4 Appendix

A) Proofs

Lemma 1: $\boldsymbol{\beta_i(c_i, q_i, \tilde{q}) = \beta_i(c_i, \tilde{q}), \forall\, i \neq 1}$

Consider equilibrium strategy $\beta_i(c_i, q_i, \tilde{q})$, as well as $\beta_i'(c_i, q_i', \tilde{q})$, where $q_i' \neq q_i$, and define β_i^* and $\beta_{i_1}^*$ as the competing firms' equilibrium bid functions. If β_i is not optimal when firm i observes q_i', then $\Pi_i^e(\beta_i', q_i', c_i, \tilde{q}) > \Pi_i^e(\beta_i, q_i', c_i, \tilde{q})$.

Writing out ex-ante expected profits,

$$\Pi_i^e = \Pr(q_i' \geq q_1) \sum_{k=1}^{N-1} B(k) * \Pr\left(\beta_i' \leq \beta_i^*, \beta_i' \leq \beta_{i_1}^*\right) [\beta_i' - c_i]$$

$$> \Pi_i^e = \Pr(q_i' \geq q_1) \sum_{k=1}^{N-1} B(k) * \Pr\left(\beta_i \leq \beta_i^*, \beta_i \leq \beta_{i_1}^*\right) [\beta_i - c_i]$$

Multiplying both sides by $\frac{\Pr(q_i \geq q_1)}{\Pr(q_i' \geq q_1)}$, we have

$$\Pi_i^e = \Pr(q_i \geq q_1) \sum_{k=1}^{N-1} B(k) * \Pr\left(\beta_i' \leq \beta_i^*, \beta_i' \leq \beta_{i_1}^*\right) [\beta_i' - c_i]$$

$$> \Pi_i^e = \Pr(q_i \geq q_1) \sum_{k=1}^{N-1} B(k) * \Pr\left(\beta_i \leq \beta_i^*, \beta_i \leq \beta_{i_1}^*\right) [\beta_i - c_i]$$

Since $\Pi_i^e(\beta_i', q_i, q_1, c_i) > \Pi_i^e(\beta_i, q_i, q_1, c_i)$, we have a contradiction, and any optimal strategy will not depend on the firm's own quality level. ■

Lemma 2: $\boldsymbol{\lambda_1, \ldots, \lambda_N}$ ***are strictly increasing functions***

Since there are only two sets of firms, we need only to check if λ_1 and λ_i, for $i \neq 1$, are strictly increasing.

From the first order conditions, $\lambda_i{}'$ can be easily verified to be strictly positive:

$$\lambda_i' = \frac{(\sum_{k=0}^{N-1} A(k)[1-F(\lambda_i)]^k)}{(\sum_{k=0}^{N-1} A(k)k[1-F(\lambda_i)]^{k-1} f(\lambda_i)[p_1-\lambda_1])} \quad (1)$$

$$\lambda_1' = \frac{\left(\sum_{k}^{N-1} B(k)[1-F(\lambda_1)][1-F(\lambda_i)]^{k-1} - \sum_{k=0}^{N-1} B(k)(k-1)\lambda_i'[1-F(\lambda_1)][1-F(\lambda_i)]^{k-2}[p-\lambda_i]\right)}{\left(\sum_{k=0}^{N-1} B(k)f(\lambda_1)\left[1-F(\lambda_i(p))\right]^{k-1}[p_i-\lambda_i]\right)} \quad (2)$$

We need to check that λ_1' is positive. Substituting (1) into (2),

$$\lambda_1' = \frac{(\sum_{k}^{N-1} B(k)[1-F(\lambda_1)][1-F(\lambda_i)]^{k-1}) * \sum A(k) * k * [1-F(\lambda_i)]^{k-1} * [p_1-\lambda_1]}{\left(\sum_{k=0}^{N-1} B(k)f(\lambda_1)\left[1-F(\lambda_i(p))\right]^{k-1}[p_i-\lambda_i]\right)\sum A(k) * k * [1-F(\lambda_i)]^{k-1} * [p_1-\lambda_1]}$$

$$\frac{(-\sum_{k=0}^{N-1} B(k)(k-1)[1-F(\lambda_1)][1-F(\lambda_i)]^{k-2}\sum A(k)[1-F(\lambda_i)]^k)[p_i-\lambda_i]}{\left(\sum_{k=0}^{N-1} B(k)f(\lambda_1)\left[1-F(\lambda_i(p))\right]^{k-1}[p_i-\lambda_i]\right)\sum A(k) * k * [1-F(\lambda_i)]^{k-1} * [p_1-\lambda_1]}$$

We may ignore the common, positive denominator of both terms for our purposes. From Proposition 2, we know that $\beta_1(c) \geq \beta_i(c), \forall\, c$.Since $p_1 = \beta_1(c)$ and $\lambda_1(\beta_1) = c$, it follows that $[p_1-\lambda_1] \geq [p_i-\lambda_i], \forall\, c$. In words, firm 1's mark-up is always at least as high as every other firms'.

Substituting this condition into λ_1' and using the fact that $(k) = B(k) * \frac{N-1}{k} * [1-G(\tilde{q})]$, we have

$$\sum B(k)[1-F(\lambda_1)][1-F(\lambda_i)]^{k-1} * \sum B(k)[1-F(\lambda_i)^{k-1} * (N-1)(1-G(\tilde{q}))$$

$$-\sum B(k) * (k-1) * [1-F(\lambda_1)][1-F(\lambda_i)]^{k-2} * \sum B(k) * \frac{1}{k} * [1-F(\lambda_i)]^k * (N-1)\left(1-G(\tilde{q})\right)$$

Dividing both terms by $(N-1) * [1-F(\lambda_1)]\left(1-G(\tilde{q})\right)$:

$$\sum B(k)[1-F(\lambda_i)]^{k-1} * \sum B(k)[1-F(\lambda_i)^{k-1}$$

$$-\sum B(k) * (k-1) * [1-F(\lambda_i)]^{k-2} * \sum B(k) * \frac{1}{k} * [1-F(\lambda_i)]^k$$

Finally, we have

$$\sum B(k)[1-F(\lambda_i)]^{2k-2} - \sum B(k)\left(\frac{k-1}{k}\right)[1-F(\lambda_i)]^{2k-2}$$

Since $\frac{k-1}{k} \leq 1\ \forall\, k \in N$, the term is positive, concluding the proof. ∎

Proposition 1: Under the assumptions of the model, an n-tuple of strategies $(\boldsymbol{\beta_1}, \dots, \boldsymbol{\beta_N})$ is a pure strategy Bayesian equilibrium if $\boldsymbol{\beta_1}, \dots, \boldsymbol{\beta_N}$ are equal to pure strategies over $[\underline{\boldsymbol{p}}, \bar{\boldsymbol{c}})$, and there exists $\underline{\boldsymbol{c}} < \bar{\boldsymbol{p}} \leq \bar{\boldsymbol{c}}$, such that the inverses $\boldsymbol{\lambda_1} = \boldsymbol{\beta_1^{-1}}, \dots, \boldsymbol{\lambda_N} = \boldsymbol{\beta_N^{-1}}$ exist,, and form a solution over $[\underline{\boldsymbol{p}}, \bar{\boldsymbol{c}})$ of the system of differential equations, satisfying the boundary conditions $\boldsymbol{\lambda_1}(\underline{\boldsymbol{c}}) = \dots = \boldsymbol{\lambda_N}(\underline{\boldsymbol{c}}) = \underline{\boldsymbol{p}}$ and $\boldsymbol{\lambda_1}(\bar{\boldsymbol{c}}) = \dots = \boldsymbol{\lambda_N}(\bar{\boldsymbol{c}}) = \bar{\boldsymbol{c}}$.

Let each firm's ex-ante profits, conditional on qualification, be written as

$$\Pi_i(\beta_i(c),.) = (p - c_i) \prod_{\substack{j=1 \\ j \neq i}}^{N} \left[1 - F_j\left(\lambda_j(p)\right)\right]$$

Where F_j is defined as the distribution of inverse bids *conditional* on qualification. That is, instead of writing every firm's profit functions as a weighted sum of the cases where it faces each number of participants, we allow ex-ante for different distributions of inverse bid functions, taking into account that the probability of losing to an unqualified firm Is zero, and the probability of losing to a qualified firm is given only by the probability of being underpriced by said firm.

The rest of the proof follows as the procurement equivalent of Lebrun (1999)'s sufficiency conditions from Theorem 1. The assumption that the maximum price $\bar{p}$ is binding guarantees that expected profits at the top of the distribution $\bar{c}$ is equal to zero, so $\beta_i(\bar{c},.) = \bar{c}$ is an equilibrium for any strategy of $j \neq i$.

The necessary conditions for the theorem are that the support of the probability measures are the same as the interval $[\underline{c}, \bar{c}]$, where $\underline{c} < \bar{c}$, and that the bid functions are differentiable in the interval $[\underline{c}, \bar{c})$. Since λ_i is increasing for $i = 1, \dots, N$ and, by assumption, the distribution F has no atoms, the final necessary condition for the existence of equilibrium is that $\lambda_i(\bar{c}) = \bar{c}$, for $i = 1, \dots, N$. As shown above, this is immediately true due to the binding maximum price rule.

As in Lebrun (1999), we use the generic system of differential equations λ_i' and the conditions $\lambda_i(\bar{c}) = \bar{c}$, $\lambda_i(\bar{c}) = \underline{p}$, for $i = 1, \dots, N$ to prove that the solution to the system is a Bayesian equilibrium, since it is a global maximum of each firms' profit functions, given the others' strategies.

$$\frac{d}{db} \sum_{j=1; j \neq i}^{n} \ln\left[1 - F_j\left(\lambda_j(p)\right)\right] = \frac{1}{p - \lambda_i(p)} \qquad (A.1)$$

for all p in $[\underline{p}, \bar{c})$ and all $1 \leq i \leq N$.

We use condition (A.1) to prove the result.

Consider $c < \bar{c}$. Since $p = \frac{c+\bar{c}}{2}$ results in strictly positive profits, bidding $p < c$ can never be a best response, and bidding $p = \bar{c}$ is not a best response when $i \neq j$, where j is the case where $\lambda_j = \bar{p}$.

Bidder i's expected payoff if he bids $p \in [\underline{p}, \bar{c})$ is is equal to $(p - c_i) \prod_{j=1, j \neq i}^{N} \left[1 - F_j\left(\lambda_j(p)\right)\right]$ and is strictly positive.

When $i = j$ as in the binding condition $\lambda_i(\bar{c}) = \bar{c}, \forall\, i$, this product is strictly positive (recall that we are considering the case where $c < \bar{c}$), and so it gives out a positive expected payoff if this firm should decide to bid $p = \bar{p}$. On the case of the benefitted firm, profits are positive if no other firm qualifies, which happens with positive probability. On the case of all others, profits are positive if no other firm qualifies other than itself, and firm 1 chooses p as its strategy[5].

Since the equation is strictly positive, we consider its logarithm and use $(A.1)$. . The derivative of this logarithm is equal to

$$\frac{1}{p-c} - \frac{1}{p-\lambda_i(p)}$$

For $c < \bar{c}$ and $p > c$.

λ_i is strictly increasing in the interval $\left[\underline{p}, \bar{c}\right]$ and we have, by the definition of inverse function, that $\lambda_i\left(\beta_i(c)\right) = c < \beta_i(c)$ when $\beta_i(p) < \bar{p}$. As such, this derivative is strictly positive if we consider an alternative bid p such that $\beta_i(c) < p < \bar{p}$. On the other hand, if we consider a deviation p such that $\beta_i(c) > p > c$, where the firm decides to deviate to a lower price, the derivative is strictly negative, since λ_j is strictly increasing for $i \neq j$, and $\lambda_i\left(\beta_i(c)\right) \leq c$.

The shifting signal for any possible deviation to lower or higher prices guarantees that $p = \beta_i(c)$ is maximizes globally the expected profit function $\Pi_i\left(\beta_i(c), .\right)$, , given β_j, for all $j \neq i$.

Note that the proof is based on two central assumptions: First, it is necessary that the expected profit for firms that observe $\bar{c}$ be equal to zero. The assumption that the maximum price $\bar{p}$ is lower than the maximum quality value $\bar{c}$ guarantees this. Second, we also require that the bid support is equal for all firms. This is guaranteed by assuming that there exists a common lower bid for all participating firms, so that $\beta_i(\underline{c}) = \underline{p}$. ■

Proposition 2: For any distributions of cost and quality $F(.)$ and $G(.)$, the two-stage procurement auction with favoritism is such that $\beta_1(c) \geq \beta_i(c), \forall c$. The inequality is holds strictly, for all $c > \underline{c}$, if $\tilde{q} > 0$.

Defining $\lambda(p)$ as the inverse bid function β^{-1}, recall that by assumption $\lambda_i\left(\underline{p}\right) = \lambda_1\left(\underline{p}\right)$, where $\underline{p}$ is the bid submitted for the lowest cost $\underline{c}$, and λ is continuous. Since β_i and β_1 share a common support, so does λ_i and λ_1. We prove that, at any point where the inverse bid functions may have the exact same value, the derivative of firm i's inverse bid function must be greater than or equal to firm 1's. In short, the result is proven if we show that for any λ_1 such that $\lambda_i(\tilde{p}) = \lambda_1(\tilde{p})$, $\lambda_i'(\tilde{p}) \geq \lambda_1'(\tilde{p})$.

Let $\lambda_i(\tilde{p}) = \lambda_1(\tilde{p})$, for some $\tilde{p} \in$ inverse image of λ_i and λ_1. Defining $\lambda \equiv \lambda_i(\tilde{p}) = \lambda_1(\tilde{p})$,

[5] Note that there needs to be some tie-breaking rule that attributes positive probability for the non-benefitted firm to win the contract. If the tie-breaking rule always awards the contract to the benefitted firm, this condition would not hold.

$$\lambda_i^{'}(\tilde{p}) = \frac{\sum A(k)[1-F(\lambda)]^k}{\sum A(k)*k*[1-F(\lambda)]^{k-1}f(\lambda)[\tilde{p}-\tilde{c}]} \quad (1)$$

$$\lambda_1^{'}(\tilde{p}) = \frac{\sum B(k)[1-F(\lambda)]^k}{\sum B(k)[1-F(\lambda)]^{k-1}f(\lambda)[\tilde{p}-\tilde{c}]} - \frac{\sum B(k)*(k-1)*[1-F(\lambda)]^{k-1}\lambda_i^{'}(\tilde{p})}{\sum B(k)[1-F(\lambda)]^{k-1}} \quad (2)$$

Where (1) and (2) are derived from the profit maximization conditions.

Substituting (1) into (2),

$$\lambda_1^{'}(\tilde{p}) = \frac{\sum B(k)[1-F(\lambda)]^k}{\sum B(k)[1-F(\lambda)]^{k-1}f(\lambda)[\tilde{p}-\tilde{c}]} - \frac{\sum B(k)*(k-1)*[1-F(\lambda)]^{k-1}\sum A(j)[1-F(\lambda)]^j}{\sum B(k)[1-F(\lambda)]^{k-1}f(\lambda)[\tilde{p}-\tilde{c}]\sum A(j)*j*[1-F(\lambda)]^{j-1}}$$

Multiplying (1) by $\left(\frac{\sum B(k)[1-F(\lambda)]^{k-1}}{\sum B(k)[1-F(\lambda)]^{k-1}}\right)$, $\lambda_i^{'}(\tilde{p})$ and $\lambda_1^{'}(\tilde{p})$ have a common and strictly positive denominator D, which can be safely ignored for our purposes.

As such, the result is proven if $R \equiv [\lambda_i'(\tilde{p}) - \lambda_1'(\tilde{p})] * D \geq 0$.

$$R = \sum A(j)[1-F(\lambda)]^j \sum B(k)[1-F(\lambda)]^{k-1} + \sum B(k)*(k-1)[1-F(\lambda)]^{k-1}\sum A(j)[1-F(\lambda)]^j$$
$$-\sum B(j)[1-F(\lambda)]^j \sum A(k)*k*[1-F(\lambda)]^{k-1}$$

The first two terms can be summed into one, such that

$$R = \sum B(k)*k*[1-F(\lambda)]^{k-1}\sum A(j)[1-F(\lambda)]^j - \sum B(j)[1-F(\lambda)]^j \sum A(k)*k*[1-F(\lambda)]^{k-1}.$$

Recall that the probabilities of the binomial distributions $A(k)$ and $B(k)$ are:

$$A(k) = \binom{N-1}{k}[1-G(q_1)]^k G(q_1)^{N-k-1}$$

$$B(k) = \binom{N-2}{k-1}[1-G(q_1)]^{k-1} G(q_1)^{N-k-1};\ B(0)=0$$

The probabilities of $A(k)$ can be rewritten as functions of $B(k)$:

$$A(k) = \frac{N-1}{k}[1-G(q_1)]B(k), for\ k \geq 1 \quad (3)$$

The rest of the proof consists of constructing a subtraction of terms that are non-negative. Let $\alpha \equiv [1 - F(\lambda)]$, where $\alpha \in [0,1]$. The key to the results are that we know that necessarily $B(0) = 0$ and $A(0) \geq 0$, with equality only when $q_1 = \underline{q}$. We can infer a simplified form of R by induction:

$$R = \sum_{j=1}^{N-1} \sum_{k=j}^{N-1} (k - j) * \alpha^{k+j-1} [B(k)A(j) - B(j)A(k)] + A(0) \sum_{j=1}^{N-1} B(j) * j * \alpha^{k-1}$$

For $N = 2$,

$$R = \sum_{j=1}^{2} \sum_{k=j}^{2} (k - j) * \alpha^{k+j-1} [B(k)A(j) - B(j)A(k)] + A(0)B(1)$$

$$= \alpha * 0 * [B(1)A(1) - B(1)A(1)] + \alpha^2 [B(2)A(1) - B(1)A(2)] + A(0)B(1)$$

$$= \sum B(k) * k * \alpha^{k-1} \sum A(j)\alpha^j - \sum B(j)\alpha^j \sum A(k) * k * \alpha^{k-1}.$$

The last equality follows from $B(0) = 0$.

If the equality holds for $N = n$, we can write:

$$\sum_{j=1}^{n-1} \sum_{k=j}^{n-1} (k - j) * \alpha^{k+j-1} [B(k)A(j) - B(j)A(k)] + A(0) \sum_{j=1}^{n-1} B(j) * j * \alpha^{k-1}$$

$$= \sum_{k=0}^{n-1} B(k) * k * \alpha^{k-1} \sum_{j=0}^{n-1} A(j)\alpha^j - \sum_{j=0}^{n-1} B(j)\alpha^j \sum_{k=0}^{n-1} A(k) * k * \alpha^{k-1}$$

For $N = n + 1$, the new terms on left side of the equation can be written as

$$A(0)B(n) * n * \alpha^{n-1} + \sum_{j=1}^{n} (n - j) * \alpha^{n+j-1} [B(n)A(j) - B(j)A(n)]$$

$$= A(0)B(n) * n * \alpha^{n-1} + \sum_{j=1}^{n} (n - j) * \alpha^{n+j-1} [B(n)A(j)] - \sum_{j=1}^{n} (n - j) * \alpha^{n+j-1} [B(j)A(n)]$$

$$= A(0)B(n) * n * \alpha^{n-1} + B(n)\alpha^n \sum_{j=1}^{n} A(j) * (n - j) * \alpha^{j-1} - A(n)\alpha^n \sum_{j=1}^{n} B(j) * (n - j) * \alpha^{j-1}$$

On the right-hand side,

$$B(n) * n * \alpha^{n-1} \sum_{j=0}^{n} A(j)\alpha^{j} - B(n)\alpha^{n} \sum_{k=1}^{n-1} A(k) * k * \alpha^{k-1} + A(n)\alpha^{n} \sum_{k=1}^{n-1} B(k) * k * \alpha^{k-1}$$

$$-A(n) * n * \alpha^{n-1} \sum_{j=0}^{n} B(j)\alpha^{j}$$

$$= B(n) * n * \alpha^{n-1} \sum_{j=0}^{n} A(j)\alpha^{j} - B(n)\alpha^{n-1} \sum_{k=1}^{n-1} A(k) * k * \alpha^{k} + A(n)\alpha^{n-1} \sum_{k=1}^{n-1} B(k) * k * \alpha^{k}$$

$$-A(n) * n * \alpha^{n-1} \sum_{j=0}^{n} B(j)\alpha^{j}$$

$$= A(0)B(n) * n * \alpha^{n-1} + B(n) * n * \alpha^{n-1} \sum_{j=1}^{n} A(j)\alpha^{j} - B(n)\alpha^{n-1} \sum_{k=1}^{n-1} A(k) * k * \alpha^{k}$$

$$+A(n)\alpha^{n-1} \sum_{k=1}^{n-1} B(k) * k * \alpha^{k} - A(n) * n * \alpha^{n-1} \sum_{j=1}^{n} B(j)\alpha^{j}$$

$$= A(0)B(n) * n * \alpha^{n-1} + B(n) * \alpha^{n-1} \left[n * \sum_{j=1}^{n} A(j)\alpha^{j} - \sum_{k=1}^{n-1} A(k) * k * \alpha^{k} \right]$$

$$+A(n)\alpha^{n-1} \left[\sum_{k=1}^{n-1} B(k) * k * \alpha^{k} - n * \sum_{j=1}^{n} B(j)\alpha^{j} \right]$$

Changing the summation indices inside the brackets,

$$= A(0)B(n) * n * \alpha^{n-1} + B(n) * \alpha^{n-1} \left[n * \sum_{k=1}^{n-1} A(k)\alpha^{k} - \sum_{k=1}^{n-1} A(k) * k * \alpha^{k} + n * A(n)\alpha^{n} \right]$$

$$+A(n)\alpha^{n-1} \left[\sum_{k=1}^{n-1} B(k) * k * \alpha^{k} - n * \sum_{k=1}^{n-1} B(k)\alpha^{k} - n * B(n)\alpha^{n} \right]$$

The $A(n)$ and $B(n)$ terms cancel out, and we can reduce the bracketed terms into one summation

$$= A(0)B(n) * n * \alpha^{n-1} + B(n) * \alpha^{n} \left[\sum_{k=1}^{n-1} (n-k)A(k)\alpha^{k-1} \right] - A(n)\alpha^{n} \left[\sum_{k=1}^{n-1} (n-k)B(k)\alpha^{k-1} \right]$$

This is equal to the left-hand side, and concludes the induction.

It is easy to see that for all $k = j$, all terms of R equal 0.

Using (3) and ignoring the positive $A(0)$ terms, we have

$$\sum_{j=1}^{N-1}\sum_{k=j+1}^{N-1}(k-j)*\alpha^{k+j-1}\left[\frac{N-1}{j}[1-G(q_1)]B(j)B(k)\ -\frac{N-1}{k}[1-G(q_1)]B(j)B(k)\right]$$

$$=\sum_{j=1}^{N-1}\sum_{k=j+1}^{N-1}(k-j)*\alpha^{k+j-1}[1-G(q_1)]B(j)B(k)\left[\frac{N-1}{j}-\frac{N-1}{k}\right]$$

Since $k > j$ for all remaining terms, all terms of the internal summation are positive, which proves that R is positive, and therefore $\lambda_i^{'}(\tilde{p}) \geq \lambda_1^{'}(\tilde{p})$ for any $\tilde{p}$ such that $\lambda_i(\tilde{p}) = \lambda_1(\tilde{p})$. In particular, $\lambda_i^{'}\left(\underline{p}\right) \geq \lambda_1^{'}\left(\underline{p}\right)$. Furthermore, if $q_1 > \underline{q}$ the $A(0)$ terms are strictly positive, so the inequality holds strictly for all $c > \underline{c}$.■

Proposition 3: There is no Pure Strategies Nash Equilibrium to the two-stage procurement game with favoritism if the maximum bid price allowed $\overline{p}$ is greater than the highest cost $\overline{c}$.

Based on Athey (2001), we show that there is no equilibrium in pure strategies to the game proposed when $\overline{p} > \overline{c}$. First, note that each firm's objective function is a weighted average of straightforward expected profit functions in a first-price procurement game. To check the single crossing, we need only to assume that there exists a bounded and atomless joint density $f(.)$ for the distribution of types, that is integrable with respect to a Lebesgue measure and that the integral is finite for all convex sets S and all non-decreasing functions $\beta: \left[\underline{c}, \overline{c}\right] \rightarrow \left[\underline{p}, \overline{p}\right]$.Reny (2011) relaxes the necessity of convexity (and therefore the necessity for single crossing) and proves that the equilibrium still exists. Finally, existence also requires affiliation between signals, which is satisfied in our case by independence.

In short, existence would require that the weighted function of log-supermodular functions is log-supermodular. Since the weighted sum of the functions amounts to a linear transformation, most other desirable properties such as continuity and almost everywhere differentiability also hold.

Finally, the last sufficient condition for existence on Athey (2001) is the continuity of the payoff function on the strategy space. To prove that there is no equilibrium, first we show that there is necessarily a discontinuity on firm 1's action space. This results in a discontinuity of payoffs for $i \neq 1$ that generates an incentive for a global deviation to bid the reservation price $\overline{p}$ instead of the bid function $\beta_i(c_i)$ implied by the system of differential equations.

First, note that for all $\tilde{q} > 0$, the probability of firm 1 being a monopolist in the market is given by

$$A(0) = F(\tilde{q})^{N-1} > 0$$

It follows directly that $\Pi_1(\overline{p}) = A(0) * [\overline{p} - c] > 0$ for *any* strategy β_i.

Since Π_1 is decreasing and continuous in c_i (with β_i fixed) for an interior solution, and $\Pi_1(\bar{c}) = 0$ if $\beta_1(\bar{c}) = \bar{c}$, it follows from the Intermediate Value Theorem that there exists some $\tilde{c}$ such that the linearly decreasing function $\Pi_1(\bar{p}, c)$ is equal to $\Pi_1(\beta_1(c), c)$, since Π_1 is continuous on β_1.

To conclude the proof we show that there will always be an incentive to deviate for firms $i \neq 1$.

From proposition 1, we know that $\beta_i(c) \leq \beta_1(c)$ for any potential interior solution to the profit maximization system. To see that firm i's payoff function is not continuous, first note that at the interior solution of $\beta_1(c)$, $\beta_1(c) < \bar{c}$. Since $\bar{p} > \bar{c}$, it follows that:

$$\lim_{n\to\infty} \Pi_i\left(\beta_1\left(\tilde{c} - \frac{1}{n}\right), \beta_i\right) < \Pi_i\big(\beta_1(\tilde{c}), \beta_i\big) = \Pi_i\big(\bar{p}, \beta_i\big)$$

To see this, note that for any n, $\beta_1\left(\tilde{c} - \frac{1}{n}\right) < \bar{c}$ and so $\lim_{n\to\infty}\left[1 - F\left(\beta_1^{-1}\left(\tilde{c} - \frac{1}{n}\right)\right)\right] > 0$ and $\left[1 - F\left(\beta_1^{-1}(\tilde{c})\right)\right] = 0$.Since there is a discontinuity in the probability of winning, the payoff function is also discontinuous on the strategy space of firm 1.

Since continuity is only a sufficient condition for existence on Athey(2001), we still need to show that this leads to non-existence.

Assume that $\beta_i(c)$ is the equilibrium bid function for all $i \neq 1$. While the probability of any firm $i \neq 1$ being alone in the market is zero, the probability of being qualified only with firm 1 is equal to $B(1) = F(\tilde{q})^{N-2} > 0$ for all $\tilde{q} > 0$. Finally, we use the previous result to argue that $B(1) * \big(1 - F(\tilde{c})\big) > 0$, which represents the strictly positive probability of firm i qualifying by itself **and** facing a firm 1 with costs such that it bids $\bar{p}$. Using the Intermediate Value Theorem once again and the fact that $\bar{p} > \bar{c}$, $\exists\ \tilde{\tilde{c}}\ and\ \epsilon > 0$ such that $\beta_i^d\big(\tilde{\tilde{c}}\big) = \bar{p} - \epsilon_1$ is a lucrative deviation from a potential symmetric equilibrium β_i. Existence of $\beta_i^d(\tilde{\tilde{c}})$ is guaranteed by the fact that $\bar{p} - \bar{c} > 0$, and $\in\ \mathbb{R}$.

Proceeding inductively, for some firm $j \neq i, 1$ there will be a deviation $\beta_j^d\big(\tilde{\tilde{c}}\big) = \bar{p} - \epsilon_2$ that is profitable, with $\bar{p} - \beta_i\big(\tilde{\tilde{c}}\big) > \epsilon_2 > \epsilon_1$

Finally, proceeding inductively it is straightforward that $\beta_i^d\big(\tilde{\tilde{c}}\big) = \beta_i(\tilde{\tilde{c}})$ as the sequence ϵ_n converges to $\bar{p} - \beta_i\big(\tilde{\tilde{c}}\big)$. Finally, we have that $\Pi_i\left(\beta_1, \beta_i, \beta_j^d\right) > \Pi(\beta_1, \beta_i, \beta_j)$, which is a contradiction to β_i being an equilibrium for all $i \neq 1$.■

Proposition 4: The second-price, two-stage procurement auction with favoritism is such that $\boldsymbol{\beta = \beta_1 = \beta_i}$***, and*** $\boldsymbol{\beta(c_i) = c_i}$

Since price competition follows a second price rule, ex-post payoffs for any firm $i = 1, \ldots N$ can be written as

$$\begin{cases} Min_{j\neq i}\, p_j - c_i, & if\ p_i < \min\limits_{j\neq i} p_j \\ 0, & if\ p_i > \min\limits_{j\neq i} p_j \end{cases}$$

Considering $p_i^{'} < c_i$, three cases are possible. If $\min_{j \neq i} p_j > c_i$, firm i would have been the winner regardless, and therefore has the same payoff. If $\min_{j \neq i} p_j < p_i^{'}$, payoff still equals zero. Finally, if $p_i^{'} < \min_{j \neq i} p_j < c_i$, payoff is strictly negative and firm i is worse off by bidding $p_i^{'} < c_i$.

Considering $p_i^{'} > c_i$, we have

$$\Pi_1\left(p_1^{'}, p_{-1}\right) = \sum A(k) \Pr\left(p_1^{'} \leq p^{*-1}\right)^k [p^{*-1} - c_i]$$

$$\Pi_i\left(p_i^{'}, p_{-i}\right) = \sum B(k) \Pr\left(p_i^{'} \leq p^{*-i}\right)^k [p^{*-i} - c_i]$$

where $p^{*-i} \equiv \min_{j \neq i} p_j^*$. Given the rivals' bid functions, probability distributions on their numbers are irrelevant; for any $A(k)$ and $B(k)$, since $\Pr\left(p_i^{'} \leq p^{*-i}\right)^k < \Pr\left(p_i \leq p^{*-i}\right)^k$, it follows that

$\Pi_i\left(p_i, p_{-i}\right) \geq \Pi_i(p_i^{'}, p_{-i})$ and $\Pi_1\left(p_1, p_{-1}\right) \geq \Pi_1(p_1^{'}, p_{-1})$.■

Proposition 5: **Let $F_1(c) \equiv \Pr\,(c_1 \leq c|\tilde{q})$ and $F_i(c) \equiv \Pr\,(c_i \leq c|q_i \geq \tilde{q})$. Expected profit functions can be rewritten as:**

$$\boldsymbol{\Pi_1 = \sum_{k=0}^{N-1} A(k) \left[1 - F_i\left(\beta_i^{-1}(p_1)\right)\right]^k * [p_1 - c_1]}$$

$$\boldsymbol{\Pi_i = \sum_{k=0}^{N-1} B(k) \left[1 - F_1\left(\beta_1^{-1}(p_i)\right)\right] * \left[1 - F_i\left(\beta_j^{-1}(p_i)\right)\right]^{k-1} * [p_i - c_i]}$$

Consider the ex-ante profit functions

$$\Pi_1 = \Pr\left(p_1 \leq p_{-1}^* | \tilde{q}, c_1\right) * \left[p_1 - c_1\right] \qquad (1)$$

$$\Pi_i = \Pr\left(p_i \leq p_{-i}^*, q_i \geq \tilde{q} \middle| \tilde{q}, c_i\right) * \left[p_1 - c_1\right] \qquad (2)$$

Note that, given c_i and common knowledge of the distributions $F(.)$ and $G(.)$, the probability of qualifying can still be considered a multiplicative constant by firms $i \neq 1$.

$$\Pi_1 = \sum A(k) \Pr\left(p_1 \leq p_1^*(k) | \tilde{q}\right) \left[p_1 - c_1\right] \qquad (1^{'})$$

$$\Pi_i = \Pr\left(q_i \geq \tilde{q}\right) \sum B(k) \Pr\left(p_i \leq p_i^*(k) | \tilde{q}\right) \left[p_i - c_i\right] \qquad (2^{'})$$

The probability of presenting the lowest priced, qualifying bid is now naturally conditioned on the quality threshold. We can now rewrite these probabilities as functions of the cost random variable in the usual fashion, so that

$$\Pi_1 = \sum A(k) \Pr\left(\beta_i^{-1}\left(p_1\right) \leq c_i \middle| q_i \geq \tilde{q}\right)^k \left[p_1 - c_1\right] \qquad (1^{''})$$

$$\Pi_i = \Pr(q_i \geq \tilde{q}) \sum B(k) \Pr\left(\beta_1^{-1}(p_i) \leq c_1 \middle| \tilde{q}\right) \Pr\left(\beta_j^{-1}(p_i) \leq c_j \middle| q_j \geq \tilde{q}\right) [p_i - c_i] \qquad (2'')$$

Substituting the conditional distributions of costs F_1 and F_i into $(1'')$ and $(2'')$ and ignoring the multiplicative probability of participation from $(2'')$ concludes the proof. ■

B) Note on Simulation Methods

A particularly useful relationship between the lower bound of the asymmetric model and the bid function of an alternative case can be derived. Consider a straightforward first-price model where the number of bidders is known, and equal to the expected value of the distribution of qualified participants in the asymmetric case. Given the system of equations derived and the fact that $\beta_i(\underline{c}) = \beta_1(\underline{c})$, where $\underline{c}$ is the lower bound of the cost distribution, it is natural that there should be no distortions at the top. For the lowest possible cost, firms do not need to worry about the distribution of the unknown number of bidders . Since $F\left(\beta_i(\underline{c})\right) = 0$, the probability of winning is always equal to 1, and the equilibrium bid is determined only by the expected number of rivals, since for any k, $[1 - F(\lambda)]^k = 1$. Considering this, we can calculate the expected number of rivals implicit on firm i's equation by defining $N^* \equiv \sum A(k) * k$ .Since the symmetric case with a known number of bidders has a well-known and easily calculated closed form solution, it automatically gives us one of the initial conditions necessary for the simulation of the equilibrium in the model with favoritism, greatly reducing the computational time required.

While this intuition facilitates the Range-Kutta algorithm of the simulations, it is not necessary. With initial conditions $\beta_i(\underline{c}) = \beta_1(\underline{c}) = \underline{p}$ and $\beta_i(\bar{c}) = \bar{c}$, we can simulate the equilibrium by using a search algorithm for the initial value $\underline{p}$ that satisfies both conditions for some ϵ such that $\beta_i(\bar{c}) - \bar{c} < \epsilon$ .This simulation method is utilized throughout the paper.

5References

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