4
Tubular Chart

In this section we show that for a given C^3 vector field, it is possible to find, for a non-periodic point $p \in M$, an open neighborhood U of p and a C^2 diffeomorphism $F : F^{-1}(U) \subset \mathbb{R}^d \to U \subset M$ with some nice properties (Theorem 4.0.15). This chart will allow us to construct the perturbed vector field in the Euclidean space and to compute the volume crushing that characterizes the non-existence of acips.

Let Leb denote the Lebesgue measure on $\mathbb{R}^d$.

Definition 4.0.13 *Given a constant $C > 1$, we say that a measure $m \ll \text{Leb}$, supported in some open subset $U \subset \mathbb{R}^d$, is C-*sliced *if its density*

$$\frac{dm}{d\text{Leb}}(x_1, \ldots, x_d) = \omega(x_1)$$

depends only on the first coordinate and is such that

1. *ω is C^1;*
2. *$\omega(t) > 0$ for all t;*
3. *$\dfrac{\omega'(t)}{\omega(t)} \leq C$ for all t.*

Definition 4.0.14 *Let $U \subset \mathbb{R}^d$ be a Borel set. We say that two measures m_1, m_2 on U are* comparable *if*

$$\frac{1}{2} \leq \frac{m_1(S)}{m_2(S)} \leq 2,$$

for all Borel subsets $S \subset U$.

Theorem 4.0.15 (Tubular Chart's Theorem) *Given a C^3 vector field X on M and a C^2 cross-section $\Sigma \subset M$, there exists a constant $C \geq 1$ with the following properties. For any non-periodic point $p \in \Sigma$ and any $T > 0$, there exists a neighborhood V of p, an open set $U \subset \mathbb{R}^d$, and a C^2-diffeomorphism $F : U \to F(U) \subset M$ such that:*

1. *$\varphi^t(V) \subset F(U)$ for all $t \in [-1, T+1]$;*
2. *$\varphi^t(p) = F(t, 0, \dots, 0)$, for all $t \in [-1, T+1]$;*
3. *the vector field X is tangent to the submanifold $F((\mathbb{R}^{d-1} \times \{0\}) \cap U)$;*
4. *$F^{-1}(\Sigma \cap V) \subset \{0\} \times \mathbb{R}^{d-1}$;*
5. *$\mathrm{NC}\big(D(F^{-1}(q)|_{T_q\Sigma}\big) \le C$, for all $q \in \Sigma \cap V$;*
6. *$\|DF(z)\| \le C$, for all $z \in U$;*
7. *$\|DF(z)e_d\| \cdot \|DF^{-1}(F(z))\| \le C$, for all $z \in U$ (where $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$);*
8. *$\|D^2F(z)(\,\cdot\,, e_d)\| \cdot \|DF^{-1}(F(z))\| \le C$, for all $z \in U$;*
9. *if m is the Riemannian volume on M then $(F^{-1})_*(m|_{F(U)})$ is comparable to a C-sliced measure $\hat{m}$ on U;*
10. *letting $\{P^t_p\}$ (resp. $\{\hat{P}^t_0\}$) be the linear Poincaré flow with base-point p (resp. 0) for the vector field X on M (resp. $\hat{X} \equiv (F^{-1})_* X$ on U), we have $\|P^{t,s}_p\| = \|\hat{P}^{t,s}_0\|$ for all $t, s \in [0, T]$.*

In order to clarify the significance of this result, we comment informally how it fits in our general strategy:

– The purpose of the Theorem 4.0.15 is to put the vector field on a neighborhood of a segment of orbit in a kind of standard form in order to make it easier to find perturbations with a (local) crushing property.

– Conditions 2, 3 and 4 mean that the chart "straightens" respectively a segment of trajectory, a codimension 1 invariant submanifold containing this trajectory, and the disk Σ; see Figure 4.1.

– The change of coordinates should be uniformly controlled in several ways; this is expressed by a single control parameter C. If C were allowed to depend on the time length T, the result would be much easier; indeed, in that case one could take a change of coordinates with stronger and simpler properties. However, it is essential to our strategy that C does not depend on T.

– The diffeomorphism F can be highly non-conformal. (In fact, we will see in the proof that the expansion rates along hyperplanes $\{t\} \times \mathbb{R}^{d-1}$ can be much smaller than along the line $\mathbb{R} \times \{0\}^{d-1}$.) Nevertheless, its restriction to $\Sigma \cap V$ is approximately conformal, as stated in condition 5.

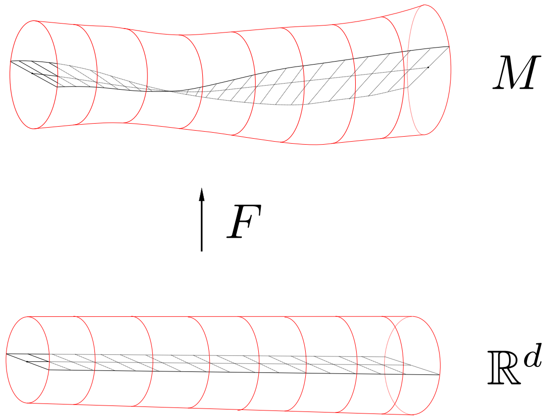

Figure 4.1: Tubular Chart

- We follow the strategy of [AB1] and try to crush volumes in one dimension only, and so to make d-dimensional objects essentially $(d-1)$-dimensional. We will crush volume towards the codimension 1 submanifold $F((\mathbb{R}^{d-1} \times \{0\}) \cap U)$ Under the change of coordinates provided by the theorem, the new vector field needs only to be perturbed along the direction of the d-th coordinate. We call such perturbations *vertical*.
- By pulling back a vertical perturbation of $\hat{X} = (F^{-1})_* X$, we should obtain a C^1-perturbation of X. Clearly, an upper bound on the C^1-distance of pulled-back vector fields should depend on the derivatives of F and F^{-1} up to second order. As we will see later, technical conditions 6, 7 and 8 are precisely what is needed to make such control possible for vertical perturbations.
- It would be nice if the map F^{-1} sent Riemannian volume in M to a Lebesgue measure in $\mathbb{R}^d$ (or a constant multiple of it); however it seems difficult to impose this extra requirement. We notice, however, that to study the crushing property we can replace a measure by a comparable one (in the sense of Definition 4.0.14.) Condition 9 in Theorem 4.0.15 means that F^{-1} sends Riemannian volume in M to something comparable to Lebesgue measure in $\mathbb{R}^d$ times a factor which varies slowly with respect to time (in the sense of the last condition in Definition 4.0.13). Those conditions will be sufficient for our strategy to work, because our crushing estimates are basically done in “time snapshots” (similarly to what happens in [AB1]).
- Condition 10 implies the norm of Poincaré flow for the new vector field $\hat{X}$ grows at most as much as fast as for X. This technical condition is needed for the construction of the crushing perturbations.

– The construction of the chart F uses the orthonormal frame flow (see § 2.4), whose class of differentiability is one less than the flow on M. Since we need F to be C^2, we ask X to be C^3. And it is of course necessary to ask Σ to be C^2, in view of condition 4.

After those remarks, let us now prove Theorem 4.0.15:

Proof: By the Whitney embedding theorem, we can assume that M is embedded in $\mathbb{R}^N$, for some large $N > d$. Moreover, by the Nash Embedding Theorem, we can assume that the Riemannian metric on M is inherited from the Euclidean metric on $\mathbb{R}^N$. (One could avoid appealing to Nash's theorem by noticing that, since M is compact, a change of Riemannian metric is absorbed by a change of the constants in the statement of Theorem 4.0.15. Alternatively, since our main theorem does not depend on the choice of the Riemannian metric, we could have fixed a priori any suitable Riemannian metric to work with.)

Fix a normal tubular neighborhood $M^\epsilon \subset \mathbb{R}^N$ of M of some width $\epsilon > 0$, and the associate bundle projection $\pi : M^\epsilon \to M$; more precisely, $M^\epsilon = \{z \in \mathbb{R}^N : d(z, M) \leq \epsilon\}$, and for each $z \in M^\epsilon$, $\pi(z)$ is the point in M which is closest to z.

Fix $X \in \mathfrak{X}^3(M)$. For any point $p \in M$ and any orthonormal frame $\mathfrak{f} = (v_1, \ldots, v_d)$ at T_pM, we will define a map $G_{p,\mathfrak{f}} : \mathbb{R}\times B_\epsilon \to M$, where B_ϵ is the closed ball in $\mathbb{R}^{d-1}$ of center 0 and radius ϵ, as follows. Let $\{(v_1(t), \ldots, v_d(t))\}_{t\in\mathbb{R}}$ be the trajectory of the orthonormal frame flow induced by X (recall § 2.4), with initial conditions

$$v_i(0) = v_i, \quad 1 \leq i \leq d.$$

Then we define

$$\begin{array}{rccl} G_{p,\mathfrak{f}}\colon & \mathbb{R} \times B_\epsilon & \to & M \\ & (x_1, x_2, \ldots, x_d) & \mapsto & \pi\left(\varphi^{x_1}(p) + \sum_{j=2}^{d} x_j v_j(x_1)\right) \end{array}$$

Since the orthonormal frame flow is C^2 (because X is C^3), this map is C^2. Moreover, by compactness of the orthonormal frame bundle, we can find a constant C_0 such that

$$\|DG_{p,\mathfrak{f}}(z)\| \leq C_0, \qquad \|D^2G_{p,\mathfrak{f}}(z)\| \leq C_0, \tag{4.1}$$

for all $p \in \Sigma$, all orthonormal frames $\mathfrak{f} \in \mathfrak{F}_p$, and all $z \in \mathbb{R} \times B_\epsilon$.

Now assume that $p \in M$ is nonsingular (i.e., $X(p) \neq 0$) and $\mathfrak{f} \in \mathfrak{F}_p$ satisfies

$$\mathfrak{f} = (v_1, \ldots, v_d) \quad \text{where} \quad v_1 = \frac{X(p)}{\|X(p)\|}\,. \tag{4.2}$$

Since $G(x_1, 0, \ldots, 0) = \varphi^{x_1}(p) \in M$, and π is a C^∞ retraction onto M, the partial derivatives of $G_{p,\mathfrak{f}}$ at $(x_1, 0 \ldots, 0)$ are given by:

$$DG_{p,\mathfrak{f}}(x_1, 0, \ldots, 0) \cdot e_j = \begin{cases} X(\varphi^{x_1}(p)) = \|X(\varphi^{x_1}(p))\| v_1(x_1) & \text{if } j = 1, \\ v_j(x_1) & \text{if } j \geq 2, \end{cases} \tag{4.3}$$

where $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{R}^d$. In particular, the map $G_{p,\mathfrak{f}}$ is a local diffeomorphism at each point in the line $\mathbb{R} \times \{0\}^{d-1}$ (under the assumptions $X(p) \neq 0$ and (4.2)).

Next, fix a C^2 cross-section $\Sigma \subset M$. Notice that the pairs $(p, \mathfrak{f})$ where $p \in \Sigma$ and $\mathfrak{f} \in \mathfrak{F}_p$ satisfies (4.2) form a compact set. Since Σ is C^2 and transverse to X, for each such p and $\mathfrak{f}$, there is a neighborhood V_p of p such that

$$G_{p,\mathfrak{f}}^{-1}(\Sigma \cap V_p) = \left\{(x, u) \in \mathbb{R} \times \mathbb{R}^{d-1} : x = g_{p,\mathfrak{f}}(u)\right\}, \tag{4.4}$$

where $g_{p,\mathfrak{f}}$ is a C^2 function on a open neighborhood of 0 in $\mathbb{R}^{d-1}$. By compactness, there is a constant C_1 such that

$$\|Dg_{p,\mathfrak{f}}(0)\| \leq C_1, \qquad \|D^2 g_{p,\mathfrak{f}}(0)\| \leq C_1, \tag{4.5}$$

for all $p \in \Sigma$ and $\mathfrak{f}$ satisfying (4.2). Also notice that $g_{p,\mathfrak{f}}(0) = 0$.

Now fix $p \in \Sigma$ and $T > 0$. The constant C that appears in the statement of the Theorem will be exhibited later, but it will not depend on p and T.

Let v_1 be given by (4.2). Choose some unit vector

$$v_d \in T_p\Sigma \cap (X(p))^\perp \tag{4.6}$$

(which is possible because we are assuming that $d \geq 3$). Next, choose vectors $v_2, \ldots, v_{d-1}$ such that $\mathfrak{f} = (v_1, v_2, \ldots, v_d)$ is an orthonormal d-frame on T_pM. (See Figure 4.2.)

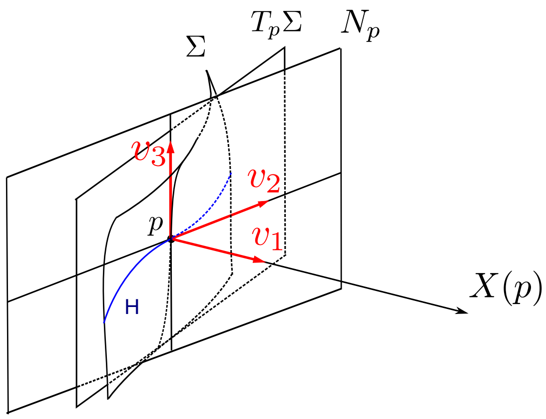

Figure 4.2: Choice of the initial orthonormal frame for $d = 3$.

Define

$$\alpha \equiv \min_{t\in[-2,T+2]} \|X(\varphi^t(p))\| \,. \tag{4.7}$$

For simplicity of notation, let $G = G_{p,\mathfrak{f}}$ and $g = g_{p,\mathfrak{f}}$. Define the following linear isomorphism

$$\begin{array}{rccc} L_\alpha \colon & \mathbb{R}^d & \to & \mathbb{R}^d \\ & (x_1, x_2, \dots, x_d) & \mapsto & (x_1, \alpha x_2, \dots, \alpha x_d) \end{array}$$

Let

$$F_1 = G \circ L_\alpha \,.$$

So (4.3) gives

$$DF_1(x_1, 0, \dots, 0) = \begin{pmatrix} \|X(\varphi^{x_1}(p))\| & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix}, \tag{4.8}$$

where the matrix is relative to the bases $(e_1, \dots, e_d)$ in $\mathbb{R}^d$ and $(v_1(x_1), \dots, v_d(x_1))$ in $T_{\varphi^{x_1}(p)}M$. By the inverse function theorem, there exists a neighborhood U_1 of $[-2, T+2] \times \{0\}^{d-1}$ such that $F_1|U_1$ is a diffeomorphism onto an open subset of M.

Notice that F_1 already satisfies property 2, that is, $F_1(t, 0, \dots, 0) = \varphi^t(p)$. The role of F_4 is basically to straighten two codimension 1 submanifolds in order to obtain properties 3 and 4.

We split $\mathbb{R}^d$ as $\mathbb{R} \times \mathbb{R}^{d-2} \times \mathbb{R}$ and take coordinates (x, w, y) with $x \in \mathbb{R}$, $w \in \mathbb{R}^{d-2}$, $y \in \mathbb{R}$.

If follows from (4.4) that for a sufficiently small neighborhood $V \ni p$,

$$F_1^{-1}(\Sigma \cap V) = \big\{(x, w, y) \in \mathbb{R} \times \mathbb{R}^{d-1} : x = g(\alpha w, \alpha y)\big\}. \tag{4.9}$$

Recalling the choice (4.7) of v_d, we obtain:

$$\frac{\partial g}{\partial y}(0,0) = Dg(0,0) \cdot e_d = 0 \,. \tag{4.10}$$

Define a diffeomorphism on a neighborhood of $[-2, T+2] \times \mathrm{dom}(g) \subset \mathbb{R}^d$ by

$$F_2(x, w, y) = \big(x - g(\alpha w, \alpha y),\ w,\ y\big).$$

So $F_2 \circ F_1^{-1}(\Sigma \cap V) \subset \{0\} \times \mathbb{R}^{d-2} \times \mathbb{R}$. Let $\{\tilde{\varphi}^t\}$ be the flow of the vector field $(F_2 \circ F_1^{-1})_* X$. Let H be a small neighborhood of 0 in $\{0\} \times \mathbb{R}^{d-2} \times \{0\}$. Then

$$\tilde{H} \equiv \bigcup_{t\in[-1,T+1]} \tilde{\varphi}^t(H)$$

is a codimension 1 submanifold of $\mathbb{R}^d$ containing the line $[-1, T+1] \times \{0\}^{d-2} \times \{0\}$.

Claim 4.0.16 *The tangent space of $\tilde{H}$ at any point of this line is $\mathbb{R} \times \mathbb{R}^{d-2} \times \{0\}$.*

Proof of the Claim: It follows from the definition of the orthonormal frame flow that

$$D\varphi^t(p) \cdot \operatorname{span}(v_1, \dots, v_{d-1}) = \operatorname{span}(v_1(t), \dots, v_{d-1}(t)).$$

Notice that the image of this space under $D(F_2 \circ F_1^{-1})(\varphi^t(p))$ is exactly the tangent space of $\tilde{H}$ at $(t, 0, 0)$. The claim follows. ■

It follows from the claim that, reducing H if necessary, the manifold $\tilde{H}$ is the graph of a function:

$$\tilde{H} = \big\{(x, w, y) : y = h(x, w)\big\},$$

where $h : \operatorname{dom}(h) \subset \mathbb{R} \times \mathbb{R}^{d-2} \to \mathbb{R}$ satisfies

$$h(x, 0) = 0 \quad \text{and} \quad Dh(x, 0) = 0\,. \tag{4.11}$$

(See Figure 4.3.)

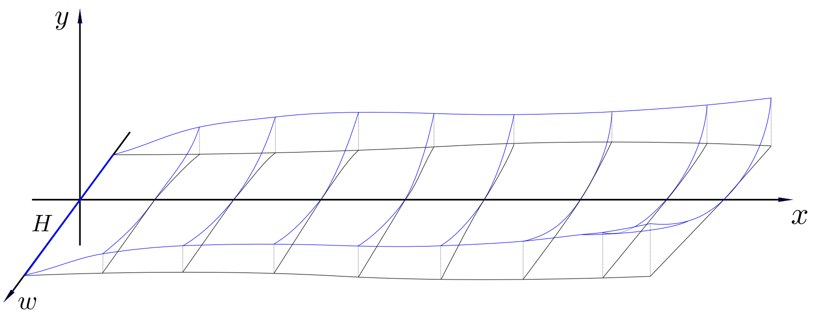

Figure 4.3: The manifold $\tilde{H}$ as a graph

Define a diffeomorphism

$$F_3(x, w, y) = \big(x,\ w,\ y - h(x, w)\big).$$

So $F_3(\tilde{H}) \subset \mathbb{R} \times \mathbb{R}^{d-1} \times \{0\}$. The compose map

$$F_3 \circ F_2(t, w, y) = \big(x - g(\alpha w, \alpha y),\ w,\ y - h(x - g(\alpha w, \alpha y), w)\big)$$

is a diffeomorphism; let $F_2 = (F_3 \circ F_2)^{-1}$, i.e.,

$$F_4(x, w, y) = \big(x + g(\alpha w, \alpha y + \alpha h(x, w)),\ w,\ y + h(x, w)\big)\,.$$

Let us check that $F = F_1 \circ F_4$ satisfies all properties in the statement of the Theorem.

We have already mentioned that F_1 satisfies property 2. Since F_4 fixes $\mathbb{R} \times \{0\}^{d-1}$, the map F will clearly inherit this property.

Properties 4 and 3 are straightforward.

It follows from Property 4, that

$$DF^{-1}(q)|_{T_q\Sigma} = (DF(F^{-1}(q)))^{-1}|_{\{0\}\times\mathbb{R}^{d-1}}. \tag{4.12}$$

Thus, in order to check Property 5, observe that $DF(0) \cdot e_j = \alpha v_j$ for $j = 2, \ldots, d$. In particular, $DF(0)$ is conformal. Taking $C \geq 2$, Property 5 follows by taking a sufficiently small neighborhood $\tilde{V} = F^{-1}(V)$ of zero.

Using (4.11) and (4.10), we see that the derivative of F_4 on the points in $\mathbb{R} \times \{0\}^{d-2} \times \{0\}$ has the following (block) matrix expression:

$$DF_4(t, 0, 0) = \begin{pmatrix} 1 & \alpha\frac{\partial g}{\partial w}(0,0) & 0 \\ 0 & \mathrm{id}_{d-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.13}$$

In particular, using (4.5) and the fact that $\alpha \leq \|X\|_{C^0}$, we obtain

$$\|(DF_4(x, 0, 0))^{\pm 1}\| \leq C_2, \tag{4.14}$$

where C_2 depends only on X and Σ. Thus, reducing U if necessary, we can assume that

$$\|(DF_4(z))^{\pm 1}\| \leq 2C_2 \quad \text{for all } z \in U. \tag{4.15}$$

It follows from (4.7) and (4.8) that $\|DF_1(x, 0, 0)\| \leq \|X\|_{C^0}$. Reducing U if necessary, we can assume that

$$\|DF_1\| \leq 2\|X\|_{C^0} \quad \text{on } F_4(U). \tag{4.16}$$

Since $F = F_1 \circ F_4$, it follows from (4.15) and (4.16) that property 6 is satisfied, provided the constant C is chosen bigger that $4C_2\|X\|_{C^0}$.

Using (4.8) and (4.13), we have

$$\begin{aligned} DF(x, 0, 0)e_d &= DF_1(F_4(x, 0, 0)) \cdot DF_4(x, 0, 0)e_d \\ &= DF_1(x, 0, 0)e_d = \alpha v_d(x). \end{aligned}$$

Reducing U, we obtain

$$\|DF(z)e_d\| \leq 2\alpha \quad \text{for all } z \in U\,. \tag{4.17}$$

It follows from (4.7) and (4.8) that $\|DF_1^{-1}(\varphi^x(p))\| = \alpha^{-1}$. So, using (4.14), we have

$$\|DF^{-1}(\varphi^x(p))\| \leq C_2\alpha^{-1},$$

for all $x \in [0,T]$. Reducing U, if necessary, we obtain

$$\|DF^{-1}(F(z))\| \leq 2C_2\alpha^{-1}, \quad \text{for all } z \in U. \tag{4.18}$$

Putting this together with (4.17), we obtain

$$\|DF(z)e_d\| \cdot \|DF^{-1}(F(z))\| \leq 4C_2\,;$$

that is, property 7 is verified, provided we choose $C \geq 4C_2$.

Let us check property 8. First observe that the linear map $D^2F(z)(e_d,\cdot)$ is the derivative of the map

$$\begin{aligned} z = (x,w,y) \mapsto\ & DF(z)\cdot e_d \\ &= DG(L_\alpha \circ F_4(z)) \circ L_\alpha \circ DF_4(z)\cdot e_d \\ &= DG(L_\alpha \circ F_4(z)) \cdot \left(\alpha\frac{\partial g}{\partial y}(w,y)\cdot e_1 + \alpha\cdot e_d\right) \\ &= \alpha\Psi(z), \end{aligned}$$

where we define Ψ as

$$\Psi(x,w,y) = DG(L_\alpha \circ F_4(z)) \cdot \left(\frac{\partial g}{\partial y}(w,y)\cdot e_1 + e_d\right)$$

Using (4.1), (4.15), (4.5), and that $\alpha \leq \|X\|_{C_0}$, we see that $\|D\Psi\| \leq C_3$, for some constant C_3 depending only on X and Σ. That is, $\|D^2F(z)(e_d,\cdot)\| \leq C_3\alpha$. Putting this together with (4.18), we conclude that property 8 is satisfied, provided $C \geq 2C_2C_3$.

Let us check Property 9. For that matter, consider the measure $\hat{m}$ defined by

$$\hat{m}(S) = \int_S \alpha^{d-1}\|X(\varphi^t(p))\|dtdx_2\ldots dx_d,$$

where $S \subset U$ is a Borel set in $\mathbb{R}^d$.

Notice that we can represent DF_1 as a matrix that sends the orthonormal base $\{e_1, e_d, \ldots, e_d\}$ of $\mathbb{R}^d$ to the orthonormal base $\{v_1(t), v_2(t), \ldots, v_d(t)\}$ of $T_{\varphi^t(p)}M$. Thus the Jacobian of F_1 is the determinant of such matrix. Using

(4.8) and (4.13), we see that the Jacobian of F along $(t, 0, \dots, 0)$ is

$$\mathrm{Jac}(F)(t, 0, \dots, 0) = \alpha^{d-1} \|X(\varphi^t(p))\|.$$

Therefore, we can reduce U if necessary, to obtain

$$\frac{1}{2} \leq \frac{\mathrm{Jac}(F)(z)}{\alpha^{d-1}\|X(\varphi^t(p))\|} \leq 2, \tag{4.19}$$

for all $z \in U$.

By the change of variables formula,

$$F_*^{-1}(m)(S) = m(F(S)) = \int_S \mathrm{Jac}(F)(t, x_1, \dots, x_d) dt dx_2 \dots dx_d,$$

which together with (4.19) leads us to conclude that $\hat{m}$ is comparable to $F_*^{-1}(m)|_{F(U)}$. In order to show that $\hat{m}$ is a C-sliced measure, observe that if $\omega(t) = \alpha^{d-1}\|X(\varphi^t(p))\|$, then

$$\begin{aligned}
\omega'(t) &\leq \alpha^{d-1} \left\| \frac{dX(\varphi^t(p))}{dt} \right\| \\
&\leq \alpha^{d-1} \|DX(\varphi^t(p)) \cdot X(\varphi^t(p))\| \\
&\leq \alpha^{d-1} \|DX\|_{C^0} \cdot \|X(\varphi^t(p))\| \\
&\leq C\omega(t),
\end{aligned}$$

provided that the constant C is chosen bigger then the C^1-norm of X.

It only remains to check Property 10. For that end, consider the canonical basis in $\mathbb{R}^d$ and the basis $(v_1(t), \dots, v_d(t))$ at the tangent space of M at $\varphi^t(p)$. We can express linear maps as matrices according to those bases. Thus:

$$\begin{aligned}
D\hat{\varphi}^s(t, 0, 0) &= (DF(t+s, 0, 0))^{-1} \circ D\varphi^s(\varphi^t p) \circ DF(t, 0, 0) \\
&= \begin{pmatrix} \|X(\varphi^{t+s}p)\|^{-1} & * \\ 0 & \alpha^{-1}\mathrm{id} \end{pmatrix} \begin{pmatrix} \frac{\|X(\varphi^{t+s}p)\|}{\|X(\varphi^t p)\|} & * \\ 0 & P_p^{t,s} \end{pmatrix} \begin{pmatrix} \|X(\varphi^t p)\| & * \\ 0 & \alpha\mathrm{id} \end{pmatrix} \\
&= \begin{pmatrix} 1 & * \\ 0 & P_p^{t,s} \end{pmatrix}.
\end{aligned}$$

So the matrices of $\hat{P}_0^{t,s}$ and $P_p^{t,s}$ coincide. Since we are taking matrices with respect to orthonormal bases, Property 10 is satisfied. ■

Remark 4.0.17 *Notice that Theorem 4.0.15 provides no uniform estimate for the C^1 norm of the new vector field $\hat{X} = (F^{-1})_* X$. It neither provides an* *estimate for the C^2 norm of F (and in fact, $\|F\|_{C^2}$ can be arbitrarily large, as shown by Example 4.0.18 below). However, no such estimates will be necessary.*

Example 4.0.18 *Let us exhibit one example where $\|F\|_{C^2}$ can be arbitrarily large. The example will be constructed in $M = \mathbb{R}^3$, but it is easy to adapt the construction to a compact manifold M of dimension $d = 3$. For $(x, w, y) \in \mathbb{R}^3$, define $X(x, w, y) = (1, 0, w^2)$. The flow induced by X is given by*

$$\varphi^t(x_0, w_0, y_0) = (x_0 + t, w_0, y_0 + w_0^2 t).$$

If $p = (0, 0, 0)$, Property 2 is already satisfied and, in particular, for any $T > 0$ we have $\alpha = 1$. Suppose Σ is a disc in $\mathbb{R} \times \{0\} \times \mathbb{R}$. By (4.2) we have $v_1 = (1, 0, 0)$; suppose we choose $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$. Then the frame $(v_1(t), v_2(t), v_3(t))$ does not depend on t and $\tilde{H}$ is the graph of $h(x, w) = xw^2$ (See Figure 4.4). Since we are already placed in $\mathbb{R}^3$ and in a context where the cross-section Σ and the base orbit are already "straight", the role of the diffeomorphism F is to straighten $\tilde{H}$, that is

$$F(x, w, y) = F_4(x, w, y) = (x, w, y + h(x, w)).$$

Observe that the curvature of the surface $\tilde{H}$ along the x-axis tends to infinity. In fact,

$$\|D^2F(x, 0, 0)\| \geq \left| \frac{\partial^2 h(x, 0)}{\partial w^2} \right| = 2|x|.$$

Therefore, the second derivative of F is unbounded.

Let $F : U \subset \mathbb{R}^d \to F(U) \subset M$ be given by Theorem 4.0.15. As explained above, we need to compare the C^1 norm of a vector field $\hat{Y} \in \mathfrak{X}^1(U)$ and its push-forward $F_*\hat{Y} \in \mathfrak{X}^1(F(U))$. Actually we will only study this problem for *vertical* vector fields $\hat{Y}$; the norm comparison is then given by the following:

Proposition 4.0.19 *Let $X \in \mathfrak{X}^3(M)$ and $F : U \subset \mathbb{R}^d \to F(U) \subset M$ be given by Theorem 4.0.15. If $\Psi : \mathbb{R}^d \to \mathbb{R}$ is a C^1 map and $\hat{Y} \in \mathfrak{X}^1(\hat{U})$ is a vector field of the form*

$$(x_1, x_2, \ldots, x_d) \to (0, \Psi(x_1, x_2, \ldots, x_d), 0, \ldots, 0),$$

then

$$\|F_*\hat{Y}\|_{C^1} \leq 2C\|\hat{Y}\|_{C^1},$$

where $C > 1$ is the constant given by Theorem 4.0.15.

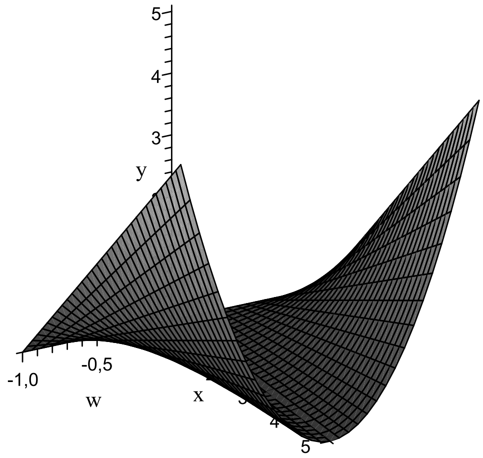

Figure 4.4: The 1-codimensional submanifold $\tilde{H} = \{(x, w, y) : y = xw^2\}$ in Example 4.0.18 is graph of a function with unbounded second derivative.

Proof: Let us denote $Y = F_* \hat{Y}$. First, note that

$$\|Y\|_{C^0} \leq \max_{z \in U} \|DF(z)\| \cdot \|\hat{Y}\|_{C^0}.$$

From property 6 in Theorem 4.0.15, we obtain

$$\|Y\|_{C^0} \leq C \|\hat{Y}\|_{C^0}.$$

Now, let us estimate the norm of the derivative. Observe that for a given $z \in U$ and v in TT_zM (which we can identify with T_zM, since M is embedded in some $\mathbb{R}^N$), we have

$$DY(z) \cdot v = A \cdot v + B \cdot v,$$

where

$$A \cdot v = DF(F^{-1}(z)) \cdot D\hat{Y}(F^{-1}(z)) \cdot DF^{-1}(z) \cdot v$$

and

$$B \cdot v = D^2F(F^{-1}(z))(DF^{-1}(z) \cdot v, \hat{Y}(F^{-1}(z))).$$

In order to estimate $\|A\|$, note that $D\hat{Y}(p) \cdot w = (D\Psi(p) \cdot w) \cdot e_d$ and $\|D\Psi(p)\| \leq \|\hat{Y}\|_{C^1}$. Then

$$\begin{aligned} \|A\| &\leq \|DF(F^{-1}(z)) \cdot D\hat{Y}(F^{-1}(z))\| \cdot \|DF^{-1}(z)\| \\ &\leq \|DF(F^{-1}(z)) \cdot e_d\| \cdot \|\hat{Y}\|_{C^1} \cdot \|DF^{-1}(z)\|. \end{aligned}$$

From property 7 in Theorem 4.0.15,

$$\|A\| \leq C\|\hat{Y}\|_{C^1}.$$

In order to estimate $\|B\|$, note that $\hat{Y}(F^{-1}(z)) = \Psi(F^{-1}(z)) \cdot e_d$ and $\|\Psi(F^{-1}(z))\| \leq \|\hat{Y}\|_{C^1}$. Then

$$\|B\| \leq \|D^2F(F^{-1}(z)(e_d, \cdot)\| \cdot \|\hat{Y}\|_{C^1} \cdot \|DF^{-1}(z)\|.$$

From property 8 of Theorem 4.0.15 we obtain

$$\|B\| \leq C\|\hat{Y}\|_{C^1}$$

and conclude that

$$\|DY(z)\| \leq 2C\|\hat{Y}\|_{C^1},$$

as claimed. ■

Now that we have presented the type of tubular chart we need in the proof of our result, we can define a κ-rectangle - a set with dimension $(d-1)$, transverse to the flow and with a specific geometry that will meet our future needs.

Definition 4.0.20 *Given $0 < \kappa < 1$ we say that $U_0 \subset \Sigma$ is a κ-rectangle if there exists $\rho > 0$ and a tubular chart $F : U \to M$ such that*

$$F(\{0\} \times [-\kappa\rho, \kappa\rho] \times [-\rho, \rho]^{d-2}) = U_0.$$

Remark 4.0.21 *The bounded eccentricity of the Euclidean κ-rectangles implies clearly that they form a Vitali Cover of $\{0\} \times \mathbb{R}^{d-1}$. By Item (5) of Theorem 4.0.15 and Remark 2.7.5, we conclude that κ-rectangles form a Vitali Cover of $\Sigma \cap V$.*