Pontifícia Universidade Católica
do Rio de Janeiro

**João Antônio Recio da Paixão**

# Analysis of Morse matchings: parameterized complexity and stable matching

**Tese de Doutorado**

Thesis presented to the Postgraduate Program in Mathematics of the Departamento de Matemática do Centro Técnico Científico da PUC–Rio, as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Thomas Lewiner

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DO RIO DE JANEIRO

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João Paixão received a Bachelor of Science degree in Mathematics from Virginia Tech University in 2007 with emphasis in Combinatorics and Algebra. In 2010, he obtained a Master of Science degree in Applied Mathematics at PUC-Rio under the supervision of Dr. Hélio Lopes and Dr. Thomas Lewiner with emphasis in vector field topology. During his doctorate at PUC-Rio, he also studied at the University of Queensland in Australia under the guidance of Benjamin Burton. His interests include computational topology, computational geometry, discrete Morse theory, vector field topology, graph theory, matchings, and complexity theory. Since 2013, he is an assistant professor in Computer Science at Departamento de Ciência da Computação, Instituto de Matemática, UFRJ.



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## Abstract

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Morse theory relates the topology of a space to the critical elements of a scalar function defined on it. This applies in both the classical theory and a discrete version of it defined by Forman in 1995. Those Morse theories permit to characterize a topological space from functions defined on it, but also to study functions based on topological constructions it implies, such as the Morse-Smale complex. While discrete Morse theory applies on general cell complexes in an entirely combinatorial manner, which makes it suitable for computation, the functions it considers are not sampling of continuous functions, but special matchings in the graph encoding the cell complex adjacencies, called Morse matchings.

When using this theory to study a topological space, one looks for *optimal* Morse matchings, *i.e.* one with the smallest number of critical elements, to get highly succinct topological information about the complex. The first part of this thesis investigates the parameterized complexity of finding such optimal Morse matching. On the one hand the ERASABILITY problem, a closely related problem to finding optimal Morse matchings, is proven to be $W[P]$-complete. On the other hand, an algorithm is proposed for computing optimal Morse matchings on triangulations of 3-manifolds which is fixed-parameter tractable in the tree-width of its dual graph.

When using discrete Morse theory to study a scalar function defined on the space, one looks for a Morse matching that captures the geometric information of that function. The second part of this thesis introduces a construction of Morse matchings based on stable matchings. The theoretical guarantees about the relation of such matchings to the geometry are established through surprisingly simple proofs that benefits from the local characterization of the stable matching. The construction and its guarantees work in any dimension. Finally stronger results are obtained if the function is discrete smooth on the complex, a notion defined in this thesis.

## Keywords

Discrete Morse theory. Optimal Morse function. Parameterized complexity. Morse-Smale decomposition. Stable matching. Computational topology.

## Resumo

Paixão, João; Lewiner, Thomas. **Análise de casamentos de Morse:complexidade parametrizada e casamento estável**. Rio de Janeiro, 2014. 86p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

A teoria de Morse relaciona a topologia de um espaço aos elementos críticos de uma função escalar definida nele. Isso vale tanto para a teoria clássica quanto para a versão discreta proposta por Forman em 1995. Essas teorias de Morse permitem caracterizar a topologia do espaço a partir de funções definidas nele, mas também permite estudar funções a partir de construções tipológicas derivadas dela, como por exemplo o complexo de Morse-Smale. Apesar da teoria de Morse discreta se aplicar para complexos celulares gerais de forma inteiramente combinatória, o que torna a teoria particularmente bem adaptada para o computador, as funções usadas na teoria não são amostragens de funções contínuas, mas casamentos especiais no grafo que codifica as adjacências no complexo celular, chamadas de casamentos de Morse. Quando usar essa teoria para estudar um espaço topológico, procura-se casamentos de Morse *ótimos*, *i.e.* com o menor número possível de elementos críticos, para obter uma informação topológica do complexo sem redundância. Na primeira parte desta tese, investiga-se a complexidade parametrizada de encontrar esses casamentos de Morse ótimos. Por um lado, prova-se que o problema Erasability, um problema fortemente relacionado à encontrar casamentos de Morse ótimos, é $W[P]$-completo. Por outro lado, um algoritmo é proposto para calcular casamentos de Morse ótimos em triangulações de 3-variedades, que é FPT no parâmetro do *tree-width* de seu grafo dual. Quando usar a teoria de Morse discreta para estudar uma função escalar definida no espaço, procura-se casamentos de Morse que capturam a informação geométrica dessa função. Na segunda parte é proposto uma construção de casamentos de Morse baseada em casamentos estáveis. As garantias teóricas sobre a relação desses casamentos com a geometria são elaboradas a partir de provas surpreendentemente simples que aproveitam da caracterização local do casamento estável. A construção e as suas garantias funcionam em qualquer dimensão. Finalmente, resultados mais fortes são obtidos quando a função for "suave discreta", uma noção definida nesta tese.

## Palavras–chave

Teoria de Morse discreta. Funções de Morse ótimas. Complexidade parametrizada. Decomposição de Morse-Smale. Casamento estável. Topologia computacional.

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# Nomenclature

| | |
|---|---|
| | **preliminaries: matchings** |
| $G = (N, A)$ | graph $G$ with its set of nodes $N$ and arcs $A$ |
| $M \subset A$ | matching |
| $\pi(x, y)$ | weight associated with arc $\{x, y\} \in A$ |
| | **preliminaries: simplicial complexes** |
| $V$ | finite set of vertices |
| $\Delta$ | simplicial complex |
| $\sigma, \tau.\rho \in \Delta$ | simplices of $\Delta$: $\tau \subset V$ |
| $\sigma \prec \tau$ | $\sigma$ is a facet $\tau$ |
| $p = \dim \tau$ | dimension of a simplex $\tau$: $\tau$ has $p + 1$ vertices |
| $d = \dim \Delta$ | dimension of $\Delta$ |
| $H$ | Hasse diagram of $\Delta$ |
| | **preliminaries: discrete Morse theory** |
| $\mathcal{V}$ | discrete vector field on $\Delta$: matching of $H$ |
| $\sigma \rightarrow \tau$ | matched simplices: $(\sigma, \tau) \in \mathcal{V}$ |
| $\blacktriangleleft \sigma_0\tau_0 \ldots \sigma_s\tau_s \blacktriangleright$ | $\mathcal{V}$-path: $\sigma_i \rightarrow \tau_i$ and $\tau_i \succ \sigma_{i+1} \neq \sigma_i$ |
| $\chi(\Delta)$ | Euler characteristic of $\Delta$ |
| $m_p$ | number of critical simplices of dimension $p$ |
| $m(\mathcal{V})$ | total number of critical simplices |
| | **complexity of optimal Morse matchings** |
| $\Delta \rightsquigarrow \tilde{\Delta}$ | $\Delta$ collapses to $\tilde{\Delta}$ |
| $\delta$ | a subcomplex of $\Delta$ with no triangle |
| $\Delta \rightsquigarrow \delta$ | $\Delta$ is erasable |
| $T$ | tree decomposition of a graph $G$ |
| $X_i, i \in I$ | bag associated to node $i$ of $T$: $X_i \subseteq N$ |
| $\mathbf{tw}(G)$ | treewidth of $G$ |
| $F_i$ | set of forgotten nodes when reaching node $i$ in the algorithm |
| $\mathbf{v}(M)$ | binary vector for checking the marching condition of $M$ |
| $\mathbf{uf}(M)$ | union-find structure for the cycle-free condition of $M$ |
| $c(x)$ | node representing the connected component of $x$ in $\mathbf{uf}(M)$ |
| $w_i$ | number of nodes in bag $X_i$ |
| $\Gamma(\mathcal{T})$ | dual graph of a simplicial triangulation of a 3-manifold $\mathcal{T}$ |

**stable Morse matchings**

| | |
|---|---|
| $f$ | scalar function on the vertices of $\Delta$ |
| $\pi_f(\sigma, \tau)$ | arc weight associated with function $f$: $\pi_f(\sigma, \tau) = f(\tau \setminus \sigma)$ |
| $>_{lex}$ | increasing lexicographic ordering of simplices |
| $\mathrm{lk}_0(\sigma)$ | vertex link of $\sigma$: $\mathrm{lk}_0(\sigma) = \{v \in V,\ \sigma \cup v \in \Delta\}$ |
| $\mathrm{v}_{\mathrm{lk}}(\sigma)$ | smallest vertex in the vertex link of $\sigma$: $\mathrm{v}_{\mathrm{lk}}(\sigma) = \min \mathrm{lk}_0(\sigma)$ |
| $\mathrm{v}_{\mathrm{m}}(\sigma)$ | smallest vertex in $\sigma$: $\mathrm{v}_{\mathrm{m}}(\sigma) = \min \sigma$ |
| $St(v)$ | star of vertex $v$: $St(v) = \{\sigma \in \Delta : v \in \sigma\}$ |
| $St_-(v)$ | lower star of vertex $v$: $St_-(v) = \{\sigma \in St(v), v' \in \sigma \Rightarrow f(v) \geq f(v')\}$ |
| $St_+(v)$ | upper star of vertex $v$: $St_+(v) = St(v) \setminus St_-(v)$ |
| $>_{revlex}$ | decreasing lexicographic ordering of simplices |
| $\Delta'$ | barycentric subdivision of $\Delta$ |
| $\Sigma = (\sigma_0, \ldots, \sigma_p)$ | simplex of $\Delta'$ |
| $L(\sigma', \sigma)$ | simplices involved in $\mathrm{lk}_0((\sigma', \sigma))$ that are not subset of $\sigma'$ |
| $l(\sigma', \sigma)$ | minimal simplex of $L(\sigma', \sigma)$ |

*A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.*

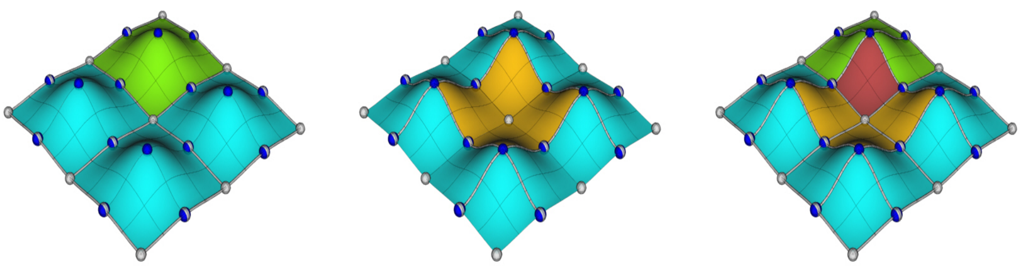
**E.T. Bell**, *Men of Mathematics (1937).*

# 1
# Introduction

Classical Morse theory [53] relates the topology of a manifold to the critical points of smooth scalar functions defined on it. This relation works in both directions: the topology of a manifold can be studied from functions defined on it and the behavior of a function can be captured by topological structures induced by this function on the manifold.

To illustrate the first direction, Morse proved that a manifold that admits a smooth function with only two critical points is homotopic to a sphere. This motivates the search for functions with a minimal number of critical points, called *optimal Morse functions*. However, identifying a topological object is known to be a computationally difficult task. We thus expect optimal Morse functions to be hard to compute, which is the topic of the first part of the present thesis.

The other way to use Morse theory is to construct topological decompositions of the manifold induced by a given function to study that function. As a fundamental example, the Morse-Smale complex (see Figure 1.1) captures the regions of the manifold where the function leads to a uniform flow. In order to compute such structures, a discrete representation of the function should be defined, hopefully preserving its critical points and dynamics. The last part of this thesis studies the behavior captured by a discrete representation of Morse functions.



(a) Critical points of the height function on a sine-shaped domain.

(b) Basin of attraction of a minimum.

(c) Intersection with the basin of repulsion of a maximum.

Figure 1.1: A Morse-Smale complex of the height function (image from A. Gyulassy [33].

All along this thesis, we use a discrete version of Morse theory introduced by Forman [26], that extended classical Morse theory to discrete objects such as triangulated manifolds, abstract simplicial complexes and arbitrary cell complexes. In this discrete version, the smooth functions on the manifold are represented as a so-called *Morse matching*, which is a particular matching in the Hasse diagram of the cell complex [17,26]. This discretization preserves and extends most of smooth Morse theory's results, in particular that a triangulation of a closed manifold admitting a Morse matching with only two unmatched (critical) elements is homotopic to a sphere [26,58]. The construction of specific Morse matchings has proven to be a powerful tool to understand topological [26,37,38,43,44], combinatorial [17,36,40] and geometrical [33,41,59,61] structures of discrete objects. This large number of applications motivates us to go back to the fundamentals, analyzing the constructions of both optimal Morse matching and geometrical Morse matching.

### Our story

Three years ago, when I started studying discrete Morse theory, reading my advisor's theses [41,43], I was struck by two intricate and weird-looking figures, (Figures 1.2 and 1.3). But I became extremely curious when he told me that those figures were proofs.

**The first figure.** Figure 1.2 is a (non-manifold) simplicial complex used in a reduction proof which shows that Erasability, a problem strongly related to finding optimal Morse matchings, is hard to compute (indeed $NP$-hard) [25]. Essentially the authors built a simplicial complex for *every* instance of a well known $NP$-hard problem, in their case Set Cover and showed that if you can solve Erasability on it, then you can solve Set Cover. Therefore Erasability is at least as hard as Set Cover. However, the authors stated that they were not able to reduce in the other direction, to show that Erasability is equivalent to Set Cover. They conjectured that Erasability should be harder than Set Cover, but left this conjecture as an open problem.

A year later, when I arrived at the University of Brisbane for my doctoral internship, I was luckily introduced to parameterized complexity theory, which has an entire hierarchy of problems to measure their difficulty. This was exactly the right tool to tackle that open question and others: Is Erasability equivalent to Set Cover? If not, how much harder is Erasability? In Chapter 3, we give a dismal answer for these questions.

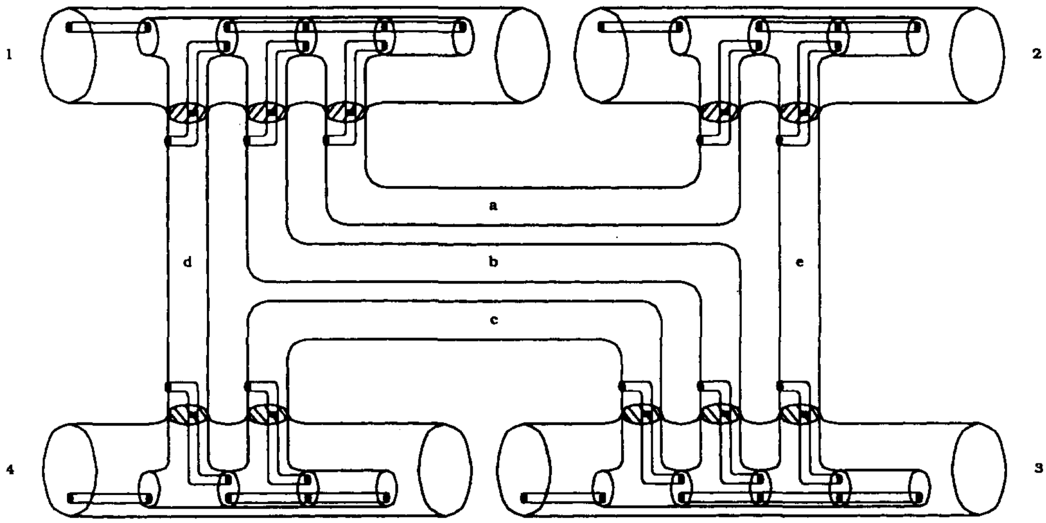



Figure 1.2: Gadget utilized to prove that ERASABILITY is $NP$-hard [25].

However, it was a puzzling result since, in practice, heuristics would, in the majority of cases, solve ERASABILITY and find optimal Morse matchings very fast, as reported in my advisor's thesis [43,44]. Therefore we wondered if there was a large class of simplicial complexes where these problems could be solved polynomial time. After the bad news, the results in Chapter 4 bring some good news, in the order most people prefer.



**The second figure.** The second figure (Figure 1.3) is part of a proof that the greedy construction of Morse matchings from a scalar function proposed by my advisor [41,42,45], correctly captures the dynamics of that function, and generates correct geometric information such as the one in Figure 1.1. It is fundamental for most applications that the constructed discrete Morse matching captures the geometric information correctly.

It is proven [41] that, for surfaces, my advisor's construction captures the critical points if the input simplicial complex is subdivided *twice*. The proof is brute force combinatorics, it requires checking a large (impractical?) amount of different cases the construction algorithm may generate. Figure 1.3 is a visualization of one particular case of saddle.

My advisor believed that there should be much easier proofs and his theorems would generalize to higher dimensions. He wrote at the end of his thesis: "The combinatorial proof of our construction of geometrical Morse complex seems to generalize to any dimension and to non–regular cases. However, the proof in itself is laborious and a generalization would require more tools from combinatorics."

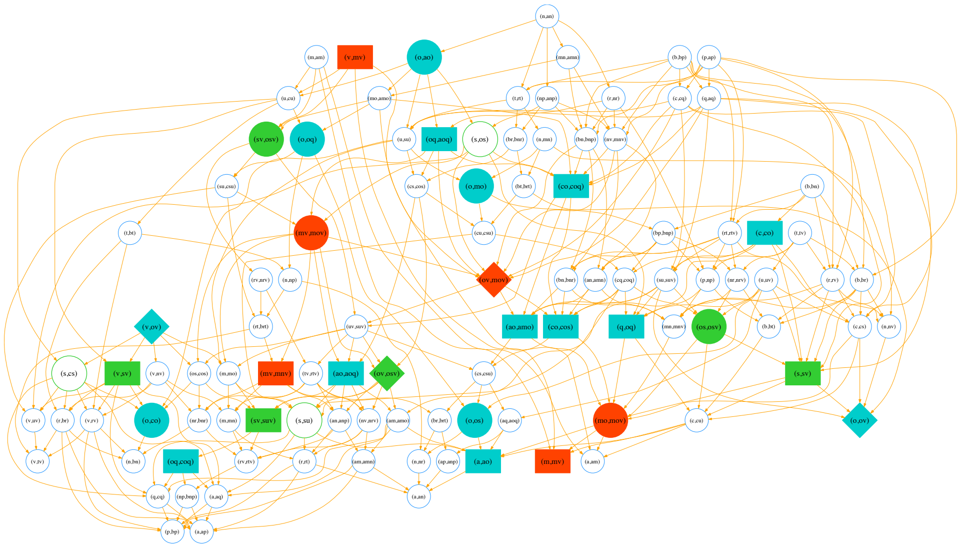


Figure 1.3: Poset used to prove that greedy construction of Morse matchings capture the geometric information correctly [41,42].

I was intrigued, but we forgot about this problem for a while and concentrated on just building discrete vector fields (also from a given geometric function), a generalization of Morse matchings which is just a graph matching problem, therefore easier to build. While reviewing the matching construction literature, we found the *stable matching*, a matching with a remarkably simple local characterization, that later on received a lot of attention since its authors won the Nobel Prize. We decided to construct discrete vector fields with these stable matchings to understand them with this local characterization.

Surprisingly in our computational experiments, every discrete vector field we built, on a surface, from the stable matching turned out to be a Morse matching. We were able to quickly prove this fact for surfaces and the proof was so simple we believed we would be able to generalize to any dimension and prove strong connections to the given geometry (see Chapter 5). We were also able to characterize the critical elements of those stable Morse matchings requiring only *one* barycentric subdivision. The proofs turned out to be much simpler using a different way of translating the input geometric function, using lexicographic orderings (which was indeed one of the first ways to build discrete Morse matchings [4]). While cleaning the proofs, I understood the smoothing effect of the barycentric subdivision, and introduced the concept of discrete smooth complexes, which seems a promising category to work with discrete geometric Morse constructions (see Chapter 6).

### Optimal Morse matchings

Morse matchings that minimize the number of critical elements are known as *optimal matchings* [43], and correspond to a well-known problem in classical Morse theory [57,65]. The number and type of the critical elements are topological (more precisely homotopy) invariants of the cell complex, just like in the case of the sphere described above. Forman's definitions [17,26] are purely combinatorial, which allows an exact interpretation of optimal matchings computations, and gave those a strong place in the computational topology [19]. Moreover, optimal Morse matchings are useful in practical applications such as volume encoding [45,64], or homology and persistence computation [30,41].

However, constructing optimal matchings is known to be NP-hard on general 2-complexes and on 3-manifolds [37,38,43]. This result follows from a reduction to this problem from the closely related to the ERASABILITY problem: how many faces must be deleted from a 2-dimensional simplicial complex before it can be completely erased, where in each erasing step only *external triangles*, *i.e.* triangles with an edge not lying in the boundary of any other triangle of the complex, can be removed [25]? Despite this hardness result, large classes of inputs – for which worst case running times suggest the problem is intractable – allow the construction of optimal Morse matchings in a reasonable amount of time using simple heuristics [44]. Such behavior suggests that, while the problem is hard to solve for some instances, it might be much easier to solve for instances which occur in practice. This motivates us to look for which *parameter* of a problem instance is responsible for the intrinsic hardness of the optimal matching problem.

To do so, we use parameterized complexity [20] to study this topological problem. More precisely, we determine the hardness of Morse type problems using the mathematically rigid framework of the $W$-hierarchy. Our first main result shows that the ERASABILITY problem is $W[P]$-complete, *i.e.* in the worst category of the $W$-hierarchy [14] (Theorems 3.13 and 3.16), when the parameter is the *natural parameter* – the number of cells that have to be removed. In other words, we prove that the ERASABILITY problem is fixed-parameter intractable in this parameter. This settles a conjecture from Eğecioğlu and Gonzalez [25] where the authors write: "We have not been able to construct an approximation preserving reduction from the SET COVER problem to the ERASABILITY problem. That is why ERASABILITY approximation problem *seems* harder than the one for SET COVER."

From a discrete Morse theory point of view, this reflects the intuition that reaching optimality in Morse matchings requires a global (at least topological) context, which is known to be computationally hard. In this way, we also show that the $W$-hierarchy as a purely complexity theoretical tool can be used in a very natural way to answer questions in the field of combinatorial topology. Although there are many results about the computational complexity of topological problems [2,15,25,49,66], to the authors' knowledge, Erasability is the first purely geometric problem shown to be $W[P]$-complete.

Our second result refines the observation that simple heuristics allow us to compute optimal matchings efficiently [44]. For general 2-complexes (and 3-manifolds), the problem reduces directly to finding a maximal alternating cycle-free matching on a spine, *i.e.* a bipartite graph representing the 1- and 2-cell adjacencies [3,38,43] (Lemma 3.8). To solve this problem, we propose an explicit algorithm for computing Morse matchings on bipartite graphs which is fixed-parameter tractable in the treewidth of the graph (Theorem 4.6).

Furthermore, we show that finding optimal Morse matchings on triangulated 3-manifolds is also fixed-parameter tractable in the treewidth of the dual graph of the triangulation (Theorem 4.8), which is a common parameter when working with triangulated 3-manifolds [15]. Our result for 3-manifold has been generalized in a surprising way very recently [13].

Finally, we use the classification of simplicial and generalized triangulations of 3-manifolds to investigate the "typical" treewidth of the respective graphs for relevant instances of Morse type problems. In this way, we give further information on the relevance of the fixed parameter results. The experiments show that the average treewidths of the respective graphs of simplicial triangulations of 3-manifolds are particularly small in the case of generalized triangulations, confirming the intuition that simple heuristics for optimal Morse matching do work on a large class of model. Furthermore, experimental data suggest a much more restrictive connection between the treewidth of the dual graph and the spine of triangulated 3-manifolds than the one stated in Theorem 4.8.

### Geometric Morse matchings

Using Morse theory in the other direction, one of the main applications of the theory is to obtain a Morse-Smale complex of a scalar function $f$, which is a decomposition of the domain into regions of uniform flow of $-\nabla f$ [16,34,45]. In Forman's discrete Morse theory, the Morse-Smale complex can be seamlessly obtained from the Morse matching [16,45], without any numerical integration or differentiation. The difficulty of the problem then concentrates on building a Morse matching which is faithful to the given geometry, *i.e.* a function $f$ sampled at the vertices.

There have been numerous geometric constructions proposed for the discrete gradient field. Lewiner *et al.* [16,45] proposed a modified greedy weighted matching algorithm with weights based on the steepest descent, *i.e.* the difference of the function values at the cells. The algorithm checks an acyclic condition at each step before adding a new edge to the matching. The geometric accuracy of the algorithm was proven robust when the triangulation is subdivided *twice* [41,42]. In addition, the proof techniques used are complicated and convoluted, making it difficult to generalize and expand the results. However this algorithm has been used in various applications [16,70,71].

Gyulassy *et al.* [34] suggested a priority-queue based algorithm to avoid checking for cycles, but his algorithm does not have any theoretical guarantees. This construction was used to build Morse-Smale complexes in several applications [32,62]. Robins *et al.* [61] developed an algorithm which uses homotopy expansions to build the Morse matching on the lower star of each vertex, and the authors are able to prove a one–to–one correspondence between the critical cells and the piecewise-linear critical points on surfaces. Because of this correspondence, this algorithm is a widely used algorithm in the literature [31,50,60].

Babson and Hersh [4] suggested a greedy algorithm with lexicographic weights and they were able to characterize the critical points in the barycentric subdivision. We were inspired in their work for the results Chapters 5 and 6.

In this thesis, we propose a construction of geometric Morse matching with a stable matching algorithm [28], which works in any dimension. The main advantage is that stable matching is characterized by a simple local stability condition, which extends and greatly simplifies the proofs compared to previous results using posets [41,42]. In Chapter 5, the local stability condition is used to prove results about the location of critical cells and regular simplexes. In particular we prove that, under mild assumptions on $f$, the acyclic condition is automatically satisfied, therefore it yields indeed a Morse matching [55]. Under those assumptions, the stable matching construction turns out to be equivalent to greedy constructions [41,42] that are already used in several applications [16,45,70,71]. We introduce the notion of discrete smoothness in Chapter 6, and fully characterize the critical cells of stable Morse matching built from discrete smooth functions. Finally we prove that one barycentric subdivision turns any function discrete smooth, generalizing previous results on greedy construction to a *single* barycentric subdivision, and in some case to any dimension.

### Summary of results

**Chapter 3.** We prove that Erasability is at the very top of hierarchy, in terms of difficulty, of parameterized complexity theory, *i.e.* Erasability is $W[P]$-complete (Theorems 3.13 and 3.16). This answers a conjecture made by Eğecioğlu and Gonzalez [25]. To the authors' knowledge, Erasability is the first purely topological problem shown to be $W[P]$-complete.

**Chapter 4.** We propose an explicit algorithm for finding optimal Morse matchings which is fixed-parameter tractable in the treewidth of the spine of the Hasse diagram (Theorem 4.6). We also extend our result and show that finding optimal Morse matchings on triangulated 3-manifolds is also fixed-parameter tractable in the treewidth of the dual graph of the triangulation (Theorem 4.8), which is a common parameter when working with triangulated 3-manifolds [15]. All the results work for Erasability as well.

To gain intuition on when the treewidth is small, *i.e.* when we can efficiently solve the previous two problems, we compute the treewidth of the relevant graphs (*i.e.* the spine and the dual graph) of all closed generalized triangulations of 3-manifolds up to 7 tetrahedra [11], and all simplicial triangulations of 3-manifolds up to 10 vertices [48] in Section 4.6. We observe that the treewidth is indeed small in the majority of cases.

The results from Chapters 3 and 4 are published [14] and are joint work with Jonathan Spreer and Benjamin Burton at University of Queensland.

**Chapter 5.** We propose an algorithm, based on stable matchings, to construct Morse matchings guided by geometric function on simplicial complex of any dimension. We prove that those stable matchings are indeed Morse matchings in Theorem 5.6. We also show that these stable Morse matchings respect the given function since they point in the direction of the steepest descent in Theorem 5.9 and the function value along its $\mathcal{V}$-paths are decreasing (Theorem 5.5). Part of this chapter was presented at the Young Researcher Forum at SoCG 2013 [55]

**Chapter 6.** We define the concept of discrete smoothness for functions in simplicial complexes. For these functions we characterize the behavior of the stable Morse matching (Corollary 6.3) and its critical simplexes (Corollary 6.5). With these results we detail the local neighborhood of critical simplexes of discrete smooth functions in Lemmata 6.6 and 6.7 and see that they look like their continuous counterparts. In addition, we prove a relationship between the critical simplicies and Banchoff's critical vertices for smooth functions in Section 6.3. In the last section, we show that any function becomes smooth after a single barycentric subdividing of the simplicial complex. Therefore all results in this chapter extend to barycentric subdivisions (Theorem 6.22).

# 2
# Preliminaries

In this chapter, the basic notions used throughout the thesis are defined. First in Section 2.1, we review the concept of matchings in graphs and specifically the notion of stable matchings. Then in Section 2.2, we define simplicial complexes, which serves in this thesis as the discretization of a manifold. With the concepts of matching and simplicial complex, the main ideas of discrete Morse theory are defined in Section 2.3. Finally in Section 2.4, the theory of parameterized complexity theory is reviewed. This theory is the main tool used in Chapters 3 and 4, to analyze the complexity of algorithms in discrete Morse theory.

## 2.1
## Matchings

We denote a graph $G$ by its set of nodes $N$ and arcs $A$ as $G = (N, A)$ [46].

**Definition 2.1** (Matching)**.** *A matching $M$ of a graph $G = (N, A)$ is a subset of arcs $M \subset A$ such that every node is incident to at most one arc in $M$.*

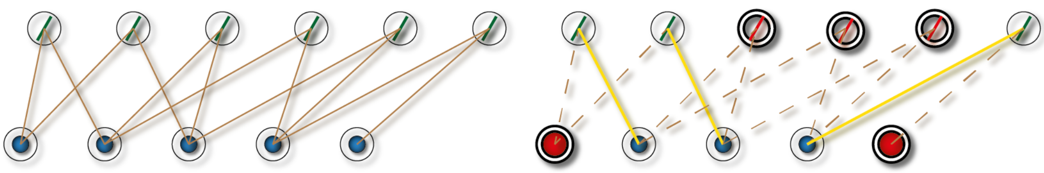


Figure 2.1: A bipartite graph (left) with a matching: the solid arcs are the matching arcs, and the red circles are the unmatched nodes (right).

Arcs in $M$ are called *matching arcs* and the nodes of the matching arcs are called *matched nodes*. Nodes and arcs which are not matched are referred to as *unmatched* (Figure 2.1). An *alternating path* is a path in $G$ in which the arcs belong alternatively to $M$ and $A \setminus M$.

The *induced M-subgraph* is the subgraph of $G$ spanned by all matched nodes and the *size* of a matching $M$ is the number of matching arcs. A matching $M$ is called a *maximum cardinality matching* of a graph $G$ if there is no matching whose size is larger than the size of $M$.

In a *weighted graph*, every arc in the graph is associated to a real number called *weight*. Let $\pi(x, y)$ be the weight associated with arc $\{x, y\} \in A$.

**Definition 2.2** (Unstable pair and stable matching)**.** *Given a matching $M$, an arc $\{x, y\} \in A \setminus M$ is unstable, if and only if both conditions below hold:*

1- *$x$ is unmatched or $\exists\, w \in N$, such that $\{x, w\} \in M$ and $\pi(x, w) > \pi(x, y)$,*

2- *$y$ is unmatched or $\exists\, z \in N$, such that $\{y, z\} \in M$ and $\pi(y, z) > \pi(x, y)$.*

*$M$ is a stable matching if there is no unstable pair (see Figure 2.2).*

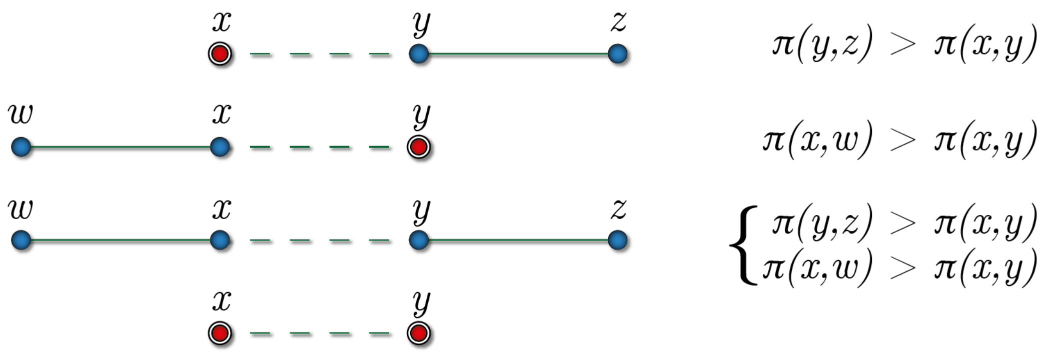



Figure 2.2: The four possibilities of an unstable pair.

A stable matching might not exist for every weighted graph, however it always exists for weighted *bipartite* graphs [46].

**Theorem 2.3** (Existence of a stable matching [28])**.** *If graph $G$ is bipartite, then there exists a stable matching $M$ of $G$. The stable matching $M$ can be found by the Gale-Shapley algorithm [28].*

There can be many stable matchings in a graph however for certain weighted graphs, the stable matching is unique.

**Theorem 2.4** (Uniqueness of a stable matching [22])**.** *If graph $G$ is bipartite and adjacent arcs have distinct weights, then the stable matching is unique.*

## 2.2 Simplicial complex

A finite simplicial complex is a finite set of vertices $V$ along with a set $\Delta$ of subsets of $V$, such that $\Delta$ satisfies the following two properties:

(i) $V \subset \Delta$,

(ii) if $\tau \in \Delta$ and $\sigma \subset \tau$, then $\sigma \in \Delta$.

We refer to the simplicial complex as $\Delta$. The elements of $\Delta$ are called simplices. A simplex $\tau \in \Delta$ is said to have *dimension* $p$, denoted by $\dim(\tau) = p$, if $\tau$ contains $p+1$ vertices. A simplex of dimension $p$ is called a *$p$-simplex*. Simplex $\sigma$ is a facet of $\tau$, denoted by $\sigma \prec \tau$, if $\sigma \subset \tau$ and $\dim(\sigma) = \dim(\tau) - 1$. The dimension of $\Delta$ is the largest dimension of its simplices.

Given a simplicial complex $\Delta$, one defines its *Hasse diagram* $H$ to be a directed graph in which the set of nodes of $H$ is the set of simplices of $\Delta$, and an arc goes from $\tau$ to $\sigma$ if and only if $\sigma \prec \tau$ (see Figure 2.3). Observed that $H$ is a bipartite graph since there is a partition into the even- and odd-dimensional simplices.

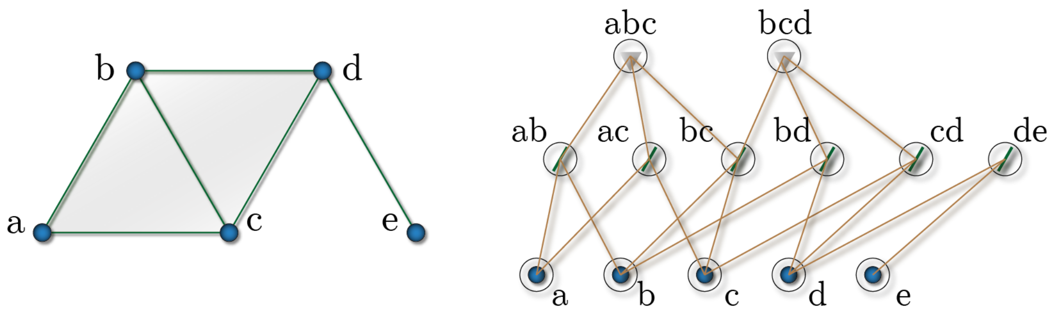



Figure 2.3: A simplicial complex $\Delta$ (left) and its Hasse diagram $H = (N, A)$ (right).

Let $H_p \subseteq H$ be the bipartite subgraph spanned by all nodes of $H$ corresponding to simplices of dimensions $p$ and $p+1$. In particular, $H_1$ describes the adjacency between the 2-simplices and 1-simplices of $\Delta$, and will be called the *spine* of the simplicial complex $\Delta$. The spine of a simplicial complex will be the main object in Chapter 4.

## 2.3 Discrete Morse theory

Consider now a simplicial complex $\Delta$.

**Definition 2.5** (Discrete vector field). *A discrete vector field $\mathcal{V}$ on $\Delta$ is a collection of pairs $(\sigma, \tau)$ of simplices of $\Delta$ with $\sigma \prec \tau$, such that each simplex is in at most one pair of $\mathcal{V}$. We write $\sigma \to \tau$ if $(\sigma, \tau) \in \mathcal{V}$, and $\sigma \not\to \tau$ otherwise. A simplex $\sigma$ is said to be* critical *for discrete vector field $\mathcal{V}$ if it does not belong to any pair of $\mathcal{V}$.*

A discrete vector field is thus a matching on the Hasse diagram and the critical faces are the unmatched nodes (see Figure 2.4).

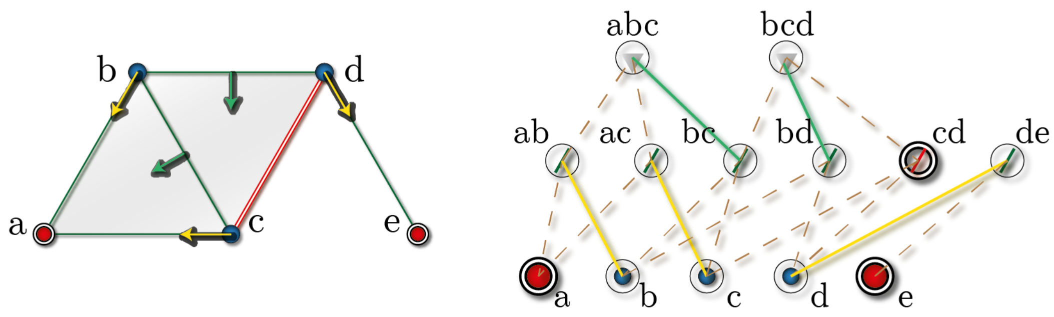



Figure 2.4: The discrete vector field $\mathcal{V}$ on $\Delta$ and on its Hasse diagram.

**Definition 2.6** (Matched below/above). *If there exists $v \in \sigma$, such that $\sigma \setminus v \to \sigma$, then $\sigma$ is matched below. If there exists $v \in \Delta$, such that $\sigma \cup v \in \Delta$ and $\sigma \to \sigma \cup v$, then $\sigma$ is matched above. If $\sigma$ is neither matched below nor above, then $\sigma$ is critical (see Figure 2.5).*

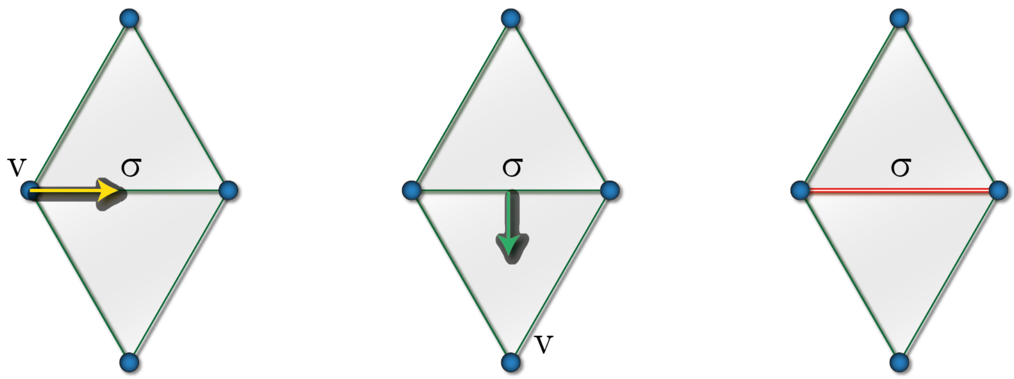



Figure 2.5: Notation: $\sigma$ matched below (left), $\sigma$ matched above (center), $\sigma$ critical (right).

**Definition 2.7** ($\mathcal{V}$-paths). *Given a discrete vector field $\mathcal{V}$ on a simplicial complex $\Delta$, a $\mathcal{V}$-path is a sequence of simplices $\blacktriangleleft\sigma_0\tau_0\sigma_1\tau_1\ldots\sigma_s\tau_s\blacktriangleright$ such that for all $i$, $\sigma_i \to \tau_i$ and $\tau_i \succ \sigma_{i+1} \neq \sigma_i$. We say that such a path is a non-trivial closed path if $s > 0$ and $\sigma_0 = \sigma_s$.*

Observe that a $\mathcal{V}$-path is an alternating path in $H$ with the restriction that all the dimensions of this simplices are $p$ or $p+1$, with $p = \dim(\sigma_0)$ (see Figure 2.6).

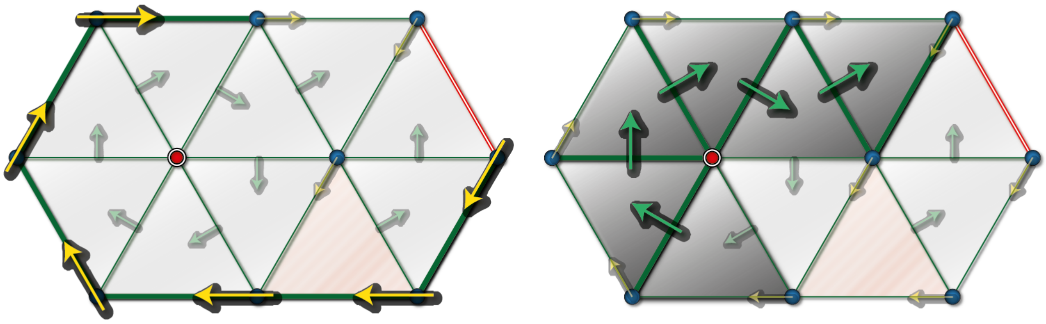


Figure 2.6: A $\mathcal{V}$-path vertices-edges (right) and edges-triangles (left).

**Definition 2.8** (Gradient vector field). *A discrete vector field $\mathcal{V}$ is the gradient vector field if and only if there is no non-trivial closed $\mathcal{V}$-path.*

We say that a matching $M$ associated to a gradient vector field is a *Morse matching* (see Figure 2.7).

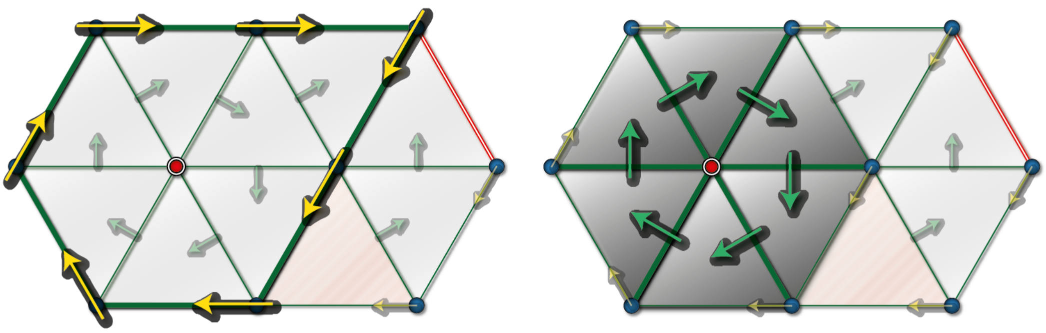


Figure 2.7: A closed $\mathcal{V}$-path on vertices-edges (right) and edges-triangles (left).

The following table is a dictionary between the graph-based concepts and the simplicial complex-based concepts.

| Graph | Simplicial complex |
|---|---|
| $\{\sigma, \tau\} \in M$ | $\sigma \rightarrow \tau$ |
| $\{\sigma, \tau\} \notin M$ | $\sigma \not\rightarrow \tau$ |
| $\sigma$ is not matched | $\sigma$ is critical |
| $\{\sigma, \tau\} \in A$ | $\sigma \prec \tau$ |
| nodes | simplices of $\Delta$ |
| arcs | incidence of simplices of $\Delta$ |
| $\sigma$ matched below | $\exists\ v \in \sigma, \sigma \setminus v \rightarrow \sigma$ |
| $\sigma$ matched above | $\exists\ \sigma \cup v \in \Delta, \sigma \rightarrow \sigma \cup v$ |
| $\sigma$ unmatched | $\sigma$ is critical |
| matching on $H$ | discrete vector field on $\Delta$ |
| Morse matching on $H$ | discrete gradient field on $\Delta$ |

Now, the main theorem of discrete Morse theory can be stated.

**Theorem 2.9.** *[26] Let $\Delta$ be a simplicial complex and $M$ be a Morse matching on $\Delta$. Then $\Delta$ is homotopy equivalent to a CW-complex* [1] *containing a cell of dimension $p$ for each critical face of dimension $p$.*

The minimization of the number of critical simplices in the Morse matching produces a succinct representation (in homotopy) of the simplicial complex. This is the main motivation for the optimization problem, which is primary focus of the next two chapters. In addition, the discrete gradient field (Morse matching) obeys the Morse inequalities, mimicking the gradient field in a smooth manifold.

**Theorem 2.10** (Weak and strong Morse inequalities [26])**.** *Let $d$ be the dimension of $\Delta$, $m_p$ be the number of critical simplices of dimension $p$, $\beta_p$ the $p^{th}$ Betti number of $\Delta$ on any field, and $\chi$ the Euler characteristic of $\Delta$ [54].*

(i) *For each $p = 0, 1, 2, \ldots, d$, we have $m_p \geq \beta_p$, and*
$m_0 - m_1 + m_2 - \ldots + (-1)^d m_d = \beta_0 - \beta_1 + \beta_2 - \ldots + (-1)^d \beta_d = \chi.$

(ii) *For each $p = 0, 1, 2, \ldots, d, d+1$,*
$m_p - m_{p-1} + \ldots \pm m_0 \geq \beta_p - \beta_{p-1} + \ldots \pm \beta_0.$

We denote by $m(\mathcal{V}) = m_0 + m_1 + \ldots + m_d$ the total number of critical simplices.

[1]Please refer to Lundell and Weingram's book [47] for the definition of CW-complexes. For our purposes, a CW-complex is only used as a reduced representation of a simplicial complex.

## 2.4 Parameterized complexity

In this thesis (Chapters 3 and 4), we study the complexity of the optimization of Morse matchings in light of parameterized complexity theory, a more refined type of complexity investigation. Following Downey and Fellows [20], an NP-complete problem is called *fixed-parameter tractable* (FPT) with respect to a *parameter* $k \in \mathbb{N}$, if for every input with parameter less or equal to $k$, the problem can be solved in $O(f(k) \cdot n^{O(1)})$ time, where $f$ is an arbitrary function independent of the problem size $n$.

For NP-complete but fixed-parameter tractable problems, we can look for classes of inputs for which fast algorithms exist, and identify which aspects of the problem make it difficult to solve. Note that the significance of an FPT result strongly depends on whether the parameter is small for large classes of interesting problem instances and easy to compute.

There are also NP-complete problems which are not fixed-parameter tractable. In order to classify fixed-parameter *intractable* NP-complete problems, Downey and Fellows [20] propose a family of complexity classes called the *W-hierarchy*:

$$\mathrm{FPT} \subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq W[P] \subseteq \mathrm{XP}.$$

The base problems in each class of the $W$-hierarchy are versions of satisfiability problems with increasing logical depth as parameter. On the left side of the $W$-hierarchy we have the complexity class FPT which contains all problems which are FPT in their natural parameter. Class $W[P]$ contains the satisfiability problems with unbounded logical depth. The rightmost complexity class XP of the $W$-hierarchy contains all problems which can be solved in $O(n^k)$ time where $k$ is the parameter of the problem. The satisfiability problems and logical depth are formally defined below. We refer to Daniel Marx's presentation[2] for further details.

**Definition 2.11** (Boolean circuit)**.** *A* Boolean circuit *consists of input gates, negation gates, AND gates, OR gates, and a single output gate. The weight of an assignment is the number of true values.*

**Definition 2.12** (Depth and weft)**.** *The* depth *of a circuit is the maximum length of a path from an input to the output. A gate is large if it has more than 2 inputs. The* weft *of a circuit is the maximum number of large gates on a path from an input to the output.*

[2] http://www.cs.bme.hu/˜dmarx/papers/marx-warsaw-fpt3

We can now define the WEIGHTED CIRCUIT SATISFIABILITY problem.

**Problem 2.13** (WEIGHTED CIRCUIT SATISFIABILITY).
INSTANCE: *A Boolean circuit.*
PARAMETER: *A non-negative integer $k$.*
QUESTION: *Is there an assignment of weight $k$ such that the output of the circuit is true?*

A notion of reduction which preserves the chosen parameter, between problems to group them in classes is defined.

**Definition 2.14** (Parameterized reduction)**.** *A parameterized problem $L$ reduces to a parameterized problem $L'$, denoted by $L \leq_{FPT} L'$, if we can transform an instance $(x, k)$ of $L$ into an instance $(x', g(k))$ of $L'$ in time $O(f(k)|x|^{O(1)})$ for arbitrary functions $f$ and $g$, such that $(x, k)$ is a yes-instance of $L$ if and only if $(x', g(k))$ is a yes-instance of $L'$.*

Finally the $W[t]$ classes can be defined, with parameterized reductions to WEIGHTED CIRCUIT SATISFIABILITY with bounded depth.

**Definition 2.15.** *Let $C[t, d]$ be the set of all circuits having weft at most $t$ and depth at most $d$. A problem $P$ is in the class $W[t]$ if there is a constant $d$ and a parameterized reduction from $P$ to* WEIGHTED CIRCUIT SATISFIABILITY *of $C[t, d]$. $W[P]$ denotes the class having unrestricted (unbounded) depth.*



### Examples in W-hierarchy

As example, we show how parameterized complexity theory can distinguish the difficulty of two NP-complete graph problems with the W-hierarchy. The following graph problems are parameterized by their natural parameter.

**Problem 2.16** (INDEPENDENT SET).
INSTANCE: *A graph G=(V,E).*
PARAMETER: *A non-negative integer $k$.*
QUESTION: *Is there a set of vertices $S$, $|S| \leq k$, such that no two vertices in $S$ are adjacent?*

**Problem 2.17** (DOMINATING SET).
INSTANCE: *A graph $G = (V, E)$.*
PARAMETER: *A non-negative integer $k$.*
QUESTION: *Is there a set of vertices $D$, $|D| \leq k$, such that every vertex not in $D$ is adjacent to at least one member of $D$?*

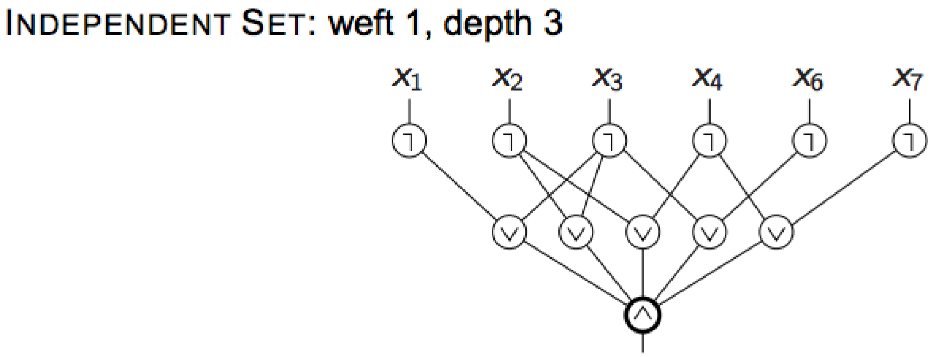



Figure 2.8: INDEPENDENT SET as a Boolean circuit (© Daniel Marx).

**Independent set.** In Figure 2.8, a parameterized reduction from INDEPENDENT SET to WEIGHTED CIRCUIT SATISFIABILITY of $C[1,3]$ is described where each vertex $v_i$ of $G$ is represented by an input $x_i$ and $x_i$ is true if and only if $v_i$ is in the independent set S, identifying the parameters of both problems. Every $v_i$ is connected to a NOT gate ($\neg$). The NOT gates associated to $x_i$ and $x_j$ are both connected to the same OR gate ($\vee$) if and only if $v_i$ and $v_j$ are adjacent in $G$. Finally, every OR gate is connected to the unique AND gate ($\wedge$) which leads to the output. Now, there exists an independent set of size $k$ in the graph $G$ if and only if there exists an assignment of weight $k$ in the circuit in Figure 2.8.

There is thus a parameterized reduction from INDEPENDENT SET to WEIGHTED CIRCUIT SATISFIABILITY of $C[1,3]$. It follows, from Definition 2.15, that INDEPENDENT SET is in $W[1]$.

In addition there is also a parameterized reduction from WEIGHTED CIRCUIT SATISFIABILITY of $C[1,3]$ to INDEPENDENT SET [20]. Therefore INDEPENDENT SET is W[1]-complete, which means that if INDEPENDENT SET can one day be shown to be in FPT, then $W[1] = \text{FPT}$. This is regarded as an unlikely collapse in parameterized complexity theory.

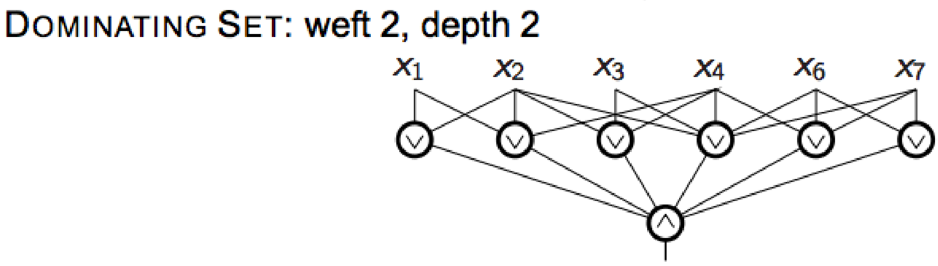



Figure 2.9: Dominating Set as a Boolean circuit (© Daniel Marx).

**Dominating Set.** Figure 2.9 shows that there is a parameterized reduction from Dominating Set to Weighted Circuit Satisfiability of $C[2, 2]$. It follows, from Definition 2.15, that Dominating Set is in $W[2]$. In addition there is also a parameterized reduction from Weighted Circuit Satisfiability of $C[2, 2]$ to Dominating Set [20], therefore Dominating Set is $W[2]$-complete. In parameterized complexity theory, in some sense, Dominating Set is harder than Independent Set

# 3
# Optimal Morse matchings: the bad news

In this chapter[1], the 2-dimensional simplicial complexes $\Delta$ we consider are pure, *i.e.* all maximal simplices are triangles (2-simplices) and *strongly connected*, *i.e.* each pair of triangles is connected by a path of triangles such that any two consecutive triangles are joined by an edge (1-simplex). In addition we consider 3-dimensional simplicial complexes which are triangulations of closed 3-manifolds, that is, simplicial complexes whose underlying topological space is a closed 3-manifold. Every 3-manifold can be represented in this way [51]. We will refer to these objects as *simplicial triangulations of* 3*-manifolds*.

The motivation to find optimal Morse matchings is given by Theorem 2.9, in the last chapter. In other words, a Morse matching with the smallest number $m(M)$ of critical simplices gives us the most compact and succinct topological representation up to homotopy. This motivates a fundamental optimization problem in discrete Morse theory, MORSE MATCHING. We write it as a decision problem in the following form:

**Problem 3.1** (MORSE MATCHING).
INSTANCE: *A simplicial complex* $\Delta$*.*
PARAMETER: *A non-negative integer* $k$*.*
QUESTION: *Is there a Morse matching* $M$ *with* $m(M) \leq k$*?*

## 3.1
## Erasability of simplicial complexes

In this chapter, instead of studying the MORSE MATCHING problem directly, we analyze a closely related problem, ERASABILITY, defined below.

Let $\Delta$ be a 2-dimensional simplicial complex. A triangle $t \in \Delta$ is called *external* if $t$ has at least one edge which is not in the boundary of any other triangle in $\Delta$; otherwise $t$ is called *internal*. Given a 2-dimensional simplicial complex $\Delta$ and a triangle $t \in \Delta$, the 2-dimensional simplicial complex obtained by removing (or *erasing*) $t$ from $\Delta$ is denoted by $\Delta \setminus t$. In addition, if $\tilde{\Delta}$ is obtained from $\Delta$ by iteratively erasing triangles such that in each step the erased triangle is external in the respective complex, we will say $\Delta$ collapses to $\tilde{\Delta}$ and write $\Delta \rightsquigarrow \tilde{\Delta}$. We say that the complex $\Delta$ is *erasable* if $\Delta \rightsquigarrow \delta$, where

[1]This chapter is a joint work with Jonathan Spreer and Benjamin Burton [14]

$\delta$ denotes a subcomplex of $\Delta$ with no triangle. Finally, for every 2-dimensional simplicial complex $\Delta$ we define $\mathrm{er}(\Delta)$ to be the size of the smallest subset $\Delta_0$ of triangles of $\Delta$ such that $\Delta \setminus \Delta_0 \rightsquigarrow \delta$. The elements of $\Delta_0$ are called *critical* triangles and hence $\mathrm{er}(\Delta)$ is sometimes also referred to as the *minimum number of critical triangles* of $\Delta$. Determining $\mathrm{er}(\Delta)$ is known as the ERASABILITY problem [25].

**Problem 3.2** (ERASABILITY).
INSTANCE: *A* 2*-dimensional simplicial complex* $\Delta$*.*
PARAMETER: *A non-negative integer* $k$*.*
QUESTION: *Is* $er(\Delta) \leq k$*?*

This operation can be defined in any dimension. Let $\sigma \prec \tau \in \Delta$, such that $\tau$ is not contained in any other facet of $\Delta$. A *collapse* is the operation of transforming $\Delta$ to $\Delta \setminus \{\sigma, \tau\}$.

## 3.2 Relationship between Erasability and Morse Matching

Forman established a connection between collapses and Morse matchings.

**Theorem 3.3.** *[26] Let* $\Delta$ *be a simplicial complex and* $\Sigma$ *a subcomplex of* $\Delta$*. Then there exists a sequence of collapses from* $\Delta$ *to* $\Sigma$ *if and only if there exists a discrete Morse matching on* $\Delta$ *such that* $\Delta \setminus \Sigma$ *contains no critical simplices.*

It follows from the theorem above that ERASABILITY can be restated as a version of MORSE MATCHING where only the number $m_2(M)$ of critical 2-simplices is counted.

**Problem 3.4** (ERASABILITY, ALTERNATE VERSION).
INSTANCE: *A* 2*-dimensional simplicial complex* $\Delta$*.*
PARAMETER: *A non-negative integer* $k$*.*
QUESTION: *Is there a Morse matching* $M$ *with* $m_2(M) \leq k$*?*

The complexity of computing optimal Morse matchings is linear on 1-complexes (graphs) [26]: A Morse matching in the Hasse diagram can be constructed with only one critical vertex by building a directed spanning tree in the actual graph where the critical vertex is the root. The correspondence is the following: there exists a directed arc from $v_1$ to $v_2$ in the directed spanning tree if and only if $(v_1, v_1v_2)$ is in the matching. Such Morse matching is optimal from the Morse inequalities, observing that there is only one critical vertex ($m_1(M) = 1$).

The complexity is also polynomial for 2-complexes that are manifolds [43], where there are at most two triangles per edge. In 2-manifolds, the adjacency between edges and triangles can also be viewed as a graph, the dual graph (the edges are the nodes and the triangles are the arcs). Therefore to build a optimal Morse matching in the 2-manifold, one builds a directed spanning tree in the dual graph and a directed spanning tree in the remainder of the primal graph as above. The Morse matching produced has one critical vertex and one critical triangle, therefore it is optimal from the Morse inequalities. The following lemma can also be proven using directed spanning trees in 2-complexes.

**Lemma 3.5.** *[38] Let $M$ be a Morse matching on a simplicial complex $\Delta$. Then we can compute a Morse matching $M'$ in polynomial time which has exactly one critical $0$-simplex, the same number of critical simplices of dimension greater or equal to $2$ as $M$, and $m(M') \leq m(M)$.*

The proof of this lemma can be found in Appendix A, and leads to the following result.

**Theorem 3.6.** *[38] Let $\Delta$ be a 2-dimensional simplicial complex. There exists a Morse matching with at most $k$ critical 2-faces if and only if there exists a Morse matching with at most $2(k+1) - \chi(\Delta)$ critical simplices altogether.*

Therefore, in a 2-dimensional simplicial complex, if one can solve ERASABILITY in polynomial time, then one can solve MORSE MATCHING in the *entire* complex in polynomial time [38,43].

We now want an analogous result for 3-manifold. Using Poincaré's duality, the dual result from Lemma 3.5 follows.

**Lemma 3.7.** *Let $M$ be a Morse matching on a closed triangulated $d$-manifold $\Delta$. Then we can compute a Morse matching $M'$ in polynomial time which has exactly one critical $d$-simplex, the same number of critical simplices of dimension less or equal to $d-2$ as $M$, and $m(M') \leq m(M)$.*

The following lemma, which has been mentioned in previous work [3,45], is a combination of Lemmata 3.5 and 3.7 for 3-manifolds together with the fact that $\chi(M) = 0$ for every 3-manifold $M$.

**Lemma 3.8.** *[3,38,45] Let $M$ be a Morse matching on a closed triangulated $3$-manifold. Then we can compute a Morse matching $M'$ in polynomial time which has exactly one critical $0$-simplex, one critical $3$-simplex, and $m_1(M') = m_2(M')$.*

An important corollary follows immediately.

**Theorem 3.9.** *Consider a simplicial triangulation of $3$-manifold. There exists a Morse matching with at most $k$ critical 2-simplices if and only if there exists a Morse matching with at most $2k+2$ critical simplices altogether.*

In other words, if one can solve ERASABILITY, then one can solve MORSE MATCHING on a 3-manifold. In Section 4.1 we show that if the spine of the simplicial complex has bounded treewidth, then we can solve ERASABILITY in linear time. Lemma 3.8 can be used to directly generalize this result to MORSE MATCHING on 3-manifolds.

## 3.3
## Fixed-parameter intractability of Erasability

In order to prove that ERASABILITY is $W[P]$-complete in the natural parameter, we first have to take a closer look at what has to be considered when proving hardness results with respect to a particular parameter.

As an example, Eğecioğlu and Gonzalez [25] reduce SET COVER to ERASABILITY to show that ERASABILITY is NP-complete. Since their reduction turns out to be a parameterized reduction, their results can be restated in the language of parameterized complexity as follows.

**Theorem 3.10.** *[25]* SET COVER $\leq_{FPT}$ ERASABILITY, *therefore* ERASABILITY *is* $W[2]$*-hard.*

This shows that, if the parameter $k$ is simultaneously bounded in both problems, ERASABILITY is *at least as hard as* SET COVER. In this section we will determine exactly how much harder ERASABILITY is than SET COVER, which is $W[2]$-complete. Namely, we will show that ERASABILITY is $W[P]$-complete in the natural parameter $k$. This will be done by (i) using a $W[P]$-complete problem as an oracle to solve an arbitrary instance of ERASABILITY (Theorem 3.13, which shows that ERASABILITY is in $W[P]$), and (ii) reducing an arbitrary instance of a suitable problem which is known to be $W[P]$-complete to an instance of ERASABILITY (Theorem 3.16, which shows that ERASABILITY is $W[P]$-hard).

There are only a few problems described in the literature which are known to be $W[P]$-complete [21, p. 473]. Amongst these problems, the following is suitable for our purposes.

**Problem 3.11** (MINIMUM AXIOM SET).
INSTANCE: *A finite set $S$ of* sentences*, and an* implication relation *$R$ consisting of pairs $(U, s)$ where $U \subseteq S$ and $s \in S$.*
PARAMETER: *A positive integer $k$.*
QUESTION: *Is there a set $S_0 \subseteq S$ (called an* axiom set*) with $|S_0| \leq k$ and a positive integer $n$, for which $S_n = S$, where we define $S_i$, $1 \leq i \leq n$, to consist of exactly those $s \in S$ for which either $s \in S_{i-1}$ or there exists a set $U \subseteq S_{i-1}$ such that $(U, s) \in R$?*

**Theorem 3.12** ( [20]). MINIMUM AXIOM SET *is* $W[P]$*-complete.*

## 3.4
## $W[P]$-completeness of Erasability

In this thesis, to show that ERASABILITY is $W[P]$-complete, we show that MINIMUM AXIOM SET is both at least (Theorem 3.13) and at most as hard (Theorem 3.16) as ERASABILITY.

**Theorem 3.13.** ERASABILITY $\leq_{FPT}$ MINIMUM AXIOM SET*, therefore* ERASABILITY *is in* $W[P]$*.*

*Proof* We show membership of ERASABILITY in $W[P]$ by reducing a given instance $(\Delta, k)$ of ERASABILITY to an instance $(S, R, k)$ of MINIMUM AXIOM SET.

Without loss of generality, we can assume that the 2-dimensional simplicial complex $\Delta$ has no external edges (if $\Delta$ has external edges we first remove these edges until no external edge exists and reduce the remaining problem instance to an instance of MINIMUM AXIOM SET). We now identify the set of triangles of $\Delta$ with the set of sentences $S$ in a one-to-one correspondence. For every edge $e \in \Delta$ we denote the set of all triangles containing $e$ by $\text{star}_\Delta(e) \subset \Delta$, and denote the corresponding set of sentences by $S_e \subset S$. We define the set of implication relations $R$ by $R = \{\ (S_e \setminus \{s\}, s)\ ,\ e \text{ edge of } \Delta,\ s \in S_e\ \}$. Note that $\Delta$ has no external edges and thus $S_e \setminus \{s\} \neq \emptyset$ for every edge $e$.

Now, we show that for all axiom sets $S_0 \subset S$ of size $k$ we have $\Delta \setminus \Delta_0 \rightsquigarrow \delta$, where $\delta$ is the 1-dimensional subcomplex of $\Delta$, for subset of triangles $\Delta_0 \subset \Delta$ of size $k$ associated to $S_0$. To see that this is true, note that for the augmenting sequence $S_0 \subset S_1 \subset \ldots \subset S_n = S$ of $S$, their corresponding subsets of triangles $\Delta_0 \subset \Delta_1 \subset \ldots \subset \Delta_n = \Delta$, and $i \in \{1, \ldots, n\}$ fixed, all sentences $s \in S_i \setminus S_{i-1}$ have to occur in a relation $(S_e \setminus \{s\}, s)$ for some edge $e$ with $S_e \setminus \{s\} \subset S_{i-1}$. For the triangle $t \in \Delta$ corresponding to $s$ this means that $\text{star}_\Delta(e) \setminus \{t\} \subset \Delta_{i-1}$. Thus, if we assume that all triangles in $\Delta_{i-1}$ are already erased, $t$ must be external and thus can be erased as well. The statement now follows by the fact that for $i = 1$, all triangles in $\Delta_0$ are already erased in $\Delta \setminus \Delta_0$ and hence $\Delta \setminus \Delta_0 \rightsquigarrow \delta$.

Conversely, let $\Delta_0 \subset \Delta$ be of size $k$ such that $\Delta \setminus \Delta_0 \rightsquigarrow \delta$. Since $\Delta$ has no external triangles but $\Delta \setminus \Delta_0 \rightsquigarrow \delta$, there must be external triangles $t \in \Delta \setminus \Delta_0$. Hence for sentence $s \in S$ corresponding to triangle $t$, there is a relation $(S_e \setminus \{s\}, s)$ with $S_e \setminus \{s\} \subset S_0$.We then define $S_1$ to be the union of $S_0$ with all sentences $s$ of the type described above. Iterating this step results in a sequence of subsets $S_0 \subset S_1 \subset \ldots \subset S_n = S$ for some $n$ that satisfies the problem. ■

In order to show that it is in fact amongst the hardest problems in this class we first need to build some gadgets.

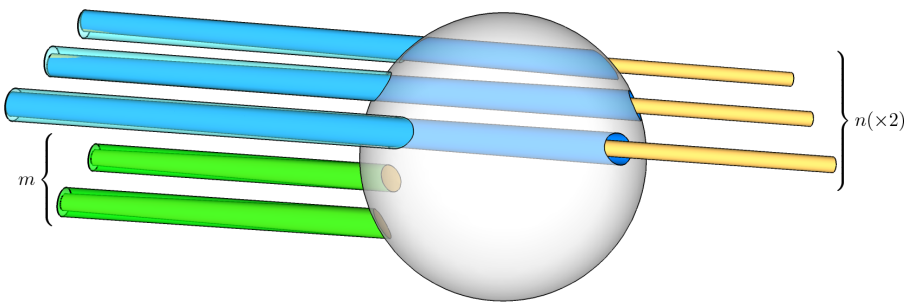



Figure 3.1: Example of a sentence gadget with $m = 2$ relations $(U, s)$ and $n = 3$ relations $(U, u)$ with additional tubes.

**Definition 3.14** (Gadgets for the hardness proof of ERASABILITY)**.** *Let $(S, R, k)$ be an instance of* MINIMUM AXIOM SET*.*

*Let $s \in S$ be a sentence. By an $s$-*gadget *or* sentence gadget *we mean a triangulated 2-dimensional sphere with $2n + m$ punctures as shown in Figure 3.1, where $m$ is the number of relations $(U, s) \in R$ and $n$ is the number of relations $(U, u) \in R$ such that $s \in U$.*

*Let $(U, s) \in R$ be a relation. A $(U, s)$-*gadget *or* implication gadget *is a collection of $|U|+1$ sentence gadgets for each sentence of $U \cup \{s\}$ together with $2|U|$ nested tubes as shown in Figure 3.2 such that (i) two tubes are attached to two punctures of the $u$-gadget for each $u \in U$ and (ii) all $2|U|$ boundary components at the other side of the tubes are identified at a single puncture of the $s$-gadget.*

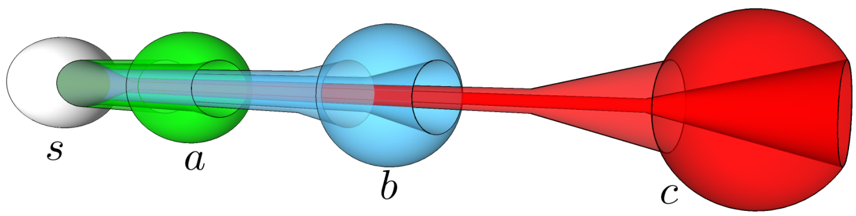



Figure 3.2: Example of a $(U, s)$-gadget with $U = \{a, b, c\}$, with sentence gadgets $\{a, b, c, s\}$.

Then, by construction the following holds for the $(U, s)$-gadget.

**Lemma 3.15.** *A $(U, s)$-gadget can be erased if and only if all sentence gadgets corresponding to sentences in $U$ are already erased.*

*Proof* On the one hand, if all sentence gadgets corresponding to sentences in $U$ are erased, the whole gadget can be erased tube by tube. If, on the other hand, one of the sentence gadgets still exists this gadget together with the two tubes connected to it builds a complex without external triangles, which thus cannot be erased. ∎

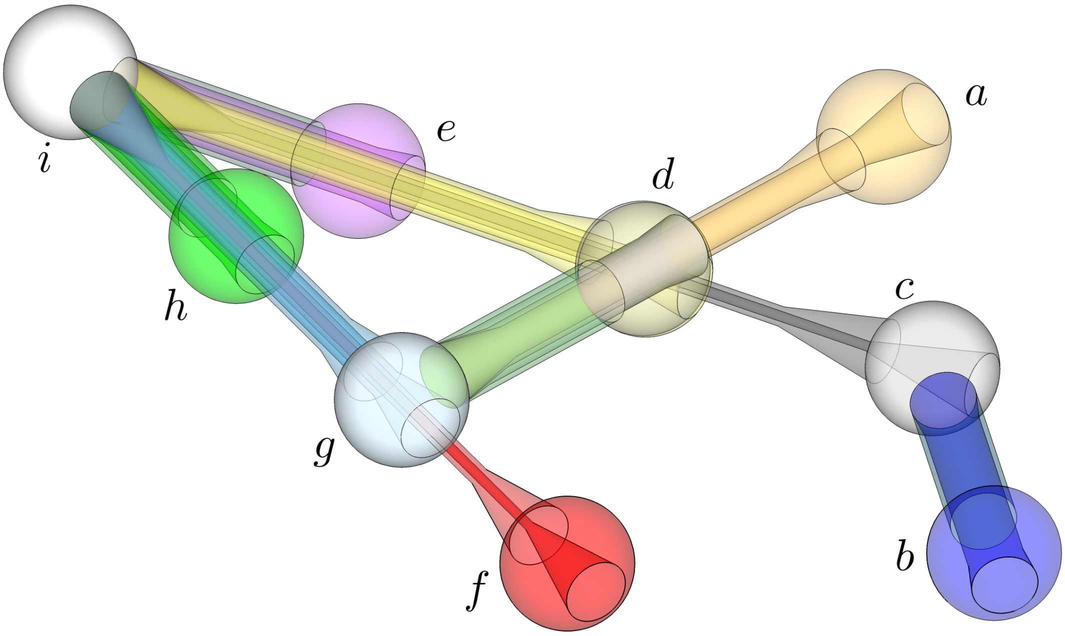



Figure 3.3: Constructing an instance of ERASABILITY from an instance $(S, R)$ of MINIMUM AXIOM SET, where $S = \{a, b, c, d, e, f, g, h, i\}$ and $R = \{(\{c, d, e\}, i), (\{f, g, h\}, i), (\{b\}, c), (\{a, d\}, g)\}$.



With these tools, we can now prove the main theorem of this section.

**Theorem 3.16.** MINIMUM AXIOM SET $\leq_{FPT}$ ERASABILITY*:* ERASABILITY *is $W[P]$-hard when the instance of* ERASABILITY *is a strongly connected pure 2-dimensional simplicial complex $\Delta$ which is embeddable in $\mathbb{R}^3$. In particular* ERASABILITY *is $W[P]$-hard.*

The simplicial complex $\Delta$ (see Figure 3.3) constructed to prove $W[P]$-hardness of ERASABILITY is in fact embeddable into $\mathbb{R}^3$. This means that, even in the relatively well-behaved class of embeddable 2-dimensional simplicial complexes, ERASABILITY when bounding the number of critical simplices is still likely to be inherently difficult.

*Proof* To show $W[P]$-hardness of ERASABILITY, we will reduce an arbitrary instance $(S, R, k)$ from MINIMUM AXIOM SET to an instance $(\Delta, k)$ of ERASABILITY. In order to do so, we will use a sentence gadget for each element of $S$ and an implication gadget for each relation $R$ (see Definition 3.14) to construct a 2-dimensional simplicial complex $\Delta$ with a polynomial number of triangles in the input size.

By construction, we can glue all sentence and implication gadgets together in order to obtain a simplicial complex $\Delta$ without any exterior triangles. Note that the only place where $\Delta$ is not a manifold surface is at the former $m$ boundary components of the sentence gadgets corresponding to the right hand sides of the relations in $R$.

For any axiom set $S_0 \subset S$ of size $k$, let $\Delta_0$ be a set of $k$ triangles, one from each sentence gadget corresponding to a sentence in $S_0$. It follows by Lemma 3.15, that $\Delta \setminus \Delta_0$ can be erased to a complex where all the sentence gadgets $s$ corresponding to relations $(U, s)$, $U \subset S_0$, have external triangles. Since $S_0$ is an axiom set, iterating this process erases the whole complex $\Delta$.

Conversely, let $\Delta_0$ be a set of $k$ triangles such that $\Delta \setminus \Delta_0 \rightsquigarrow \delta$. First, note that erasing a triangle of any tube of an implication gadget always allows us to remove the sentence gadget at the right end of this tube. Hence, without loss of generality, we can assume that all $k$ triangles in $\Delta_0$ are triangles of some sentence gadget in $\Delta$. Now, if any sentence gadget contains more than one triangle of $\Delta_0$ we delete all additional triangles obtaining a set $\Delta_0'$ of $k'$ triangles, $k' \leq k$, such that $\Delta \setminus \Delta_0' \rightsquigarrow \delta$ and thus the corresponding set of sentences is an axiom set of size $k' \leq k$.

The result now follows by the observation that $\Delta$ can be realized by at most a quadratic number of triangles in the input size of $(S, R, k)$.

■

The $W[P]$-completeness result implies that if ERASABILITY turns out to be fixed parameter tractable, then $W[P] = \mathrm{FPT}$, *i.e.* every problem in $W[P]$ including the ones lower in the hierarchy would turn out to be fixed parameter tractable, an unlikely and unexpected collapse in parameterized complexity. Also, it would imply that the $n$-variable satisfiability problem could be solved in time $2^{o(n)}$, that is, better than a brute force search [1]. With respect to this result, if we want to prove fixed parameter tractability of ERASABILITY, the parameter must be different from the natural parameter.

# 4
# Optimal Morse matchings: the good news

In this chapter[1], we prove that there is still hope to find an efficient algorithm to solve Morse Matching for certain complexes. We give positive results for the field of discrete Morse theory by proving that Erasability and Morse Matching are fixed parameter tractable in the treewidth (Section 4.1) of the spine of the input simplicial complex, where this problem reduces to the Alternating Cycle-Free Matching problem (Section 4.2). We provide an explicit algorithm for this result in Sections 4.3 and 4.4. Then, we extend this result to the treewidth of the dual graph of simplicial triangulation of a 3-manifold (Section 4.5). Finally, in Section 4.6, we run some numerical experiments to analyze the treewidth in various 2-complexes and 3-manifolds.

## 4.1
## Treewidth

The treewidth is a fundamental concept for several parameterized complexity problems. Its definition relies on tree decompositions of a graph as follows.

**Definition 4.1** (Treewidth)**.** *A tree decomposition of a graph $G$ is a tree $T$ where each node $i \in I$ is associated with a bag $X_i$. Each bag $X_i$ is a subset of nodes of $G$ such that:*

(i) node coverage: *every node of $G$ is contained in at least one bag $X_i$*

(ii) arc coverage: *for each arc of $G$, some bag $X_i$ contains both its endpoints*

(iii) coherence: *for all bags $X_i$, $X_j$ and $X_k$ of $T$, if $X_j$ lies on the unique simple path from $X_i$ to $X_k$ in $T$, then $X_i \cap X_k \subseteq X_j$.*

*The* width *of a tree decomposition is defined as* $\max\{|X_i| - 1, i \in I\}$*, and the* treewidth $\mathbf{tw}(G)$ *of $G$ is the minimum width over all tree decompositions of $G$.*

In terms of complexity, for graphs of bounded treewidth, computing a tree decomposition of width up to $k$ has running time $O(f(k)|V|)$, using an algorithm by Bodlaender [8]. Regarding the size of $f(k)$: using the improved algorithm by

[1] This chapter is a joint work with Jonathan Spreer and Benjamin Burton [14]

Perković and Reed [56], at most $O(k^2)$ recursive calls of Bodlaender's improved linear time fixed-parameter tractable algorithm for bounded treewidth [9] are needed. This latter algorithm in turn is said to have a constant factor $f(k)$ which is "at most singly exponential in $k$". Details on the running times of tree decomposition algorithms are available in the literature [7,39].

We often use the bag $X_i$ to refer to node $i$. In fact, it is sufficient to look for *nice* tree decompositions to compute the treewidth.

**Definition 4.2** (Nice tree decomposition)**.** *A tree decomposition* $(X_i \,|\, i \in I, T)$ *is called a* nice tree decomposition *if the following conditions are satisfied:*

– *every bag of the tree $T$ has at most two children,*

*root bag* – *there is a fixed bag $X_r$ with $|X_r| = 1$ acting as the root of $T$,*

*forget bag* – *if bag $X_j$ has no children, then $|X_j| = 1$,*

*join bag* – *if a bag $X_i$ has two children $X_j$ and $X_k$, then $X_i = X_j = X_k$,*

– *if a bag $X_i$ has a single child $X_j$, then either*

*introduce bag* – $|X_i| = |X_j| + 1$ *and* $X_j \subset X_i$*,*

*or*

*forget bag* – $|X_j| = |X_i| + 1$ *and* $X_i \subset X_j$*.*



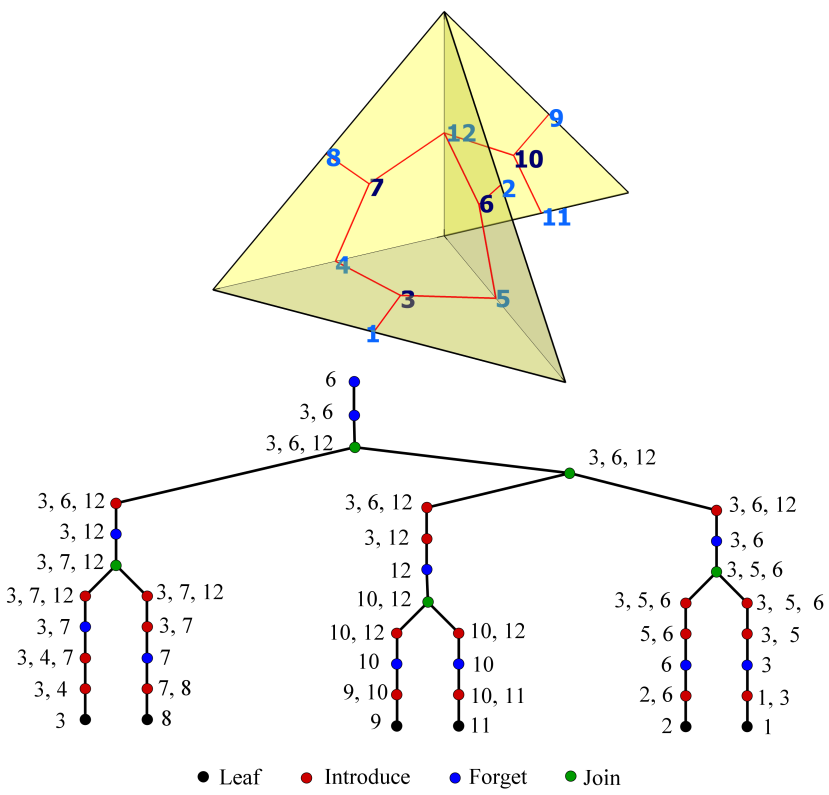



Figure 4.1: Example of a nice tree decomposition (bottom) of the spine of a non-manifold 2-dimensional simplicial complex made of 4 triangles (top).

A given tree decomposition can be transformed into a nice tree decomposition (see Figure 4.1) in linear time:

**Lemma 4.3.** *[39] Given a graph $G$ with $n$ nodes, and a tree decomposition of $G$ with width $w$ and $O(n)$ bags, we can find a nice tree decomposition of $G$ that also has width $w$ and $O(n)$ bags in time $O(n)$.*

## 4.2 Alternating cycle-free matchings

Given a graph $G = (N, A)$ and a matching $M \subset A$ on $G$, an *alternating path* is a sequence of pairwise adjacent arcs such that each matching arc in the sequence is followed by an unmatching arc and conversely. An *alternating cycle* of $M$ is a closed alternating path. Matchings that do not have any such alternating cycle are called an *alternating cycle-free matchings.* From the definition of Morse matching, we can state Erasability in the language of alternating cycle-free matchings as follows:

**Problem 4.4** (Alternating Cycle-Free Matching).
Instance: *A bipartite graph $G = (N_1 \cup N_2, A)$.*
Parameter: *A non-negative integer $k$.*
Question: *Does $G$ have an alternating cycle-free matching $M$ with at most $k$ unmatched nodes in $N_1$?*

Specifically, if $G = H_1$ is the spine of some simplicial complex $\Delta$, then Erasability is equivalent to the Alternating Cycle-Free Matching problem [17].

With this graph-based definition, Courcelle's theorem [18] can be used to show that Alternating Cycle-Free Matching and Erasability are fixed parameter tractable in the treewidth of the associated graph. Indeed, Courcelle's celebrated theorem [18] states that all graph properties that can be defined in *Monadic Second-Order Logic* (MSOL) can be decided in linear time when the graph has bounded treewidth. However, it is not obvious how to *directly* state Erasability and Morse Matching in MSOL. Instead, we apply Courcelle's theorem to Alternating Cycle-Free Matching which by the previous comment is a graph theoretical problem equivalent to Erasability.

**Theorem 4.5.** *Let $w \geq 1$. Given a bipartite graph with* $\mathbf{tw}(G) \leq w$, Alternating Cycle-Free Matching *can be solved in linear time.*

*Proof* Let $G = (N_1 \cup N_2, A)$ be a bipartite graph and let $N = N_1 \cup N_2$ be the set of nodes of $G$. We write an MSOL formulation of Alternating Cycle-Free Matching based on the fact that $M \subset A$ is an alternating cycle-free matching if and only if $M$ is a matching and every induced $M$-subgraph contains a node of degree 1 [29]:

$$\begin{aligned} \max M : \forall x \in N[\neg\exists\, a_1, a_2 \in M(a_1 \neq a_2 \wedge inc(x, a_1) \wedge inc(x, a_2))] \\ \wedge \forall M' \subseteq M(\exists\, a \in M', \exists\, x \in N[inc(x, a) \wedge (\forall x_1 \in N, \\ (\neg\exists\, a_1 \in M'(x \neq x' \wedge adj(x, x_1) \wedge inc(x_1, a_1))))]), \end{aligned}$$

where $inc(x, a)$ is the incidence predicate between node $x$ and arc $a$ and $adj(x, x')$ is the adjacency predicate between node $x$ and node $x'$. The above statement can be translated to plain English as follows: "Find the largest matching $M$ of $G$, where each node is incident to at most one arc, such that in every subset $M'$ of the matching $M$ there exists a matched node $x$ in $M'$ such that its only neighbor matched in $M'$ is the other endpoint of the unique matching arc incident to $x$".

∎

## 4.3 FPT algorithm for the Alternating Cycle-Free Matching problem

The result stated by Theorem **??** is a purely theoretical result, since the complexity stated by Courcelle's theorem contains towers of exponents in the parameter function. In order to obtain a more precise notion of the complexity of Morse Matching, we focus here on the construction of a linear time algorithm to solve Alternating Cycle-Free Matching for inputs of bounded treewidth, and we obtain a significantly faster running time.

**Theorem 4.6.** *Let $G = (N = N_1 \cup N_2, A)$ be a simple bipartite graph with a given nice tree decomposition. Then the size of a maximum alternating cycle-free matching of $G$ can be computed in $O(4^{w^2+w} \cdot w^3 \cdot \log(w) \cdot n)$ time, where $n = |N|$ and $w$ denotes the width of the tree decomposition.*

### Algorithm overview

Our algorithm constructs alternating cycle-free matchings of $G$ along a nice tree decomposition $T$ of $G$, from the leaves of $T$ up to its root, visiting each bag $X_i$ exactly once. In the following we denote by $F_i$ the set of nodes which are already processed and forgotten by the time node $i$ is reached; we call $F_i$ the *set of forgotten nodes*. At each bag $X_i$ of the decomposition, we construct a set $\mathcal{M}(i)$ representing *all* valid alternating cycle-free matchings in the subgraph of $G$ induced by the nodes in $X_i \cup F_i$.

Each leaf bag contains a single node of $G$, and the only matching is thus empty. At each introduce bag $X_i = X_j \cup \{x\}$, each matching $M$ of $\mathcal{M}(j)$ can be extended to several matchings as follows. The newly introduced node $x$

can be either left unmatched, or matched with one of its neighbors as long as it generates a valid and cycle-free matching with $M$. At each join bag $X_i = X_j = X_k$, $\mathcal{M}(i)$ is build from the valid combinations of pairs of matchings from $\mathcal{M}(j)$ and $\mathcal{M}(k)$. The final list of valid matchings is then evaluated at the root bag $r$.

However, this final list $\mathcal{M}(r)$ contains an exponential number of matchings. Fortunately, the nice tree decomposition allows us to group together, at each step, all matchings $M$ that coincide on the nodes of $X_i$. Indeed, the algorithm takes the same decisions for all the matchings of the group. We can thus store and process a much smaller list $\mathcal{M}(j)$ of matchings containing only one representative $\tilde{M}$ of each group. In each group, we choose the one with the smallest number of unmatched nodes so far. This grouping takes place at the forget and join bags. This makes the algorithm exponential in the bag size, instead of the input size.

### Matching data structure

The structure storing an alternating cycle-free matching $M$ in a set $\mathcal{M}(i)$ must be suitable for checking the matching validity whenever a matching is extended at an introduce bag or a join bag. It must store which nodes are already matched in $M$ to avoid matching a node of $G$ twice (*matching condition*). We use a binary vector $\mathbf{v}(M)$, where the $x$-th coordinate is 1 if node $x \in X_i$ is matched and 0 otherwise. Checking the matching condition and updating when nodes are matched has thus a constant execution time $O(1)$.

Also, the structure must store which nodes are connected by an alternating path in a matching to avoid closing a cycle when extending or combining matchings (*cycle-free condition*). When matching two nodes $x$ and $y$, an alternating cycle is created if there exists an alternating path from a neighbor of $x$ to a neighbor of $y$. To test this, we use a union-find structure $\mathbf{uf}(M)$ [68], storing for each matched node $x$ the index of a matched node $c(x)$ connected to $x$ by an alternating path in $M$. For a subset of matched nodes which are all connected to each other, a component index $c$ is chosen. For each unmatched node, we store the ordered list of component indexes of *matched* neighbor nodes. The cycle-free condition check reduces to find calls on the adjacent lists, and the update of the structure when increasing the matching size reduces to union calls, both executing in near-constant time in amortized analysis. All the matchings are stored in a hash structure to allow faster search for duplicates. Finally, we can return not only the maximal cycle-free matching size, but an actual maximal cycle-free matching by storing, along with each representative matching, a binary vector of size $|X_i \cup F_i|$ with all the matched nodes so far.

### Grouping

Traversing the nice tree decomposition in a bottom-up fashion, each node appears in a set of bags that form a subtree of the tree decomposition (*coherence* requirement of the tree decomposition). This means that, whenever a node is forgotten, it is never introduced again in the bottom-up traversal.

A naïve version of the algorithm described above would build the complete list of valid alternating cycle-free matchings: the set $\mathcal{M}(i)$ would contain all valid matchings in the graph induced by the nodes in $X_i \cup F_i$. In particular, for each matching $M \in \mathcal{M}(i)$ the algorithm would store the binary vector $\mathbf{v}(M)$ and the union-find structure $\mathbf{uf}(M)$ on $X_i \cup F_i$. However, it is sufficient to store the essential information about each $M$ by restricting the union-find structure $\mathbf{uf}(M)$ and the binary vector $\mathbf{v}(M)$ *only* to the nodes in the bag $X_i$ (for any matched node $x \in X_i$, node $c(x)$ of the union-find structure is then chosen inside $X_i$). More precisely, we define an equivalence relation $\sim_i$ on the matchings of $\mathcal{M}(i)$ such that $M \sim_i M'$ if and only if $\mathbf{v}(M)_{|X_i} = \mathbf{v}(M')_{|X_i}$ and $\mathbf{uf}(M)_{|X_i} = \mathbf{uf}(M')_{|X_i}$ on the nodes of $X_i$. Since two equivalent matchings only differ on the forgotten nodes $F_i$, and the forgotten nodes do not appear later in the algorithm, the validation of the matching and cycle-free conditions of any extension of $M$ or $M'$ (or any combination with a third equivalent matching $M''$) will be equal from now on.

Since we are interested in the alternating cycle-free matching with the minimum number of unmatched nodes, for each equivalence class we will choose a matching $\tilde{M}$ with the minimum number $m(\tilde{M})$ of unmatched forgotten nodes as class representative. This number $m(\tilde{M})$ is stored together with $(\mathbf{v}, \mathbf{uf})$ for each equivalence class of $\tilde{\mathcal{M}}(i) = \mathcal{M}(i)/\sim_i$. In addition, we can compute the alternating cycle-free matching of maximum size by storing the complete binary vector $\mathbf{v}$ along with $m(\tilde{M})$ (since the matching is cycle-free, this is sufficient to recover the set of arcs defining the matching).

### Execution time complexity

To measure the running time we need to bound the number of equivalence classes of $\tilde{\mathcal{M}}(i)$. Let $w_i$ be the number of nodes in $X_i$. The number of equivalence classes of $\tilde{\mathcal{M}}(i)$ is then bounded above by the number of possible pairs $(\mathbf{v}, \mathbf{uf})$ on $w_i$ nodes. The union-find stores for each matched node $x$ a component node $c(x) \in X_i$, and for each unmatched node a list of at most $w_i$ component nodes, leading to at worst $2^{w_i}$ different lists per node, giving $2^{w_i^2}$ possible combinations of lists. Also there are $2^{w_i}$ possible binary vectors $\mathbf{v}$ of length $w_i$, therefore there are at worst $2^{w_i^2}2^{w_i}$ elements in $\tilde{\mathcal{M}}(i)$. Observe that this enumeration includes invalid matchings and incoherent pairs $(\mathbf{v}, \mathbf{uf})$.

The time complexity is dominated by the execution at the join bag where pairs of equivalences classes from $\tilde{\mathcal{M}}(j)$ and $\tilde{\mathcal{M}}(k)$ have to be combined. The maximum number of combinations is the square of the number of equivalence classes in each set, and the complexity for a join bag will be $O(4^{w^2+w} \cdot w^3 \cdot \log(w))$ (please refer to the next section for details). Since there are $O(n)$ bags in a nice tree decomposition, the total execution time is in $O(4^{w^2+w} \cdot w^3 \cdot \log(w) \cdot n)$. Finally, as already stated in Lemma 4.3, for bounded treewidth computing a tree decomposition and a nice tree decomposition is linear. So the whole process from the bipartite graph to the resulting maximal alternating cycle-free matching is fixed-parameter tractable in the treewidth. Note that neither the decomposition nor the algorithm use the fact that the graph is bipartite.

## 4.4 Algorithm for Alternating Cycle-Free Matching: step by step

The algorithm visits the bags of the nice tree decomposition bottom-up from the leaves to the root evaluating the corresponding mappings in each step according to the following rules (see Figure 4.5).

### Leaf bag

The set of matchings $\tilde{\mathcal{M}}(i)$ of a leaf bag $X_i = \{x\}$ is trivial with a unique empty matching $\tilde{M}$ represented by $\mathbf{v}(\tilde{M}) = [0, \ldots, 0]$, and $\mathbf{uf}(\tilde{M})(x)$ defined by as an empty list, associated with $m(\tilde{M}) = 0$.

### Introduce bag

Let $X_i = X_j \cup \{x\}$ be an introduce bag with child bag $X_j$. The set of valid matchings $\tilde{\mathcal{M}}(i)$ is built from $\tilde{\mathcal{M}}(j)$ by introducing $x$ in each matching $\tilde{M} \in \tilde{\mathcal{M}}(j)$, generating several possible matchings $\tilde{M}'$ (see Figure 4.2).

We can always introduce $x$ as an unmatched node, then $\tilde{M}$ is extended on $x$ by setting $\mathbf{v}(\tilde{M}')_{|x} = 0$ and updating $\mathbf{uf}(\tilde{M}')$ with the ordered list of components for each matched neighbor of $x$.

In addition, for each unmatched neighbor $y \in X_j$, we can introduce $x$ as a matched node in the following way. We match both $x$ and $y$ in $\tilde{M}$ and set $\mathbf{v}(\tilde{M}')_{|x} = 1$ and $\mathbf{v}(\tilde{M}')_{|y} = 1$. If the intersection of the list of neighbor components of $x$ and $y$ is empty, then the matching of $x$ and $y$ does not create cycle. In this case $\tilde{M}'$ is a valid extension of $\tilde{M}$. The update of the union-find structure must then reflect the extensions of all alternating paths through arc $\{x, y\}$. We perform in $\mathbf{uf}(\tilde{M}')$ a union operation for $x$ and all its matched neighbors (including $y$), and for $y$ and all its matched neighbors. We also add the merged component index $c(x)$ to the list of neighbor components of each unmatched neighbor of $x$ and $y$. Then we include all valid extensions $\tilde{M}'$ to $\tilde{\mathcal{M}}(i)$, reducing $\mathbf{v}(\tilde{M}')$ by calling find for each node and neighbor component list entry, and we set $m(\tilde{M}') = m(\tilde{M})$ for all extensions $\tilde{M}'$ of $\tilde{M}$.

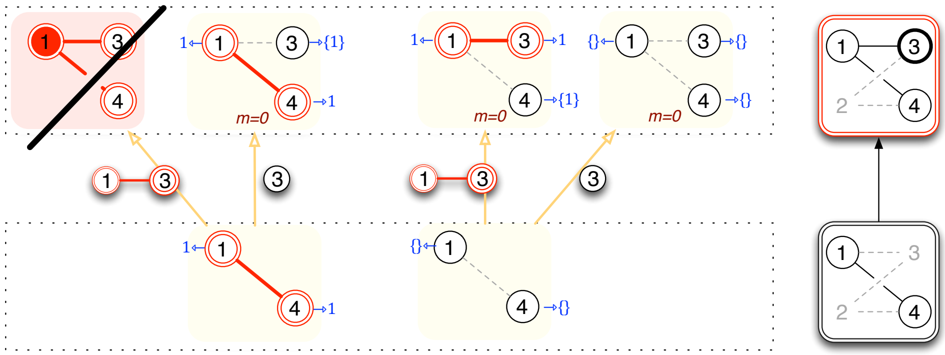


Figure 4.2: Detail of the decisions at an introduce bag.

*Running time.* There are at most $2^{w_i^2+w_i}$ extended matchings $\tilde{M}'$ for bag $X_i$ (including all invalid ones), where $w_i = |X_i| = |X_j| + 1$ (a new possible matching can be generated only once). Each new matching is validated by a direct lookup at $\mathbf{v}(\tilde{M}')$ and ordered list comparison, leading to a linear time $w_i$. The update of each structure requires constant time for each matched neighbor of $x$ and almost linear time $O(w_i)$ plus the sorted insertion $O(w_i \cdot \log(w_i))$ for each unmatched neighbor, and there are at most $w_i$ neighbors in the bag. Thus, the total running time of an introduce bag is in $O(2^{w_i^2+w_i} \cdot w_i^2 \cdot \log(w_i))$.

#### Forget bag

Let $X_i = X_j \setminus \{x\}$ be a forget bag with child bag $X_j \ni x$ (see Figure 4.3). While the set of all possible matchings on $X_i \cup F_i$ does not change ($\mathcal{M}(j) = \mathcal{M}(i)$), the equivalence relation $\sim_i$ possibly identifies more matchings than $\sim_j$. For each matching $\tilde{M} \in \tilde{\mathcal{M}}(j)$, a new matching $M'$ is obtained by deleting coordinate $x$ of $\mathbf{v}(\tilde{M})$. If $c(x) = x$, $\mathbf{uf}(\tilde{M})$ needs to be updated. To do so, the set of nodes $X_i$ is traversed twice, once to look for node $y \neq x$ of minimal index such that $c(y) = c(x)$ ($y$ might be empty), and a second time to replace $x$ by $y$ each time $x$ is used as a component index. If $x$ was unmatched in $\tilde{M}$ (*i.e.* $\mathbf{v}(\tilde{M})_{|x} = 0$), then we set $m(M') = m(\tilde{M}) + 1$, otherwise we set $m(M') = m(\tilde{M})$.

Once the set $\mathcal{M}(j)'$ of all the generated $M'$ is computed, $\tilde{\mathcal{M}}(i)$ is obtained as the quotient of $\mathcal{M}(j)'$ by $\sim_i$, the equivalence relation on $X_i$. More precisely, each pair $(M', M'') \in \mathcal{M}(j)'^2$ is tested for equality on both $\mathbf{v}$ and $\mathbf{uf}$. If they are equal, one with the lowest $m$ is defined to be the new representative in $\tilde{\mathcal{M}}(i)$.

*Running time.* Each new matching $M'$ is obtained from a single element of $\tilde{\mathcal{M}}(j)$ in worst-case time $O(w_i^2 \cdot \log(w_i))$. Equivalent matchings are detected

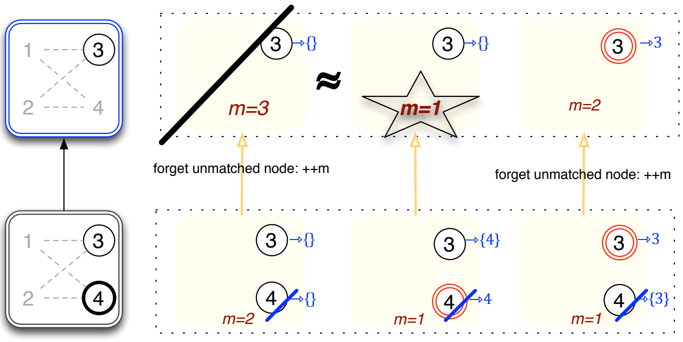



Figure 4.3: Detail of the decisions at a forget bag.

on-the-fly when filling the hash structure of $\tilde{\mathcal{M}}(i)$, and each equivalence test is linear in $w_i^2$. The complexity is thus in $O(2^{w_j^2+w_j} \cdot w_j^2 \cdot \log(w_j))$.

### Join bag

Let $X_i = X_j = X_k$ be a join bag with child bags $X_j$ and $X_k$ (see Figure 4.4). The matchings of $\mathcal{M}(i)$ are generated by combining all the pairs of matchings $(M, M') \in \mathcal{M}(j) \times \mathcal{M}(k)$. A combination is valid if and only if it satisfies both the matching and cycle-free conditions. The matching condition says that a node cannot be matched in both $M$ and $M'$, which is checked by a logical AND operation ($\mathbf{v}(M)$ AND $\mathbf{v}(M')$). The cycle-free condition is checked with the union-find structures $M$ and $M'$: the combination is valid if no node of the component of a matched node in $\mathbf{uf}(M)$ is neighbor of the same component in $\mathbf{uf}(M')$ and vice versa, each test requiring $O(w_i^2)$ per component.

If a combination is valid, its structure $M''$ is defined by $\mathbf{v}(M'') = \mathbf{v}(M)$ OR $\mathbf{v}(M')$. The union-find structure is initialized from $\mathbf{uf}(M)$, and updated as the introduce bag for each matched nodes of $M'$. Finally, $m(M'') = m(M) + m(M')$.

As in the forget bag, two combinations may result in equivalent matchings, and we must compare them pairwise and choose the representative with the lowest number of unmatched forgotten bags. Note that the sets of forgotten nodes of $X_j$ and forgotten nodes of $X_k$ have to be disjoint by the *coherence* requirement of Definition 4.1 and hence no forgotten node can be counted twice in this setting. Furthermore, all possible combinations of matched and unmatched nodes are enumerated in $\mathcal{M}(j)$ and $\mathcal{M}(k)$ and hence no possible matching is overseen.

*Running time.* Each pair of matchings is validated and updated in time $O(w_i \cdot w_i^2 \cdot \log(w_i))$. The comparison and the choice of representative is done on-the-fly when filling the hash structure of $\mathcal{M}(i)$. There are at worst $(2^{w_i^2+w_i})^2$ pairs. Thus, the complexity of the join bag dominates all other running times.

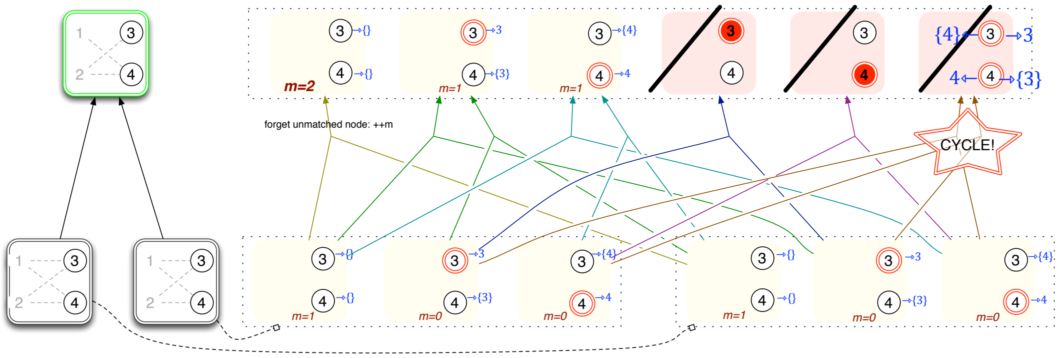



Figure 4.4: Detail of the decisions at a join bag.

Therefore, the complexity of the algorithm is in $O(4^{w_i^2+w_i} \cdot w_i^3 \cdot \log(w_i))$ per bag.

### Root bag

Let $X_r = \{x\}$ be the root of $T$. $\mathcal{M}(r)$ consists of at most two matchings $\mathbf{v}(\tilde{M}) = [0]$ or $\mathbf{v}(\tilde{M}') = [1]$, where $\mathbf{uf}(\tilde{M})$ is an empty list and $\mathbf{uf}(\tilde{M}')$ defined by $c(x) = x$. It follows that the minimum number of unmatched nodes for any alternating cycle-free matching of $G$ is given by $m = \min\left\{m(\tilde{M}) + 1, m(\tilde{M}')\right\}$, and the maximum size of an alternating cycle-free matching is given by $(n-m)/2$ where $n = |N|$ denotes the number of nodes of $G$.

*Running time.* The total time complexity of the algorithm is bounded above by the running time of the join bag. Since there is a linear number of bags, and since for every bag $X_i$ we have $|X_i| \leq \mathbf{tw}(G) + 1 = w + 1$, the total time complexity of the algorithm described above is

$$O\left(4^{w^2+w} \cdot w^3 \cdot \log(w) \cdot n\right).$$

### Correctness of the algorithm

We must check that the algorithm, without the grouping, considers every possible alternating cycle-free matching in $G$ and that the grouping occurring at the forget and join bags does not discard the maximal matching.

The *node coverage* and *arc coverage* properties of nice tree decompositions (Definition 4.1) ensure that each node is processed and each arc is considered for inclusion in the matching at one introduce node. Since the introduce node discards only matchings that violate either the matching or the cycle condition, and these violations cannot be legalized by further extensions or combinations of the matchings, all possible valid matchings are considered.

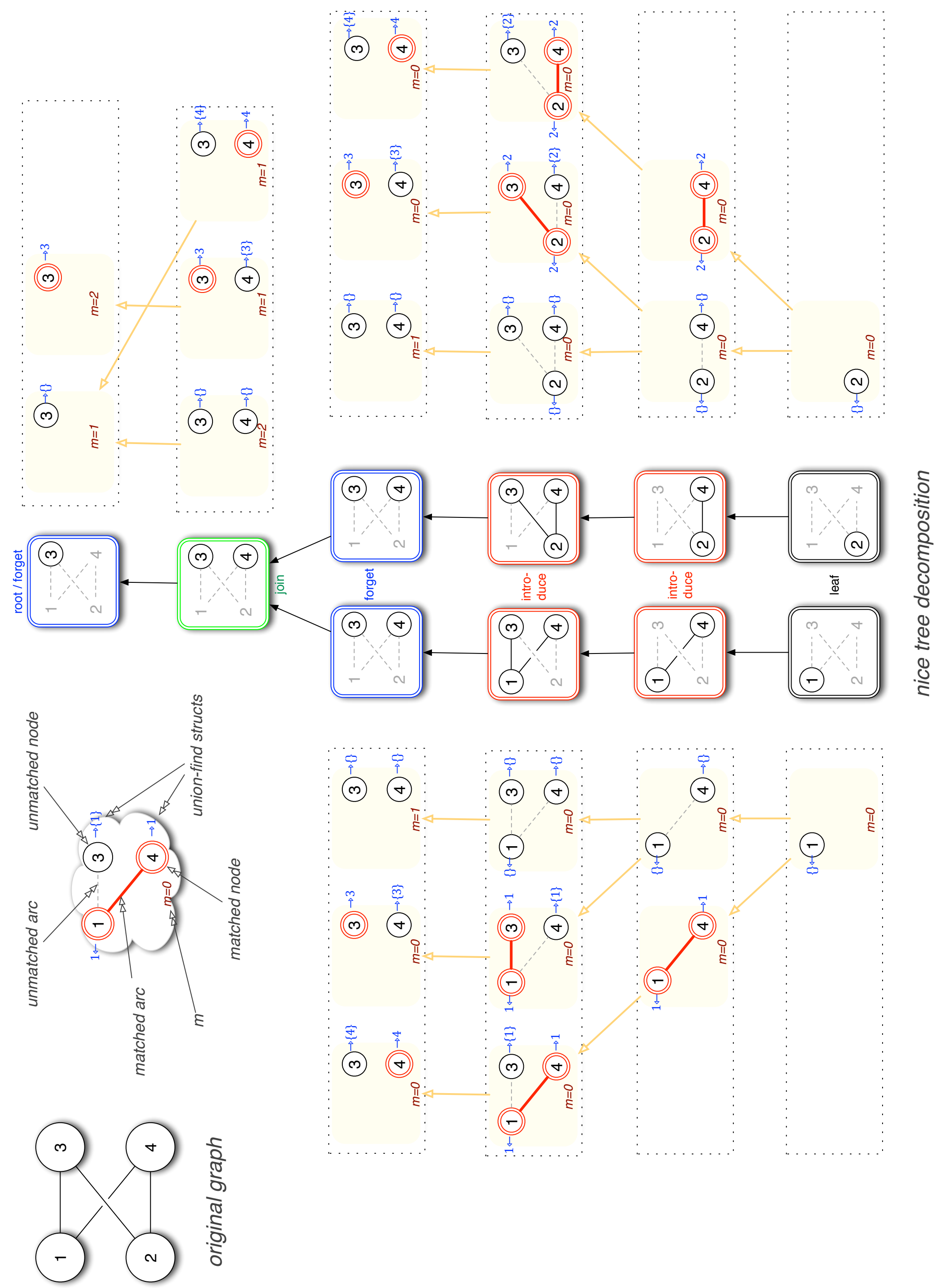



Figure 4.5: Algorithm execution on a small bipartite cycle (top left) with its nice tree decomposition (center). At each bag, a set of matchings $\tilde{\mathcal{M}}(i)$ is generated according to the bag type. $\tilde{\mathcal{M}}(i)$ is represented on the side of each bag, with the nomenclature illustrated at the top of the figure.

Now, consider two matchings $M$ and $M'$ that are grouped together and represented by $\tilde{M}$ at a forget or join bag $X_i$. In the further course of the algorithm, the representative $\tilde{M}$ is then extended or combined with other matchings to form new valid matchings $\tilde{M}'$. The *coherence* property of Definition 4.1 assures that no neighbor of a newly introduced node can be a forgotten node, so the extension or combination only modifies matchings $M$ and $\tilde{M}$ on nodes of $X_i$, which are represented in the structure of $\tilde{M}'$. Hence, the valid matchings $\tilde{M}'$ actually represent all the valid extensions and combinations of $M$ and $\tilde{M}$. The grouping thus generates all valid and relevant representatives of matchings in order to find a maximal alternating cycle-free matching. Moreover, in case $M$ and $M'$ are equivalent and both with the lowest number of forgotten unmatched nodes, choosing $M$ or $M'$ as representative leads to the exact same extensions and combinations.

Finally, let $M_m$ be the alternating cycle-free matching of maximum size of $G$. In each bag the corresponding matching must be one of the matchings with the lowest number of unmatched nodes within its equivalence class $\tilde{M}_m \in \tilde{\mathcal{M}}(i)$. Otherwise, a matching in the same class $\tilde{M}_m$, extended and combined as $M_m$ in the sequel of the algorithm would give rise to a matching with fewer unmatched nodes. Therefore, the choice of the representative at the forget and join bags never discards the future alternating cycle-free matching of maximum size.



## 4.5 Treewidth of the dual graph

Up to this point, we have been dealing primarily with simplicial complexes and their spines. We now turn our attention to simplicial triangulations of 3-manifolds and a more natural parameter associated to them.

**Definition 4.7** (Dual graph)**.** *The* dual graph *of a simplicial triangulation of a* 3*-manifold* $\mathcal{T}$*, denoted* $\Gamma(\mathcal{T})$*, is the graph whose nodes represent tetrahedra of* $\mathcal{T}$*, and whose arcs represent pairs of tetrahedron faces that are joined together.*

The following theorem states that, if the treewidth of the dual graph is bounded, so is the treewidth of the spine.

**Theorem 4.8.** *Let $G$ be the spine of a simplicial triangulation of a* 3*-manifold* $\mathcal{T}$*. If* $\mathbf{tw}(\Gamma(\mathcal{T})) \leq k$*, then* $\mathbf{tw}(G) \leq 10k + 9$*.*

*Proof* Let $T$ be a tree decomposition of the dual graph, where each bag $X_i$ contains at most $k+1$ tetrahedra. We show how to construct a tree decomposition $T'$ of the spine of $\mathcal{T}$, modeled on the same underlying tree as

$T$, in which each bag $X'_i$ contains less or equal $10(k+1)$ edges and triangles.

For each bag $X_i$ of $T$, we simply define the bag $X'_i$ to contain all edges and triangles of all tetrahedra in $X_i$. It remains to verify the three properties of a tree decomposition (Definition 4.1).

*Node coverage.* It is clear that every edge or triangle in the spine belongs to some bag $X'_i$, since every edge or triangle is contained in some tetrahedron $\tau$, which belongs to some bag $X_i$.

*Arc coverage.* Consider some arc in the spine. This must join a triangle $t$ to an edge e that contains it. Let $\tau$ be some tetrahedron containing $t$; then $\tau$ contains both $t$ and e, and so if $X_i$ is a bag containing $\tau$ then the corresponding bag $X'_i$ contains the chosen arc in the spine (joining $t$ with e).

*Coherence.* Here we treat edges and triangles separately. Let $t$ be some triangle in the simplicial complex. We must show that the bags containing $t$ correspond to a connected subgraph of the underlying tree.

If $t$ is a boundary triangle, then $t$ belongs to a unique tetrahedron $\tau$, and the bags $X'_i$ that contain $t$ correspond precisely to the bags $X_i$ that contain $\tau$. Since the tree decomposition T satisfies the coherence property, these bags correspond to a connected subgraph of the underlying tree. If $t$ is an internal triangle, then $t$ belongs to two tetrahedra $\tau_1$ and $\tau_2$, and the bags $X'_i$ that contain $t$ correspond to the bags $X_i$ that contain *either* $\tau_1$ or $\tau_2$. By the coherence property of the original tree, the bags containing $\tau_1$ describe a connected subgraph of the tree, and so do the bags containing $\tau_2$. Furthermore, there is an arc in the dual graph from $\tau_1$ to $\tau_2$, and so some bag $X_i$ contains *both* $\tau_1$ and $\tau_2$. Thus the union of these two connected subgraphs is another connected subgraph, and we have established the coherence property for $t$.

Now let $e$ be some edge of the simplicial complex. Again, we must show that the bags containing $e$ correspond to a connected subgraph of the underlying tree. This is simply an extension of the previous argument. Suppose that e belongs to the tetrahedra $\tau_1, \ldots, \tau_m$ (ordered cyclically around $e$). Then for each $\tau_j$, the bags $X_i$ that contain $\tau_j$ describe a connected subgraph of the underlying tree, and the bags $X'_j$ containing e describe the union of these subgraphs, which we need to show is again connected. This follows because there is a sequence of arcs in the dual graph $(\tau_1, \tau_2)$, $(\tau_2, \tau_3)$ and so on; from the tree decomposition $T$ it follows that the subgraph for $\tau_1$ meets the subgraph for $\tau_2$, the subgraph for $\tau_2$ meets the subgraph for $\tau_3$, and so on. Therefore the union of these subgraphs is itself connected. ∎

## 4.6 Experimental results

In this section, in addition to simplicial complexes, we briefly concentrate on a slightly more general notion of a *generalized triangulation of a 3-manifold*, which is a collection of tetrahedra all of whose faces are affinely identified or "glued together" such that the underlying topological space is a 3-manifold. Generalized triangulations use far fewer tetrahedra than simplicial complexes, which makes them important in computational 3-manifold topology (where many algorithms run in exponential time). Every simplicial triangulation is a generalized triangulation, and the second barycentric subdivision of a generalized triangulation is a simplicial triangulation [51], hence both objects are closely related.

In Chapter 3 we have seen that the problem of finding optimal Morse matchings is hard to solve in general. In this chapter, we proved that in the case of a small treewidth of the spine of a 2-dimensional complex or, equivalently, in the case of a bounded treewidth of the dual graph of a simplicial triangulation of a 3-manifold, finding an optimal Morse matching becomes easier. The results stated in Section 4.5 hold for generalized triangulations as well (also, note that the notion of a spine or the dual graph can be extended in a straightforward way to generalized triangulations).

Given this situation, a natural question to ask is the following: What is a *typical* value for the treewidth of the respective graphs of (i) small generic generalized triangulations of 3-manifolds, and (ii) small generic simplicial triangulations of 3-manifolds?

In a series of computer experiments we computed the treewidth of the relevant graphs (*i.e.* the spine and the dual graph) of all closed generalized triangulations of 3-manifolds up to 7 tetrahedra [11], and all simplicial triangulations of 3-manifolds up to 10 vertices [48]. The computer experiments were done using LibTW [69] to compute the treewidth / upper bounds for the treewidth, with the help of the GAP package simpcomp [23,24] and the 3-manifold software Regina [10,12]. We report the minimal, maximal and average treewidths of all triangulations with the same number of tetrahedra in Table 4.1 and of all simplicial triangulations with the same number of vertices in Table 4.2. Furthermore, in Table 4.3 we list the tree-width of the spines of a number of small 2-dimensional simplicial complexes with various properties which have the potential of interfering with the computation of optimal Morse functions.

On the one hand, regarding the treewidth of generalized triangulations of 3-manifolds, we observe that there is a large difference between the average

| $n$ | # triangulations | | min | max | avg. | | min | max | avg. | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | (3) | 1 | 2 | 1.50 | (1.67) | 0 | 0 | 0.00 | |
| 2 | 17 | (12) | 1 | 3 | 2.47 | (2.42) | 1 | 1 | 1.00 | |
| 3 | 81 | (63) | 1 | 3 | 2.51 | (2.49) | 1 | 2 | 1.60 | (1.52) |
| 4 | 577 | (433) | 1 | 5(4) | 2.77 | (2.73) | 1 | 3 | 1.91 | (1.87) |
| 5 | 5184 | (3961) | 1 | 6(5) | 2.95 | | 1 | 4 | 2.16 | (2.18) |
| 6 | 57753 | (43584) | 1 | 6 | 3.16 | (3.19) | 1 | 4 | 2.31 | (2.35) |
| 7 | 722765 | (538409) | 1 | 7 | 3.35 | (3.40) | 1 | 4 | 2.45 | (2.50) |

Table 4.1: Treewidths of the spine (left) and of the dual graphs (right) of closed generalized triangulations up to 7 tetrahedra. The values in brackets are for 1-vertex triangulations.

treewidth and the maximal treewidth for both the dual graph and the spine. In particular, the average treewidth appears to be relatively small. Moreover, there is only a slight difference between the data for general closed triangulations and 1-vertex triangulations. This fact is somehow in accordance with our intuition since the number of 0-dimensional simplices should neither directly affect the spine nor the dual graph of a generalized triangulation.

On the other hand, the gap between the maximum treewidth and the average treewidth in the case of simplicial triangulations of 3-manifolds is relatively small compared to the data for generalized triangulations. At this point it is important to note that, while the data concerning the spines for simplicial complexes only consists of upper bounds, experiments applying the algorithm for the upper bound to smaller graphs and then computing their real treewidths suggest that these upper bounds (in average) are reasonably close to the exact treewidth.

Further analysis shows that the average treewidth of the spines for both generalized and simplicial triangulations of 3-manifolds is mostly less than twice the treewidth of the dual graph, and hence much below the theoretical upper bound given by Theorem 4.8. Also, the ratio between these two numbers appears to be more or less stable for all values shown in Tables 4.1 and 4.2. This can be seen as experimental evidence that for triangulated 3-manifolds the treewidth of the dual graph is responsible for the inherent difficulty to solve ERASABILITY and related problems.

Despite the small values of $n$ in our tables, there are theoretical reasons to believe that the patterns of small treewidth will continue for larger $n$. For instance, the conjectured minimal triangulations of Seifert fibered spaces over the sphere have dual graphs with $O(1)$ treewidth for arbitrary $n$. Moreover, following recent results of Gabai *et al.* [27] there are reasons to believe that large infinite classes of topological 3-manifolds admit triangulations whose treewidths are below provable upper bounds.

| $n$ | # triangulations | min | max | avg. | min | max | avg. |
|---|---|---|---|---|---|---|---|
| 5 | 1 | 6 | 6 | 6.00 | 4 | 4 | 4.00 |
| 6 | 2 | $\leq$ 7 | $\leq$ 8 | $\leq$ 7.50 | 4 | 5 | 4.50 |
| 7 | 5 | $\leq$ 8 | $\leq$ 11 | $\leq$ 9.40 | 4 | 6 | 5.00 |
| 8 | 39 | $\leq$ 8 | $\leq$ 14 | $\leq$ 11.23 | 4 | 7 | 5.74 |
| 9 | 1297 | $\leq$ 8 | $\leq$ 18 | $\leq$ 13.55 | 4 | 9 | 7.01 |
| 10 | 249015 | $\leq$ 8 | $\leq$ 22 | $\leq$ 16.33 | $\leq$ 4 | $\leq$ 13 | $\leq$ 8.99 |

Table 4.2: Upper bounds and exact values for the treewidths of the spine (left) and of the dual graph (right) of simplicial triangulations of 3-manifolds up to 10 vertices.

| complex | shell. | ext. s. | constr. | vtx. dec. | $f$-vector | $w$ |
|---|---|---|---|---|---|---|
| Dunce hat, Zeeman [72] | | | No | | $(8, 24, 17)$ | $\leq 6$ |
| Björner [6, Exerc. 7.37] | | No | | | $(6, 15, 11)$ | 5 |
| Hachimori 1 [35] | Yes | No | | | $(7, 19, 13)$ | $\leq 5$ |
| Hachimori 2 [35] | No | | Yes | | $(12, 37, 26)$ | $\leq 5$ |
| Hachimori 3 [35] | No | | Yes | | $(13, 39, 27)$ | $\leq 5$ |
| Hachimori 4 [35] | No | | Yes | | $(10, 31, 22)$ | $\leq 5$ |
| Simon 1 [63] | | No | | | $(7, 20, 14)$ | $\leq 4$ |
| Simon 2 [63] | Yes | | | No | $(6, 15, 10)$ | 4 |
| Moriyama-Takeuchi 1 [52] | Yes | No | | | $(6, 14, 9)$ | 3 |
| Moriyama-Takeuchi 2 [52] | Yes | No | | | $(6, 14, 9)$ | 3 |
| Moriyama-Takeuchi 3 [52] | Yes | No | | | $(6, 15, 10)$ | 4 |
| Moriyama-Takeuchi 4 [52] | Yes | No | | | $(6, 15, 10)$ | 4 |
| Moriyama-Takeuchi 5 [52] | Yes | No | | | $(6, 15, 10)$ | 4 |
| Moriyama-Takeuchi 6 [52] | Yes | No | | | $(6, 15, 10)$ | 4 |
| Moriyama-Takeuchi 7 [52] | Yes | No | | | $(6, 15, 10)$ | 4 |
| Moriyama-Takeuchi 8 [52] | Yes | No | | | $(6, 15, 11)$ | 5 |
| Moriyama-Takeuchi 9 [52] | Yes | No | | | $(6, 15, 11)$ | 5 |
| Moriyama-Takeuchi 10 [52] | Yes | No | | | $(7, 17, 11)$ | 4 |
| Moriyama-Takeuchi 11 [52] | Yes | No | | | $(7, 18, 12)$ | $\leq 4$ |
| Moriyama-Takeuchi 12 [52] | Yes | | | No | $(6, 15, 10)$ | 4 |
| Moriyama-Takeuchi 13 [52] | Yes | | | No | $(6, 15, 10)$ | 3 |

Table 4.3: Treewidths (w) of the spine of some 2-dimensional simplicial complexes of particular interest for discrete Morse theory, together with information on their shellability (sh.), extended shellability (ext. s.), vertex decomposability (vtx. dec.), and $f$-vector (the $i$-th entry of the *$f$-vector* of a simplicial complex denotes the number of $i$-dimensional simplices in the complex) [35].

# 5
# Stable matchings and discrete Morse theory

In this chapter, we study how Morse matchings can be constructed from a "geometric" function $f$ defined at the vertices of a simplicial complex $\Delta$, using stable matchings (Definition 2.2). Once the matching is constructed, the related topological structures are easily extracted through efficient algorithms [41].

To do so, we propose define the weight of an arc in the Hasse diagram of $\Delta$ according to function $f$, and construct a discrete vector field as a stable matching (Section 5.1). Theorem 5.5 shows that the $\mathcal{V}$-paths of this discrete vector field follows decreasing directions of $f$. This essentially proves that such stable discrete vector fields are in fact discrete *gradient* vector fields (Theorem 5.9), and we thus call them *stable Morse matchings*. Since the definition of stable matchings is local, this framework provides simple tools to analyze the local dynamics of stable Morse matching (much simpler than previous results [41,42]!). In particular, we characterize most pairs of any stable Morse matching in Section 5.3, and complete this characterization under a discrete smoothness assumption in the next chapter.



## 5.1
## Stable matchings in the Hasse diagram

First of all, to define a stable matching in the Hasse diagram of $\Delta$, the graph needs to be a weighted graph. The weight of each arc is given from a geometric function $f$ by the following definition.

**Definition 5.1** (Geometric arc weights)**.** *Given a function $f$ defined at the vertices of $\Delta$, the weight of arc $\{\sigma, \tau\} \in H$ is defined as $\pi_f(\sigma, \tau) = f(\tau \setminus \sigma)$.*

In the previous definition, $(\sigma, \tau)$ is an arc of $H$, so $\dim(\tau) = \dim(\sigma) + 1$, and thus $\tau \setminus \sigma$ contains a single vertex, which is where $f$ is evaluated. We say that $\pi_f$ is *tie-free* if adjacent arcs have distinct weights, which implies the uniqueness of the stable matching (see Theorem 2.4). Observe that some cell complexes do not admit such tie-free weight (see Figure 5.1). However, they do not fit into our definition of simplicial complex.

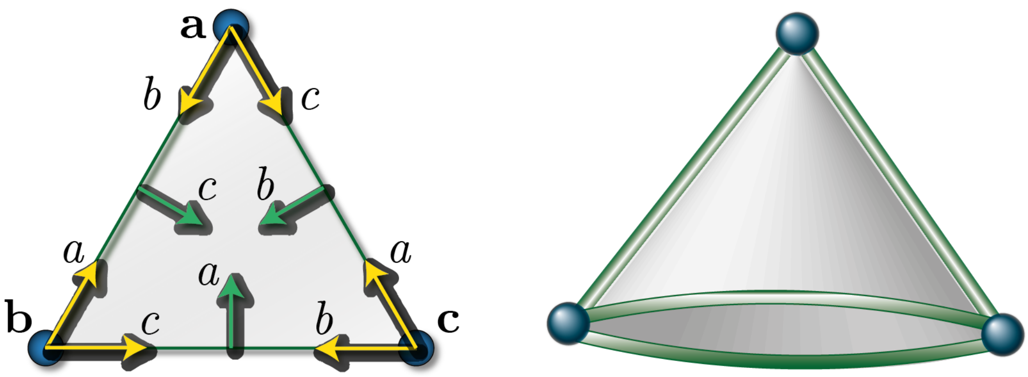



Figure 5.1: Arc weights from the function values (left) and a cell (not simplicial) complex which does not admit any tie-free weight (right).

**Definition 5.2** (Stable Morse matching)**.** *Let $f$ be a function defined on the vertices $V$ a simplicial complex $\Delta$ such that the associated weight $\pi_f$ is tie-free. The stable Morse matching $M$ of $f$ is the unique stable matching in the Hasse diagram of $\Delta$ weighted by $\pi_f$.*

The existence and uniqueness of stable matching $M$ are guaranteed by Theorems 2.3 and 2.4. It can be efficiently computed by the Gale-Shapley algorithm [28]. Usually $f$ is a real-valued function, however the definition of the stable matching only requires the values of $f$ to be comparable.

In the remainder of this thesis, we consider that function $f$ leads to a tie-free weight, and denote by $\pi$ the associated weight, $M$ the resulting stable Morse matching and $\mathcal{V}$ the corresponding discrete vector field. The next section justifies why this stable matching is indeed a Morse matching.



## 5.2 $\mathcal{V}$-paths of stable discrete vector fields

The main theorem of this section states that stable Morse matchings are decreasing (Theorem 5.5) with respect to $f$. Indeed, function $f$ induces an ordering on the simplices of $\Delta$ using lexicographic order of the vertices. More precisely, consider two simplices $\sigma = \{v_0, \ldots, v_p\} \in \Delta$ and $\sigma' = \{v'_0, \ldots, v'_p\} \in \Delta$ of same dimension $p$, where the vertices of each simplex are ordered by increasing $f$-values: $f(v_0) < f(v_1) < \ldots < f(v_p)$ and $f(v'_0) < f(v'_1) < \ldots < f(v'_p)$.

**Definition 5.3** (Lexicographic ordering)**.** *With the notation above, $\sigma >_{lex} \sigma'$ if and only if $\exists\ i < p$ such that $\forall j < i$, $f(v_j) = f(v'_j)$ and $f(v_i) > f(v'_i)$.*

By an abuse of notation, we denote $v > w$ whenever $f(v) > f(w)$ and $\sigma > \tau$ whenever $\sigma >_{lex} \tau$.

With the simple and local definition of an unstable pair (Definition 2.2), the following lemma states that a $\mathcal{V}$-path cannot increase twice in a row.

**Lemma 5.4** (Stability of $\mathcal{V}$-path). *If* $\blacktriangleleft\sigma_1\tau_1\sigma_2\tau_2\blacktriangleright$ *is a* $\mathcal{V}$*-path, then either* $\tau_1 \setminus \sigma_2 > \tau_1 \setminus \sigma_1$ *or* $\tau_1 \setminus \sigma_2 > \tau_2 \setminus \sigma_2$.

*Proof* Since $\blacktriangleleft\sigma_1\tau_1\sigma_2\tau_2\blacktriangleright$ is a $\mathcal{V}$-path, $(\sigma_1, \tau_1) \in M$ and $(\sigma_2, \tau_2) \in M$. Now $M$ is a stable matching, so $(\sigma_2, \tau_1) \in A \setminus M$ is not unstable. Therefore one of the conditions of Definition 2.2 is false: $\pi(\sigma_2, \tau_1) > \pi(\sigma_1, \tau_1)$ or $\pi(\sigma_2, \tau_1) > \pi(\sigma_2, \tau_2)$. This can be written in terms of vertices: $\tau_1 \setminus \sigma_2 = \pi(\sigma_2, \tau_1) > \pi(\sigma_1, \tau_1) = \tau_1 \setminus \sigma_1$ or $\tau_1 \setminus \sigma_2 = \pi(\sigma_2, \tau_1) > \pi(\sigma_2, \tau_2) = \tau_2 \setminus \sigma_2$ (see Figure 5.2). ∎

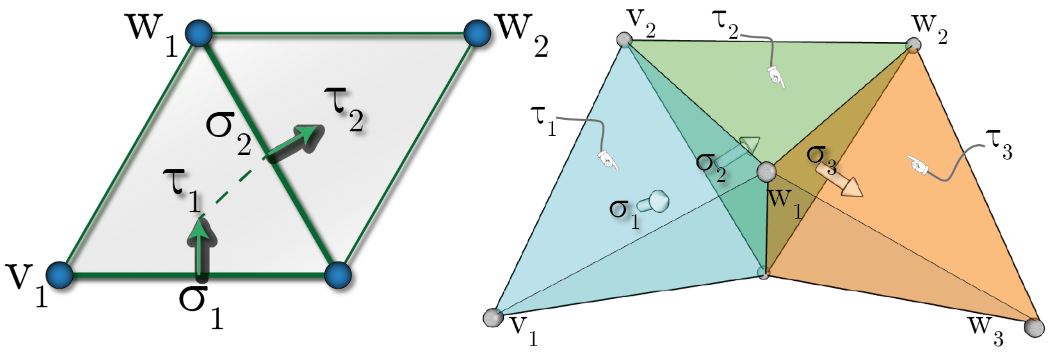



Figure 5.2: Notation for Lemma 5.4 and Theorem 5.5: observe that $w_1 = \tau_1 \setminus \sigma_1$, $w_2 = \tau_2 \setminus \sigma_2$, and $v_1 = \tau_1 \setminus \sigma_2$. With this notation, Lemma 5.4 states that either $v_1 > w_1$ or $v_1 > w_2$.

We can state this lemma with the lexicographic order: $\tau_1 \setminus \sigma_2 > \tau_1 \setminus \sigma_1$ can be written as $\sigma_1 > \sigma_2$ and $\tau_1 \setminus \sigma_2 > \tau_2 \setminus \sigma_2$ can be written as $\tau_1 > \tau_2$. The lemma ensures that along the $\mathcal{V}$-path $\blacktriangleleft\sigma_1\tau_1\sigma_2\tau_2\blacktriangleright$, either the $\sigma's$ or the $\tau's$ decreases at each step. For longer $\mathcal{V}$-paths, we can state the following theorem.

**Theorem 5.5** (Decreasing $\mathcal{V}$-paths). *Let* $\blacktriangleleft\sigma_1\tau_1\sigma_2\tau_2 \ldots \sigma_s\tau_s\blacktriangleright$ *be a* $\mathcal{V}$*-path, and denote* $w_k = \tau_k \setminus \sigma_k$*, and* $v_k = \tau_k \setminus \sigma_{k+1}$*. If* $\sigma_1$ *is the minimal simplex of the* $\mathcal{V}$*-path for the lexicographic order, then* $\forall k \in \{2, \ldots, d\}$*, there exists* $\rho_k \in \Delta$ *such that* $\sigma_1 = \{v_1, v_2, \ldots, v_{k-1}\} \cup \rho_k$*,* $\sigma_k = \{w_1, w_2, \ldots, w_{k-1}\} \cup \rho_k$*, and* $w_1 > v_1 > w_2 > \ldots > w_{k-1} > v_{k-1} > w_k$*.*

*Proof by induction.* With the notation of the theorem, Lemma 5.4 ensures that $v_k > w_k$ or $v_k > w_{k+1}$ (see Figure 5.2). Since weight $\pi$ is tie-free, $\tau_k \setminus \sigma_k \neq \tau_k \setminus \sigma_{k+1}$, and in particular $w_k \neq v_k$. We will now check the existence of $\rho_k$ and the ordering of the $v's$ and the $w's$ by induction.

Base case. Let $k = 2$ and $\rho_2 = \sigma_1 \setminus v_1$. Then $\sigma_1 = v_1 \cup \rho_2$ and $\sigma_2 = \tau_1 \setminus v_1 = (\sigma_1 \cup w_1) \setminus v_1 = \rho_2 \cup w_1$. Since $\sigma_1$ is minimal, $\sigma_2 > \sigma_1$. Therefore

$\rho_2 \cup w_1 > \rho_2 \cup v_1$, and thus $w_1 > v_1$. The first alternative of Lemma 5.4 is false: $v_1 \not> w_1$ (recall that $v_1 \neq w_1$). Then the other one is true: $v_1 > w_2$. All together, we have $w_1 > v_1 > w_2$.

Induction step. Suppose that there exists $\rho_k$ satisfying the theorem for a given $k \geq 2$ with $w_1 > v_1 > w_2 > \ldots > w_{k-1} > v_{k-1} > w_k$.

First assume as a contradiction hypothesis that $v_k \in \{w_1, \ldots, w_{k-1}\}$. Since $\sigma_1$ is minimal, $\sigma_{k+1} = \rho_k \cup (\{w_1, \ldots, w_{k-1}\} \setminus v_k) \cup w_k > \sigma_1 = \rho_k \cup \{v_1, v_2, \ldots, v_{k-1}\}$. This would imply that $(\{w_1, \ldots, w_{k-1}\} \setminus v_k) \cup w_k > \{v_1, v_2, \ldots, v_{k-1}\}$, but this contradicts the induction hypothesis $w_1 > v_1 > w_2 > \ldots > w_{k-1} > v_{k-1} > w_k$. Therefore $v_k \notin \{w_1, \ldots, w_{k-1}\}$.

Since $v_k \in \sigma_k$, then $v_k \in \sigma_k \setminus \{w_1, \ldots, w_{k-1}\} = \rho_k$ and we can define $\rho_{k+1}$ as $\rho_{k+1} = \rho_k \setminus v_k$. We check $\sigma_1 = \{v_1, \ldots, v_{k-1}\} \cup \rho_k = \{v_1, \ldots, v_k\} \cup \rho_{k+1}$ and $\sigma_{k+1} = \rho_k \cup \{w_1, \ldots, w_k\} \setminus v_k = \rho_{k+1} \cup \{w_1, \ldots, w_k\}$.

Now we check the ordering of $v_k$ and $w_k$. Since $\sigma_1$ is minimal, $\rho_{k+1} \cup \{w_1, \ldots, w_k\} = \sigma_{k+1} > \sigma_1 = \{v_1, \ldots, v_k\} \cup \rho_{k+1}$. Therefore $\{w_1, \ldots, w_k\} > \{v_1, \ldots, v_k\}$, and the first comparison of the lexicographic order implies

$$\min\{w_1, \ldots, w_k\} \geq \min\{v_1, \ldots, v_k\} \quad .$$

From the induction hypothesis $w_1 > v_1 > \ldots > w_{k-1} > v_{k-1} > w_k$, so

$$w_k = \min\{w_1, \ldots, w_k\} < \min\{v_1, \ldots, v_{k-1}\} = v_{k-1} \; .$$

As observed at the beginning of the proof, $w_k \neq v_k$, thus $w_k > v_k$. Since $v_k \neq w_k$, using Lemma 5.4, $v_k \not> w_k$ implies that $v_k > w_{k+1}$. All together $w_k > v_k > w_{k+1}$.

■

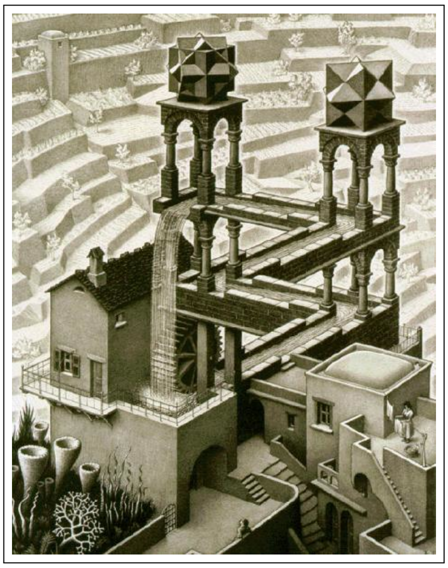


Figure 5.3: Paradox in *Waterval*, M. C. Escher (1961).

Since in a stable discrete vector field, any $\mathcal{V}$-path is decreasing in the sense of the previous theorem, the field admits no closed $\mathcal{V}$-path (it would be a paradox analogous the one in Figure 5.3).

**Theorem 5.6.** *The discrete vector field $\mathcal{V}$ associated to the stable matching $M$ is a discrete gradient vector field.*

*Proof by contradiction.* Assume $\mathcal{V}$ is not a discrete gradient vector field. Then, by Definition 2.8, there exists a non-trivial closed $\mathcal{V}$-path $\blacktriangleleft\sigma_1\tau_1\sigma_2\tau_2\ldots\sigma_s\blacktriangleright$ where $s \geq 2$ and $\sigma_1 = \sigma_s$. Since it is a closed $\mathcal{V}$-path, we can assume without loss of generality that the minimal simplex is $\sigma_1$. Then, Theorem 5.5 leads to a contradiction, since we have, for any $k \geq 2$, $\sigma_1 = \{v_1, v_2, \ldots, v_{k-1}\} \cup \rho_k$, $\sigma_k = \{w_1, w_2, \ldots, w_{k-1}\} \cup \rho_k$, with vertices $v's$ and $w's$ all distinct. Therefore $\sigma_1 \neq \sigma_k$ for any $k \geq 2$. ■

The Gale-Shapley algorithm [28] used to compute stable matching is then equivalent to the greedy constructions of geometric discrete Morse function [33, 41], provided the weight is tie-free. However, the stable matching formulation provides effective tools to analyze the generated function and to optimize the algorithm.

## 5.3
## Local dynamics of stable Morse matchings

In this section, we analyze the local behavior of the discrete gradient vector field obtained by stable matching. We show that the discrete gradient vector field "points to the steepest descent". The main elements of this analysis rely on the following definitions and some immediate facts about them.

**Definition 5.7.** *The vertex link of $\sigma$ denoted by $\mathrm{lk}_0(\sigma) = \{v \in V \mid \sigma \cup v \in \Delta\}$ is the set of vertices that are adjacent to $\sigma$. If $\mathrm{lk}_0(\sigma) \neq \varnothing$, let $\mathrm{v_{lk}}(\sigma) = \min \mathrm{lk}_0(\sigma)$ be the smallest vertex in the vertex link of $\sigma$. Also let $\mathrm{v_m}(\sigma) = \min \sigma$, the smallest vertex in $\sigma$.*

The following lemma states direct relations between the definitions above.

**Lemma 5.8.** *For all $v \in \sigma$, with $\dim(\sigma) > 0$ and $\mathrm{lk}_0(\sigma) \neq \varnothing$:*

(i) $\mathrm{lk}_0(\sigma) \cup v \subseteq \mathrm{lk}_0(\sigma \setminus v)$,

(ii) $\mathrm{v_{lk}}(\sigma) \geq \mathrm{v_{lk}}(\sigma \setminus v)$,

(iii) $v \geq \mathrm{v_{lk}}(\sigma \setminus v)$,

(iv) $\mathrm{v_m}(\sigma \setminus v) \geq \mathrm{v_m}(\sigma)$,

(v) $\mathrm{v_m}(\sigma \setminus v) > \mathrm{v_m}(\sigma) \Leftrightarrow v = \mathrm{v_m}(\sigma)$.

*Proof.* Item (i) follows from the definition of simplex as subset of vertices: if $w \in \mathrm{lk}_0(\sigma)$, then $\sigma \cup w$ is a simplex and so is its face $(\sigma \setminus v) \cup w$. Also $(\sigma \setminus v) \cup v = \sigma \in \Delta$, so $v \in \mathrm{lk}_0(\sigma \setminus v)$.

Item (ii) follows from item (i), since the minimum of subset $\mathrm{lk}_0(\sigma)$ is larger that the minimum of the larger set $\mathrm{lk}_0(\sigma \setminus v)$.

Item (iii) follows from $v \in \mathrm{lk}_0(\sigma \setminus v)$, so it is larger than the minimum of $\mathrm{lk}_0(\sigma \setminus v)$.

Item (iv) follows similarly from $\sigma \setminus v \subset \sigma$: the minimum of the subset $\sigma \setminus v$ is larger than the minimum of the larger set $\sigma$.

Finally, item (v) follows from item (iv), observing that the minima of the two set differ if the vertex removed from $\sigma$ is indeed the minimum of $\sigma$. ∎

Now we can state and prove the Steepest Descent theorem which essentially says that if the smallest vertex in the vertex link of $\sigma$ is smaller then every vertex in $\sigma$, then the stable Morse matching points in that direction.

**Theorem 5.9** (Steepest Descent)**.** *Let $M$ be a stable matching. If $\mathrm{lk}_0(\sigma) \neq \varnothing$ and $\mathrm{v_m}(\sigma) > \mathrm{v_{lk}}(\sigma)$, then $\sigma \to \sigma \cup \mathrm{v_{lk}}(\sigma)$.*

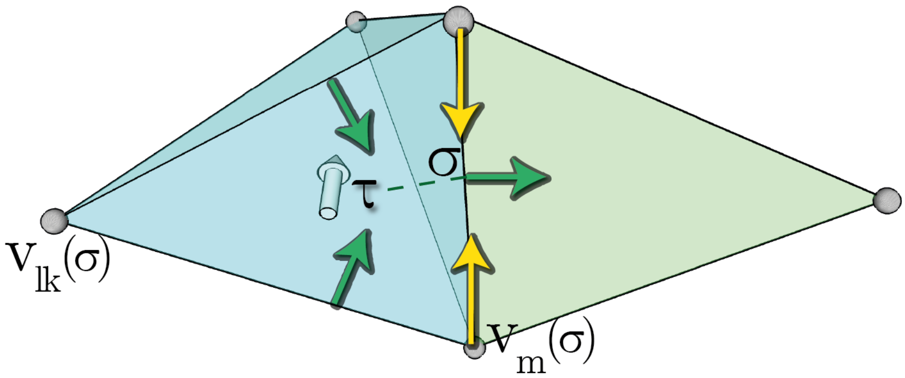



Figure 5.4: Instabilities around $\sigma \to \sigma \cup \mathrm{v_{lk}}(\sigma)$ in Theorem 5.9.

*Proof by contradiction.* Let $\tau = \sigma \cup \mathrm{v_{lk}}(\sigma)$. Assuming by contradiction that $(\sigma, \tau) \notin M$, we show that $(\sigma, \tau)$ is unstable in $M$. Let's check Condition 1 of Definition 2.2: "$\tau$ is unmatched or $\exists\ w \in N$, such that $(\tau, w) \in M$ and $\pi(\tau, w) > \pi(\sigma, \tau)$".

Case 1a. $\tau$ is critical in $M$. Then Condition 1 is true.

Case 1b. $\tau$ matched above in $M$. $\exists\, v' \in \mathrm{lk}_0(\tau), v' \neq \mathrm{v_{lk}}(\sigma)$ such that $(\tau, \tau \cup v') \in M$. It follows that $v' \geq \mathrm{v_{lk}}(\tau) \geq \mathrm{v_{lk}}(\sigma)$. Since $v' \neq \mathrm{v_{lk}}(\sigma)$, $v' > \mathrm{v_{lk}}(\sigma)$. Therefore $\pi(\tau, \tau \cup v') = v' > \mathrm{v_{lk}}(\sigma) = \pi(\sigma, \tau)$. Condition 1 holds.

Case 1c. $\tau$ matched below in $M$. $\exists\ v' \in \tau, v' \neq \mathrm{v_{lk}}(\sigma)$ such that $(\tau \setminus v', \tau) \in M$.

Therefore $v' \in \tau \setminus \mathrm{v_{lk}}(\sigma) = \sigma$. It follows that $v' \geq \mathrm{v_m}(\sigma) > \mathrm{v_{lk}}(\sigma)$. Therefore $\pi(\tau, \tau \cup v') = v' > \mathrm{v_{lk}}(\sigma) = \pi(\sigma, \tau)$. Condition 1 is true.

Therefore, if $(\sigma, \tau) \notin M$, then Condition 1 of Definition 2.2 is always true. Similarly, let's check Condition 2 of Definition 2.2: "$\sigma$ is unmatched or $\exists\, w \in N$, such that $(\sigma, w) \in M$ and $\pi(\sigma, w) > \pi(\sigma, \tau)$".

Case 2a. $\sigma$ is critical in $M$. Then Condition 2 is true.

Case 2b. $\sigma$ matched above in $M$. $\exists\, v' \in \mathrm{lk}_0(\sigma), v' \neq \mathrm{v_{lk}}(\sigma)$ such that $(\sigma, \sigma \cup v') \in M$. It follows that $v' \geq \mathrm{v_{lk}}(\sigma)$. Since $v' \neq \mathrm{v_{lk}}(\sigma)$, $v' > \mathrm{v_{lk}}(\sigma)$. Therefore $\pi(\sigma, \sigma \cup v') = v' > \mathrm{v_{lk}}(\sigma) = \pi(\sigma, \tau)$. Condition 2 holds.

Case 2c. $\sigma$ matched below in $M$. $\exists\, v' \in \sigma, v' \neq \mathrm{v_{lk}}(\sigma)$ such that $(\sigma \setminus v', \sigma) \in M$. It follows that $v' \geq \mathrm{v_m}(\sigma) > \mathrm{v_{lk}}(\sigma)$. Therefore $\pi(\tau, \tau \cup v') = v' > \mathrm{v_{lk}}(\sigma) = \pi(\sigma, \tau)$. Condition 2 is true.

Therefore, if $(\sigma, \tau) \notin M$, then Condition 2 of Definition 2.2 is always true. Since both Conditions 1 and 2 are true, then $(\sigma, \tau)$ is unstable. This contradicts the hypothesis that $M$ is a stable matching, therefore $(\sigma, \tau) \in M$. ∎

The Steepest Descent theorem gives a sufficient condition for a simplex $\sigma$ to match above with $\tau = \sigma \cup \mathrm{v_{lk}}(\sigma)$. It can be re-formulated from the point of view of $\tau$: every $\tau \setminus \mathrm{v_m}(\tau) \in \Delta$ is matched above.

**Lemma 5.10.** *For every $\tau \in \Delta$ with $\dim \tau > 0$, $\tau \setminus \mathrm{v_m}(\tau) \to (\tau \setminus \mathrm{v_m}(\tau)) \cup \mathrm{v_{lk}}(\tau \setminus \mathrm{v_m}(\tau))$.*

*Proof.* Using $\sigma = \tau \setminus \mathrm{v_m}(\tau)$ in the Steepest Descent theorem, we only need to check that $\mathrm{v_m}(\sigma) > \mathrm{v_{lk}}(\sigma)$. From items (iv) and (iii) of Lemma 5.8, $\mathrm{v_m}(\tau \setminus \mathrm{v_m}(\tau)) > \mathrm{v_m}(\tau)$ and $\mathrm{v_m}(\tau) \geq \mathrm{v_{lk}}(\tau \setminus \mathrm{v_m}(\tau))$. Therefore $\mathrm{v_m}(\sigma) = \mathrm{v_m}(\tau \setminus \mathrm{v_m}(\tau)) > \mathrm{v_{lk}}(\tau \setminus \mathrm{v_m}(\tau)) = \mathrm{v_{lk}}(\sigma)$, and Theorem 5.9 applies to $\sigma$. ∎

The next corollary, which gives sufficient conditions for $\sigma$ to be matched below, follows directly from the previous lemma.

**Corollary 5.11.** *If $\mathrm{v_m}(\sigma) = \mathrm{v_{lk}}(\sigma \setminus \mathrm{v_m}(\sigma))$, then $\sigma \setminus \mathrm{v_m}(\sigma) \to \sigma$.*

The Steepest Descent theorem gives some sufficient conditions for a simplex to be matched above and Corollary 5.11 gives sufficient condition for a simplex to be matched below. The results of this section can be summarized in the following table.

| Function $\Rightarrow$ | Matching | Matched | Result |
|---|---|---|---|
| $\mathrm{v_m}(\sigma) > \mathrm{v_{lk}}(\sigma)$ | $\sigma \to \sigma \cup \mathrm{v_{lk}}(\sigma)$ | Above | Steepest Descent theorem |
| $\mathrm{v_m}(\sigma) = \mathrm{v_{lk}}(\sigma \setminus \mathrm{v_m}(\sigma))$ | $\sigma \setminus \mathrm{v_m}(\sigma) \to \sigma$ | Below | Corollary 5.11 |

When we combine the contrapositive of both results, we obtain necessary conditions for a critical simplex.

**Lemma 5.12.** *If $\sigma$ is critical, then* $\mathrm{v}_{\mathrm{lk}}(\sigma) > \mathrm{v}_{\mathrm{m}}(\sigma) > \mathrm{v}_{\mathrm{lk}}(\sigma \setminus \mathrm{v}_{\mathrm{m}}(\sigma))$.

The proof is direct since $\mathrm{v}_{\mathrm{m}}(\sigma) \neq \mathrm{v}_{\mathrm{lk}}(\sigma)$ and $\mathrm{v}_{\mathrm{m}}(\sigma) \geq \mathrm{v}_{\mathrm{lk}}(\sigma \setminus \mathrm{v}_{\mathrm{m}}(\sigma))$

The Steepest Descent theorem gives sufficient conditions for pairs in stable Morse matchings, and a natural question is whether its converse is true. However the example of Figure 5.5 shows that the converse is not always true. Let $\Delta = \{a, b, c, d, ab, bc, cd\}$ and $a > b > c > d$. The stable Morse matching is $b \to ab$, $c \to bc$, and $d \to cd$. We have $b > \mathrm{v}_{\mathrm{m}}(c) = c$ but $c \to bc$ (not steepest descent!).

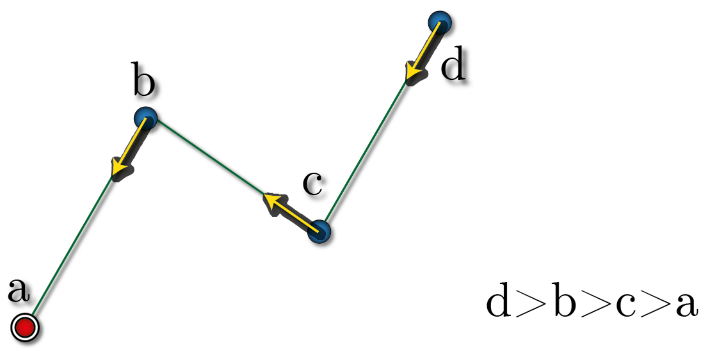



Figure 5.5: One-dimensional counterexample for the converse of the Steepest Descent theorem: matching $M = \{(b, ab), (c, bc), (d, cd)\}$ is stable and the conclusion of Theorem 5.9 $c \to c \cup b$ is true, but its premise is false: $\mathrm{v}_{\mathrm{m}}(c) = c \not> \mathrm{v}_{\mathrm{lk}}(c) = b$.

If the converse of the Steepest Descent theorem were true, we would have a full characterization of the pairs of stable Morse matchings. In particular, we would locally define the critical simplices which is fundamental in computational applications [4,61]. The next chapter studies properties of simplicial complexes and functions $f$ where the converse is true.

# 6
# Discrete smoothness and stable Morse matchings

In this chapter, we introduce a concept of discrete smoothness for functions on a simplicial complex and then prove that for discrete smooth functions, the converse of the Steepest Descent theorem holds (Section 6.1). In that case, we have a full characterization of the stable Morse matching and its critical simplices. In Section 6.2, we show that the dynamics around critical simplices are similar to their counterparts from the smooth theory. In Section 6.3, we relate them to critical points of piecewise-linear interpolation of function $f$, as defined by Banchoff [5]. Our results on discrete smoothness generalize several results stated with barycentric subdivision [4,41,42], and we prove at the end of the chapter that any function defined on a simplicial complex is discrete smooth after one barycentric subdivision.

## 6.1
## Discrete smoothness

As in the previous part of this thesis, function $f$ is defined on the vertices of a simplicial complex $\Delta$, and is used to define an ordering of those.

**Definition 6.1** (Discrete smoothness)**.** *A function $f$ is discrete smooth on simplicial complex $\Delta$ if for every $\sigma \in \Delta$ with $\mathrm{lk}_0(\sigma) \neq \varnothing$, the following holds:*

$$\textit{if}\;\; \mathrm{v}_{\mathrm{lk}}(\sigma) > \mathrm{v}_{\mathrm{m}}(\sigma), \quad \textit{then} \quad \forall v \in \mathrm{lk}_0(\sigma),\; \mathrm{v}_{\mathrm{lk}}(\sigma \setminus \mathrm{v}_{\mathrm{m}}(\sigma) \cup v) = \mathrm{v}_{\mathrm{m}}(\sigma) \quad .$$

Note that this definition depends on the function as well as the structure of the simplicial complex, and we write that the simplicial complex $\Delta$ is discrete smooth when the function defined on $\Delta$ is discrete smooth. In this entire section we assume that $\Delta$ is discrete smooth. Observe that the simplicial complexes in the right of Figure 6.1 are not discrete smooth.

With this definition, we can obtain necessary conditions for a simplex to be matched above, showing that if a simplicial complex is discrete smooth, then the converse of Steepest Descent theorem holds.

**Lemma 6.2** (Converse of the steepest descent theorem)**.** *Consider a discrete smooth complex $\Delta$. If $\sigma \to \sigma \cup v$, then $\mathrm{v}_{\mathrm{m}}(\sigma) > \mathrm{v}_{\mathrm{lk}}(\sigma) = v$.*

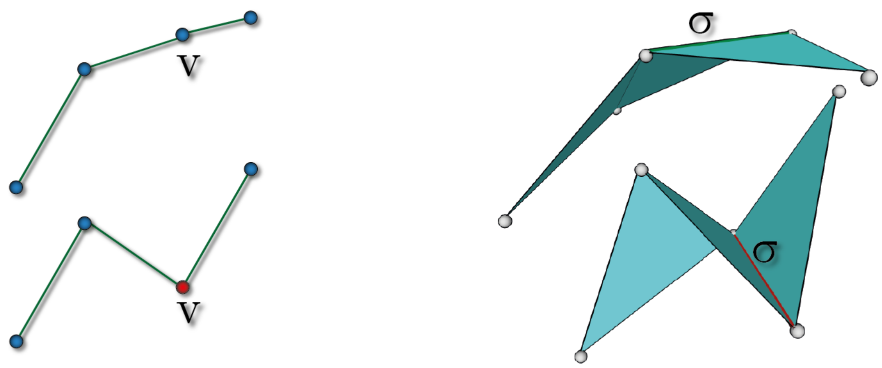



Figure 6.1: Examples of a smooth vertex $v$ (top left), a non-smooth vertex $v$ (bottom left), a smooth edge $\sigma$ (top right), and non-smooth edge $\sigma$ (bottom right). The function $f$ is the height.

This is indeed the converse of Theorem 5.9 since $\mathrm{v_m}(\sigma) \notin \mathrm{lk}_0(\sigma)$, therefore $\mathrm{v_m}(\sigma) \neq \mathrm{v_{lk}}(\sigma)$.

*Proof by contradiction.* Assume by contradiction that either $\mathrm{v_{lk}}(\sigma) > \mathrm{v_m}(\sigma)$ or $v \neq \mathrm{v_{lk}}(\sigma)$. We separate the proof in two cases, depending on the order of $\mathrm{v_m}(\sigma)$ and $\mathrm{v_{lk}}(\sigma)$:

Case 1: $\mathrm{v_m}(\sigma) > \mathrm{v_{lk}}(\sigma)$. It follows that $v \neq \mathrm{v_{lk}}(\sigma)$. Therefore, by Theorem 5.9, we have that $\sigma$ is matched to $\sigma \cup \mathrm{v_{lk}}(\sigma)$ and not $\sigma \cup v$ as stated in the hypothesis.

Case 2: $\mathrm{v_{lk}}(\sigma) > \mathrm{v_m}(\sigma)$. Since $\sigma \rightarrow \sigma \cup v$, we have $v \in \mathrm{lk}_0(\sigma)$. Therefore $v \geq \mathrm{v_{lk}}(\sigma) > \mathrm{v_m}(\sigma)$. Let $\sigma' = \sigma \setminus \mathrm{v_m}(\sigma) \cup v$ the simplex appearing in the definition of discrete smoothness. Since $v > \mathrm{v_m}(\sigma)$, it follows that $\mathrm{v_m}(\sigma') > \mathrm{v_m}(\sigma)$. The discrete smoothness implies that $\mathrm{v_{lk}}(\sigma') = \mathrm{v_m}(\sigma)$ (Definition 6.1). Therefore $\mathrm{v_m}(\sigma') > \mathrm{v_m}(\sigma) = \mathrm{v_{lk}}(\sigma')$. It follows from Theorem 5.9, that $\sigma' \rightarrow \sigma' \cup \mathrm{v_{lk}}(\sigma')$. However $\sigma' \cup \mathrm{v_{lk}}(\sigma') = \sigma' \cup \mathrm{v_m}(\sigma) = \sigma \cup v$. Therefore $\sigma' \rightarrow \sigma \cup v$. This is a contradiction since $\sigma \cup v$ is already matched with $\sigma$ by hypothesis. ■

We now combine the previous lemma with the Steepest Descent theorem to give the full characterization of the stable Morse matching.

**Corollary 6.3.** *Consider a discrete smooth complex $\Delta$. Then:*

$$\sigma \rightarrow \sigma \cup v \quad \Leftrightarrow \quad \mathrm{v_m}(\sigma) > \mathrm{v_{lk}}(\sigma) = v \quad .$$

This characterization gives an important and extremely fast, linear and naturally parallel algorithm to compute stable Morse matching (in the discrete smooth case): the algorithm only needs to check every $\sigma \in \Delta$ and $\mathrm{lk}_0(\sigma)$ to see if $\sigma$ is matched above.

With the converse of the steepest descent theorem, we can now prove the converse of Corollary 5.11, which is essentially applying the converse of the steepest descent theorem from the dimension below.

**Corollary 6.4.** *Consider a discrete smooth complex $\Delta$ and $v \in \tau \in \Delta$. Then:*

$$\tau \setminus v \to \tau \quad \Leftrightarrow \quad \mathrm{v}_{\mathrm{lk}}(\tau \setminus \mathrm{v}_{\mathrm{m}}(\tau)) = \mathrm{v}_{\mathrm{m}}(\tau) = v \quad .$$

*Proof.* ($\Rightarrow$) Using Lemma 6.2 on $\sigma = \tau \setminus v$, we get $\mathrm{v}_{\mathrm{m}}(\sigma) > \mathrm{v}_{\mathrm{lk}}(\sigma) = v$. Therefore $\mathrm{v}_{\mathrm{m}}(\tau \setminus v) > \mathrm{v}_{\mathrm{lk}}(\tau \setminus v) = v$. Then it follows that $v = \mathrm{v}_{\mathrm{m}}(\tau)$ by Lemma 5.8.

($\Leftarrow$) Corollary 5.11. ∎

We summarize the results when $\Delta$ is discrete smooth in the following table. It is exactly corresponds to the table of Section 5.3.

| Matching $\Leftrightarrow$ | Function | Matched | Result |
|---|---|---|---|
| $\sigma \setminus v \to \sigma$ | $\mathrm{v}_{\mathrm{lk}}(\sigma \setminus \mathrm{v}_{\mathrm{m}}(\sigma)) = \mathrm{v}_{\mathrm{m}}(\sigma) = v$ | Below | Corollary 6.4 |
| $\sigma \to \sigma \cup v$ | $\mathrm{v}_{\mathrm{m}}(\sigma) > \mathrm{v}_{\mathrm{lk}}(\sigma) = v$ | Above | Lemma 6.2 |

Since $\sigma$ is critical if and only if $\sigma$ is neither matched above nor below, critical simplices are characterized by the following corollary.

**Corollary 6.5.** *$\sigma$ is critical if and only if both conditions hold:*

(i) (unmatched below) $\dim(\sigma) = 0 \quad or \quad \mathrm{v}_{\mathrm{lk}}(\sigma) > \mathrm{v}_{\mathrm{m}}(\sigma)$ ,

(ii) (unmatched above) $\mathrm{lk}_0(\sigma) = \varnothing \quad or \quad \mathrm{v}_{\mathrm{m}}(\sigma) > \mathrm{v}_{\mathrm{lk}}(\sigma \setminus \mathrm{v}_{\mathrm{m}}(\sigma))$ .

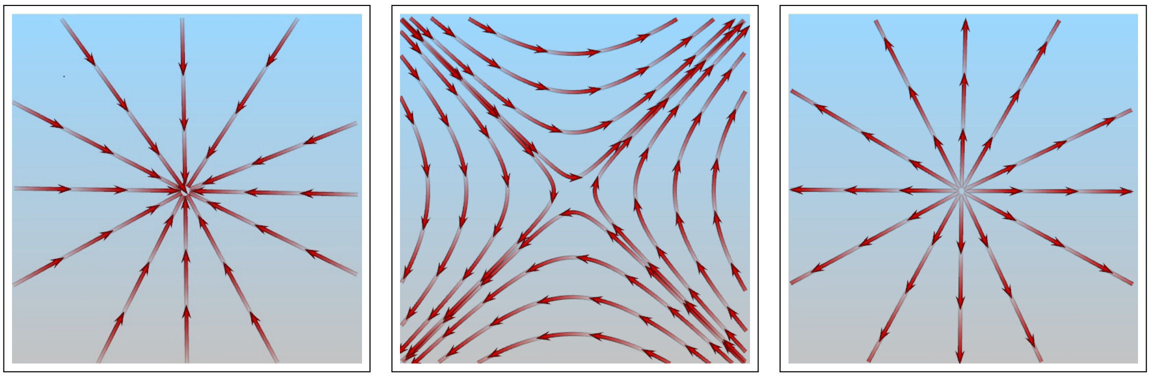


Figure 6.2: Critical (singular) points in a *continuous* gradient vector field: a sink (left), a saddle (middle), and a source (right).

## 6.2
## Nice-looking critical simplices

Now that we have characterized the critical simplices, we want to understand the behavior of the stable Morse matching in their neighborhood. We show that this behavior is similar to the local dynamics around the continuous critical points (see Figure 6.2). In this entire section we assume that $\Delta$ is discrete smooth.

**Lemma 6.6** (Critical Above). *If $\tau$ is a critical simplex, then all facets are matched above: $\forall v \in \tau$, $\tau \setminus v$ is matched above. Moreover, $\mathrm{v_m}(\tau) > \mathrm{v_{lk}}(\tau \setminus v)$.*

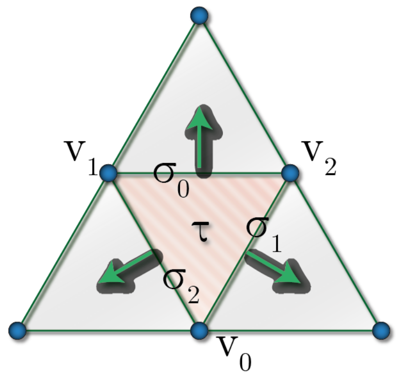



Figure 6.3: Notation used in Lemma 6.6: $\sigma_k = \tau \setminus v_k$.

*Proof.* Let's write the vertices of $\tau = (v_0, v_1, \ldots, v_k)$ in decreasing order: $v_k > v_{k-1} > \ldots > v_0$, and let $\sigma_k = \tau \setminus v_k$ be the facet of $\tau$ opposed to $v_k$ (see Figure 6.3). Since two adjacent unmatched nodes (critical simplices) is an unstable pair (Definition 2.2), none of the facets of $\tau$ is critical.

To prove the lemma, it remains to show that the facets are not matched below. Since $\tau = \sigma_0 \cup v_0$ is critical then $\sigma_0 \not\to \sigma_0 \cup v_0$. By Corollary 6.3, either $\mathrm{v_m}(\sigma_0) \not> \mathrm{v_{lk}}(\sigma_0)$ or $\mathrm{v_{lk}}(\sigma_0) \neq v_0$. Since $\mathrm{v_m}(\sigma_0) = v_1 > v_0$, it follows that $v_0 > \mathrm{v_{lk}}(\sigma_0)$, and thus $\sigma_0$ is matched above from the Steepest Descent theorem.

Now let $l \geq 1$. Observe that $\mathrm{v_{lk}}(\sigma_0) \in \mathrm{lk}_0(\sigma_0) = \mathrm{lk}_0(\tau \setminus v_0) \subseteq \mathrm{lk}_0(\sigma_l \setminus v_0)$ and $v_0 \in \mathrm{lk}_0(\sigma_l \setminus v_0)$. Therefore, $v_0 > \mathrm{v_{lk}}(\sigma_0)$ and thus $v_0 \neq \mathrm{v_{lk}}(\sigma_l \setminus v_0)$. It follows, from Corollary 6.3, that $\sigma_l \setminus v_0 \not\to \sigma_l$.

Now let $k \geq 1, k \neq l$. It follows from $v_k > v_{k-1} > \ldots > v_0$, that $\mathrm{v_m}(\sigma_l \setminus v_k) = v_0$. Since $v_k > v_0 = \mathrm{v_m}(\sigma_l \setminus v_k)$, then, by Corollary 6.3, $\sigma_l \setminus v_k \not\to \sigma_l$: $\sigma_l$ is not matched with any of its facets.

Therefore no facet $\sigma_l$ of $\tau$ is matched below: they are all matched above. It follows, from Corollary 6.3, that $\forall l, \mathrm{v_m}(\sigma_l) > \mathrm{v_{lk}}(\sigma_l)$. Thus $\mathrm{v_m}(\tau) > \mathrm{v_{lk}}(\sigma_l)$. ■

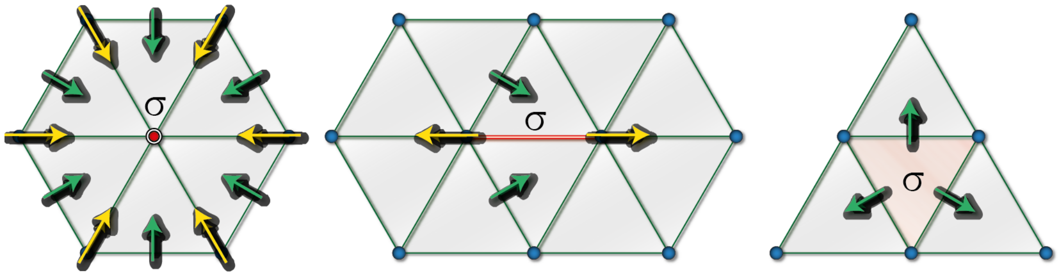



Figure 6.4: Results of Lemmata 6.6 and 6.7 on a critical vertex (left), a critical edge (center), and a critical triangle (right). In a smooth complex, they are similar to minima (sinks), saddles, and maxima (sources).

**Lemma 6.7** (Critical Below). *Let $\sigma$ be a critical simplex. If $\sigma$ is a facet of $\tau$, then $\tau \setminus \mathrm{v_m}(\sigma) \to \tau$.*

*Proof.* Let $\sigma$ be critical, so $\mathrm{v_{lk}}(\sigma) > \mathrm{v_m}(\sigma)$ by Corollary 6.3. Let $v \in \mathrm{lk}_0(\sigma)$ and $\tau = \sigma \cup v$. We have $v \geq \mathrm{v_{lk}}(\sigma) > \mathrm{v_m}(\sigma)$, so $\mathrm{v_m}(\tau) = \mathrm{v_m}(\sigma)$. The smoothness of $\Delta$ (Definition 6.1) implies that $\mathrm{v_{lk}}(\tau \setminus \mathrm{v_m}(\sigma)) = \mathrm{v_m}(\sigma)$. All together, $\mathrm{v_m}(\tau \setminus \mathrm{v_m}(\sigma)) > \mathrm{v_m}(\sigma) = \mathrm{v_{lk}}(\tau \setminus \mathrm{v_m}(\sigma))$. Then the Steepest Descent theorem applies: $\tau \setminus \mathrm{v_m}(\sigma) \to \tau \setminus \mathrm{v_m}(\sigma) \cup \mathrm{v_{lk}}(\tau \setminus \mathrm{v_m}(\sigma)) = \tau$. ■

Figure 6.4 shows the combined result of both previous lemmata on surfaces (2-manifolds). In particular, Lemma 6.7 states that all edges adjacent to a minimum points towards the minimum, and Lemma 6.6 states that all facets of a maximum points outwards that maximum.

## 6.3 Relation to Banchoff's critical points

In this entire section, assuming that $\Delta$ is a discrete smooth triangulated manifold, we relate Banchoff's critical points for embedded polyhedra theory and to critical simplices of our stable Morse matchings. More precisely, we show that Banchoff's piecewise linear critical vertices are faces of critical simplices of the stable Morse matchings. Banchoff's critical points can be defined using the notions of star of a vertex.

**Definition 6.8** (Stars). *The star of a vertex $v$ denoted $St(v)$ is the set of simplices in $\Delta$ that contain $v$: $St(v) = \{\sigma \in \Delta | v \in \sigma\}$.*
*The lower star of $v$ is the set of simplices of $St(v)$ whose maximal vertex is $v$: $St_-(v) = \{\sigma \in St(v) | v' \in \sigma \Rightarrow f(v) \geq f(v')\}$.*
*The upper star of $v$ is the complement: $St_+(v) = St(v) \setminus St_-(v)$.*

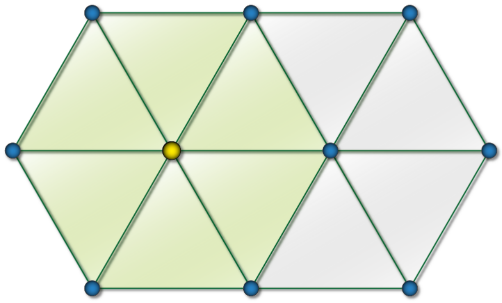


Figure 6.5: The green area is the star of yellow vertex

Now we can define Banchoff's critical points [5] for triangulated manifolds in any dimension (see Figure 6.6 for the two-dimensional case).

**Definition 6.9** (Banchoff's classification)**.** *Let $v$ be a vertex of $\Delta$.*

(i) *$v$ is a regular vertex if $St_{-}(v)$ is a disk.*

(ii) *$v$ is a Banchoff minimum if $St_{-}(v) = \emptyset$.*

(iii) *$v$ is a Banchoff maximum if $St_{-}(v) = St(v)$.*

(iv) *$v$ is a Banchoff 1-saddle of multiplicity $m$ ($m \geq 1$) if $St_{-}(v)$ has $m + 1$ connected components.*



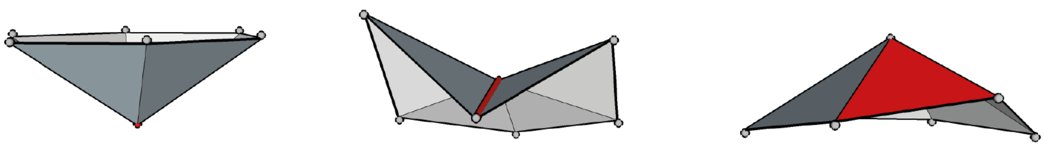


Figure 6.6: Definition of Banchoff's minimum (left), saddle (center), and maximum (right), with the critical simplex of stable Morse matching in red.

The first result in this section proves equivalence between Banchoff's minima and critical vertices in stable Morse matching on discrete smooth triangulated manifolds.

**Lemma 6.10** (Minimum)**.** *Vertex $v$ is a Banchoff minimum for $f \Leftrightarrow v$ is a critical vertex of the stable Morse matching.*

*Proof.* Vertex $v$ is a minimum $\Leftrightarrow St_-(v) = \emptyset \Leftrightarrow \mathrm{v_{lk}}(v) > v = \mathrm{v_m}(v) \Leftrightarrow$ vertex $v$ is critical, by Corollary 6.5. ■

The second result relating Banchoff's 1-saddles and critical edges is weaker, since it is not an equivalence. However, it says that not only there is always an adjacent critical edge to each Banchoff's 1-saddle but the number of adjacent critical edges is equal to the multiplicity of the saddle: Morse matchings split 1-saddles of multiplicity $m$ into $(m-1)$ simple saddles (see Figure 6.7).

**Lemma 6.11** (Saddle)**.** *If vertex $v$ is a Banchoff 1-saddle of multiplicity $m$, then $\exists\, \sigma_1, \sigma_2, \ldots, \sigma_m \in St(v)$ such that $\sigma_1, \sigma_2, \ldots, \sigma_m$ are critical edges.*

*Proof.* We know from the definition of a Banchoff's 1-saddle, that $St_-(v)$ has $m+1$ connected components $C_0, \ldots, C_m$: $St_-v = C_0 \sqcup C_1 \sqcup \ldots \sqcup C_m$. Let $v_i = \min C_i$ and $\sigma_i = (v, v_i)$ for $i = 1, \ldots, m$, and assume without loss of generality that $v_m > v_{m-1} > \ldots > v_0$. Since $v_i \in St_-(v)$, then $v > v_m > v_{m-1} > \ldots > v_0$, and $v_0 = \mathrm{v_{lk}}(v)$.

Since the components are disjoint, $\mathrm{lk}_0(\sigma_i) \subseteq (C_i \setminus v_i) \cup St_+(v)$. Now, $C_i \setminus v_i \subset St_-(v)$, and thus $\min(C_i \setminus v_i) \cup St_+(v) = \min(C_i \setminus v_i)$. It follows that $\mathrm{v_{lk}}(\sigma_i) \geq \min(C_i \setminus v_i) \cup St_+(v) = \min(C_i \setminus v_i) > v_i > v_0 = \mathrm{v_{lk}}(v) = \mathrm{v_{lk}}(\sigma_i \setminus v_i)$. Since $\mathrm{v_m}(\sigma_i) = v_i$, then $\mathrm{v_{lk}}(\sigma_i) > \mathrm{v_m}(\sigma_i) > \mathrm{v_{lk}}(\sigma_i \setminus \mathrm{v_m}(\sigma_i))$. It follows from Corollary 6.5, that edge $\sigma_i$ is critical. ■

We can also show that all Banchoff's maximum vertices are faces of a top dimensional critical simplex. This holds when the maximum occurs in the interior of the triangulated manifold $\Delta$, *i.e.* $\forall v \in \sigma, |\,\mathrm{lk}_0(\sigma \setminus v)| > 1$.

**Lemma 6.12** (Maximum)**.** *If vertex $v_0$ is a Banchoff maximum from the interior of $\Delta$, then $\exists\, \sigma \in St(v_0)$ such that $\sigma$ is a critical and* $\dim(\sigma) = d = \dim \Delta$.

*Proof.* Consider the following $d$ vertices in $\mathrm{lk}_0(v_0)$, all adjacent to each other: $v_k = \max \mathrm{lk}_0(\{v_0, v_1, \ldots, v_{k-1}\})$ for $k = 1, \ldots, d$, in particular $v_1 > \ldots > v_d$. Since $St_-(v) = St(v)$, then $v_0 > v_1 > \ldots > v_d$. Now let $\sigma = \{v_0, v_1, \ldots, v_d)\}$. Since $v_0$ is in the interior of $\Delta$, then $|\,\mathrm{lk}_0(\sigma \setminus v_d)| > 1$. It follows that $\mathrm{v_m}(\sigma) = v_d = \max \mathrm{lk}_0(\sigma \setminus v_d) > \min \mathrm{lk}_0(\sigma \setminus v_d) = \mathrm{v_{lk}}(\sigma \setminus v_d)$. Therefore $\sigma$ is critical by Corollary 6.5. ■

Observe that, in the last lemma, $v_0$ may be on the boundary if $|\,\mathrm{lk}_0(\sigma \setminus v_d)| > 1$.

Note that the definition of $k$-saddle, the generalization of 1-saddle, was not included in Definition 6.9 since we do not prove any result about them in relation to the stable Morse matching. However, we conjecture that there is a relationship between the critical simplices and the $k$-saddle in every dimension.

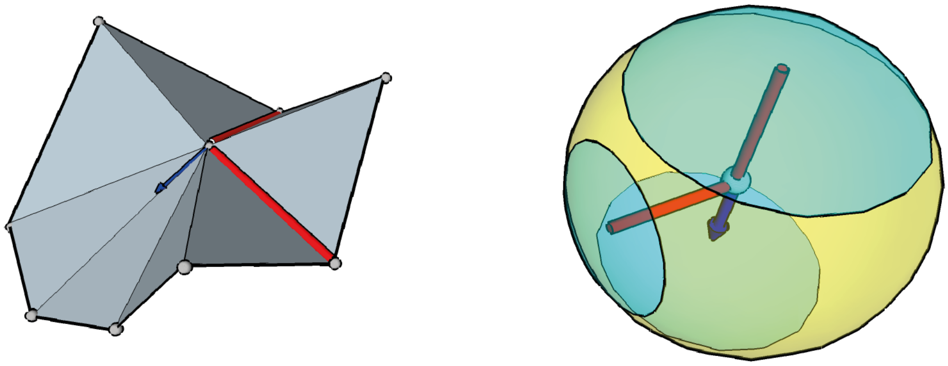


Figure 6.7: Money saddles (saddles of multiplicity 2) in dimensions 2 and 3: the saddle is split into two simple saddles.

## 6.4 Smoothness of the barycentric subdivision

The goal of this section is to prove that, for any function $f$ defined on a simplicial complex $\Delta$, the function induced by $f$ on the barycentric subdivision of $\Delta$ is discrete smooth. This proves that the results of the previous sections apply to barycentric subdivision, generalizing previous results on similar greedy constructions of Morse matchings [4,41,42].

Let $f$ be a function defined on the vertices of $\Delta$. This function induces an ordering on the simplices of $\Delta$ using the reverse lexicographic ordering of their vertices. More precisely, let $\sigma = \{v_0, \ldots, v_p\} \in \Delta$ and $\sigma' = \{v'_0, \ldots, v'_{p'}\} \in \Delta$, where the vertices are ordered by decreasing $f$-values: $f(v_0) > f(v_1) > \ldots f(v_p)$ and $f(v'_0) > f(v'_1) > \ldots > f(v'_{p'})$.

**Definition 6.13** (Reverse lexicographic ordering)**.** *With the notation above, $\sigma >_{revlex} \sigma'$ if one of the following statement is true:*

(i) $\exists\, i < \min(p, p'), \forall j < i, f(v_j) = f(v'_j)$ *and* $f(v_i) > f(v'_i)$*,*

(ii) $p < p'$ *and* $\forall j \leq p, f(v_j) = f(v'_j)$*.*

Observe that this definition is similar to Definition 5.3 except that $f$-values are taken in the decreasing order and there is a tie-breaking rule when comparing simplices of different dimensions (in case of tie, the lower dimension has a greater position in the ordering). We also have $\sigma =_{revlex} \sigma' \Leftrightarrow \sigma = \sigma'$.

To illustrate the ordering, consider $\Delta$ as a single triangle $abc$ and its faces, with $f(a) > f(b) > f(c)$. The reverse lexicographic ordering orders the faces of $\Delta$ as: $a >_{\mathbf{revlex}} ab >_{\mathbf{revlex}} abc >_{revlex} ac >_{revlex} b >_{\mathbf{revlex}} bc >_{revlex} c$. The comparisons in boldface ($>_{\mathbf{revlex}}$) are the one resulting from the tie-breaking rule. The notation $>_{revlex}$ is dropped for the remainder of the section.

**Lemma 6.14.** *The reverse-lexicographic ordering relates to the set operations on simplices as follows:*

(i) *If $\sigma \subset \tau$, then $\tau > \sigma \Leftrightarrow \max \tau > \max \sigma$.*

(ii) *If $\sigma \not\subset \tau$, then $\tau > \sigma \Leftrightarrow \max\{\tau \setminus (\sigma \cap \tau)\} > \max\{\sigma \setminus (\sigma \cap \tau)\}$.*

(iii) *$\sigma > \sigma \cup v \Leftrightarrow \min \sigma > v \Leftrightarrow v = \min(\sigma \cup v)$.*

The reverse-lexicographic ordering allows to extend the ordering given by $f$ on $\Delta$ to its barycentric subdivision $\Delta'$ (see Figure 6.8).

**Definition 6.15** (Barycentric subdivision)**.** *The barycentric subdivision $\Delta'$ of a simplicial complex $\Delta$ is a simplicial complex constructed as follows:*

– *for every simplex $\sigma \in \Delta$, there is a vertex $b(\sigma) \in \Delta'$,*

– *for every sequence $\{\sigma_0, \ldots, \sigma_p\} \subset \Delta$, such that $\sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_p$, there is a $p$-simplex $\Sigma = \{b(\sigma_0), \ldots, b(\sigma_p)\} \in \Delta'$.*

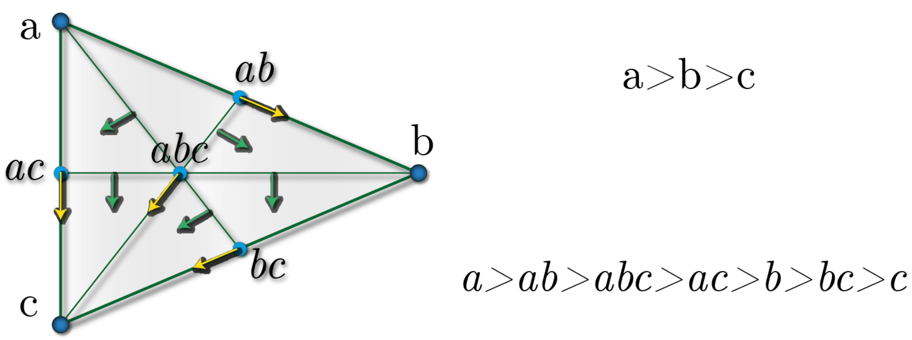



Figure 6.8: Barycentric subdivision of simplex *abc*, indicating the pairs that are always stable in the stable Morse matching.

Since each vertex $b(\sigma)$of the subdivision corresponds to an original simplex $\sigma$ of $\Delta$, we extend function $f$ to the vertices in $\Delta'$ ordering vertex $b(\sigma)$ according to the reverse-lexicographic ordering of the vertices of $\sigma$:

**Definition 6.16** (Extension of $f$)**.** *The extension $f'$ of function $f$ on the vertices of $\Delta'$ is defined by $f'(b(\sigma)) = \sigma$. The values of $f'$ are totally ordered using the reverse lexicographic ordering.*

In order to prove the smoothness of $f'$ on $\Delta'$, we need to write the vertex link $\mathrm{lk}_0(\Sigma)$ of a simplex $\Sigma$ in the barycentric subdivision explicitly. Looking at Figure 6.8, we observe that the vertex link of edge $\{a, abc\}$ are vertices $ab$ and $ac$, which are exactly the only two possibilities to insert between $a$ and $abc$. The vertex link of vertex $\{ab\}$ is composed of vertices $a$, $b$, and $abc$. The two first are the possible insertion before $ab$ that form a simplex of $\Delta'$, and the last one is a possible insertion after $ab$. This is the general situation, as stated below.

**Lemma 6.17** (lk$_0$ in the subdivision)**.** *Let* $\Sigma = \{b(\sigma_0), \ldots, b(\sigma_p)\} \in \Delta'$*. Then* $\mathrm{lk}_0(\Sigma) = \{b(\tau), \tau \in \Delta, \sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_i \subset \tau \subset \sigma_{i+1} \subset \ldots \subset \sigma_p$ *or* $\sigma_p \subset \tau\}$

*Proof.* This follows directly from Definitions 5.7 and 6.15: $\mathrm{lk}_0(\Sigma) = \{b(\tau), \Sigma \cup b(\tau) \in \Delta'\}$ and $\Sigma \cup b(\tau) \in \Delta'$ if and only if $\{\tau, \sigma_0, \ldots, \sigma_p\}$ can be totally arranged in a sequence of inclusions, that is $\sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_i \subset \tau \subset \sigma_{i+1} \subset \ldots \subset \sigma_p$ or $\sigma_p \subset \tau$. ∎

With an abuse of notation, we write $\Sigma = \{b(\sigma_0), \ldots, b(\sigma_p)\}$ as a sequence $\Sigma = (\sigma_0, \ldots, \sigma_p)$, omitting the $b()$.

The vertex link of any simplex can be defined from the (simpler) vertex link of some edges, drawing attention to the case where $\Sigma = (\sigma', \sigma)$ is an edge of the subdivision. We denote by $L(\sigma', \sigma)$ the set of simplices involved in $\mathrm{lk}_0((\sigma', \sigma))$ not contained in $\sigma'$. On the example of Figure 6.8, for edge $\{a, abc\}$, $L(a, abc)$ is exactly $\mathrm{lk}_0(\{a, abc\}) = \{ab, ac\}$, and its minimum is $l(a, abc) = ac$. In case the triangle of Figure 6.8 is the base of a tetrahedron $\{a, b, c, d\}$, the vertex link of $\{ab, abc\}$ is $\{a, b, abcd\}$, while $L(ab, abc) = \{abcd\}$ since it discards the faces of $ab$.

**Definition 6.18.** *Let* $L(\sigma', \sigma) = \{\tau | \sigma' \subset \tau \subseteq \sigma\} \cup \{\tau | \sigma \subset \tau\}$*. If* $\sigma' \subset \sigma$*, we denote by* $l(\sigma', \sigma)$ *its minimal simplex:* $l(\sigma', \sigma) = \min L(\sigma', \sigma)$*.*

Observe that, if $\sigma', \sigma \in \Sigma$, then $L(\sigma', \sigma) \subseteq \Sigma \cup \mathrm{lk}_0(\Sigma)$. If $\sigma' \subset \sigma$, $l(\sigma', \sigma)$ can be explicitly computed.

**Lemma 6.19.** *If* $\sigma' \subset \sigma$*, then* $l(\sigma', \sigma) = \sigma' \cup \min(\sigma \setminus \sigma')$*.*

*Proof.* Let $v = \min \sigma \setminus \sigma'$. First observe that $\sigma' \subset \sigma' \cup v \subseteq \sigma$, therefore $\sigma' \cup v \in \{\tau \mid \sigma' \subset \tau \subseteq \sigma\}$.

In order to check the minimality of $\sigma' \cup v$, we compare with the elements $\tau$ in each set defining $L(\sigma', \sigma)$:

$\sigma' \subset \tau \subseteq \sigma$. Then $\tau \setminus \sigma' \subseteq \sigma \setminus \sigma'$, and thus $\max \tau \setminus \sigma' \geq \min \tau \setminus \sigma' \geq \min \sigma \setminus \sigma' = v$. Item (i) of Lemma 6.14 then implies $\tau \setminus \sigma' \geq v$. It follows that $\tau = \sigma' \cup (\tau \setminus \sigma') \geq \sigma' \cup v$.

$\sigma \subseteq \tau$. Then $\sigma \setminus \sigma' \subseteq \tau \setminus \sigma'$, and thus $\max \tau \setminus \sigma' \geq \max \sigma \setminus \sigma' \geq v$. Using again Lemma 6.14, $\tau = \sigma' \cup (\tau \setminus \sigma') \geq \sigma' \cup v$.

It follows that $\sigma' \cup v = l(\sigma', \sigma)$. ∎

In the example of the tetrahedron, we had $L(ab, abc) = \{abcd\}$, so $l(ab, abc) = abcd > ab$. The definition of smoothness involves the vertex link of $ab \setminus \min ab = ab \setminus b = a$. In that case, $l(ab \setminus b, abc) = l(a, abc) = ab$ as exemplified above. This is indeed the general case, as stated below.

**Lemma 6.20.** *If $\sigma' \subset \sigma$ and $l(\sigma', \sigma) > \sigma'$, then $l(\sigma' \setminus \min(\sigma'), \sigma) = \sigma'$.*

*Proof.* Let $v = \min \sigma \setminus \sigma'$. From Lemma 6.19, $l(\sigma', \sigma) = \sigma' \cup v$. The hypothesis then writes $\sigma' \cup v > \sigma'$, and Item (iii) of Lemma 6.14 guarantees that $v > \min(\sigma')$. Then $\min(\sigma \setminus \sigma' \cup \min(\sigma')) = \min(\min(\sigma \setminus \sigma'), \min(\sigma')) = \min(v, \min(\sigma')) = \min(\sigma')$.

Using Lemma 6.19 again, $l(\sigma' \setminus v, \sigma) = \sigma' \setminus v \cup \min(\sigma \setminus (\sigma' \setminus v)) = \sigma' \setminus v \cup \min(\sigma \setminus \sigma' \cup v) = \sigma' \setminus v \cup v = \sigma'$. ■

In the next lemma, we identify some subdivided simplices where the Steepest Descent theorem apply directly (so that they also respect Definition 6.1). The idea of the statement relies on the position of the minimum $\mathrm{v_m}(\Sigma)$ of $\Sigma$ and the position where vertices of the link can be inserted. If the minimum appears after a "hole in the dimension sequence of $\Sigma$" where a vertex of the link can be inserted, then Lemma 6.19 ensures there will be a vertex of the link $\mathrm{v_{lk}}(\Sigma)$ smaller than $\mathrm{v_m}(\Sigma)$, and the Steepest Descent theorem applies.

**Lemma 6.21.** *If $\exists\, \tau \in \mathrm{lk}_0(\Sigma)$ such that $\tau \subset \mathrm{v_m}(\Sigma)$, then $\mathrm{v_m}(\Sigma) > \mathrm{v_{lk}}(\Sigma)$.*

*Proof.* Let $\sigma_m = \mathrm{v_m}(\Sigma)$ and write $\Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$. If $\exists\, \tau \in \mathrm{lk}_0(\Sigma)$ such that $\tau \subset \sigma_m$, then, by Lemma 6.17, $\exists\, \sigma_{j-1}, \sigma_j \in \Sigma$ such that $\sigma_{j-1} \subset \tau \subset \sigma_j$.

Let $\rho = \sigma_{j-1} \cup \min \sigma_j \setminus \sigma_{j-1} = l(\sigma_{j-1}, \sigma_j)$ by Lemma 6.19. Since $\sigma_{j-1} \subset \tau \subset \sigma_j \subseteq \sigma_m$, we have $\sigma_m \in L(\sigma_{j-1}, \sigma_j)$, and thus $\sigma_m \geq \rho = l(\sigma_{j-1}, \sigma_j)$.

Since $\sigma_{j-1} \subset \rho \subset \sigma_j$, then $\rho \in \mathrm{lk}_0(\Sigma)$ by Lemma 6.17. Therefore $\rho \geq \mathrm{v_{lk}}(\Sigma)$.

All together, $\mathrm{v_m}(\Sigma) = \sigma_m \geq \rho \geq \mathrm{v_{lk}}(\Sigma)$. ■

Now we can prove that the extension of $f$ on the barycentric subdivision is discrete smooth, the main result of the section. To do so, check that every simplex in the subdivision respects the smoothness definition (Definition 6.1). The previous lemma handles the cases where the minimum $\mathrm{v_m}(\Sigma)$ appears after a "hole in the dimension sequence of $\Sigma$", so we are left with the case where $\mathrm{v_m}(\Sigma)$ appears in the first vertices of $\Sigma$, where the dimensions of the sequence are exactly $0, 1, \ldots$. In that case, the "hole" left by $\mathrm{v_m}(\Sigma)$ in $\Sigma \setminus \mathrm{v_m}(\Sigma) \cup v$ is exactly where $\mathrm{v_{lk}}\left(\Sigma \setminus \mathrm{v_m}(\Sigma) \cup v\right)$ must be inserted, and we state below that $\mathrm{v_{lk}}\left(\Sigma \setminus \mathrm{v_m}(\Sigma) \cup v\right) = \mathrm{v_m}(\Sigma)$, which is the smoothness condition.

We can check on the example of Figure 6.8 with $\Sigma = \{a, abc\}$: we have $\mathrm{v_m}(\Sigma) = abc$ and $\mathrm{lk}_0(\Sigma) = \{ab, ac\}$. For all $v \in \mathrm{lk}_0(\Sigma)$, $\Sigma \setminus \mathrm{v_m}(\Sigma) \cup v$ is either $\{a, ab\}$ or $\{a, ac\}$, and $\mathrm{v_{lk}}(\{a, ab\} = \mathrm{v_{lk}}(\{a, ac\} = abc$.

**Theorem 6.22.** *The extension of $f$ is discrete smooth on the subdivided complex $\Delta'$.*

*Proof.* The smoothness definition states that if $\mathrm{v}_{\mathrm{lk}}(\Sigma) > \mathrm{v}_{\mathrm{m}}(\Sigma)$, then we must have $\forall \sigma \in \mathrm{lk}_0(\Sigma)$, $\mathrm{v}_{\mathrm{lk}}(\Sigma \setminus \mathrm{v}_{\mathrm{m}}(\Sigma) \cup \sigma) = \mathrm{v}_{\mathrm{m}}(\Sigma)$.

Let $\sigma_m = \mathrm{v}_{\mathrm{m}}(\Sigma)$. By Lemma 6.21, since $\mathrm{v}_{\mathrm{lk}}(\Sigma) > \mathrm{v}_{\mathrm{m}}(\Sigma)$, $\nexists \tau \in \mathrm{lk}_0(\Sigma)$ such that $\tau \subset \sigma_m$. In particular, $\exists\, v_m \in \sigma_m, \sigma_m = \sigma_{m-1} \cup v_m$ (with the convention that $\sigma_{-1} = \varnothing$). Since $\sigma_{m-1} > \mathrm{v}_{\mathrm{m}}(\Sigma) = \sigma_m$, then Item (iii) of Lemma 6.14 implies that $v_m = \min \sigma_m$.

Moreover, $\forall \tau \in \mathrm{lk}_0(\Sigma \setminus \sigma_m)$, $\tau \not\subset \sigma_{m-1}$. Otherwise $\exists\, k \leq m$, such that $\sigma_k \subset \tau \subset \sigma_{k+1} \subseteq \sigma_{m-1}$, so $\tau \in \mathrm{lk}_0(\Sigma)$ which contradicts the fact that $\nexists \tau \in \mathrm{lk}_0(\Sigma), \tau \subset \sigma_m$.

We first separate the case where $\sigma_m \neq \sigma_p$.

$\sigma_m \subset \sigma_{m+1}$. Since $L(\sigma_m, \sigma_{m+1}) \subseteq \Sigma \cup \mathrm{lk}_0(\Sigma)$, as previously observed, and $\mathrm{v}_{\mathrm{lk}}(\Sigma) > \sigma_m$ by hypothesis, we have $l(\sigma_m, \sigma_{m+1}) \geq \min \Sigma \cup \mathrm{lk}_0(\Sigma) = \sigma_m$. Since $\sigma_m \notin L(\sigma_m, \sigma_{m+1})$, then the inequality is strict: $l(\sigma_m, \sigma_{m+1}) > \sigma_m$. We can then use Lemma 6.20: $l(\sigma_{m-1}, \sigma_{m+1}) = l(\sigma_m \setminus \min \sigma_m, \sigma_{m+1}) = \sigma_m$.

As stated above, $\mathrm{lk}_0(\Sigma \setminus \sigma_m)$ does not contain any face of $\sigma_{m-1}$. Therefore $\mathrm{lk}_0(\Sigma \setminus \sigma_m) \subseteq L(\sigma_{m-1}, \sigma_{m+1})$, and $\mathrm{v}_{\mathrm{lk}}(\Sigma \setminus \sigma_m) \geq l(\sigma_{m-1}, \sigma_{m+1}) = \sigma_m$.

Now, for any $\sigma \in \mathrm{lk}_0(\Sigma)$, $\mathrm{v}_{\mathrm{lk}}(\Sigma \setminus \sigma_m \cup \sigma) \geq \mathrm{v}_{\mathrm{lk}}(\Sigma \setminus \sigma_m)$ (Item (ii) of Lemma 5.8). Therefore $\mathrm{v}_{\mathrm{lk}}(\Sigma \setminus \sigma_m \cup \sigma) \geq \sigma_m$, but $\sigma_m \in \mathrm{lk}_0(\Sigma \setminus \sigma_m \cup \sigma)$. We conclude that $\mathrm{v}_{\mathrm{lk}}(\Sigma \setminus \sigma_m \cup \sigma) = \sigma_m$.

$\sigma_m = \sigma_p$. Since $\mathrm{lk}_0(\Sigma)$ does not contain any face of $\sigma_m$, $\forall \sigma \in \mathrm{lk}_0(\Sigma), \sigma_m = \sigma_p \subset \sigma$. Then $\mathrm{lk}_0(\Sigma) = \{\tau, \sigma_m \subset \tau\} = L(\sigma_m, \sigma)$. Therefore $l(\sigma_m, \sigma) = \mathrm{v}_{\mathrm{lk}}(\Sigma) > \sigma_m$.

We can now apply Lemma 6.20, since $\min(\sigma_m) = v_m$. Therefore $l(\sigma_{m-1}, \sigma) = l(\sigma_m \setminus v_m, \sigma) = l(\sigma_m \setminus \min(\sigma_m), \sigma) = \sigma_m$.

Also we know that $\mathrm{lk}_0(\Sigma \setminus \sigma_m \cup \sigma) = \{\tau, \sigma_{m-1} \subset \tau \subset \sigma\} \cup \{\tau, \sigma \subset \tau\} \subset \{\tau, \sigma_{m-1} \subset \tau \subseteq \sigma\} \cup \{\tau, \sigma \subseteq \tau\} = l(\sigma_{m-1}, \sigma)$.

Therefore $\mathrm{v}_{\mathrm{lk}}\,(\Sigma \setminus \sigma_m \cup \sigma) \geq l(\sigma_{m-1}, \sigma) = \sigma_m$. Since $\sigma_m \in \mathrm{lk}_0(\Sigma \setminus \sigma_m \cup \sigma)$, then $\sigma_m = \mathrm{v}_{\mathrm{lk}}(\Sigma \setminus \sigma_m \cup \sigma)$. ∎

# 7
# Conclusions and future work

This thesis introduced new results in the construction and analysis of algorithms in discrete Morse theory. First we analyzed how hard it is to find an optimal Morse matching, which gives a succinct topological representation of a discrete object. We studied the problem through the lenses of parameterized complexity, a refined complexity analysis categorization. The result is very bleak, ERASABILITY, a problem closely related to finding optimal Morse matching turns out to be in the worst possible complexity class: $W[P]$. The result is interesting because there are not many problems in this class and MORSE MATCHING is, to our knowledge, the first geometrical / topological problem in it.

Since the problem is extremely difficult, we turned our attention to finding a category of simplicial complexes where the problem becomes easy. In Chapter 4, we designed a polynomial time algorithm to find optimal Morse matching in 2-dmensional simplicial complexes and 3-manifolds whose adjacent graph has bounded tree-width. Very recently Burton and Downey [13] were able to generalize our result for 3-manifolds from Chapter 4 in very encompassing way.

Both of these results start to pinpoint where the topological difficulty of the problem lies. However, we are left with the question if there could be a larger category, where the problem is still easy. Computational topology problems tend to shift rapidly from easy to very hard (even sometimes not computable). This opens fundamental questions like what topological property on what class of objects lies at the limit between easy and hard problems?

To further analyze this question, instead of looking for specific categories where MORSE MATCHING is easy, we can ask a related but simpler question: Is the simplicial complex $\Delta$ collapsible? The problem can be written as:

**Problem 7.1** (COLLAPSIBILITY).
INSTANCE: *A simplicial complex $\Delta$.*
QUESTION: *Is there a Morse matching $M$ with only one critical vertex and no other critical simplices?*

This problem turns out to be $NP$-complete if $\dim(\Delta) = 3$ [67] and can be solved in polynomial time if $\dim(\Delta) = 2$ [43]. Therefore is natural to ask: "on which side of the fence does the problem sit on" if $\Delta$ is a 3-manifold or even if $\Delta$ is a 3-ball.

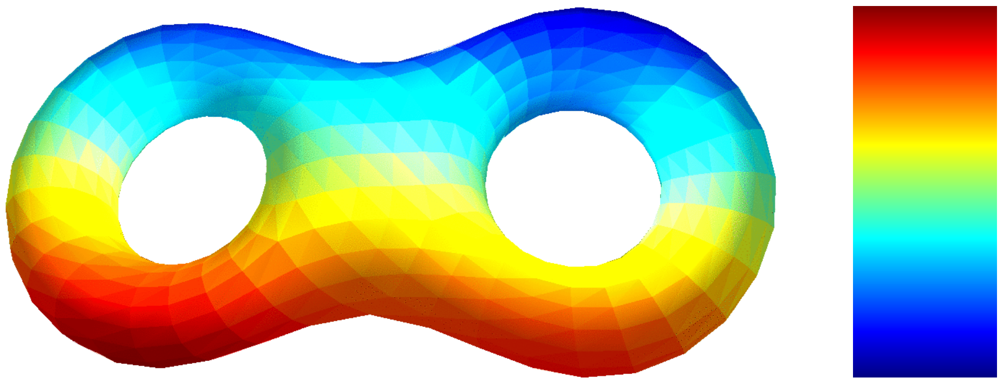


Figure 7.1: A vase model with a random height function defined on its vertices (function values increase from blue to red color).



In the second part of thesis, we focused to a more geometric side of discrete Morse theory. Instead of finding optimal Morse matchings, we constructed Morse matchings from stable matchings, which have to be related to some given geometric information. With the simple and local definition of a stable matching, we were able to prove various relationships in a concise and simple manner and more surprisingly we were able to do it in any dimension.

With these simple proofs, the behavior of the stable Morse matching was succinctly understood and we were motivated to ask if there were types of functions where deeper connections to the given geometric information could be established. A natural definition of a smooth discrete function emerged and provided a full characterization of the stable Morse matching. Then we were able to characterize the critical simplices, connect them to Banchoff's piecewise-linear discretization, and show that their neighborhood looks similar to their continuous counterparts. At the end, we showed that this discrete smooth definition complies with our intuition when we showed that a function can be "smoothed" on a simplicial complex through a barycentric subdivision of the complex.

This definition of smoothness for simplicial complexes raises many interesting questions, to be explore in the near future:

- Does every simplicial complex admit a discrete smooth function?
- Are the actual models (2-manifolds) used in practice, such as the Vase model with a height function (see Figure 7.1), discrete smooth?

- Can we make a non-smooth function smooth if we apply a standard Gaussian filter on it? Or another specific filter?
- Given a planar point set with a function defined on vertices, is there a discrete smooth planar triangulation of these vertices?
- Is the Morse Smale complex simpler to construct for smooth functions?
- There are many results in topology and geometry which are true in the barycentric subdivision of a simplicial complex. Which of these results also hold if the complex is smooth?

We hope our work has shed new light in discrete Morse theory, a theory that we believed is not even close to reaching its full potential and hope that our techniques can be used in other fields such as complexity theory, combinatorics, and graph theory.

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# A
# Proof of Joswig and Pfetsch's lemma

The proof of Lemma 3.8 actually follows directly from Joswig and Pfetsch's proof of the following lemma.

Their proof builds a Morse matching from a spanning tree of the *primal graph*, *i.e.* the graph obtained considering only the vertices and edges of $\Delta$. For a 3-manifold $\Delta$, the proof of the previous lemma can be applied exactly the same way on the *dual graph*, *i.e.* the graph whose nodes represent tetrahedra of $\Delta$, and whose arcs represent common triangles of $\Delta$ joined together (see Definition 4.7), to obtain the following result.

Here, we simply reproduce the proof of Joswig and Pfetsch [38] verbatim applying it to 3-manifold complexes, using Poincaré's duality.

First consider a Morse matching $M$ for a connected 3-manifold $\Delta$. Let $\gamma(M)$ be obtained from the primal graph of $\Delta$ by removing all arcs (edges of $\Delta$) matched with triangles and let $\gamma^*(M)$ be derived from the dual graph of $\Delta$ by removing all arcs corresponding to triangles matched with tetrahedra of $\Delta$. Note that $\gamma(M)$ contains all vertices (0-simplices) and $\gamma^*(M)$ contains all tetrahedra of $\Delta$ as nodes.

**Lemma A.1.** *The graphs $\gamma(M)$ and $\gamma^*(M)$ are connected.*

*Proof.* Suppose that $\gamma(M)$ is disconnected. Let $N$ be the set of nodes in a connected component of $\gamma(M)$, and let $C$ be the set of cut edges, that is, edges of $\Delta$ with one vertex in $N$ and one vertex in its complement. Since $\Delta$ is connected, $C$ is not empty. By definition of $\gamma(M)$, each edge in $C$ is matched to a unique 2-simplex.

Consider the directed subgraph $D$ of the Hasse diagram consisting of the edges in $C$ and their matching 2-simplices. The standard direction of arcs in the Hasse diagram (from the higher to the lower dimensional simplices) is reversed for each matching pair of $M$, *i.e.* $D$ is a subgraph of $H(M)$. We construct a directed path in $D$ as follows. Start with any node of $D$ corresponding to a cut edge $e_1$ and consider the node of $D$ determined by the unique 2-simplex $\tau_1$ matched with $e_1$. Then $\tau_1$ contains at least one other cut edge $e_2$, otherwise $e_1$ cannot be a cut edge. Now iteratively go to $e_2$, then to its unique matching 2-simplex $\tau_2$, choose another cut edge $e_3$, and so on. We observe that we obtain a directed path $e_1, \tau_1, e_2, \tau_2, \dots$ in $D$, *i.e.* the arcs are directed in the correct direction. Since we have a finite graph at some point the path must arrive at a node of $D$ which we have visited already. Hence, $D$ (and therefore also $H$) contains a directed cycle, which is a contradiction since $M$ is a Morse matching.

To prove that $\gamma^*(M)$ is connected, we repeat the proof above on the dual graph. ■

*Proof of Lemma 3.8.* Since $\gamma(M)$ and $\gamma^*(M)$ are connected, they both admit spanning trees, and we will use them to build the Morse matching. First pick an arbitrary node $r_1$ and any spanning tree of $\gamma(M)$ and direct all arcs away from $r_1$. Then pick an arbitrary tetrahedron (node in the dual graph) $r_2$ and any spanning tree of $\gamma^*(M)$ and direct all triangles (arcs in dual graph) away from $r_2$. This yields a maximum Morse matching on $\gamma(M)$ and $\gamma^*(M)$. Now, replacing the part of $M$ on $\gamma(M)$ and $\gamma^*(M)$ with this matching yields a Morse matching. This Morse matching has only one critical vertex (the root $r_1$) and one critical tetrahedron (the root $r_2$). Note that Morse inequalities imply that every Morse matching in a triangulated 3-manifold contains at least one critical vertex and at least one critical tetrahedron. Furthermore, the total number of critical simplices can only decrease, since we computed an optimal Morse matching on $\gamma(M)$ and $\gamma^*(M)$. ■