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Corporate ALM: A Multistage Linear Stochastic Programming Model for Optimal Bond Issuance

Debt management is one of the main tasks of a large corporation. It consists on the dynamic issuance of bonds with the purpose of funding its capital, its investments in new projects and its operational expenses. Debt portfolios are structured as a mix of securities with differing indexations, denominations, maturities and amortization schedules, in an attempt to balance the expected cost of servicing the debt with risks inherent to interest rates, corporate revenues and costs. In addition to corporate and regulatory operational constraints, debt management must take into account fluctuations in total debt, asset, cash savings along with other financial performance measures affecting the company's stock price and credit rating. In face of the required modeling flexibility, there is a firmly established literature with applications of *Multistage Stochastic Programming* (MSP) techniques to debt management and, more generally, to *Asset Liability Management* (ALM) problems. Starting with (BRADLEY; CRANE, 1972), ALM models have been developed for several different applications including insurance companies (CARINO; ZIEMBA, 1998) and pension funds (KOUWENBERG, 2001; DERT, 1998; HILLI et al., 2007). More recently, similar techniques were especialized for an optimal sovereign bond issuance, also dealing with the trade-off between minimum expected cost and minimum risk (BALIBEK; MURAT, 2009; CONSIGLIO; STAINO, 2010; DATE P., 2011). For the corporate case, (XU; BIRGE, 2006) introduce a simplified model that maximizes shareholder value over production strategy and dividend distribution policy, considering a single short term debt instrument. However, their model requires the availability of known risk neutral probabilities, an unrealistic assumption especially for companies without a portfolio of tradable assets. To the best of our knowledge, the literature lacks models describing corporate bond issuance under uncertainty, dealing with the detailed coupon payment schedule observed in practice and the complexity of the dynamic decision process in consideration of the trade-off among expected costs, risks and financial performance measures.

In this article, we present an MSP model for a corporation financing a predetermined set of projects, considering a universe of fixed and floating rates debt instruments whose predetermined payment schedule is modeled in detailed.[1] Uncertainty is represented by an event tree followed by a independent path information structure, used to avoid exponential growth of complexity with the number of stages. In the first part of the horizon, we build a detailed event tree with a full range of debt instruments available to the decision maker. For the other portion of the planning horizon, the event tree is formed by a subsample approximation of uncertainty realizations characterized by an independent scenario structure with a predetermined decision rule. Our optimization model describes the dynamic decision process where, at every yearly stage, the state of the system is represented by the current cash holdings and the past debt portfolio. It takes into account the mean-risk trade-off between expected cost of debt service and expected insolvency value. Additional operational constraints express corporate debt valuation and the current asset value used to compute the leverage ratio at each stage. (LEWELLEN; EMERY, 1986) asserts that most reasonable characterizations of corporate debt management policies adopt a borrowing strategy organized around leverage ratio targets. We integrate this performance measure into the objective function modeling it as a convex piecewise linear penalty of the computed excess leverage. An illustrative application shows numerical results of the model, including a stability analysis with respect to sampling variation. We compute the efficient frontier for the mean-risk trade-off performing a sensitivity analysis with respect to the insolvency penalty. Computations were performed with a financial planning software tool implemented for a financial and risk management group at Brazilian oil company Petrobras. In our illustration we consider a fictitious, although realistic, project data set.

The remaining content of this chapter is organized as follows. Section 4.1 describes our multistage stochastic programming model, with a comprehensive presentation of all elements in the formulation. In Section **??**, we perform a series of sensitivity analyses of the optimal solution considering an illustrative example. Section 4.3 outlines the assumptions underlying the application of the multistage stochastic optimization model to corporate finance problems, in particular, features specific to the oil industry. We also present computational results for an extended example, comparing cases resulting from different values for model parameters.

4.1

[1] Note that we do not envision debt renegotiation.

Multistage Stochastic Programming Model

Multistage stochastic programming is a natural framework for long-term financial planning problems, corporate debt management in particular. The model must describe a dynamic setting where, at a given stage, a decision is taken facing an unknown future. Once decisions are implemented, the next period information is revealed and the process is repeated for the next stage.

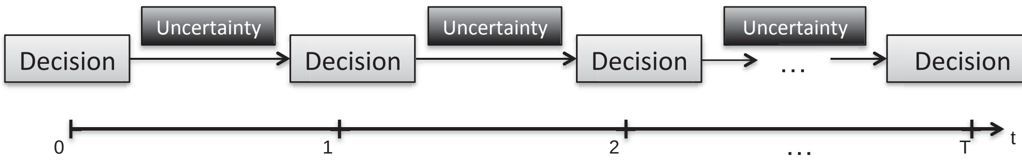


Figura 4.1: Dynamic decision process

A standard approach in MSP models is to represent uncertainty by a discrete event tree, where nodes indicate the state of the process at decision points and arcs the realizations of uncertainty before the next stage. Formally, the information structure given by an event tree can be understood as a filtered probability space (CONSIGLIO; STAINO, 2010) generating a deterministic equivalent of the MSP model. A complete path in the event tree is called a *scenario* and a policy is defined as the set of decisions for all stages and scenarios. This information structure requires that decisions be based solely on past information, expressed in the MSP model formulation by the *non-anticipativity* constraints, which stipulate decision variables at a given stage must be equal if their scenarios share the same node in the event tree. For instance, given a generic policy $X_t(s), \forall\, t \in \{0,1,2\}$, $s \in \{1,2,3,4\}$ and the information structure in Figure 4.2, we would include $X_0(1) = X_0(2) = X_0(3) = X_0(4)$, $X_1(1) = X_1(2)$ and $X_1(3) = X_1(4)$ as non-anticipativity constraints.

Given this tree structure, we can immediately observe that the size of the deterministic equivalent grows exponentially with the number of stages. Some authors have dealt with this *curse of dimensionality* applying large scale optimization techniques (PEREIRA; PINTO, 1991; ROCKAFELLAR; WETS, 1991), while others approximate the original multistage problem by reducing the number decision variables with the adopting single policy rule (RUSH; MULVEY, 2000). A policy rule is a function of the uncertainty realization that generates a unique sequence of feasible decisions for each time of the planning horizon. This framework fits into the independent scenario structure as stated in (RUSH; MULVEY, 2000), however it usually leads to a suboptimal solution

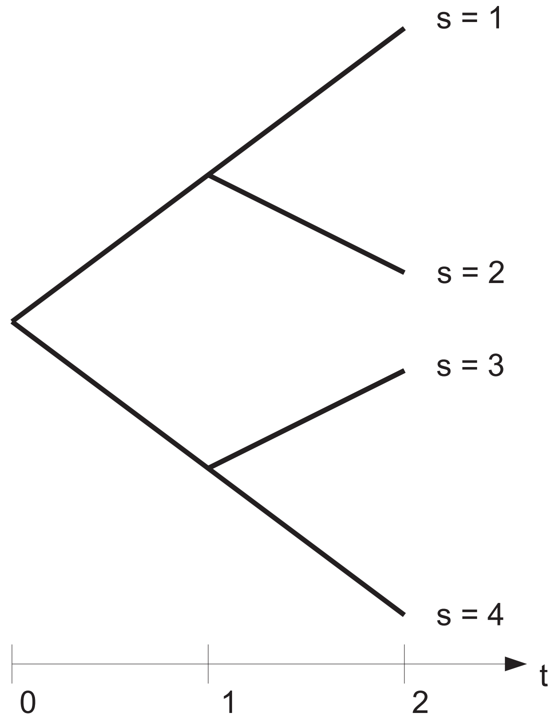


Figura 4.2: Information structure given by an event tree

when compared to the original multistage one. Indeed, one could define a set of policy rules generally leading to a non-convex optimization problem.

With the purpose of reducing the high dimensionality of our final formulation, we propose a hybrid approach comprising a traditional multistage model for the first T^* periods and an independent-scenario structure with simple fixed-policy rule for $t > T^*$. For the latter, we represent uncertainty by a subsample of the full event tree structured as independent scenarios. In our model, a full set of securities is considered for $t \leq T^*$, while for $t > T^*$ we allow only short term bonds to ensure the minimum cash threshold. This framework is motivated by the assumption that most investments take place at the first part of the planning horizon where the decision process is described in more detail.

4.1.1 Definitions

Preparing a complete formal statement of the model, let us first define parameters, risk factors and decision variables used in the formulation.

Scalar Parameters

T: Planning horizon T^*: Detailed planning horizon

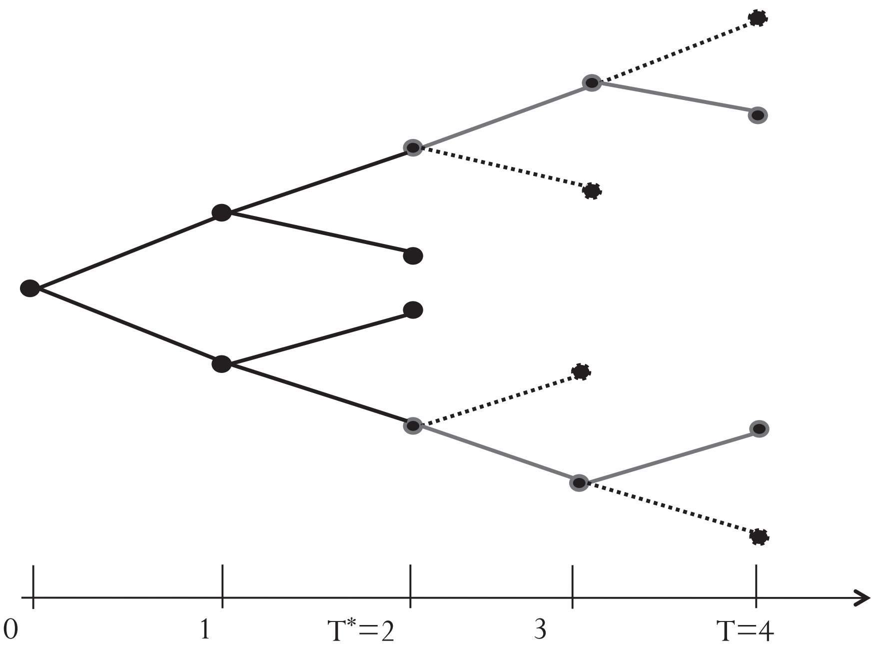


Figura 4.3: Hybrid model for $T^* = 2$ and $T = 4$

S: Number of scenarios
c: Initial cash
nX: Number of fixed rate bonds
K: Number of leverage targets
ω: Weighted average cost of capital
p: Risk aversion parameter
nY: Number of floating rate bonds

Sets

$\mathcal{H} = \{0, \ldots, T-1\}$
$\mathcal{S} = \{1, \ldots, S\}$
$\mathcal{X} = \{1, \ldots, nX\}$
$\mathcal{H}^* = \{0, \ldots, T^*-1\}$
$\mathcal{K} = \{1, \ldots, K\}$
$\mathcal{Y} = \{1, \ldots, nY\}$

Vector parameters

$\hat{c}_t$: Minimum cash at $t \in \mathcal{H}$

γ_k: Leverage ratio for target $k \in \mathcal{K}$

θ_k: Penalty for excess leverage exceeding target $k \in \mathcal{K}$

x_t: Payment at $t \in \mathcal{H} \cup \{T\}$ of pre-existing fixed-rate bonds

y_t: Outstanding face value at $t \in \mathcal{H} \cup \{T\}$ of pre-existing floating rate bonds

Δy_t: Amortization at $t \in \mathcal{H} \cup \{T\}$ of pre-existing floating rate bonds

M_X^i: Maturity of fixed rate bond $i \in \mathcal{X}$, defined when $M_X^i \leq T - T^* + 1$

M_Y^i: Maturity of floating rate bond $i \in \mathcal{Y}$, defined when $M_Y^i \leq T - T^* + 1$

ΔX_j^i: Amortization schedule of fixed rate bond $i \in \mathcal{X}$, for payment $j \in \{1, \ldots, M_X^i\}$, where $\sum_{j=1}^{M_X^i} \Delta X_j^i = 1$

ΔY_j^i: Amortization schedule of floating rate bond $i \in \mathcal{Y}$, $j \in \{1, \ldots, M_Y^i\}$, where $\sum_{j=1}^{M_Y^i} \Delta Y_j^i = 1$

Risk factors

$f_t(s)$: Cash flow at time $t \in \mathcal{H} \cup \{T\}$ and scenario $s \in \mathcal{S}$

$r_{t,\tau}(s)$: Annual effective yield at time $t \in \mathcal{H}$, for maturity τ and scenario $s \in \mathcal{S}$

$\rho_t(s)$: Cash account return at time $t \in \mathcal{H}$ and scenario $s \in \mathcal{S}$

$\alpha_t^i(s)$: Coupon of fixed rate bond $i \in \mathcal{X}$, at time $t \in \mathcal{H}$ and scenario $s \in \mathcal{S}$

$\psi_{t,k}(s)$: Risk premium at time $t \in \mathcal{H}$, for maturity k and scenario $s \in \mathcal{S}$

4.1.2 Decision variables

These sets of variables represent implementable policies such as the amount issued for each bond, as well as auxiliary variables defining the state of the firm, such as cash account, asset and debt values. Note that we implicitly assume the non-negativity of decision variables unless specified.

$X_{t,j}^i(s)$: Outstanding face value at time $t+j$ of fixed rate bond $i \in \mathcal{X}$ issued at $t \in \mathcal{H}^*$ and scenario $s \in \mathcal{S}$, where $j \in \{0, \ldots, \min(t, M_X^i - 1)\}$

$Y_{t,j}^i(s)$: Outstanding face value at time $t+j$ of floating rate bond $i \in \mathcal{Y}$ issued at $t \in \mathcal{H}^*$ and scenario $s \in \mathcal{S}$, where $j \in \{0, \ldots, \min(t, M_Y^i - 1)\}$

$C_t(s)$: Cash savings at time $t \in \mathcal{H}$ and scenario $s \in \mathcal{S}$

$C_T^+(s)$: Positive part of terminal cash savings under scenario $s \in \mathcal{S}$

$C_T^-(s)$: Negative part of terminal cash savings under scenario $s \in \mathcal{S}$

$D_t(s)$: Debt value at time $t \in \mathcal{H}$ and scenario $s \in \mathcal{S}$

$\widetilde{D}_t(s)$: Debt value at time $t \in \mathcal{H}$ and scenario $s \in \mathcal{S}$, excluding bonds issued at t

$A_t(s)$: Asset value at time $t \in \mathcal{H}$ and scenario $s \in \mathcal{S}$, where $A_t(s) \in \mathbb{R}$

$I_{t,k}(s)$: Excess leverage above target $k \in \mathcal{K}$, at time $t \in \mathcal{H}$ and scenario $s \in \mathcal{S}$

4.1.3 Balance constraints

Amortization These constraints update the outstanding face value of each bond after amortization payments. For each bond i issued at t under scenario s, the outstanding face value at $t+j$ is the outstanding value at $t+j-1$ minus the j-th amortization payment.

For fixed rate bonds, $\forall i \in \mathcal{X} \setminus \{1\}$, $\forall t \in \mathcal{H}^*$, $\forall s \in \mathcal{S}$ and $\forall j \in \{1, \ldots, M_X^i - 1\}$, we have

$$X_{t,j}^i(s) = X_{t,j-1}^i(s) - \Delta X_j^i . X_{t,0}^i(s).$$

For floating rate bonds, $\forall i \in \mathcal{Y}$, $\forall t \in \mathcal{H}^*$, $\forall s \in \mathcal{S}$ and $\forall j \in \{1, \ldots, M_Y^i - 1\}$, we have

$$Y_{t,j}^i(s) = Y_{t,j-1}^i(s) - \Delta Y_j^i . Y_{t,0}^i(s).$$

Minimum Cash: For $t \in \mathcal{H}$ and $\forall s \in \mathcal{S}$, $C_t(s) \geq \hat{c}_t$.

Cash balance

Cash balance constraints keep track of inflows and outflows at every stage of the system. We define them differently for the four portions of the planning horizon.

For $t = 0$ and $\forall s \in \mathcal{S}$,

$$C_t(s) = c + f_t(s) - x_t - y_t \rho_{t-1}(s) - \Delta y_t + \sum_{i \in \mathcal{X}} X_{t,0}^i(s) + \sum_{i \in \mathcal{Y}} Y_{t,0}^i(s).$$

Total cash at the end of the first period is the initial value updated with new cash flow, minus payments and amortization for pre-existing debt, plus borrowing income, all at $t = 0$. Observe that the total current issuance is composed of two summations, adding all types of fixed and floating rate bond.

For $t \in \mathcal{H}^* \setminus \{0\}, \forall s \in \mathcal{S}$,

$$
\begin{aligned}
C_t(s) = {} & \big(1+\rho_{t-1}(s)\big)C_{t-1}(s) + f_t(s) - x_t - \big(y_t\rho_{t-1}(s) + \Delta y_t\big) \\
& + \sum_{i\in\mathcal{X}} X^i_{t,0}(s) + \sum_{i\in\mathcal{Y}} Y^i_{t,0}(s) \\
& - \sum_{i\in\mathcal{X}} \sum_{j=1}^{\min(t,M^i_X)} \big(\alpha^i_{t-j}(s)X^i_{t-j,j-1}(s) + \Delta X^i_j.X^i_{t-j,0}(s)\big) \\
& - \sum_{i\in\mathcal{Y}} \sum_{j=1}^{\min(t,M^i_Y)} \Big(\big(\rho_{t-1}(s) + \psi_{t-j,M^i_Y}(s)\big)Y^i_{t-j,j-1}(s) + \Delta Y^i_j.Y^i_{t-j,0}(s)\Big).
\end{aligned}
$$

As in $t = 0$, total cash is the previously accrued value updated with all current inflows and outflows, but also including payments and amortization of new fixed and floating rate bonds.

For $t \in \mathcal{H} \setminus \mathcal{H}^*, \forall s \in \mathcal{S}$,

$$
\begin{aligned}
C_t(s) = {} & \big(1+\rho_{t-1}(s)\big)C_{t-1}(s) + f_t(s) - x_t - \big(y_t\rho_{t-1}(s) + \Delta y_t\big) \\
& + X^1_{t,0}(s) - \big(\alpha^1_{t-1}(s)X^1_{t-1,0}(s) + \Delta X^1_1.X^1_{t-1,0}(s)\big) \\
& - \sum_{i\in\overline{\mathcal{X}}_t} \sum_{j=t-T^*+1}^{\min(t,M^i_X)} \big(\alpha^i_{t-j}(s)X^i_{t-j,j-1}(s) + \Delta X^i_j.X^i_{t-j,0}(s)\big) \\
& - \sum_{i\in\overline{\mathcal{Y}}_t} \sum_{j=t-T^*+1}^{\min(t,M^i_Y)} \Big(\big(\rho_{t-1}(s) + \psi_{t-j,M^i_Y}(s)\big)Y^i_{t-j,j-1}(s) + \Delta Y^i_j.Y^i_{t-j,0}(s)\Big),
\end{aligned}
$$

where $\overline{\mathcal{X}}_t = \{i \mid i \in \mathcal{X} \setminus \{1\}, M^i_X \geq t - T^* + 1\}$ and $\overline{\mathcal{Y}}_t = \{i \mid i \in \mathcal{Y}, M^i_Y \geq t - T^* + 1\}$.

For the simplified portion of the horizon ($T^* \leq t < T$), cash balance constraints differ as new issuances are limited to short term bonds. Note also that summations limits in the terms corresponding to payments and amortization of long term bonds account for only those issued during the detailed horizon.

For $t = T, s \in \mathcal{S}$,

$$
\begin{aligned}
C_t^+(s) - C_t^-(s) = \big(1 + \rho_{t-1}(s)\big)C_{t-1}(s) + f_t(s) - x_t - \big(y_t\rho_{t-1}(s) + \Delta y_t\big) \\
- \big(\alpha_{t-1}^1(s)X_{t-1,0}^1(s) + \Delta X_1^1.X_{t-1,0}^1(s)\big) \\
- \sum_{i\in\overline{\mathcal{X}}_t}\sum_{j=t-T^*+1}^{\min(t,M_X^i)} \big(\alpha_{t-j}^i(s).X_{t-j,j-1}^i(s) + \Delta X_j^i.X_{t-j,0}^i(s)\big) \\
- \sum_{i\in\overline{\mathcal{Y}}_t}\sum_{j=t-T^*+1}^{\min(t,M_Y^i)} \Big(\big(\rho_{t-1}(s) + \psi_{t-j,M_Y^i}(s)\big)Y_{t-j,j-1}^i(s) + \Delta Y_j^i.Y_{t-j,0}^i(s)\Big).
\end{aligned}
$$

At the end of the planning horizon, we do not consider new debt issuance. Infeasibility is avoided by expressing the left hand side with two components and allowing for negative values in the final cash balance. Under scenario s, $C_T^+(s)$ represents the terminal cash savings, while $C_T^-(s)$ the outstanding obligations at the end of the horizon. We can also interpret $C_T^-(s)$ as the cash requirement to avoid insolvency. Note that we construct the objective function in such way that $C_T^+(s)\, C_T^-(s) = 0$.

Asset valuation

Net asset value at time t under scenario s is the conditional expectation of the present value of future project cash flows. Based on (MILLER; MODIGLIANI, 1958; MILLER; MODIGLIANI, 1963), we use the weighted average cost of capital (WACC) of the firm denoted by ω as the discount rate.

Then, for $t \in \mathcal{H}, s \in \mathcal{S}$:

$$
A_t(s) = C_t(s) + \frac{1}{|\mathcal{S}(t,s)|}\sum_{\tilde{s}\in\mathcal{S}(t,s)}\sum_{k=1}^{T-t}\frac{f_{t+k}(\tilde{s})}{(1+\omega)^k},
$$

where $\mathcal{S}(t,s) = \{\tilde{s} \in \mathcal{S} \mid N(t,s) = N(t,\tilde{s})\}$. Note that $\mathcal{S}$ stands for $\mathcal{S}(0,s)$, $\forall s \in \mathcal{S}$.

Debt valuation

The market value of total outstanding debt at time t under scenario s is defined as the face value of current bond issues plus the market value of the previously issued debt. For fixed rate bonds, the market value is the net present value of their payments, discounted by the interest rate associated with each instrument. The outstanding face value defines the marked value for previously issued floating rate bonds.

For $t \in \mathcal{H}^*, s \in \mathcal{S}$,

$$D_t(s) = \sum_{i \in \mathcal{X}} X^i_{t,0}(s) + \sum_{i \in \mathcal{Y}} Y^i_{t,0}(s) + \widetilde{D}_t(s)$$

where, for $t = 0$,

$$\widetilde{D}_t(s) = \sum_{k=1}^{T-t} \frac{x_{t+k}}{(1 + r_{t,k}(s))^k} + y_t$$

while for $t \in \mathcal{H}^* \setminus \{0\}$,

$$\widetilde{D}_t(s) = \sum_{i \in \mathcal{X}/\{1\}} \sum_{k=1}^{(M^i_X - 1)} \sum_{j=k+1}^{\min(t+k, M^i_X)} \frac{\alpha^i_{t+k-j}(s)\, X^i_{t+k-j,j-1}(s) + \Delta X^i_j\, X^i_{t+k-j,0}(s)}{(1 + r_{t,k}(s))^k}$$
$$+ \sum_{k=1}^{T-t} \frac{x_{t+k}}{(1 + r_{t,k}(s))^k} + \left(\sum_{i \in \mathcal{Y}} \sum_{j=1}^{\min(t, M^i_Y - 1)} Y^i_{t-j,j}(s) \right) + y_t$$

As in the cash balance constraints, for the detailed horizon, the value of currently issued bonds are computed as summations over all types of fixed and floating rate instruments. The value of previously issued debt, $\widetilde{D}_t(s)$, has different definitions for the initial stage and the remainder of the detailed horizon.

For the simplified horizon, the total debt value is the currently issued short term bond, plus the market value of the all other instruments issued during the detailed horizon.

Then, for $t \in \mathcal{H} \setminus \mathcal{H}^*, s \in \mathcal{S}$,

$$D_t(s) = X^1_{t,0}(s) + \widetilde{D}_t(s)$$

where,

$$\widetilde{D}_t(s) = \sum_{i \in \widetilde{\mathcal{X}}_t} \sum_{k=1}^{(M^i_X - 1)} \sum_{j=t+k-T^*+1}^{\min(t+k, M^i_X)} \frac{\alpha^i_{t+k-j}(s) X^i_{t+k-j,j-1}(s) + \Delta X^i_j . X^i_{t+k-j,0}(s)}{(1 + r_{t,k}(s))^k}$$
$$+ \sum_{k=1}^{T-t} \frac{x_{t+k}}{(1 + r_{t,k}(s))^k} + \left(\sum_{i \in \widetilde{\mathcal{Y}}_t} \sum_{j=t-T^*+1}^{\min(t, M^i_Y - 1)} Y^i_{t-j,j}(s) \right) + y_t$$

and

$$\widetilde{\mathcal{X}}_t = \{ i \mid i \in \mathcal{X}, M^i_X \geq t - T^* + 2 \},$$

$$\widetilde{\mathcal{Y}}_t = \{ i \mid i \in \mathcal{Y}, M^i_Y \geq t - T^* + 2 \}.$$

Non-anticipativity

Thus far, our model formulation described only the relationships of decision variables within each scenario. The non-anticipativity constraints preserve the dynamic structure of the model by stating the equality of variables across different scenarios when they share the same history, or, equivalently, are associated with the same node in the event tree. This guarantees implementable optimal policies, where it is possible to state the corresponding dynamic programming equations. First, we define $\mathcal{N}$ as the set of nodes in the tree and function $N(t,s) : \mathcal{H} \times \mathcal{S} \to \mathcal{N}$, mapping stage t in scenario s into its corresponding node. Then, we define the subsets of decision variable indexes for each node, $\mathcal{U}_n = \{(t,s) \mid N(t,s) = n\} \, \forall n \in \mathcal{N}$. For each non-singleton subset $\mathcal{U}_{n^*}$, we select a canonical element (t^*, s^*) and build the equality constraints linking corresponding decision variables with their counterparts associated with the other elements of the set.

For $n^* \in \{n \in \mathcal{N} \mid |\mathcal{U}_n| > 1\}$,

$$X^i_{t^*,0}(s^*) = X^i_{t,0}(s), \forall i \in \mathcal{X}, \forall (t,s) \in \mathcal{U}_{n^*}/(t^*, s^*),$$

$$Y^i_{t^*,0}(s^*) = Y^i_{t,0}(s), \forall i \in \mathcal{Y}, \forall (t,s) \in \mathcal{U}_{n^*}/(t^*, s^*).$$

Note that it is sufficient to consider only the constraints corresponding to $X^i_{t,0}(s)$ and $Y^i_{t,0}(s)$ since all other decision variable are consequently determined.

4.1.4 Objective function

The objective function in our model includes two contrasting components. The first measures the mean-risk trade-off between expected terminal cash savings and risk of default at the end of the horizon, expressed by a utility function on the terminal cash. For a risk neutral agent, since all accrued borrowing costs are accounted in the cash balance, we can easily establish that maximizing the expected terminal cash is equivalent to minimizing the expected future cost of servicing the debt. The latter quantity is commonly used as part of the objective in the debt management literature (BALIBEK; MURAT, 2009; CONSIGLIO; STAINO, 2010; DATE P., 2011), combined with Conditional Value-at-Risk (CVaR) (ROCKAFELLAR; URYASEV, 2000) as a risk aversion measure to be minimized or constrained. As shown in (SHAPIRO, 2009), CVaR is not time consistent. Akin to non-anticipativity which forces identical decisions for scenarios sharing the same past, time consistency requires that optimality and feasibility should not depend on future scenarios that cannot happen when conditioned by the state at the moment of the decision. We argue that CVaR is inappropriate as a risk aversion measure for dynamic multistage stochastic programming models. As a matter of fact, including CVaR in the objective function may lead to suboptimality of the first stage decisions as illustrated in (RUDLOFF; STREET; VALLADÃO, 2011).

The second component of the objective function takes into consideration the company's debt worthiness based on financial performance measures available to market agents. Ideally, we would have included in the model an adjustment in interest rates reflecting the company's credit rating. However not only estimation of these corrections would not be possible with the available data, but it would greatly increase complexity, prohibitive even in moderately sized instances of MSP models. We propose instead a practical approach where a penalty function increasingly discourages excess leverage at intermediate stages of the planning horizon.

Terminal cash utility function

The utility function $U(C_T)$ assigns a value to a scenario at the end of the horizon based of the final cash balance. For the sake of ease in economic interpretation, we propose a piecewise linear function

$$U\left(C_T\right) = C_T^+ - p\, C_T^-,$$

where a negative terminal cash value is penalized by the risk aversion parameter $p >= 1$. The expected value of the utility function is

$$\mathbb{E}\big[U\left(C_T\right)\big] = \mathbb{E}\left[C_T^+\right] - p\,\mathbb{E}\left[C_T^-\right],$$

combining the expected terminal cash savings with the penalized expected value of insolvency.

This approach for measuring risk aversion is closely related to integrated chance constraints, which have also been used in financial planning problems (HANEVELD; VLERK, 2006), in particular in ALM models (HANEVELD; STREUTKER; VLERK, 2010). Observe in definition of $U(C_T)$ that coefficient p is a risk aversion parameter, with $p = 1$ representing a risk neutral agent.

Excess leverage penalty

The second part of our objective function deals with the effect of market perception on a company's bond issuance policy. As recommended by (LEWELLEN; EMERY, 1986) in a comparison of corporate debt management policies, firms should manage their debt by following a target on the Debt-

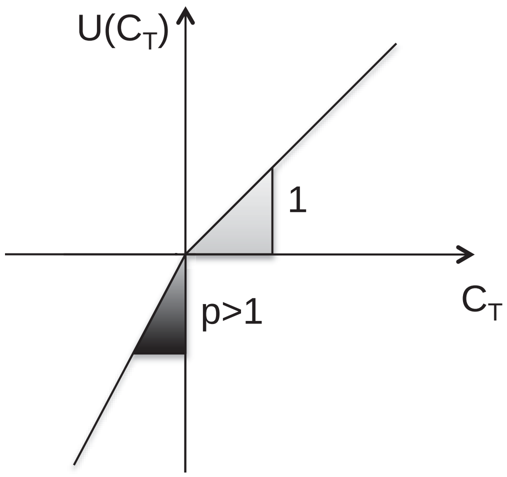


Figura 4.4: Terminal cash utility function

to-Asset ratio. This ratio is also used frequently by market analysts as an indicator of the company's financial performance. Given this background, our model guides the optimal policies by including in the objective a penalty for highly leveraged debt positions. We propose a piecewise linear function that increasingly penalizes the excess leverage based on a sequence of threshold targets for the Debt-to-Asset ratio. Denoted by $\gamma_1 \leq \ldots \leq \gamma_K$, these values correspond to critical leverage levels established by debt managers. In the objective function, we impose a cumulative penalty for violating each one of the leverage targets in each scenario, at each time period. First, we define the amount of excess leverage above each target,

$$I_{t,k}(s) = [D_t(s) - \gamma_k A_t(s)]^+ = \max\big(0, D_t(s) - \gamma_k A_t(s)\big), \quad \forall t \in \mathcal{H},\ s \in \mathcal{S},\ k \in \mathcal{K}.$$

In the linear programming formulation of the model, this last expression is stated by initially adding as constraints the following inequalities,

$$I_{t,k}(s) \geq 0,\ I_{t,k}(s) \geq D_t(s) - \gamma_k A_t(s), \forall t \in \mathcal{H},\ s \in \mathcal{S},\ k \in \mathcal{K}.$$

The equality in the definition of variables $I_{t,k}(s)$ is guaranteed only in the optimal solution from the construction of the objective function which includes penalties on each excess leverage variable,

$$\theta_k I_{t,k}(s)\ \forall t \in \mathcal{H},\ s \in \mathcal{S},\ k \in \mathcal{K},$$

where $\theta_1 \leq \ldots \leq \theta_K$ are positive penalty factors also assigned by debt managers.

The total excess leverage penalty sums the future values of the violations above each target in all scenarios,

$$\sum_{k \in \mathcal{K}} \theta_k \sum_{t \in \mathcal{H}} I_{t,k}(s) \prod_{\tau=t+1}^{T} \big(1 + \rho_\tau(s)\big) \; \forall s \in \mathcal{S}.$$

In its final form, we state the full objective by adding the two components and taking the expected value,

$$\max \; S^{-1} \sum_{s \in \mathcal{S}} \Big(C_T^+(s) - pC_T^-(s) - \sum_{k \in \mathcal{K}} \theta_k \sum_{t \in \mathcal{H}} I_{t,k}(s) \prod_{\tau=t+1}^{T} \big(1 + \rho_\tau(s)\big) \Big),$$

noting that the penalty functions take negative signs in the maximization objective.

4.2 Illustrative example

In this section, we illustrate some key features of the model by building a simplified example where uncertainty is considered only on the term structure of the interest rates. We assume a project portfolio with a deterministic cash flow stream repeated for all scenarios, setting the values for risk factor $f_t(s)$. With a planning horizon $T = 15$, we assign high capital expenses for the initial 5 stages and constant revenues there after. We also assume a null return for the cash account, with $\rho_t(s) = 0$ for all time periods and scenarios.

Based on this uncertainty framework, the model is further specified by the portfolio of available debt instruments and the generation of the event tree. The resulting problem is solved for the maximization of the terminal cash utility only. When compared to a complete instance of the implemented model, this example allows for a larger number of scenarios in the event tree. The low computational effort in the solution of each instance also permits building the efficient frontier for the risk aversion parameter.

4.2.1 Debt instruments

We consider nine types of bonds of varying maturities and amortization schedules:

Fixed Rate Bonds

Short Term: 1-year short-term bond

Fixed-5 Final: 5-year bond with full amortization at the end

Fixed-10 Final: 10-year bond with full amortization at the end

Fixed-5 Constant: 5-year bond with constant amortization bond

Fixed-10 Constant: 10-year bond with constant amortization bond

Floating Rate Bonds

Floating-5 Final: 5-year bond with full amortization at the end

Floating-10 Final: 10-year bond with full amortization at the end

Floating-5 Constant: 5-year bond with constant amortization

Floating-10 Constant: 10-year bond with constant amortization

Expressing these definitions as parameters of the model, we have $nY = 4$ and $nX = 5$. Note that the set of fixed-rate bonds include short-term instruments indexed by $i = 1$ in $\mathcal{X} = \{1, 2, 3, 4, 5\}$. Amortization schedules are defined as:

For $i \in \{1, 2, 3\}$,

$$\Delta X^i_j = \begin{cases} 1, & \text{for } j = M^i_X \\ 0, & \text{otherwise} \end{cases}$$

For $i \in \{4, 5\}$,

$$\Delta X^i_j = \frac{1}{M^i_X}, \forall j = 1, \ldots, M^i_X.$$

Maturities are defined as:

$$M^i_X = \begin{cases} 1, & \text{for } i = 1 \\ 5, & \text{for } i \in \{2, 4\} \\ 10, & \text{for } i \in \{3, 5\} \end{cases}$$

For the floating rate bonds, we have $\mathcal{Y} = \{1, 2, 3, 4\}$, with the amortization schedules defined as:

For $i \in \{1, 2\}$,

$$\Delta Y^i_j = \begin{cases} 1, & \text{for } j = M^i_Y \\ 0, & \text{otherwise} \end{cases}$$

For $i \in \{3, 4\}$,

$$\Delta Y^i_j = \frac{1}{M^i_Y}, \forall j = 1, \ldots, M^i_X.$$

Maturities are defined as follows:

$$M_Y^i = \begin{cases} 1, & \text{for } i = 1 \\ 5, & \text{for } i \in \{2,4\} \\ 10, & \text{for } i \in \{3,5\} \end{cases}$$

4.2.2
Scenario tree generation

Scenarios for the MSP are generated by the forecasting model presented in (VEREDA, 2011), which supplies the estimated parameters for Brazilian and American term structure of the interest rates in the following state space framework:

$$\begin{aligned} \eta_t &= A + B\,\xi_t \\ \xi_t &= \Phi\,\xi_{t-1} + \Sigma^{1/2}\varepsilon_t, \quad \varepsilon_t \sim N(0, I). \end{aligned}$$

Based on the *Adjusted Random Sampling* of (KOUWENBERG, 2001), we compute an event tree that approximates the original stochastic vector ε_t, using antithetic values along with a variance adjustment. For the sake of implementation efficiency, we generate the residual tree nodewise, i.e., $\epsilon(n)$, $\forall n \in \mathcal{N}$.

Given a node n, let us denote $\mathcal{Q}(n) = \{q_1, q_2, \ldots\}$ the set of all possible successor nodes, with $Q(n) = |\mathcal{Q}(n)|$. Using antithetic values, we match the zero mean and all null higher odd moments for each univariate stochastic component of $\epsilon_i(q)$, $\forall q \in \mathcal{Q}(n)$. After initializing $\varepsilon_i(q) = 0$, $\forall q \in \mathcal{Q}(n)$, we sample via Monte Carlo simulation the first $k = 1, \ldots, \lfloor Q(n)/2 \rfloor$ elements and generate the antithetic values for the remainder for $j = Q(n) - k$,

$$\epsilon(q_j) = -\epsilon(q_k).$$

Note that this procedure ensures null conditional odd moments of the simulated $\epsilon_i(n)$ for each component i, i.e.,

$$\mathbb{E}\left[\epsilon_i^p | n \in \mathcal{N}\right] = \frac{1}{Q(n)} \sum_{q \in \mathcal{Q}(n)} \epsilon_i^p(q) = 0, \quad \forall p = 1, 3, 5, \ldots\ .$$

Returning to the original notation, we define the unadjusted residuals as

$$\tilde{\varepsilon}_t(s) = \epsilon(n), \forall t \in \mathcal{H} \cup \{T\},\ s \in \mathcal{S} \text{ such that } n = N(t,s).$$

Then, we adjust the variance of $\tilde{\varepsilon}_{i,t}$ for each stage t and for each component i. Indeed, to suit the hybrid tree structure, our procedure matches the unconditional variances in opposition to the conditional approach of (KOUWENBERG, 2001). Therefore the adjusted residuals are given by

$$\varepsilon_{i,t}(s) = \frac{\tilde{\varepsilon}_{i,t}(s)}{\sqrt{\frac{1}{S}\sum_{s\in\mathcal{S}}\tilde{\varepsilon}^2_{i,t}(s)}}, \quad \forall s \in \mathcal{S}.$$

4.2.3
Solution and Sensitivity analysis

Sensitivity analysis examines the robustness and stability of the optimal solution vis-à-vis changes in the input parameters and data uncertainty. With the objective function limited to the terminal cash utility function, without the excess leverage penalty, this experiment builds the efficient frontier for the risk aversion parameter p. Since the proposed model is only tractable for relatively small event trees, the optimal solution is subject to estimation errors. Given a value for p, we generate N independent event trees whose sets of scenarios are denoted by $\mathcal{S}_i$, $\forall i = 1, \ldots, N$. Then, we solve the problem for each scenario set, using the optimal solution to compute the two components of the objective function, $\mathcal{C}_i^+ = S^{-1}\sum_{s\in\mathcal{S}_i} C_T^+(s)$ and $\mathcal{C}_i^- = S^{-1}\sum_{s\in\mathcal{S}_i} C_T^-(s)$, $\forall i = 1, \ldots, N$.

Ultimately, we build the efficient frontier corresponding to each scenario set $\mathcal{S}_i$ by solving the problem for each value of p and linearly interpolating the observed points to compute the curve $\mathcal{C}_i^+$ vs $\mathcal{C}_i^-$, $\forall i \in \{1, \ldots, N\}$. In our experiment, we assumed $N = 1000$ and $p \in \{1, 50, 100, 200, 500, 1000, 2000\}$, generating 7000 instances of the model. The resulting efficient frontier is represented in Figure 4.5, where the average and 95% percentile are obtained from the distribution of all possible values of $\mathcal{C}^+$ given a fixed level of risk $\mathcal{C}^-$.

From the results of this experiment, we can also develop a sensitivity analysis for the first stage decision with respect to the risk aversion parameter p. For each $i = 1, \ldots, N$, we take the first stage optimal solution for each scenario set $\mathcal{S}_i$, $Z_i = (X^1_{0,0}, \ldots, X^{nX}_{0,0}, Y^1_{0,0}, \ldots, Y^{nY}_{0,0})$, indicating the amounts issued in fixed and floating rate bonds, for all maturities and amortization schedules. Then, we compute the sample average approximation $\overline{Z} = N^{-1}\sum_{i=1}^{N} Z_i$. for all values of p. The stacked bar graph in Figure 4.6 indicates the amounts corresponding to each available debt instrument, expressing the behavior of the optimal solution with respect to risk aversion level. Note that for the risk neutral case, $p = 1$, the short term bond is preferred while the risk averse case, $p > 1$, long term bonds are increasingly more attractive. This behavior is explained by the locked cost of long term debt, while a short term portfolio

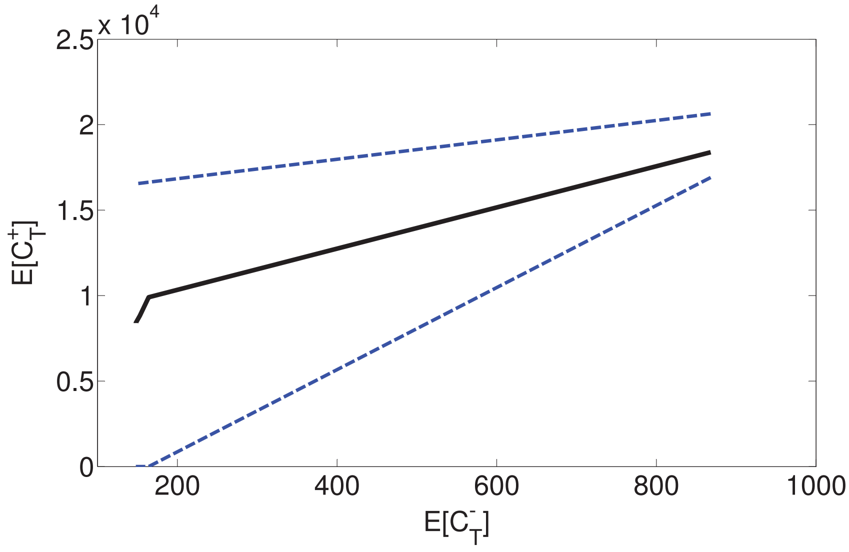


Figura 4.5: Efficient Frontier

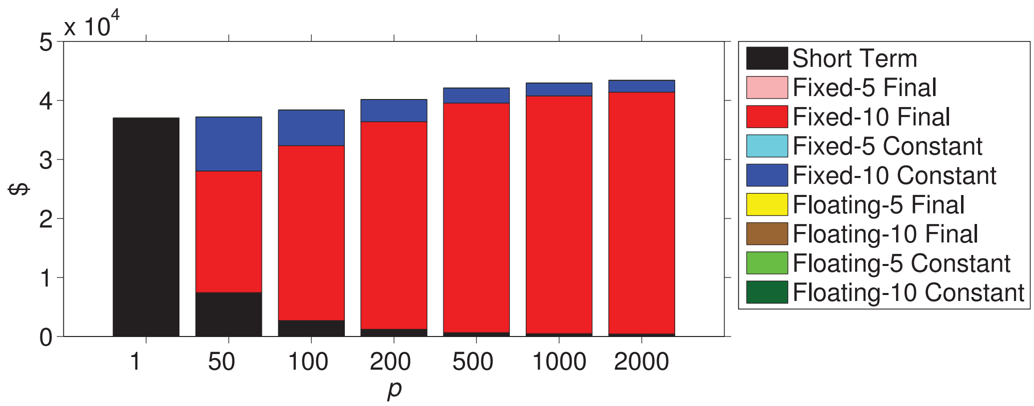


Figura 4.6: First Stage Decision X Risk Aversion Parameter

would imply in periodic refinancing, subject to higher risk in borrowing cost.

4.3 Application to the oil industry

Application of the proposed model to a real-world problem requires further assumptions, complete specification of risk factors and inclusion of the excess leverage penalty into the objective function. These features were implemented in a financial planning software tool deployed in a risk management organization at Brazilian oil company Petrobras. Two auxiliary modules have been developed for the generation of risk factor scenarios: An integrated interest and exchange rates forecasting model and a simulator for future spot prices of crude oil, its byproducts and natural gas. Considering the same debt instruments as before, we specialize the various elements of our multistage stochastic programming model for this application and present results based on a fictitious, although realistic, project data set.

4.3.1 Risk factors

The model formulation constraints refer explicitly to scenarios for interest rates and risk premiums. We call those *Financial Risk Factors*. In addition, *Project Risk Factors* are reflected indirectly in the scenarios for cash flows generated by the project portfolio, including market prices for crude oil and natural gas. In an attempt to approximate continuous flow of revenues and expenditures distributed over each planning stage, we assume the average values during the period. The detailed description of the forecast models is available in (VEREDA, 2011) and (FERNANDES; LIMA, 2011).

Financial Risk Factors An integrated forecast model provides scenarios for interest and exchange rates.

$r_{t,\tau}(s)$: Annual effective yield for bonds denominated in US$, issued by the company

$\rho_t(s)$: Risk free interest rate, assumed to be US government bond 1-year yield

$\psi_{t,k}(s)$: Risk premium associated with the company for the corporate bonds denominated in US$

Note that there is a unique mapping between the fixed rate coupons and the term structure of the interest rate. Therefore, $\alpha_t^i(s)$ must be derived from the corresponding term structure $r_{t,j}(s)$, $\forall j = 1, \ldots, M_X^i$, assuming the net present value of future payments is equal to its face value. Without loss of generality, we compute the coupon using a unit face value, i.e.,

$$1 = \alpha_t^i(s) \left(\frac{1}{1 + r_{t,1}(s)} + \sum_{j=2}^{M_X^i} \frac{1 - \sum_{k=1}^{j-1} \Delta X_k^i}{(1 + r_{t,j}(s))^j} \right) + \sum_{j=1}^{M_X^i} \frac{\Delta X_j^i}{(1 + r_{t,j}(s))^j}.$$

Then, the coupon is defined as

$$\alpha_t^i(s) = \left(1 - \sum_{j=1}^{M_X^i} \frac{\Delta X_j^i}{(1 + r_{t,j}(s))^j} \right) \left(\frac{1}{1 + r_{t,1}(s)} + \sum_{j=2}^{M_X^i} \frac{1 - \sum_{k=1}^{j-1} \Delta X_k^i}{(1 + r_{t,j}(s))^j} \right)^{-1}.$$

Project Risk Factors We assume all stochastic cash flows generated by the project portfolios to be an affine functions of project risk factors. Besides market prices for crude oil and its byproducts, exchange rates are also considered risk factors in this category, as the project portfolio includes multi-currency investments. Based on these risk factors, investments and production data, a preprocessor to the optimization model computes scenarios for the cash streams $f_t(s)$, $d_t(s)$ and $l_t(s)$.

4.3.2 Computational experiments

The financial planning software tool implemented from our model uses Matlab to perform all of the data preparation and solution presentation. The linear programming formulation was implemented with the MOSEL modeling language, using the Xpress optimization suite as the solver. The computational experiments were carried out on an Intel(R) Core(TM) i7 CPU based computer with 24Gb RAM and 8 processors.

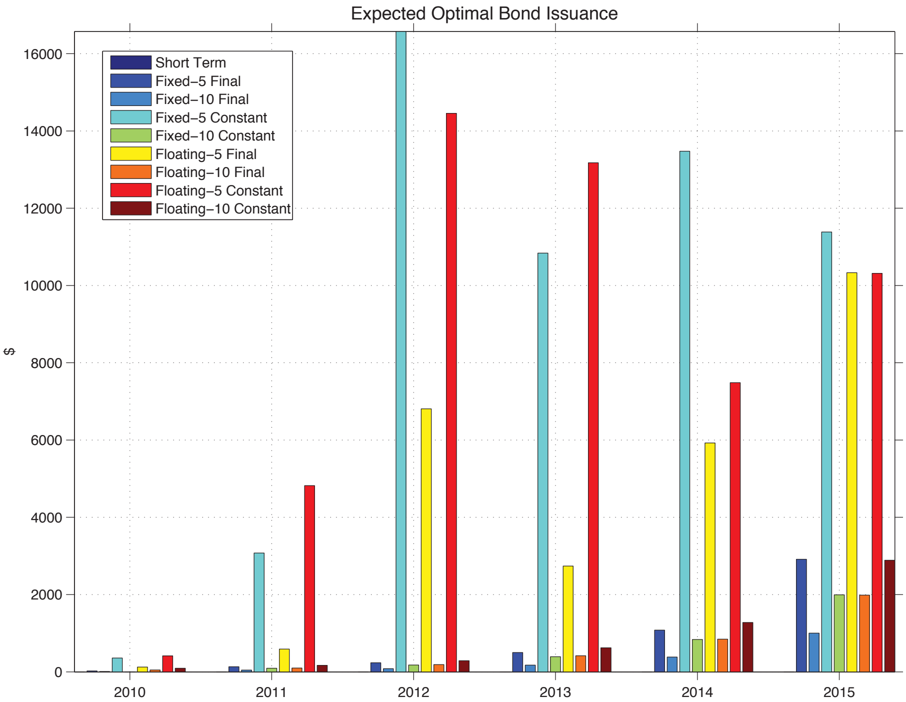
As opposed to the illustrative example presented in Section 4.2 where we assume a null return for the cash account, savings are now invested in short term US government bonds, subject to stochastic returns. This key assumption closely matches the actual corporate financial strategy. However, it increases the probability of outlier scenarios where the income generated by the cash account is greater than the costs of some debt instruments. The optimal policy for these scenarios would be a highly leveraged, possibly unbounded, debt portfolio. Under these circumstances, intermediate excess leverage penalties are added to the objective function, avoiding unrealistic solutions. The parameters for the forecasting model were tuned to simulate an environment of growing interest rates and well-behaved term-structures, emphasizing the effect of our multi-criteria objective function.

In this experiment, we consider a base case with horizon $T = 48$ starting in 2010, detailed horizon ($T^* = 6$), initial cash saving $c = 5.00$, minimum cash $\hat{c}_t = 5.00$, risk aversion parameter $p = 10$, weighted average cost of capital $\omega = 8.8$, number of leverage targets $K = 3$ and number of scenarios $S = 1024$. With this specification, the resulting equivalent deterministic linear program has 820534 rows, 813056 columns and 4559468 non-zero matrix elements.

We compare the solutions of the problem under two assumptions for the excess leverage penalties. In Case 1 we assume zero penalties and an arithmetic progression in Case 2, with leverage targets $\gamma = \{35\%, 50\%, 100\%\}$ and incremental leverage penalties $\theta = \{1, 1, 1\}$. Examining the optimal solutions for both cases, we first compare the expected bond issuances for the detailed portion of the horizon. Figure 4.7 shows the average amounts issued for each bond on each stage $t \in \mathcal{H}^*$. For Case 1, the total debt issued is much higher than the amount required to fulfill the minimum cash requirement of the firm, indicating the presence of scenarios with cash saving earnings above debt costs. This effect disappears in Case 2.

The impact of imposing intermediate penalties is further illustrated by analyzing the behavior of the stochastic leverage ratio, $D_t(s)/A_t(s)$. Figure 4.8 displays in different colors, for Case 2, the probability that the leverage ratio belongs to each range of target values, noting that all scenarios have the same solution in the first stage.

The solution for Case 1 counters the intuitive premise that a firm with a fixed project portfolio should not be unnecessarily exposed to risk from uncertain financial returns. In Case 2, the intermediate penalties discourage risky policies with high leverage ratios occurring in a small subset of scenarios. The last leverage range, $D_t(s)/A_t(s) \geq 100\%$, defines the insolvency state at each stage, when debt exceeds total assets. This experiment suggests that debt managers use the proposed model interactively, tuning risk aversion parameters in the intermediate excess leverage penalties to avoid highly leveraged portfolios.



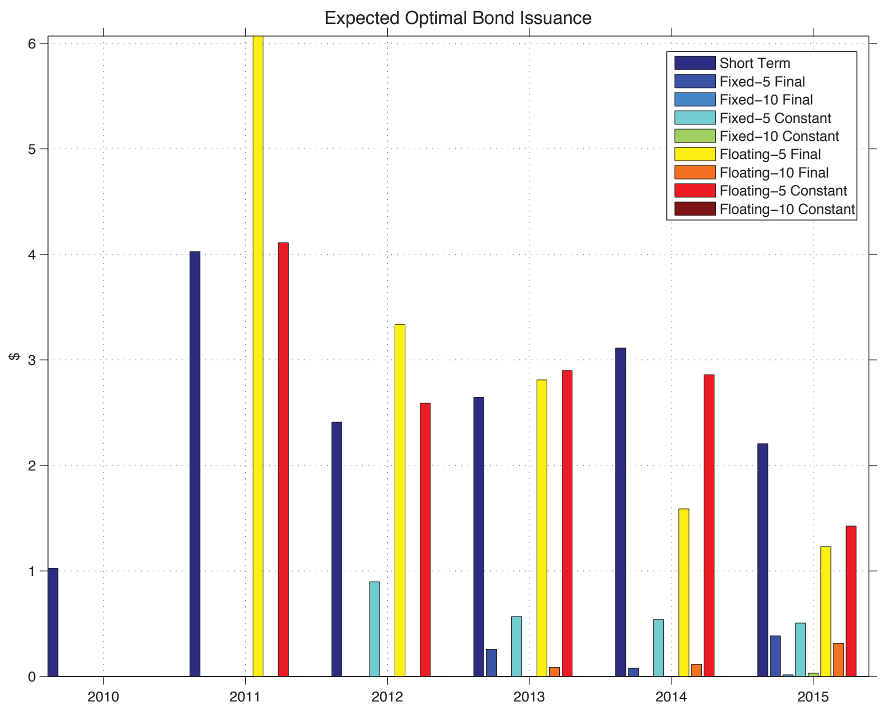


Figura 4.7: Expected Optimal Bond Issuance - Case 1 and Case 2

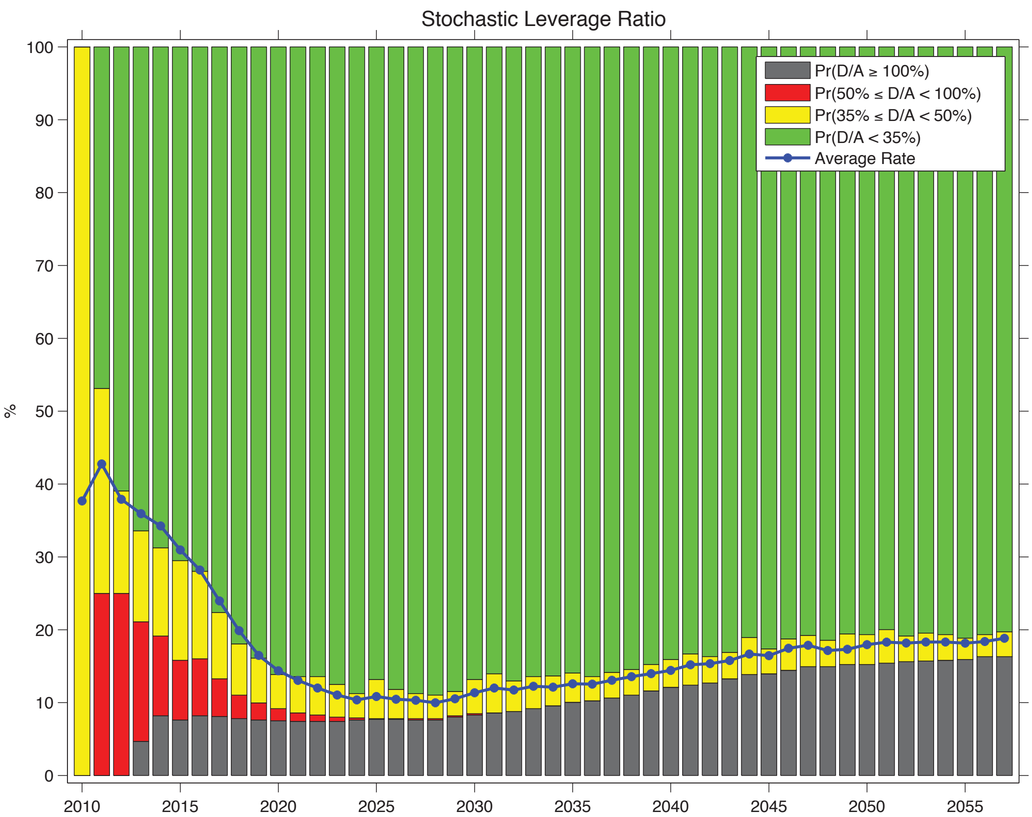


Figura 4.8: Stochastic Leverage Ratio - Case 2