3
Preliminaries

Let M be a smooth manifold and $\pi : TM \to M$ the canonical projection. The slit tangent bundle $\widetilde{TM} := TM \setminus 0$, where 0 is a notation for the zero section. Similarly, consider the dual tangent bundle T^*M and the corresponding objects $\bar{\pi} : T^*M \to M$, $\widetilde{T^*M}$.

The pulled-back vector bundle π^*TM over $\widetilde{TM}$ is given by

$$\pi^*TM := \bigcup_{(x,v)\in\widetilde{TM}} T_xM.$$

In the same way we define π^*T^*M.

Remark 3.0.1. *In what follows,* $(x^1, ..., x^n) = (x^i) : U \subset M \to \mathbb{R}^n$ *is a local coordinate chart on a open set* U*. As usual,* $\{\frac{\partial}{\partial x^i}\}$ *and* $\{dx^i\}$ *are, respectively, the induced coordinate basis of* TM *and* T^*M*. The coordinate chart* (x^i) *give rise to a coordinate chart* (x, y) *on* $\pi^{-1}(U) \subset TM$*, where* $y = y^i \frac{\partial}{\partial x^i}$*.*

Throughout this text we will use the notation $G_{x^i} = \frac{\partial G}{\partial x^i}$ *and* $G_{y^j} = \frac{\partial G}{\partial y^j}$ *for the partial derivatives of a function* $G : V \subset TM \to \mathbb{R}$*.*

Definition 3.0.2. *A Finsler manifold* (M, F) *is a smooth manifold* M *equipped with a Finsler metric* F*. A* C^k $(k \geq 2)$ *Finsler metric is continuous map,* $F : TM \to [0, \infty)$, C^k *on* $\widetilde{TM}$ *such that*

i) F *is positively homogeneous, that is,* $F(x, \lambda v) = \lambda F(x, v)$ *for all* $\lambda > 0$ *and* $(x, v) \in TM$*.*

ii) If $F(x, v) = 0$ *then* $v = 0$*.*

iii) Legendre condition. The coefficients

$$g_{ij}(x, v) = \frac{1}{2}(F^2)_{y^i y^j}(x, v)$$

form a positive definite matrix for all $(x, v) \in \widetilde{TM}$*.*

In addition to the items above, if the metric F satisfies $F(x, v) = F(x, -v)$ then it is called *reversible*.

The coefficients g_{ij} define a natural Riemannian metric on the pulled-back vector bundle π^*TM by

$$g = g_{ij}\, dx^i \otimes dx^j.$$

There is another important quantity associated with the Finsler structure called the Cartan tensor. Define it by

$$C := C_{ijk}\, dx^i \otimes dx^j \otimes dx^k,$$

where

$$C_{ijk} = \frac{1}{4}(F^2)_{y^i y^j y^k}.$$

It is a symmetric section of $\otimes^3 \pi^*TM$ and if it vanishes then the Finsler metric is Riemannian, that is, $F = \sqrt{\alpha}$ with α a Riemannian metric on M.

Given a Lipschitz continous curve $c : [a, b] \to M$ we define the *length* of c by

$$\ell_F(c) = \int_a^b F(c(t), \frac{dc}{dt}(t))\, dt.$$

This give rise to a pseudo-metric by

$$d_F(p, q) = \inf_c \ell_F(c)$$

where the infimum is taken over all the Lipschitz continuous curves c such that $c(a) = p$ and $c(b) = q$. If F is reversible then d_F is a metric.

Definition 3.0.3. *A curve $\sigma : [a, b] \to M$ is said to be minimizing if*

$$\ell_F(\sigma) = d_F(\sigma(a), \sigma(b)).$$

3.1 Lagrangian viewpoint

The main source for this section is [10] (see also [1] and [12]). Along this section, we will call $L = \frac{1}{2}F^2$ and, unless explicitly said, every curve will be piecewise C^2.

We are interested in necessary conditions on $c : [a, b] \to M$ such that it realize the Finslerian distance between its extremals, that is,

$$\ell_F(c) = d_F(c(a), c(b)).$$

Define the *energy* of the curve c by

$$\mathcal{E}(c) = \int_a^b L(c(t), \frac{dc}{dt}(t))\, dt.$$

The next lemma shows that we can study minimizers of the energy in order to find curves that realizes the Finslerian distance between two points.

Lemma 3.1.1. *Suppose that $\sigma : [a,b] \to M$ a curve joining two points of M such that for every $c : [a,b] \to M$, $c(a) = \sigma(a)$ and $c(b) = \sigma(b)$, $\mathcal{E}(\sigma) \leq \mathcal{E}(c)$. In this case,*

$$\ell_F(\sigma) = d_F(\sigma(a), \sigma(b)).$$

A piecewise C^2 variation of a smooth curve $\sigma : [a,b] \mapsto M$ is a continuous map

$$\bar{\sigma} : [a,b] \times (-\varepsilon, \varepsilon) \to M$$

which is C^2 on each $[t_{i-1}, t_i] \times (-\varepsilon, \varepsilon)$ and $\bar{\sigma}(t,0) = \sigma(t)$.

The first order term of the Taylor expansion of $E(s) = \mathcal{E}(\bar{\sigma}_s)$ around $s = 0$ is called the *first variation of the energy*. It is given by

$$\delta\mathcal{E}(c)[\eta] = \left.\frac{dE(s)}{ds}\right|_{s=0}$$

where $\eta = \left.\frac{\partial\sigma}{\partial s}\right|_{s=0}$ is a piecewise C^1 vector field. If we suppose that each part of c is contained in a given coordinate chart, then

$$\delta\mathcal{E}(c)[\eta] = -\sum_k \int_{t_{k-1}}^{t_k} \left(\frac{d}{dt}(L_{y^i}) - L_{x^i}\right) \eta^i\, dt.$$

The system of equations

$$\left(\frac{d}{dt}(L_{y^i}) - L_{x^i}\right) = 0 \tag{3.1.1}$$

is called *Euler-Lagrange* equations associated with the Lagrangian L. In fact, this system of equations does not depend on the coordinate chart chosen, that is, the system behaves well under a change of normal coordinates[1].

[1]Actually, these equations define a *spray* on $\widetilde{TM}$, that is, a vector field locally defined by $y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where the functions G^i are 2-homogeneous on the second coordinate and behave under change of coordinates as $2\widetilde{G^i} = \frac{\partial \widetilde{x^i}}{\partial x^j} 2G^j - \frac{\partial \widetilde{y^i}}{\partial y^j} y^j$. The orbits of this vector field are the lifts to $\widetilde{TM}$ of the solutions of the Euler-Lagrange equation. In the case of a Finsler metric, $2G^i(x,y) = \frac{1}{2} g^{ij} [\frac{\partial^2 F^2}{\partial x^k \partial y^j}(x,y) y^k - \frac{\partial F^2}{\partial x^j}]$. For details about sprays in Finsler geometry see [5].

Lemma 3.1.2. *Given $x \in M$ and $v \in T_xM$ there exists $\varepsilon > 0$ and a unique solution*

$$\sigma : [0, \varepsilon) \to M$$

of the system (3.1.1) *such that $\sigma(0) = x$ and $\sigma'(0) = v$.*

Given a curve σ, if it satisfies the Euler-Lagrange equations then the curve is called a *constant speed geodesic* or simply a *geodesic*. By constant speed we mean that $F(\sigma(t), \sigma'(t)) = k$ for all t, where $k > 0$.

Remark 3.1.3. *We should mention that if, instead of the energy, we had considered the length function then the solutions of the Euler-Lagrange equations would not, necessarily, had constant speed. All in all, they would be reparametrization of the constant speed ones.*

Definition 3.1.4. *The flow ϕ_t of the vector field over $\widetilde{TM}$ given by* (3.1.1) *is called geodesic flow. For a point $(x, v) \in \widetilde{TM}$, $\phi_t(x, v) = (\sigma(t), \sigma'(t))$ where σ is a geodesic such that $\sigma(0) = x$ and $\sigma'(0) = v$.*

Lemma 3.1.5. *Let $\sigma : [a, b] \to M$ be a curve connecting p and q. We have the following equivalence: for all piecewise C^1 vector field η along σ such that $\eta(a) = \eta(b) = 0$*

$$\delta\mathcal{E}(\sigma)[\eta] = 0$$

if and only if σ is a geodesic.

Suppose that σ is geodesic. In this case, the first order term of the Taylor expansion relative to the variation $\bar{\sigma}$ vanishes and it is the second order term of the Taylor expansion of $E(s)$ that gives information about minimizing properties of σ.

The second order term is given by

$$\delta^2\mathcal{E}_{p,q}(\sigma)[\eta] := \left.\frac{d^2E(s)}{ds^2}\right|_{s=0}$$

and, supposing that each part of σ is contained in a coordinate chart, we have

$$\delta^2\mathcal{E}_{p,q}(c)[\eta] = -\sum_k \int_{t_{k-1}}^{t_k} \left(\frac{d}{dt}\left(L_{y^iy^j}\frac{d\eta^i}{dt} + L_{x^iy^j}\eta^i\right) + \left(L_{y^ix^j}\frac{d\eta^i}{dt} + L_{x^ix^j}\eta^i\right)\right)\eta^j \, dt.$$

Under a coordinate change, the equation

$$\frac{d}{dt}\left(L_{y^iy^j}\frac{d\eta^i}{dt} + L_{x^iy^j}\eta^i\right) + \left(L_{y^ix^j}\frac{d\eta^i}{dt} + L_{x^ix^j}\eta^i\right)$$

does not change. This leads to the following definition.

Definition 3.1.6. *Let $\mathfrak{X}_p^k(\sigma)$ be the space of piecewise C^k vector fields along the geodesic σ. Define the differential operator $J : \mathfrak{X}_p^2(\sigma) \to \mathfrak{X}_p^0(\sigma)$, called the Jacobi operator, in terms of its local expression by*

$$J_{ij}\eta^i = \frac{d}{dt}\left(L_{y^iy^j}\frac{d\eta^i}{dt} + L_{x^iy^j}\eta^i\right) + \left(L_{y^ix^j}\frac{d\eta^i}{dt} + L_{x^ix^j}\eta^i\right). \tag{3.1.2}$$

A vector field $\eta \in \mathfrak{X}_p^2(\sigma)$ is called a Jacobi field if it satisfies the Jacobi equation

$$J\eta = 0.$$

Jacobi fields appears when we vary geodesics by geodesics. In other words, consider a piecewise C^2 variation $\bar{\sigma}$ of the geodesics σ such that $\bar{\sigma}_s(t) = \bar{\sigma}(t, s)$ is a geodesic for every s. The vector field obtained by $\eta = \frac{\partial \bar{\sigma}}{\partial s}\big|_{s=0}$, called the *variational field*, is a Jacobi field.

Definition 3.1.7. *Given $p, q \in M$. We say that p is conjugated to q if there exists a geodesic $\sigma : [a, b] \to M$, with $\sigma(a) = p$ and $\sigma(b) = q$, and a non-vanishing Jacobi field η along σ such that $\eta(a) = \eta(b) = 0$.*

Conjugate points plays an important role in the study of minimizing geodesics.

Proposition 3.1.8. *Let $\sigma : [a, b] \to M$ be a geodesic and suppose that there is $\bar{b} \in (a, b)$ such that $\sigma(a)$ is conjugated to $\sigma(\bar{b})$. Then there exists a curve $c : [a, b] \to M$, with $c(a) = \sigma(a)$ and $c(b) = \sigma(b)$, such that*

$$\ell_F(c) < \ell_F(\sigma).$$

Therefore, to prove that a geodesic do not realize the distance between two points we can study its Jacobi equation and try to find a non-trivial Jacobi field over the geodesic that has two zeros.

3.2 Geometrical viewpoint

Now we will introduce some concepts of Finsler geometry. The main sources for this section are [13], [20] and [4]. For a open neighborhood $U \subset M$ let $\Gamma(TU)$ and $\Gamma(\widetilde{TU})$ be the space of smooth sections of these vector bundles. By a *affine connection* ∇^V we mean a bi-linear map

$$\nabla^V : \Gamma(TU) \times \Gamma(TU) \to \Gamma(TU),$$

$(X, Y) \mapsto \nabla_X^V Y$, satisfying

$$\nabla_X^V(fY) = X(f)Y + f\nabla_X^V Y$$

for all $f \in C^\infty(U)$ and $X, Y \in \Gamma(TU)$.

Theorem 3.2.1 (Chern)**.** *Let (M, F) be a Finsler manifold. There is a map*

$$\nabla : \Gamma(\widetilde{TU}) \times \Gamma(TU) \times \Gamma(TU) \to \Gamma(TU) \qquad (3.2.1)$$

$(V, X, Y) \mapsto \nabla_X^V Y$ *with the following properties*

i) For every $V \in \Gamma(\widetilde{TU})$, the map ∇^V is an affine connection;

ii) ∇^V is torsion free, that is,

$$\nabla_X^V Y - \nabla_Y^V X = [X, Y];$$

iii) ∇^V is almost metric, that is,

$$X\left(g_V(Y, Z)\right) = g_V(\nabla_X^V Y, Z) + g_V(Y, \nabla_X^V Z) + 2C_V(\nabla_X^V V, Y, Z).$$

Moreover we have

$$\begin{aligned} 2g_V(\nabla_X^V Y, Z) =& X\left(g_V(Y, Z)\right) + Y\left(g_V(Z, X)\right) - Z\left(g_V(X, Y)\right) \\ &+ g_v([X, Y], Z) - g_v([Y, Z], X) + g_v([Z, X], Y) \\ &- 2C_V(\nabla_X^V V, Y, Z) - 2C_V(\nabla_Y^V V, Z, X) + 2C_V(\nabla_Z^V V, X, Y) \end{aligned}$$

for all vector fields $X, Y, Z \in \Gamma(TU)$. This equation, called the generalized Koszul formula, uniquely determines V.

The Chern connection the neighborhood U define a family of functions[2] $\Gamma_{ij}^k : \widetilde{TU} \to \mathbb{R}$ by

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(p) = \Gamma_{ij}^k(p, V_p)\frac{\partial}{\partial x^k}.$$

Using the functions Γ_{ij}^k we introduce the *covariant derivative* along a curve $\sigma : [a, b] \to M$. For a vector field $X(t)$ along σ, define locally

[2]These are almost the Christofell symbols $\gamma_{ij}^k = \frac{g^{ks}}{2}(\frac{\partial g_{si}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s} + \frac{\partial g_{js}}{\partial x^i})$ from the Riemannian geometry but calculated with respect to the fundamental tensor g_{ij}. In fact we have the following identity

$$\Gamma_{ij}^k(x, y)y^i y^j = \gamma_{ij}^k(x, y)y^i y^j.$$

See [4] for details.

$$D_{\dot{\sigma}}X := \left(\frac{dX^k}{dt} + \Gamma^k_{ij}(\sigma(t), \dot{\sigma}(t))\,\dot{\sigma}^i X^j\right) \frac{\partial}{\partial x^k}\bigg|_{\sigma(t)}. \tag{3.2.2}$$

In fact, since the Chern connection is globally defined, the definition of the covariant derivative extends over the whole curve. Observe that $D_{\sigma'}X = \nabla^{\sigma'}_{\sigma'}X$, where on the right-hand side one has to take extensions of σ' and X.

Lemma 3.2.2. *The Euler-Lagrange equations are equivalent to*

$$D_{\sigma'}\sigma' = 0.$$

So, a curve that satisfies the equation above is a constant speed geodesic.

If we had use the length functional to obtain the Euler-Lagrange equations, see 3.1.3, then the geodesics would be described by

$$D_{\sigma'}\left(\frac{\sigma'}{F(\sigma')}\right).$$

Definition 3.2.3. *A Finsler manifold (M, F) is said to be forward geodesically complete if every geodesic $\sigma(t)$, $t \in [a, b)$, can be extended to a geodesic defined on $[a, \infty)$. If the metric is reversible then the manifold is forward geodesically complete if and only if it is geodesically complete, that is, every geodesic can be extended to a geodesic defined on whole line.*

As a consequence of the Hopf-Rinow theorem, which is also valid in the Finsler case (cf. [4], p. 168), every compact Finsler manifold is forward geodesically complete, as well as its universal covering endowed with the pullback metric Finsler metric. Another consequence of Hopf-Rinow is that every pair of points in M, or in $\widetilde{M}$, can be joined by a *minimizing geodesic* $\sigma : [a, b] \to M$.

We will end these section with a concept that will be important ahead. For this, consider the universal covering $\widetilde{M}$ of the manifold M and let $p : \widetilde{M} \to M$ be the covering map. Given a curve c and a point x in the manifold M let $\widetilde{x}$ be contained in the fiber over x and let $\widetilde{c}$ be the lift of c through $\widetilde{x}$. Consider $\widetilde{F}$ the lifted Finsler metric and $d_{\widetilde{F}}$ its corresponding distance function. It is easy to see that $p : (\widetilde{M}, d_{\widetilde{F}}) \to (M, d_F)$ is a local isometry.

Definition 3.2.4. *A geodesic $\sigma \subset M$ of the Finsler metric F is called forward globally minimizing if for some lift $\widetilde{\sigma} \subset \widetilde{M}$ of σ we have that*

$$\ell_{\widetilde{F}}(\widetilde{\sigma}) = d_{\widetilde{F}}(\widetilde{\sigma}(s), \widetilde{\sigma}(t))$$

for every $s, t \in \mathbb{R}$ and $s \leq t$.

It is important to notice that $\widetilde{\sigma}$ in the definition above is also a geodesic of $\widetilde{F}$ and that if F is reversible then we can drop the term forward.

Lemma 3.2.5. *A geodesic σ is forward globally minimizing if and only if σ has no conjugate points along it.*

Given a nonzero vector field V and its associated connection ∇^V, one can consider the curvature tensor R^V defined by

$$R^V(X,Y)Z = \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z.$$

If the vector field V is geodesic, that is, every orbit of its flow is a geodesic of the Finsler metric, then it follows that

$$R^V(Y) = R^V(Y,V)V = -\nabla^V_V \nabla^V_Y V - \nabla^V_{[Y,V]} V.$$

Definition 3.2.6. *For a Finsler manifold (M,F) and a flag (V,π) consisting of a nonzero tangent vector $V \in T_xM$ and a plane $\pi \in T_xM$ spanned by V and some other tangent vector U, the flag curvature is defined by*

$$K(V;\pi) := \frac{g_V\left(R^V(U),U\right)}{||V||_V^2\,||U||_V^2 - g_V\left(V,U\right)}.$$

This definition depends neither on the flag or the plane chosen. When M is a surface, the flag curvature depends only on the point of $\widetilde{TM}$ taken.

If $\dim M = 2$ then, if $u,v \in \widetilde{T_xM}$, the curvature tensor has the following simple expression

$$R^v(u) = K(v)\left(||v||_v\, u - g_v(v,u)\, v\right), \tag{3.2.3}$$

where $K(v)$ is the flag curvature on the point (x,v) with flagpole v.

Recall that in definition 3.1.6 we introduced the concept of Jacobi fields along a geodesic. In the geometrical setting they appear as a solution of a second order linear equation which the first order term depends on the flag curvature along the geodesic. Both definitions coincide but we will enunciate again for the sake of completeness.

Definition 3.2.7. *On a Finsler manifold (M,F), a vector field $\eta = \eta(t)$ along a geodesic $\sigma : [a,b] \to M$ which satisfies the equation*

$$D_{\sigma'} D_{\sigma'} \eta + R^{\sigma'} \eta = 0 \tag{3.2.4}$$

is called a Jacobi field (we shal denote as usual $\eta' = D_{\sigma'}\eta$ and $\eta'' = D_{\sigma'} D_{\sigma'}\eta$).

Remark 3.2.8. *This equation appears when we use the covariant derivative to calculate the second variation of energy. Using this formalism, the geometrical character of the second variation become more clear.*

Lemma 3.2.9. *Let $\sigma : [a, b] \to M$ be a unit speed geodesic and η a Jacobi field along σ. Then, we have that η satisfy: for a $t_0 \in [a, b]$ we have that*

$$g_{\sigma'}(\sigma'(t), \eta(t)) = (t - t_0)g_{\sigma'}(\sigma'(t_0), \eta'(t_0)) + g_{\sigma'}(\sigma'(t_0), \eta(t_0)).$$

In the case of surfaces, the lemma above together with (3.2.3) implies that, if σ is a unit speed geodesic and $g_{\sigma'}(\sigma'(t_0), \eta(t_0))g_{\sigma'}(\sigma'(t_0), \eta'(t_0)) = 0$ for some $t_0 \in [a, b]$ then the Jacobi equation (3.2.4) has the form

$$\eta'' + K(\sigma')\eta = 0. \tag{3.2.5}$$

The Finslerian exponential map will play a important role in the subsequent sections, so we will define it. Denote by $\sigma_{(x,v)}$ the geodesic such that $\sigma_{(x,v)}(0) = x$ and $\sigma'_{(x,v)}(0) = v$. Given a positive constant λ, by uniqueness we have that $\sigma_{(x,\lambda v)}(t) = \sigma_{(x,v)}(\lambda t)$.

Definition 3.2.10. *Define the exponential map $\exp_x : U \subset T_xM \to M$, where U is a neighborhood of the origin, by*

$$\exp_x(v) = \sigma_{(x,v)}(1)$$

and $\exp_x(0) = x$. When the Finsler manifold is forward geodesically complete the map $\exp_x$ is defined at all points of T_xM.

The map $\exp_x$ is C^1 and if we denote the origin in T_xM by 0_x we have $d(\exp_x)_{0_x} = Id$.

Remark 3.2.11. *Actually, the exponential function $\exp_x$ in Finsler geometry is C^∞ outside the origin but, in general, only C^1 at it. This phenomenon shows some rigidy in Finsler geometry. In fact, if the exponential map is C^2 over the zero section of TM then a result from Akbar-Zadeh says that the metric is of Berwald type (cf. [4], p. 128).*

3.3 On Lagrangian graphs

The Legendre condition on the definition of Finsler metrics induces a diffeomorphism

$$\mathcal{L}_F : \widetilde{TM} \to \widetilde{T^*M}$$

given by $\mathcal{L}_F(x,v) = (x, g_{v_x}(v_x,\cdot))$. This map is called the *Legendre transform* of associated with the Finsler metric F.

Briefly, lets shift our view to the slit cotangent bundle $\widetilde{T^*M}$. A *symplectic structure* on a manifold N is a closed, non-degenerated 2-form. The $\widetilde{T^*M}$ admits a natural symplectic strure

$$\omega_0 = -d\lambda_0$$

such that $(\lambda_0)_{(x,p)} = p \circ (d\bar{\pi})_{(x,p)}$. The form λ_0 has a special name, it is called *canonical* 1-form. Pull back ω_0 by $\mathcal{L}_F$ to obtain a symplectic form

$$\omega^F = \mathcal{L}_F^* \omega_0$$

on $\widetilde{TM}$. The definition of ω^F implies that $(\widetilde{TM}, \omega^F)$ and $(\widetilde{T^*M}, \omega_0)$ are *symplectomorphic*, and so, symplectic objects, such as Lagrangian spaces, are carry out by $\mathcal{L}_F$.

Lemma 3.3.1. *The geodesic flow ϕ_t of the Finsler metric F preserves ω_F. In other words,*

$$\phi_t^* \omega^F = \omega^F.$$

Recall that $\widetilde{T^*M}$ admits a natural symplectic structure, that is, a closed, non-degenerated 2-form ω_0 given by

$$\omega_0 = -d\lambda_0$$

where $(\lambda_0)_{(x,p)} = p \circ (d\bar{\pi})_{(x,p)}$. The form λ_0 is called *canonical 1-form.*

For $\theta \in \widetilde{TM}$, a n dimensional subspace L_θ of the $2n$ dimensional space $T_\theta \widetilde{TM}$ is called the *lagrangian subspace* if $\omega_\theta^F\big|_{L_\theta} = 0$.

Similarly, a C^1 smooth submanifold $N \subset \widetilde{TM}$ is said to be *Lagrangian* if, for every $\theta \in N$, $T_\theta N$ is a lagrangian subspace of $T_\theta \widetilde{TM}$.

Definition 3.3.2. *A subset $\Sigma \subset \widetilde{TM}$ is called a C^k graph if $\bar{\pi}|_\Sigma$ is a C^k diffeomorfism. Similarly, it is called a continuous, or C^0, graph, if the restriction is an homeomorphism. When Σ is C^k graph then there is a C^k vector field X such that, considering $X : M \to \widetilde{TM}$ as a section of the slit tangent bundle, $X(M) = \Sigma$. The graph Σ is invariant by the geodesic flow ϕ_t, or simply invariant, if $\phi_t(\Sigma) = \Sigma$ for all t.*

The proof of the following result can be found in [9], p. 82.

Lemma 3.3.3. *Every C^0 invariant Lagrangian graph Σ is Lipschitz, that is, there exists $K > 0$ such that, for every $x, y \in \Sigma$, $d(x,y) \le K\, d(\bar{\pi}(x), \bar{\pi}(y))$.*

Since every graph is Lipschitz, it is differentiable almost everywhere. So, it is possible to study properties, such as being Lagrangian, even for a continuous graph.

Lemma 3.3.4. *Let $\Sigma \subset \widetilde{TM}$ be a Lagrangian graphs. The following holds:*

i) If Σ is invariant then Σ is contained in a level set of the Finsler metric F, that is, $F(\Sigma) = k$, for some $k > 0$;

ii) If is contained in some level set of F then Σ is invariant and the geodesics which are lifted to orbits in Σ are globally minimizing.

Proof. Let $\theta \in \Sigma$. Since ω^F is invariant by the geodesic flow, we have that

$$d_\theta F^2 = \omega^F_\theta(X_F, \cdot),$$

where $X_F = \frac{d}{dt}\phi_t(\theta)\big|_{t=0}$. But Σ is also invariant, therefore $X_F(\theta) \in T_\theta\Sigma$. From the fact that Σ is Lagrangian,

$$\omega^F_\theta(X_F, v) = 0,$$

for every $v \in T_\theta\Sigma$, so $d_\theta F^2(v) = 0$. In this way, F is constant in a neighbourhood of θ. Connectedness finishes the argument.

In order to prove the second part, we will work on the universal covering $\widetilde{M}$. If we endow the universal covering with the pull back metric

$$\widetilde{F} = p^* F,$$

$p : \widetilde{M} \to M$ the covering map, we have that the symplectic form $\omega^{\widetilde{F}}$ associated to $\widetilde{F}$ induces a local symplectomorphism between the slit tangent bundle of $\widetilde{M}$ and the slit tangent bundle of M. So, the lifted graph $\widetilde{\Sigma}$ is Lagrangian with respect to $\omega^{\widetilde{F}}$. Now, lets treat $\widetilde{\Sigma}$ as a vector field $\widetilde{X}$. Using the legendre transform we obtain a 1-form μ on $\widetilde{M}$ by $\upsilon = \mathcal{L}_{\widetilde{F}}(\widetilde{X})$. Since the lifted graph $\widetilde{\Sigma}$ is Lagrangian then the graph $\Upsilon = \upsilon(M)$ in the slit cotangent bundle of $\widetilde{M}$ is also Lagrangian. It follows from a standard fact of symplectic linear algebra (cf. [19], p. 99) that the form υ must be closed and therefore exact because $\widetilde{M}$ is simply connected. Then there is a function $f : \widetilde{M} \to \mathbb{R}$ such that $\upsilon = df$ and, using the lifted fundamental tensor $\widetilde{g}$, we conclude that $\widetilde{X}$ is the gradient vector field of f.

Assume that our graph Σ is contained in F_k. Given an orbit $\gamma : [a, b] \to \widetilde{M}$ of $\widetilde{X}$, let $c : [a, b] \to \widetilde{M}$ be a curve such that $\gamma(a) = c(a)$ and $\gamma(b) = c(b)$. Since

$F(\gamma, \gamma') = k$,

$$\ell_{\widetilde{F}}(c) = \int_a^b F(c, c')\, dt = \frac{1}{k} \int_a^b F(\gamma, \gamma') F(c, c')\, dt.$$

From the Cauchy-Schwartz theorem for Finsler metrics (cf. [4]),

$$F(\gamma, \gamma') F(c, c') \geq |g_{\gamma'}(\gamma', c')| = |df(c')|$$

and so

$$\ell_{\widetilde{F}}(c) \geq \frac{1}{k} \left| \int_a^b (f \circ c)'\, dt \right| = \frac{1}{k} \left| f(c(b)) - f(c(a)) \right|.$$

On the other hand, if $c = \gamma$, then the Cauchy-Schwartz inequality becomes an equality and we have that

$$\ell_{\widetilde{F}}(\gamma) = \frac{1}{k} \left| f(\gamma(b)) - f(\gamma(a)) \right| = \frac{1}{k} \left| f(c(b)) - f(c(a)) \right|.$$

And then

$$\ell_{\widetilde{F}}(c) \geq \ell_{\widetilde{F}}(\gamma)$$

for all curves c joining $\gamma(a)$ and $\gamma(b)$. From the previous sections, γ has to be minimizing geodesic and, since the extreme points were taken by chance, it realizes the distance between any consecutive point chosen in its orbit. And so, $p \circ \gamma$ is globally minimizing.

□

As a direct corollary of the lemma 3.3.4 and lemma 3.2.5 we have:

Corollary 3.3.5. *Consider the Finsler manifold (M, F). If a geodesic σ has conjugate points then its lift to $\widetilde{TM}$ is not contained in any Lagrangian graph.*

This leads us to the following conclusion.

If there is a point $p \in M$ such that every geodesic passing through p has conjugate points then there are no Lagrangian graphs.