\omega} := \bar{\omega}_{-n} = \omega_{-n} \dots \omega_{-1}$, where $n \geq 1$ and $\omega_i \in \{1, \dots, k\}$, we define the *relative cylinder* by

$$\mathcal{C}_{\bar{\omega}}(\zeta) \stackrel{\text{def}}{=} \{\xi \in W^s_{loc}(\zeta; \tau) : \xi_{-i} = \omega_i,\ \text{for } i = 1, \dots, n\}. \tag{9.1}$$

Recall that $\mathcal{S} = \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}$ is the set of symbolic skew product maps in Definition 1.7. Let us observe that in what follows $\nu^\alpha < \lambda < 1$, $\alpha > 0$, and there is no restriction on β. In the next lemma we estimate the distance between the backward orbits of a point x when iterated by different maps ψ_ξ^{-1}.

Lemma 9.1. *Consider $\Psi = \tau \ltimes \psi_\xi \in \mathcal{S}$, a word $\bar{\omega} = \omega_{-n} \dots \omega_0 \dots \omega_n$, and a point $x \in \overline{D}$ such that for every $\zeta \in \mathcal{C}_{\bar{\omega}}$ one has that $\psi^{-j}_{\tau^{-1}(\zeta)}(x) \in \overline{D}$ for every $1 \leq j \leq n$. Then it holds*

$$\left\| \psi^{-i}_{\tau^{-1}(\xi)}(x) - \psi^{-i}_{\tau^{-1}(\zeta)}(x) \right\| < C_\Psi\, \nu^{\alpha(n-i)} \sum_{j=0}^{i-1} (\lambda^{-1} \nu^\alpha)^j,$$

for all $1 \leq i \leq n$ and all $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$.

Proof. The proof is by induction. For $i = 1$, the Hölder inequality (1.4) and $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$ imply that

$$\|\psi^{-1}_{\tau^{-1}(\xi)}(x) - \psi^{-1}_{\tau^{-1}(\zeta)}(x)\| \leq C_\Psi d_{\Sigma_k}(\tau^{-1}(\xi), \tau^{-1}(\zeta))^\alpha \leq C_\Psi\, \nu^{\alpha(n-1)}.$$

We argue inductively. Suppose that the lemma holds for $i - 1$, $i < n$:

$$\left\| \psi^{-(i-1)}_{\tau^{-1}(\xi)}(x) - \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x) \right\| < C_\Psi\, \nu^{\alpha(n-i+1)} \sum_{j=0}^{i-2} (\lambda^{-1} \nu^\alpha)^j, \tag{9.2}$$

for every $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$. We will see that the estimate also holds for i. By the triangle inequality, one has that

$$\begin{aligned}\left\|\psi^{-i}_{\tau^{-1}(\xi)}(x) - \psi^{-i}_{\tau^{-1}(\zeta)}(x)\right\| &\leq \left\|\psi^{-i}_{\tau^{-1}(\xi)}(x) - \psi^{-1}_{\tau^{-i}(\xi)} \circ \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x)\right\| + \\ &+ \left\|\psi^{-1}_{\tau^{-i}(\xi)} \circ \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x) - \psi^{-i}_{\tau^{-1}(\zeta)}(x)\right\|.\end{aligned}$$

Since the inverse of these functions expand at most $1/\lambda$, we get that the above equation is less than or equal to

$$\frac{1}{\lambda}\left\|\psi^{-(i-1)}_{\tau^{-1}(\xi)}(x) - \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x)\right\| + \left\|\psi^{-1}_{\tau^{-i}(\xi)}(y) - \psi^{-1}_{\tau^{-i}(\zeta)}(y)\right\|,$$

where $y = \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x) \in \overline{D}$. By induction hypothesis (9.2) we obtain

$$\frac{1}{\lambda}\left\|\psi^{-(i-1)}_{\tau^{-1}(\xi)}(x) - \psi^{-(i-1)}_{\tau^{-1}(\zeta)}(x)\right\| \leq C_\Psi\, \lambda^{-1}(\nu^\alpha)^{n-i+1} \sum_{j=0}^{i-2} (\lambda^{-1}\,\nu^\alpha)^j.$$

As $y \in \overline{D}$ applying the Hölder inequality (1.4) and since $\xi, \zeta \in \mathcal{C}_{\bar{\omega}}$ we get

$$\|\psi^{-1}_{\tau^{-i}(\xi)}(y) - \psi^{-1}_{\tau^{-i}(\zeta)}(y)\| \leq C_\Psi\, \nu^{\alpha(n-i)}.$$

Putting togheter the previous inequalities we get

$$C_\Psi\, \lambda^{-1}(\nu^\alpha)^{n-i+1} \sum_{j=0}^{i-2} (\lambda^{-1}\,\nu^\alpha)^j + C_\Psi\, \nu^{\alpha(n-i)} = C_\Psi\, \nu^{\alpha(n-i)} \sum_{j=0}^{i-1} (\lambda^{-1}\,\nu^\alpha)^j,$$

ending the proof of the lemma. □

9.1 Proof of Theorem C

Consider a one-step map $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k) \in \mathcal{S}$ and an open subset B of D. Recall that we need to prove the following:

> B has the covering property for $\mathcal{G}_{\phi_1,\ldots,\phi_k} \iff$ there are $\delta > 0$ and a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}$ such that $\Gamma^+_\Psi(\Sigma_k \times B) \cap H^s \neq \emptyset$ for every $\Psi \in \mathcal{V}$ and every δ-horizontal disk H^s in $\Sigma_k \times B$.

$\Longleftarrow$ We see that if the covering property is not satisfied then intersection (1.10) is also not satisfied. If B does not satisfy the covering property then there is $x \in \overline{B}$ such that $x \notin \phi_i(B)$ for all $i = 1, \ldots, k$. First note that we can assume that $x \in B$. Otherwise, we can take an arbitrarily small perturbation $\Psi = \tau \ltimes (\psi_1, \ldots, \psi_k)$ of Φ such that the covering property in B for

IFS$(\psi_1, \dots, \psi_k)$ is not satisfied for a point in B. The condition $x \notin \phi_i(B)$ for all $i = 1, \dots, k$ implies that $\Phi^{-1}(\xi, x) \notin \Sigma_k \times \overline{B}$ for all $\xi \in \Sigma_k$ and hence

$$(\xi, x) \notin \bigcap_{n \geq 0} \Phi^n(\Sigma_k \times \overline{B}) \stackrel{\text{def}}{=} \Gamma^+_\Phi(\Sigma_k \times B) \quad \text{for all } \xi \in \Sigma_k.$$

Therefore $\Gamma^+_\Phi(\Sigma_k \times B)$ does not meet the horizontal disk $H^s = W^s_{loc}(\xi; \tau) \times \{x\}$, and thus the intersection property (1.10) is not verified.

$\Longrightarrow$ We split the proof of the fact that the covering property implies the intersection condition into two steps.

Choice of the neighborhood $\mathcal{V}$ of Φ. First recall that given an open covering $\mathcal{C}$ of a compact set X of a metric space there is a constant $L > 0$, a *Lebesgue number* of $\mathcal{C}$, such that every subset of X with diameter less than L is contained in some member of $\mathcal{C}$.

Let $2\,L > 0$ be a Lebesgue number of the covering $\{\phi_1(B), \dots, \phi_k(B)\}$ of the set $\overline{B}$. Note that there are C^0-neighborhoods $\mathcal{U}_i$ of ϕ_i such that the family

$$B_i = \operatorname{int}\Big(\bigcap_{\psi \in \mathcal{U}_i} \psi(B) \Big), \qquad i = 1, \dots, k, \tag{9.3}$$

is an open covering of $\overline{B}$. By shrinking the size of the sets $\mathcal{U}_i$ we can assume that L is a Lebesgue number of this covering. We can also assume that any $\psi \in \mathcal{U}_i$ is a C^0-(λ, β)-Lipschitz map on $\overline{D}$ for all $i = 1, \dots, k$.

We take a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}$ such that if $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ then $\psi_\xi \in \mathcal{U}_{\xi_0} = \mathcal{U}_i$. In that case, by (9.3), we get that

$$\psi^{-1}_{\tau^{-1}(\xi)}(\overline{B}_{\xi_0}) \subset B \quad \text{for all } \xi \in \Sigma_k. \tag{9.4}$$

Since Φ is a one-step map then $\phi_\xi = \phi_\zeta$ if $\xi_0 = \zeta_0$, hence we can take the Hölder constant $C_\Phi = 0$. The definition of the distance in (1.6) implies that C_Ψ is close to $C_\Phi = 0$. Since, by hypothesis, $\nu^\alpha < \lambda$, by shrinking the neighborhood $\mathcal{V}$ we can assume that for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds

$$C_\Psi \sum_{i=0}^{\infty} (\lambda^{-1} \nu^\alpha)^i < L/2. \tag{9.5}$$

This completes the choice of the neighborhood $\mathcal{V}$ of Φ.

Existence of a point in $\Gamma^+_\Psi(\Sigma_k \times B) \cap H^s$. The main step is the following proposition.

Proposition 9.2. *Let $\mathcal{V}$ the neighborhood of Φ above. Consider small $\delta > 0$ and a δ-horizontal disk H^s associated to $W^s_{loc}(\zeta; \tau) \times \{z\}$ for some $(\zeta, z) \in \Sigma_k \times B$. Then there are a sequence of nested compact subsets $\{V_n\}$ of B contained in $\mathscr{P}(H^s)$ and a sequence $\xi \in W^s_{loc}(\zeta; \tau)$ such that for all $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it* *holds*

$$\psi^{-n}_{\tau^{-1}(\xi)}(V_n) \subset B \quad \textit{and} \quad \operatorname{diam}\big(\psi^{-n}_{\tau^{-1}(\xi)}(V_n)\big) \to 0.$$

Let us see how the implication ($\Longrightarrow$) follows from this proposition. Let $\{x\} = \cap_n V_n$. By the first part of the proposition $\psi^{-n}_{\tau^{-1}(\xi)}(x) \in B$ for all $n \in \mathbb{N}$ and thus $\Psi^{-n}(\xi, x) \in \Sigma_k \times B$ for all $n \in \mathbb{N}$ and hence $(\xi, x) \in \Gamma^+_\Psi(\Sigma_k \times B)$. Note that since $\xi \in W^s_{loc}(\zeta;\tau)$ and $x \in \mathscr{P}(H^s)$ then we also have $(\xi, x) \in H^s$. Thus $\Gamma^+_\Psi(\Sigma_k \times B) \cap H^s \neq \emptyset$.

To complete the proof of Theorem C it remains to prove the proposition.

Proof of Proposition 9.2. Consider $\delta > 0$ such that $\lambda^{-1}\delta < L/2$ for the δ-horizontal disk H^s associated to $W^s_{loc}(\zeta;\tau) \times \{z\}$ and the (α, C)-Hölder graph map h (see Definition 1.9). The construction of the nested sequence of sets $\{V_n\}$ and the point $\xi \in W^s_{loc}(\zeta;\tau)$ is done inductively. Let

$$V := \mathscr{P}(H^s) \subset B.$$

Note that $\operatorname{diam}(V) \leq 2\delta < L$. Thus, by the definiton of the Lebesgue number, $V \subset B_{i_1}$ for some $i_1 \in \{1, \ldots, k\}$. Recall the definition of the relative cylinder in (9.1) associated to $\zeta \in \Sigma_k$ and the word $\bar{\omega}_{-1} = i_1$ and consider the set

$$V_1 := \mathscr{P}\big(H^s \cap (\mathcal{C}_{\bar{\omega}_{-1}}(\zeta) \times V)\big).$$

By construction, $V_1 \subset V \subset B_{i_1}$. Thus, by (9.4), for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ one has that

$$\psi^{-1}_{\tau^{-1}(\xi)}(V_1) \subset B. \tag{9.6}$$

Claim 9.3. $\operatorname{diam}(V_1) \leq \delta_1 \stackrel{\text{def}}{=} C\,\nu^{2\,\alpha}$.

Proof. Given x and y in V_1 there are ξ and η in $\mathcal{C}_{\bar{\omega}_{-1}}(\zeta)$ such that $x = h(\xi)$ and $y = h(\eta)$. Since h is (α, C)-Hölder continuous we have

$$\|x - y\| = \|h(\xi) - h(\eta)\| \leq C d_{\Sigma_k}(\xi, \eta)^\alpha \leq C\,\nu^{2\,\alpha} = \delta_1,$$

proving the claim. □

By Claim 9.3 and since for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ the maps ψ_ξ are (λ, β)-Lipschitz we have that

$$\operatorname{diam}\big(\psi^{-1}_{\tau^{-1}(\xi)}(V_1)\big) \leq \lambda^{-1}\delta_1 \quad \text{for all } \xi \in \mathcal{C}_{\bar{\omega}_{-1}}(\zeta).$$

Recalling that $C\nu^\alpha < \delta$ (see Definition 1.9) we get

$$\lambda^{-1}\delta_1 = \lambda^{-1}C\nu^{2\alpha} \leq \lambda^{-1}C\nu^\alpha < \lambda^{-1}\delta \leq L/2.$$

Therefore

$$\operatorname{diam}\big(\psi^{-1}_{\tau^{-1}(\xi)}(V_1)\big) \le \lambda^{-1}\delta_1 \le L/2.$$

Arguing inductively, suppose that we have constructed words $\bar{\omega}_{-n} := \omega_{-n}\dots\omega_{-1}$ (the word $\bar{\omega}_{-i}$ is obtained adding the letter ω_{-i} to the word $\bar{\omega}_{-i+1}$) and closed sets $V_n \subset V_{n-1} \subset \cdots \subset V_1$ with

$$\operatorname{diam}(V_n) \le C\nu^{(n+1)\alpha} \stackrel{\text{def}}{=} \delta_n \tag{9.7}$$

and such that for every $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ one has that for all $\xi \in \mathcal{C}_{\bar{\omega}_{-n}}(\zeta)$ it holds

$$\psi^{-n}_{\tau^{-1}(\xi)}(V_n) \subset B \quad \text{and} \quad \operatorname{diam}\big(\psi^{-n}_{\tau^{-1}(\xi)}(V_n)\big) \le \lambda^{-n}\delta_n. \tag{9.8}$$

We now construct the word $\bar{\omega}_{-(n+1)}$ and the closed set $V_{n+1} \subset V_n$ satisfying analogous inclusions and inequalities. By (9.8) we have that

$$A_n \stackrel{\text{def}}{=} \bigcup_{\xi \in \mathcal{C}_{\bar{\omega}_{-n}}(\zeta)} \psi^{-n}_{\tau^{-1}(\xi)}(V_n) \subset B.$$

Claim 9.4. $\operatorname{diam}(A_n) < L$.

Proof. Given $\bar{x}$ and $\bar{y}$ in A_n there are $x, y \in V_n$ and $\xi, \eta \in \mathcal{C}_{\bar{\omega}_{-n}}(\zeta)$ such that $\bar{x} = \psi^{-n}_{\tau^{-1}(\xi)}(x)$ and $\bar{y} = \psi^{-n}_{\tau^{-1}(\eta)}(y)$. Then

$$\begin{aligned}
\|\bar{x} - \bar{y}\| &= \|\psi^{-n}_{\tau^{-1}(\xi)}(x) - \psi^{-n}_{\tau^{-1}(\eta)}(y)\| \\
&\le \|\psi^{-n}_{\tau^{-1}(\xi)}(x) - \psi^{-n}_{\tau^{-1}(\eta)}(x)\| + \|\psi^{-n}_{\tau^{-1}(\eta)}(x) - \psi^{-n}_{\tau^{-1}(\eta)}(y)\| \\
&\le C_\Psi \sum_{j=0}^{n-1} (\lambda^{-1}\nu^\alpha)^j + \lambda^{-n}\delta_n \qquad (9.9) \\
&\le L/2 + \lambda^{-n}\delta_n, \qquad (9.10)
\end{aligned}$$

where (9.9) follows from Lemma 9.1 and induction hypothesis (9.8), and the last inequality (9.10) follows from (9.5). Note also that

$$\lambda^{-n}\,\delta_n = \lambda^{-n}\,C(\nu^\alpha)^{n+1} \le C\,(\lambda^{-1}\nu^\alpha)^n \le C\,\lambda^{-1}\nu^\alpha < \lambda^{-1}\delta < L/2.$$

Therefore for every pair of points $\bar{x}, \bar{y} \in A_n$ we have $\|\bar{x} - \bar{y}\| < L$ and thus $\operatorname{diam}(A_n) < L$, proving the claim. □

As L is a Lebesgue number of the covering $\{B_i\}_{i=1}^k$, the claim implies that there is $i_{n+1} \in \{1, \dots, k\}$ such that $A_n \subset B_{i_{n+1}}$. We let

$$\bar{\omega}_{-(n+1)} = i_{n+1}\omega_{-n}\dots\omega_{-1} \quad \text{and} \quad V_{n+1} = \mathscr{P}\big(H^s \cap (\mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta) \times V_n)\big).$$

Note that by construction $V_{n+1} \subset V_n$.

Claim 9.5. $\mathrm{diam}(V_{n+1}) \leq C\,\nu^{(n+2)\,\alpha} \stackrel{\mathrm{def}}{=} \delta_{n+1}$.

Proof. Just note that given $x, y \in V_{n+1}$ there are $\xi, \eta \in \mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta)$ such that $x = h(\xi)$ and $y = h(\eta)$. From the (α, C)-Hölder continuity of h and since $\xi, \eta \in \mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta)$ we get

$$\|x - y\| \leq C d_{\Sigma_k}(\xi, \eta)^{\alpha} \leq C\nu^{(n+2)\alpha}.$$

Thus $\mathrm{diam}(V_n) \leq C\,\nu^{(n+2)\alpha} = \delta_{n+1}$. □

Using $V_{n+1} \subset V_n$, $\mathrm{diam}(V_n) \leq \delta_{n+1}$, and Equations (9.4) and (9.8) we get that for all $\Psi = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds

$$\psi_{\tau^{-1}(\xi)}^{-(n+1)}(V_{n+1}) \subset B \quad \text{and} \quad \mathrm{diam}\Big(\psi_{\tau^{-1}(\xi)}^{-(n+1)}(V_{n+1})\Big) \leq \lambda^{-(n+1)}\delta_{n+1}, \qquad (9.11)$$

for every $\xi \in \mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta)$. Therefore (9.8) holds for $n+1$-step and we can continue arguing inductively. This completes the construction of the sequence of nested sets V_n in the proposition. Observe that the sequence ξ whose positive part is ζ and whose negative part satisfies $\xi_{-n} = \omega_{-n}$ belongs to $\mathcal{C}_{\bar{\omega}_{-(n+1)}}(\zeta) \subset W^s_{loc}(\zeta; \tau)$. This completes the proof of the proposition. □

The proof of Theorem C is now complete. □