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Proof of the main theorem

In order to use the methods developed in the first section we have to reduce the formula of the Jacobi equation in corollary 5.1.4 to a first order equation. So, consider the Fermi coordinates along a radial geodesic α_v passing through p .

Given a vector field along α_v of the type $\eta(t) = x(t)V(t)$ where $x : (-\rho - \delta, \rho + \delta) \rightarrow \mathbb{R}$ and V is the parallel vector field along α_v introduced in the definition of the Fermi coordinates. This vector field was chosen to be perpendicular along α_v .

Lemma 7.0.3. *If $x(-\rho) = 0$ and x is not trivial then, in the Fermi coordinates of corollary 5.1.4, the Jacobi operator applied on the perpendicular vector field $\eta(t) = x(t)V(t)$ is given by*

$$J^\sigma \eta = \left(x''(t) - \frac{f'(t)}{\Lambda_0} x'(t) + (K_v(t) + \frac{\lambda^2(t)}{\Lambda_0^2} \frac{f'(t)}{|t|} - \frac{f'(t)}{2\Lambda_0} (C_{112})_{x^2} x(t)) \right) V(t), \quad (7.0.1)$$

where λ is a smooth positive function defined on $[-\rho, \rho]$ with $\lambda(0) = 1$.

Proof. First of all, a Jacobi field η along a geodesic α_v satisfies the relation

$$g_{\alpha'_v}(\alpha'_v(t), \eta(t)) = g_{\alpha'_v}(\alpha'_v(-\rho), \eta(-\rho)) + (t + \rho) g_{\alpha'_v}(\alpha'_v(-\rho), \eta'(-\rho)). \quad (7.0.2)$$

If we multiply both sides of the equation above by $e^{2\sigma}$ we obtain the same relation for the Finsler metric F_σ . Then, the vector field η is perpendicular to α_v for both metrics, g and g^σ . Therefore, given a parallel vector field W along α_v associated with the metric g^σ such that $g_{\alpha'_v}^\sigma(\dot{\alpha}'_v(s), W(s)) = 0$, there exists a function $y = y(s)$ such that $\eta = yW$, where the dot corresponds to the derivative in the new time parameter. Now, using the Chern-Rund connection formalism

$$J^\sigma \eta = D_{\dot{\alpha}_v} D_{\dot{\alpha}_v} \eta + R^\sigma(\eta, \dot{\alpha}_v) \dot{\alpha}_v$$

but, from the parallelism of W , $D_{\dot{\alpha}_v} W = 0$ and

$$R^\sigma(\eta, \dot{\alpha}_v) \dot{\alpha}_v = K_v^\sigma(s) y(s) W(s).$$

So, the vector $J^\sigma \eta$ is parallel to W which is parallel to V because of equation (7.0.2).

Since $\eta = x(t)V(t)$ we have that $J_{1i}^\sigma \eta^1 = 0$. From what was done above we have that $J_{21}^\sigma \eta^2 = 0$ because the vector $J^\sigma \eta$ is parallel to V . After all this reductions we conclude that

$$J^\sigma \eta = J_{22}^\sigma \eta^2 V(t).$$

To finish the proof we have to calculate the derivative $\sigma_{x^2 x^2}$. According to the definition of σ , using the Fermi coordinate system defined on equation (5.1.2) on a square contained inside the ball $\mathcal{B}_p(\rho)$,

$$\sigma(x^1, x^2) = \sigma_0((\exp_p)^{-1}(x^1, x^2)),$$

where $(x^1(s, t), x^2(s, t)) = \pi \circ \phi_s(\alpha_v(t), V(t))$. Consider the vector field along tv defined by $\bar{V}(t) = d(\exp_p^{-1})_{\alpha_v(t)} V(t)$. Gauss lemma implies that $\bar{V}(t) = \lambda(t)\bar{V}_0$, where $\bar{V}_0$ is a unitary vector perpendicular to v and $\lambda > 0$ such that $\lambda(0) = 1$.

Therefore, we have that

$$\begin{aligned} \sigma_{x^2 x^2}(x^1(t, 0), x^2(t, 0)) &= \frac{d^2}{ds^2} \Big|_{s=0} (\sigma_0(tv + s\bar{V}(t))) \\ &= -\frac{\lambda^2(t) f'(t)}{2\Lambda_0^2 |t|}. \end{aligned}$$

□

Lemma 7.0.4. *If there is a non trivial solution y of equation*

$$y''(t) + \left(K_v(t) + \frac{f'(t)}{|t|} \right) y(t) = 0 \quad (7.0.3)$$

such that $y(-\rho) = y(t_0) = 0$, $t_0 \in (-\rho, \rho]$, then the geodesic α_v has conjugate points.

Proof. The equation

$$x''(t) - \frac{f'(t)}{\Lambda_0} x'(t) + (K_v(t) + \frac{\lambda^2(t) f'(t)}{\Lambda_0^2 |t|} - \frac{f'(t)}{2\Lambda_0} (C_{112})_{x^2}) x(t) = 0 \quad (7.0.4)$$

can be seen as a first order linear system of the type

$$X'(t) = F(t, X, \epsilon), \quad (7.0.5)$$

where $X = (x, p)$, $p = x'$, and F is linear in the X variable. Observe that, although the dependence on the parameter ϵ is no explicit in (7.0.4), it appears if we look to the definition of f . Moreover, $f, f' \rightarrow 0$ uniformly when $\epsilon \rightarrow 0$.

Change the ϵ dependence associated with the terms $\frac{f'(t)}{\Lambda_0}$ and $\frac{f'(t)}{2\Lambda_0}(C_{112})_{x^2}$ by $\bar{\epsilon}$ such that $\bar{\epsilon} \in [0, \epsilon)$. Call these new terms $g_i(t, \bar{\epsilon})$, $i = 1, 2$, respectively. It is clear that this term also goes uniformly to zero when $\bar{\epsilon} \rightarrow 0$. We will obtain another second order equation (now with parameters $(\epsilon, \bar{\epsilon})$)

$$x''(t) - g_1(t, \bar{\epsilon})x'(t) + \left(K_v(t) + \frac{\lambda^2(t)}{\Lambda_0^2} \frac{f'(t)}{|t|} - g_2(t, \bar{\epsilon}) \right) x(t) = 0 \quad (7.0.6)$$

which will linearize to

$$X'(t) = \bar{F}(t, X, \epsilon, \bar{\epsilon}). \quad (7.0.7)$$

Recall that we have to choose the constant ρ with respect to ϵ and this will only depend on the infimum of the curvatures inside the coordinate ball chosen.

Consider the solution $\bar{X}(t) = \bar{X}(t; \epsilon, \bar{\epsilon})$ of (7.0.7) with initial conditions $\bar{X}(-\rho) = (0, 1)$. The continuous dependence on initial conditions and parameters (cf. [8], p. 58) implies that, if, for some $t_0 \in (-\rho, \rho)$ and $\bar{\epsilon} = 0$, $\bar{X}(t_0) = (0, -\tau)$ then there exists $\delta > 0$ such that for $\bar{\epsilon} \in [0, \delta)$ there is $\bar{t}_0 = \bar{t}_0(\bar{\epsilon})$ with $\bar{X}(\bar{t}_0; \epsilon, \bar{\epsilon}) = (0, -\bar{\tau})$. Actually, by continuity, this will work also for $\bar{\epsilon} = \delta$. Then, applying several times the continuous dependence theorem we can cover the interval $[0, \epsilon)$ and finally conclude that the solution $X(t) = X(t; \epsilon)$ of 7.0.5 with the same initial conditions of $\bar{X}$ also has a zero at some $t_1 > -\rho$.

To finish the proof, just observe that $\frac{\lambda^2}{\Lambda_0^2} \geq 1$ and apply Sturm comparison theorem. $\square$

7.1

Proof of theorem 1.0.1

Let $\epsilon > 0$. If $\epsilon > \bar{\epsilon} > 0$, let $\beta \in (0, 1)$, $\rho(\bar{\epsilon}, \beta)$ and $\alpha(\bar{\epsilon}, \beta)$ be the ones obtained in proposition 4.0.11. These constants together with the Finsler metric F determine the functions σ_0 and σ . Rename σ as σ_β . Because of 6.1.2 we can suppose that

$$\|1 - e^{\sigma_\beta}\|_1 < \frac{\epsilon}{\|F\|_1}.$$

Define $\bar{F} = \frac{\rho}{a}F$. Since $K_v(t) \geq K_0^\rho$, we have that $K_v(t) + \frac{f'(t)}{|t|} \geq K_{\epsilon, \alpha, \beta}^\rho$ and, therefore, by proposition 4.0.11, equation (7.0.3) has conjugate points. So, lemma 7.0.4, implies that the geodesic has conjugate points for every $v \in T_p^1 M$.

Given a geodesic γ of $\bar{F}_\beta = e^{\sigma_\beta} \bar{F}$ such that $\gamma(0) = p$, there exist $0 < t_0 \leq \rho$ such that the length of gamma restricted to $[-t_0, t_0]$ is strictly bigger than $d(\gamma(-t_0), \gamma(t_0))$. This fact is invariant by rescaling, so, all the geodesics of $e^{\sigma_\beta} F = F_\beta$ containing p have conjugate points.

Finally, observe that

$$\|F - e^\sigma F\|_1 \leq \|1 - e^\sigma\|_1 \|F\|_1 < \epsilon.$$

To finish the proof, Lemma 6.1.2 implies that $\|\sigma_\beta\| < \epsilon$.