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A Linear Stochastic Programming Model for Optimal Leveraged Portfolio Selection

The portfolio selection theory has an extensive literature exploring different objective functions and constraints to represent the well known risk-return trade-off presented by Markowitz in (MARKOWITZ, 1952). In particular, many works have focused in studying risk measures (see (ARTZNER et al., 1999; ROCKAFELLAR; URYASEV, 2000; ROCKAFELLAR; URYASEV, 2002; Bion-Nadal, 2008; DETLEFSEN; SCANDOLO, 2005; RIEDEL, 2004; CHERIDITO; DELBAEN; KUPPER, 2006; ROORDA; SCHUMACHER, 2007; KOVACEVIC; PFLUG, 2009)) and their consequences to the optimal investment decisions (see (SHAPIRO, 2009; RUDLOFF; STREET; VALLADÃO, 2011)). However, little attention is given to loan modeling, in particular to borrowing costs and credit limits. In practice, there is a finite number of lenders and each one offers a limited amount of money for a fixed borrowing rate greater than the risk free interest rate due to the credit risk involved.

In static models, it is common to assume a fixed borrowing rate such as in the classical mean-variance approach. Indeed, (MARKOWITZ, 1952) assumes an unbounded credit limit and risk-free borrowing rate allowing a short selling position in the risk free asset. A more complex, but still unrealistic, assumption is to define the borrowing rate as the risk free one plus a *fixed* positive risk premium, as we see in many dynamic models for asset and liability management (see (ZIEMBA; MULVEY, 1998; CARINO; ZIEMBA, 1998; KOUWENBERG, 2001; MULVEY; SHETTY, 2004; HILLI et al., 2007; BIRGE; LOUVEAUX, 1997)) and debt management (see (BALIBEK; MURAT, 2009; CONSIGLIO; STAINO, 2010; DATE P., 2011)).

In this work, we propose a portfolio and leverage selection optimization model with a piecewise linear borrowing cost function with the purpose of representing multiple lenders as in practice.

This chapter is organized as follows: In section 3.1, we develop a two-stage stochastic programming model with the proposed cost function. In section 3.2, we motivate our modeling choice with a numerical example showing the practical consequences of sub-optimal decisions obtained using the usual linear borrowing cost approximations. Moreover, we develop in section 3.3 a multistage extension with a proportional credit limit and argue that, for stage-wise independent returns and proportional credit limits (see (BLOMVALL; SHAPIRO, 2006) for details), the optimal policy is myopic and the first stage decision can be obtained by solving the proposed two-stage model on section 3.1. Moreover, we show that proportional credit limits are equivalent to upper bounds, imposed by each lender, on the incremental leverage ratio of the borrower.

3.1 The optimization model

In this section, we describe a model that optimizes the expected portfolio return under a risk constraint. At the beginning of the period ($t = 0$) an investor has an initial wealth W_0 and wants to determine his asset allocation, $\mathbf{x} = (x_1, \ldots, x_N)$, and the amount borrowed, d. Let us denote $\mathcal{X}(W_0)$ the set of all feasible strategies, $R(\mathbf{x}, d)$ the stochastic portfolio return, $\mathbb{E}[.]$ the unconditional expectation and $\mathbb{D}[.]$ a deviation measure. Then, we define the following problem:

$$\max_{(\mathbf{x},d)\in\mathcal{X}(W_0)} \{\mathbb{E}\left[R(\mathbf{x}, d)\right] \mid \mathbb{D}\left[R(\mathbf{x}, d)\right] \le \nu\} \tag{3-1}$$

where ν is a risk averse parameter.

The set of all feasible strategies depending on the initial wealth is defined as

$$\mathcal{X}(W_0) = \left\{ (\mathbf{x}, d) \in \mathbb{R}_+^{N+1} \middle| \sum_{i=1}^{N} x_i - d = W_0 \right\}.$$

In addition, we consider the probability space $(\Omega, \mathcal{F}, P)$ and define the stochastic portfolio return $R(\mathbf{x}, d)$ for each uncertainty realization ω as

$$R(\mathbf{x}, d)(\omega) = \frac{W_1(\mathbf{x}, d)(\omega) - W_0}{W_0}, \quad \forall \omega \in \Omega. \tag{3-2}$$

where $W_1(\mathbf{x}, d)$ is the stochastic terminal wealth depending on the asset and debt allocation.

After that, let us denote r_i to be the stochastic return of asset i, $\forall i = 1, \ldots, N$ and then define the terminal wealth $W_1(\mathbf{x}, d)$ as

$$W_1(\mathbf{x}, d)(\omega) = \sum_{i=1}^{N} \big(1 + r_i(\omega)\big)x_i - f(d), \quad \forall \omega \in \Omega,$$

where $f : \mathbb{R} \to \mathbb{R}$ is the borrowing cost function.

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Usually, past works have assumed a borrowing cost function with a fixed rate, nonetheless in practice, one has to choose among a finite number of lenders and each one of them has a different borrowing rate and credit limit. This model approximate the actual borrowing cost by a linear function such as $f(d) = \left(1 + \overline{rd}\right) d$, where $\overline{rd}$ would be a "representative" borrowing rate. But how should we choose this rate? And given a certain rate, how far are we from the actual optimal solution?

For the purpose of avoiding sub-optimality, we model the borrowing cost function exactly as in practice. For a given amount d, a borrower minimizes his cost accounting for all available lenders. Let us consider K lenders and denote δ_k the amount borrowed from a lender k whose rate and credit limit are given rd_k and $\bar{\delta}_k$, respectively. Then, we denote $\delta = (\delta_1, \ldots, \delta_K)$ and $\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_K)$, and define the cost function as

$$f(d) = \min_{\delta} \left\{ \sum_{k=1}^{K} (1 + rd_k)\, \delta_k \;\middle|\; \sum_{k=1}^{K} \delta_k = d;\; 0 \le \delta \le \bar{\delta} \right\}. \tag{3-3}$$

The cost function $f(d)$ proposed in (3-3) is convex and piecewise linear as illustrated in Figure 3.1 for $K = 3$ and $rd_1 \le rd_2 \le rd_3$. Indeed, this problem can be interpreted as a continuous knapsack problem where the solution of (3-3) has straightforward intuition of borrowing as much as you can from the cheapest lender. For instance, let $rd_1 \le \ldots \le rd_K$ and $\sum_{k=1}^{i} \bar{\delta}_k \le d \le \sum_{k=1}^{i+1} \bar{\delta}_k$ with $1 \le i < K$, the optimal solution is

$$\delta_k^* = \begin{cases} \bar{\delta}_k, & \forall k \le i \\ d - \sum_{k=1}^{i} \bar{\delta}_k, & k = i+1 \\ 0, & \forall k > i+1. \end{cases}$$

From this definition, one can see that the slopes and segment sizes can be respectively interpreted as unit borrowing costs and credit limits of K different lenders.

By virtue of solving (3-1) efficiently using (3-3), we need to rewrite it as a linear stochastic programming model. Then, we must redefine the feasible set, the terminal wealth and the portfolio return as

$$\bar{\mathcal{X}}(W_0) = \left\{ (\mathbf{x}, \delta) \in \mathbb{R}_+^{N+K} \;\middle|\; \sum_{i=1}^{N} x_i - \sum_{k=1}^{N} \delta_k = W_0,\; \delta \le \bar{\delta} \right\},$$

$$\bar{W}_1(\mathbf{x}, \delta)(\omega) = \sum_{i=1}^{N} (1 + r_i(\omega))\, x_i - \sum_{k=1}^{K} (1 + rd_k)\, \delta_k, \quad \forall \omega \in \Omega,$$

$$\bar{R}(\mathbf{x}, \delta)(\omega) = \frac{W_1(\mathbf{x}, \delta)(\omega) - W_0}{W_0}, \quad \forall \omega \in \Omega.$$

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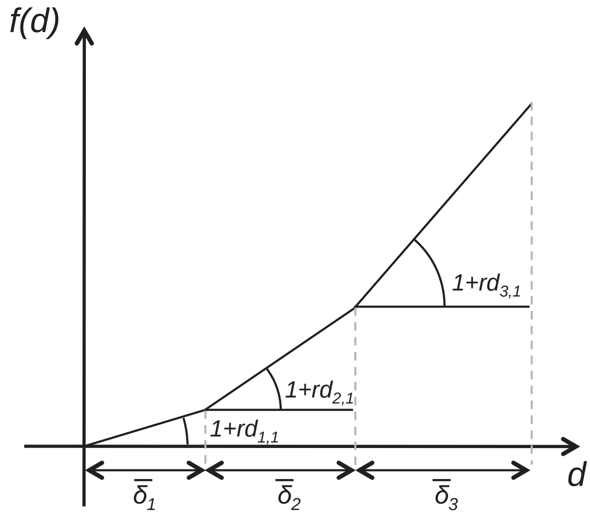

Figura 3.1: Borrowing cost function

Then, we can finally define the equivalent problem as

$$\max_{(\mathbf{x},\delta)\in\bar{\mathcal{X}}(W_0)} \left\{\mathbb{E}\left[\bar{R}(\mathbf{x},\delta)\right] \,\middle|\, \mathbb{D}\left(\bar{R}(\mathbf{x},\delta)\right) \leq \nu\right\}. \quad (3\text{-}4)$$

Now, to have a full description of our problem, we still need to choose a deviation measure. In this paper, we choose a CVaR based deviation measure defined in (ROCKAFELLAR; URYASEV; ZABARANKIN, 2006) and illustrate in Figure 3.1. We choose the CVaR deviation since it is a coherent risk measure (see (ARTZNER et al., 1999)) with a suitable economic interpretation (see (STREET, 2009; BEN-TAL; TEBOULLE, 2007)) that can be written as a linear stochastic programming problem as in (ROCKAFELLAR; URYASEV, 2000). Letting $R = \bar{R}(\mathbf{x},\delta)$ and $W_1 = \bar{W}_1(\mathbf{x},\delta)$, we define

$$\mathbb{D}(R) = \mathbb{E}[R] - \phi_\alpha(R)$$

where $\phi_\alpha(R) = -CVaR_\alpha(R) = \sup_z \left\{z - (1-\alpha)^{-1}\,\mathbb{E}\left[(R-z)^-\right]\right\}$ and α is the significance level of the CVaR.

Note that maximizing the expected portfolio return is equivalent to maximizing the expected terminal wealth since W_0 is a constant and $\mathbb{E}[R] = (\mathbb{E}[W_1] - W_0)/W_0$. Note also that $\mathbb{D}(R) = \mathbb{D}(W_1)/W_0$ since $\mathbb{E}[.]$ and $\phi(.)$ are both positively homogeneous and translation invariant, see (STREET, 2009) for details. Then, we can write the following equivalent stochastic programming model

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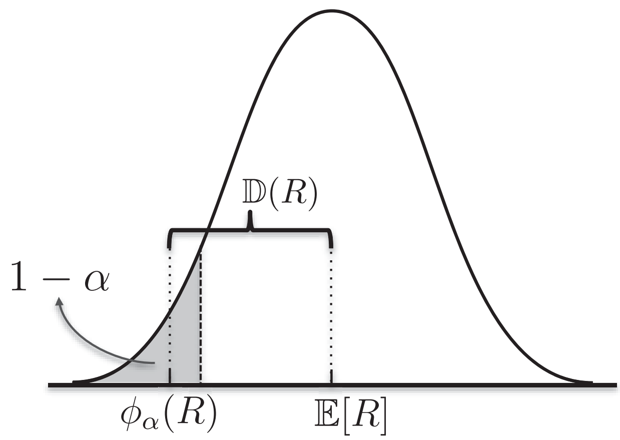

Figura 3.2: CVaR based deviation measure

$$
\begin{aligned}
\underset{W_1,\mathbf{x},\delta,z}{\text{maximize}} \quad & \mathbb{E}\left[W_1\right] \\
\text{subject to} \quad & \textstyle\sum_{i=1}^{N} x_i - \sum_{k=1}^{K} \delta_k = W_0 \\
& W_1 - \textstyle\sum_{i=1}^{N} \left(1+r_i\right) x_i + \sum_{k=1}^{K} \left(1+rd_k\right) \delta_k = 0 \\
& \mathbb{E}\left[W_1 - \left(z - (1-\alpha)^{-1} \left(W_1 - z\right)^{-}\right)\right] \leq \nu W_0 \\
& \delta \leq \bar{\delta} \\
& \mathbf{x}, \delta \geq 0.
\end{aligned}
\tag{3-5}
$$

Usually in a continuous distribution case, one would use Monte Carlo simulation to approximate the *true problem* by its sample average approximation (SAA), see for instance (PAGNONCELLI; AHMED; SHAPIRO, 2009) and (KLEYWEGT; SHAPIRO; MELLO, 2002). It is well known that the optimal value of the SAA problem is a consistent estimator of the "true problem". Said so, let us solve the proposed model for a numerical example and show how bad it would be the possible fixed borrowing rate approximations.

For a discrete distribution, which embodies the SAA problem, we can write the deterministic equivalent linear program as follows:

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$$
\begin{aligned}
\underset{W_1,\mathbf{x},\delta,z,q}{\text{maximize}} \quad & \textstyle\sum_{\omega\in\Omega} p_\omega W_1(\omega) \\
\text{subject to} \quad & \textstyle\sum_{i=1}^{N} x_i - \sum_{k=1}^{N} \delta_k = W_0 \\
& W_1(\omega) - \textstyle\sum_{i=1}^{N} (1 + r_i(\omega))\, x_i + \sum_{k=1}^{K} (1 + rd_k)\, \delta_k = 0, \quad \forall\omega \in \Omega \\
& \textstyle\sum_{\omega\in\Omega} p_\omega \left[W_1(\omega) - (z - (1-\alpha)^{-1} q(\omega))\right] \leq \nu W_0 \\
& q(\omega) \geq z - W_1(\omega), \quad \forall\omega \in \Omega \\
& q(\omega) \geq 0, \quad \forall\omega \in \Omega \\
& \delta \leq \bar{\delta} \\
& \mathbf{x}, \delta \geq 0,
\end{aligned}
\tag{3-6}
$$

where p_ω is the probability of scenario $\omega \in \Omega$.

Note that the linear approximation approach solves problem (3-6) for $K = 1$ using $rd_1 = \overline{rd}$ which is a "representative" borrowing rate and for a certain credit limit. The approximated problem is defined as

$$
\begin{aligned}
\underset{W_1,\mathbf{x},\delta_1,z,q}{\text{maximize}} \quad & \textstyle\sum_{\omega\in\Omega} p_\omega W_1(\omega) \\
\text{subject to} \quad & \textstyle\sum_{i=1}^{N} x_i - \delta_1 = W_0 \\
& W_1(\omega) - \textstyle\sum_{i=1}^{N} (1 + r_i(\omega))\, x_i + \left(1 + \overline{rd}\right) \delta_1 = 0, \quad \forall\omega \in \Omega \\
& \textstyle\sum_{\omega\in\Omega} p_\omega \left[W_1(\omega) - (z - (1-\alpha)^{-1} q(\omega))\right] \leq \nu W_0 \\
& q(\omega) \geq z - W_1(\omega), \quad \forall\omega \in \Omega \\
& q(\omega) \geq 0, \quad \forall\omega \in \Omega \\
& \delta_1 \leq \bar{\delta}_1 \\
& \mathbf{x}, \delta \geq 0.
\end{aligned}
\tag{3-7}
$$

The model proposed in (3-6) reflects the actual situation of leveraged investment decision process while (3-7) is only a approximation that leads to a suboptimal strategy. Even though (3-7) is smaller optimization problem than (3-6), we argue that the complexity of problems (3-6) and (3-7) are almost the same since, for practical applications, K is much smaller than the usual number of scenarios S. Thus, our model generates the *actual* optimal solution with very low extra computational cost when compared to the pre-existing alternatives. We illustrate this advantage through a numerical example.

3.2

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Numerical example

Let us consider $N = 4$ assets, where the first one is risk free with $r_1(\omega) = 0$, $\forall \omega \in \Omega$. In addition, we assume that the remaining assets have log excess returns that follow a multivariate normal with mean and covariance matrix estimated with historical data. The three risky assets are investments in large, medium and small companies represented by the indexes S&P 500, S&P Midcap 400 and S&P Smallcap 600, respectively. We used a monthly database starting from February 1990 until December 2010, summing up 251 samples. The mean and covariance matrix estimates are

$$\mu = \begin{bmatrix} 0.003334853 \\ 0.007157464 \\ 0.006317372 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 0.001899971 & 0.001980483 & 0.001900386 \\ 0.001980483 & 0.002552800 & 0.002563507 \\ 0.001900386 & 0.002563507 & 0.002993681 \end{bmatrix}.$$

Let us assume $K = 3$ lenders and their monthly borrowing rates and credit limits are given by Table 3.1.

	Borrowing Rate	Credit Limit
Lender 1	0.10%	25% of W_0
Lender 2	0.25%	25% of W_0
Lender 3	0.50%	50% of W_0

Tabela 3.1: Borrowing Rates and Credit Lines

To solve the numerical example, we sample $S = 1000$ scenario via Monte Carlo simulation and formulate (3-6) with $\Omega = \{1, \ldots, S\}$. Table 3.2 gives the descriptive statistics for the simulated (arithmetic) returns.

Asset	1	2	3	4
Mean	0.00%	0.93%	0.79%	0.56%
StdDev	0.00%	7.64%	3.50%	2.02%
V@R$_\alpha$	0.00%	11.11%	4.96%	2.68%
CV@R$_\alpha$	0.00%	14.74%	6.35%	3.59%
Max	0.00%	23.04%	12.04%	6.98%
Min	0.00%	-27.29%	-10.53%	-5.29%

Tabela 3.2: Asset returns - Descriptive Statistics for $\alpha = 95\%$.

For comparison purposes, let us solve 4 models where the first three are fixed-rate approximations and the forth is our modeling choice with a piecewise linear cost function. The first approximation is to assume the borrowing rate as the cheapest one. The second approximation is to assume an weighted average based on the credit limits. Finally, the third one is to assume the most expensive rate. For all these approximations, we assume a 100% leverage limit which means a credit limit equal to W_0. The initial wealth is assumed to be $W_0 = 1$ without loss of generality.

Formally speaking, to obtain the optimal solution of model 1, 2 and 3 we solve (3-7) for $\bar{\delta}_1 = W_0$. In particular, we have for model 1: $\overline{rd} = 0.0010$, for model 2: $\overline{rd} = 0.25 \cdot 0.001 + 0.25 \cdot 0.0025 + 0.5 \cdot 0.005$ and for model 3: $\overline{rd} = 0.005$.

Moreover, for model 4 we solve (3-6) for $K = 3$ and

$$\bar{\delta} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix} W_0, \quad \mathbf{rd} = \begin{bmatrix} 0.0010 \\ 0.0025 \\ 0.0050 \end{bmatrix}. \tag{3-8}$$

where $\mathbf{rd} = (rd_1, rd_2, rd_3)$.

We obtain the optimal solution of each model and for different values of the risk parameter ν. Then, given the optimal strategies we evaluate the portfolio return $R = R(\mathbf{x}, d)$ considering the cost function described by (3-3) for the actual parameters as in (3-8). We compare the risk-return trade-off ($\mathbb{D}[R]$ vs $\mathbb{E}[R]$) of each approximation to the efficient frontier given by our model. This comparison is given by Figure 3.3.

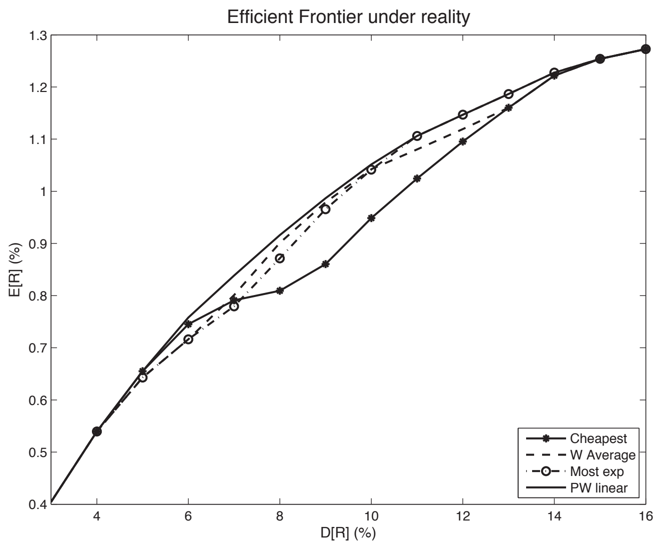

Figura 3.3: Efficient Frontier under the actual cost function.

Model 1 - Cheapest: $\overline{rd} = 0.0010$; Model 2 - W. Average: $\overline{rd} = 0.25 \cdot 0.001 + 0.25 \cdot 0.0025 + 0.5 \cdot 0.005$; Model 3 - Most exp.: $\overline{rd} = 0.005$; Model 4 - PW linear: $\bar{\delta} = W_0 \cdot (0.25, 0.25, 0.50)'$ and $\mathbf{rd} = (0.0010, 0.0025, 0.0050)'$.

We can see that all approximations are equally good when the agent is too risk averse (small ν) or almost risk neutral (large ν). However, for the

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central region ($4\% \leq \mathbb{D}(R) \leq 15\%$), the approximations are worse than the piecewise linear model. This means that, for the same level of risk, choosing any approximation for the borrowing cost function would give a lower expected return of the portfolio in comparison to our model. Moreover, none of the approximations perform better than the others for all values of ν.

For instance, the optimal strategies for $\nu = 5\%$ are given by in Table 3.3. We argue, using Figure 3.3 and Table 3.3, that the cheapest approximation is the best proxy and it has the same solution as the piecewise linear as opposed to the weighted average and the most expensive ones. This happens because for this value of ν we have a high risk aversion and it is only worthwhile to borrow with a cheap rate, e.g., the cheapest proxy and the first segment of the piecewise linear. So, one could conclude that the cheapest approximation is a good choice and our model does not have a significant improvement in comparison to the existent literature.

Model	x(1)	x(2)	x(3)	x(4)	d
Cheapest	0.00	0.00	0.10	1.06	0.16
Weighted Average	0.00	0.00	0.36	0.64	0.00
Most expensive	0.00	0.00	0.36	0.64	0.00
Piecewise linear	0.00	0.00	0.10	1.06	0.16

Tabela 3.3: Optimal Solutions for $\nu = 5\%$

However, note in Table 3.4 that we have completely different solutions for $\nu = 10\%$. In this case, we argue that the weighted average and the most expensive are better approximations, see Figure 3.3, than the cheapest one whose strategies are completely different in comparison to our model.

Model	x(1)	x(2)	x(3)	x(4)	d
Cheapest	0.00	0.00	0.73	1.27	1.00
Weighted Average	0.00	0.00	1.26	0.29	0.55
Most expensive	0.00	0.00	1.40	0.00	0.40
Piecewise linear	0.00	0.00	1.31	0.19	0.50

Tabela 3.4: Optimal Solutions for $\nu = 10\%$

Hence, using a linear borrowing cost approximation is not sufficient to represent the complexity of the borrowing cost function in a leveraged investment decision process. Indeed, it is not possible to choose a "representative" borrowing rate that consistently approximates the optimal strategy. For this reason we argue that our model must be used instead of these proxies because it is still tractable and guarantees optimality of the decisions.

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3.3 Multistage extension

Let us assume a multistage setting with a finite planning horizon T, where $\mathcal{H} = \{0, \ldots, T-1\}$. We also consider the stochastic process $\mathbf{r_t}(\omega)$ and the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a related filtration $\mathcal{F}_0 \subseteq \ldots \subseteq \mathcal{F}_T$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$. Then, we denote by $r_{i,t}$ the $\mathcal{F}_t$-adapted stochastic process for the return of asset $i \in \{1, \ldots, N\}$. Moreover, we also develop the following notation extension:

- W_t: wealth at time t.
- $x_{i,t}$: amount invested in asset $i \in \{1, \ldots, N\}$ at time t.
- $\delta_{k,t}$: amount borrowed from lender $k \in \{1, \ldots, K\}$ at time t.
- $\bar{\delta}_{k,t}$: credit limit of lender $k \in \{1, \ldots, K\}$ at time t.

For the multistage extension, we assume that the credit limit of each lender is a proportion of the current wealth, i.e.,

$$\bar{\delta}_{k,t} = \bar{\gamma}_k W_t, \quad \forall k \in \{1, \ldots, K\},\, t \in \{1, \ldots, T\}.$$

Note that this assumption is equivalent to a fixed upper bound, given by each lender, on the leverage ratio increment. Let the leverage ratio increment of lender k be $\gamma_{k,t} = \delta_{k,t}/W_t$. Then, $\gamma_{k,t} \leq \bar{\gamma}_k$ is equivalent to $\delta_{k,t} \leq \bar{\gamma}_k W_t$, since $W_t \geq 0$. We argue that it is a reasonable assumption since, in practice, credit limit do depend on the current wealth of the borrower.

Then, we define a dynamic stochastic programming model where the value function $V_t(W_t)$ for $t = T-1$ is defined as follows:

$$\begin{aligned}
\underset{\mathbf{x_{T-1}}, \boldsymbol{\delta}_{\mathbf{T-1}}, z}{\text{maximize}} \quad & \mathbb{E}\left[\sum_{i=1}^{N} (1 + r_{i,T})\, x_{i,T-1} - \sum_{k=1}^{K} (1 + rd_k)\, \delta_{k,T-1} \,\middle|\, \mathcal{F}_{T-1}\right] \\
\text{subject to} \quad & \sum_{i=1}^{N} x_{i,T-1} - \sum_{k=1}^{K} \delta_{k,T-1} = W_{T-1} \\
& \mathbb{E}\left[W_T - \left(z - (1-\alpha)^{-1} (W_T - z)^{-}\right)\right] \leq \nu W_{T-1} \\
& \delta_{T-1} \leq \bar{\gamma}\, W_{T-1} \\
& \mathbf{x_t}, \boldsymbol{\delta}_\mathbf{t} \geq 0.
\end{aligned}$$

For $t \in \mathcal{H}$, $V_t(W_t)$ is defined as

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$$
\begin{aligned}
\underset{\mathbf{x_t},\delta_\mathbf{T},z}{\text{maximize}} \quad & \mathbb{E}\left[V_{t+1}\left(\textstyle\sum_{i=1}^{N}\left(1+r_{i,t+1}\right)x_{i,t}-\sum_{k=1}^{K}\left(1+rd_k\right)\delta_{k,t}\right)\middle|\mathcal{F}_t\right] \\
\text{subject to} \quad & \textstyle\sum_{i=1}^{N} x_{i,t}-\sum_{k=1}^{K}\delta_{k,t}=W_t \\
& \mathbb{E}\left[W_{t+1}-\left(z-(1-\alpha)^{-1}\left(W_{t+1}-z\right)^{-}\right)\right]\leq \nu W_t \\
& \delta_t \leq \bar{\gamma}\, W_t \\
& \mathbf{x_t}, \delta_\mathbf{t} \geq 0,
\end{aligned}
\tag{3-9}
$$

For stage-wise independent returns and homogeneous feasible sets, we argue that this dynamic problem is easily solved since it has a myopic optimal policy as described in (BLOMVALL; SHAPIRO, 2006). Indeed, the optimal solution of the two-stage problem (3-5) is also the optimal for its multistage extension (3-9). It is worth mentioning that, even though we use the CVaR in a dynamic setting, our model generates time consistent optimal policies, see (RUDLOFF; STREET; VALLADÃO, 2011; SHAPIRO, 2009) for details.

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