2
A geometric approach

In this section, we will make some further developments of Bangert's idea in [3]. This is new in the literature and shall motivate the analytic construction for Finsler metrics.

Our aim is to prove the following.

Theorem 2.0.2. *Let g be a C^∞, non flat Riemannian metric in the two torus T^2. Given $\epsilon > 0$ there is a C^∞ metric $\bar{g}$ with the properties:*

i) $||g - \bar{g}||_1 < \epsilon$ *and* $||g - \bar{g}||_{1,\frac{1}{3}} < C$*, where* $C > 0$ *does not depend on* $\bar{g}$*;*

ii) $\bar{g}$ *admits no continuous field of minimizers.*

As an immediate corollary we obtain:

Corollary 2.0.3. *Let g be any C^∞ metric in the two torus T^2. Arbitrarily close to g in the C^1 topology there exists a metric $\bar{g}$ without continuous field of minimizers and with finite $C^{1,\frac{1}{3}}$ norm.*

We shall begin by showing the above theorem for a metric g with a neighbourhood where the sectional curvature is positive and constant. To be more specific, there exists $p \in T^2$, $r > 0$, $\rho > 0$, and $\delta > 0$ with $\rho \gg \delta$ such that the sectional curvature K satisfies

$$K(q) = \frac{1}{r^2},$$

for all q in the geodesic ball $\mathcal{B}_p(\rho + \delta)$.

If $\gamma : (-\rho - \delta, \rho + \delta) \to \mathcal{B}_p(\rho + \delta)$ is a unit speed geodesic, then introduce the polar coordinates $P : [0, \rho + \delta) \times (-\pi, \pi) \to \mathcal{B}_p(\rho + \delta)$ by

$$P(R, \tau) = \exp_p(R \cos \tau \gamma'(0) + R \sin \tau (\gamma'(0))^\perp).$$

Define the set

$$S_{(\tau,\theta)} = P([0, \rho + \delta) \times (\tau, \theta)),$$

where $\tau, \theta \in (-\pi, \pi)$ and $\tau < \theta$. Consider the set

$$B_\theta = \mathcal{B}_p(\rho + \delta) \setminus P([0, \rho + \delta) \times \{0\}).$$

Using Cartan's theorem ([6], p.174) the set $S_{(-\theta,0)}$ is isometric to $S_{(0,\theta)}$. Let C_θ^0 be the spherical cone obtained by identifying B_θ with this isometry. It is clear that C_θ^0 is a smooth Riemannian manifold with the metric inherited from B_θ. Define $C_\theta = \overline{C_\theta^0}$. The process is illustrated in the Figure 2.1. The Riemannian distance in $C_\theta^0 \subset C_\theta$ can be extended to make C_θ a complete metric space.

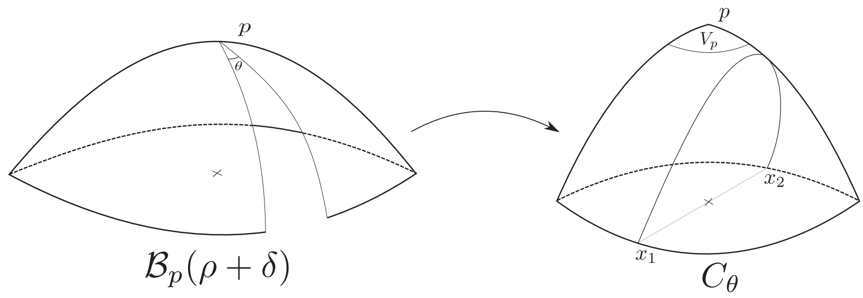

Figure 2.1: In the right, the curve connecting x_1 and x_2 is a minimizing geodesic of length 2ρ that not intersect V_p.

Lemma 2.0.4. *Suppose that $\theta \in (0,\pi)$. If $x_1, x_2 \in C_\theta$ satisfy*

$$d(p,x_1), d(p,x_2) \geq \rho$$

then

$$d(x_1,x_2) < 2\rho.$$

Proof. Let γ_1 and γ_2 be geodesics in C_θ joining x_1 to p and p to x_2, respectively. At p the angle between them is

$$\angle(-\gamma_1'(\rho), \gamma_2'(0)) = \pi - \theta.$$

If we consider a comparison triangle in $\mathbb{R}^2$ determined by the line segments $(\sigma_1, \sigma_2, \sigma_2)$ such that σ_1 and σ_2 have length ρ and $\angle(-\sigma_1'(\rho), \sigma_2'(0)) = \pi - \theta$ then the length of σ_3 is

$$\ell(\sigma_3) = 2\rho\cos(\frac{\theta}{2}) < 2\rho.$$

By Toponogov's theorem ([7], p. 35),

$$d(x_1,x_2) \leq \ell(\sigma_3).$$

Therefore, $d(x_1,x_2) < 2\rho$. $\square$

Denote by $\mathcal{A}_\delta$ the set of points in $x \in C_\theta$ such that $\rho + \delta > d(x,p) \geq \rho$. The previous lemma implies that the minimizing geodesics connecting points in $\mathcal{A}_\delta$ avoid a neighborhood around the vertex p. In the following, we shall estimated the size of this neighborhood in order to smooth the cone.

2.1
Perturbations by cones

Let $a > 0$ such that $a \ll \rho$. The cone C_θ we defined previously has constant sectional curvature $\frac{1}{r^2}$. Let $S_{\rho+\delta} = C_\theta \setminus C_\theta^0 \cup \{p\}$.

The singular surface C_θ could be obtained as a surface of revolution but we prefer to work with the surface that is the revolution of

$$\alpha(x) = (x, 0, f_a(x)),$$

where

$$f_a(x) = \left(r^2 - (x+a)^2\right)^{\frac{1}{2}} \tag{2.1.1}$$

and $x \in [0, r_{\rho+\delta}]$, around the z axis. The constant $r_{\rho+\delta} < r$ is given implicitly by $\rho + \delta = \int_0^{r_{\rho+\delta}} ||\alpha'(s)||\, ds$. Since ρ is close to zero, both singular manifolds are close in the C^1 topology. So, from now on we will consider C_θ to be the spherical cone generated by the revolution of the curve α around z axis.

Let V_p be the geodesic ball around p with radius $\delta_a > 0$. We wish to estimate δ_a in terms of the constant a in the function f such that every minimizing geodesic with end points in $\mathcal{A}_\delta$ avoid V_p.

First, we will relate δ_a with θ. Consider an Euclidean geodesic triangle $(\sigma_1, \sigma_2, \sigma_3)$ such that $\ell(\sigma_1) = \ell(\sigma_2) = \rho$ and the angle at the vertex $\bar{p} = \sigma_1(\rho) = \sigma_2(0)$ given by $\angle(-\sigma_1'(\rho), \sigma_2'(0)) = \pi - \theta$. If d_0 is the Euclidean distance, then

$$d_0(\bar{p}, \sigma_3) = \rho \sin(\frac{\theta}{2}).$$

By the Toponogov's theorem, the distance from p to any minimizing geodesic of C_θ connecting points in $\mathcal{A}_\delta$ is greater than $d_0(\bar{p}, \sigma_3) = \rho \sin(\frac{\theta}{2})$. Set

$$\delta_a = \rho \sin(\frac{\theta}{2}).$$

If θ is small then

$$\delta_a \simeq \frac{\rho\,\theta}{2}. \tag{2.1.2}$$

Now we will relate δ_a and a. Since $\rho = \ell(\alpha|_{[0,r_\rho]})$ then

$$\rho = r \text{ arcsin}\left(\frac{r_\rho + a}{r}\right) - r \text{ arcsin}\left(\frac{a}{r}\right). \tag{2.1.3}$$

Expand arcsin up to order 4 at $x = -a$ to obtain

$$\rho = r\left(\frac{r_\rho + a}{r} - \frac{a}{r} + \frac{1}{3!r^6}(r_\rho^3 + 3r_\rho^2 a + 3r_\rho a) + O(5)\right)$$
$$\geq r_\rho + \frac{7a^3}{6r^5} + O(5) \simeq r_\rho + \frac{7a^3}{6r^5}, \tag{2.1.4}$$

because $a \ll r_\rho$. On the other hand, it is possible to calculate the perimeter $2\pi\, r_\rho$ by means of Jacobi fields. Given a unit speed geodesic $\gamma : [0, \rho] \to C_\theta$, $\gamma(0) = p$, let $\gamma_s : [0, \rho] \to C_\theta$, $s \in [0, 2\pi - \theta)$ be a polar parametrization of all radial geodesics of length ρ such that $\gamma_s(0) = 0$ and $\gamma_0 = \gamma$. If $E_s(t)$ is a perpendicular parallel vector field of norm 1 and the sectional curvature K of C_θ is $\frac{1}{r^2}$ then

$$J_s(t) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}t)\, E_s(t)$$

is a perpendicular Jacobi field. The perimeter is given by

$$2\pi r_\rho = \int_0^{2\pi-\theta} ||J_s(\rho)||\, ds$$
$$= (2\pi - \theta)\, \frac{\sin(\sqrt{K}\rho)}{\sqrt{K}}.$$

We have the following estimative for the perimeter

$$2\pi r_\rho \simeq (2\pi - \theta)\, \rho. \tag{2.1.5}$$

With the results obtained from (2.1.4) and (2.1.5) we conclude that

$$2\pi(\rho - \frac{7a^3}{6r^5}) \geq 2\pi r_\rho \simeq (2\pi - \theta)\, \rho.$$

Therefore,

$$\theta\, \rho \geq 2\pi \frac{7a^3}{6r^5}. \tag{2.1.6}$$

Substituting this equation on (2.1.2) we have

$$\delta_a \geq \frac{7\pi a^3}{6r^5}. \tag{2.1.7}$$

Consider the spherical cone $\Phi_a : B_0(r_\rho) \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $\Phi_a(x, y) = (x, y, f_a(\sqrt{x^2 + y^2}))$. Define the singular metric g_a^s in $B_0(r_\rho)$ by

$$g_a^s = \Phi_a^* g_{euc},$$

where g_{euc} is the standard Euclidean metric in $\mathbb{R}^3$. In the same way, the metric of the sphere of radius r is given by $g^r = \Phi_0^* g_{euc}$.

Lemma 2.1.1. *There exists a sequence of C^1 metrics g_a in $B_0(r_\rho)$, such that* *i)* g_a *has no continuous field of minimizers;*

ii) g_a *converges to* g^r *in the* C^1 *topology;*

iii) for $a > 0$ *sufficiently small,*

$$||g_a - g^r|| < \left(\frac{7\pi\, K}{6}\right)^{-\frac{1}{3}}.$$

Proof. Let r_a be given implicitly by $\delta_a = \int_0^{r_a} ||\alpha'(s)||\, ds$. Since $a \ll \rho$, and, therefore, $a \ll r$, we have that $r_a \simeq \delta_a$.

The neighbourhood V_p of radius δ_a which the minimizing geodesics avoid is open but there exist a geodesic that realizes the infimum of

$$\{d(p,\gamma)|\gamma \text{ is a minimizing geodesic with endpoints in } \mathcal{A}_\delta\}.$$

Then we can extend V_p to a neighbourhood of radius $\delta_a + \omega$.

Let $r_\omega > r_a$ be given implicitly in the same way r_a, but with respect to $\delta_a + \omega$. Consider the bump function β such that $\beta(t) = 1$ for $[0, r_a]$ and $\beta(t) = 0$ for $t \in [r_\omega, r_\rho]$.

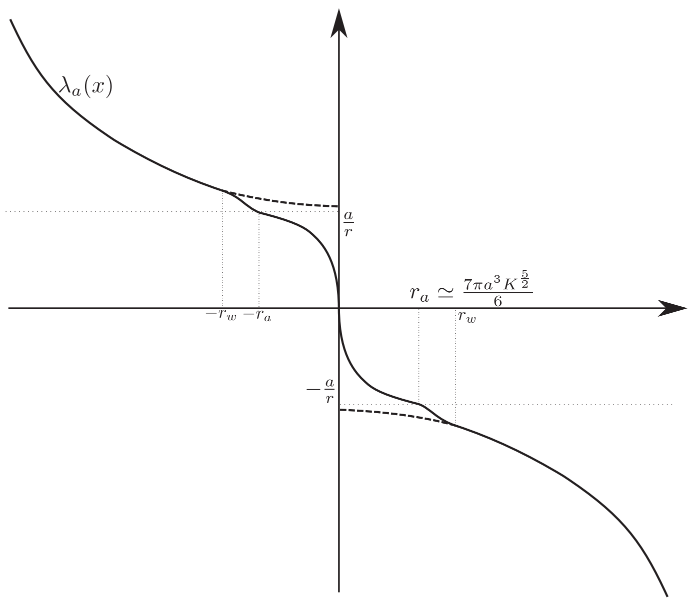

Figure 2.2: On the interval $[-r_a, r_a]$ we used a function of order $x^{\frac{1}{3}}$ to smooth f_a'.

Define $\lambda_a : [0, r_{\rho+\delta}] \to \mathbb{R}$ by

$$\lambda_a(x) = -\beta(x)\left(\frac{7\pi\, K}{6}\right)^{-\frac{1}{3}} x^{1/3} + (1 - \beta(x)) f_a'(x), \tag{2.1.8}$$

where $K = \frac{1}{r^2}$ is the sectional curvature of the metric g^r. In Figure 2.2, it is shown the graph of λ_a.

Now, define the function $\Lambda_a : [0, r_{\rho+\delta}] \to \mathbb{R}$ by

$$\Lambda_a(x) = \sqrt{r^2 - (r_\rho + a)^2} - \int_0^{r_\rho} \lambda_a(s)\, ds + \int_0^{x} \lambda_a(l)\, dl. \tag{2.1.9}$$

The metric

$$g_a = \Psi_a^* g_{euc},$$

where $\Psi_a(x, y) = (x, y, \Lambda(\sqrt{x^2 + y^2}))$, is C^1 in $B_0(r_\rho)$ and C^∞ in $B_0(r_\rho) \setminus \{0\}$. When $a \to 0^+$, the metric g_a converges in the C^1 topology to the spherical metric g^r because $f_a \overset{C^\infty}{\to} f_0$, $r_a \to 0$ and $\lambda_a \to f_0'$. The convergence of the derivatives happens because $|\lambda_a|$ is bounded above by $|f_a'|$. More precisely, on $[0, r_a]$,

$$\lambda_a(t) \geq \lambda_a(r_a) \simeq \lambda_a(\frac{7\pi a^3}{6r}) = -a\sqrt{K} \geq f_a'(0).$$

The Hölder norm of $\lambda_a - f_a'$ is given by

$$\begin{aligned} ||\lambda_a - f_a'||_{\frac{1}{3}} &= \sup_{x \neq y;\ x,y \in [0, r_w]} \frac{|\lambda_a(x) - \lambda_a(y)|}{|x - y|^{\frac{1}{3}}} \\ &= \left(\frac{7\pi\, K}{6}\right)^{-\frac{1}{3}}, \end{aligned}$$

where $K = \frac{1}{r^2}$.

Outside the neighbourhood V_p, the manifold $(B_0(r_{\rho+\delta}), g_a)$ is isometric to the cone C_θ. Therefore, every minimizing geodesic on length greater than 2ρ do not intersect V_p, then these geodesics do not contain p. We conclude that g_a cannot have a continuous field of minimizers. □

2.2
Proof of theorem 2.0.2

Let g be any Riemannian C^2 metric in the $2-$torus. If g is not flat, by the Gauss-Bonnet theorem there exists p such that the sectional curvature $K(p) > 0$. Then, given $\epsilon > 0$, there exists a metric g_0, in an ϵ neighbourhood of g such that the sectional curvature of g_0 satisfies

$$K_0(q) = K(p)$$

for every q in a neighbourhood of p of radius $\rho + \delta$. When g is flat, we can perturb g in order to obtain a non-flat metric and apply the same argument. Anyway, given $\epsilon > 0$, there is, $\epsilon-$close to g in the C^2 topology a metric with positive constant curvature in a neighbourhood around a point.

Now we can use the preceding construction. Let $\mathcal{B}_p(\rho + \delta)$ and let $f_0 : B_0(r_{\rho+\delta}) \to \mathcal{B}_p(\rho + \delta)$ be given by (2.1.1). Choose $\theta \in (0, \pi)$ such that

$2\pi r_\rho - \theta\rho \in [2\pi r_\rho, 2\pi r_{\rho+\delta}]$. So, θ and $a > 0$ are related by (2.1.2) and $\theta(a) \to 0$ as $a \to 0$.

The metrics g_a in $B_0(r_\rho)$ obtained in lemma 2.1.1 can be smoothed, by means of a bump function, in a small neighbourhood of radius $r_0 \ll r_a$ without changing the convergence in the $C^{1,\frac{1}{3}}$ topology. Let

$$(\Phi_0)_* g_a$$

be a metric in $\mathcal{B}_p(\rho)$.

Finally, if Δ is a bump function in $\mathcal{B}_p(\rho+\delta)$ such that $\Delta(q) = 1$ in $\mathcal{B}_p(\rho)$ and $\Delta(q) = 0$ in $\mathcal{B}_p(\rho+\delta) \setminus \mathcal{B}_p(\rho_\theta)$, where ρ_θ is the radius of $(B_0(r_{\rho+\delta}, g_a)$. Define G_a by

$$G_a = \Delta\,(\Phi_0)_* g_a + (1-\Delta) g.$$

The result follows from lemma 2.1.1.