4
C^1-Generic setting

In this section we gather some generic properties of diffeomorphisms in $\text{Diff}^1(M)$. We say that a property $\mathbf{P}$ is C^1*-generic* if there is a residual (G_δ and dense) subset $\mathcal{R}$ of $\text{Diff}^1(M)$ such that the property $\mathbf{P}$ holds for every diffeomorphisms in $\mathcal{R}$.

This section is divided in 4 subsections. In the first one we introduce the concept of *homoclinic classes* and state some generic properties of them. The second one stablish the relationship between homoclinic classes and transitive sets. In Subsection 3.3 we give an extension of the partial hyperbolicity to a neighborhood of a transitive attractor. Finally, in Subsection 3.4 we study the Lebesgue measure of these sets in the generic context.

4.1
C^1-Generic homoclinic classes

Definition 4.1 (Homoclinic class) Let p be a hyperbolic periodic point of a diffeomorphism f. A *homoclinic point* x of p is a point whose forward and backward iterates converge to the orbit $\mathcal{O}_f(p)$ of p (i.e., $x \in W^s(\mathcal{O}_f(p)) \cap W^u(\mathcal{O}_f(p))$). If the stable and unstable manifolds of the orbit of p meet transversely at x, we say that x is a *transverse homoclinic point*. Otherwise, we say that x is a *homoclinic tangency*.

The homoclinic class of p, denoted by $H(p, f)$, is the closure of the set of all transverse homoclinic points of p, that is,

$$H(p, f) = \overline{W^s(\mathcal{O}_f(p)) \pitchfork W^u(\mathcal{O}_f(p))}.$$

Remark 4.2 A homoclinic class is a transitive set and the subset of its periodic points is a dense subset. By the persistence of transverse intersections one easily deduce that homoclinic class vary lower semicontinuously. We refer to Chapter 10.4 of (13) for a more detailed discussion about homoclinic classes.

We now summarize the results in (18, 28) about C^1-generic diffeomorphisms.

Theorem 4.3 (C^1-generic properties) *There is a residual subset $\mathcal{R}_0$ of* $\mathrm{Diff}^1(M)$ *such that, for every $f \in \mathcal{R}_0$, the following holds:*

1. *The diffeomorphism f is Kupka-Smale: every periodic point of f is hyperbolic and their invariant manifolds met transversely.*
2. *The set* $\mathrm{Per}(f)$ *of periodic points of f is dense in the non-wandering set $\Omega(f)$ of f. In particular, any isolated transitive compact set has a dense subset of periodic points.*
3. *For every $p \in \mathrm{Per}(f)$, the homoclinic class of p satisfies*

$$H(p, f) = \overline{W^s(\mathcal{O}_f(p))} \cap \overline{W^u(\mathcal{O}_f(p))}.$$

4. *For every $p \in \mathrm{Per}(f)$, the closure of the stable and unstable manifold and the homoclinic class of p_g depend continuously on $g \in \mathcal{R}_0$ in a neighborhood of f.*
5. *Any transitive set intersecting a homoclinic class is contained in it (i.e., homoclinic classes are maximal transitive sets). In particular, any pair of homoclinic class are either disjoint or coincide.*

Items (1) and (2) is the main theorem in (28). Items (3), (5), and the continuity of $H(p, f)$ in item (4) are, respectively, Lemma 3.5, item 1 , and item 3 of Theorem A in (18). The continuity of $\overline{W^u(\mathcal{O}_f(p))}$ and $\overline{W^s(\mathcal{O}_f(p))}$ in $\mathcal{R}_0$ follows from the continuity of $W^u_\varepsilon(\mathcal{O}_f(p))$ and $W^s_\varepsilon(\mathcal{O}_f(p))$ in $\mathrm{Diff}^1(M)$ for any fixed $\varepsilon > 0$. This gives a semicontinuous dependence of $\overline{W^u(\mathcal{O}_f(p))}$ and $\overline{W^s(\mathcal{O}_f(p))}$ in $\mathrm{Diff}^1(M)$, and consequently a continuous dependence of these sets on a residual subset of $\mathrm{Diff}^1(M)$.

4.2
C^1-Generic transitive sets

Next we establish in Propositions 4.4 and 4.9 the connection between homoclinic classes and transitive sets.

Proposition 4.4 *There is a residual subset $\mathcal{R}_1$ of* $\mathrm{Diff}^1(M)$ *such that, if $f \in \mathcal{R}_1$ and $\Lambda_f(U)$ is an isolated subset of M, then the following properties hold:*

1. *If $\Lambda_f(U)$ is a transitive attractor, then there is a neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{R}_1 \cap \mathcal{U}$, the set $\Lambda_g(U)$ is a transitive attractor. In other words, the set $\Lambda_f(U)$ is a generically transitive attractor.*

2. *If $\Lambda_f(U)$ is non-hyperbolic then it contains a pair of (hyperbolic) saddles of different indices.*

Item (1) is Theorem B of (1). To prove item (2), given any non-hyperbolic diffeomorphism, one can perform a small perturbation to create a non-hyperbolic periodic point whose orbit is contained in U (see Theorem B in (24)). The bifurcation of this point yields two hyperbolic periodic points with different indices. By construction the orbit of these two points are in U. The existence of these two hyperbolic points with different indices is an open property in $\mathrm{Diff}^1(M)$.

Definition 4.5 Let $\Lambda_f(U)$ be an isolated set and $p \in \Lambda_f(U)$ a hyperbolic periodic point. The *relative homoclinic class* of p in U, denoted by $H_U(p, f)$, is the closure of transverse homoclinic points of p whose orbit remains in U, that is,

$$H_U(p, f) = \mathrm{closure}\{z \in W^s(\mathcal{O}_f(p)) \pitchfork W^u(\mathcal{O}_f(p)) \,|\, \mathcal{O}_f(z) \subset U\}.$$

Remark 4.6 Note that $H_U(p, f)$ is an invariant subset of U. Hence it is contained in the maximal invariant set $\Lambda_f(U)$. In the case that $\Lambda_f(U)$ is an attractor, $H_U(p, f) = H(p, f)$, since $W^u(\mathcal{O}_f(p))$ is an invariant subset of U (recall Remark 3.7) that contains $H(p, f)$.

Definition 4.7 (heterodimensional cycle) Let p and q be hyperbolic periodic points with different indices for $f \in \mathrm{Diff}^1(M)$. Suppose that the stable manifold of each point meets the unstable manifold of the other. Then we say that there is an *heterodimensional cycle* associated to the points p and q. This cycle is *far from homoclinic tangencies* if for every g in a neighborhood of f there are no homoclinic tangencies associated to the continuations p_g and q_g. That is, for every g close to f, the intersection points of $W^s(p_g)$ and $W^u(p_g)$ are transverse (similarly for the invariant manifolds of q_g).

Remark 4.8 Let $\Lambda_f(U)$ be a generically transitive set having periodic points p and q of different indices. In Proposition 1.1 of (8) it is shown how to create, by an arbitrarily small pertubation g of f, an heterodimensional cycle associated to the continuations p_g and q_g. In the case of $(s, 1, u)$-partially hyperbolic sets, this cycle is far from homoclinic tangencies. Indeed, in this case one of the invariant manifolds, say $W^i(p, f)$, of a hyperbolic periodic point p in $\Lambda_f(U)$ coincides with the corresponding strong leaf $\mathcal{F}^i(p, f)$ at p, with $i = s$ or u. By the continuity of the strong foliations, for $\varepsilon > 0$ sufficiently small, every intersection point of $\mathcal{F}^i(p, f)$ with the local invariant manifold $W^i_\varepsilon(p, f)$ is a transverse intersection. This prevents the existence of homoclinic tangencies of p in the local manifold $W^i_\varepsilon(p,f)$ and, consequently, globally.

Proposition 4.9 summarizes to our setting some useful results about the unfolding of heterodimensional cycles in (11, 8).

Proposition 4.9 *There is a residual subset $\mathcal{R}_2$ of $\mathrm{Diff}^1(M)$, with $\mathcal{R}_2 \subset \mathcal{R}_1$, consisting of diffeomorphisms f satisfying the following properties:*

Let $\Lambda_f(U)$ be a transitive isolated set of f that is $(s,1,u)$-partially hyperbolic. Then for every pair of hyperbolic periodic points $p,q \in \Lambda_f(U)$, with indices s and $s+1$ respectively, there is an open set $\mathcal{V}_{p,q} \subset \mathrm{Diff}^1(M)$, with $f \in \overline{\mathcal{V}_{p,q}}$, such that for all $g \in \mathcal{V}_{p,q}$:

1. *$W^s(\mathcal{O}_g(q_g),g) \subset \overline{W^s(\mathcal{O}_g(p_g),g)}$ and $W^u(\mathcal{O}_g(p_g),g) \subset \overline{W^u(\mathcal{O}_g(q_g),g)}$,*
2. *$H_U(p_g,g) = H_U(q_g,g)$, and*
3. *if $\Lambda_g(U)$ is transitive then $\Lambda_g(U) \subset \overline{W^s(\mathcal{O}_g(p_g),g)} \cap \overline{W^u(\mathcal{O}_g(p_g),g)}$.*

Let us indicate how to obtain Proposition 4.9 from (11, 8). By item (2) of Proposition 4.4 and Remark 3.5, the set $\Lambda_f(U)$ is generically transitive and contains saddles p and q of indices s and $s+1$, respectively. Remark 2.1 says that, for every g close to f, one has $p_g, q_g \in \Lambda_g(U)$. By Remark 4.8 we can take g having an heterodimensional cycle far from homoclinic tangencies associated to p_g and q_g.

Now, item (1) is just Proposition 2.6 of (12). This property of the closure of an invariant manifold containing a bigger dimensional one is obtained by the unfold of some heterodimensional cycle and is a key argument to construct some robustly transitive diffeomorphisms (see (9)).

Item (2) follows from Proposition 1.1 of (11) and the comments therein.

Item (3) is a straightforward adaptation of Proposition 2.3 of (12) to the case of generically transitive sets. We reproduce the proof of item (3) for the sake of completeness.

ProofProof of item (3): We use the following claim.

Claim 4.10 *For every g close to f such that $\Lambda_g(U)$ is transitive it holds that $\Lambda_g(U) \subset \overline{W^u(\mathcal{O}_g(p_g),g)} \cap \overline{W^s(\mathcal{O}_g(q_g),g)}$.*

Proof: Let us prove that $\Lambda_g(U) \subset \overline{W^u(\mathcal{O}_g(p_g),g)}$. The inclusion $\Lambda_g(U) \subset \overline{W^s(\mathcal{O}_g(q_g),g)}$ follows identically considering g^{-1}.

Consider $x \in \Lambda_g(U)$ whose forward orbit is dense. Then there is some positive iterate $g^n(x)$ of x that is sufficiently close to p_g so that its strong stable leaf meets transversally the local unstable manifold of p_g, say at a point z (recall Remark 3.8 and that index$(p) = s$). Hence, by Remark 3.9, the closure of the forward orbit of z contains $\Lambda_g(U)$. As $z \in W^u(p_g, g)$, the conclusion follows. ■

By the claim and item (1) we have

$$\Lambda_g(U) \subset \overline{W^s(\mathcal{O}_g(q_g), g)} \cap \overline{W^u(\mathcal{O}_g(p_g), g)} \subset \overline{W^s(\mathcal{O}_g(p_g), g)} \cap \overline{W^u(\mathcal{O}_g(p_g), g)},$$

proving item (3). ■

4.3
An extention of the partially hyperbolic splitting.

A partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ defined over a compact invariant set Λ can always be extended to a continuous splitting in a neighborhood of Λ, but its not always possible to make this extension invariant.

Next remark is a consequence of Corollary 1.13 of (15) and the results in (14) (Chapter 3).

Remark 4.11 There is a residual subset $\mathcal{R}_3 \subset \mathrm{Diff}^1(M)$ with the following property. Let $f \in \mathcal{R}_3$ and $\Lambda = H(p, f)$ be an isolated partially hyperbolic homoclinic class. Then we can extend (not uniquely) the splitting on Λ to a partially hyperbolic splitting on a compact neighborhood U of Λ that is invariant in the following sense: for every $x \in U$ such that $f(x) \in U$, we have that $Df_x(E^i(x)) = E^i(f(x))$, for $i \in \{s, c, u\}$. Moreover, there is a neighborhood of f in $\mathrm{Diff}^1(M)$ for which such invariant splitting is defined.

Remark 4.12 *In the case that the homoclinic class is also an attractor, the previous remark holds more generally (that is, for every $f \in \mathrm{Diff}^1(M)$). The reason is that attractors that contain a periodic point is always a chain recurrent class, while for homoclinic classes it holds only generically.*

In Lemma 3.6 of (14), they prove the following invariant property about the tangent space of the stable manifolds of the periodic points inside U.

Lemma 4.13 ((14)) *Let $p \in U$ be a periodic point of index s (resp. $s+1$). Then, for any $x \in W^s(p) \cap U$, the stable manifold $W^s(p)$ is tangent at x to $E^s(x)$ (resp. $E^s(x) \oplus E^c(x)$).*

The next proposition is an adaptation of this lemma to ensure the same property for the strong stable leaf of any point in U.

Proposition 4.14 *Let $f \in \mathcal{R}_3$ and $\Lambda = H(p,f)$ be an isolated partially hyperbolic homoclinic class admitting an extension of the splitting to a compact neighborhood U. For every $x \in \Lambda$ it holds that $\mathcal{F}^s(x)$ is tangent to E^s at every point in U.*

Proof: Since $x \in H(p,f)$, there exist a sequence $\{x_n\}_{n\in\mathbb{N}}$ of homoclinic points of p that accumulates at x. By the continuity of the strong stable foliation in Λ, for every $r > 0$, the disk sequence $\mathcal{F}^s_r(x_n)$ accumulates (with the Hausdorff distance) at the disk $\mathcal{F}^s_r(x)$. By Lemma 3.6 of (14), the proposition holds for periodic points, so the disks $\mathcal{F}^s_r(x_n) \cap U$ are tangent to the bundle E^s. Hence, the accumulation of these disks to $\mathcal{F}^s_r(x)$ implies that it is also tangent to E^s. By the arbitrary choice of r, we conclude the statement of this proposition. ∎

Remark 4.15 *Observe that proposition 4.14 implies that, if $\mathcal{F}^s(x)$ accumulates on a hyperbolic periodic point $q \in \Lambda$ of index s, then it must intersect the local unstable manifold of q.*

Proposition 4.16 *If $\mathcal{F}^s(x)$ accumulates at $y \in \Lambda$, then $\mathcal{F}^s(y) \subset \overline{\mathcal{F}^s(x)}$.*

Proof:

Consider a sequence $\{x_k\}_{k\in\mathbb{N}}$ of points in $\mathcal{F}^s(x)$ converging to the point y, and fix $r > 0$ sufficiently small so that any disk of radius r lies inside U. Consider the sequence of disks $\mathcal{F}^s_r(x_k)$. By Proposition 4.14, the disks $\mathcal{F}^s_r(x_k)$ are tangent to E^s at every point.

Let D be the limit of a subsequence of $\{\mathcal{F}^s_r(x_k)\}_{k\in\mathbb{N}}$ and observe that $y \in D$.

Claim 4.17 $D = \mathcal{F}^s_r(y)$.

ProofProof of the claim: By C^1-continuity, the set D inherits the strong contraction of E^s and $f^n(D)$ converges exponentially fast to q. From the Hirsh-Pugh-Shub theory (see (23), Theorem 5.4) the set $\mathcal{F}^s_r(y)$ characterises the points near y with this sharp asymptotic behaviour. Hence, $D \subset \mathcal{F}^s_r(y)$. On the other hand, the fact that D is a topological manifold of dimension s and radius r inside $\mathcal{F}^s_r(y)$ implies that we actually have $D = \mathcal{F}^s_r(y)$. ∎

By this claim, $\mathcal{F}^s_r(y) \subset \overline{\mathcal{F}^s(x)}$. Consider $n \in \mathbb{N}$, $f^n(x)$, and $f^n(y)$. Note that $\mathcal{F}^s(f^n(x))$ must accumulate at $f^n(y)$, so Claim 4.3 also gives that $\mathcal{F}^s_r(f^n(y)) \subset \overline{\mathcal{F}^s(f^n(x))}$. Taking the n-th pre-image of this inequality, and using the invariance of the foliation, we obtain that

$$f^{-n}\mathcal{F}^s_r(f^n(y)) \subset \overline{\mathcal{F}^s(x)}.$$

As this holds for every $n \in \mathbb{N}$, Lemma 4.14 implies that $\mathcal{F}^s(y) \subset \overline{\mathcal{F}^s(x)}$. ∎

In what follows, we fix the notation $\mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$ and assume that the isolating block U of an attractor $\Lambda_f(U)$ is always endowed with an extended partially hyperbolic splitting.

4.4 Lebesgue measure and genericity

In what follows we consider the manifold M endowed with a Lebesgue measure Leb.

Proposition 4.18 *Let f be a diffeomorphism in* $\mathrm{Diff}^1(M)$, *$\Lambda_f(U)$ be an isolated set of f, and $\mathcal{U}$ be a compatible neighborhood of f. Consider the map φ defined by*

$$\varphi\colon \mathcal{U} \to \mathbb{R}, \quad g \in \mathcal{U} \mapsto \varphi(g) = \mathrm{Leb}(\Lambda_g(U)).$$

The map φ is upper semicontinuous. As a consequence, the continuity points of the map φ form a residual subset of $\mathcal{U}$.

Proof: Fix $g \in \mathcal{U}$ and consider the nested sequence of open sets

$$\Lambda(g,k) = \bigcap_{n=-k}^{k} g^n(U),$$

satisfying $\Lambda(g,k) \searrow \Lambda_g(U)$ as $k \to \infty$. Since the Lebesgue measure is regular, one has that $\mathrm{Leb}(\Lambda(g,k)) \to \mathrm{Leb}(\Lambda_g(U))$.

given $\varepsilon > 0$, there is $N = N(g,\varepsilon) \in \mathbb{N}$ such that

$$\mathrm{Leb}(\Lambda(g,k)) < \mathrm{Leb}(\Lambda_g(U)) + \varepsilon, \quad \text{for all } k \geq N.$$

Note that there is some $N_0 \in \mathbb{N}$ such that the closure of $\Lambda(g, N+N_0)$ is contained in the open set $\Lambda(g,N)$. Then, for every h sufficiently close to g, it holds that $\Lambda(h, N+N_0) \subset \Lambda(g,N)$. Hence,

$$\mathrm{Leb}(\Lambda_h(U)) \leq \mathrm{Leb}(\Lambda(h, N+N_0)) \leq \mathrm{Leb}(\Lambda(g,N)) \leq \mathrm{Leb}(\Lambda_g(U)) + \varepsilon.$$

This means that $\varphi(h) \leq \varphi(g) + \varepsilon$, implying the proposition. ∎

Scholium 4.19 *Isolated sets vary upper semicontinuously.*

Proof: In the scope of the proof of Proposition 4.18 we get that every h sufficiently close to g satisfies that $\Lambda(h, N+N_0) \subset \Lambda(g,N)$. Since $\Lambda_h(U) \subset$

$\Lambda(h, N + N_0)$, we get that $\Lambda_h(U) \subset \Lambda(g, N)$. Letting ε goes to zero, and consequently $N = N(g, \varepsilon)$ goes to infinity, we obtain $\Lambda_h(U) \subset \Lambda_g(U)$ for every h sufficiently close to g, which proves the upper semicontinuity of the isolated sets. ■

Corollary 4.20 *Under the hypotheses and the notation of Proposition 4.18, if there is a dense subset $\mathcal{W}$ of $\mathcal{U}$ such that $\varphi(g) = 0$ for all $g \in \mathcal{W}$, then there is a residual subset $\mathcal{G}$ of $\mathcal{U}$ consisting of diffeomorphisms g such that $\varphi(g) = 0$.*

Proof: By the semicontinuity in Proposition 4.18, for each $n \in \mathbb{N}$ the set $\mathcal{Z}_n$ of diffeomorphisms g with $\varphi(g) < 1/n$ is open and dense in $\mathcal{U}$. Now it is enough to let $\mathcal{G} = \bigcap_{n\in\mathbb{N}} \mathcal{Z}_n$. ■

Remark 4.21 Proposition 4.18 and Corollary 4.20 hold for attractors, as any attractor is an isolated set.