8
Final Considerations

In view of Theorem 6.1, we need to show that both items 1) and 2) may occur, so the Theorem is optimal in the sense that it can not be reduced to only one of the two possibilities.

The main examples of non-hyperbolic robustly transitive diffeomorphisms are the following:

1. skew-products: there are two types. The Shub's example in (30) is derived from the product map $(A, B) : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ of two Anosov diffeomorphisms. The Bonatti-Díaz example in $\mathbb{T}^n \times N$, where N is any compact manifold. In this later, the non-hyperbolic direction may be chosen to be tangent to the fibers $\{x\} \times N$, so $\dim(E^c) = \dim(N)$.

2. DA diffeomorphism (Mañé's example in in $\mathbb{T}^3$ (24), and the generalizations of Bonatti-Viana in $\mathbb{T}^4$ (15)). This last one exhibit the first example where the partial hyperbolicity has only one hyperbolic direction, and the first one with no hyperbolic direction at all.

3. perturbations of the time-1 map of Anosov flows, (9). It was the first time that the geometric model called *blender* appears. It was used to makes some lower dimensional manifolds behaves as bigger ones.

An easy way to produce examples of proper robustly transitive attractors is to consider the product of robustly transitive diffeomorphisms with a north-south dynamics in the circle S^1.

Let $f : M \to M$ be a robustly transitive diffeomorphism and $\phi : S^1 \to S^1$ given by $\phi(x) = 2x^3 - 3x^2 + 2x$. Consider the diffeomorphism $F : S^1 \times M \to S^1 \times M$ defined by $F(x, y) = (\phi(x), f(y))$.

Clearly, the set $\Lambda_F = \{1/2\} \times M$ is a proper attractor of F. Let G be a C^1-perturbation of F. By the theory of normal hyperbolicity (23), the continuation Λ_G is homeomorphic to Λ_G and normal to the circles $S^1 \times \{y\}$, $y \in M$. In addition, this homeomorphism can be obtained by a projection $h : \Lambda_G \to \Lambda_F$ along these circles.

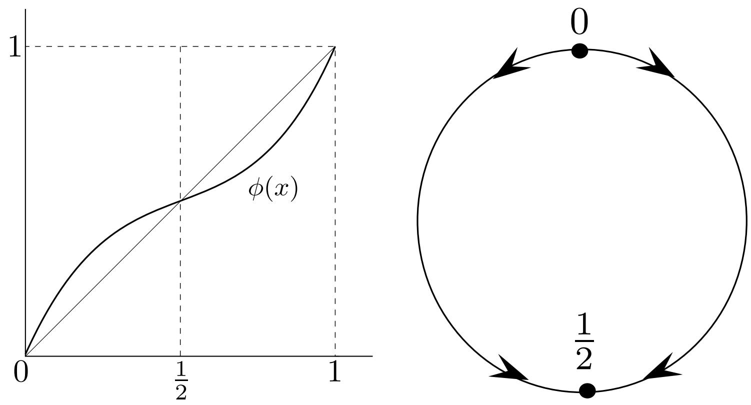

Figure 8.1: The map $\phi : S^1 \to S^1$.

The map $h^{-1} \circ G \circ h$ is C^1-close to F, provided G is close enough to F. Observe also that this map leaves Λ_F invariant and then it induces a homeomorphism g on $\times M$ close to f. Since f is robustly transitive, the map g is also transitive, and consequently Λ_F is a transitive attractor of $h^{-1} \circ G \circ h$. This implies that Λ_G is transitive to G. As G was chosen arbitrarily in a small neighborhood of F, we conclude that F is robustly transitive. Clearly, the same argument holds in the case that f is generically transitive.

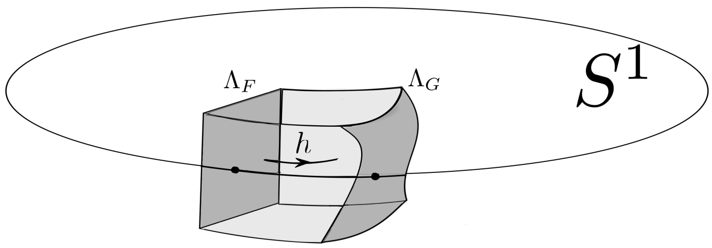

Figure 8.2: The projection $h : \Lambda_F \to \Lambda_G$

Now, by Corollary 5.17, if $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) is minimal, then Λ_F is robustly s-minimal (resp. u-minimal). By considering f^{-1} instead of f, we obtain another attractor $\Lambda_{\tilde{F}}$ for the map $\tilde{F} = \phi \times f^{-1}$. The attractors Λ_F and $\Lambda_{\tilde{F}}$ provides examples of u and s-minimal proper attractors in the same manifold $S^1 \times M$.

Concerning possible generalizations of Theorem 6.1, one may ask if this result could be obtained to the broader setting of robustly/generically transitive sets (instead of attractors). Next we show that, by an example in (7), it is not possible.

In (7) it is proved that every manifold with dimension bigger than 2 admits a generically transitive set that is not a robustly transitive set. The construction gives a diffeomorphism f and a dense subset of a neighborhood of f for which the semicontinuation $\Lambda_g(U)$ of the isolated set $\Lambda_f(U)$ has an isolated point (so it is not transitive). In addition, $\Lambda_f(U)$ is strongly partially hyperbolic with one dimensional central bundle, which is the case we treat in this thesis.

In their construction there are two hyperbolic periodic points p and q, with $\mathrm{index}(p) = \mathrm{index}(q) + 1$, that lie in the "corner" of the set $\Lambda_f(U)$ (see the precise definition of *cuspidal point* in (7)). These points have the property that $\mathcal{F}^s(p)$ just meet $\Lambda_f(U)$ at p, and $\mathcal{F}^s(q)$ just meet $\Lambda_f(U)$ at q, which prevent both foliations to be minimal. Moreover, being a cuspidal point is a robust property, so the continuations $\Lambda_g(U)$ also do not have minimal foliations. Hence, the example in (7) shows that Theorem 6.1 cannot be generalized to transitive sets.