



Luis Antonio Gómez Ardila

**Deep Morin Singularities of the
McKean-Scovel Operator**

Tese de Doutorado

Thesis presented to the Programa de Pós-graduação em Matemática, do Departamento de Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Carlos Tomei

Rio de Janeiro
September 2021



Luis Antonio Gómez Ardila

**Deep Morin Singularities of the
McKean-Scovel Operator**

Thesis presented to the Programa de Pós-graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática. Approved by the Examination Committee:

Prof. Carlos Tomei

Advisor

Departamento de Matemática – PUC-Rio

Prof. Bernhard Ruf

Unimi

Prof. Nicolau Corção Saldanha

Departamento de Matemática – PUC-Rio

Prof. Ederson Moreira dos Santos

USP-São Carlos

Prof. Nivaldo de Góes Grulha Júnior

USP-São Carlos

Prof. Sergey Galkin

Departamento de Matemática – PUC-Rio

Prof. João Marcos Bezerra do Ó

UFPB

Rio de Janeiro, September the 29th, 2021

All rights reserved.

Luis Antonio Gómez Ardila

Electronic Engineer by Universidad Industrial de Santander (Bucaramanga, Colombia). Master in Mathematics by Universidad de los Andes (Bogotá, Colombia). Research interest: Operator Theory.

Bibliographic data

Gómez Ardila, Luis Antonio

Deep Morin Singularities of the McKean-Scovel Operator / Luis Antonio Gómez Ardila; advisor: Carlos Tomei. – 2021.

54 f: il. color. ; 30 cm

Tese (doutorado) - Pontifícia Universidade Católica do Rio de Janeiro, Departamento de Matemática, 2021.

Inclui bibliografia

1. Matemática – Teses. 2. Análise – Teses. 3. Singularidades – Teses. 4. Operador. 5. Pontos Críticos. 6. Singularidades. 7. McKean-Scovel. 8. Morin. 9. Profundidade Arbitrária. I. Tomei, Carlos. II. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Matemática. III. Título.

CDD: 510

Acknowledgments

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance code 001. Also by the Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro - FAPERJ.

Also, I am very grateful to...

Prof. PhD. Carlos Tomei, my teacher and adviser, by share with me the problem that this thesis is about. Also by his wonderful talks about Spectral Theory.

The Department of Mathematics of PUC-Rio, by giving me the opportunity to do my doctoral studies.

Profs. Ricardo Monturiol, Edilberto Reyes, Rafael Isaacs, Rafael Castro, Gildardo Guzmán and Sonia Sabogal at Universidad Industrial de Santander (Bucaramanga, Colombia).

Prof. Monika Winklmeier at Universidad de los Andes (Bogotá, Colombia).

Yunelsy, by her great help during 2017-2. Thanks!

Abstract

Gómez Ardila, Luis Antonio; Tomei, Carlos (Advisor). **Deep Morin Singularities of the McKean-Scovel Operator**. Rio de Janeiro, 2021. 54p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

The McKean-Scovel operator is the simplest nonlinear Sturm-Liouville operator acting on functions satisfying Dirichlet boundary conditions: its nonlinearity is just taking the square of the incoming function. This text contains a proof of a conjecture from the late '80: its critical set consists only of Morin singularities, which attain arbitrary depth.

Keywords

Operator; Critical Points; Singularities; McKean-Scovel; Morin; Arbitrary Depth.

Resumo

Gómez Ardila, Luis Antonio; Tomei, Carlos. **Singularidades de Morin Profundas do Operador McKean-Scovel**. Rio de Janeiro, 2021. 54p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

O operador de McKean-Scovel agindo sobre funções que satisfazem condições de Dirichlet é o operador não-linear de Sturm-Liouville mais simples: a não-linearidade é elevar ao quadrado. Nesse texto, demonstra-se uma conjectura que de mais de trinta anos: seu conjunto crítico só contém singularidades de Morin, que podem ter profundidade arbitrária.

Palavras-chave

Operador; Pontos Críticos; Singularidades; McKean-Scovel; Morin; Profundidade Arbitrária.

Table of contents

0	Basic Notation	10
1	Introduction	12
2	Morin Singularities	15
2.1	Spectral Properties of DF	15
2.2	The Critical Set \mathcal{C}	16
2.3	Morin Singularities of F	16
2.3.1	The Thom-Boardman Stratification of the Critical Set	16
2.3.2	Morin Singularities as Level Sets	17
2.3.3	A Characterization of Functions in $\mathcal{Z}(M_k)$	20
3	Every Singularity of F is of Morin type	26
3.1	Computing $\text{grad } \tilde{I}_j(u)$ on $\mathcal{Z}(M_k)$ for $j \leq k$	26
3.2	Zero is a Regular Value of M_k	31
3.3	Every critical point belongs to some $\mathcal{Z}(M_k) \setminus \mathcal{Z}(M_{k+1})$	33
4	Constructing Singularities of Arbitrary Depth	35
4.1	Extending M_k	35
4.2	Operating with singularities	39
4.2.1	Scaling	40
4.2.2	Juxtaposition	41
4.2.3	Symmetrization: from odd depth k to depth $k + 1$	42
4.3	Some deep singularities in X^δ	43
4.4	Adjustable Functions	44
4.4.1	From even k to $k + 1$	45
4.4.2	Existence of Adjustable Functions	48
4.5	There exist Morin singularities of arbitrary depth	50
4.6	An application of Theorem 1.0.2	51
	Bibliography	52

*Functions, just like living beings,
are characterized by their
singularities.*

Paul Montel

0

Basic Notation

- \mathbb{N} , \mathbb{N}_0 and \mathbb{R} are the *positive integers*, *non-negative integers* and *real numbers*, respectively.
- $\|\cdot\|_{sup}$ denotes the *supremum norm* (or *uniform norm*) on $C([a, b])$.
- Norm and inner product on $L^2([a, b])$ are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$.
- If V is an inner product space, $\langle \cdot, \cdot \rangle_V$ denotes its inner product and $\|\cdot\|_V$ the induced norm.
- A function $w : [a, b] \rightarrow \mathbb{R}$ is Dirichlet if it satisfies Dirichlet boundary conditions, $w(a) = w(b) = 0$.
- The function $u : [a, b] \rightarrow \mathbb{R}$ is *absolutely continuous* if and only if there exists $v \in L^1([a, b])$ such that

$$u(x) = u(a) + \int_a^x v(t) dt, \quad x \in [a, b].$$

The function v is denoted by u' .

- $AC([a, b]) = \{u : [a, b] \rightarrow \mathbb{R} \mid u \text{ is absolutely continuous}\}$.
- $H^1([a, b]) = \{u \in AC([a, b]) \mid u' \in L^2([a, b])\}$.
- $H_0^1([a, b]) = \{u \in H^1([a, b]) \mid u(a) = u(b) = 0\}$ is a Hilbert space with the inner product $\langle u, v \rangle_{H_0^1} := \langle u', v' \rangle_{L^2}$. The norm $\|\cdot\|_{H_0^1}$ induced by $\langle u, v \rangle_{H_0^1}$ is equivalent to $\|\cdot\|_{H^1}$ on $H_0^1([a, b])$.
- $H_D^2([a, b]) = H^2([a, b]) \cap H_0^1([a, b]) = \{u \in H_0^1([a, b]) \mid u' \in H^1([a, b])\}$.
- For real Banach spaces E_1, E_2 ,

$$\mathcal{B}(E_1, E_2) = \{T : E_1 \rightarrow E_2 \mid T \text{ is linear continuous}\}$$

with the usual operator norm, $\|T\| = \sup_{\|v\|=1} \|Tv\|$.

- For a differentiable map $G : E_1 \rightarrow E_2$, the derivative (Jacobian) of G at the point $u \in E_1$ is denoted by $DG(u) \in \mathcal{B}(E_1, E_2)$. For every nonzero $v \in E_1$, the directional derivative of G at $u \in E_1$ along v is $\partial_v G(u) = DG(u) \cdot v$.

- For a real Hilbert space H and a differentiable functional $\varphi : H \rightarrow \mathbb{R}$, the gradient $\text{grad } \varphi(u) \in H$ is the only element of H such that $D\varphi(u) \cdot v = \langle \text{grad } \varphi(u), v \rangle_H$ for every $v \in H$.
- For a real valued map f , $\mathcal{Z}(f)$ denotes the zero set of f ,

$$\mathcal{Z}(f) = \{u \in \text{Dom}(f) \mid f(u) = 0\}.$$

- For M a smooth manifold and $p \in M$, $T_p M$ stands for the tangent space of M at the point p .
- The prime $'$ stands for the derivative in the real variable x (i.e., $' = \frac{d}{dx}$).
- A function f is an $o(x^n)$ -function, $f = o(x^n)$, as $x \rightarrow 0$ if and only if

$$\lim_{x \rightarrow 0} \left| \frac{f(x)}{x^n} \right| = 0.$$

- For C^1 functions $f, g : [a, b] \rightarrow \mathbb{R}$, the *Wronskian* of f and g is the function $[f, g] : [a, b] \rightarrow \mathbb{R}$ defined by $[f, g](x) = f(x)g'(x) - f'(x)g(x)$.

1 Introduction

The Ambrosetti-Prodi theorem [1] has been the starting point of active research for decades. After contributions by Manes-Micheletti [2] and Berger-Podolak [3], it may be stated as follows. For $\Omega \subset \mathbb{R}^n$, a bounded domain with smooth boundary, consider the Dirichlet Laplacian

$$-\Delta : X = H^2(\Omega) \cap H_0^1(\Omega) \rightarrow Y = L^2(\Omega)$$

with eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots .$$

Given a smooth, strictly convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$a = \lim_{x \rightarrow -\infty} f'(x) < \lambda_1 < \lim_{x \rightarrow +\infty} f'(x) = b < \lambda_2,$$

define

$$F : X \rightarrow Y, \quad u \mapsto -\Delta u + f(u).$$

Theorem 1.0.1 *The map F is a global fold. More explicitly, there are global homeomorphisms defined on X and Y which convert F to the map*

$$G : Z \times \mathbb{R} \rightarrow Z \times \mathbb{R} \\ (z, t) \mapsto (z, t^2),$$

for some real Banach space Z .

It is a wonderful fact that a nonlinear differential operator may admit such a simple description of its global geometry, possibly the simplest if we exclude homeomorphisms. The different arguments leading to the result emphasized the importance of local theory: the map is not a local homeomorphism at each point of its domain. But such critical points turned out to be the simplest possible deviations from the inverse function theorem: within the classification from singularity theory, they are *folds*.

The search for additional contexts led to further examples in which global geometry may still be described in simple terms (Church-Timourian folds [4])

and others in which different singularities were identified (Church-Timourian cusps [4, 5, 14, 15]).

One interesting case was suggested by McKean and Scovel [6]. Set $X = H_D^2([0, 1])$ and $Y = L^2([0, 1])$. The *McKean-Scovel operator* $F : X \rightarrow Y$ is

$$F(u) = -u'' + \frac{1}{2}u^2.$$

Let \mathcal{C} denotes the critical set of F ,

$$\mathcal{C} = \{u \in X \mid DF(u) \in \mathcal{B}(X, Y) \text{ is non-invertible}\}.$$

In opposition to the situation in the Ambrosetti-Prodi theorem, the nonlinearity $f(x) = \frac{1}{2}x^2$ interacts with all eigenvalues of the linear term. The one-dimensional context ensures that critical points (i.e., points in \mathcal{C}) are potentially simple: the dimension of the kernel of the derivative $DF(u) : X \rightarrow Y$ is never larger than one. With an additional requirement of transversality described in detail in Section 2.3, such critical points are called *Morin singularities* ([17, 21]).

Morin singularities are classified by their *depth*: after local changes of variable (see [9, 14] for the infinite-dimensional case), the function F near a singularity of depth k takes the form

$$\begin{aligned} (z, s_1, \dots, s_{k-1}, t) \in Z \times \mathbb{R}^k &\mapsto \\ (z, s_1, \dots, s_{k-1}, t^{k+1} + s_1 t^{k-1} + s_2 t^{k-2} + \dots + s_{k-1} t) &\in Z \times \mathbb{R}^k. \end{aligned} \quad (1.1)$$

In [7] McKean presented a proof that F has Morin singularities of arbitrary depth. Unfortunately, Ruf identified an error [8], and the original claim became a conjecture. In this thesis we vindicate it.

Theorem 1.0.2 *A. Every critical point of F is a Morin singularity.*

B. Any dense subspace $D \subset X$ contains singularities of arbitrary depth.

Deeper singularities did not receive as much attention as folds and cusps, the first two cases in the hierarchy of Morin singularities. Theoretical aspects – specifically, the counterpart of the usual characterization for infinite dimensional spaces — have been clarified in [9] and [14]. Conditions yielding existence of arbitrarily deep singularities for operators $F(u) = -u' + f(u)$ acting on periodic functions were obtained in [14], solving a conjecture of Cafagna-Donati [10].

The first step in the proof of Theorem 1.0.2 is identifying Morin singularities of depth k in a concrete fashion. As we shall see in Sections 2.3 and 3.2,

there are real smooth maps $M_i : X \rightarrow \mathbb{R}^i$, $i = 1, 2, \dots$, for which $u \in X$ is a Morin singularity of depth k if and only if

- (i) $M_k(u) = 0$, $M_{k+1}(u) \neq 0$,
- (ii) Zero is a regular value of M_k , i.e., $DM_k(u) : X \rightarrow \mathbb{R}^k$ is surjective.

In Section 2.3.2 we prove that for each critical point u there is $j \in \mathbb{N}$ such that $M_j(u) \neq 0$. Thus every singularity of F is indeed of Morin type.

In Chapter 4, we show that every M_k has a root in X . The maps M_k admit real smooth extensions $\hat{M}_k : H^{-1}([0, 1]) \rightarrow \mathbb{R}^k$ (Section 4.1). We are then led to consider solutions defined by finite sums of deltas $\sum_j c_j \delta_{x_j}$, $x_j \in (0, 1)$. In Proposition 2.3.4 we present *juxtaposition*: from two functions in adjacent intervals which are zeros of the \hat{M}_k 's, we juxtapose domains and rescale the resulting interval back to $[0, 1]$ so as to obtain not only another zero of \hat{M}_k , but a curve of zeros by inserting a δ of arbitrary weight at the juncture of both domains. Iteration of this process leads to zeros of \hat{M}_k with additional degrees of freedom. The difficulty now lies in identifying appropriate weights to such degrees of freedom to obtain a root of \hat{M}_{k+1} . Odd values k are easier to handle (Proposition 4.2.3). When k is even, there are specific locations $x_s \in (0, 1)$ - the *points of adjustment* of Section 4.4 - for which the insertion of a weighted δ_{x_s} suffices to obtain a root u_* of \hat{M}_{2k+1} . A transversality argument then implies that u_* may be perturbed so as to obtain a root of M_{2k+1} on arbitrary dense subspaces of X , and those are indeed Morin singularities of depth larger than k of F . A substantial amount of symbolic computation led us to the construction of deep singularities. Some examples are shown in Section 4.3.

Propositions 2.3.5 and 3.1.1 are of a strong algebraic nature. Extending the result for additional nonlinearities would require a different approach.

2 Morin Singularities

For $X = H_D^2([0, 1])$ and $Y = L^2([0, 1])$, the *McKean-Scovel operator* is

$$F : X \rightarrow Y, \quad F(u) = -u'' + \frac{1}{2}u^2.$$

Morin singularities of the McKean-Scovel operator are described in terms of eigenvalues and eigenvectors of its Jacobians $DF(u)$.

In Section 2.1 we encapsulate some standard spectral properties of $DF(u)$ (see [6, 20, 22, 23]). In Section 2.2, we introduce the critical set \mathcal{C} of F . Deeper singularities are defined in Section 2.3.

2.1 Spectral Properties of DF

Let $M_u : X \rightarrow Y$ be the operator of multiplication by u .

Proposition 2.1.1 *The map $F : X \rightarrow Y$ is proper, (real) analytic and its derivative at each $u \in X$ is the bounded operator*

$$DF(u) : X \rightarrow Y, \quad DF(u) = -\frac{d^2}{dx^2} + M_u.$$

$DF(u) : X \subset Y \rightarrow Y$ is a self-adjoint operator and its spectrum $\sigma(DF(u))$ consists of simple eigenvalues converging to infinity,

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) < \cdots \rightarrow +\infty.$$

Let $\phi_n(u)$ be the eigenfunction associated with $\lambda_n(u)$, normalized so as $\|\phi_n(u)\| = 1$ (in the L^2 -norm) and $\phi_n(u)'(0) > 0$. It has $n - 1$ simple zeros in the open interval $(0, 1)$. The operator $DF(u) : X \rightarrow Y$ is Fredholm of index zero, with $\dim \ker(DF(u) - \lambda_n(u)) = 1$. The eigenmaps $u \in X \mapsto \lambda_n(u) \in \mathbb{R}$ and $u \in X \mapsto \phi_n(u) \in X$ are smooth and, for each $u, v \in X, v \neq 0$,

$$\partial_v \lambda_n(u) = \langle \phi_n(u)^2, v \rangle, \tag{2.1}$$

$$(DF(u) - \lambda_n(u)) \cdot \partial_v \phi_n(u) = -v \phi_n(u) + [\partial_v \lambda_n(u)] \phi_n(u). \tag{2.2}$$

On lines of X , the eigenvalue maps restrict to real analytic maps.

2.2

The Critical Set \mathcal{C}

Let \mathcal{C} denote the *critical set* of F ,

$$\begin{aligned}\mathcal{C} &= \{u \in X \mid DF(u) \in \mathcal{B}(X, Y) \text{ is non-invertible}\} \\ &= \{u \in X \mid \dim \ker DF(u) = 1\} \\ &= \{u \in X \mid \exists \varphi \in X \setminus \{0\}, -\varphi'' + u\varphi = 0\}.\end{aligned}$$

Define the zero set $\mathcal{C}_n = \{u \in X \mid \lambda_n(u) = 0\}$ of the eigenmap λ_n . The following result is a small variation of a theorem in [6].

Proposition 2.2.1 *Each \mathcal{C}_n is a smooth submanifold of X of codimension one. The sets are disjoint and isolated from each other. The critical set $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ also is a smooth submanifold of X of codimension one.*

Proof. The set \mathcal{C}_n is nonempty and connected – as is shown in Theorem C [13], it is a graph over the orthogonal complement of $\sin(\pi x)$. For $u \in \mathcal{C}_n$, from Proposition 2.1.1, the eigenfunction $\phi_n(u)$ has $n + 1$ (simple) zeros in $[0, 1]$: the sets \mathcal{C}_n are disjoint. As $\phi_n(u) \neq 0$, from equation (2.1) we have that $D\lambda_n(u)$ is nonzero and therefore, by the Implicit Function Theorem, \mathcal{C}_n is a smooth submanifold of X of codimension one. In particular, close to $u \in \mathcal{C}_n$, one cannot have $\lambda_m(u) = 0$ for $m \neq n$: the sets are isolated, so that \mathcal{C} is also a manifold. ■

2.3

Morin Singularities of F

We recall the definition of a Morin singularity of depth k . The geometric definition yields a set of equations characterizing such singularities for the map F .

2.3.1

The Thom-Boardman Stratification of the Critical Set

Given a smooth map $F : X \rightarrow Y$, we stratify its critical set according to the Thom-Boardman symbols [21]. Here, X and Y are real Banach spaces, and we suppose that Jacobians $DF(u) : X \rightarrow Y$ are Fredholm operators of index zero. Define the *Thom set* Σ^J for a multi-index J with entries in \mathbb{N}_0 . For $i \in \mathbb{N}_0$,

$$\Sigma^i = \Sigma^i(F) = \{u \in X \mid \dim \ker DF(u) = i\}.$$

For $i_1, i_2 \in \mathbb{N}_0$ with $i_1 \geq i_2$, if Σ^{i_1} is a manifold, set

$$\Sigma^{i_1, i_2} = \Sigma^{i_2}(F|_{\Sigma^{i_1}}).$$

Inductively, suppose $J = (i_1, \dots, i_k)$ is a non-increasing sequence in \mathbb{N}_0 and Σ^J is a manifold. Then, for $i_{k+1} \in \mathbb{N}_0$ with $i_k \geq i_{k+1}$,

$$\Sigma^{i_1, \dots, i_k, i_{k+1}} = \Sigma^{i_1, \dots, i_k, i_{k+1}}(F) = \Sigma^{i_{k+1}}(F|_{\Sigma^J}).$$

Clearly, if Σ^{i_1, \dots, i_k} is a manifold and $i_k \geq i_{k+1}$ then

$$\Sigma^{i_1, \dots, i_k} \supseteq \Sigma^{i_1, \dots, i_k, i_{k+1}}.$$

We are interested in very special multi-indices: for the McKean-Scovel operator F , we have $\dim \ker DF(u) \leq 1$. Since $\Sigma^1 = \mathcal{C}$ is a manifold, $\Sigma^{1,1}$ makes sense.

Set $[k] = (1, \dots, 1) \in \mathbb{R}^k$. For $k \geq 1$, if $\Sigma^{[k]}$ is a manifold, *the set of Morin singularities of depth k* is

$$\mathcal{S}_k = \Sigma^{[k]} \setminus \Sigma^{[k+1]}.$$

The set of all Morin singularities is $\mathcal{S} = \bigcup_{k \in \mathbb{N}} \mathcal{S}_k$. Clearly $\mathcal{S} \subseteq \mathcal{C}$.

In the notation above, Theorem 1.0.2 becomes

Theorem 2.3.1 (A) *Every critical point of F is a Morin singularity, $\mathcal{C} = \mathcal{S}$.*

(B) *There are singularities of arbitrary depth: $\mathcal{S}_k \neq \emptyset$ for every $k \in \mathbb{N}$.*

2.3.2

Morin Singularities as Level Sets

Define $\Sigma^{[0]} = X$. The Thom-Boardman stratification may be described as the zero-levels of a set of functionals: $\Sigma^{[k]} = \mathcal{Z}(I_k)$, the zero set of appropriate $I_k : \Sigma^{[k-1]} \rightarrow \mathbb{R}$.

To verify if $u_* \in X$ is a Morin singularity of the map F , we first check if u_* is critical, i.e., if u_* is a zero of the eigenvalue map $u \mapsto \lambda_n(u)$ for some $n \in \mathbb{N}$. Set $\lambda = \lambda_n$ and $\phi = \phi_n$. If $\Sigma^{[1]} = \mathcal{C}$ is a manifold (which is the case for the map F), we proceed to verify if $u_* \in \Sigma^{[2]}$, i.e., if $\phi(u_*)$ lies in $T_{u_*}\mathcal{C}$, i.e., if u_* is a zero of the map $u \mapsto \partial_{\phi(u)}\lambda(u)$. Iteration of this argument suggests the definitions

$$I_1 : \Sigma^{[0]} \rightarrow \mathbb{R}, \quad u \mapsto I_1(u) = \lambda(u),$$

and, for $k \geq 1$, if $\Sigma^{[k]}$ is a manifold,

$$I_{k+1} : \Sigma^{[k]} \rightarrow \mathbb{R}, \quad u \mapsto I_{k+1}(u) = \partial_{\phi(u)} I_k(u) = \partial_{\phi(u)}^{(k)} \lambda(u).$$

The above definition makes sense: $u \in \Sigma^{[k]}$ implies that $\phi(u)$ belongs to $T_u \Sigma^{[k-1]}$ and so directional derivatives of the restrictions I_k are well defined.

Proposition 2.3.2 *For $k \geq 0$, if $\Sigma^{[k]}$ is a manifold then*

$$\Sigma^{[k+1]} = \mathcal{Z}(I_{k+1}) = \{u \in X \mid \partial_{\phi(u)}^{(0)} \lambda(u) = \cdots = \partial_{\phi(u)}^{(k)} \lambda(u) = 0\}.$$

Proof. The statement is true for $k = 0$ since $\Sigma^{[1]} = \{u \in X \mid \lambda(u) = 0\} = \mathcal{Z}(I_1)$. Assume it is true for some $k \geq 0$ and suppose $\Sigma^{[k]}$ is a manifold. Then

$$\begin{aligned} \Sigma^{[k+1]} &= \{u \in \Sigma^{[k]} \mid \phi(u) \in T_u \Sigma^{[k]}\} = \{u \in \Sigma^{[k]} \mid \phi(u) \in T_u \mathcal{Z}(I_k)\} \\ &= \{u \in \Sigma^{[k]} \mid DI_k(u) \cdot \phi(u) = 0\} = \{u \in \Sigma^{[k]} \mid \partial_{\phi(u)} I_k(u) = 0\} \\ &= \{u \in \Sigma^{[k]} \mid I_{k+1}(u) = 0\} = \mathcal{Z}(I_{k+1}). \end{aligned}$$

■

At some point, we must prove that the sets $\Sigma^{[k]}$ are indeed (nested) manifolds. A simple modification of the above construction gives a geometric perspective. Since the maps $I_1 = \lambda : X \rightarrow \mathbb{R}$ and $\phi : X \rightarrow X$ are smooth, globally defined, the functionals $I_k : \Sigma^{[k-1]} \rightarrow \mathbb{R}$ admit an obvious extension $\tilde{I}_k : X \rightarrow \mathbb{R}$,

$$\tilde{I}_1 = I_1, \quad \tilde{I}_{k+1}(u) = \partial_{\phi(u)}^{(k)} \lambda(u) = \partial_{\phi(u)} \tilde{I}_k(u).$$

Set

$$M_k : X \rightarrow \mathbb{R}^k, \quad u \mapsto (\tilde{I}_1(u), \dots, \tilde{I}_k(u)),$$

and $\mathcal{Z}(M_k) = M_k^{-1}(0)$. If zero is a regular value of M_k , then $\mathcal{Z}(M_k)$ is a submanifold of X of codimension k , and then clearly $\mathcal{Z}(M_k) = \Sigma^{[k]}$. We already know that $\mathcal{Z}(M_1) = \mathcal{C}_n$. Clearly

$$\mathcal{Z}(M_1) \supset \mathcal{Z}(M_2) \supset \mathcal{Z}(M_3) \supset \cdots.$$

Theorem 1.0.2 in this notation corresponds to the following statements.

Theorem 2.3.3 *For each $u \in X$:*

- i) *Zero is a regular value of M_k .*
- ii) *$u \in \mathcal{S}_k$ if and only if $M_k(u) = 0$ and $M_{k+1}(u) \neq 0$.*
- iii) *If $u \in \mathcal{C}$, there exists $j \in \mathbb{N}$ such that $M_j(u) \neq 0$.*
- iv) *For each $k \in \mathbb{N}$, $\mathcal{Z}(M_k) \neq \emptyset$.*

From (i), the sets $\mathcal{Z}(M_k)$ are manifolds of codimension k and $\mathcal{Z}(M_k) = \Sigma^{[k]}$. Item (ii) characterizes Morin singularities of depth k in terms of M_k and M_{k+1} . Items (iii) and (iv) are respectively parts (A) and (B) of Theorem 1.0.2.

To obtain a more concrete description of the maps M_k , take successive directional derivatives along $\phi(u)$ of the eigenvalue equation

$$DF(u) \cdot \phi(u) = -\phi(u)'' + u \phi(u) = \lambda(u)\phi(u). \quad (2.3)$$

To simplify notation define the maps $z_k : X \rightarrow X$,

$$z_1(u) = \phi(u), \quad z_{k+1}(u) = \partial_{\phi(u)}^{(k)} z_1(u) = \partial_{\phi(u)} z_k(u),$$

together with the functions $r_k : X \rightarrow X$,

$$r_1 \equiv 0, \quad r_{k+1}(u) = z_1(u)z_k(u) + \partial_{\phi(u)} r_k(u).$$

The functions $z_j(u)$ are Dirichlet.

Proposition 2.3.4 *For each $u \in X$, and all $j \in \mathbb{N}$,*

$$-z_j(u)'' + u z_j(u) = -r_j(u) + \sum_{i=0}^{j-1} \binom{j-1}{i} \tilde{I}_{i+1}(u) z_{j-i}(u), \quad (2.4)$$

where

$$r_{j+1}(u) = \sum_{i=1}^j \binom{j}{i} z_i(u) z_{j+1-i}(u). \quad (2.5)$$

If $u \in \mathcal{Z}(M_k)$ then

$$\begin{cases} -z_j(u)'' + u z_j(u) & = -r_j(u), \quad j = 1, \dots, k, \\ -z_{k+1}(u)'' + u z_{k+1}(u) & = -r_{k+1}(u) + \tilde{I}_{k+1}(u) \phi(u). \end{cases} \quad (2.6)$$

Moreover,

$$\tilde{I}_{k+1}(u) = \langle r_{k+1}(u), \phi(u) \rangle. \quad (2.7)$$

Proof. The formula for r_p is clear for $p = 2$. We drop the obvious dependence on u and verify the formula for r_{p+1} from r_p :

$$\begin{aligned} r_{p+1} &= z_1 z_p + \partial_{\phi} r_p = z_1 z_p + \partial_{\phi} \left(\sum_{i=1}^{p-1} \binom{p-1}{i} z_i z_{p-i} \right) \\ &= z_1 z_p + \sum_{i=1}^{p-1} \binom{p-1}{i} z_{i+1} z_{p-i} + \sum_{i=1}^{p-1} \binom{p-1}{i} z_i z_{p+1-i} \\ &= z_1 z_p + \sum_{i=2}^p \binom{p-1}{i-1} z_i z_{p+1-i} + \sum_{i=1}^{p-1} \binom{p-1}{i} z_i z_{p+1-i} \\ &= z_1 z_p + \sum_{i=2}^{p-1} \left[\binom{p-1}{i-1} + \binom{p-1}{i} \right] z_i z_{p+1-i} + p z_1 z_p \\ &= z_1 z_p + \sum_{i=2}^{p-1} \binom{p}{i} z_i z_{p+1-i} + p z_1 z_p = \sum_{i=1}^p \binom{p}{i} z_i z_{p+1-i}. \end{aligned}$$

The differential equation is obtained by taking successive derivatives of the eigenvalue equation (2.3) at u along $\phi(u)$.

For $u \in \mathcal{Z}(M_k)$, the system (2.6) is obtained from (2.4) making $\tilde{I}_1(u) = \dots = \tilde{I}_k(u) = 0$. We now consider the expression for $\tilde{I}_{k+1}(u)$. Recall that $DF(u) : X \subset Y \rightarrow Y$ is a self-adjoint, Fredholm operator of index 0 and $\ker DF(u)$ is spanned by $z_1(u) = \phi(u)$. Solvability of

$$DF(u) \cdot z_{k+1}(u) = -r_{k+1}(u) + \tilde{I}_{k+1}(u) \phi(u)$$

is equivalent to $\langle -r_{k+1}(u) + \tilde{I}_{k+1}(u) \phi(u), \phi(u) \rangle = 0$. As $\|\phi(u)\| = 1$, equation (2.7) holds. ■

A brief summary of the constructions in this section is convenient. We take iterated directional derivatives along of the L^2 -normalized eigenfunction $\phi(u)$ of two functions, the eigenvalue map $\lambda : X \rightarrow \mathbb{R}$ and the eigenfunction map $\phi : X \rightarrow X$, defining the maps \tilde{I}_j and z_j . From z_1, \dots, z_j , we obtained r_{j+1} and there are differential equations relating \tilde{I}_j , z_j and r_j ; the Dirichlet functions z_j solve a system of equations. Uniqueness of solution definitely does not hold at a critical point u , as the equation for $j = 1$ is solved by multiples of the eigenfunction $z_1(u) = \phi(u)$.

In the next section, we parameterize the set of solutions to this system in terms of an initial value problem for the same equations, leading to Proposition 2.3.8, a characterization of functions $u \in \mathcal{Z}(M_k)$ in terms of the existence of Dirichlet solutions for the system.

2.3.3

A Characterization of Functions in $\mathcal{Z}(M_k)$

Proposition 2.3.4 leads us to consider a system of differential equations, which we now study as an initial value problem. More precisely, we relate vector solutions w and \tilde{w} of the (nonlinear) IVP

$$\begin{cases} -y_j'' + u y_j = -q_j, & j = 1, \dots, k+1, \\ q_1 = 0, & q_{j+1} = \sum_{i=1}^j \binom{j}{i} y_i y_{j+1-i}, & j = 1, \dots, k, \\ y_j(0) = 0, & j = 1, \dots, k+1, \end{cases} \quad (2.8)$$

with additional conditions on the derivative,

$$y'(0) = e_1, \quad \tilde{y}'(0) = a \in \mathbb{R}^{k+1}, \quad a_1 \neq 0. \quad (2.9)$$

Here e_1, \dots, e_{k+1} are the canonical vectors of \mathbb{R}^{k+1} . We write the system in matrix form. The expressions for q_j are quadratic forms on y , $q_j = q_j(y) = \langle Q_j y, y \rangle$: $Q_1 = 0$ and, for $j \geq 2$, the nonzero entries of the symmetric matrix

Q_j are

$$(Q_j)_{s,j-s} = \frac{1}{2} \left[\binom{j-1}{s} + \binom{j-1}{j-s} \right]. \quad (2.10)$$

The range of Q_j is $\text{span}\{e_1, \dots, e_{j-1}\}$. Write $(v_j)_j$ for $v = (v_1, \dots, v_{k+1})$, so that $q = q(y) = (\langle Q_j y, y \rangle)_j$. In matrix notation, system (2.8) is

$$-y'' + uy = -\left(\langle Q_j y, y \rangle\right)_j = -q(y), \quad y(0) = 0.$$

Proposition 2.3.5 *Let w and \tilde{w} be the two solutions of the IVP (2.8). Then there is a constant lower triangular matrix M for which*

$$\tilde{w} = Mw, \quad q(\tilde{w}) = Mq(w),$$

where

$$(a) \text{ for } j = 1, \dots, k+1, m_{j,j} = a_1^j, \quad (b) \quad Me_1 = a.$$

$$(c) \text{ for } j \geq 2 \text{ and } s \in \{2, \dots, j\}, \text{ and any } t \in \{1, \dots, s-1\},$$

$$m_{j,s} = \frac{1}{\binom{s-1}{t}} \sum_{i=t}^{j-s+t} \binom{j-1}{i} m_{i,t} m_{j-i,s-t}.$$

The proof requires preparation. The lemma below is strictly algebraic.

Lemma 2.3.6 *Write $q = q(w) = (\langle Q_j w, w \rangle)_j$ for Q_j as above. Then there is a lower triangular $(k+1) \times (k+1)$ matrix M for which, for all $w \in \mathbb{R}^{k+1}$,*

$$Mq(w) = q(Mw).$$

The matrix M is uniquely determined by its first column, provided $m_{1,1} \neq 0$. Moreover, $m_{j,j} = a_1^j \neq 0$ and thus M is invertible.

We say M is an *IVP matrix*. Let $m_i = Me_i$ be the i -th column of M .

Proof. For the bilinear extension $b(y, z) = (\langle Q_j y, z \rangle)_j$ of the quadratic form q ,

$$Mb(y, z) = b(My, Mz), \quad y, z \in \mathbb{R}^{k+1},$$

and, for arbitrary canonical vectors $y = e_s$ and $z = e_r$, we must then obtain

$$Mb(e_s, e_r) = b(m_s, m_r), \quad r, s = 1, \dots, k+1.$$

For $r = 1$ and arbitrary s , the expressions become

$$m_{1+s} = \frac{1}{(Q_{s+1})_{1,s}} b(m_s, m_1). \quad (2.11)$$

Thus m_2 is obtained from (the given first column) m_1 , and m_3 from m_2 , successively up to m_{k+1} . These formulae define M as a triangular matrix. Indeed, for some number c , $m_{1+s} = c b(m_1, m_s) = c b(m_s, m_1)$. For $j = 1$, we have $Q_j = 0$ and $(m_2)_1 = 0$. Now induce and use that only the first $j - 1$ coordinates of $Q_j m_1$ are possibly nonzero. From equation (2.11), induction also yields $m_{j,j} = a_1^j$.

The rest of the proof consists of showing that the other restrictions, obtained by setting $r \neq 1$ and arbitrary s , are automatically satisfied. We must prove

$$m_{r+s} = \frac{1}{(Q_{r+s})_{r,s}} b(m_s, m_r), \quad r + s \leq k + 1. \quad (2.12)$$

The restrictions follow from those we have chosen to define M if we prove that, for $r = 1, \dots, k$, $s = 2, \dots, k + 1$,

$$\frac{1}{(Q_{r+s})_{r,s}} b(m_r, m_s) = \frac{1}{(Q_{r+s})_{r+1,s-1}} b(m_{r+1}, m_{s-1})$$

so that such expressions with constant sum $r + s$ are all equal to the case $r = 1$, for arbitrary s . Using equation (2.11) twice, we are left with proving

$$\frac{1}{(Q_{r+s})_{r,s}} (Q_s)_{s-1,1} b(m_r, b(m_{s-1}, m_1)) = \frac{1}{(Q_{r+s})_{r+1,s-1} (Q_{r+1})_{i,1}} b(m_{s-1}, b(m_i, m_1)).$$

We check the equality of each coordinate. Again by bilinearity, it suffices to prove the result by replacing the vectors m_r and m_{s-1} for canonical vectors e_R and e_{S-1} , for indices corresponding to the nontrivial entries of m_r and m_{s-1} (recall the ranges of i and j in the induction hypothesis),

$$k + 1 \geq R \geq r, \quad S - 1 \geq s - 1 \geq 1.$$

We must prove

$$\begin{aligned} & \frac{1}{(Q_{r+s})_{r,s} (Q_j)_{s-1,1}} \left\langle Q_j e_R, \left(\langle Q_\ell e_{S-1}, m_1 \rangle \right)_\ell \right\rangle \\ &= \frac{1}{(Q_{r+s})_{r+1,s-1} (Q_{r+1})_{i,1}} \left\langle Q_j e_{S-1}, \left(\langle Q_\ell e_R, m_1 \rangle \right)_\ell \right\rangle. \end{aligned}$$

The argument splits in four cases: $R < k + 1$ or not, $S < k + 2$ or not.

(I) $R \geq k + 1, S \geq k + 2$

Only the first k first columns of Q_j are nonzero. Thus $Q_j e_R = Q_j e_{S-1} = 0$.

(II) $R \geq k + 1, 1 < S < k + 2$

Again, the left hand side is zero. We consider the right hand side. We

have $Q_j e_{S-1} = c e_{k+1-S+1}$, for $c \neq 0$. But then, up to a nonzero constant, the rhs is $\langle Q_{k+1-S+1} e_R, m_1 \rangle$, and $Q_{k+1-S+1} e_R$ is zero if $R > k + 1 - S + 1$.

(III) $R < k + 1, S \geq k + 2$

This case is similar to the previous one.

(IV) $R < k + 1, 1 < S < k + 2$

Here, we have to take into account the constants. We use extensively the fact that $Q_m e_i = (Q_m)_{m-i, i} e_{m-i}$. Simplifying the left hand side,

$$\begin{aligned} \langle Q_j e_R, (\langle Q_\ell e_{S-1}, m_1 \rangle)_\ell \rangle &= (Q_j)_{k+1-R, R} \langle Q_{k+1-R} e_{S-1}, m_1 \rangle \\ &= (Q_j)_{k+1-R, R} (Q_{k+1-R})_{k+2-R-S, S-1} (m_1)_{k+2-R-S}. \end{aligned}$$

Similarly, in the right hand side,

$$\langle Q_j e_{S-1}, (\langle Q_\ell e_R, m_1 \rangle)_\ell \rangle = (Q_j)_{k+2-S, S-1} (Q_{k+2-S})_{k+2-R-S, R} (m_1)_{k+2-R-S}.$$

Getting rid of the term $(m_1)_{k+2-R-S}$, we are left with verifying an expression in the entries of matrices Q_m . From the formulas for their entries, we arrive at the proof of a combinatorial identity: the product

$$\left[\binom{k}{k+1-R} + \binom{k}{R} \right] \left[\binom{k-R}{k+2-R-S} + \binom{k-R}{S-1} \right] \left[\binom{R+S-1}{R+1} + \binom{R+S-1}{S-1} \right] \left[\binom{R}{R} + \binom{R}{1} \right]$$

should equal

$$\left[\binom{k}{k+2-S} + \binom{k}{S-1} \right] \left[\binom{R+S-1}{R} + \binom{R+S-1}{S} \right] \left[\binom{k+1-S}{k+2-R-S} + \binom{k+1-S}{R} \right] \left[\binom{S-1}{S-1} + \binom{S-1}{1} \right].$$

Start by changing variables $\tilde{S} = S - 1$ (and then rename $S = \tilde{S}$), obtaining an expression which is symmetric in $R, S = 1, \dots, k$. Now use the fact that each pair of binomials is of the form below, which is easily simplified: for $m + 1 = a + b$,

$$\binom{m}{a} + \binom{m}{b} = \frac{m!}{(a-1)!(b-1)!} \left[\frac{1}{a} + \frac{1}{b} \right].$$

After simplifying factorials, one is left with a cumbersome expression with simple fractions, whose equality is easily verified. \blacksquare

We prove Proposition 2.3.5: IVP matrices exist.

Proof. Take w and \tilde{w} as above, so that

$$-w'' + uw = -q(w), \quad w(0) = 0, \quad w'(0) = e_1,$$

$$-\tilde{w}'' + u\tilde{w} = -q(\tilde{w}), \quad \tilde{w}(0) = 0, \quad \tilde{w}'(0) = a.$$

Let M be the IVP matrix with $m_1 = a$. To get $\tilde{w} = Mw$, first multiply the differential equation for w by M ,

$$M(-w'' + uw) = -Mq(w).$$

Since $Mq(w) = q(Mw)$, $-(Mw)'' + uMw = -q(Mw)$. As $(Mw)'(0) = Mw'(0) = Me_1 = m_1 = a$ then, by uniqueness of solutions of IVP's, $\tilde{w} = Mw$. ■

Lemma 2.3.7 *Let w_1, \dots, w_{t+1} , and $\tilde{w}_1, \dots, \tilde{w}_{t+1}$ be solutions of the IVP (2.8). Then, for $j = 1, \dots, t+1$, w_1, \dots, w_j are Dirichlet functions if and only if the $\tilde{w}_1(u), \dots, \tilde{w}_j(u)$ also are.*

Proof. The IVP matrix relating both sets of solutions is lower triangular. ■

We characterize $u \in \mathcal{Z}(M_k)$ in terms of differential equations.

Proposition 2.3.8 *For $k \geq 1$, $u \in \mathcal{Z}(M_k)$ if and only if the system*

$$\begin{cases} -w_j'' + u w_j = -h_j, & j = 1, \dots, k, \\ h_1 = 0, \quad h_j = \sum_{i=1}^{j-1} \binom{j-1}{i} w_i w_{j-i}, & j = 2, \dots, k, \end{cases} \quad (2.13)$$

has Dirichlet solutions w_j , $j = 1, \dots, k$, with $w_1 \neq 0$.

Proof. Suppose $u \in \mathcal{Z}(M_k)$. From Proposition 2.3.4, we obtain the system

$$-z_j(u)'' + u z_j(u) = -r_j(u), \quad j = 1, \dots, k, \quad (2.14)$$

showing that system (2.13) has the required solutions. For the converse, take Dirichlet solutions $w_1(u), \dots, w_k(u)$ of system (2.13) with $w_1(u) \neq 0$. We have to show that $u \in \mathcal{Z}(M_k)$. Clearly $u \in \mathcal{Z}(M_1)$.

Let $k \geq 2$ and $u \in \mathcal{Z}(M_\ell)$ for some $\ell = 1, \dots, k-1$. Then, as above, $-z_i(u)'' + u z_i(u) = -r_i(u)$, for $i = 1, \dots, \ell$. Consider the slightly altered system

$$\begin{cases} -z_i(u)'' + u z_i(u) & = -r_i(u), \quad i = 1, \dots, \ell, \\ -\tilde{z}_{\ell+1}(u)'' + u \tilde{z}_{\ell+1}(u) & = -r_{\ell+1}(u), \quad \tilde{z}(u)(0) = 0, \quad \tilde{z}'(u)(0) = 0. \end{cases}$$

From Lemma 2.3.7, as $w_1(u), \dots, w_{\ell+1}(u)$ are Dirichlet functions, a solution $\tilde{z}_{\ell+1}(u)$ of the above system (having the same potential u as system (2.13)) is also a Dirichlet function, and therefore $\tilde{I}_{\ell+1}(u) = \langle r_{\ell+1}(u), z_1(u) \rangle = 0$. Thus $u \in \mathcal{Z}(M_{\ell+1})$ and $u \in \mathcal{Z}(M_k)$ follows by induction. ■

The next result will simplify computations considerably later.

Proposition 2.3.9 (i) Given Dirichlet solutions w_1, \dots, w_k, w_{k+1} of system (2.8) with $w_1 \neq 0$, there exist Dirichlet solutions $\tilde{w}_1, \dots, \tilde{w}_k, \tilde{w}_{k+1}$, for arbitrary initial conditions $\tilde{w}'(1) = a$, for $a_1 \neq 0$.

(ii) Let w be a vector solution of the IVP (2.8) satisfying $w'(0) = e_1$. Then there is \tilde{w} satisfying (2.8) such that, at a fixed point x_0 for which $w'_1(x_0) \neq 0$, $\tilde{w}'(x_0) = e_1$.

Proof. The first item is immediate. We consider (ii). From Proposition 2.3.5, any solution \tilde{w} of (2.8) with $\tilde{w}(0) = 0$ and $\tilde{w}'(0) = a$, $a_1 \neq 0$, is of the form $\tilde{w} = Mw$ where M is a constant invertible lower triangular matrix M , which is determined by its first column $m_1 = a$. We search for an IVP matrix M_0 for which $e_1 = M_0 w'(x_0)$, i.e., such that the first column of its inverse matrix M_0^{-1} is $w'(x_0)$. We show that there is a unique such matrix.

From equation (2.12) which specifies column m_j in terms of m_{j-1} , one obtains entry $i+1$ of m_j from the first i entries of m_{j-1} . The diagonal entries of M_0 and of M_0^{-1} are computed from the first entry of $w'(x_0)$. These facts suffice to compute both M_0 and M_0^{-1} simultaneously. Indeed, assume by induction that the top $j \times j$ blocks of M_0 and M_0^{-1} are computed. Then the last row of the $(j+1) \times (j+1)$ block of M_0 is known from (2.12), with the exception of the entry $(j+1, 1)$, which is obtained from equating entry $(j+1, 1)$ of the equality $M_0 M_0^{-1} = I$. Inverting, one obtains the $(j+1) \times (j+1)$ block of M_0^{-1} .

Finally set $\tilde{w} = M_0 w$: \tilde{w} solves the IVP (2.8) and $\tilde{w}'(x_0) = M_0 w'(x_0) = M_0 M_0^{-1} e_1 = e_1$. ■

3

Every Singularity of F is of Morin type

In this chapter we prove the first item of Theorem 2.3.3: zero is a regular value of M_k . To obtain DM_k at its zeros, we compute in Section 3.1 the directional derivatives of the functionals \tilde{I}_j , the components of M_k . Linear independence of the gradients is shown in Section 3.2.

3.1

Computing $\text{grad } \tilde{I}_j(u)$ on $\mathcal{Z}(M_k)$ for $j \leq k$

Proposition 3.1.1 *For $u \in \mathcal{Z}(M_k)$, there are $a_{1,k}, \dots, a_{k,k} \in \mathbb{R}$, with $a_{k,k} = 1$, such that, for every nonzero $v \in X$,*

$$\sum_{i=1}^k a_{i,k} \partial_v \tilde{I}_i(u) = \langle r_{k+1}(u), v \rangle. \quad (3.1)$$

There are $b_{1,k}, \dots, b_{k,k} \in \mathbb{R}$, with $b_{k,k} = 1$, such that for every nonzero $v \in X$,

$$\partial_v \tilde{I}_k(u) = \sum_{i=1}^k b_{i,k} \langle r_{i+1}(u), v \rangle. \quad (3.2)$$

The proof is *not* by induction. We present the cases $k = 1$ and $k = 2$ separately, in order to present arguments employed along the computations.

Proof. Taking the derivative of (2.4) at $u \in \mathcal{Z}(M_k)$ along v ,

$$\begin{aligned} DF(u) \cdot \partial_v z_k(u) &= -v z_k(u) - \partial_v r_k(u) \\ &\quad + \sum_{i=0}^{k-1} \binom{k-1}{i} \left[\partial_v \tilde{I}_{i+1}(u) z_{k-i}(u) + \tilde{I}_{i+1}(u) \partial_v z_{k-i}(u) \right] \\ &= -v z_k(u) - \partial_v r_k(u) + \sum_{i=0}^{k-1} \binom{k-1}{i} \left[\partial_v \tilde{I}_{i+1}(u) \right] z_{k-i}(u). \end{aligned}$$

Taking the L^2 -inner product with $\phi(u)$,

$$\begin{aligned} \sum_{i=0}^{k-1} \binom{k-1}{i} \left[\partial_v \tilde{I}_{i+1}(u) \right] \langle z_{k-i}(u), \phi(u) \rangle & \\ = \langle v, z_1(u) z_k(u) \rangle + \langle \partial_v r_k(u), z_1(u) \rangle. & \end{aligned} \quad (3.3)$$

For $k = 1$ the above equation is $\partial_v \tilde{I}_1(u) = \langle r_2(u), v \rangle$: the equation (3.1) then holds with $a_{1,1} = 1$. For $k \geq 2$,

$$\begin{aligned} \partial_v r_k(u) &= \sum_{i=1}^{k-1} \binom{k-1}{i} [[\partial_v z_i(u)] z_{k-i}(u) + z_i(u) [\partial_v z_{k-i}(u)]] \\ &= \sum_{i=1}^{k-1} \binom{k-1}{i} [\partial_v z_i(u)] z_{k-i}(u) + \sum_{i=1}^{k-1} \binom{k-1}{i} z_i(u) [\partial_v z_{k-i}(u)] \\ &= \sum_{i=1}^{k-1} \binom{k-1}{i} [\partial_v z_i(u)] z_{k-i}(u) + \sum_{i=1}^{k-1} \binom{k-1}{k-i} [\partial_v z_i(u)] z_{k-i}(u) \\ &= \sum_{i=1}^{k-1} \binom{k}{i} [\partial_v z_i(u)] z_{k-i}(u), \end{aligned}$$

so that

$$\langle \partial_v r_k(u), z_1(u) \rangle = \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i(u), z_1(u) z_{k-i}(u) \rangle$$

and now equation (3.3) becomes

$$\begin{aligned} \sum_{i=0}^{k-1} \binom{k-1}{i} [\partial_v \tilde{I}_{i+1}(u)] \langle z_{k-i}(u), \phi(u) \rangle \\ = \langle v, z_1(u) z_k(u) \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i(u), z_1(u) z_{k-i}(u) \rangle. \end{aligned} \tag{3.4}$$

Moreover, for $k = 2$,

$$\begin{aligned} & [\partial_v \tilde{I}_1(u)] \langle z_2(u), z_1(u) \rangle + [\partial_v \tilde{I}_2(u)] \langle \phi(u), \phi(u) \rangle \\ &= \langle v, z_1(u) z_2(u) \rangle + 2 \langle \partial_v z_1(u), z_1(u) z_1(u) \rangle \\ &= \langle v, z_1(u) z_2(u) \rangle + 2 \langle \partial_v \phi(u), r_2(u) \rangle \\ &= \langle v, z_1(u) z_2(u) \rangle + 2 \langle \partial_v \phi(u), -DF(u) \cdot z_2(u) \rangle \\ &= \langle v, z_1(u) z_2(u) \rangle + 2 \langle -DF(u) \cdot \partial_v \phi(u), z_2(u) \rangle \\ &= \langle v, z_1(u) z_2(u) \rangle + 2 \langle v \phi(u) - \partial_v \tilde{I}_1(u) \phi(u), z_2(u) \rangle \\ &= \langle v, z_1(u) z_2(u) \rangle + 2 \langle v z_1(u), z_2(u) \rangle \quad (\text{as } \langle \phi(u), z_2(u) \rangle = 0) \\ &= \langle v, 3z_1(u) z_2(u) \rangle \\ &= \langle v, r_3(u) \rangle. \end{aligned}$$

Thus (3.1) holds with $a_{1,2} = \langle z_2(u), z_1(u) \rangle$ and $a_{2,2} = \langle \phi(u), \phi(u) \rangle = 1$.

Let $k \geq 3$. We drop the dependence on u . We first decrease the largest index k of the expected answer:

$$\begin{aligned}
\langle v, r_{k+1} \rangle &= \left\langle v, \sum_{i=1}^k \binom{k}{i} z_i z_{k+1-i} \right\rangle = \sum_{i=1}^k \binom{k}{i} \langle v z_i, z_{k+1-i} \rangle \\
&= k \langle v z_1, z_k \rangle + \langle v z_k, z_1 \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle v z_i, z_{k+1-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle \\
&\quad + \sum_{i=2}^{k-1} \binom{k}{i} \left\langle -DF \cdot \partial_v z_i - \partial_v r_i + \sum_{j=0}^{i-1} \binom{i-1}{j} [\partial_v \tilde{I}_{j+1}] z_{i-j}, z_{k+1-i} \right\rangle \\
&= (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle -DF \cdot \partial_v z_i, z_{k+1-i} \rangle \\
&\quad - \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v r_i, z_{k+1-i} \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \sum_{j=0}^{i-1} \binom{i-1}{j} [\partial_v \tilde{I}_{j+1}] \langle z_{i-j}, z_{k+1-i} \rangle.
\end{aligned}$$

Rearranging terms,

$$\begin{aligned}
\langle v, r_{k+1} \rangle &- \sum_{i=2}^{k-1} \binom{k}{i} \sum_{j=0}^{i-1} \binom{i-1}{j} [\partial_v \tilde{I}_{j+1}] \langle z_{i-j}, z_{k+1-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle -DF \cdot \partial_v z_i, z_{k+1-i} \rangle \\
&\quad - \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v r_i, z_{k+1-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v z_i, -DF \cdot z_{k+1-i} \rangle \\
&\quad - \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v r_i, z_{k+1-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v z_i, r_{k+1-i} \rangle \\
&\quad - \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v r_i, z_{k+1-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \left\langle \partial_v z_i, \sum_{l=1}^{k-i} \binom{k-i}{l} z_l z_{k+1-i-l} \right\rangle \\
&\quad - \sum_{i=2}^{k-1} \binom{k}{i} \left\langle \sum_{l=1}^{i-1} \binom{i}{l} (\partial_v z_l) z_{i-l}, z_{k+1-i} \right\rangle \\
&= (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \sum_{l=1}^{k-i} \binom{k}{i} \binom{k-i}{l} \langle \partial_v z_i, z_l z_{k+1-i-l} \rangle \\
&\quad - \sum_{i=2}^{k-1} \sum_{l=1}^{i-1} \binom{k}{i} \binom{i}{l} \langle \partial_v z_l, z_{i-l} z_{k+1-i} \rangle.
\end{aligned}$$

Now use $\sum_{i=2}^{k-1} \sum_{l=1}^{i-1} = \sum_{l=1}^{k-2} \sum_{i=l+1}^{k-1}$ and $\binom{k}{i} \binom{i}{l} = \binom{k}{l} \binom{k-l}{k-i}$:

$$\begin{aligned}
& \langle v, r_{k+1} \rangle - \sum_{i=2}^{k-1} \binom{k}{i} \sum_{j=0}^{i-1} \binom{i-1}{j} [\partial_v I_{j+1}] \langle z_{i-j}, z_{k+1-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle \\
&\quad + \sum_{i=2}^{k-1} \sum_{l=1}^{k-i} \binom{k}{i} \binom{k-i}{l} \langle \partial_v z_i, z_l z_{k+1-i-l} \rangle \\
&\quad - \sum_{l=1}^{k-2} \sum_{i=l+1}^{k-1} \binom{k}{l} \binom{k-l}{k-i} \langle \partial_v z_l, z_{i-l} z_{k+1-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle \\
&\quad + \sum_{i=2}^{k-1} \sum_{l=1}^{k-i} \binom{k}{i} \binom{k-i}{l} \langle \partial_v z_i, z_l z_{k+1-i-l} \rangle \\
&\quad - \sum_{i=1}^{k-2} \sum_{l=i+1}^{k-1} \binom{k}{i} \binom{k-i}{k-l} \langle \partial_v z_i, z_{l-i} z_{k+1-l} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle \\
&\quad + k \langle \partial_v z_{k-1}, z_1 z_1 \rangle \\
&\quad + \sum_{i=2}^{k-2} \sum_{l=1}^{k-i} \binom{k}{i} \binom{k-i}{l} \langle \partial_v z_i, z_l z_{k+1-i-l} \rangle \\
&\quad - k \sum_{l=2}^{k-1} \binom{k-1}{k-l} \langle \partial_v z_1, z_{l-1} z_{k+1-l} \rangle \\
&\quad - \sum_{i=2}^{k-2} \sum_{l=i+1}^{k-1} \binom{k}{i} \binom{k-i}{k-l} \langle \partial_v z_i, z_{l-i} z_{k+1-l} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle \\
&\quad + k \langle \partial_v z_{k-1}, z_1 z_1 \rangle \\
&\quad + \sum_{i=2}^{k-2} \sum_{l=1}^{k-i} \binom{k}{i} \binom{k-i}{l} \langle \partial_v z_i, z_l z_{k+1-i-l} \rangle \\
&\quad - k \sum_{l=2}^{k-1} \binom{k-1}{k-l} \langle \partial_v z_1, z_{l-1} z_{k+1-l} \rangle \\
&\quad - \sum_{i=2}^{k-2} \sum_{l=1}^{k-i-1} \binom{k}{i} \binom{k-i}{k-l-i} \langle \partial_v z_i, z_l z_{k+1-l-i} \rangle \\
&= (k+1) \langle v, z_1 z_k \rangle \\
&\quad + k \langle \partial_v z_{k-1}, z_1 z_1 \rangle - k \left\langle \partial_v z_1, \sum_{l=2}^{k-1} \binom{k-1}{k-l} z_{l-1} z_{k+1-l} \right\rangle \\
&\quad + \sum_{i=2}^{k-2} \binom{k}{i} \left\langle \partial_v z_i, \sum_{l=1}^{k-i} \binom{k-i}{l} z_l z_{k+1-l-i} \right\rangle \\
&\quad - \sum_{i=2}^{k-2} \binom{k}{i} \left\langle \partial_v z_i, \sum_{l=1}^{k-i-1} \binom{k-i}{k-i-l} z_l z_{k+1-l-i} \right\rangle.
\end{aligned}$$

$$\begin{aligned}
& \text{As } \binom{k-i}{k-i-l} = \binom{k-i}{l}, \\
& \langle v, r_{k+1} \rangle - \sum_{i=2}^{k-1} \binom{k}{i} \sum_{j=0}^{i-1} \binom{i-1}{j} [\partial_v \tilde{I}_{j+1}] \langle z_{i-j}, z_{k+1-i} \rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + k \langle \partial_v z_{k-1}, z_1 z_1 \rangle \\
& \quad - k \left\langle \partial_v z_1, \sum_{l=2}^{k-1} \binom{k-1}{k-l} z_{l-1} z_{k+1-l} \right\rangle + \sum_{i=2}^{k-2} \binom{k}{i} \langle \partial_v z_i, \sum_{l=1}^{k-i} \binom{k-i}{l} z_l z_{k+1-l-i} \rangle \\
& \quad - \sum_{i=2}^{k-2} \binom{k}{i} \langle \partial_v z_i, \sum_{l=1}^{k-i-1} \binom{k-i}{l} z_l z_{k+1-l-i} \rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + k \langle \partial_v z_{k-1}, z_1 z_1 \rangle - k \left\langle \partial_v z_1, \sum_{l=2}^{k-1} \binom{k-1}{k-l} z_{l-1} z_{k+1-l} \right\rangle \\
& \quad + \sum_{i=2}^{k-2} \binom{k}{i} \langle \partial_v z_i, z_{k-i} z_1 \rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_{k-i} z_1 \rangle - k \left\langle \partial_v z_1, \sum_{l=1}^{k-1} \binom{k-1}{k-1-l} z_l z_{k-l} \right\rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_{k-i} z_1 \rangle - k \left\langle \partial_v z_1, \sum_{l=1}^{k-1} \binom{k-1}{l} z_l z_{k-l} \right\rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle \\
& \quad - k \left\langle \partial_v z_1, -z_1 z_{k-1} + \sum_{l=1}^{k-1} \binom{k-1}{l} z_l z_{k-l} \right\rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=2}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle \\
& \quad + k \langle \partial_v z_1, z_1 z_{k-1} \rangle - k \left\langle \partial_v z_1, \sum_{l=1}^{k-1} \binom{k-1}{l} z_l z_{k-l} \right\rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle - k \left\langle \partial_v z_1, \sum_{l=1}^{k-1} \binom{k-1}{l} z_l z_{k-l} \right\rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle - k \langle \partial_v z_1, r_k \rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle - k \langle \partial_v \phi, -DF \cdot z_k + I_k \phi \rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle - k \langle \partial_v z_1, -DF \cdot z_k \rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle - k \langle -DF \cdot \partial_v z_1, z_k \rangle \\
& = (k+1) \langle v, z_1 z_k \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle - k \langle v z_1, z_k \rangle \\
& = \langle v, z_1 z_k \rangle + \sum_{i=1}^{k-1} \binom{k}{i} \langle \partial_v z_i, z_1 z_{k-i} \rangle.
\end{aligned}$$

By (3.4),

$$\begin{aligned} & \sum_{i=0}^{k-1} \binom{k-1}{i} [\partial_v \tilde{I}_{i+1}] \langle z_{k-i}, \phi \rangle \\ &= \langle v, r_{k+1} \rangle - \sum_{i=2}^{k-1} \sum_{j=0}^{i-1} \binom{k}{i} \binom{i-1}{j} [\partial_v \tilde{I}_{j+1}] \langle z_{i-j}, z_{k+1-i} \rangle. \end{aligned}$$

So, equation (3.1) holds and $a_{k,k} = 1$.

To prove the last statement, consider matrices $A = (a_{i,j})$ and $B = (b_{i,j})$. Clearly $B = A^{-1}$. Since A is triangular with diagonal entries equal to 1, the same is true of B . \blacksquare

The directional derivatives computed above employ the L^2 -inner product. Equivalent formulae for the inner product of the ambient space X are obtained by pulling back the inner product in Y with the isomorphism $-\frac{d^2}{dx^2} : X \rightarrow Y$.

3.2

Zero is a Regular Value of M_k

There is a well known formula for the solutions of the IVP (2.8) when $w'(0) = e_1$ in terms of the right hand sides and (independent) solutions φ and ψ of the homogenous equation

$$\begin{cases} -\varphi'' + u\varphi = 0, & \varphi(0) = 0, & \varphi'(0) = 1, \\ -\psi'' + u\psi = 0, & \psi(0) = 1, & \psi'(0) = 0. \end{cases}$$

Clearly $w_1 = \varphi$. For $j \geq 2$,

$$w_j(x) = \left(\int_0^x h_j \psi \right) \varphi(x) - \left(\int_0^x h_j \varphi \right) \psi(x). \quad (3.5)$$

In Proposition 3.2.2 we prove the linear independence of the set $\{h_2, \dots, h_{k+1}\}$, by considering the local behavior of the functions near $x = 0$. Let $\varepsilon > 0$ be small enough such that $[0, \varepsilon] \cap \varphi^{-1}(0) = \{0\}$. As $\varphi'(0) = 1$ then $\varphi(x) > 0$ on $(0, \varepsilon]$. The expressions

$$\varphi(x) = x + \int_0^x \left(\int_0^s u(t) \varphi(t) dt \right) ds, \quad \psi(x) = 1 + \int_0^x \left(\int_0^s u(t) \psi(t) dt \right) ds$$

yield

$$\varphi(x) = x + o(x^2) \quad \text{and} \quad \psi(x) = 1 + o(x) \quad \text{as} \quad x \rightarrow 0. \quad (3.6)$$

Lemma 3.2.1 *Let w_j and h_j be solutions and right hand sides of system (2.8) with initial conditions $w'(0) = e_1$. Then there are strictly positive numbers $\alpha_1, \dots, \alpha_k$ and $\beta_2, \dots, \beta_{k+1}$ such that, as $x \rightarrow 0$, for $j = 1, \dots, k$,*

$$w_j(x) = \alpha_j x^{3j-2} + o(x^{3j-1}) \quad \text{and} \quad h_{j+1}(x) = \beta_{j+1} x^{3j-1} + o(x^{3j}). \quad (3.7)$$

Proof. For $k = 1$, from (3.6), as $x \rightarrow 0$,

$$w_1(x) = \varphi(x) = x + o(x^2), \quad h_2(x) = [w_1(x)]^2 = x^2 + o(x^3).$$

So (3.7) holds for $k = 1$ with $\alpha_1 = \beta_2 = 1$.

Suppose the result for $p \geq 1$: we show it for $p + 1$. From the induction hypothesis, there are nonzero numbers $\alpha_1, \dots, \alpha_p$ and $\beta_2, \dots, \beta_{p+1}$ such that (3.7) holds for $j = 1, \dots, p$. As $x \rightarrow 0$,

$$\begin{cases} h_{p+1}(x)\psi(x) = \beta_{p+1}x^{3p-1} + o(x^{3p}), \\ h_{p+1}(x)\varphi(x) = \beta_{p+1}x^{3p} + o(x^{3p+1}). \end{cases}$$

Thus

$$\begin{aligned} w_{p+1}(x) &= \left(\int_0^x h_{p+1}\psi \right) \varphi(x) - \left(\int_0^x h_{p+1}\varphi \right) \psi(x) \\ &= \left[\frac{\beta_{p+1}}{3p} x^{3p} + o(x^{3p+1}) \right] [x + o(x^2)] - \left[\frac{\beta_{p+1}}{3p+1} x^{3p+1} + o(x^{3p+2}) \right] [1 + o(x^1)] \\ &= \beta_{p+1} \left(\frac{1}{3p} - \frac{1}{3p+1} \right) x^{3p+1} + o(x^{3p+2}) = \alpha_{p+1} x^{3(p+1)-2} + o(x^{3(p+1)-1}) \end{aligned}$$

and

$$\begin{aligned} h_{p+2}(x) &= \sum_{i=1}^{p+1} \binom{p+1}{i} w_i(x) w_{p+2-i}(x) \\ &= \sum_{i=1}^{p+1} \binom{p+1}{i} \left[\alpha_i x^{3i-2} + o(x^{3i-1}) \right] \left[\alpha_{p+2-i} x^{3(p+2-i)-2} + o(x^{3(p+2-i)-1}) \right] \\ &= \sum_{i=1}^{p+1} \binom{p+1}{i} \left[\alpha_i \cdot \alpha_{p+2-i} x^{3(p+1)-1} + o(x^{3(p+1)}) \right] \\ &= \beta_{p+2} x^{3(p+1)-1} + o(x^{3(p+1)}), \end{aligned}$$

where $\alpha_{p+1} = \beta_{p+1} \left(\frac{1}{3p} - \frac{1}{3p+1} \right) > 0$ and $\beta_{p+2} = \sum_{i=1}^{p+1} \binom{p+1}{i} \alpha_i \cdot \alpha_{p+2-i} > 0$. Thus (3.7) also holds for $k = p + 1$. \blacksquare

Proposition 3.2.2 *For each $k \in \mathbb{N}$, zero is a regular value of $M_k : X \rightarrow \mathbb{R}^k$. Hence the sets $\mathcal{Z}(M_j)$ are nested manifolds and $\Sigma^{[k]} = \mathcal{Z}(M_k)$. For all $u \in X$, $u \in \mathcal{S}_k$ if and only if $M_k(u) = 0$ and $M_{k+1}(u) \neq 0$.*

Proof. From Proposition 3.1.1, the set $\{\text{grad } \tilde{I}_1(u), \dots, \text{grad } \tilde{I}_k(u)\}$ is linearly independent if and only if the right hand sides $\{r_2(u), \dots, r_{k+1}(u)\}$ of system (2.14) are. Since IVP matrices are invertible, this in turn is equivalent to the independence of the right hand sides $\{h_2(u), \dots, h_{k+1}(u)\}$ associated with solutions w_j satisfying

$$w(0) = 0, \quad w'(0) = e_1, \quad j = 1, \dots, k + 1.$$

We prove the linear independence of $\{h_2(u), \dots, h_{k+1}(u)\}$. The case $k = 1$ is clear as $h_2(u) = [w_1(u)]^2 \neq 0$. For $k \geq 2$, let $c_2, c_3, \dots, c_{k+1} \in \mathbb{R}$ be such that

$$c_2 h_2 + c_3 h_3 + \dots + c_{k+1} h_{k+1} = 0.$$

By (3.7), as $x \rightarrow 0$,

$$c_2[\beta_2 x^2 + o(x^3)] + c_3[\beta_3 x^5 + o(x^6)] + \dots + c_{k+1}[\beta_{k+1} x^{3k-1} + o(x^{3k})] = 0,$$

where $\beta_2, \beta_3, \dots, \beta_{k+1}$ are positive numbers. Clearly $c_2 = 0$ otherwise x^2 is an $o(x^3)$ -function, a contradiction. Similarly, $c_3 = \dots = c_{k+1} = 0$.

From Proposition 2.3.2, $\Sigma^{[k]} = \mathcal{Z}(M_k)$. The last statement is easy. ■

At this point we completed the proof of the first two items of Theorem 2.3.3. We next prove item (iii). In Proposition 4.4.2 we prove (iv).

3.3

Every critical point belongs to some $\mathcal{Z}(M_k) \setminus \mathcal{Z}(M_{k+1})$

We prove item (iii) of Theorem 2.3.3.

Proposition 3.3.1 *For each $u \in X$ there is $j \in \mathbb{N}$ such that $M_j(u) \neq 0$.*

Proof. Suppose by contradiction that there is $u_0 \in X$ such that $M_k(u_0) = 0$ for all $k \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}_0$,

$$\partial_{\phi(u_0)}^{(k)} \tilde{I}_1(u_0) = \tilde{I}_{k+1}(u_0) = 0.$$

From Proposition 2.1.1, $\tilde{I}_1 : X \rightarrow \mathbb{R}$ is a real analytic map when restricted to the line $\{u_0 + t\phi(u_0) \mid t \in \mathbb{R}\}$, and thus the function

$$t \in \mathbb{R} \mapsto \tilde{I}_1(u_0 + t\phi(u_0)) \in \mathbb{R}$$

is identically zero: the line $u_0 + \mathbb{R}\phi(u_0)$ is contained in \mathcal{C} . For the curve of eigenfunctions $t \in \mathbb{R} \mapsto \omega(t) = \phi(u_0 + t\phi(u_0)) \in X$, for $t \in \mathbb{R}$,

$$-\omega(t)'' + [u_0 + t\phi(u_0)]\omega(t) = 0.$$

Setting $\omega^{(j)}(t) = \frac{\partial^j}{\partial t^j} \omega(t)$ for $j \in \mathbb{N}_0$, for every $k \in \mathbb{N}$,

$$-\omega^{(k)}(t)'' + [u_0 + t\phi(u_0)]\omega^{(k)}(t) = -k\phi(u_0)\omega^{(k-1)}(t). \quad (3.8)$$

Define $\omega_1 = \omega(0) = \phi(u_0)$ and $\omega_{k+1} = \omega^{(k)}(0)$. Evaluating (3.8) in $t = 0$,

$$-\omega_{k+1}'' + u_0 \omega_{k+1} = -k\phi(u_0)\omega_k. \quad (3.9)$$

As $\phi(u_0 + t\phi(u_0)) = z_1(u_0 + t\phi(u_0))$, we have $\omega^{(k)}(0) = \partial_{\phi(u_0)}^{(k)} z_1(u_0)$ and

$$\omega_{k+1} = \partial_{\phi(u_0)}^{(k)} z_1(u_0) = z_{k+1}(u_0) \quad (k \in \mathbb{N}).$$

By (3.9), $r_{k+1}(u_0) = k\phi(u_0)z_k(u_0)$ and setting $k = 2$, $3\phi(u_0)z_2(u_0) = 2\phi(u_0)z_2(u_0)$. Thus $\phi(u_0)z_2(u_0) = 0$. Since the number of zeros of $\phi(u_0)$ is finite, $z_2(u_0) = 0$ and then $r_2(u_0) = z_2(u_0)'' - u_0z_2(u_0) = 0$. As $r_2(u_0) = z_1(u_0)^2 = \phi(u_0)^2$, we derive $\phi(u_0) = 0$, a contradiction. ■

4

Constructing Singularities of Arbitrary Depth

We prove the second part of Theorem 1.0.2: for $k \geq 1$, the operator F admits a Morin singularity of depth k , i.e., $\mathcal{Z}(M_k) = \Sigma^{[k]}$ is nonempty. The process is inductive: we show how to obtain a singularity of depth $k+1$ from a singularity of depth k . Section 4.2.3 considers the case k odd. The subsequent sections handle k even, which is harder.

The argument is indirect. From Section 2.3, a Morin singularity of depth k is a zero of $M_k : X \rightarrow \mathbb{R}^k$. We first show that this map extends smoothly to $\hat{M}_k : \hat{Y} \rightarrow \mathbb{R}^k$, where $\hat{Y} = H^{-1}([0, 1]) = (H_0^1([0, 1]))^*$ is the *dual space* of $\hat{X} = H_0^1([0, 1])$ (see [24]). We construct a zero of \hat{M}_{k+1} and finally a density argument provides the required singularity in X .

In a nutshell, the problem is discretized: we find zeros of the maps \hat{M}_k given by sums of deltas. Notice that we do not consider extensions of $F : X \rightarrow Y$: the space \hat{Y} contains distributions and interpretation of the term $f(u) = \frac{1}{2}u^2$ would be required. On the other hand, as it's shown in the next section, the relevant spectral properties of the Jacobian $DF(u) : X \rightarrow Y$ hold for more general potentials $u \in \hat{Y} = H^{-1}([0, 1])$.

4.1

Extending M_k

The definition of the maps $M_k : X \rightarrow \mathbb{R}^k$ required two steps. For $u \in X$, first solve sequentially the initial value problems,

$$-w_j(u)'' + u w_j(u) = -h_j(u), \quad w(u)(0) = 0, \quad w(u)'(0) = a_j,$$

where $h_1(u) = 0$, $a_1 \neq 0$ and $h_j(u) = \sum_{i=1}^{j-1} \binom{j-1}{i} w_i(u) w_{j-i}(u)$ for $j = 2, \dots, k$. If $w_1(u)$ is Dirichlet then, for $j \geq 2$, set $I_j(u) = \langle h_j(u), \varphi(u) \rangle$.¹ Here $\varphi(u)$ is the eigenfunction associated to some eigenvalue λ of $DF(u) : X \rightarrow Y$ for which $\varphi(0) = 0$, $\varphi'(0) = 1$. We now give meaning to both steps for $u \in \hat{Y} = H^{-1}([0, 1])$.

¹Originally, the functionals I_j were described in terms of a different normalization, $I_j = \langle r_j, z_1 \rangle$. This is not relevant, since both expressions differ by a nonzero multiplicative factor, so that the common roots $\mathcal{Z}(M_j)$ are the same for both definitions.

Sturm-Liouville problems with potentials in \hat{Y} , and the related spectral theory, have been considered recently by a number of authors ([18, 19]). We provide a sketch of the necessary arguments.

For a smooth function u , the IVP

$$-w'' + u w = -h, \quad w(0) = 0, \quad w'(0) = a,$$

admits an integral counterpart

$$w(x) = ax + \int_0^x ds \int_0^s dt [u(t)w(t) + h(t)].$$

For general $u \in \hat{Y}$, we have to give meaning to the expression $\int_0^s u(t)w(t) dt$. Back to the smooth context, we can write, for a primitive U (so that $U' = u$),

$$\int_0^s u(t)w(t) dt = U(s)w(s) - \int_0^s U(t)w'(t) dt,$$

where replacing U by $U + \text{const}$ has no effect in the expression. In particular, we may take U (L^2 -) orthogonal to 1.

Set $\mathcal{W} = \{w \in H^1([0, 1]) \mid w(0) = 0\}$.

Lemma 4.1.1 *Let $h \in \mathcal{W}$ and $a \in \mathbb{R}$. For each $u \in \hat{Y}$ the equation*

$$w(x) = ax + \int_0^x ds \left[U(s)w(s) - \int_0^s U(t)w'(t) dt \right] + \int_0^x ds \int_0^s h(t) dt$$

has a unique solution $w = w(u) \in \mathcal{W}$, defining a function

$$w : \hat{Y} \rightarrow \mathcal{W}, \quad u \mapsto w(u),$$

which is weakly analytic when restricted to lines of \hat{Y} .

Proof. We leave the details to the reader. The key point is that $u \in \hat{Y}$ is the (distributional) derivative of a function $U \in L^2([0, 1])$ orthogonal to the constant 1 (orthogonality specifies U uniquely as a smooth function of u). Then all terms in sight belong to L^1 and the standard estimates for the iteration follow. ■

Thus, functions $u \in \hat{Y} \mapsto w_j(u), h_{j+1}(u) \in \mathcal{W}$ are well defined and weakly analytic on lines of \hat{Y} (recall $w_j \in \mathcal{W}$, in which multiplication is defined and continuous, so that $h_j \in \mathcal{W} \subset \hat{Y}$).

Let $u \in \hat{Y}$. To give sense to an eigenfunction $\varphi(u)$, instead of the usual eigenvalue equation for the Jacobians $DF(u) : X \rightarrow Y, v \mapsto -v'' + uv$, we

consider

$$(T(u) - \lambda(u)) \cdot \varphi(u) = 0$$

for nonzero $\varphi(u) \in \hat{X} \subset \hat{Y}$ and $\lambda(u) \in \mathbb{R}$, where

$$\begin{aligned} T(u) : \hat{X} &\rightarrow \hat{Y} \\ w &\mapsto T(u) \cdot w = \langle w, \cdot \rangle_{\hat{X}} + \langle wu, \cdot \rangle. \end{aligned}$$

For $w \in \hat{X}$, wu is the element in \hat{Y} such that $\langle wu, v \rangle = \langle u, wv \rangle$ for all $v \in \hat{X}$.

We prove smoothness of the eigenvalue and eigenfunction maps in the larger setting $u \in \hat{Y}$ as a consequence of the following result, Proposition 16 in [11] (see also [25]) for the current situation.

Proposition 4.1.2 *Let $T(u_*) \in \mathcal{B}(\hat{X}, \hat{Y})$ have eigenvalue $\lambda_* = 0 \in \mathbb{R}$ and eigenvector $\varphi_* \in \hat{X}$, so that $(T(u_*) - \lambda_* I)\varphi_* = 0$. Assume that $T(u_*) - \lambda_* I$ is a Fredholm operator of index zero with one dimensional kernel, and that $\varphi_* \notin \text{Ran}(T(u_*) - \lambda_* I)$. Let $\ell \in \hat{Y} = \hat{X}^*$ be a linear functional for which $\ell(\varphi_*) = 1$ and set $W = \varphi_* + \ker \ell$. Then there is an open neighborhood $V \subset \mathcal{B}$ of $T(u_*)$ and unique smooth maps $\lambda : V \rightarrow \mathbb{R}$ and $\varphi : V \rightarrow W$ for which $(T - \lambda(T)I)\varphi(T) = 0$ and $\lambda(T(u_*)) = \lambda_*$, $\varphi(T(u_*)) = \varphi_*$ with the usual analytic properties when restricted to lines of \hat{Y} .*

In the case of interest, the operators in V are of the form $T(u)$, $u \in B \subset \hat{Y}$, where B is an open neighborhood of u . The functional ℓ provides a linear normalization of the eigenfunction, which simplifies the argument. Clearly, different choices of a normalization of the eigenvector are innocuous.

Proof. In order to prove smoothness, it suffices to check the hypotheses of the proposition.

- $T(u_*) \in \mathcal{B}(\hat{X}, \hat{Y})$.

Begin by the following fact. The *multiplication map*

$$\begin{aligned} \hat{X} \times \hat{Y} = H_0^1([0, 1]) \times H^{-1}([0, 1]) &\rightarrow H^{-1}([0, 1]) = \hat{Y} \\ (w, u) &\mapsto wu \end{aligned}$$

is bi-linear and continuous. Bi-linearity is clear. To show continuity take $(w, u) \in \hat{X} \times \hat{Y}$. For every $v \in \hat{X}$,

$$\begin{aligned} |\langle wu, v \rangle| &= |\langle u, wv \rangle| \leq \|u\|_{\hat{Y}} \|wv\|_{\hat{X}} = \|u\|_{\hat{Y}} \|(wv)'\|_{L^2} \\ &= \|u\|_{\hat{Y}} \|wv' + w'v\|_{L^2} \\ &\leq \|u\|_{\hat{Y}} (\|wv'\|_{L^2} + \|w'v\|_{L^2}) \\ &\leq \|u\|_{\hat{Y}} (\|w\|_{sup} \|v'\|_{L^2} + \|w'\|_{L^2} \|v\|_{sup}) \\ &\leq 2\|u\|_{\hat{Y}} \|w\|_{\hat{X}} \|v\|_{\hat{X}}. \end{aligned}$$

Thus

$$\|wu\|_{\hat{Y}} \leq 2\|w\|_{\hat{X}}\|u\|_{\hat{Y}}.$$

Therefore $T(u_*) : \hat{X} \rightarrow \hat{Y}$ is well defined, linear and bounded:

$$\begin{aligned} \|T(u_*) \cdot w\|_{\hat{Y}} &\leq \|\langle w, \cdot \rangle_{\hat{X}}\|_{\hat{Y}} + \|wu\|_{\hat{Y}} \leq \|w\|_{\hat{X}} + 2\|w\|_{\hat{X}}\|u_*\|_{\hat{Y}} \\ &\leq (1 + 2\|u_*\|_{\hat{Y}}) \|w\|_{\hat{X}}. \end{aligned}$$

- $T(u_*) - \lambda_*$ is a Fredholm operator of index zero.

For fixed $u \in \hat{Y}$, the linear operator $w \in \hat{X} \mapsto wu \in \hat{Y}$ is compact with norm bounded by $2\|u\|_{\hat{Y}}$. Compactness is clear for $u \in C^0$, as $wu \in C^0$ and the inclusion $C^0 \hookrightarrow \hat{Y}$ is compact. For $u \in \hat{Y}$, approximate by continuous functions and recall that the uniform limit of compact operators is also compact.

Since $w \in \hat{X} \mapsto \langle w, \cdot \rangle_{\hat{X}} \in \hat{Y}$ is an isomorphism and $w \in \hat{X} \mapsto wu_* \in \hat{Y}$ is compact, $T(u_*) - \lambda_*$ is a Fredholm operator of index zero.

- $\dim \ker(T(u_*) - \lambda_*) = 1$.

The argument is standard: if there are two independent kernel elements, there is a kernel element solving the (second order) homogeneous differential equation with trivial initial conditions at 0.

- $\varphi_* \notin \text{Ran}(T(u_*) - \lambda_*)$.

First note that for $u \in \hat{Y}$, $T(u)$ is symmetric in the sense that, for every $w, v \in \hat{X}$, $\langle T(u) \cdot w, v \rangle = \langle T(u) \cdot v, w \rangle$:

$$\begin{aligned} \langle T(u) \cdot w, v \rangle &= \langle w, v \rangle_{\hat{X}} + \langle wu, v \rangle = \langle w, v \rangle_{\hat{X}} + \langle u, wv \rangle \\ &= \langle v, w \rangle_{\hat{X}} + \langle u, vw \rangle = \langle v, w \rangle_{\hat{X}} + \langle vu, w \rangle = \langle T(u) \cdot v, w \rangle. \end{aligned}$$

Thus, if $w \in \hat{X}$ is such that $(T(u_*) - \lambda_*) \cdot w = \varphi_*$ then

$$0 \neq \langle \varphi_*, \varphi_* \rangle = \langle (T(u_*) - \lambda_*) \cdot w, \varphi_* \rangle = \langle (T(u_*) - \lambda_*) \cdot \varphi_*, w \rangle = 0.$$

The proposition then applies, and smoothness of eigenvalue and eigenvector maps is proved. ■

We finally extend $M_k : X \rightarrow \mathbb{R}^k$.

Corollary 4.1.3 *Let $k \in \mathbb{N}$. For $u_* \in \hat{Y}$ for which $\lambda_n(u_*) = 0$, consider $B \subset \hat{Y}$ defined in Proposition 4.1.2. For $j = 1, \dots, k$, the map*

$$\hat{I}_j : B \subset \hat{Y} \rightarrow \mathbb{R}, \quad u \mapsto \langle r_j(u), \varphi(u) \rangle$$

is smooth. $\hat{M}_k = (\hat{I}_1, \dots, \hat{I}_k) : B \subset \hat{Y} \rightarrow \mathbb{R}^k$ is well defined and each coordinate is analytic on lines of $\hat{Y} \cap B$.

Proof. Let $u \in \hat{Y}$. From the lemma above, for arbitrary choices of initial conditions a_j , we can solve the recursive integral system, obtaining functions $w_j(u), r_j(u) \in \hat{X}$. ■

We introduce the subspace $X^\delta \subset \hat{Y}$:

$$X^\delta = \left\{ u \in \hat{Y} \mid u = \sum_{i=1}^N c_i \delta_{x_i}, N \in \mathbb{N}, c_i \in \mathbb{R}, 0 < x_1 < \dots < x_N < 1 \right\}.$$

For $u \in H^1 + X^\delta$, the differential equations in the definition of $\mathcal{Z}(\hat{M}_k)$ are easily interpreted without referring to their weak versions. More precisely, for $u = \tilde{u} + \sum_{i=1}^N c_i \delta_{x_i}$,

$$-w'' + uw = -h$$

is a standard differential equation in $(0, 1) \setminus \{x_1, \dots, x_N\}$ and, at x_i ,

$$w'(x_i^+) - w'(x_i^-) = c_i w(x_i).$$

Elements of X^δ contain deltas, but we still call them functions, to distinguish them from possibly more complicated distributions in \hat{Y} .

Proposition 4.1.4 *The Jacobians of \hat{M}_k satisfy the same formula than those of M_k . For functions $u \in X^\delta \cap \mathcal{Z}(\hat{M}_k)$, zero is a regular value and, near such points, $\mathcal{Z}(\hat{M}_k)$ are manifolds.*

Proof. The formulae for the directional derivatives of I_j extend by smoothness of \hat{M}_k . Transversality (Proposition 3.2) has been proved by verifying properties of functions near zero: they are unaltered for $u \in X^\delta$. ■

We will construct elements of $\mathcal{Z}(\hat{M}_{k+1})$ in X^δ . The space \hat{Y} is required because the density argument obtaining a deep singularity $u \in \mathcal{Z}(M_{k+1}) = \Sigma^{[k+1]} \subset X$ cannot start from a space which is not complete, as X^δ .

4.2 Operating with singularities

We introduce some techniques to generate singularities.

4.2.1 Scaling

The McKean-Scovel operator acts on functions defined in $[0, 1]$. We consider the obvious variation: for $I = [a, b]$, $X_I = H^2(I) \cap H_0^1(I)$ and $Y_I = L^2(I)$, $F_I : X_I \rightarrow Y_I$, $u \mapsto -u'' + \frac{1}{2}u^2$. Statements about F convert easily to F_I : the critical set \mathcal{C}_I of F_I consists only of Morin singularities, which are characterized as zeros of a map $M_{k,I} : X_I \rightarrow \mathbb{R}^k$, $\Sigma_I^{[k]} = \mathcal{Z}(M_{k,I})$. There are obvious counterparts $\hat{X}_I, \hat{Y}_I, X_I^\delta$.

Proposition 4.2.1 *The natural scaling map $\hat{S} : \hat{X}_I \rightarrow \hat{X}$ (resp. the restricted scaling $S : X_I \rightarrow X$),*

$$u \mapsto v = \hat{S}(u), \quad v(x) = (b-a)^2 u(a + (b-a)x) \text{ for } x \in [0, 1],$$

is a diffeomorphism between $\mathcal{Z}(\hat{M}_{k,I})$ and $\mathcal{Z}(\hat{M}_k)$ (resp. $\mathcal{Z}(M_{k,I})$ and $\mathcal{Z}(M_k)$).

Proof. We prove the statement for the restricted scaling, to avoid cluttering the text with integrals. If $u \in X_I$ belongs to $u \in \mathcal{Z}(M_{k,I})$ then there are functions $w_1(u), \dots, w_k(u) \in X_I$, with $w_1(u) \neq 0$, satisfying the recursive system

$$\begin{cases} -w_j(u)'' + u w_j(u) = -h_j(u), & j = 1, \dots, k, \\ h_1(u) = 0, \quad h_j(u) = \sum_{i=1}^{j-1} \binom{j-1}{i} w_i(u) w_{j-i}(u), & j \geq 2. \end{cases}$$

Set $v = S(u)$. We must find $w_1(v), \dots, w_k(v) \in X$, with $w_1(v) \neq 0$, for which

$$\begin{cases} -w_j(v)'' + v w_j(v) = -h_j(v), & j = 1, \dots, k, \\ h_1(v) = 0, \quad h_j(v) = \sum_{i=1}^{j-1} \binom{j-1}{i} w_i(v) w_{j-i}(v), & j \geq 2. \end{cases}$$

Define $w_j(v) : [0, 1] \rightarrow \mathbb{R}$ by

$$w_j(v)(x) = \left(\frac{1}{b-a}\right)^{3j-2} w_j(u)(a + (b-a)x).$$

If $k \geq 2$, for $j = 2, \dots, k$, a simple computation gives,

$$\begin{aligned} h_j(v)(x) &= \sum_{i=1}^{j-1} \binom{j-1}{i} w_i(v)(x) w_{j-i}(v)(x) \\ &= \left(\frac{1}{b-a}\right)^{3j-4} h_j(u)(a + (b-a)x). \end{aligned}$$

So $-w_j(v)'' + v w_j(v) = -h_j(v)$, $j = 1, \dots, k$, and therefore $v \in \mathcal{Z}(M_k)$.

Similar computations apply to the inverse scaling $S^{-1} : X \rightarrow X_I$: S is indeed a diffeomorphism between the sets $\mathcal{Z}(M_{k,I})$ and $\mathcal{Z}(M_k)$. \blacksquare

Scaling will be frequently used in the sequel, without explicit mention.

4.2.2

Juxtaposition

Let $u = \sum_{i=1}^N c_i \delta_{x_i} \in X_{[a,c]}^\delta$ and $v = \sum_{i=1}^M d_i \delta_{y_i} \in X_{[c,b]}^\delta$. The *juxtaposition* $(uv) \in X_{[a,b]}^\delta$ is

$$(uv) = \sum_{i=1}^{N+M} e_i \delta_{z_i},$$

where

$$e_i, z_i = \begin{cases} c_i, x_i & \text{for } i = 1, \dots, N, \\ d_{i-N}, y_{i-N} & \text{for } i = N+1, \dots, N+M. \end{cases}$$

We extend accordingly. For example,

$$w_j(uv) = \begin{cases} w_j(u) & \text{if } x \in [a, c] \\ w_j(v) & \text{if } x \in [c, b] \end{cases}.$$

From now on, whenever the information is obvious, we drop the reference to the underlying interval.

Proposition 4.2.2 *Suppose $u \in \mathcal{Z}(\hat{M}_k) \cap X_{[a,c]}^\delta$ and $v \in \mathcal{Z}(\hat{M}_k) \cap X_{[c,b]}^\delta$.*

1. $(uv) + \alpha \delta_c \in \mathcal{Z}(\hat{M}_k) \cap X_{[a,b]}^\delta$ for any $\alpha \in \mathbb{R}$.
2. If $(uv) \in \mathcal{Z}(\hat{M}_{k+1}) \cap X_{[a,b]}^\delta$ then either $u, v \in \mathcal{Z}(\hat{M}_{k+1}) \cap X^\delta$ or $u, v \notin \mathcal{Z}(\hat{M}_{k+1}) \cap X^\delta$.

Proof. To prove (1), we solve the differential system (2.8) for $w_j(uv)$ piecewise, starting from point c in both cases, as in Proposition 2.3.9. More precisely, in $[a, c]$, consider Dirichlet solutions $w_j(u) \in \hat{X}_{[a,c]}$ of system (2.8) with $w'_1(u)(c) = 1$, $w'_j(u)(c) = 0$, $j = 2, \dots, k$. In $[c, b]$, solve the system for Dirichlet solutions $w_j(v) \in \hat{X}_{[c,b]}$ satisfying $w'_j(v)(c) = w'_j(u)(c)$. The derivatives of w_j are the same at both sides of c and, close to c , both u and v belong to X (indeed, they are zero near c). The functions $w_j(uv)(x)$ then solve the system in $[a, b]$ for the potential (uv) . Furthermore, the same functions solve the system also for the potential $(uv) + \alpha \delta_c$. Indeed, at c , both $w_j(u)$ and $w_j(v)$ are zero, and the term $\alpha \delta_c$ is irrelevant in the verification that $w_j(uv)$ solves $-w_j(uv) + ((uv) + \alpha \delta_c)w_j(uv) = -h_j(uv)$.

To prove (2), let $w_1(uv), \dots, w_k(uv), w_{k+1}(uv) \in \hat{X}_{[a,b]}$, $w_1(uv) \neq 0$, solve the usual system for the potential (uv) . Note that the restrictions of these functions to the intervals $[a, c]$ and $[c, b]$ solve the usual system for the potentials u and v , respectively, and since $u, v \in \mathcal{Z}(\hat{M}_k)$ then such restrictions

are Dirichlet for $j = 1, \dots, k$. Now

$$\begin{aligned} u \in \mathcal{Z}(\hat{M}_{k+1}) &\Leftrightarrow (w_{k+1}(uv) \upharpoonright_{[a,c]})(c) = 0 \\ &\Leftrightarrow (w_{k+1}(uv) \upharpoonright_{[c,b]})(c) = 0 \Leftrightarrow v \in \mathcal{Z}(\hat{M}_{k+1}). \end{aligned}$$

■

Thus juxtaposition of two functions in $\mathcal{Z}(\hat{M}_k) \cap X^\delta$ provides a *line* in $\mathcal{Z}(\hat{M}_k) \cap X^\delta$. Successive juxtapositions yield additional degrees of freedom, which may be adjusted to obtain a function z satisfying an additional scalar request, $z \in \mathcal{Z}(\hat{M}_{k+1}) \cap X^\delta$. For odd k , the process is much simpler: this is the content of Proposition 4.2.3.

4.2.3

Symmetrization: from odd depth k to depth $k + 1$

Let $u = \sum_{i=-N}^{-1} c_i \delta_{x_i} \in X_{[-1,0]}^\delta$. The *symmetrization* $\hat{u} \in X_{[-1,1]}^\delta$ of u is the juxtaposition of u with its reflection u_r , $\hat{u} = (uu_r)$:

$$\hat{u} = \sum_{i=-N}^{-1} c_i \delta_{x_i} + \sum_{i=1}^N c_i \delta_{x_i},$$

where, for $i = 1, \dots, N$, $c_i = c_{-i}$ and $x_i = -x_{-i}$.

Proposition 4.2.3 *Let k be odd. If $u \in \mathcal{Z}(\hat{M}_k)$ then $\hat{u} \in \mathcal{Z}(\hat{M}_{k+1})$.*

Proof. From Proposition 2.3.5 and Corollary 2.3.9, for $u \in \mathcal{Z}(\hat{M}_k)$ the IVP starting from $x = 0$ in $[-1, 0]$ admits Dirichlet solutions

$$-w_j(u)'' + u w_j(u) = -h_j(u), \quad j = 1, \dots, k,$$

$$h_1(u) = 0, \quad h_j(u) = \sum_{i=1}^{j-1} \binom{j-1}{i} w_i(u) w_{j-i}(u), \quad j = 2, \dots, k,$$

$$w_1(u)(0) = w_j(u)(0) = 0, \quad w_1(u)(-1) = w_j(u)(-1) = 0, \quad j = 2, \dots, k,$$

$$w_1(u)'(0) = 1, \quad w_j(u)'(0) = 0, \quad j = 2, \dots, k.$$

A right hand side h_{k+1} is obtained from the known functions w_1, \dots, w_k . Using Proposition 2.3.4, we show that $\hat{I}_{k+1}(\hat{u}) = \langle h_{k+1}(\hat{u}), \phi(\hat{u}) \rangle = 0$, which then implies $\hat{u} \in \mathcal{Z}(\hat{M}_{k+1})$.

For $j = 1, \dots, k$, define the odd extensions $w_j(\hat{u}) : [-1, 1] \rightarrow \mathbb{R}$ by

$$w_j(\hat{u})(x) = \begin{cases} w_j(u)(x) & \text{if } x \in [-1, 0] \\ (-1)^j w_j(u)(-x) & \text{if } x \in [0, 1] \end{cases}.$$

From the formulas for h_j ,

$$h_j(\hat{u})(x) = \begin{cases} h_j(u)(x) & \text{if } x \in [-1, 0] \\ (-1)^j h_j(u)(-x) & \text{if } x \in [0, 1] \end{cases}.$$

Thus, changes of sign of $w_j(\hat{u})$ and $h_j(\hat{u})$ are simultaneous and satisfy

$$-w_j(\hat{u})'' + u w_j(\hat{u}) = -h_j(\hat{u}), \quad j = 1, \dots, k$$

in $[-1, 1]$, as the derivatives of $w_j(\hat{u})$ at both sides of zero are equal for $j = 1$ (as w_1 is odd) or, even more, are equal to zero, by the prescribed initial conditions, for $j > 1$. Thus, $\hat{u} \in \hat{M}_k \cap X_{[-1,1]}^\delta$. Moreover, from the formula, $h_{k+1}(\hat{u})$ is even. Since $w_1(\hat{u})$ is an odd function, we have $\hat{I}_{k+1}(\hat{u}) = 0$, as desired. ■

4.3

Some deep singularities in X^δ

The results in this chapter were inspired by extensive symbolic computation. Two key ingredients have been presented: (a) for k odd, symmetrization allows to obtain singularities of depth $k + 1$ from singularities of depth k , (b) juxtaposition generates a line of singularities of depth k , and possibly a choice of a point in such line is of depth $k + 1$.

From these ingredients, by sheer computation, potentials $u_k \in \mathcal{Z}(\hat{M}_k)$ up to $k = 11$ were derived out of the simplest possible eigenfunction (the case $k = 1$). More precisely, split $I_k = [-2^k, 2^k]$ in 2^k equal intervals, and assign $2^k + 1$ deltas at their endpoints (zero at the extremes -2^k and 2^k). We list the weights of the deltas, for depth k from 1 to 5.

$$(0, -2, 0)$$

$$(0, -2, 0, -2, 0)$$

$$(0, -2, 0, -2, 0, -2, -106/21, -2, 0)$$

$$(0, -2, 0, -2, 0, -2, -106/21, -2, 0, -2, -106/21, -2, 0, -2, 0, -2, 0)$$

$$(0, -2, 0, -2, 0, -2, -106/21, -2, 0, -2, -106/21, -2, 0, -2, 0, -2, 0, -2, 0,$$

$$-2, -30971960/2300571, -2, -106/21, -2, 0, -2, -106/21, -2, 0, -2, 0, -2, 0)$$

The adjusting points (which may be guessed from the values) are at entries 6, 20, 88, 336, ... (entries are counted from 0). Going from k odd to $k + 1$ is just symmetrization, other transitions require adjustment. We do not know if the process continues indefinitely: up to $k = 11$, the equation for the

point to choose in the line of potentials in $\mathcal{Z}(\hat{M}_k)$ is a linear polynomial (!) with nonzero leading coefficient, and the computation of the adjusting value is trivial.

In order to complete the proof of Theorem 1.0.2, we must show that appropriate juxtapositions yield lines giving rise to such simple, solvable equations. The argument is of an existential nature.

An alternative scenario may serve as a warning. Suppose one searches for a normal form of deep Morin singularities in \mathbb{R}^n for which all Jacobians are symmetric matrices. The same ingredients used above, starting with

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the potential $u = (-2, 0, -2)$ (in perfect analogy with the case $k = 2$ above, as extremal coordinates do not come up in the Dirichlet discretization) gives rise to a sequence of potentials u_k , $k = 2, \dots, 6$, yielding in turn normal forms (in \mathbb{R}^{2^k-1}) of the first Morin singularities. But the computations break down for $k = 7$: the linear equation to be solved has zero leading coefficient.

Still, the proof of Theorem 1.0.2 can be adapted to show the existence of these special normal forms.

In the next section, we introduce the appropriate transversality hypothesis which guarantees that arbitrarily deep singularities may be obtained from the two basic ingredients.

4.4

Adjustable Functions

We consider *adjustable functions*. Section 4.4.1 shows how to get functions in $\mathcal{Z}(\hat{M}_{2\ell+1})$ from an adjustable function $\vartheta \in \mathcal{Z}(\hat{M}_{2\ell}) \cap X^\delta$. Section 4.4.2 proves the existence of such ϑ .

Set $\ell \in \mathbb{N}$. A function $\vartheta \in \mathcal{Z}(\hat{M}_{2\ell}) \cap X_{[a,b]}^\delta$ is *adjustable* at $x_* \in (a, b)$ if and only if there exist $u \in \mathcal{Z}(\hat{M}_\ell) \cap X_{[a,x_*]}^\delta \setminus \mathcal{Z}(\hat{M}_{\ell+1})$ and $v \in \mathcal{Z}(\hat{M}_\ell) \cap X_{[x_*,b]}^\delta \setminus \mathcal{Z}(\hat{M}_{\ell+1})$ such that $\vartheta = (uv)$. We then write $\vartheta = (u|v)$. Both u and v have no term $c\delta_{x_*}$. As we will see, juxtaposition of u and v then yields a line in $\mathcal{Z}(\hat{M}_{2\ell}) \cap X_{[a,b]}^\delta$ through ϑ containing an element $\vartheta^* \in \mathcal{Z}(\hat{M}_{2\ell+1})$. The number x_* is called the *adjustment point* of ϑ .

4.4.1

From even k to $k + 1$

The whole section is dedicated to the proof of Proposition 4.4.2.

Lemma 4.4.1 *Let M be an IVP matrix with first column given by*

$$m_1 = (1, 0, \dots, 0, m_{\ell+1,1}, \dots, m_{2\ell+1,1}) .$$

Then

1. $m_{j,j} = 1, \quad j = 1, \dots, 2\ell + 1.$
2. $m_{j,s} = 0, \quad s = 2, \dots, 2\ell, \quad j = s + 1, \dots, \min\{s + \ell - 1, 2\ell + 1\}.$
3. $m_{\ell+p+1,p+1} = c_{\ell,p} \cdot m_{\ell+1,1}, \quad c_{\ell,p} \neq 0, \quad p = 1, \dots, \ell.$

Recall that M is a lower triangular matrix. For $\ell \geq 2$,

$$M = \begin{pmatrix} 1 & & & & \\ 0 & \ddots & & & \\ m_{\ell+1,1} & & 1 & & \\ \vdots & \ddots & 0 & \ddots & \\ m_{2\ell+1,1} & \cdots & m_{2\ell+1,\ell+1} & 0 & 1 \end{pmatrix}_{(2\ell+1) \times (2\ell+1)} .$$

Proof. From Proposition 2.3.5, for $s \geq 2$ and $j \geq s$,

$$m_{j,s} = \frac{1}{s-1} \sum_{i=1}^{j-s+1} \binom{j-1}{i} m_{i,1} m_{j-i,s-1} .$$

Recall that $m_{i,1} = 0$ for $2 \leq i \leq \ell$.

1. Indeed, $m_{j,j} = (m_{1,1})^j.$
2. For $2 \leq s \leq \ell + 1$ or $\ell + 2 \leq s \leq 2\ell$ we have $2 \leq j - s + 1 \leq \ell$. Then

$$m_{j,s} = \frac{1}{s-1} \sum_{i=1}^{j-s+1} \binom{j-1}{i} m_{i,1} \cdot m_{j-i,s-1} = \frac{j-1}{s-1} m_{j-1,s-1} .$$

Iterate: $m_{j,s} = 0$, from $m_{i,1} = 0$ for $2 \leq i \leq \ell$.

3. If $p = 1, \dots, \ell$,

$$\begin{aligned} m_{\ell+p+1,p+1} &= \frac{1}{p} \sum_{i=1}^{\ell+1} \binom{\ell+p}{i} m_{i,1} \cdot m_{\ell+p+1-i,p} \\ &= \frac{\ell+p}{p} m_{1,1} \cdot m_{\ell+p,p} + \frac{1}{p} \binom{\ell+p}{\ell+1} m_{\ell+1,1} \cdot m_{p,p} \\ &= \frac{\ell+p}{p} m_{\ell+p,p} + \frac{1}{p} \binom{\ell+p}{\ell+1} m_{\ell+1,1} . \end{aligned}$$

By induction, $m_{\ell+p+1,p+1} = (p+1) \left(\prod_{i=2}^{p+1} \frac{\ell+i}{i} \right) m_{\ell+1,1} = c_{\ell,p} \cdot m_{\ell+1,1}$, where $c_{\ell,p} = (p+1) \left(\prod_{i=2}^{p+1} \frac{\ell+i}{i} \right) \neq 0$ only depends of ℓ and p .

■

Proposition 4.4.2 *If there is an adjustable function in $\mathcal{Z}(\hat{M}_{2\ell})$, then $\mathcal{Z}(\hat{M}_{2\ell+1})$ is not empty.*

Proof. Set $\ell \geq 1$ and let $\vartheta \in \mathcal{Z}(\hat{M}_{2\ell}) \cap X^\delta$ be an adjustable function with adjustment point $x_* \in (0, 1)$. Let also

$$w(\vartheta) = (w_1(\vartheta), \dots, w_{2\ell}(\vartheta), w_{2\ell+1}(\vartheta)) : [0, 1] \rightarrow \mathbb{R}^{2\ell+1},$$

with $w_1(\vartheta) \neq 0$, solving

$$-w(\vartheta)'' + \vartheta w(\vartheta) = -q(w(\vartheta)), \quad w(\vartheta)(0) = 0. \quad (4.1)$$

Since $\vartheta \in \mathcal{Z}(\hat{M}_{2\ell})$, by Proposition 2.3.7, the first 2ℓ components of $w(\vartheta)$ are Dirichlet. Note that $w_{2\ell+1}(\vartheta)(1)$ is not necessarily zero: we only know $\vartheta \in \mathcal{Z}(\hat{M}_{2\ell})$. Moreover, as ϑ is adjustable and different solutions of the system (4.1) are related by (lower triangular) IVP matrices, we have

$$w_1(\vartheta)(x_*) = \dots = w_\ell(\vartheta)(x_*) = 0 \quad \text{and} \quad w_{\ell+1}(\vartheta)(0) \neq 0.$$

Set $\alpha = w(\vartheta)(x_*)$: we have $\alpha_1 = \dots = \alpha_\ell = 0$ and $\alpha_{\ell+1} \neq 0$. Also, from Proposition 2.3.9, $w(\vartheta)$ can be chosen so that

$$w(\vartheta)'(x_*) = e_1 \in \mathbb{R}^{2\ell+1}.$$

Consider the line

$$C \in \mathbb{R} \mapsto \vartheta(C) = \vartheta + C \cdot \delta_{x_*} \in X^\delta.$$

We have that to find $w(\vartheta(C)) : [0, 1] \rightarrow \mathbb{R}^{2\ell+1}$, with $w_1 \neq 0$ and Dirichlet components $w_j, j = 1, \dots, 2k$, which solves

$$-w(\vartheta(C))'' + \vartheta(C) w(\vartheta(C)) = -q(w(\vartheta(C))), \quad w(\vartheta(C))(0) = 0. \quad (4.2)$$

Clearly, the solution must be continuous and also must take into account the jump in the derivative at x_* ,

$$w(\vartheta(C))'(x_*^+) = w(\vartheta(C))'(x_*^-) + C \cdot w(\vartheta(C))(x_*). \quad (4.3)$$

We define $w(\vartheta(C))$ such that $w(\vartheta(C)) = w(\vartheta)$ in $[0, x_*]$. The required jump of the derivative at x_* is given by

$$w(\vartheta(C))'(x_*^+) = e_1 + C \cdot \alpha.$$

Let M be the $(2\ell + 1) \times (2\ell + 1)$ IVP matrix such that $Mw(\vartheta)'(x_*) = e_1 + C \cdot \alpha$. As $w(\vartheta)'(x_*) = e_1$ then $m_1 = Me_1 = e_1 + C \cdot \alpha$, i.e.,

$$m_1 = (1, 0, \dots, 0, C \cdot \alpha_{\ell+1}, \dots, C \cdot \alpha_{2\ell+1}), \quad \alpha_{\ell+1} \neq 0,$$

and, in particular, $m_{\ell+1,1} = C \cdot \alpha_{\ell+1}$. In the interval $[x_*, 1]$, the functions $w(\vartheta)$ and $Mw(\vartheta)$ at x_* may be different. From the form of the IVP matrix M (Lemma 4.4.1) and since $m_{2\ell+1,\ell+1} = c_{\ell,\ell} \cdot m_{\ell+1,1}$, $c_{\ell,\ell} \neq 0$ (item (3)),

$$\begin{aligned} Mw(\vartheta)(x_*) - w(\vartheta)(x_*) &= M\alpha - \alpha \\ &= m_{2\ell+1,\ell+1} \cdot \alpha_{\ell+1} e_{2\ell+1} \\ &= c_{\ell,\ell} \cdot m_{\ell+1,1} \cdot \alpha_{\ell+1} e_{2\ell+1} \\ &= c_{\ell,\ell} \cdot (\alpha_{\ell+1})^2 \cdot C e_{2\ell+1} \\ &= A_{2\ell+1} \cdot C e_{2\ell+1}, \end{aligned}$$

where $A_{2\ell+1} = c_{\ell,\ell} \cdot (\alpha_{\ell+1})^2 \neq 0$ does not depend of C .

Now define

$$w(\vartheta(C)) = \begin{cases} w(\vartheta) & \text{if } x \in [0, x_*] \\ Mw(\vartheta) - A_{2\ell+1} \cdot C \cdot \psi(\vartheta) e_{2\ell+1} & \text{if } x \in [x_*, 1] \end{cases},$$

where $\psi(\vartheta) : [x_*, 1] \rightarrow \mathbb{R}$ solves

$$-\psi(\vartheta)'' + \vartheta \psi(\vartheta) = 0, \quad \psi(\vartheta)(x_*) = 1, \quad \psi(\vartheta)'(x_*) = 0.$$

Clearly $w(\vartheta(C))$ solves (4.2), is continuous and the jump (4.3) of the derivative is satisfied, as $\psi(\vartheta)'(x_*) = 0$. Moreover, the first 2ℓ components of $w(\vartheta(C))$ are Dirichlet (Proposition 2.3.7).

To handle the last component, we start by proving the following fact: $w_{2\ell+1}(\vartheta(C))(1)$ is a non-constant linear polynomial in the variable C .

First recall that $\psi(\vartheta)$ solves $-z'' + \vartheta z = 0$ in $[x_*, 1]$, with $\psi(\vartheta)(x_*) = 1$, $\psi(\vartheta)'(x_*) = 0$. Also, $w_1(\vartheta)$ is a Dirichet solution to the same equation. From the Wronskian identity, $\psi(\vartheta)(1) \neq 0$. Therefore $A_{2\ell+1} \cdot \psi(\vartheta)(1) \neq 0$ and

$$\begin{aligned}
w_{2\ell+1}(\vartheta(C))(1) &= \langle e_{2\ell+1}, Mw(\vartheta)(1) \rangle - A_{2\ell+1} \cdot C \cdot \psi(\vartheta)(1) \\
&= \langle Me_{2\ell+1}, w(\vartheta)(1) \rangle - A_{2\ell+1} \cdot C \cdot \psi(\vartheta)(1) \\
&= \langle e_{2\ell+1}, w(\vartheta)(1) \rangle - A_{2\ell+1} \cdot C \cdot \psi(\vartheta)(1). \\
&= w_{2\ell+1}(\vartheta)(1) - A_{2\ell+1} \cdot C \cdot \psi(\vartheta)(1).
\end{aligned}$$

As $w_{2\ell+1}(\vartheta)(1)$ is a constant which does not depend of C , we are done.

We are ready to complete the proof of the proposition. From the fact above, we can choose $C = C^*$ such that $w_{2\ell+1}(\vartheta(C^*))(1) = 0$ and therefore, for $\vartheta^* = \vartheta(C^*) = \vartheta + C^* \cdot \delta_{x_*}$, all the components of $w(\vartheta^*)$ are Dirichlet: ϑ^* belongs to $\mathcal{Z}(\hat{M}_{2\ell+1}) \cap X^\delta$. \blacksquare

4.4.2 Existence of Adjustable Functions

In Proposition 4.4.5 we obtain adjustable functions in $\mathcal{Z}(\hat{M}_{2\ell}) \cap X^\delta$ for $\ell \in \mathbb{N}$. As an example, which is also the first step in the inductive argument, we show how to obtain an adjustable function in $\mathcal{Z}(\hat{M}_2) \cap X^\delta$. In other words, we show that there are $u, v \in \mathcal{Z}(\hat{M}_1) \cap X^\delta \setminus \mathcal{Z}(\hat{M}_2)$ such that $(u|v) \in \mathcal{Z}(\hat{M}_2)$.

Set $u = -2 \delta_{-1} \in X_{[-2,0]}^\delta$ and let $w_1(u) : [-2, 0] \rightarrow \mathbb{R}$ solves the IVP

$$-w_1(u)'' + u w_1(u) = 0, \quad w_1(u)(-2) = 0, \quad w_1(u)'(-2) = 1.$$

Thus

$$w_1(u)(x) = \begin{cases} 2 + x & \text{if } x \in [-2, -1] \\ -x & \text{if } x \in [-1, 0] \end{cases}.$$

Since $w_1(u)$ also satisfies the Dirichlet condition, $u \in \mathcal{Z}(\hat{M}_1)$ and $w_1(u)$ is an eigenfunction associated to $\lambda = 0$. Moreover, $w_1(u)$ does not change sign in $[-2, 0]$, so that $\hat{I}_2(u) = \int_{-2}^0 w_1(u)^3 \neq 0$ and therefore $u \notin \mathcal{Z}(\hat{M}_2)$. Let $u_r \in \mathcal{Z}(\hat{M}_1) \cap X_{[0,2]}^\delta$ be the reflection of u . Clearly $u_r \notin \mathcal{Z}(\hat{M}_2)$ and $\vartheta = \hat{u} = (u|u_r) \in X_{[-2,2]}^\delta$. By symmetrization (Proposition 4.2.3), $\vartheta \in \mathcal{Z}(\hat{M}_2)$ and is adjustable at $x_* = 0$.

The following result is standard.

Lemma 4.4.3 X^δ is dense in \hat{Y} .

We also employ the following density theorem ([14, 12]).

Theorem 4.4.4 Let X be a Banach space, D a dense subspace of X and M a smooth submanifold of finite codimension of X . Then $M \cap D$ is dense in M .

If $\vartheta = (u|v) \in \mathcal{Z}(\hat{M}_{2\ell})$ is adjustable, we also write $\vartheta^* \in \mathcal{Z}(\hat{M}_{2\ell+1})$ obtained in the previous section as $\vartheta^* = (u \uparrow v)$.

Proposition 4.4.5 *For $\ell \in \mathbb{N}$, if $\mathcal{Z}(M_{2\ell})$ is nonempty, there exists an adjustable function $\vartheta \in \mathcal{Z}(\hat{M}_{2\ell}) \cap X^\delta$.*

Proof. The proof is by induction. The case $\ell = 1$ was shown above. Suppose the statement holds for $\ell = 1, \dots, p-1$ ($p \geq 2$). Let $\mathcal{Z}(M_{2p}) \neq \emptyset$. We must find an adjustable function $\vartheta_{2p} \in \mathcal{Z}(\hat{M}_{2p}) \cap X^\delta$.

As the $\mathcal{Z}(M_j)$'s are nested then $\mathcal{Z}(M_p)$ is non-empty and therefore $\mathcal{Z}(M_p) \setminus \mathcal{Z}(M_{p+1}) \neq \emptyset$. Let $\tilde{u} \in \mathcal{Z}(M_p) \setminus \mathcal{Z}(M_{p+1})$. For concreteness, suppose that \tilde{u} is defined in the interval² $[-1, 0]$. Since X^δ is dense in \hat{Y} , by density (Theorem 4.4.4), there is $u \in \mathcal{Z}(\hat{M}_p) \setminus \mathcal{Z}(\hat{M}_{p+1}) \cap X^\delta$ near \tilde{u} . Let u_r be the reflection of u . Then $u_r, (uu_r) \in \mathcal{Z}(\hat{M}_p)$. We first prove that, for some $v_1^* \in X^\delta$, we have $(uv_1^*) \in \mathcal{Z}(\hat{M}_{p+1}) \cap X^\delta$.

- For odd p , $(uu_r) \in \mathcal{Z}(\hat{M}_{p+1}) \cap X^\delta$ (symmetrization, Proposition 4.2.3). Take $v_1^* = u_r$.
- For even p , we must work harder. As $p/2 \in \{1, \dots, p-1\}$, from the inductive hypothesis, take an adjustable $\vartheta_p = (u_p|v_p) \in \mathcal{Z}(\hat{M}_p) \cap X^\delta$ with $u_p, v_p \in \mathcal{Z}(\hat{M}_{\frac{p}{2}}) \cap X^\delta \setminus \mathcal{Z}(\hat{M}_{\frac{p}{2}+1})$. Then, by the properties of juxtaposition (Proposition 4.2.2),

$$\tilde{u}_p = ((uu_r)u_p) = (u(u_ru_p)) \in \mathcal{Z}(\hat{M}_{\frac{p}{2}}) \cap X^\delta \setminus \mathcal{Z}(\hat{M}_{\frac{p}{2}+1})$$

and, as $(uu_r), \vartheta_p \in \mathcal{Z}(\hat{M}_p)$,

$$((uu_r)\vartheta_p) = (((uu_ru_p)v_p) = (\tilde{u}_p v_p) \in \mathcal{Z}(\hat{M}_p) \cap X^\delta.$$

Thus $(\tilde{u}_p|v_p) \in \mathcal{Z}(\hat{M}_p) \cap X^\delta$ is adjustable. From Proposition 4.4.2, we obtain $(\tilde{u}_p \uparrow v_p) = (uvu_p \uparrow v_p) \in \mathcal{Z}(\hat{M}_{p+1}) \cap X^\delta$.

In both cases, for $v_1^* = u_r \in X^\delta$ or $((vu_p) \uparrow v_p) \in X^\delta$, respectively, we have $(uv_1^*) \in \mathcal{Z}(\hat{M}_{p+1}) \cap X^\delta$.

In order to obtain the adjustable function $\vartheta_{2p} \in \mathcal{Z}(M_{2p}) \cap X^\delta$, we obtain sequentially, for $q = 1, \dots, p$, adjustable functions $\vartheta_{p+q} \in \mathcal{Z}(M_{p+q}) \cap X^\delta$. The starting point of the induction $q = 1$ is the first step above.

Suppose by induction that, for $q = 1, \dots, p-1$, there is $v_q^* \in X^\delta$ such that $(u|v_q^*) \in \mathcal{Z}(\hat{M}_{p+q}) \cap X^\delta$. We follow closely the argument above.

²Still, we drop references to it: new operations will naturally change the domains of definition of the functions of interest, and we systematically invoke implicitly the invariance of singularity depth under scaling (Proposition 4.2.1).

- If $p + q$ is odd,

$$((uv_q^*)\theta_q) = (u(v_q^*\theta_q)) \in \mathcal{Z}(\hat{M}_{p+q+1}) \cap X^\delta,$$

where $\theta_q = (uv_q^*)_r$ is the reflection of (uv_q^*) .

- If $p + q$ is even, take an adjustable $\vartheta_{p+q} = (u_{p+q}|v_{p+q}) \in \mathcal{Z}(\hat{M}_{p+q}) \cap X^\delta$ for $u_{p+q}, v_{p+q} \in \mathcal{Z}(\hat{M}_{\frac{p+q}{2}}) \cap X^\delta \setminus \mathcal{Z}(\hat{M}_{\frac{p+q}{2}+1})$. Then

$$\tilde{u}_{p+q} = ((uv_q^*)u_{p+q}) = (u(v_q^*u_{p+q})) \in \mathcal{Z}(\hat{M}_{\frac{p+q}{2}}) \cap X^\delta \setminus \mathcal{Z}(\hat{M}_{\frac{p+q}{2}+1})$$

and

$$((uv_q^*)\vartheta_{p+q}) = (((uv_q^*)u_{p+q})|v_{p+q}) = (\tilde{u}_{p+q}|v_{p+q}) \in \mathcal{Z}(\hat{M}_{p+q}) \cap X^\delta.$$

Thus $(\tilde{u}_{p+q}|v_{p+q}) \in \mathcal{Z}(\hat{M}_{p+q}) \cap X^\delta$ is adjustable and therefore $(\tilde{u}_{p+q} \uparrow v_{p+q}) = (u((v_q^*u_{p+q}) \uparrow v_{p+q})) \in \mathcal{Z}(\hat{M}_{p+q+1}) \cap X^\delta$.

Again, for $v_{q+1}^* = (uv_q^*)_r \in X^\delta$ or $((v_q^*u_{p+q}) \uparrow v_{p+q}) \in X^\delta$, we have $(uv_{q+1}^*) \in \mathcal{Z}(\hat{M}_{p+q+1}) \cap X^\delta$. The proof of the inductive step is complete.

By setting $q = p - 1$, we obtain a function $v_p^* \in X^\delta$ such that $(uv_p^*) \in \mathcal{Z}(\hat{M}_{2p}) \cap X^\delta$. Since $u \in \mathcal{Z}(\hat{M}_p) \cap X^\delta \setminus \mathcal{Z}(\hat{M}_{p+1})$, by Proposition 4.2.2, we also have $v_p^* \in \mathcal{Z}(\hat{M}_p) \cap X^\delta \setminus \mathcal{Z}(\hat{M}_{p+1})$: $\vartheta_{2p} = (u|v_p^*) \in \mathcal{Z}(\hat{M}_{2p}) \cap X^\delta$ is adjustable. ■

4.5

There exist Morin singularities of arbitrary depth

We are ready to prove the second part of Theorem 1.0.2: F admits Morin singularities of arbitrary depth, i.e., $\mathcal{Z}(M_k) \neq \emptyset$ for all $k \geq 1$. As $\mathcal{Z}(M_1) \neq \emptyset$ all we need is the inductive step below.

Proposition 4.5.1 *For each $k \in \mathbb{N}$, $\mathcal{Z}(M_k) \neq \emptyset$ implies $\mathcal{Z}(M_{k+1}) \neq \emptyset$.*

Proof. Let $k \in \mathbb{N}$ and suppose $\mathcal{Z}(M_k) \neq \emptyset$. Since $\mathcal{Z}(M_k) \subset X \subset \hat{Y}$, then $\mathcal{Z}(\hat{M}_k) \cap X^\delta \neq \emptyset$ by density (Theorem 4.4.4).

If k is odd, $\mathcal{Z}(\hat{M}_{k+1}) \cap X^\delta \neq \emptyset$, by symmetrization (Proposition 4.2.3).

If k is even, there exists an adjustable function in $\mathcal{Z}(\hat{M}_k) \cap X^\delta$ (Proposition 4.4.5) which in turn yields a function $u \in \mathcal{Z}(\hat{M}_{k+1}) \cap X^\delta$ (Proposition 4.4.1).

The density argument (Theorem 4.4.4) applies since, near u , $\mathcal{Z}(\hat{M}_{k+1})$ is a submanifold of \hat{Y} of finite codimension (Proposition 4.1.4) and X is dense in \hat{Y} : $X \cap \mathcal{Z}(\hat{M}_{k+1}) = \mathcal{Z}(M_{k+1})$ is nonempty. ■

The proof of Theorem 1.0.2 is complete.

4.6

An application of Theorem 1.0.2

McKean and Scovel [6] proved that each $g \in Y$ has at most a finite number of pre-images in X under F . This number can be arbitrarily large, as shown by a more general result by Ruf and Solimini [16]. Here, we show that an arbitrarily large number of solutions may be arbitrarily close to each other.

Corollary 4.6.1 *Given $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $g \in Y$ such that*

$$-u'' + \frac{1}{2}u^2 = g, \quad u \in X,$$

has $k + 1$ solutions which are at most ε apart.

Proof. Let $u_* \in X$ be a Morin singularity of depth k and consider the normal form of F near u_* , given in equation (1.1). Now choose small numbers s_1, \dots, s_{k-1} for which the polynomial $t^{k+1} + s_1 t^{k-1} + s_2 t^{k-2} + s_{k-1} t$ admits some value with $k + 1$ preimages. ■

Bibliography

- [1] AMBROSETTI, A.; PRODI, G. **On the inversion of some differentiable mappings with singularities between Banach spaces.** Ann. Mat. Pura Appl., 93:231–246, 1972.
- [2] MANES, A.; MICHELETTI, A. M. **Un estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine.** Boll. Un. Mat. Ital., 7:285–301, 1973.
- [3] BERGER, M. S.; PODOLAK, E. **On the solutions of a nonlinear Dirichlet problem.** Indiana Univ. Math. Journal, 24:837–846, 1975.
- [4] BERGER, M. S.; CHURCH, P. T.; TIMOURIAN, J. G. **Folds and cusps in Banach spaces, with applications to nonlinear partial differential equations. i.** Indiana Univ. Math. Journal, 34:1–19, 1985.
- [5] BERGER, M. S.; CHURCH, P. T.; TIMOURIAN, J. G. **Folds and cusps in Banach spaces, with applications to nonlinear partial differential equations. ii.** Transactions of the American Mathematical Society, 307:225–244, 1988.
- [6] MCKEAN, H. P.; SCOVEL, J. C. **Geometry of some simple nonlinear differential operators.** Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 13 (2):299–346, 1985.
- [7] MCKEAN, H. P. **Singularities of a simple elliptic operator.** J. Differential Geometry, 25:157–165, 1987.
- [8] MCKEAN, H. P. **Correction to: Singularities of a simple elliptic operator.** J. Differential Geometry, 36:255, 1992.
- [9] BALBONI, F.; DONATI, F. **Singularities of Fredholm maps with one-dimensional kernels, I: A complete classification.** arXiv: Functional Analysis, 1:1–67, 2014.
- [10] CAFAGNA, V.; DONATI, F. **Un résultat global de multiplicité pour un problème différentiel non linéaire du premier ordre.** C. R. Acad. Sc. Paris, 300:523–526, 1985.

- [11] CALANCHI, M.; TOMEI, C.; ZACCUR, A. **Abundance of cusps and a converse to the Ambrosetti-Prodi theorem.** arXiv:1508.01436v1, p. 1–26, 2015.
- [12] BURGHELEA, D.; SALDANHA, N. C.; TOMEI, C. **Results on infinite dimensional topology and applications to the structure of the critical set of nonlinear Sturm-Liouville operators.** J. Differential Equations, 188:569–590, 2003.
- [13] BUENO, H.; TOMEI, C. **Critical sets of nonlinear Sturm-Liouville operators of Ambrosetti-Prodi type.** Nonlinearity, 15-4:1073–1077, 2002.
- [14] MALTA, I.; SALDANHA, N. C.; TOMEI, C. **Morin singularities and global geometry in a class of ordinary differential operators.** Topological Methods in Nonlinear Analysis, 10:137–169, 1997.
- [15] RUF, B. **Singularity theory and bifurcation phenomena in differential equations.** In: TOPOLOGICAL NONLINEAR ANALYSIS II, p. 315–395, Boston, 1997.
- [16] RUF, B.; SOLIMINI, S. **On a class of superlinear Sturm–Liouville problems with arbitrarily many solutions.** SIM J. Math. Anal. <https://doi.org/10.1137/0517055>, 17-4:761–771, 1986.
- [17] MORIN, B. **Formes canoniques de singularités d’une application différentiable.** C.R. Acad. Sc. Paris, 260:5662–5665, 6503–6506, 1965.
- [18] SAVCHUK, A.M.; SHKALIKOV, A.A. **Sturm-Liouville operators with distribution potentials.** Trans. Moscow Math. Soc., p. 143–192, 2003.
- [19] ECKHARDT, J.; GESZTESY, F.; NICHOLS, R.; TESCHL, G. **Weyl-Titchmarsh theory for sturm-liouville operators with distributional potentials.** Opuscula Math. <http://dx.doi.org/10.7494/OpMath.2013.33.3.467>, 33-3:467–563, 2013.
- [20] PÖSCHEL, J.; TRUBOWITZ, E. **Inverse Spectral Theory.** AP, Orlando, Florida (US), 1st edition, 1987.
- [21] ARNOLD, V. I.; GORYUNOV, V. V.; LYASHKO, O. V.; VASIL’EV, V. **Singularity Theory I.** Springer, Berlin (Ge), 1st edition, 1998.
- [22] SCHMÜDGEN, K. **Unbounded Self-adjoint Operators on Hilbert Space.** Springer, Berlin (Ge), 1st edition, 2012.

- [23] GOHBERG, I.; GOLDBERG, S.; KAASHOEK, M. A. **Basic Classes of Linear Operators**. Birkhäuser, Berlin (Ge), 1st edition, 2000.
- [24] BREZIS, H. **Functional Analysis, Sobolev Spaces and Partial Differential Equations**. Springer-Universitext, 1st edition, 2011.
- [25] ZEIDLER, E. **Nonlinear Functional Analysis and its Applications, IV**. Springer, New York, 1997.