2
Time consistency and risk averse dynamic decision models: Definition, interpretation and practical consequences

In a stochastic programming context, the Conditional Value at Risk (CVaR) became one of the most widely used risk measures for three reasons: first, it is a coherent risk measure (see (ARTZNER et al., 1999)); second, it has a clear and suitable economic interpretation (see (ROCKAFELLAR; URYASEV, 2000) and (STREET, 2009)); and last, but not least, it can be written as a linear stochastic programming model as shown in (ROCKAFELLAR; URYASEV, 2000). For these three reasons, the CVaR has been applied to static and even to dynamic models. However, to choose a coherent risk measure as objective function of a dynamic model is not a sufficient condition to obtain suitable optimal policies. In the recent literature, time consistency is shown to be one basic requirement to get suitable optimal decisions, in particular for multistage stochastic programming models. Papers on time consistency are actually divided in two different approaches: the first one focuses on risk measures and the second one on optimal policies.

The first approach states that, in a dynamic setting, if some random payoff A is always riskier than a payoff B conditioned to a given time $t+1$, than A should be riskier than B conditioned to t. It is well known that this property is achieved using a recursive setting, leading to so called time consistent dynamic risk measures proposed by various authors, e.g., (Bion-Nadal, 2008; DETLEFSEN; SCANDOLO, 2005; RIEDEL, 2004; CHERIDITO; DELBAEN; KUPPER, 2006; ROORDA; SCHUMACHER, 2007; KOVACEVIC; PFLUG, 2009). Other weaker definitions, like acceptance and rejection consistency, are also developed in these works (see (CHERIDITO; DELBAEN; KUPPER, 2006; KOVACEVIC; PFLUG, 2009) for details).

The second approach, formally defined by (SHAPIRO, 2009), is on time consistency of optimal policies in multistage stochastic programming models. The interpretation of this property given by the author is the following: "at every state of the system, our optimal decisions should not depend on scenarios which we already know cannot happen in the future". This interpretation is an indirect consequence of solving a sequence of problems whose objective functions can be written recursively as in the formerly cited time consistent dynamic risk measures. It is shown in (SHAPIRO, 2009) for instance that if, for every state of the system, we want to minimize the CVaR of a given quantity at the end of the planning horizon, we would obtain a time *inconsistent* optimal policy. Indeed, this sequence of problems does not have recursive objective functions and the optimal decisions at particular future states might depend on scenarios that "we already know cannot happen in the future". However, if for $t = 0$ we want to minimize the CVaR of a given quantity at the end of the planning horizon and for $t > 0$ we actually follow the dynamic equations of the first stage problem, then we obtain a time *consistent* optimal policy even though it depends on those scenarios we already know cannot happen. On the other hand, one can argue that this policy is not reasonable because for $t > 0$ the objective function has no economic interpretation.

In this paper, we use a direct interpretation for time consistency of optimal policies based on its formal definition. We actually state that *a policy is time consistent if and only if the future planned decisions are actually going to be implemented*. In the literature, time inconsistent optimal policies have been commonly proposed, in particular (BÄUERLE; MUNDT, 2009) at section 3 and 4.1 and (FÁBIÁN; VESZPRÉMI, 2008) have developed portfolio selection models using CVaR in a time inconsistent way. In our work, we show with a numerical example that a time inconsistent CVaR based portfolio selection model can lead to a suboptimal sequence of implemented decisions and may not take risk aversion into account at some intermediate states of the system. We propose a time consistent alternative with a recursive objective function and compare its optimal policy to the time inconsistent one. Other alternatives have been proposed by (BODA; FILAR, 2006) and (CUOCO; HE; ISSAENKO, 2008), however none of them used the recursive set up of time consistent dynamic risk measures. Since the lack of a suitable economic interpretation for this recursive set up is one of the main reasons why it is not commonly proposed, we prove for a more general set of problems that this objective function is the certainty equivalent w.r.t. the time consistent dynamic utility defined as the composed form of one period preference functionals. We show that our application fits into this general set of problems and develop the interpretation for the numerical example.

2.0.1 Assumptions and notation

In this paper, we assume a multistage setting with a finite planning horizon T. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a related filtration

$\mathcal{F}_0 \subseteq \ldots \subseteq \mathcal{F}_T$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$.

Since our application is on portfolio selection, we use a unique notation for all models developed here. This section includes definition of sets, stochastic processes, decision and state variables.

Let us define the set of assets, $\mathcal{A} = \{1, \ldots, A\}$, the set stages, $\mathcal{H} = \{0, \ldots, T-1\}$, and the set of stages starting from τ, $\mathcal{H}(\tau) = \{\tau, \ldots, T-1\}$, $\forall \tau \in \mathcal{H}$. In addition, we define the excess return of asset $i \in \mathcal{A}$, between stages $t \in \{1, \ldots, T\}$ and $t-1$, under scenario $\omega \in \Omega$, as the stochastic process $r_{i,t}(\omega)$ where we denote $\mathbf{r_t}(\omega) = (r_{1,t}(\omega), \ldots, r_{A,t}(\omega))'$ and, for $s \leq t$, $\mathbf{r_{[s,t]}}(\omega) = (\mathbf{r_s}(\omega), \ldots, \mathbf{r_t}(\omega))'$.

Let us also denote the state variable $W_t(\omega)$ to be the wealth at stage $t \in \mathcal{H} \cup \{T\}$ under scenario $\omega \in \Omega$ and the decision variable $x_{i,t}(\omega)$ to be the amount invested in asset $i \in \mathcal{A}$, at stage $t \in \mathcal{H}$ under scenario $\omega \in \Omega$ where $\mathbf{x_t}(\omega) = (x_{1,t}(\omega), \ldots, x_{A,t}(\omega))'$ and, for $s \leq t$, $\mathbf{x_{[s,t]}}(\omega) = (\mathbf{x_s}(\omega), \ldots, \mathbf{x_t}(\omega))'$.

Without loss of generality, we assume that there is a risk free asset, indexed by $i = 1$, with null excess return for each state of the system, i.e., $r_{1,t}(\omega) = 0$, $\forall t \in \mathcal{H} \cup \{T\}, \omega \in \Omega$. Moreover, we assume that W_t, $r_{i,t}$, $x_{i,t} \in L^\infty(\mathcal{F}_t)$, $\forall t \in \mathcal{H} \cup \{T\}$.

Let W be a $\mathcal{F}$ measurable function and consider a realization sequence $\bar{\mathbf{r}}_{[1,t]} = (\bar{\mathbf{r}}_1, \ldots, \bar{\mathbf{r}}_t)'$ of the asset returns. Then, we denote the conditional and unconditional expectations by $\mathbb{E}\left[W \mid \bar{\mathbf{r}}_{[1,t]}\right] = \mathbb{E}\left[W \mid \mathbf{r}_{[1,t]} = \bar{\mathbf{r}}_{[1,t]}\right]$ and $\mathbb{E}[W]$, respectively.

We also use the negative of the CVaR developed by (ROCKAFELLAR; URYASEV, 2000) as an "acceptability" measure (see (KOVACEVIC; PFLUG, 2009) for details) whose conditional and unconditional formulations are defined respectively as

$$\phi_t^\alpha\left(W, \bar{\mathbf{r}}_{[1,t]}\right) = -CVaR_\alpha\left(W \mid \bar{\mathbf{r}}_{[1,t]}\right) = \sup_{z \in \mathbb{R}} \left\{ z - \frac{\mathbb{E}\left[(W-z)^- \mid \bar{\mathbf{r}}_{[1,t]}\right]}{1-\alpha} \right\} \tag{2-1}$$

and

$$\phi_0^\alpha(W) = -CVaR_\alpha(W) = \sup_{z \in \mathbb{R}} \left\{ z - \frac{\mathbb{E}\left[(W-z)^-\right]}{1-\alpha} \right\},$$

where $x^- = -\min(x, 0)$.

Note that, $\mathbb{E}\left[\cdot \mid \bar{\mathbf{r}}_{[1,t]}\right], \mathbb{E}[\cdot], \phi_t^\alpha\left(\cdot, \bar{\mathbf{r}}_{[1,t]}\right)$ and $\phi_t^\alpha(\cdot)$ are real valued functions, i.e, $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}$. It is also important to note that all constraints represented in this paper are defined for almost every $\omega \in \Omega$, in the $\mathbb{P}$ a.s. sense, that affects the objective function. For instance, if the objective function of a particular optimization problem is a conditional expectation $\mathbb{E}\left[\cdot \mid \bar{\mathbf{r}}_{[1,t]}\right]$, then the constraints of this problem are defined for almost every

$\omega \in \left\{ \bar{\omega} \in \Omega \mid \mathbf{r}_{[1,t]} \left(\bar{\omega} \right) = \bar{\mathbf{r}}_{[1,t]} \right\}.$

2.1 Experimental

The major reason for developing dynamic (multistage) models instead of static (two-stage) ones is the fact that we can incorporate the flexibility of dynamic decisions to improve our objective function. In other words, the possibility of changing a policy after the realization of some random variables increases the objective function (for a maximization problem) and allows the first stage decisions to be less conservative than their counterpart in the static case. However, it doesn't make any sense to incorporate this flexibility if the intermediate decisions are not actually going to be implemented.

As we stated before, *an optimal policy is time consistent if and only if the future planned decisions are actually going to be implemented.* Only under this property we guarantee that the flexibility and optimality of a dynamic policy will not be polluted by any spurious future planned decisions. Said so, one can even argue that the first stage decisions of a time inconsistent policy are, for practical reasons, suboptimal considering that the optimal policy would not be followed in the future.

In a multistage stochastic programming context, a policy is a sequence of decisions for each stage and for each scenario (a realization of the uncertainty). As in (SHAPIRO, 2009), one has to define which (multistage) optimization problem should be solved when the current time is a particular stage $t \in \mathcal{H}$ of the planning horizon. Said that, when the current time is $t = 0$, we solve the corresponding optimization problem and obtain the first stage optimal decision and the future planned optimal policy. This policy is time consistent if and only if these future planned decisions for each scenario are also optimal for each problem when the current time is $t > 0$.

In order to motivate this discussion, we develop a CVaR based portfolio selection model which incorporates the well known mean-risk trade-off presented by (MARKOWITZ, 1952). As a coherent risk measure, the CVaR should be a suitable way to assess risk, however we want to point out the fact that if one chooses a dynamic model, time consistency should also be take into account. Assessing risk in a time inconsistent way may lead to a time inconsistent policy and therefore to a suboptimal sequence of implemented decisions.

For an illustrative purpose, we apply the CVaR in a time inconsistent way to the portfolio selection problem and show some practical consequences of the related optimal policy.

2.1.1
Example of a time inconsistent policy

The portfolio selection problem is normally formulated to consider the mean-risk trade-off. Some models use the expected value as the objective function with a risk constraint while others minimize risk with a constraint on the expected value. In this paper, we combine these two approaches defining our objective function as a convex combination of the expected value and the acceptability measure previously stated. In other words, the investor wants to maximize its expected return and also minimize risk, given his current state. It is very important to note that the planning horizon is a fixed date in the future and, depending on the investor's current state, he / she solve a different optimization problem.

Then, we define the problem $Q_\tau\left(W_\tau, \bar{\mathbf{r}}_{[1,\tau]}\right)$ solved by the investor, given his / her current stage τ and the current realization $\bar{\mathbf{r}}_{[1,\tau]}$ of the random process, as

$$\begin{aligned}
\underset{W_{[\tau+1,T]},\mathbf{x}_{[\tau,T-1]}}{\text{maximize}} \quad & (1-\lambda)\,\mathbb{E}\left[W_T \;\middle|\; \bar{\mathbf{r}}_{[1,\tau]}\right] + \lambda\,\phi_\tau^\alpha\left(W_T, \bar{\mathbf{r}}_{[1,\tau]}\right) \\
\text{subject to} \quad & W_{t+1} = \textstyle\sum_{i\in\mathcal{A}} (1+r_{i,t+1})\,x_{i,t}, \quad \forall\, t \in \mathcal{H}(\tau) \\
& \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = W_t, \quad \forall\, t \in \mathcal{H}(\tau) \\
& \mathbf{x}_t \geq 0,
\end{aligned}$$

where $\lambda \in [0,1]$.

Using (2-1), the problem can be equivalently formulated as

$$\begin{aligned}
\underset{W_{[\tau+1,T]},\mathbf{x}_{[\tau,T-1]},z}{\text{maximize}} \quad & \mathbb{E}\left[(1-\lambda)\,W_T + \lambda\left(z - \frac{(W_T - z)^-}{1-\alpha}\right) \;\middle|\; \bar{\mathbf{r}}_{[1,\tau]}\right] \\
\text{subject to} \quad & W_{t+1} = \textstyle\sum_{i\in\mathcal{A}} (1+r_{i,t+1})\,x_{i,t}, \quad \forall\, t \in \mathcal{H}(\tau) \\
& \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = W_t, \quad \forall\, t \in \mathcal{H}(\tau) \\
& \mathbf{x}_t \geq 0.
\end{aligned}$$

Note that, the first stage problem $Q_0\,(W_0)$ is defined equivalently as follows:

$$\begin{aligned}
\underset{W_{[1,T]},\mathbf{x}_{[0,T-1]},z}{\text{maximize}} \quad & \mathbb{E}\left[(1-\lambda)\,W_T + \lambda\left(z - \frac{(W_T - z)^-}{1-\alpha}\right)\right] \\
\text{subject to} \quad & W_{t+1} = \textstyle\sum_{i\in\mathcal{A}} (1+r_{i,t+1})\,x_{i,t}, \quad \forall\, t \in \mathcal{H}(\tau) \\
& \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = W_t, \quad \forall\, t \in \mathcal{H}(\tau) \\
& \mathbf{x}_t \geq 0.
\end{aligned} \tag{2-2}$$

In order to have a numerical example, Let us assume our probability

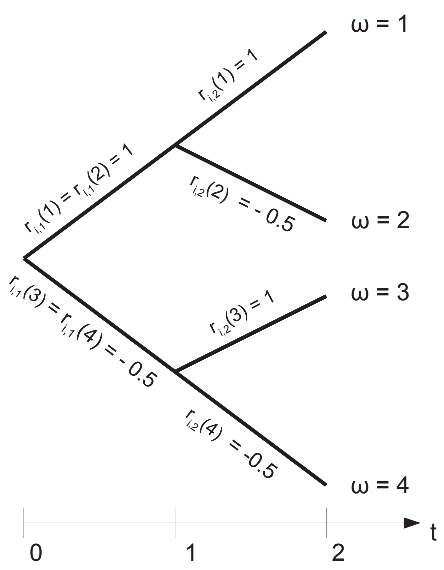

Figura 2.1: Return tree for $i = 2$

space to be represented by a discrete event tree. For instance, consider $T = 2$ and the tree represented in Figure 2.1, where the scenarios $\omega \in \Omega = \{1, 2, 3, 4\}$ are numbered by the terminal nodes. In our notation, a node is a subset of Ω, e.g., the root node is defined as $\Omega = \{1, 2, 3, 4\}$, the intermediate nodes as $\{1, 2\}$ and $\{3, 4\}$ and the terminal nodes as $\{1\}, \{2\}, \{3\}, \{4\}$. Now, let us denote $\mathcal{N}_t$ the set of nodes at stage t and $\mathcal{F}_t$ the σ-algebra generated by it. In our example, $\mathcal{N}_1 = \{\Omega\}$, $\mathcal{N}_2 = \{\{1, 2\}, \{3, 4\}\}$ and $\mathcal{N}_3 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$.

For sake of simplicity, we consider a two-asset model, i.e., $\mathcal{A} = \{1, 2\}$, and a probability measure defined as $\mathbb{P}(\omega) = 0.25$, $\forall \omega \in \Omega = \{1, 2, 3, 4\}$. The first asset indexed by $i = 1$ is risk free and it has null excess return for every state of the system, i.e, $r_{1,t}(\omega) = 0$, $\forall t \in \{1, 2\}$, $\omega \in \Omega$. The second one is assumed to have iid returns given by

$$r_{2,t}(\omega) = \begin{cases} 1, & \text{for } t = 1, \omega \in \{1, 2\} \\ -0.5, & \text{for } t = 1, \omega \in \{3, 4\} \\ 1, & \text{for } t = 2, \omega \in \{1\} \\ -0.5, & \text{for } t = 2, \omega \in \{2\} \\ 1, & \text{for } t = 2, \omega \in \{3\} \\ -0.5, & \text{for } t = 2, \omega \in \{4\}. \end{cases}$$

and graphically represented in Figure 2.1. It is straightforward to see that the risky asset has greater expected return and higher risk than the risk free one. This represents the mean-risk trade-off of a typical portfolio selection problem.

Now, we write an equivalent deterministic linear programming model for the problem $Q_0(W_0)$ defined in (2-2) assuming, without loss of generality, that $W_0 = 1$. Then we have the following:

$$\begin{aligned}
\underset{q,W_{[1,2]},\mathbf{x}_{[0,1]},z}{\text{maximize}} \quad & \frac{1}{4}\sum_{\omega=1}^{4}\left[(1-\lambda)\,W_2(\omega)+\lambda\left(z-\frac{q(\omega)}{1-\alpha}\right)\right] \\
\text{subject to} \quad & W_{t+1}(\omega)=\textstyle\sum_{i\in\mathcal{A}}\left(1+r_{i,t+1}(\omega)\right)x_{i,t}(\omega), \quad \forall\, t\in\mathcal{H},\ \omega\in\Omega \\
& \textstyle\sum_{i\in\mathcal{A}} x_{i,t}(\omega)=W_t(\omega), \quad \forall\, t\in\mathcal{H},\ \omega\in\Omega \\
& \mathbf{x}_t(\omega)\geq 0, \quad \forall t\in\mathcal{H},\ \omega\in\Omega \\
& q(\omega)\geq z-W_2(\omega), \quad \forall\omega\in\Omega \\
& q(\omega)\geq 0, \quad \forall\omega\in\Omega.
\end{aligned} \tag{2-3}$$

where $\mathbf{x}_t$ is $\mathcal{F}_t$-adapted, i.e., $\mathbf{x}_0(1) = \mathbf{x}_0(2) = \mathbf{x}_0(3) = \mathbf{x}_0(4)$, $\mathbf{x}_1(1) = \mathbf{x}_1(2)$ and $\mathbf{x}_1(3) = \mathbf{x}_1(4)$, which are the well known non-antecipativity constraints. Note that q is a $\mathcal{F}_T$-adapted auxiliar variable to represent the CVaR as developed in (ROCKAFELLAR; URYASEV, 2000).

Solving this problem for $\alpha = 95\%$ and $\lambda = 0.5$, we have the following optimal solution:

$$x^*_{1,t}(\omega)=\begin{cases}0.5, & \text{for } t=0,\omega\in\Omega\\ 0, & \forall t=1,\omega\in\{1,2\}\\ 0.75, & \forall t=1,\omega\in\{3,4\},\end{cases}$$

$$x^*_{2,t}(\omega)=\begin{cases}0.5, & \text{for } t=0,\omega\in\Omega\\ 1.5, & \forall t=1,\omega\in\{1,2\}\\ 0, & \forall t=1,\omega\in\{3,4\}.\end{cases} \tag{2-4}$$

At the root node, it is optimal to split evenly the investment, while at node $\{1,2\}$ everything is invested in the risky asset and at node {3,4} everything is invested in the risk free one.

Now, let us suppose one period has passed and the current state is at time $\tau = 1$ and at node $\{1,2\}$. Let us write an equivalent deterministic problem for $Q_1(W_1, \bar{\mathbf{r}}_1)$, for $W_1 = 1.5$ and $\bar{\mathbf{r}}_1 = (0,1)'$ as

$$\begin{aligned}
\underset{q,W_2,\mathbf{x}_1,z}{\text{maximize}} \quad & \frac{1}{2}\sum_{\omega=1}^{2}\left[(1-\lambda)\,W_2(\omega)+\lambda\left(z-\frac{q(\omega)}{1-\alpha}\right)\right] \\
\text{subject to} \quad & W_2(\omega)=\textstyle\sum_{i\in\mathcal{A}}(1+r_{i,2}(\omega))\,x_{i,1}, \quad \forall\,\omega\in\{1,2\} \\
& \textstyle\sum_{i\in\mathcal{A}} x_{i,1}=W_1 \\
& \mathbf{x}_1\geq 0 \\
& q(\omega)\geq z-W_T(\omega)\,, \quad \forall\omega\in\{1,2\} \\
& q(\omega)\geq 0, \quad \forall\omega\in\{1,2\}.
\end{aligned} \tag{2-5}$$

This problem reflects what the investor would do at $\tau=1$ and at node $\{1,2\}$ if the optimal decision $\mathbf{x}_0^*$ in (2-4) had been implemented. In other words, given $x_{1,t}^*$ and $x_{2,t}^*$ for $t=0$, the optimal solution of (2-5) is the decision implemented at $\tau=1$ and at node $\{1,2\}$ of an agent that maximizes the chosen acceptability measure of terminal wealth.

We want to show that the optimal solutions for this problem at node $\{1,2\}$ are different from the ones in (2-4), meaning that at $t=0$ the future planned decisions for $t=1$ are different from the ones that are actually going to be implemented. It is also important to understand why it happens and what kind of error a investor would do with this time inconsistent policy. The optimal solution of (2-5) is given by the following:

$$\begin{aligned}
\tilde{x}_{1,t}^*(\omega) &= 1.5, \quad \forall t=1,\omega\in\{1,2\}, \\
\tilde{x}_{2,t}^*(\omega) &= 0, \quad \forall t=1,\omega\in\{1,2\}.
\end{aligned} \tag{2-6}$$

The optimal planned strategy at node $\{1,2\}$ obtained by solving (2-3) is to invest everything in the risky asset, while the solution of problem (2-5) (the one that is actually going to be implemented) is to invest everything in the risk free asset (see equation (2-6)). This happens because, in problem (2-3), the $CVaR_{95\%}$ is the worst case loss at scenario $\omega=4$ given by $-W_2(4)$. Then, at node $\{1,2\}$, it is optimal for first stage problem to choose the investment strategy with the highest expected return since this decision will not affect the terminal wealth at scenario $\omega=4$.

This example points out that a time inconsistent policy may lead to a sequence of optimal decisions where a risk-averse decision maker shows a risk neutral preference at some intermediate state. In other words, risk aversion may not be taken into account at some intermediate states of the system. Therefore, we propose a time consistent alternative that has significant advantages over the time inconsistent one since it incorporates the flexibility of a dynamic decision model ensuring that the future planned decisions are actually going to be implemented.

2.1.2 Time consistent alternative

In this section, we propose an alternative to the previous time inconsistent policy. We base our formulation on (RUSZCZYNSKI; SHAPIRO, 2006) and develop dynamic equations. For $t = T-1$, we define the problem $V_{T-1}\left(W_{T-1}, \bar{\mathbf{r}}_{T-1}\right)$ as follows:

$$\begin{aligned} \underset{W_T, \mathbf{x}_{T-1}}{\text{maximize}} \quad & (1-\lambda)\, \mathbb{E}\left[W_T \;\middle|\; \bar{\mathbf{r}}_{[1,T-1]}\right] + \lambda\, \phi^{\alpha}_{T-1}\left(W_T, \bar{\mathbf{r}}_{[1,T-1]}\right) \\ \text{subject to} \quad & W_T = \textstyle\sum_{i\in\mathcal{A}} \left(1 + r_{i,T}\right) x_{i,T-1} \\ & \textstyle\sum_{i\in\mathcal{A}} x_{i,T-1} = W_{T-1} \\ & \mathbf{x}_{T-1} \geq 0. \end{aligned}$$

Using the definition of $\phi^{\alpha}_{T-1}\left(W, \bar{\mathbf{r}}_{[1,T-1]}\right)$ given in (2-1), we rewrite the problem as follows:

$$\begin{aligned} \underset{W_T, \mathbf{x}_{T-1}, z}{\text{maximize}} \quad & \mathbb{E}\left[(1-\lambda)\, W_T + \lambda \left(z - \frac{\left(W_T - z\right)^-}{1-\alpha}\right) \;\middle|\; \bar{\mathbf{r}}_{[1,T-1]}\right] \\ \text{subject to} \quad & W_T = \textstyle\sum_{i\in\mathcal{A}} \left(1 + r_{i,T}\right) x_{i,T-1} \\ & \textstyle\sum_{i\in\mathcal{A}} x_{i,T-1} = W_{T-1} \\ & \mathbf{x}_{T-1} \geq 0. \end{aligned}$$

For the last period, our proposed model is to maximize the convex combination of the expected terminal wealth and the acceptability measure $\phi^{\alpha}_{T-1}\left(W, \bar{\mathbf{r}}_{[1,T-1]}\right)$. Now, for $t < T-1$, we propose a nested value function, based on the conditional version of the same convex combination. Then, $V_t\left(W_t, \bar{\mathbf{r}}_{[1,t]}\right)$, $\forall t = 0, \ldots, T-2$, is defined as follows:

$$\begin{aligned} \underset{W_{t+1}, \mathbf{x}_t}{\text{maximize}} \quad & (1-\lambda)\, \mathbb{E}\left[V_{t+1} \;\middle|\; \bar{\mathbf{r}}_{[1,t]}\right] + \lambda\, \phi^{\alpha}_{t}\left(V_{t+1}, \bar{\mathbf{r}}_{[1,t]}\right) \\ \text{subject to} \quad & W_{t+1} = \textstyle\sum_{i\in\mathcal{A}} \left(1 + r_{i,t+1}\right) x_{i,t} \\ & \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = W_t \\ & \mathbf{x}_t \geq 0. \end{aligned} \tag{2-7}$$

where V_{t+1} stands for $V_{t+1}\left(W_{t+1}, \mathbf{r}_{[1,t+1]}\right)$.

Equivalently to $t = T-1$, we rewrite problem (2-7) as follows:

$$\begin{aligned}
\underset{W_{t+1},\mathbf{x}_t,z}{\text{maximize}} \quad & \mathbb{E}\left[(1-\lambda)\, V_{t+1} + \lambda\left(z - \frac{(V_{t+1}-z)^-}{1-\alpha}\right) \,\middle|\, \bar{\mathbf{r}}_{[1,t]}\right] \\
\text{subject to} \quad & W_{t+1} = \textstyle\sum_{i\in\mathcal{A}} (1+r_{i,t+1})\, x_{i,t} \\
& \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = W_t \\
& \mathbf{x}_t \geq 0.
\end{aligned} \tag{2-8}$$

For comparison purposes, we solve this model for the numerical example proposed in section 2.1.1. To do so, we use the result shown in (BLOMVALL; SHAPIRO, 2006) that, for stage-wise independent returns such problem has a myopic optimal policy which is obtained as the solution of the following two-stage problem for $t \in \mathcal{H}$:

$$\begin{aligned}
\underset{W_{t+1},\mathbf{x}_t,z}{\text{maximize}} \quad & \mathbb{E}\left[(1-\lambda)\, W_{t+1} + \lambda\left(z - \frac{(W_{t+1}-z)^-}{1-\alpha}\right)\right] \\
\text{subject to} \quad & W_{t+1} = \textstyle\sum_{i\in\mathcal{A}} (1+r_{i,t+1})\, x_{i,t} \\
& \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = W_t \\
& \mathbf{x}_t \geq 0.
\end{aligned} \tag{2-9}$$

For $W_0 = 1$, the (time consistent) optimal policy obtained by solving problem (2-8) is the following:

$$x^*_{1,t}(\omega) = W_t = 1, \quad \forall t \in \mathcal{H}, \omega \in \Omega,$$

$$x^*_{2,t}(\omega) = 0, \quad \forall t \in \mathcal{H}, \omega \in \Omega.$$

The optimal policy is always to invest the total wealth in the risk free asset. Note that this strategy is more conservative compared to the time inconsistent one, because it takes risk into account at every state of the system. Since this difference may lead to a sub-optimal solution, we develop in the following section a systematic way of measuring this effect on the objective function.

2.1.3
The time inconsistent sub-optimality gap

In section 2.1.1, we show a time inconsistent example where planned and implemented policies are different. However, what we do want to know is how a time inconsistent policy impacts our objective function. The way of measuring it is computing the sub-optimality gap that concerns the disparity of the objective function we expect to obtain with our planning policy, denoted by OF_{plan}, and the one we actually get when evaluating the policy to be implemented in the future, denoted by OF_{imp}. Then, let us define

$$gap = \frac{OF_{plan} - OF_{imp}}{OF_{plan}},$$

For instance, take the portfolio selection problem defined in (2-2), in particular the numerical example in (2-3). The optimal value $Q_0(W_0)$ defines OF_{plan}, i.e.,

$$OF_{plan} = Q_0(W_0).$$

On the other hand, OF_{imp} is obtained by computing the wealth distribution $\widehat{W}_T$ at $t = T$ using the implemented decisions and then evaluate the objective function

$$OF_{imp} = \mathbb{E}\left[(1-\lambda)\,\widehat{W}_T + \lambda\left(z - \frac{\left(\widehat{W}_T - z\right)^-}{1-\alpha}\right)\right],$$

The terminal wealth $\widehat{W}_T$ is obtained by the following procedure:

for $\tau \in \mathcal{H}$, $\omega \in \Omega$: **do**

$\widehat{\mathbf{x}}_\tau = (\hat{x}_{1,\tau}, \ldots, \hat{x}_{A,\tau})' \leftarrow$ the first stage solution of $Q_\tau\left(\widehat{W}_\tau(\omega), \bar{\mathbf{r}}_{[1,\tau]}(\omega)\right)$

Compute $\widehat{W}_{\tau+1}(\omega) = \sum_{i\in\mathcal{A}}(1 + r_{i,T}(\omega))\,\hat{x}_{i,T-1}$

end for

where, for $\tau = 0$, $\widehat{W}_\tau(\omega) = W_0$ and $Q_\tau\left(\widehat{W}_\tau(\omega), \bar{\mathbf{r}}_{[1,\tau]}(\omega)\right) = Q_0(W_0)$, $\forall\omega \in \Omega$.

In this numerical example where $\lambda = 0.5$ and $T = 2$ we compute a gap of 9.09%. For completeness, we run a sensitivity analysis varying $\lambda \in \{0, 0.1, \ldots, 0.9, 1\}$ and $T \in \{2, 3, \ldots, 9, 10\}$, where the results are presented in Table 2.1. From this results we have a better assessment on how a time inconsistent policy would affect the decision process in practice.

We observe that gap can be significantly different depending on planning horizon size T and risk aversion level λ. On the one hand, we observe a zero gap for $\lambda \in \{0, 1\}$ and for all $T \in \{2, \ldots, 10\}$. For $\lambda = 0$ our results are validated by the fact that a risk neutral formulation ensures time consistency. For $\lambda = 1$, the preference function is too conservative in this particular example and, therefore, planned and implemented decisions are to invest everything in the risk free asset. On the other hand, for $\lambda \in \{0.4, 0.5, 0.6, 0.7, 0.9\}$ we observe significantly high gap values for different planning horizon sizes T. We illustrate this behavior in Figure 2.2 where we plot for each λ, the gap as a function of T. For a given λ, we observe that sub-optimality gap increases with the planning horizon until it stabilizes around a certain value. Nonetheless the gap for a smaller λ increases faster with T, the case with a larger λ stabilizes at a higher gap value.

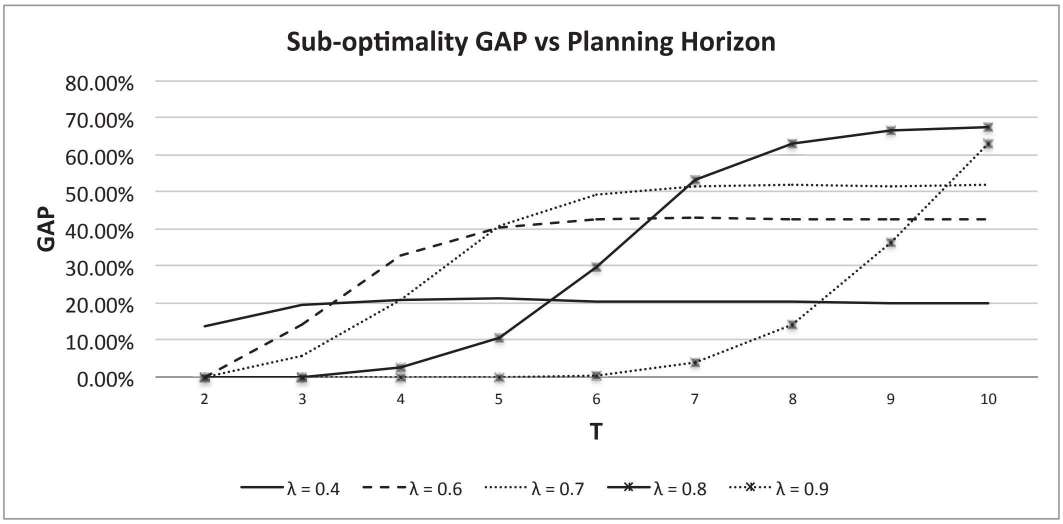

Figura 2.2: Sub-Optimality gap due to Time Inconsistency

To sum up, for the most aggressive and the most conservative allocation strategies, time inconsistency would not be a problem. On the other hand, for $0 < \lambda < 1$, time consistency do matter and can have a significant impact on the objective function. For this reason, we reassure the importance of time consistency which can be achieved by the recursive formulation illustrated in Section 2.1.2. The major problem of utilizing this formulation in practice is the difficulty to economically interpret it. Therefore, we develop a suitable economic interpretation for this recursive objective function based on certain equivalent of the related preference function.

2.2 Results and discussion

The problem of choosing the proposed recursive set up is usually the lack of a suitable economic interpretation for the objective function. How can a investor choose a policy if he / she does not know what is actually going to be optimized? For this reason, we prove that the objective function is the certainty equivalent w.r.t. the time consistent dynamic utility generated by one period preference functionals.

Let us consider a generic one period preference functional $\psi_t : L^\infty(\mathcal{F}_{t+1}) \to L^\infty(\mathcal{F}_t)$ and, for a particular realization sequence of the uncertainty $\bar{\mathbf{r}}_{[1,t]}$, the related real valued function $\psi_t\left(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\right) : L^\infty(\mathcal{F}_{t+1}) \to \mathbb{R}$. Moreover, let us define important properties and concepts of $\psi_t\left(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\right)$ used to develop our main results:

Translation invariance: $\psi_t\big(W_{t+1}+m \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big) = \psi_t\big(W_{t+1} \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big) + m$, where $W_{t+1} \in L^\infty(\mathcal{F}_{t+1})$ and $m \in \mathbb{R}$.

Monotonicity: $\psi_t\big(X_T \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big) \geq \psi_t\big(Y_T \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big)$ for all $X_T, Y_T \in L^\infty(\mathcal{F}_T)$, such that $X_T(\omega) \geq Y_T(\omega)$, $\forall \omega \in \Omega$.

Normalization: Let us also assume that $\psi_t\left(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\right)$ is normalized to zero, i.e., $\psi_t\left(0 \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\right) = 0$.

Definição 2.1 *The certainty equivalent $\tilde{C}_t(W_{t+1} \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]})$ of $W_{t+1} \in L^\infty(\mathcal{F}_{t+1})$ with respect to $\psi_t\big(W_{t+1} \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big)$ is a real value $m \in \mathbb{R}$ such that $\psi_t\big(m \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big) = \psi_t\big(W_{t+1} \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big)$.*

Let us also denote $(U_t)_{t\in\mathcal{H}}$ as the time consistent dynamic utility function generated by ψ_t (see (CHERIDITO; KUPPER, 2009) for details). Formally speaking, $U_t : L^\infty\left(\mathcal{F}_T\right) \to L^\infty\left(\mathcal{F}_t\right)$ is defined as follows:

$$U_T\left(W_T\right) = W_T \quad \text{and} \quad U_t\left(W_T\right) = \psi_t\left(U_{t+1}\left(W_T\right)\right), \forall t \in \mathcal{H},$$

where $W_T \in L^\infty\left(\mathcal{F}_T\right)$. Note that we can also use a conditional version of U_t as follows:

$$U_t\left(W_T \;\middle|\; \bar{\mathbf{r}}_{[1,t]}\right) = \psi_t\left(U_{t+1}\left(W_T\right) \;\middle|\; \bar{\mathbf{r}}_{[1,t]}\right), \forall t \in \mathcal{H}.$$

The concept of certainty equivalent is also developed for the time consistent dynamic utility U_t:

Definição 2.2 *The certainty equivalent $C_t(W_T \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]})$ of $W_T \in L^\infty(\mathcal{F}_T)$ with respect to $U_t\big(W_T \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big)$ is a real value $m \in \mathbb{R}$ such that $U_t\big(m \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big) = U_t\big(W_T \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]}\big)$.*

Now, let us define the following dynamic stochastic programming model where the value function at time t depends on the decisions at $t-1$ and the realization sequence of the uncertainty until t. Thus, for $t = T$ we define it as follows:

$$\mathcal{V}_T\left(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[1,T]}\right) = W_T\left(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[1,T]}\right),$$

where $W_T = W_T\left(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[1,T]}\right)$ is a real valued function.

For $t \in \mathcal{H}$, we define the following:

$$\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) = \sup_{\mathbf{x}_t \in \mathcal{X}_t} \psi_t\left(\mathcal{V}_{t+1}\left(\mathbf{x}_t, \mathbf{r}_{[1,t+1]}\right) \;\middle|\; \bar{\mathbf{r}}_{[1,t]}\right),$$

where $\mathcal{X}_t = \mathcal{X}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right)$ is the feasible set for each time t. Note that for $t = 0$, we have

$$\mathcal{V}_0 = \sup_{\mathbf{x}_0 \in \mathcal{X}_0} \psi_0\left(\mathcal{V}_1\left(\mathbf{x}_0, \mathbf{r}_1\right)\right),$$

where $\mathcal{X}_0$ is a deterministic set.

Then, we develop the following results.

Proposição 2.3 *If ψ_t is a translation invariant, monotone functional normalized to zero, then for $t \in \mathcal{H}$ the value function can be written as*

$$\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) = \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau,\, \forall \tau = t,\ldots,T-1} C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right),$$

where $C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right)$ is the certainty equivalent of W_T w.r.t. U_t conditioned on the realization sequence $\bar{\mathbf{r}}_{[1,t]}$.

Prova.

By definition we have

$$\begin{aligned}
\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) &= \sup_{\mathbf{x}_t \in \mathcal{X}_t} \psi_t\left(\mathcal{V}_{t+1}\left(\mathbf{x}_t, \mathbf{r}_{[1,t+1]}\right) \mid \bar{\mathbf{r}}_{[1,t]}\right) \\
&= \sup_{\mathbf{x}_t \in \mathcal{X}_t} \psi_t\left(\ldots \sup_{\mathbf{x}_{T-1} \in \mathcal{X}_{T-1}} \psi_{T-1}\left(W_T\right) \,\middle|\, \bar{\mathbf{r}}_{[1,t]}\right).
\end{aligned}$$

Using the monotonicity of ψ_t and the definition of U_t we have the following:

$$\begin{aligned}
\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) &= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau,\, \forall \tau = t,\ldots,T-1} \psi_t\left(\ldots \psi_{T-1}\left(W_T\right) \mid \bar{\mathbf{r}}_{[1,t]}\right) \\
&= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau,\, \forall \tau = t,\ldots,T-1} U_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right).
\end{aligned} \tag{2-10}$$

By the certainty equivalent definition, we have that $C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right)$ satisfies $U_t\left(C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right) \mid \bar{\mathbf{r}}_{[1,t]}\right) = U_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right)$. It is easy to show that $U_t\left(\cdot \mid \bar{\mathbf{r}}_{[1,t]}\right)$ is translation invariant and normalized to zero, since its generators ψ_t have the same properties. Then, $U_t\left(C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right) \mid \bar{\mathbf{r}}_{[1,t]}\right) = C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right)$ and consequently, $U_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right) = C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right)$.

Finally we have that

$$\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) = \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau,\, \forall \tau = t,\ldots,T-1} C_t\left(W_T \mid \bar{\mathbf{r}}_{[1,t]}\right).$$

■

Corolário 2.4 *If ψ_t is a translation invariant, monotone functional normalized to zero, then for $t \in \mathcal{H}$ the value function can be written as*

$$\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) = \sup_{x_\tau \in \mathcal{X}_\tau,\, \forall \tau = t,\ldots,T-1} \tilde{C}_t\left(\ldots \tilde{C}_{T-1}\left(W_T\right) \,\middle|\, \bar{\mathbf{r}}_{[1,t]}\right),$$

where $\tilde{C}_t$ and $\tilde{C}_t(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]})$ are the certainty equivalent w.r.t. ψ_t and $\psi_t(\cdot \mid \bar{\mathbf{r}}_{[\mathbf{1},\mathbf{t}]})$, respectively.

Prova. By the certainty equivalent definition we have that $\tilde{C}_t\left(\cdot \mid \bar{\mathbf{r}}_{[1,t]}\right)$ satisfies $\psi_t\left(C_t\left(\cdot \mid \bar{\mathbf{r}}_{[1,t]}\right) \mid \bar{\mathbf{r}}_{[1,t]}\right) = \psi_t\left(\cdot \mid \bar{\mathbf{r}}_{[1,t]}\right)$ and using the assumption that ψ_t is translation invariant and normalized to zero, we have $\psi_t = \tilde{C}_t$. Note that this property also holds true for the conditional version. Then, from equation (2-10) we have the following:

$$\begin{aligned}\mathcal{V}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) &= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau,\, \forall \tau=t,\ldots,T-1} \psi_t\left(\ldots \psi_{T-1}\left(W_T\right) \mid \bar{\mathbf{r}}_{[1,t]}\right) \\ &= \sup_{\mathbf{x}_\tau \in \mathcal{X}_\tau,\, \forall \tau=t,\ldots,T-1} \tilde{C}_t\left(\ldots \tilde{C}_{T-1}\left(W_T\right) \Big| \bar{\mathbf{r}}_{[1,t]}\right).\end{aligned}$$

■

Note that we could also include intermediate "costs" as in (RUSZCZYNSKI; SHAPIRO, 2006) and our results would still hold true for a more general set of problems. It is worth mentioning that we define the feasible sets, X_t, $\forall t \in \mathcal{H}$, and the terminal wealth function, $W_T\left(x_{T-1}, r_{[1,T]}\right)$ generically depending on the application. For the portfolio selection problem, we define them to fit the original constraints. Then, we have that

$$\mathcal{X}_t\left(\mathbf{x}_{t-1}, \bar{\mathbf{r}}_{[1,t]}\right) = \left\{\mathbf{x}_t \in \mathbb{R}^A : \textstyle\sum_{i\in\mathcal{A}} x_{i,t} = \sum_{i\in\mathcal{A}} \left(1+\bar{r}_{i,t}\right) x_{i,t-1}\right\},$$

$$\mathcal{X}_0 = \left\{x_0 \in \mathbb{R}^A : \textstyle\sum_{i\in\mathcal{A}} x_{i,0} = W_0\right\},$$

$$W_T(\mathbf{x}_{T-1}, \bar{\mathbf{r}}_{[1,T]}) = \textstyle\sum_{i\in\mathcal{A}} \left(1+\bar{r}_{i,T}\right) x_{i,T-1}.$$

For the proposed portfolio selection model, we define our one period translation invariant, monotone and normalized utility functional ψ_t as the convex combination of the expected value and the CVaR based acceptability measure, formally defined as

$$\psi_t\left(\mathcal{V}_{t+1}\right) = (1-\lambda)\,\mathbb{E}\left[\mathcal{V}_{t+1} \mid \mathbf{r}_{[1,t]}\right] + \lambda\, \phi_t^\alpha\left(\mathcal{V}_{t+1}, \mathbf{r}_{[1,t]}\right),$$

which is again a coherent acceptability measure. As before, $\mathcal{V}_{t+1} \in L^\infty\left(\mathcal{F}_{t+1}\right)$ and we can write the conditional version as the real valued function

$$\psi_t\left(\mathcal{V}_{t+1} \mid \bar{\mathbf{r}}_{[1,t]}\right) = (1-\lambda)\,\mathbb{E}\left[\mathcal{V}_{t+1} \mid \bar{\mathbf{r}}_{[1,t]}\right] + \lambda\, \phi_t^\alpha\left(\mathcal{V}_{t+1}, \bar{\mathbf{r}}_{[1,t]}\right).$$

The objective function of the proposed model at t is the certainty equivalent w.r.t. the time consistent dynamic utility function generated by the one period preference functional of the investor. This recursive formulation ensures time consistent optimal policies and it is also motivated by Corollary 2.4. The objective at $t = T-1$ is to maximize the certainty equivalent (CE) of

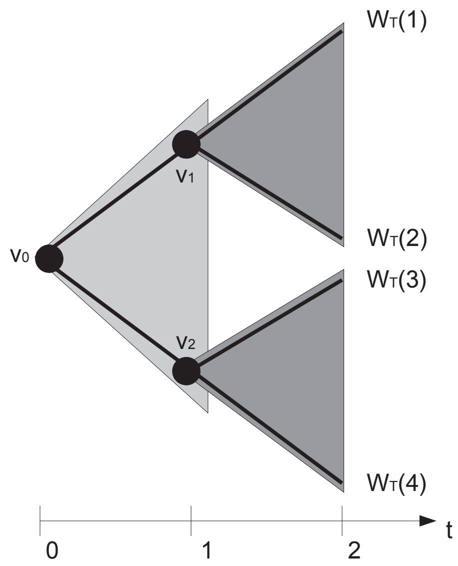

Figura 2.3: Conditional certainty equivalents

terminal wealth w.r.t. the one period preference functional ψ_{T-1}. Indeed, we can interpret the optimal CE as the portfolio value since it is the deterministic amount of money the investor would accept instead of the (random) terminal wealth obtained by his / her optimal trading strategy. At $t = T - 2, \ldots, 0$, the preference functional ψ_t is applied to the (random) portfolio value whose realizations are given by all possible optimal CE's at $t+1$. Thus, the problem at time t is to maximize the CE of the portfolio value w.r.t. the one period preference functional ψ_t of the investor.

For instance, in our numerical example the (random) portfolio value at $t = 1$ is given by the realizations v_1 and v_2 in Figure 2.3 obtained by solving problem (2-9) for nodes $\{1, 2\}$ and $\{3, 4\}$, respectively. The portfolio value v_0 (see also Figure 2.3) obtained by solving (2-9) for $t = 0$ is the optimal certainty equivalent of the (random) portfolio value at $t = 1$.

Tabela 2.1: Results of the sensitivity analysis

λ	T								
	2	3	4	5	6	7	8	9	10
0	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
0.1	0.00%	0.00%	0.12%	0.00%	0.00%	0.02%	0.00%	0.00%	0.01%
0.2	0.00%	0.78%	0.63%	0.19%	0.09%	0.11%	0.04%	0.02%	0.03%
0.3	2.60%	2.18%	1.80%	0.63%	0.40%	0.32%	0.13%	0.09%	0.08%
0.4	13.64%	19.42%	20.85%	21.20%	20.36%	20.24%	20.26%	20.09%	20.06%
0.5	9.09%	22.97%	27.95%	29.24%	28.99%	28.40%	28.35%	28.24%	28.12%
0.6	0.00%	14.29%	32.98%	40.49%	42.83%	42.94%	42.54%	42.61%	42.57%
0.7	0.00%	5.62%	20.90%	40.78%	49.37%	51.66%	51.95%	51.73%	51.80%
0.8	0.00%	0.00%	2.54%	10.81%	29.67%	53.15%	63.03%	66.62%	67.58%
0.9	0.00%	0.00%	0.00%	0.00%	0.41%	3.85%	14.17%	36.41%	63.09%
1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%