ON THE SOLUTION OF H^2/H^∞ OPTIMAL PROBLEM

M. A. DA SILVEIRA

Abstract. Quadratic convergence is know for the Galerkin approximation sequence used to solve the H^2/H^∞ optimal control problem. We show in this paper that is also converges in the H^∞ topology, at least for Laguerre-type generator sets. Some comments are made enlightening the Hilbertian context used to solve the problem.

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PII.

1. Introduction. A question was raised in [1] about uniform convergence for a Galerkin approximation of the optimal controller solving the mixed H^2/H^∞ optimal control problem. In the present paper this uniform convergence is proved, at least for Laguerre-type complete sets. The proof is based on a result from Szegö [2]. Also, some comments are made enlightening the Hilbertian context used to solve the problem.

More explicitly, in [1] the mixed H^2/H^∞ optimal control problem for continuous time systems - a quadratic criterium to be minimized under H^∞ constraints - was formulated in a Hilbertian context in such a way that:

1) the proof of existence and unicity of the optimal solution was obtained;

2) it was remarked that, as a consequence of [3], the optimal solution was not finite-dimensional, neither exponentially stable, except for "trivial" cases - but the sense of "trivial" was not completely explaned;

3) it was proved that the solution transfer function is continuous in the extended imaginary axis, which means that it can be approached by rational transfer function in the H^∞ topology [4];

4) an approximating sequence for the optimal control transfer function was defined by a Galerkin method.

The convergence of the approximating sequence was proved in a Hilbert space cointaining the space H^∞, but with a strictly coarse topology. But the uniform convergence (in the closed right complex semiplane) was not proved, which raises the question about conditions for such property.

The strong H^∞ convergence is necessary to the usual robustness considerations, i. e., to warrant the stabilizing property for the finite-dimensional controller approaching the optimal one, dating from some order. See [5] for connections between this question and the spillover problem. Moreover, as a consequence of the Arzela-Ascoli Theorem, if the H^∞ convergence is not verified, a subsequence will present undesirable spikes in its Bode diagrams. The examples in [1], [6] and [7] did not present such spikes in spite of their complexity, a situation claiming more research.

In the next section, some preliminary Lemmas will be shown, as an introdutory material for the main results. In the third section conditions for the announced H^∞ convergence will be presented. The fourth section gives an example enlightening the sense of "trivial" in point 3 above.

Notations. The set of real polynomiais in the variable z will be denoted by $\boldsymbol{R}[z]$. The class of bounded analytic functions on the open right complex semiplan will be denoted as H_+^∞, and the class of bounded analytic functions on the complex unit disk as H_∞. The usual sup norm for both spaces will be represented by $\|f(.)\|_\infty$, the supremum taken on the imaginary axis or in the complex unit circle, respectively. Following the same denotation principle, the usual quadratic Hardy class on the open right complex semiplan will be represented by H_+^2, and the corresponding function class on the complex unit disk will be represented by H_2 (see [8] or [9] for the theory of Hardy spaces). The inner product in H_2 will be explicitly given by:

$$\int_{-\pi}^{\pi} f(e^{i\theta})\overline{g}(e^{i\theta})d\theta = \int_{-\pi}^{\pi} f(e^{i\theta})g[(e^{i\theta})^{-1}]d\theta$$

The norm in H_+^2 wil be denoted as $\|f(s)\|_2$, and the norm in H_2 as $\|f(z)\|_2$, the variable expliciting the space. The class of H_+^∞ functions continuous on the completed imaginary axis will be denoted by A. It is the closure of stable proper rational functions in the H_+^∞ topology [4]. The class of H_∞ functions continuous on the unit circle will be denoted by C. It is the closure of $\boldsymbol{R}[z]$ in H_∞ [4].

Let $H_+^{2,-1}$ be the weighted Hardy space defined in [1] as the class of functions $f(s)$ such that $(s+a)f(s)$ belongs to H_+^2 for some strictly positive real number a, with a inner product defined by:

$$\int_{-\infty}^{\infty} f(i\omega)\frac{2a}{\omega^2+a^2}\overline{g}(i\omega)d\omega = \int_{-\infty}^{\infty} f(i\omega)\frac{2a}{\omega^2+a^2}g(-i\omega)d\omega$$

and an associated Hilbertian norm defined by

$$\|f(s)\|_{2,-1} = \left\|\frac{\sqrt{2a}}{s+a}f(s)\right\|_2 .$$

This function space was introduced to merge H_+^2 and H_+^∞ in a Hilbert space where the H^2/H^∞ is well posed.

Finally, to simplify some statements, the real rational stable and proper functions will be denoted by S, and the subset of real rational stable and strictly proper functions will be denoted by S_{sp}.

Remark 1. In [1], the inner product and the norm of $H_+^{2,-1}$ are defined with $a = 1$ and without the factor $\sqrt{2a}$. These changes define equivalent norms, as it was showed in the same reference, Theorem 5. The present choice eases some statements in the following.

2. Some preliminary Lemmas. The first Lemma collects some results from [1] (Theorems 2 and 3) and from [4] (pp. 640, 668).

LEMMA 1. *The function classes defined above satisfy the following set-inclusions:*

(a) $H_+^{2,-1} \supset H_+^\infty \supset A \supset S$.

(b) $H_+^{2,-1} \supset H_+^2 \supset S_{sp}$.

(c) $H_2 \supset H_\infty \supset C \supset \boldsymbol{R}[z]$.

A and C are closed subsets of H_+^∞ and H_∞, respectively. For all the others inclusions, the smaller set is dense in the largest.

The next statement shows that the role of the space $H_+^{2,-1}$ for continuous-time systems is the same as H_2 for discrete-time systems, when the domain, the open unit disk, is bounded. It is interesting to recall that H_+^2 corresponds to a closed subspace of H_2 for sampled systems, because its functions are null at infinity - which it is not necessary for $H_+^{2,-1}$ functions.

LEMMA 2. *Let "a" be a strictly positive real number. Define a bilinear transformation in the complex plane by* $z = \phi(s) = (a-s)(a+s)^{-1}$. *Then* $z = \phi(s)$ *defines a unitary transformation between* $H_+^{2,-1}$ *and* H_2, *and a isometry between* H_+^{∞} *and* H_∞, *which induces a isometry between A and C.*

Proof outline. A straightforward calculation shows that $z = \phi(s)$ is invertible, with $s = \phi^{-1}(z) = a(1-z)(1+z)^{-1}$, transforming the completed (with a point at infinity) closed right complex semiplan onto the closed unit disk, and the completed imaginary axis onto the unit circle, with $\phi(\infty) = -1$. Using polar coordinates, the last transformation can be described by

$$\cos(\theta) = \frac{a^2-\omega^2}{a^2+\omega^2},\ \sin(\theta) = \frac{-2a\omega}{a^2+\omega^2},\ \omega = -a.tg(\frac{\theta}{2}),\ \frac{d\omega}{d\theta} = -a\frac{1-\cos(\theta)}{\sin^2(\theta)}.$$

Therefore, the inner product on $H_+^{2,-1}$ can be written as

$$\int_{-\pi}^{\pi} f(i\omega)\frac{2a}{\omega^2+a^2}g(-i\omega)d\omega = \int_{-\pi}^{\pi} f[\phi^{-1}(s)]\bar{g}[\phi^{-1}(s)]\,|_{z=e^{i\theta}}\, d\theta =$$

$$\int_{-\pi}^{\pi} f[-a.tg(\frac{\theta}{2})i]g[a.tg(\frac{\theta}{2})i]d\theta$$

because

$$\frac{2a}{\omega^2+a^2} = \frac{2\sin^2(\theta)}{a\{\sin^2(\theta)+[1-\cos(\theta)]^2\}} = \frac{\sin^2(\theta)}{a[1-\cos(\theta)]} = \left[\frac{d\omega}{d\theta}\right]^{-1},$$

which proves Lemma 2. □

This bilinear transformation sends Laguerre-type rational function to polynomials in z. Explicitly, $f(s) = \sum_{k=1}^{n}\frac{b_k}{(s+a)^k}$ is transformed into $f[\phi^{-1}(z)] = \sum_{k=1}^{n}\frac{b_k}{a^k}(z+1)^k$. In particular, the Laguerre functions $L_n(s) = \sqrt{2a}(s-a)^{n-1}(s+a)^{-n}$ are transformed into $L_n[\phi^{-1}(z)] = \sqrt{2a^{-3}}(-z)^{n-1}(z+1)$. These comments prove the next Lemma.

LEMMA 3. *Let* $f(s)$ *be a rational function with all its poles at* $s = a$. *Then,* $f[\phi^{-1}(z)]$ *is a polynomial in* $\boldsymbol{R}[z]$.

The fourth Lemma repeats Theorem 13.1.3 from [2] pg. 314.

LEMMA 4. *Let* $m(\theta)$ *be a positive weighting function in the unit circle* $e^{i\theta}$ *which satisfies the Bernstein condition* $|m(\theta+\delta)-m(\theta)| < M|\ln(\delta)|^{-(\lambda+1)}$ *for some* $M > 0$ *and some* $\lambda > 1$, δ *sufficiently small. Let* $\{p_k(z)\}$ *be a set of orthonormal polynomials in the unit circle associated to* $m(\theta)$, *complete on C. Let* $f(z)$ *be a function on C, and let* $f_n(z)$ *and* $F_n(z)$ *be the nth-partial sum in its expansion in the orthonormal set of polynomials and the nth-partial sum of its Taylor series, respectively. Then,* $\lim_{n\to\infty}\{f_n(z) - F_n(z)\} = 0$, *uniformly in the closed unit disk.*

Remark 2. In the case of a (eventually redundant) complete polynomial set in C, but not orthonormal with respect to $m(\theta)$, the Gramm-Schmidt algorithm produces an associated orthonormal polynomial set with exactly the same partial sums. In other words, Lemma 4 needs only a complete polynomial set in C (the completeness on H_2 being a consequence), the orthonormality assumption being superfluous.

3. Main results. After some parametrizations and calculations [1], the H^2/H^∞ optimal control problem for finite-dimensional linear systems can be reduced to minimize a quadratic functional

$$J_2[K(s)] = \left\|\Phi(s)[K(s)-\breve{K}(s)]\right\|_2 \text{ subject to } J_\infty[K(s)] = \left\|A(s)K(s)-B(s)\right\|_\infty \leq \lambda;$$

where $\Phi(s)$ is a real-rational, stable, strictly proper and miniphase function, $A(s)$ and $B(s)$ are proper real-rational functions, λ is a strictly positive real number such that the constraint set is not void, and $\breve{K}(s)$ is the unconstrained optimal solution, which can be calculated by an usual formula [1], [10]. Here, it will be considered the cases where the optimal solution $\hat{K}(s)$ belongs to $H_+^{2,-1}$, or to H_+^2. For the last case, $\Phi(s)$ needs to be biproper. Conditions for the existence of such solutions are given in [1], Theorems 7 and 11.

Similarly, a Galerkin algorithm was proposed to find the optimal solution $\hat{K}(s)$. The problem was solved on the n-dimensional subspace defined by the first n functions in a complete (but not necessarily orthogonal or independent) set on $H_+^{2,-1}$, being suggested the use of Laguerre-type functions. This means that the denominators of those functions are in the form $(s+a)^k$, for a strictly positive real number a. Denoting by $\hat{K}_n(s)$ the optimal solution for the H^2/H^∞ problem restricted to the n-dimensional subspace defined above, it was proved that the sequence $\{\hat{K}_n(s)\}$ converges to $\hat{K}(s)$ in the $H_+^{2,-1}$ topology ([1], Theorem 9). Moreover, $\hat{K}(s)$ belongs to A, i. e., it is continuous on the extended imaginary axis [1], Theorem 12. The following Theorem goes beyond these results.

Theorem 1. Consider the optimal solution of the H^2/H^∞ problem studied in [1]. *Suppose it verifies $\hat{K}(s) \in H_+^{2,-1}$ (i.e., the conditions on* [1], *Theorem 9, are verified), and also that the generator set $\{p_k(s)\}$ is Lagrange-type. If $\{\hat{K}_n(s)\}$ denotes the approximating sequence calculated by the Galerkin method associated to this generator set, then it converges to $\hat{K}(s)$ also in H_+^∞.*

Proof. As $\{\hat{K}_n(s)\}$ converges to $\hat{K}(s)$ in $H_+^{2,-1}$ by Lemma 2, $\hat{K}_n[\phi^{-1}(z)]$ converges to $\hat{K}[\phi^{-1}(z)]$ in the H_2 topology. In this case the weigthing function is $m(\theta) \equiv 1$, trivially verifying the Bernstein condition. Also, by Lemma 3, the corresponding complete set on C is polynomial. Thus, $\hat{K}_n[\phi^{-1}(z)] \equiv \sum_{k=1}^{n}\alpha_k p_k[\phi^{-1}(z)]$ defines a *nth*-partial sum of the Fourier series of $\hat{K}[\phi^{-1}(z)]$ for a complete (in C) set of polynomials. Therefore, Lemma 4 and Remark 2 apply.

Let β_k be the k^{th}-Taylor coefficient for $\hat{K}[\phi^{-1}(z)]$. Then

$$\left|\hat{K}[\phi^{-1}(z)]-\hat{K}_n[\phi^{-1}(z)]\right| = \left|\hat{K}[\phi^{-1}(z)]-\sum_{k=1}^{n}\beta_k z^k + \sum_{k=1}^{n}\beta_k z^k - \hat{K}_n[\phi^{-1}(z)]\right|$$

$$\leq \left|\hat{K}[\phi^{-1}(z)]-\sum_{k=1}^{n}\beta_k z^k\right| + \left|\hat{K}_n[\phi^{-1}(z)]-\sum_{k=1}^{n}\beta_k z^k\right|$$

At the last row, the first term converges uniformly in the closed unit disk by the Weierstrass Theorem, and the second term by Lemma 4. Therefore, $\hat{K}_n[\phi^{-1}(z)]$ converges to $\hat{K}[\phi^{-1}(z)]$ in H_∞.

Finally, by Lemma 2 we can return to functions in A with the H_+^∞ topology, which proves that $\hat{K}_n(s)$ converges to the optimal solution $\hat{K}(s)$ in the H_+^∞ topology. □

The H_+^2 situation is readily solved if we remember that this space is a subspace of $H_+^{2,-1}$ with a finer topology. Then, if $\hat{K}_n(s)$ converges on H_+^2, it converges also in $H_+^{2,-1}$, and Theorem 1 applies.

Corollary 1. Consider the optimal solution of the H^2/H^∞ problem studied in [1]. *Suppose it is such that $\hat{K}(s) \in H_+^2$ (i.e., the conditions on* [1], *Theorem 11, are verified), and also that the generator set is Lagrange-type. If $\{\hat{K}_n(s)\}$ denotes the approximating sequence calculated by the Galerkin method associated to this generator set, then it converges to $\hat{K}(s)$ also in H_+^∞.*

Remark 3. The topological arguments used to proof Theorem 1 hide the inner product relations, an interesting information for convergence estimates. To better understand these relations, it is necessary to apply the bilinear transformation directly to the quadratic functional. First, the functional $J_2[K(s)]$ can be rewritten as

$$J_2[K(s)] = \left\| \frac{\sqrt{2a}}{s+a}\left[\frac{s+a}{\sqrt{2a}}\Phi(s)\right]\left[K(s)-\breve{K}(s)\right]\right\|_2 = \left\| \frac{s+a}{\sqrt{2a}}\Phi(s)\left[K(s)-\breve{K}(s)\right]\right\|_{2,-1}.$$

The function $h(s) = [\sqrt{2a}]^{-1}(s+a)\Phi(s)$ is a real-rational biproper function with all poles and zeros in the open left complex semi-plan. Thus, $h[\phi^{-1}(z)]$ is also a real-rational biproper function, but with all poles and zeros at the exterior of the unit disk. Therefore,

$$m(\theta) = h[\phi^{-1}(e^{-i\theta})]h[\phi^{-1}(e^{i\theta})]$$

is a positive weighting function as in Lemma 4, bounded and differentiable in the unit circle, with a bounded derivative in the same set. An usual argument shows that this last property implies the Bernstein condition necessary to apply Lemma 4.

Second, assume that the generating set $\{p_k(s)\}$ used in the Galerkin method is a Laguerre-type generator set orthonormal for the weighted inner product in $H_+^{2,-1}$ defined by

$$\langle f(s), g(s)\rangle_h = \int_{-\infty}^{\infty} \bar{f}(i\omega)g(i\omega)[h(i\omega)\bar{h}(i\omega)]\frac{2a}{\omega^2+a^2}\,d\omega.$$

By Lemma 2 and the comments above, this inner product is transformed in a inner product on the unit circle weighted by the positive function $m(\theta)$. Also, $\hat{K}_n[\phi^{-1}(z)]$ converges (in the weighted quadratic mean) to $\hat{K}[\phi^{-1}(z)]$, a function belonging to C.

From these comments, $\hat{K}_n[\phi^{-1}(z)] = \sum_{k=1}^{n}\alpha_k p_k[\phi^{-1}(z)]$ defines a *nth*-partial sum of the Fourier series of $\hat{K}[\phi^{-1}(z)]$ for a set of polynomials in z, orthonormal in respect to the weighting function $m(\theta)$. Then, Lemma 4 implies the uniform convergence of $\hat{K}_n[\phi^{-1}(z)]$ to $\hat{K}[\phi^{-1}(z)]$. Lemma 2 and Remark 2 complete an alternative proof for Theorem 1.

Remark 4. The use of Cèsaro sums to obtain the uniform convergence for the optimal solution Fourier series do not apply, because the Cèsaro partial sum $\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\alpha_j p_j(s)$ does not coincide with $\hat{K}_n(s)$.

Remark 5. More general conditions can be given on the norms of Dirichlet kernels for the generating original set, as indicated in [11]. But the explicit connexion with usual orthonormal rational functions, as Laguerre functions, is not clear.

4. A finite dimensional solution for a particular H^2/H^∞ problem.

Finite dimensional solutions for the H^2/H^∞ problem here considered are obtained if the unconstrained solution $\breve{K}(s)$ verifies the H^∞ constraint. In the H_+^2 context there is no other possibility [3]. In the $H_+^{2,-1}$ context there are another possibilities: when the function $A(s)\breve{K}(s) - B(s)$ (see the beginning of section 3) is all-pass, $A(s)$ is miniphase and $A^{-1}(s)B(s)$ is stable and proper.

Indeed, it was showed in [1] that the H^2/H^∞ problem can be see as a best approximation problem: to find the convex projection of $\breve{K}(s)$ on the constraint defined by $J_\infty[K(s)] = \|A(s)K(s) - B(s)\|_\infty \le \lambda$ according to the $H_+^{2,-1}$ metric. If $A(s)\breve{K}(s) - B(s)$ is all-pass, with $\|A(s)\breve{K}(s) - B(s)\|_\infty = \mu$ and $\mu > \lambda$ (otherwise $\breve{K}(s)$ verifies itself the constraint), a variational argument on the quadratic metric shows that the optimal solution $\hat{K}(s)$ will be obtained by reducing $\breve{K}(s)$ at each frequence to give $\|A(s)\hat{K}(s) - B(s)\|_\infty = \lambda$.

Now, assume that $|A(i\omega)\hat{K}(i\omega) - B(i\omega)| < \lambda$ in a real subset E with positive measure. Let $\tilde{K}(i\omega)$ be a version of $\hat{K}(s)$ modified on E to perform the equality. Therefore, $\tilde{K}(i\omega)$ is bounded on the imaginary axis, verifies the constraint, and, necessarily, $J_2(\tilde{K}) < J_2(\hat{K})$. Thus, by the contrapositive implication, if $\hat{K}(s)$ is optimal, $|A(i\omega)\hat{K}(i\omega) - B(i\omega)| = \lambda$ on E.

In general, the function so defined is not racional, because a rational function cannot be constant on a finite and open imaginary axis interval. But, if $A(s)\breve{K}(s) - B(s)$ is all-pass, with $\|A(s)\breve{K}(s) - B(s)\|_\infty = \mu$ and $\mu > \lambda$, set set E coincides with all the imaginary axis. Therefore, if $A(s)$ and $B(s)$ satisfy the above assumptions, the function

$$\hat{K}(s) = (\lambda/\mu)\,(\lambda/\mu)\breve{K}(s) - [(\mu - \lambda)/\mu]A^{-1}(s)B(s)$$

satisfy the wished equality on the imaginary axis and is a proper real-rational function in H_+^∞. In conclusion, the proposed $\hat{K}(s)$ is the optimal solution.

An example is the problem defined by the minimization of

$$J_2[K(s)] = \int_{-\infty}^{\infty} \{K(-i\omega)\frac{1}{a^2 + \omega^2}K(i\omega) - 2K(-i\omega)\frac{1}{(i\omega + a)^2}U(i\omega)\}d\omega$$

$$\text{subject to } J_\infty[K(s)] = \|K(s)\|_\infty \le \lambda,$$

where $U(s)$ is any all-pass rational function. Here, $\Phi(s) = \frac{1}{s+a}$, $\breve{K}(s) = \frac{a-s}{a+s}U(s)$, and,if $\|\breve{K}(s)\|_\infty = \mu > \lambda$, the optimal solution is $\hat{K}(s) = (\lambda/\mu)\frac{a-s}{a+s}U(s)$, a rational function.

This comment enlightes the "trivial cases" in [1] Theorem 12.

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