6
Construction of conformal radial perturbations

Choose any point $p \in M$ and consider the totally geodesic ball $\mathcal{B}_p(a+\delta)$ centered at p with radius $a+\delta$. It follows that the exponential map $\exp_p$ restricted to $B(a+\delta) \subset \mathbb{R}^2$ is a diffeomorphism over $\mathcal{B}_p(a+\delta)$.

Given $v \in T^1_p M$ consider the unitary geodesic $\alpha_v(t) = \exp_p(tv)$, where $t \in [-a, a]$. For each of these geodesics define

$$K_v(t) = K(\alpha_v(t), \alpha'_v(t)),$$

where $K(\alpha_v(t), \alpha'_v(t))$ is the flag curvature of (M, F) at $\alpha_v(t)$ with flagpole $\alpha'_v(t)$.

In order to define the perturbation consider

$$K_0 := \inf\{K_v(t) | v \in T^1_p M,\, t \in [-a, a]\}$$

and, since we are inside a geodesic ball, all geodesic of α_v type has no conjugate points in the interval $[-a, a]$ we have that, by proposition 4.0.11, there exists $\rho(\epsilon) > 0$ such that for every $\beta \in (0, 1)$ there is an $\alpha(\rho) > 0$ and a function $K^{\rho}_{\epsilon,\alpha,\beta}$ such that the equation

$$x''(t) + K^{\rho}_{\epsilon,\alpha,\beta} x(t) = 0$$

has conjugate points on $[-\rho, \rho]$.

We will stretch out the Finsler metric F to obtain a new metric $\bar{F}$ and prove the result for the metric. The metric $\bar{F}$ is given by

$$\bar{F} = \frac{\rho}{a} F.$$

6.1
Building the σ function

For every geodesic α_v consider the parallel vector field V along α_v such that $F(\alpha_v, V) = 1$ and $\{\alpha'_v, V\}$ is a positive basis. If $\bar{V}(t) = d(\exp_v^{-1})_{\alpha_v} V(t)$ then, by the Gauss lemma, there is a function $\lambda_v(t)$ such that $\bar{V}(t) = \lambda_v(t)\bar{V}_0$

where $\bar{V}_0 \in T_pM$ is a unitary vector perpendicular to v. Set the constant Λ_0 as

$$\Lambda_0 = \inf_{v \in T^1_pM} \{|\lambda_v(t)|\,;\, t \in [-\rho, \rho]\}. \tag{6.1.1}$$

It follows that this constant is positive.

Here, we will also define a singular perturbation σ_0 in the spirit of lemma 2.1.1 in chapter 2. The definition of both functions is very similar. So, we will do them together.

First, for $i = 0, 1$, define the functions $\widetilde{\sigma}_i : B(\rho + \delta) \to \mathbb{R}$ by

$$\widetilde{\sigma}_i = -\frac{1}{2\Lambda_0^2} f_i\left(\sqrt{(x^1)^2 + (x^2)^2}\right),$$

where

$$f_0(t) = \int_0^t \epsilon\lambda(s)\, ds - \epsilon\frac{(\rho + \frac{\delta}{2})^{1+\beta}}{(\beta + 1)},$$

$$\lambda_0(t) = \begin{cases} t^\beta, & t > 0; \\ -(-t)^\beta, & t < 0, \end{cases}$$

and

$$f_1(t) = \int_0^t \lambda(s)\, ds - \epsilon\frac{(\rho + \frac{\delta}{2})^{1+\beta}}{(\beta + 1)}, \tag{6.1.2}$$
$$\lambda(t) = \epsilon\, \alpha^{\beta - 1}\delta_\alpha(t)t + (1 - \delta_\alpha(t))\lambda_0(t),$$

where $\alpha = \alpha(\epsilon)$ is given in lemma 4.1.6 and δ_α in (4.1.8).

Now, define the function $\widetilde{\sigma}_i : B(\rho + \delta) \to \mathbb{R}$ by

$$\widetilde{\sigma}_i(x^1, x^2) = -\frac{1}{2\Lambda_0^2} f_i\left(\sqrt{(x^1)^2 + (x^2)^2}\right). \tag{6.1.3}$$

We are ready to define the functions σ and σ_0 necessary to the main theorem.

Let $\sigma_i^1 : \mathcal{B}_p(\rho + \delta) \to \mathbb{R}$ defined by

$$\sigma_i^1(q) = \widetilde{\sigma}_i\left((\exp_p)^{-1}(q)\right).$$

Define $\sigma_i : M \to \mathbb{R}$ by

$$\sigma_i(q) = \begin{cases} \beta_{\frac{\delta}{2}}(q)\sigma_i^1(q) & \text{if } q \in \mathcal{B}_p(\rho + \delta); \\ 0 & \text{if } q \in M \setminus \mathcal{B}_p(\rho + \delta), \end{cases} \tag{6.1.4}$$

where $\beta_{\frac{\delta}{2}}$ is a bump function which is 1 for $q \in \mathcal{B}_p(\rho)$ and zero for $q \in \mathcal{B}_p(\rho + \delta) \setminus \mathcal{B}_p(\rho + \frac{\delta}{2})$.

To set notations, let

$$\sigma_1 = \sigma.$$

This is the smoothed version of the singular σ_0.

Remark 6.1.1. *Recall that the exponential function is only C^1 around the origin. In order to avoid this kind of problems here and keep the generality, the function κ is zero in a neighbourhood of zero and consequentially the function σ is also zero in a neighbourhood of the origin. So, the problems with differentiability of σ at p won't bother us.*

Lemma 6.1.2. $||\sigma_0||_{1,\beta} \to 0$ *when* $\epsilon \to 0$.

Proof. Outside the ball $\mathcal{B}_p(\rho+\delta)$, σ_0 is zero. Then the C^0 of σ_0 is at most

$$\frac{\epsilon(\rho+\delta)^{1+\beta}}{2(1+\beta)\Lambda_0^2}.$$

The norm of $\nabla\sigma_0$ satisfies

$$||\nabla\sigma_0|| \leq \frac{\epsilon\,\rho^\beta\,h}{2(1+\beta)\Lambda_0^2},$$

where h is constant, uniform on M, that relates the Riemannian metric g that our manifold is equipped and some Finsler F that we will use to make our construction. And finally, we have to analyse the $\beta-$Hölder norm of $\nabla\sigma_0$. The function $l(t) = |t|^\beta$ is the prototype of $\beta-$Hölder function and has $||l||_\beta \leq 2^{1-\beta}$ on the interval $[-\rho-\delta, \rho+\delta]$. Thefore,

$$||\nabla\sigma_0||_\beta \leq 2^{1-\beta}\frac{\epsilon^\beta h^\beta}{2^\beta(1+\beta)^\beta\Lambda_0^{2\beta}}.$$

Since ρ is of order $\epsilon^{-\frac{1}{2+\beta}}$ we have that $\epsilon\rho^{1+\beta} \to 0$ and $\epsilon\rho^\beta \to 0$. □