Pontifícia Universidade Católica
do Rio de Janeiro

**Edison Fausto Cuba Huamani**

# Existence and regularity of solutions: nonlocal and nonlinear models

**Tese de doutorado**

Thesis presented to the Programa de Pós–graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor : Prof. Edgard Almeida Pimentel
Co-advisor: Prof. Ricardo José Alonso Plata

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To my grandmother Delia (in memoriam), with love.

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# Abstract

Cuba Huamani, Edison Fausto; Almeida Pimentel, Edgard (Advisor); Alonso Plata, Ricardo José (Co-Advisor). **Existence and regularity of solutions: nonlocal and nonlinear models**. Rio de Janeiro, 2021. 80p. Tese de Doutorado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

We consider two classes of partial differential equations. Namely: the radiative transfer equation and a doubly nonlinear model. The former concerns a nonlocal problema, driven by a scattering operator. We study the well-posedness of solutions in the peaked regime, for the half-space. A new averaging lemma yields interior regularity for the solutions and improved fractional regularization for the time derivatives. The second model we examine is a Trudinger equation with distinct nonlinearities degrees. Inspired by ideas launched by L. Caffarelli, we resort to approximation methods and prove improved regularity results for the solutions. The strategy is to relate our equation with p-caloric functions.



## Keywords

Radiative transfer equation; Initial-boundary value problem; Doubly nonlinear equation; Degenerate equation; Regularity of the solutions; Hölder regularity; Average lemma; Existence ; Uniqueness of solutions.

# Resumo

Cuba Huamani, Edison Fausto; Almeida Pimentel, Edgard; Alonso Plata, Ricardo José. **Existência e regularidade de soluções: modelos não locais e não lineares**. Rio de Janeiro, 2021. 80p. Tese de Doutorado – Departamento de Matemática , Pontifícia Universidade Católica do Rio de Janeiro.

Estudamos duas classes de equações diferenciais parciais, nomeadamente: uma equação de transferência radiativa e uma equação do calor duplamente não-linear. O primeiro modelo envolve uma equação não-local, na presença de um operador de espalhamento. Estuda-se a boa colocação do problema no semi-plano, no regime 'peaked'. Prova-se um lema de 'averaging', que produz regularidade interior para o problema, além de regularização fracionária para as derivadas temporais da solução. O segundo conjunto de resultados da tese trata de uma equação de Trudinger com graus de não-linearidade distintos. Aproxima-se este problema pela $p$-equação do calor e importa-se regularidade da última para a primeira. Como consequência, mostra-se um resultado de regularidade melhorada no contexto não-homogêneo.



## Palavras-chave

Equação de transferência radiativa; Problema com valor inicial e valor na fronteira; Equação duplamente não linear; Equação degenerada; Regularidade de Hölder; Regularidade das soluções; Lema da media; Existência; Unicidade de soluções.

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# 1
# Introduction

This thesis is devoted to the study of some problems in partial differential equations (PDEs), including nonlocal and nonlinear models. Specifically, we consider two classes of problems. First, we prove the well-posedness of the radiative transfer equation in the half-space. We also study a doubly nonlinear degenerate parabolic equation in the parabolic cylinder. In this context we produce a regularity theory for the solutions. To be precise, we prove Hölder regularity for its solutions.

## 1.1
## Introductory remarks

The aim of this work is to study regularity, existence and uniqueness of solutions for certain classes of local and nonlocal equations. Specifically, we study the following PDEs: the radiative transfer equation and the doubly nonlinear degenerate parabolic equation.

Radiative transfer is the physical phenomenon of energy transfer in the form of electromagnetic radiation. The radiative transfer equation (RTE) serves as a model for physical phenomena associated with wave propagation in random media. For example propagation of high frequency waves that are weakly coupled due to heterogeneity, or regimes associated with "long range" propagation of waves in weakly heterogeneous media. Related to the latter, the classical example is the highly forward-peaked regime commonly found in neutron transport, atmospheric radiative transfer, and optical imaging, see [33, 35, 39]. The RTE model in the forward-peaked regime is now commonly used for medical imaging inversion since it describes fairly well the propagation of waves through biological tissue in such conditions.

The first part of this thesis can be seen as a natural continuation of the regularity, existence and uniqueness results initiated in [1] for the radiative transfer equation. In this way, we study the well-posedness and regularity theory for the Radiative Transfer equation in the peaked regime posed in the half-space. An average lemma for the transport equation in the half-space is established and used to generate interior regularity for solutions of the model. The averaging also shows a fractional regularisation gain up to the boundary for the spatial derivatives [2]. In this regard, the first part of the present work is based on the following research paper:

> R. Alonso, E. Cuba. *Radiative Transfer with long-range interactions in the half-space.* **Journal of Differential Equations.** Vol. 269, Issue 10: 8801-8837 (2020).

The second half of this thesis examines the smoothness of solutions to a local, doubly nonlinear PDE. We establish improved regularity, through approximation methods, and relate our model with the $p$-caloric equation. Doubly nonlinear parabolic equations arise to model several physical phenomena. For instance in plasma physics or in the analysis of turbulent filtration of a gas through porous media. Here, the regularity of solutions of the doubly nonlinear degenerate problem is studied. We begin with some comments on doubly nonlinear parabolic equations and the history of the problem. The doubly nonlinear parabolic equation in the literature is called as before due to the nonlinearity depending on both the solution and its spatial gradient. These structures affect the time derivative and the second order term driving the PDE. The interest in such equation comes from mixed types of degeneracies and singularities, ultimately resonating in the regularity properties. We continue with a preliminary description of our models.



## 1.2 Preliminaries

In what follows we introduce the equations studied in this thesis and detail our main assumptions. Preliminary notions and basic results close this section.

The radiative transfer equation (RTE) in the half-space can be written as

$$\begin{cases} \partial_t u + \theta \cdot \nabla_x u = \mathcal{I}(u) & \text{in} \quad (0,T) \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}, \\ u = u_0 & \text{on} \quad \{t=0\} \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}, \\ u = g & \text{on} \quad (0,T) \times \{x_d = 0\} \times \{\theta_d > 0\}, \end{cases} \tag{1.1}$$

where $T$ is an arbitrary time, $u = u(t,x,\theta)$ is the radiation intensity distribution in $(0,T) \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}$, and $\mathbb{R}^d_+ = \{x \,|\, x_d > 0\}$ denotes the half-plane, or half-space. The initial radiation distribution $u_0(x,\theta)$ and the boundary radiation intensity $g(t,\bar{x},\theta)$ are assumed nonnegative most of the time[1]. We adopt the bar notation $\bar{x}$ to represent points in $\partial\mathbb{R}^d_+ = \{x \mid x_d = 0\} \sim \mathbb{R}^{d-1}$ for any $d \geq 3$.

The scattering operator is defined as

$$\mathcal{I}(u) := \mathcal{I}_{b_s}(u) = \int_{\mathbb{S}^{d-1}} (u(\theta') - u(\theta)) b_s(\theta,\theta')\,\mathrm{d}\theta'. \tag{1.2}$$

In this work we are interested in the highly forward-peaked regime in the half-space where the angular scattering kernel takes the form

$$b_s(\theta,\theta') = \frac{b(\theta\cdot\theta')}{(1-\theta\cdot\theta')^{\frac{d-1}{2}+s}}, \qquad s \in \left(0, \min\left\{1, \tfrac{d-1}{2}\right\}\right), \tag{1.3}$$

where $b(z) \geq 0$ has some smoothness in the neighbourhood of $z = 1$. More precisely, we will consider in the sequel its decomposition into two nonnegative components

$$b(z) = b(1) + \tilde{b}(z), \quad \text{where} \quad b(1) > 0 \quad \text{and} \quad \frac{\tilde{b}(z)}{(1-z)^{1+s}} =: h(z) \in L^1(-1,1). \tag{1.4}$$

The weak formulation of this operator is given, for any sufficiently regular test function $\psi$, by

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)(\theta)\psi(\theta)\,\mathrm{d}\theta \\ := -\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} (u(\theta')-u(\theta))\,(\psi(\theta')-\psi(\theta))\,b_s(\theta,\theta')\,\mathrm{d}\theta\,\mathrm{d}\theta'.\end{aligned} \tag{1.5}$$

The RTE (1.1) has been studied in the whole domain using slightly different approaches, based on hypo-ellipticity techniques [6], in the references [1, 21]. References treating problems with boundaries are scarce in the context of kinetic equations with singular scattering, however, for the classical kinetic Fokker-Planck equation with absorbing boundary we refer to [26, 27].

Next, we present the second model under analysis in this thesis. The doubly nonlinear parabolic equation has the following form:

$$\partial_t\left(|u|^{\beta-2}u\right) - \mathrm{div}|Du|^{p-2}Du = f \quad \text{in} \quad Q_1, \tag{1.6}$$

where $f \in L^{q,r}(Q_1)$, $p \geq \beta \geq 2$ and $Q_1 := B_1 \times (-1,0]$. We recall that $L^{q,r}(Q_1)$ is the space of functions satisfying an $r$-integrability condition in time and a $q$-integrability condition in space. As our focus sits on interior regularity, we are not interested in the associated Dirichlet problem. As a result, (1.6) is not equipped with a initial datum.

[1]The $L^2$ theory of the equation will not require non negativity of the data. However, *a priori* estimates based on the $L^1$ integrability of solutions will require it.

Roughly speaking, (1.6) is modeled after the $p$-heat equation, taking also into account a nonlinearity in the time derivative. As a practical effect, such nonlinearity introduces a further layer of degeneracy/singularity in the model. Namely, not only the set of critical points for the solutions jeopardizes ellipticity, but their zero level-sets also affect the evolution character of the problem.

The doubly nonlinear parabolic equation (1.6) deserved the attention of several authors along the last decades. In the simpler case, $\beta = 2$, Eq. (1.6) becomes: the heat equation, for $p = 2$, and the evolutionary $p$-Laplace equation, for $1 < p \neq 2$, which possesses a degeneracy in the principal part for $p > 2$, coming from the fact that its so-called modulus of ellipticity $|Du|^{p-2}$ vanishes at points where $|Du| = 0$ and the quantity diverges (blows up) if $1 < p < 2$ as $|Du| \to 0$. Often, the parabolic $p-$Laplacian equations are classified as degenerate $(p > 2)$ and singular $(1 < p < 2)$ and are studied separately. In particular, the local Hölder regularity can be found in [12] for $\beta = 2$. A similar type of behavior is displayed by the Trudinger's equation which agrees with (1.6) for $\beta = p - 1$. In this case, local Hölder continuity for the weak solutions of the degenerate Trudinger's equation was established in [18] while the complementary case $\beta = p - 1$ and $1 < p < 2$ that corresponds to the singular one was proven in [31]. Moreover, local Hölder continuity for the weak solutions in the case $0 < \beta < 1$ and $p > 2$ was studied in [25].

### 1.2.1
### Main assumptions and former notions

We start by gathering some preliminaries concerning the RTE. Fix $T > 0$. Take boundary data

$$0 \leq \left(u_0, \sqrt{\theta_d}\, g\right) \in L^1(\mathbb{R}^d_+ \times \mathbb{S}^{d-1}) \times L^2\Big((0,T); L^2(\Gamma^-)\Big).$$

A nonnegative function

$$0 \leq u \in L^\infty\Big([0,T]; L^1(\mathbb{R}^d_+ \times \mathbb{S}^{d-1})\Big) \cap L^2\Big((0,T); L^2_x \cap H^s_\theta(\mathbb{R}^d_+ \times \mathbb{S}^{d-1})\Big)$$

is a weak solution of the RTE provided (1.1) is satisfied weakly for all test functions in $H^1_{t,x} \cap H^s_\theta\Big((0,T) \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}\Big)$. Here $H^1_{t,x}$ is the usual Sobolev space for time and space variables and $H^s_\theta$ for the angular variable. A precise definition for $H^s_\theta$ is given below in Section 2.1.2. We remark that it is implicit in the definition of weak solutions that such must have well defined traces in order to have a valid weak formulation through the Green's formula.

In order to simplify notation, we introduce the set $\Gamma^-$ for the boundary data. In fact, we will consider throughout the document the sets

$$\Gamma^{\pm} = \left\{ (\bar{x}, \theta) \in \partial\mathbb{R}^d_+ \times \mathbb{S}^{d-1} \,\middle|\, \pm\theta_d < 0 \right\}, \qquad \Sigma^T_{\pm} = (0,T) \times \Gamma^{\pm},$$

and the spaces $L^2(\Sigma^T_{\pm}\ ;\ |\theta_d|\ \mathrm{d}\bar{x}\ \mathrm{d}\theta\ \mathrm{d}t)$ to be the set of square integrable functions in $\Sigma^T_{\pm}$ with respect to the measure $|\theta_d|\ \mathrm{d}\bar{x}\ \mathrm{d}\theta\ \mathrm{d}t$. Of course, $\mathrm{d}\bar{x}$ represents the standard Lebesgue measure in $\partial\mathbb{R}^d_+$. In the sequel, we may simply use the shorthand $L^2(\Sigma^T_{\pm})$ for such spaces. In addition, it will be common to use the shorthand notation $L^2_{t,x,\theta}$ when the domain of the functions is clear from the context.

Now, we turn our attention to the doubly nonlinear parabolic equation. In this part, we characterize the treated problem and specify some notations that we use in the Chapter 3. Moreover, we explain our notion of weak solutions, and also describe our main strategy. Throughout the Chapter 3 we work under specific conditions on the source term $f$. We detail the anisotropic Lebesgue space to which the source term $f$ belongs in the form of an assumption.

**A 1 (Integrability of the source term $f$)** *The source term $f : Q_1 \to \mathbb{R}$ is such that $f \in L^{q,r}(Q_1)$. In addition,*

$$\|f\|_{L^{q,r}(Q_1)} := \left( \int_{-1}^{0} \left| \int_{B_1} |f(x,t)|^q \, dx \right|^{r/q} dt \right)^{1/r} \leq \varepsilon_0,$$

*where $C$ is a positive constant.*

Concerning the integrability of the source $f$, we might assume that

$$\frac{1}{r} + \frac{d}{pq} < 1, \tag{1.7}$$

which is the minimal integrability condition that guarantees the existence of bounded weak solutions. The next one defines the borderline setting for the optimal Hölder regularity regime,

$$\frac{\beta}{r} + \frac{d}{q} > 1. \tag{1.8}$$

Our goal is to quantify this qualitative behaviour. In other words, for a weak bounded solution to (1.6) and for a given source $f \in L^{q,r}(Q_1)$, our findings ensure that solutions belong to $C^{\gamma}_{loc}$ in space and $C^{\gamma/\theta}_{loc}$ in time with

$$\gamma \leq \frac{(pq-d)r - pq}{q[(p-1)r - (p-2)]} = \frac{p\left(1 - \frac{d}{pq} - \frac{1}{r}\right)}{\left(\frac{2}{r} + \frac{d}{q} - 1\right) + \left(1 - \frac{1}{r} - \frac{d}{pq}\right)}. \tag{1.9}$$

Notice that if (1.7) and (1.8) hold, then we have that $0 < \gamma < 1$. Let us define $\theta$ as follows

$$\theta := p - \gamma(p - \beta) = (1 - \gamma)p + \gamma\beta.$$

Clearly, $\beta < \theta < p$ since $0 < \gamma < 1$. In other words, $\theta$ is an interpolation between $p$ and $\beta$. In view of this, we are in the context of the intrinsic $\theta$-parabolic cylinder $Q_\rho$ given by

$$Q_\rho(x_0, t_0) \, := \, B_\rho(x_0) \, \times \, \left(t_0 - \rho^\theta, t_0\right), \qquad \rho > 0.$$

Now, we proceed with the definition of weak solution to (1.6) in the distributional sense.

**Definition 1.2.1 (Weak solution)** *A measurable function*

$$u \in C_{loc}(0, T; L^\beta_{loc}(B_1)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(B_1))$$

*is said to be a weak (distributional) solution to* (1.6) *if*

$$\int_0^T \int_{B_1} -|u|^{\beta-2} u \, \varphi_t \, dxdt \, + \, |Du|^{p-2} Du \cdot D\varphi \, dxdt \, = \, \int_0^T \int_{B_1} f \, \varphi \, dxdt,$$

*for every* $\phi \in \mathcal{C}^\infty_0(Q_1)$*. A normalized weak solution to* (1.6) *is a weak solution that satisfies*

$$\|u\|_{L^\infty(Q_1)} \, \leq \, 1.$$

In [18] the authors propose an alternative definition involving the Steklov averages of $u$. It allows us to use a Caccioppoli estimate. Hence, we proceed with an equivalent definition of local weak solution, involving the discrete time derivative of $|u|^{\beta-2}u$. For $0 < h < T$, the Steklov average of a function $v \in L^1(Q_T)$ is defined by

$$v_h(x,t) = \begin{cases} \frac{1}{h}\int_t^{t+h} v(x,\tau)\, d\tau & \text{if} \quad 0 < t \leq T - h \\ \qquad 0 & \text{if} \quad T - h < t \leq T. \end{cases} \tag{1.10}$$

**Definition 1.2.2** *A measurable function*

$$u \in C(0, T; L^\beta_{loc}(B_1)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(B_1))$$

*is a local weak solution of* (1.6) *if, for every* $K \subset\subset B_1$ *and every* $0 < t < T - h$*, we have*

$$\int_{K\times\{t\}} -\partial_t([|u|^{\beta-2}u]_h) \, \varphi \, dx + \int_{K\times\{t\}} [|Du|^{p-2}Du]_h \cdot D\varphi \, dx \, = \, \int_{K\times\{t\}} f_h \, \varphi \, dx,$$

*for all* $\varphi \in W_0^{1,p}(B_1) \cap L^\infty_{loc}(B_1)$.

The former definition is instrumental in producing energy estimates for the solutions to (1.6). In the next section we put forward our main results.

## 1.3 Main results

In Chapter 2, we study the existence, uniqueness and regularity theory for the Radiative Transfer equation in the peaked regime posed in the half-space. To do this, we first stablished an average lemma for the transport equation in the half-space. Then, we use it to generate interior regularity for solutions of the radiative transfer equation. The averaging also shows a fractional regularisation gain up to the boundary for the spatial derivatives. Our main result concerning the RTE are the following

**Theorem 1.3.1 (Averaging lemma)** *Fix* $d \geq 3$ *and boundary data* $\theta_d\, g \in L^2((t_0,t_1)\times\Gamma^-)$. *Assume that* $u \in \mathcal{C}\Big([t_0,t_1); L^2(\mathbb{R}^d_+ \times \mathbb{S}^{d-1})\Big)$ *solves* (1.1) *for* $t \in (t_0,t_1)$. *Then, for any* $s \in (0,1)$, *there exists a constant* $C := C(d,s)$ *such that*

$$\|(-\Delta_x)^{s_0/2} u\|_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} \leq C\Big(\|u(t_0)\|_{L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})} + \|u\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})} \\ + \|(-\Delta_v)^{s/2} U_{\mathcal{J}}\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{R}^{d-1})} + \|\theta_d\, g\|_{L^2((t_0,t_1)\times\Gamma^-)}\Big),$$

*where* $s_0 = \frac{s/8}{2s+1}$ *and* $U_{\mathcal{J}}$ *is defined in Section 2.1.1.*

The following result ensures the uniqueness of weak solutions to the radiative transfer equation

**Theorem 1.3.2 (Absorbing boundary)** *Consider* $u_0 \in L^2\Big(\mathbb{R}^d_+ \times \mathbb{S}^{d-1}\Big)$ *and* $f \in L^2\Big((0,T)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1}\Big)$. *Then, the problem*

$$\begin{cases} \partial_t u + \mathcal{L}(u) = f & in \quad (0,T)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1}, \\ u = u_0 & on \quad \{t=0\}\times\mathbb{R}^d_+\times\mathbb{S}^{d-1}, \\ u = 0 & on \quad (0,T)\times\partial\mathbb{R}^d_+\times\mathbb{S}^{d-1} \;\; and \;\; -(\theta\cdot n(\bar{x})) > 0, \end{cases}$$

*has a unique weak solution*

$$u \in \mathcal{C}\Big([0,T]; L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})\Big) \cap L^2\Big((0,T); L^2_x \cap H^s_\theta(\mathbb{R}^d_+\times\mathbb{S}^{d-1})\Big).$$

*Its trace satisfies* $\gamma^+(u) \in L^2\Big((0,T)\times\Gamma^+, |\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\Big)$. *Furthermore,* $u \geq 0$ *if* $u_0 \geq 0$ *and* $f \geq 0$.

In Chapter 3, we provide sharp regularity estimates for locally bounded solutions of the doubly nonlinear degenerate parabolic equation. We work under a proximity regime on the exponent governing the nonlinearity of the problem. More precisely, we show that solutions are locally of class $\mathcal{C}^{\gamma,\frac{\gamma}{\theta}}$ where $\gamma$ depends explicitly only on the integrability of $f$ in space and time, and the nonlinearity parameters $\beta$ and $p$. We have

**Theorem 1.3.3 (Improved regularity)** *Let $u$ be a weak solution to (1.6). Suppose that $f \in L^{q,r}(Q_1$ with $\|f\|_{L^{q,r}(Q_1)} \leq \varepsilon_0$. Given*

$$0 < \gamma < \frac{(pq-d)r - pq}{q[(p-1)r - (p-2)]},$$

*there exists $\varepsilon = \varepsilon(d,\gamma)$ such that, if $0 < \beta - 2 < \varepsilon$. Then $u$ is locally of class $C^{0,\gamma}$ in space and of class $C^{0,\gamma/\theta}$ in time where*

$$\theta := p - \gamma(p - \beta).$$

*In addition, there exists $C > 0$, depending only on $\gamma$ and the dimension $d$, for which*

$$\sup_{B_r(x_0)\times(t_0-r^\theta,t_0)} |u(x,t) - u(x_0,t_0)| \\ \leq C\left(|x-x_0|^\gamma + |t-t_0|^{\gamma/\theta}\right)\left(\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{q,r}(Q_1)}\right).$$

In other words, we obtain new (sharp) regularity results for the solutions to (1.6).

The underlying motivation for this analysis is to import regularity from *degenerate p-parabolic equations* back to the doubly nonlinear model. This is done through a set of techniques inspired by ideas introduced by L. Caffarelli in [8].Indeed, we combine approximation methods with a localization argument and, in line with the techniques in [43] and [18], we establish improved regularity of the solutions .

More precisely, the proof of Theorem 1.3.3 is based on approximation methods, combined with a scaling type argument. In fact, the approximation regime allows us to transmit information from the parabolic $p$-laplace equation back to the doubly nonlinear parabolic equation. It translates into an oscillation control in cylinders of a universal, fixed, radius. Then, the scaling procedure localizes the oscillation estimate, establishing the result.

The next chapter puts forward our findings concerning the radiative transfer equation (1.1).

# 2
# Radiative Transfer equation in the half-space

This chapter presents our findings about the radiative transfer equation. In Section 2.1, we recall the basics of the work. We detail the proof of Theorem 1.3.1 in Section 2.2. In Section 2.3, *a priori* estimates leading to interior smoothness of solutions are established. Finally, in Section 2.4 we develop the well-posedness theory of the model and prove Theorem 1.3.2. The argument in this last section is based on a modification of the classical literature of the radiative transfer in convex domains using semigroup theory.

The radiative transfer equation (RTE) in the half-space can be written as

$$\begin{cases} \partial_t u + \theta \cdot \nabla_x u = \mathcal{I}(u) & \text{in} \quad (0,T) \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}, \\ u = u_0 & \text{on} \quad \{t=0\} \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}, \\ u = g & \text{on} \quad (0,T) \times \{x_d = 0\} \times \{\theta_d > 0\}, \end{cases} \tag{2.1}$$

where $T$ is an arbitrary time, $u = u(t,x,\theta)$ is the radiation intensity distribution in $(0,T) \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}$, and where the half-plane has been defined as $\mathbb{R}^d_+ = \{x \,|\, x_d > 0\}$. The initial radiation distribution $u_0(x,\theta)$ and the boundary radiation intensity $g(t,\bar{x},\theta)$ are assumed nonnegative most of the time and the bar notation $\bar{x}$ accounts for points in $\partial\mathbb{R}^d_+ = \{x \,|\, x_d = 0\} \sim \mathbb{R}^{d-1}$, for $d \geq 3$.

The scattering operator is defined as

$$\mathcal{I}(u) \coloneqq \mathcal{I}_{b_s}(u) = \int_{\mathbb{S}^{d-1}} (u(\theta') - u(\theta)) b_s(\theta,\theta') \, \mathrm{d}\theta'. \tag{2.2}$$

We are interested in the highly forward-peaked regime in the half-space where the angular scattering kernel takes the form

$$b_s(\theta,\theta') = \frac{b(\theta\cdot\theta')}{(1-\theta\cdot\theta')^{\frac{d-1}{2}+s}}, \qquad s \in \left(0, \min\left\{1, \frac{d-1}{2}\right\}\right), \tag{2.3}$$

where $b(z) \geq 0$ has some smoothness in the neighbourhood of $z = 1$. More precisely, we will consider in the sequel its decomposition into two nonnegative components

$$b(z) = b(1) + \tilde{b}(z), \quad \text{where} \quad b(1) > 0 \quad \text{and} \quad \frac{\tilde{b}(z)}{(1-z)^{1+s}} =: h(z) \in L^1(-1,1). \tag{2.4}$$

The weak formulation of this operator is given, for any sufficiently regular test function $\psi$, by

$$\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)(\theta)\psi(\theta)\,\mathrm{d}\theta \\ := -\frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} \left(u(\theta') - u(\theta)\right)\left(\psi(\theta') - \psi(\theta)\right) b_s(\theta,\theta')\,\mathrm{d}\theta\,\mathrm{d}\theta'. \tag{2.5}$$

Next, we gather some technical preliminaries.

## 2.1
## Further preliminary material

### 2.1.1
### Representation of the projected scattering operator

Our analysis is based on the stereographic projection of the kinetic variable $\theta \in \mathbb{S}^{d-1}$ on the plane $v \in \mathbb{R}^{d-1}$, see [1]. The benefit of such approach is the fact that the geometry of the kinetic space is replaced by Bessel weights which are manageable with Fourier methods and render explicit formulas for numerical implementation.

Recall that the stereographic projection $\mathcal{S} : \mathbb{S}^{d-1} \to \mathbb{R}^{d-1}$ is given by

$$v_i = \mathcal{S}(\theta)_i := \frac{\theta_i}{1-\theta_d}, \quad 1 \le i \le d-1\,. \tag{2.6}$$

Its inverse $\mathcal{J} : \mathbb{R}^{d-1} \to \mathbb{S}^{d-1}$ is given by

$$\mathcal{J}_i(v) = \frac{2v_i}{\langle v\rangle^2}, \quad 1 \le i \le d-1 \qquad \text{and} \qquad \mathcal{J}_d(v) = \frac{|v|^2-1}{\langle v\rangle^2}, \tag{2.7}$$

where $\langle\cdot\rangle := \sqrt{|\cdot|^2+1}$ is the Japanese bracket. The Jacobian of such transformations can be computed as

$$\mathrm{d}v = \frac{\mathrm{d}\theta}{(1-\theta_d)^{d-1}} \qquad \text{and} \qquad \mathrm{d}\theta = \frac{2^{d-1}\,\mathrm{d}v}{\langle v\rangle^{2(d-1)}}\,.$$

Using the shorthanded notation $\theta = \mathcal{J}(v)$ and $\theta' = \mathcal{J}(v')$ we obtain that

$$1 - \theta\cdot\theta' = 2\,\frac{|v-v'|^2}{\langle v\rangle^2\langle v'\rangle^2}. \tag{2.8}$$

In the sequel we use the shorthand $u_{\mathcal{J}} := u \circ \mathcal{J} : \mathbb{R}^{d-1} \to \mathbb{S}^{d-1} \to \mathbb{R}$ for pull back functions, and with a capital letter we introduce the function

$U_{\mathcal{J}} := \frac{u_{\mathcal{J}}}{\langle\cdot\rangle^{d-1-2s}}$ that will be important along this document.

**Proposition 2.1.1** *For any sufficiently regular function $u$ in the sphere the stereographic projection of the operator $\mathcal{I}_{b(1)}$ is given by*

$$\frac{\left[\mathcal{I}_{b(1)}(u)\right]_{\mathcal{J}}}{\langle\cdot\rangle^{d-1+2s}} = \frac{2^{\frac{d-1}{2}-s}b(1)}{c_{d-1,s}}\left(-(-\Delta_v)^s U_{\mathcal{J}} + u_{\mathcal{J}}\,(-\Delta_v)^s\left(\frac{1}{\langle\cdot\rangle^{d-1-2s}}\right)\right) \\ = \frac{2^{\frac{d-1}{2}-s}\,b(1)}{c_{d-1,s}}\left(-(-\Delta_v)^s U_{\mathcal{J}} + c_{d,s}\,\frac{u_{\mathcal{J}}}{\langle\cdot\rangle^{d-1+2s}}\right). \tag{2.9}$$

*As a consequence, we have that*

$$\frac{1}{b(1)}\int_{\mathbb{S}^{d-1}} \mathcal{I}_{b(1)}(u)(\theta)\,\overline{u(\theta)}\,\mathrm{d}\theta = -c_{d,s}\,\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|^2_{L^2(\mathbb{R}^{d-1})} + C_{d,s}\|u\|^2_{L^2(\mathbb{S}^{d-1})}, \tag{2.10}$$

*for some explicit positive constants $c_{d,s}$ and $C_{d,s}$. Furthermore, defining the differential operator $(-\Delta_\theta)^s$ acting on functions on the sphere by the formula*

$$\left[(-\Delta_\theta)^s u\right]_{\mathcal{J}} := \langle\cdot\rangle^{d-1+2s}\,(-\Delta_v)^s U_{\mathcal{J}}, \tag{2.11}$$

*the scattering operator, $\mathcal{I}_{b_s} = \mathcal{I}_{b(1)} + \mathcal{I}_h$, is given as the sum of a singular part and $L^2_\theta-$ bounded part*

$$\mathcal{I}_{b_s} = -D\,(-\Delta_\theta)^s + c_{s,d}\,I + \mathcal{I}_h, \tag{2.12}$$

*where $D = 2^{\frac{d-1}{2}-s}\frac{b(1)}{c_{d-1,s}}$ is the diffusion constant.*

*Proof.* The proof is the same as [1, Proposition 2.1]. But for the sake of completeness, all details are provided. Given the decomposition of the scattering kernel $b_s$ assumed in (2.4) we can write the scattering operator as $\mathcal{I} = \mathcal{I}_{b(1)} + \mathcal{I}_h$. The operator $\mathcal{I}_h$ is a bounded operator in $L^2(\mathbb{S}^{d-1})$. Indeed, assumption (2.4) implies that

$$\theta' \to h(\theta\cdot\theta') = \frac{\tilde{b}(\theta\cdot\theta')}{(1-\theta\cdot\theta')^{\frac{d-1}{2}+s}} \in L^1(\mathbb{S}^{d-1})\,.$$

Then, using Cauchy-Schwarz inequality it follows that

$$\left\|\mathcal{I}_h(u)\right\|_{L^2(\mathbb{S}^{d-1})} \leq 2\left\|h\right\|_{L^1(\mathbb{S}^{d-1})}\|u\|_{L^2(\mathbb{S}^{d-1})}\,. \tag{2.13}$$

The details can be found in the [1, Lemma A.1]. Let us concentrate on the leading term $\mathcal{I}_{b(1)}$ using the Stereographic projection and (2.8)

$$\begin{aligned}
\Big[\mathcal{I}_{b(1)}(u)\Big]_{\mathcal{J}}(v) &= 2^{\frac{d-1}{2}-s}\, b(1)\, \langle v\rangle^{d-1+2s} \int_{\mathbb{R}^{d-1}} \frac{u_{\mathcal{J}}(v') - u_{\mathcal{J}}(v)}{|v-v'|^{d-1+2s}} \frac{\mathrm{d}v'}{\langle v'\rangle^{d-1-2s}} \\
&= 2^{\frac{d-1}{2}-s}\, b(1)\, \langle v\rangle^{d-1+2s} \Bigg( \int_{\mathbb{R}^{d-1}} \frac{U_{\mathcal{J}}(v') - U_{\mathcal{J}}(v)}{|v-v'|^{d-1+2s}} \mathrm{d}v' \\
&\qquad + U_{\mathcal{J}}(v) \int_{\mathbb{R}^{d-1}} \frac{\frac{1}{\langle v\rangle^{d-1-2s}} - \frac{1}{\langle v'\rangle^{d-1-2s}}}{|v-v'|^{d-1+2s}} \mathrm{d}v' \Bigg) \\
&= \frac{2^{\frac{d-1}{2}-s}\, b(1)}{c_{d-1,s}} \langle v\rangle^{d-1+2s} \Big( -(-\Delta_v)^s U_{\mathcal{J}} + u_{\mathcal{J}} (-\Delta_v)^s \frac{1}{\langle\cdot\rangle^{d-1-2s}} \Big) \\
&= \frac{2^{\frac{d-1}{2}-s}\, b(1)}{c_{d-1,s}} \langle v\rangle^{2s} \Big( -(-\Delta_v)^s w_{\mathcal{J}} + c_{d,s} \frac{u_{\mathcal{J}}}{\langle v\rangle^{d-1+2s}} \Big).
\end{aligned} \tag{2.14}$$

For the last inequality, we have used a result based on Bessel potentials obtained in [1, Lemma A.2]. This result is the following computation

$$(-\Delta_v)^s \frac{1}{\langle\cdot\rangle^{d-1-2s}}(v) = \frac{c_{d,s}}{\langle v\rangle^{d-1+2s}}\,.$$

This proves (2.9) and as a direct consequence, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \mathcal{I}_{b(1)}(u)(\theta)\, \overline{u(\theta)}\, \mathrm{d}\theta \\
&\qquad = 2^{d-1} \int_{\mathbb{S}^{d-1}} \Big[\mathcal{I}_{b(1)}(u)\Big]_{\mathcal{J}}(v)\, \overline{U_{\mathcal{J}}(v)}\, \frac{\mathrm{d}v}{\langle v\rangle^{2(d-1)}} \\
&\qquad = 2^{\frac{3(d-1)}{2}-s} \frac{b(1)}{c_{d-1,s}} \Big[ -\big\|(-\Delta)^{s/2} u_{\mathcal{J}}\big\|^2_{L^2(\mathbb{R}^{d-1})} + \frac{c_{d,s}}{2^{d-1}} \|u\|^2_{L^2(\mathbb{S}^{d-1})} \Big].
\end{aligned} \tag{2.15}$$

■

The next section details some of the functional spaces used in the remainder of this chapter.

## 2.1.2
## Functional spaces

Recalling the notation $U_{\mathcal{J}} := \frac{u_{\mathcal{J}}}{\langle v\rangle^{d-1-2s}}$, the fractional Sobolev space $H^s(\mathbb{S}^{d-1})$ defined as

$$H^s_\theta := \Big\{ u \in L^2_\theta : (-\Delta_v)^{s/2} U_{\mathcal{J}} \in L^2_v \Big\}, \qquad s \in (0,1),$$

endowed with the inner product

$$\langle u, w\rangle_{H^s_\theta} := \langle(-\Delta_v)^{s/2}U_{\mathcal{J}}, (-\Delta_v)^{s/2}W_{\mathcal{J}}\rangle_{L^2(\mathbb{R}^{d-1})}\,. \tag{2.16}$$

The following useful representation of the inner product norm in $H^s(\mathbb{S}^{d-1})$ which follows directly from (2.10) and (2.5) will be important in the next sections. Then, we have

$$\begin{aligned}\|u\|^2_{H^s_\theta(\mathbb{S}^{d-1})} &\sim -\int_{\mathbb{S}^{d-1}} \mathcal{I}_{b(1)}(u)(\theta)\,\overline{u(\theta)}\,\mathrm{d}\theta + \int_{\mathbb{S}^{d-1}} |u(\theta)|^2\,\mathrm{d}\theta \\ &\sim \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} \frac{(u(\theta')-u(\theta))^2}{|\theta'-\theta|^{d-1+2s}}\,\mathrm{d}\theta'\,\mathrm{d}\theta + \|u\|^2_{L^2(\mathbb{S}^{d-1})}.\end{aligned} \tag{2.17}$$

where, for the second equivalence, we used that $2(1-\theta\cdot\theta') = |\theta'-\theta|^2$ valid for any two unitary vectors. We also have, by a straightforward computation, the Sobolev embedding $H^s_\theta \hookrightarrow L^{p_s}_\theta$

$$\|u\|_{L^{p_s}_\theta} \le C_{d,s}\|u\|_{H^s_\theta}\,, \qquad \frac{1}{p_s} = \frac{1}{2} - \frac{s}{d-1}\,. \tag{2.18}$$

### 2.1.3
### Natural a priori energy estimates

Assume the existence of a sufficiently smooth nonnegative solution $u$ to the RTE problem (2.1). Direct integration in $(x,\theta)\in\mathbb{R}^d_+\times\mathbb{S}^{d-1}$ and time $0\le t'\le t\le T$, together with the divergence theorem and the fact that $\mathcal{I}$ is a mass preserving operator $\int_{\mathbb{S}^{d-1}}\mathcal{I}(u)\,\mathrm{d}\theta = 0$ gives that

$$\begin{aligned}&\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u(t,x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x + \int_{t'}^{t}\int_{\Gamma+} u\,(\theta\cdot n(\bar{x}))\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau \\ &\quad= \int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u(t',x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x + \int_{t'}^{t}\int_{\Gamma-} g\big|\theta\cdot n(\bar{x})\big|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau\,.\end{aligned} \tag{2.19}$$

This identity describes the mass in the system. The boundary terms, from left to right, represent the out flux and in flux of mass through the boundary.

Now, in order to obtain the description of the energy multiply the RTE equation by $u$ and integrate in the variables $(x,\theta)\in\mathbb{R}^d_+\times\mathbb{S}^{d-1}$ to obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^2\,\mathrm{d}\theta\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \theta\cdot\nabla_x u^2\,\mathrm{d}\theta\,\mathrm{d}x \\ &\quad= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^2\,\mathrm{d}\theta\,\mathrm{d}x + \frac{1}{2}\int_{\partial\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^2\,(\theta\cdot n(\bar{x}))\,\mathrm{d}\theta\,\mathrm{d}\bar{x} \\ &\quad= \int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)\,u\,\mathrm{d}\theta\,\mathrm{d}x\,,\end{aligned}$$

where we used, again, the divergence theorem in the second step. Therefore,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^2\,\mathrm{d}\theta\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma^+} u^2\left(\theta\cdot n(\bar{x})\right)\,\mathrm{d}\theta\,\mathrm{d}\bar{x} \\ = \frac{1}{2}\int_{\Gamma^-} g^2\big|\theta\cdot n(\bar{x})\big|\,\mathrm{d}\theta\,\mathrm{d}\bar{x} + \int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)\,u\,\mathrm{d}\theta\,\mathrm{d}x.\end{aligned} \tag{2.20}$$

Since $\int_{\mathbb{S}^{d-1}}\mathcal{I}(u)\,u\,\mathrm{d}\theta \leq 0$, we can integrate in time $0 < t' \leq t < T$ to conclude that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d_+}\|u(t)\|^2_{L^2_\theta}\,\mathrm{d}x + \frac{1}{2}\int_{t'}^{t}\int_{\Gamma^+} u^2(\theta\cdot n(\bar{x}))\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau \\ \leq \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}}\|u(t')\|^2_{L^2_\theta}\,\mathrm{d}x + \frac{1}{2}\int_{t'}^{t}\int_{\Gamma^-} g^2\big|\theta\cdot n(\bar{x})\big|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau\,.\end{aligned} \tag{2.21}$$

This estimate can be upgraded to add the diffusion term in the scattering angle using relation (2.17). We are led to

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^d_+}\|u(t)\|^2_{L^2_\theta}\,\mathrm{d}x + D_0\int_{t'}^{t}\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau \\ + \frac{1}{2}\int_{t'}^{t}\int_{\Gamma^+} u^2(\theta\cdot n(\bar{x}))\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau \leq \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}}\|u(t')\|^2_{L^2_\theta}\,\mathrm{d}x \\ + D_1\int_{t'}^{t}\int_{\mathbb{R}^d_+}\|u(\tau)\|^2_{L^2_\theta}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_{t'}^{t}\int_{\Gamma^-} g^2\big|\theta\cdot n(\bar{x})\big|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau\,.\end{aligned} \tag{2.22}$$

Here $D_0$ depends on $d, s$ and $b(1)$, while $D_1$ depends on $d, s, b(1)$ and the integrable scattering kernel $h(\cdot)$. Estimate (2.22) shows the diffusion nature of the equation in the scattering variable $\theta$. We will complete, using the Proposition 2.3.2 below, such estimate to include the spatial diffusion nature of the model as well.



## 2.2
## Averaging lemma in the half-space

In this section, we give a regularisation mechanism for the RTE. It is related to the fact that the diffusion in the kinetic variable $\theta$ is propagated to the spatial variable by means of the advection operator $\theta\cdot\nabla_x$. We follow the framework developed in [6, 1] and adapt it to the fact that we are considering half-space on the spatial variable.

In the sequel, the Fourier transform in time and spatial variables for a suitable function $\varphi(t,x)$ defined in $(0,\infty)\times\mathbb{R}^d_+$ is given by

$$\widehat{\varphi}(w,k) = \mathcal{F}_{t,x}\{\varphi\}(w,k) = \int_0^\infty\int_{\mathbb{R}^d_+}\varphi(t,x)\,\mathrm{e}^{-\mathrm{i}w\,t}\mathrm{e}^{-\mathrm{i}k\cdot x}\mathrm{d}x\,\mathrm{d}t\,,\quad (w,k)\in\mathbb{R}\times\mathbb{R}^d\,. \tag{2.23}$$

The partial Fourier transforms in time $\mathcal{F}_t\{\cdot\}$ and space $\mathcal{F}_x\{\cdot\}$ for $\varphi(t,x)$ are defined in obvious manner. The fractional differentiation in the spatial variable for a sufficiently smooth function $\varphi(x)$ with domain in $\mathbb{R}^d_+$ is defined through its Fourier transform

$$\mathcal{F}_x\Big\{(-\Delta_x)^{s/2}\varphi\Big\}(k) = |k|^s \int_{\mathbb{R}^d_+} \varphi(x)\, \mathrm{e}^{-\mathrm{i}k\cdot x}\mathrm{d}x\,, \qquad k \in \mathbb{R}^d\,, \quad s > 0\,. \tag{2.24}$$

Note that this definition agrees with the classical one using the extension of $\varphi$ by zero in $\{x \,|\, x_d \leq 0\}$. As a consequence, $(-\Delta_x)^{s/2}\varphi$ is a tempered distribution defined in $\mathbb{R}^d$ and have the information of the trace of $\varphi$ on $\partial\mathbb{R}^d_+$ encoded.

In what follows, we prove the first main result in this part of the thesis. Namely, Theorem 1.3.1. For completeness, we recall it next.

**Theorem 2.2.1 (Averaging lemma)** *Let $d \geq 3$ and fix the boundary data $\theta_d\, g \in L^2((t_0,t_1)\times\Gamma^-)$. Assume that $u \in \mathcal{C}\Big([t_0,t_1); L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})\Big)$ solves the RTE on the half-space (2.1) for $t \in (t_0,t_1)$. Then, for any $s \in (0,1)$, there exists a constant $C := C(d,s)$ such that*

$$\begin{aligned}\|(-\Delta_x)^{s_0/2}u\|_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} \leq C\,\Big(&\|u(t_0)\|_{L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})} + \|u\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\\ &+\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{R}^{d-1})} + \|\theta_d\, g\|_{L^2((t_0,t_1)\times\Gamma^-)}\Big)\,,\end{aligned} \tag{2.25}$$

*with $s_0 = \frac{s/8}{2s+1}$.*

*Proof.* Start with an approximation of the identity in the sphere $\{\rho_\epsilon\}_{\epsilon>0}$ defined through an smooth function $\rho \in \mathcal{C}(-1,1)$, which satisfies

$$\int_{-1}^{1}\rho(z)z^{\frac{d-3}{2}}\,\mathrm{d}z = 1, \qquad 0 < \rho(z) \lesssim \frac{1}{z^{\frac{d-1}{2}+s}}. \tag{2.26}$$

Introduce the quantity

$$C_\epsilon = |\mathbb{S}^{d-2}|\int_{-1}^{1}\rho(z)z^{\frac{d-3}{2}}(2-\epsilon z)^{\frac{d-3}{2}}\,\mathrm{d}z, \qquad \epsilon \in (0,1]\,, \tag{2.27}$$

and note that $\inf_{\epsilon\in(0,1]} C_\epsilon > 0$. Thus, define the approximation of the identity as

$$\rho_\epsilon(z) = \frac{1}{C_\epsilon \epsilon^{\frac{d-1}{2}}}\rho\left(\frac{z}{\epsilon}\right). \tag{2.28}$$

We can see that

$$\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\,\mathrm{d}\theta' = 1, \qquad \epsilon > 0. \tag{2.29}$$

We understand the convolution in the sphere, for any real function $\psi$ defined on the sphere, as

$$(\rho\star\psi)(\theta) = \int_{\mathbb{S}^{d-1}}\rho(1-\theta\cdot\theta')\,\psi(\theta')\,\mathrm{d}\theta'. \tag{2.30}$$

Now, consider a sufficiently smooth solution $u$ of the RTE on the half-space, in the interval $[t_0,t_1]$ for any $0 < t_0 < t_1 < \infty$. The Fourier transform of $\partial_t u$

can be computed as

$$\widehat{\partial_t u}(w,k,\theta) = -\mathrm{e}^{-\mathrm{i}wt_0}\mathcal{F}_x\{u\}(t_0,k,\theta) + \mathrm{i}w\,\widehat{u}(w,k,\theta)\,,$$

where the boundary component at $t_1$ is disregarded by the causality of the equation. In the same spirit we can compute the Fourier transform of $\theta \cdot \nabla_x u$. To understand the spectral transformation one considers the problem

$$\begin{cases} \theta \cdot \nabla_x u = f & \text{in } \mathbb{R}^d_+ \times \mathbb{S}^{d-1}\,, \\ \quad u = g & \text{on } \Gamma^-\,. \end{cases}$$

The characteristics $x + t\theta$ imply that for $\theta_d > 0$, both the boundary $g$ and the interior values $f = \theta \cdot \nabla_x u$ contribute to $u$, while for $\{\theta_d < 0\}$ only $f$ contributes to $u$. In particular, the value of $u$ at $\Gamma^+$ is fully determined by the knowledge of $\theta \cdot \nabla_x u$ in $\mathbb{R}^d_+ \times \mathbb{S}^{d-1}$, see [34].

Keeping this in mind, set $x = (\bar{x}, x_d)$ with $\bar{x} = (x_1, x_2, \cdots\cdots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $x_d > 0$. In our coordinate system we will consider $\mathbb{R}^d_+ = \{x \,|\, x_d > 0\}$. For the spatial Fourier variable we perform a similar decomposition $k = (\bar{k}, k_d)$ with $\bar{k} \in \mathbb{R}^{d-1}$ and $k_d \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{F}_x\Big\{\theta \cdot \nabla_x u\Big\}(t,k,\theta) &= \int_{\mathbb{R}^d_+} \theta \cdot \nabla_x u \;\; \mathrm{e}^{-\mathrm{i}k\cdot x}\,\mathrm{d}x \\
&= \int_0^{+\infty}\int_{\mathbb{R}^{d-1}} \Big(\bar{\theta}\cdot\nabla_{\bar{x}} u + \theta_d \cdot \partial_{x_d} u\Big)\;\; \mathrm{e}^{-\mathrm{i}\bar{x}\cdot\bar{k}}\,\mathrm{d}\bar{x}\;\; \mathrm{e}^{-\mathrm{i}x_d\,k_d}\,\mathrm{d}x_d \\
&= \int_0^{+\infty}\Big(\int_{\mathbb{R}^{d-1}} \bar{\theta}\cdot\nabla_{\bar{x}} u \;\; \mathrm{e}^{-\mathrm{i}\bar{x}\cdot\bar{k}}\,\mathrm{d}\bar{x} + \int_{\mathbb{R}^{d-1}} \theta_d \cdot \partial_{x_d} u \;\; \mathrm{e}^{-\mathrm{i}\bar{x}\cdot\bar{k}}\,\mathrm{d}\bar{x}\Big)\mathrm{e}^{-\mathrm{i}x_d\,k_d}\,\mathrm{d}x_d \\
&= \int_0^{+\infty}\Big((\mathrm{i}\,\bar{\theta}\cdot\bar{k})\,\mathcal{F}_{\bar{x}}\{u\}(t,\bar{k},x_d,\theta) + \theta_d\cdot\mathcal{F}_{\bar{x}}\{\partial_{x_d}u\}(t,\bar{k},x_d,\theta)\Big)\mathrm{e}^{-\mathrm{i}x_d\,k_d}\,\mathrm{d}x_d \\
&= (\mathrm{i}\,\bar{\theta}\cdot\bar{k})\mathcal{F}_x\{u\}(t,k,\theta) + \theta_d\int_0^{+\infty} \partial_{x_d}\mathcal{F}_{\bar{x}}\{u\}(t,\bar{k},x_d,\theta)\;\; \mathrm{e}^{-\mathrm{i}x_d\,k_d}\,\mathrm{d}x_d \\
&= (\mathrm{i}\,\bar{\theta}\cdot\bar{k})\;\mathcal{F}_x\{u\}(t,k,\theta) + \theta_d\Big[-\mathcal{F}_{\bar{x}}\{u\}(t,\bar{k},0,\theta) + \mathrm{i}k_d\,\mathcal{F}_x\{u\}(w,k,\theta)\Big] \\
&= (\mathrm{i}\,\theta\cdot k)\;\mathcal{F}_x\{u\}(t,k,\theta) - \mathcal{F}_{\bar{x}}\{G\}(t,\bar{k},\theta),
\end{aligned}$$

where for the boundary $\{x \,|\, x_d = 0\}$, we introduced

$$G(t,\bar{x},\theta) := \begin{cases} 0 & \text{if } \;\theta_d < 0\,, \\ \theta_d\, g(t,\bar{x},\theta) & \text{if } \;\theta_d > 0\,, \end{cases} \tag{2.31}$$

because $u = g$ on $\Sigma^T_-$. Meanwhile, $G$ vanishes in $\Sigma^T_+$ because the term $\mathrm{i}\,\theta\cdot k\,\mathcal{F}_x\{u\}$ uniquely defines $u$ at $\Gamma^+$. As a consequence,

$$\widehat{\theta\cdot\nabla_x u}(w,k,\theta) = (\mathrm{i}\,\theta\cdot k)\;\widehat{u}(w,k,\theta) - \widehat{G}(w,\bar{k},\theta)\,,$$

where $\widehat{G} = \mathcal{F}_{t,\bar{x}}\{G\}$. Overall, we conclude the Fourier transform of (2.1) is given by

$$\mathrm{i}(w + \theta \cdot k)\ \widehat{u}(w,k,\theta) = \mathcal{I}(\widehat{u})(w,k,\theta) + \widehat{u}(t_0,k,\theta)\ \mathrm{e}^{-iwt_0} + \widehat{G}(w,\bar{k},\theta). \quad (2.32)$$

A key step in the proof is to decompose $\widehat{u}$, for any fixed $(w,k)$, as

$$\widehat{u}(w,k,\theta) = (\rho \star \widehat{u})(w,k,\theta) + \Big[\widehat{u}(w,k,\theta) - (\rho \star \widehat{u})(w,k,\theta)\Big]. \quad (2.33)$$

From (2.26)-(2.32) and Proposition 2.1.1, the error can be estimated similarly as in [1, Theorem 3.2] by

$$\begin{aligned}
\|\widehat{u}(w,k,\cdot) - &(\rho_\epsilon \star \widehat{u})(w,k,\cdot)\|^2_{L^2(\mathbb{S}^{d-1})} \\
&= \int_{\mathbb{S}^{d-1}} \Big| \int_{\mathbb{S}^{d-1}} \rho_\epsilon(1-\theta\cdot\theta')\ (\widehat{u}(w,k,\theta) - \widehat{u}(w,k,\theta'))\ \mathrm{d}\theta' \Big|^2 \mathrm{d}\theta \\
&\le \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \rho_\epsilon(1-\theta\cdot\theta')\ |\widehat{u}(w,k,\theta) - \widehat{u}(w,k,\theta')|^2\ \mathrm{d}\theta'\ \mathrm{d}\theta \\
&\lesssim \frac{\epsilon^s}{C_\epsilon} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{|\widehat{u}(w,k,\theta) - \widehat{u}(w,k,\theta')|^2}{(1-\theta\cdot\theta')^{\frac{d-1}{2}+s}}\ \mathrm{d}\theta'\ \mathrm{d}\theta \\
&= -\frac{2\epsilon^s}{C_\epsilon b(1)} \int_{\mathbb{S}^{d-1}} \mathcal{I}_{b(1)}(\widehat{u})(w,k,\theta)\overline{\widehat{u}(w,k,\theta)}\ \mathrm{d}\theta \\
&\lesssim \frac{\epsilon^s}{C_\epsilon} \Big\|(-\Delta_v)^{s/2}\ \widehat{U_{\mathcal{J}}}(w,k,\cdot)\Big\|^2_{L^2(\mathbb{S}^{d-1})}.
\end{aligned} \quad (2.34)$$

Now, we estimate the term $\rho \star \widehat{u}$ in (2.33) for each fixed $(w,k)$. Using (2.32)

$$\widehat{u} = \frac{\lambda \widehat{u} + \mathcal{I}(\widehat{u}) + \widehat{u}(t_0,k,\theta)\mathrm{e}^{-\mathrm{i}wt_0} + \widehat{G}(w,\bar{k},\theta)}{\lambda + \mathrm{i}(w + \theta\cdot k)}, \quad (2.35)$$

where $\lambda > 0$ is an interpolation parameter depending only on $|k|$ (the parameter $\epsilon$ will depend only on $|k|$ as well). Formulas (2.12) and (2.35) lead to

$$\begin{aligned}
&(\rho\star\widehat{u})(w,k,\theta)\\
&\quad=\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\frac{\lambda\widehat{u}(w,k,\theta')+\mathcal{I}(\widehat{u})(w,k,\theta')+\widehat{u}(t_0,k,\theta')\,\mathrm{e}^{-\mathrm{i}wt_0}}{\lambda+\mathrm{i}(w+k\cdot\theta')}\,\mathrm{d}\theta'\\
&\quad=\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\frac{\widehat{u}(w,k,\theta')+\frac{1}{\lambda}\,\mathcal{K}(\widehat{u})(w,k,\theta')}{1+\mathrm{i}(w+k\cdot\theta')/\lambda}\,\mathrm{d}\theta'\\
&\qquad-\frac{D}{\lambda}\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\frac{(-\Delta_{\theta'})^s\,\widehat{u}(w,k,\theta')}{1+\mathrm{i}(w+k\cdot\theta')/\lambda}\,\mathrm{d}\theta'\\
&\qquad+\frac{1}{\lambda}\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\frac{\widehat{u}(t_0,k,\theta)\,\mathrm{e}^{-\mathrm{i}wt_0}}{1+\mathrm{i}(w+k\cdot\theta')/\lambda}\,\mathrm{d}\theta'\\
&\qquad+\frac{1}{\lambda}\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\frac{\widehat{G}(w,k,\theta)}{1+\mathrm{i}(w+k\cdot\theta')/\lambda}\,\mathrm{d}\theta'\\
&\quad\triangleq T_1+T_2+T_3+T_4,
\end{aligned}$$

where $\mathcal{K}\coloneqq c_{s,d}\mathbf{1}+\mathcal{I}_h$ is the bounded part of $\mathcal{I}$.

The terms $T_i$ for $i=1,2,3$ have been estimated in [1] in formula (3.33) for $T_1$, formulas (3.40), (3.41), (3.44) for $T_2$ and formula (3.47) for $T_3$. For the sake of completeness we shall give the proof

*Estimating the term* $T_1$. Note that

$$\begin{aligned}
|T_1(w,k,\theta)|\le&\left(\int_{\mathbb{S}^{d-1}}\frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1+\mathrm{i}(w+k\cdot\theta')/\lambda|^2}\,\mathrm{d}\theta'\right)^{\frac{1}{2}}\\
&\times\left[\left(\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')|\hat{u}(w,k,\theta')|^2\mathrm{d}\theta'\right)^{\frac{1}{2}}\right.\\
&\left.+\frac{1}{\lambda}\left(\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')|\mathcal{K}(\hat{u})(w,k,\theta')|^2\mathrm{d}\theta'\right)^{\frac{1}{2}}\right].
\end{aligned}\tag{2.36}$$

The first integral in (2.36) is estimated using that

$$\rho_\epsilon\lesssim\frac{1}{C_\epsilon\epsilon^{\frac{d-1}{2}}}\mathbb{1}_{|z|\le\epsilon}\qquad\text{and}\qquad|\theta-\theta'|^2=2(1-\theta\cdot\theta').$$

Choosing $\hat{k}$ as the north pole of $\mathbb{S}^{d-1}$ we can descompose any vector $\theta\in\mathbb{S}^{d-1}$ as $\theta=(\theta\cdot\hat{k})\hat{k}+\theta_\perp$ with $\theta_\perp\in\mathbb{R}^{d-1}$ and $\theta_\perp\perp\hat{k}$. Hence,

$$\begin{aligned}
\rho_\epsilon(1-\theta\cdot\theta')&\lesssim\frac{1}{C_\epsilon\epsilon^{\frac{d-1}{2}}}\mathbb{1}_{\{|\theta-\theta'|^2\le2\epsilon\}}\\
&=\frac{1}{C_\epsilon\epsilon^{\frac{d-1}{2}}}\mathbb{1}_{\{|\theta\cdot\hat{k}-\theta'\cdot\hat{k}|^2+\left|\theta_\perp-\theta'_\perp\right|^2\le2\epsilon\}}\\
&\le\frac{1}{C_\epsilon\epsilon^{\frac{d-1}{2}}}\mathbb{1}_{\{|\theta\cdot\hat{k}-\theta'\cdot\hat{k}|^2\le2\epsilon\}}\,\mathbb{1}_{\{\left|\theta_\perp-\theta'_\perp\right|^2\le2\epsilon\}}
\end{aligned}\tag{2.37}$$

Thus, using (2.37) we can establish that

$$\int_{\mathbb{S}^{d-1}} \frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1+\mathrm{i}(w+k\cdot\theta')/\lambda|^2}\,\mathrm{d}\theta'$$
$$\leq \frac{1}{C_\epsilon \epsilon^{\frac{d-1}{2}}}\int_0^\pi \frac{\mathbb{1}_{\{|\theta\cdot\hat{k}-\cos(\alpha)^2|\leq 2\epsilon\}}}{|1+\mathrm{i}(w+|k|\cos(\theta))/\lambda|^2}\int_{\mathbb{S}^{d-2}}\mathbb{1}_{\{|\theta_\perp-\sin(\alpha)\sigma|^2\leq 2\epsilon\}}\mathrm{d}\sigma\,\sin^{d-2}(\alpha)\mathrm{d}\alpha$$
$$\lesssim \frac{1}{C_\epsilon\sqrt{\epsilon}}\int_0^\pi \frac{\mathbb{1}_{\{|\theta\cdot\hat{k}-\cos(\alpha)^2|\leq 2\epsilon\}}}{|1+\mathrm{i}(w+|k|\cos(\theta))/\lambda|^2}\,\mathrm{d}\alpha. \tag{2.38}$$

The above integral can be estimated using Parseval's theorem. Setting $\cos\alpha = z$, then $\mathrm{d}\alpha = \frac{-1}{\sqrt{1-z^2}}\mathrm{d}z$

$$\begin{aligned}
&\frac{1}{C_\epsilon\sqrt{\epsilon}}\int_0^\pi \frac{\mathbb{1}_{\{|\theta\cdot\hat{k}-\cos(\alpha)^2|\leq 2\epsilon\}}}{|1+\mathrm{i}(w+|k|\cos(\theta))/\lambda|^2}\,\mathrm{d}\alpha\\
&\qquad\lesssim \frac{1}{C_\epsilon\sqrt{\epsilon}}\int_0^\pi \frac{1}{1+((w+|k|\cos(\theta))/\lambda)^2}\,\frac{1}{\sqrt{1-z^2}}\mathrm{d}z\\
&\qquad\lesssim \frac{1}{C_\epsilon\sqrt{\epsilon}}\int_0^\pi \frac{1}{1+((w+|k|\cos(\theta))/\lambda)^2}\left(\frac{1}{|1-z|^{\frac12}}+\frac{1}{|1+z|^{\frac12}}\right)\mathrm{d}z\\
&\qquad\lesssim \frac{C}{C_\epsilon\sqrt{\epsilon}}\frac{\lambda}{2|k|}\int_\infty^\infty \mathrm{e}^{-\mathrm{i}\frac{w}{|k|}}\mathrm{e}^{-\frac{\lambda}{|k|}}\frac{\cos(\xi)}{|\xi|^{\frac12}}\mathrm{d}\xi \lesssim \frac{1}{C_\epsilon\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}.
\end{aligned} \tag{2.39}$$

Using (2.39) in (2.36) one obtains that the $L^2_\theta$-norm of $T_1$ is estimated by

$$\begin{aligned}
&\|T_1(w,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})}\\
&\qquad\leq C\left(\frac{1}{\sqrt{\epsilon}}\frac{\lambda}{|k|}\right)^{\frac12}\left(\|\hat{u}(w,k,\cdot\|_{L^2(\mathbb{S}^{d-1})}+\frac{1}{\lambda}\|\mathcal{K}(\hat{u})(w,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})}\right).
\end{aligned} \tag{2.40}$$

*Estimating the term $T_2$.* Using the stereographic projection and the definition of the operator $(-\Delta_\theta)^s$ we have

$$\begin{aligned}
T_2(w,k,\theta) &= -2^{d-1}\frac{D}{\lambda}\int_{\mathbb{R}^{d-1}}\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))\frac{[(-\Delta_{\theta'})^s\hat{u}]_{\mathcal{J}}(w,k,v)}{1+i(w+k\cdot\mathcal{J}(v))/\lambda}\frac{\mathrm{d}v}{\langle v\rangle^{d-1}}\\
&= -2^{d-1}\frac{D}{\lambda}\int_{\mathbb{R}^{d-1}}\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))\frac{[(-\Delta_{\theta'})^s\hat{u}]_{\mathcal{J}}(w,k,v)}{(1+i(w+k\cdot\mathcal{J}(v))/\lambda)\langle v\rangle^{\frac{d-1}{2}-s}}\\
&\qquad\times(-\Delta_v)^s\widehat{U_{\mathcal{J}}}(w,k,v)\mathrm{d}v\\
&= -2^{d-1}\frac{D}{\lambda}\int_{\mathbb{R}^{d-1}}\nabla_v\left[\frac{\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))}{(1+i(w+k\cdot\mathcal{J}(v))/\lambda)\langle v\rangle^{\frac{d-1}{2}-s}}\right]\\
&\qquad\cdot\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)\mathrm{d}v,
\end{aligned} \tag{2.41}$$

where the fractional gradient operator $\nabla^{2s-1}$ is defined by Fourier transform as

$$\mathcal{F}\{\nabla_v^{2s-1}\psi\}(\xi) = -i|\xi|^{2s-1}\hat{\xi}\mathcal{F}\{\psi\}(\xi).$$

Now, we explicitly compute the gradient inside the last integral in (2.41) to obtain 3 terms, namely,

$$\begin{aligned}\nabla_v\left[\frac{\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))}{(1+i(w+k\cdot\mathcal{J}(v))/\lambda)\,\langle v\rangle^{\frac{d-1}{2}-s}}\right] &= \frac{\nabla_v\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))}{(1+i(w+k\cdot\mathcal{J}(v))/\lambda)\,\langle v\rangle^{\frac{d-1}{2}-s}}\\ &\quad+\frac{\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))}{\langle v\rangle^{\frac{d-1}{2}-s}}\nabla_v\frac{1}{(1+i(w+k\cdot\mathcal{J}(v))/\lambda)}\\ &\quad+\frac{\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))}{(1+i(w+k\cdot\mathcal{J}(v))/\lambda)}\nabla_v\frac{1}{\langle v\rangle^{\frac{d-1}{2}-s}},\end{aligned}$$

which give us the decomposition $T_2 = T_2^1 + T_2^2 + T_2^3$ respectively. Additionally, note that for any vector $x \in \mathbb{R}^{d-1}$

$$|\nabla_v(x\cdot\mathcal{J}(v)| \lesssim \frac{|x|}{v}, \tag{2.42}$$

which leads to the estimate for $T_2^1$

$$\begin{aligned}T_2^1(w,k,\theta) &\lesssim \frac{D}{\lambda}\int_{\mathbb{R}^{d-1}}\frac{|\rho'_\epsilon(1-\theta\cdot\mathcal{J}(v))|}{|1-i(w+k\cdot\mathcal{J}(v))/\lambda|\,\langle v\rangle^{\frac{d-1}{2}-s+1}}\frac{|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)|}{}\mathrm{d}v\\ &= \frac{D}{\lambda}\left(\int_{\mathbb{R}^{d-1}}\frac{|\rho'_\epsilon(1-\theta\cdot\mathcal{J}(v))|}{|1-i(w+k\cdot\mathcal{J}(v))/\lambda|^q\,\langle v\rangle^{(\frac{d-1}{2}-s+1)q}}\mathrm{d}v\right)^{\frac{1}{q}}\\ &\qquad\times\left(\int_{\mathbb{R}^{d-1}}|\rho'_\epsilon(1-\theta\cdot\mathcal{J}(v))|\left|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)\right|^p\mathrm{d}v\right)^{\dfrac{1}{p}},\end{aligned} \tag{2.43}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$. Using a Sobolev embedding inequality one has

$$\begin{aligned}\left\|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^p(\mathbb{R}^{d-1})} &\le C_{d,s}\left\|\nabla_v^{s}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^p(\mathbb{R}^{d-1})}\\ &= C_{d,s}\left\|(-\Delta_v)^{s/2}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^p(\mathbb{R}^{d-1})},\end{aligned} \tag{2.44}$$

for $\dfrac{1}{2}-\dfrac{1-s}{d-1}=\dfrac{1}{p}$. This defines our choice of $p := p(d,s) > 2$ in (2.43). In this way,

$$\left(\frac{d-1}{2}-s+1\right)q = d-1$$

and estimate (2.43) reduces to

$$\left\|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^p(\mathbb{R}^{d-1})} = \frac{D}{\lambda}\left(\int_{\mathbb{R}^{d-1}}\frac{|\rho'_\epsilon(1-\theta\cdot\theta')|}{|1-i(w+k\cdot\theta')/\lambda|^q}\mathrm{d}\theta'\right)^{\frac{1}{q}} \times\left(\int_{\mathbb{R}^{d-1}}|\rho'_\epsilon(1-\theta\cdot\mathcal{J}(v))|\left|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)\right|^p\mathrm{d}v\right)^{\frac{1}{p}}. \tag{2.45}$$

Similar to what was done in $T_1$, we have that

$$\int_{\mathbb{R}^{d-1}}\frac{|\rho'_\epsilon(1-\theta\cdot\theta')|}{|1-i(w+k\cdot\theta')/\lambda|^q}\mathrm{d}\theta' \le \frac{c}{\epsilon\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}. \tag{2.46}$$

Furthermore, estimates (2.44),(2.45) and (2.46) give

$$\left\|T_2^1(w,k,\cdot)\right\|_{L^p(\mathbb{S}^{d-1})} \le \frac{D}{\lambda\epsilon}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}}\left\|(-\Delta_v)^{s/2}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^2(\mathbb{R}^{d-1})}. \tag{2.47}$$

In the same manner, the term $T_2^2$ can be estimated as

$$\begin{aligned}
&T_2^2(w,k,\theta)\\
&\lesssim \frac{D|k|}{\lambda^2}\int_{\mathbb{R}^{d-1}}\frac{\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))}{|1-i(w+k\cdot\mathcal{J}(v))/\lambda|^2\left\langle v\right\rangle^{\frac{d-1}{2}-s+1}}\frac{|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)|}{}\mathrm{d}v\\
&\lesssim \frac{D|k|}{\lambda^2}\left(\int_{\mathbb{R}^{d-1}}\frac{\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))}{|1-i(w+k\cdot\mathcal{J}(v))/\lambda|^{2q}}\mathrm{d}\theta'\right)^{\frac{1}{q}}\\
&\qquad\times\left(\int_{\mathbb{R}^{d-1}}\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))\left|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)\right|^p\mathrm{d}v\right)^{\frac{1}{p}}\\
&\lesssim \frac{D|k|}{\lambda^2}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^{d-1}}\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))\left|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)\right|^p\mathrm{d}v\right)^{\frac{1}{p}},
\end{aligned} \tag{2.48}$$

where the exponents $p$ and $q$ are those of the term $T_2^1$. Previous estimate lead us to the bound

$$\left\|T_2^2(w,k,\cdot)\right\|_{L^p(\mathbb{S}^{d-1})} \le \frac{D|k|}{\lambda^2}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}}\left\|(-\Delta_v)^{s/2}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^2(\mathbb{R}^{d-1})}. \tag{2.49}$$

For the final term $T_2^3$ note that

$$\nabla_v\frac{1}{\left\langle v\right\rangle^{\frac{d-1}{2}-s}} = -\left(\frac{d-1}{2}-s\right)\frac{2v}{\left\langle v\right\rangle^{\frac{d-1}{2}-s+1}},$$

therefore the stereographic projection leads to

$$T_2^3(w,k,\theta) \lesssim \frac{D}{\lambda}\left(\int_{\mathbb{S}^{d-1}}\frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1-i(w+k\cdot\theta')/\lambda|^q}\left(\frac{|\theta^\perp|}{1-\theta^\perp}\right)^q\mathrm{d}\theta'\right)^{\frac{1}{q}} \times\left(\int_{\mathbb{R}^{d-1}}\rho_\epsilon(1-\theta\cdot\mathcal{J}(v))\left|\nabla_v^{2s-1}\widehat{U_{\mathcal{J}}}(w,k,v)\right|^p\mathrm{d}v\right)^{\frac{1}{p}}. \tag{2.50}$$

Note that $\frac{|\theta^{\perp}|}{1-\theta^{\perp}} \leq \frac{1}{\sin(\alpha)}$, with $\alpha$ the polar angle. Hence, the following estimate is valid for $d \geq 3$ (recall that $q \in (0,2)$)

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1+i(w+k\cdot\theta')/\lambda|^q}\left(\frac{|\theta^{\perp}|}{1-\theta^{\perp}}\right)^q \mathrm{d}\theta' \\
&\lesssim \frac{1}{C_\epsilon \epsilon^{\frac{d-1}{2}}}\int_0^\pi \frac{\mathbb{1}_{\{|\theta\cdot\hat{k}-\cos(\alpha)^2|\leq 2\epsilon\}}}{|1+\mathrm{i}(w+|k|\cos(\theta))/\lambda|^q}\left(\int_{\mathbb{S}^{d-2}} \mathbb{1}_{\{|\theta_\perp-\sin(\alpha)\sigma|^2\leq 2\epsilon\}}\,\mathrm{d}\sigma\,\sin^{d-2-q}(\alpha)\right)\mathrm{d}\alpha \\
&\lesssim \frac{1}{C_\epsilon\epsilon}\int_0^\pi \frac{\mathbb{1}_{\{|\theta\cdot\hat{k}-\cos(\alpha)^2|\leq 2\epsilon\}}}{|1+\mathrm{i}(w+|k|\cos(\theta))/\lambda|^q}\,\frac{\mathrm{d}\alpha}{\sin(\alpha)^{q-1}} \lesssim \frac{1}{C_\epsilon\epsilon}\left(\frac{\lambda}{|k|}\right)^{1-\frac{q}{2}}.
\end{aligned} \tag{2.51}$$

Finally, one concludes that

$$\left\|T_2^3(w,k,\cdot)\right\|_{L^p(\mathbb{S}^{d-1})} \leq \frac{D}{\lambda\sqrt{\epsilon}}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{2-q}{q}}\left\|(-\Delta_v)^{s/2}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^2(\mathbb{R}^{d-1})}. \tag{2.52}$$

*Estimating the term $T_3$.* As in the previous calculations we have

$$\begin{aligned}
|T_3(w,k,\theta)| &= \frac{1}{\lambda}\int_{\mathbb{S}^{d-1}} \rho_\epsilon(1-\theta\cdot\theta')\frac{|\hat{u}(t_0,k,\theta')|}{|1+i(w+k\cdot\theta')/\lambda|}\mathrm{d}\theta' \\
&\leq \frac{1}{\lambda}\left(\int_{\mathbb{S}^{d-1}} \frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1+i(w+k\cdot\theta')/\lambda|^{2(1-s_0)}}\mathrm{d}\theta'\right)^{\frac{1}{2}} \\
&\qquad \times\left(\int_{\mathbb{R}^{d-1}} \rho_\epsilon(1-\theta\cdot\theta')\frac{|\hat{u}(t_0,k,\theta')|^2}{|1+i(w+k\cdot\theta')/\lambda|^{2s_0}}\,\mathrm{d}\theta'\right)^{\frac{1}{2}},
\end{aligned} \tag{2.53}$$

where $s_0 \in \left(\frac{1}{2},1\right)$ will be chosen in a moment. Note that we can estimate the first integral of the above equation as

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1+i(w+k\cdot\theta')/\lambda|^{2(1-s_0)}}\mathrm{d}\theta' \\
&\qquad \lesssim \frac{1}{C_\epsilon\sqrt{\epsilon}}\int_{-1}^1 \frac{1}{|1+((w+|k|z)/\lambda)^2|^{1-s_0}}\frac{1}{\sqrt{1-z^2}}\,\mathrm{d}z \\
&\qquad \lesssim \frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\int_{-\infty}^{\infty} \mathcal{F}\{\mathcal{B}_{2(1-s_0)}\}(\xi)\,\frac{1}{|\xi|^{\frac{1}{2}}}\mathrm{d}\xi \lesssim \frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}.
\end{aligned} \tag{2.54}$$

Recall that $\mathcal{B}_{2(1-s_0)}$ is the Bessel potential of order $2(1-s_0)$, so, previous estimate is valid for $s_0$ sufficiently close to $1/2$ and such that the singularity at $\xi = 0$ becomes integrable. More precisely, from the discussion in the appendix about Bessel potentials one can see that any $s_0 \in (\frac{1}{2},\frac{3}{4})$ will do. Plug estimate (2.54) in (2.53) and integrating in $(w,\theta)$ to obtain

$$\|T_3(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} \le \frac{C}{\lambda}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}} \|\widehat{u}(t_0,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})}. \tag{2.55}$$

Let us summarize the above computations, namely $T_1, T_2$ and $T_3$. First,

$$\begin{aligned}&\|T_1(w,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})}\\ &\quad\le C\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}\left(\|\widehat{u}(w,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})}+\frac{1}{\lambda}\|\mathcal{K}(\widehat{u})(w,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})}\right).\end{aligned} \tag{2.56}$$

The term $T_2$ satisfies

$$\begin{aligned}\|T_2(w,k,\cdot)\|_{L^p(\mathbb{S}^{d-1})} \le{}& \frac{D}{\lambda\epsilon}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}}\left\|(-\Delta_v)^{s/2}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^2(\mathbb{R}^{d-1})}\\ &+\frac{D|k|}{\lambda^2}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}}\left\|(-\Delta_v)^{s/2}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^2(\mathbb{R}^{d-1})}\\ &+\frac{D}{\lambda\sqrt{\epsilon}}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{2-q}{q}}\left\|(-\Delta_v)^{s/2}\widehat{U_{\mathcal{J}}}(w,k,\cdot)\right\|_{L^2(\mathbb{R}^{d-1})},\end{aligned} \tag{2.57}$$

where $p := p(s,d)$ and $q := q(s,d)$, which are convex dual, are given by the formulas

$$\frac{1}{2}-\frac{1-s}{d-1}=\frac{1}{p} \quad\text{and}\quad \frac{1}{2}+\frac{1-s}{d-1}=\frac{1}{q}, \quad d\ge 3.$$

The term $T_3$ is estimates as

$$\|T_3(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} \le \frac{C}{\lambda}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}} \|\widehat{u}(t_0,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})}. \tag{2.58}$$

Finally, we estimate the boundary term $T_4$ as follows. Note that

$$\begin{aligned}|T_4(w,k,\theta)| &= \frac{1}{\lambda}\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\frac{|\widehat{G}(w,\bar{k},\theta')|}{|1+i(w+k\cdot\theta')/\lambda|}\mathrm{d}\theta'\\ &\le\frac{1}{\lambda}\left(\int_{\mathbb{S}^{d-1}}\frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1+i(w+k\cdot\theta')/\lambda|^2}\mathrm{d}\theta'\right)^{\frac{1}{2}}\\ &\qquad\times\left(\int_{\mathbb{S}^{d-1}}\rho_\epsilon(1-\theta\cdot\theta')\,|\widehat{G}(w,\bar{k},\theta')|^2\,\mathrm{d}\theta'\right)^{\frac{1}{2}}.\end{aligned} \tag{2.59}$$

From (2.38) and (2.39), we have

$$\int_{\mathbb{S}^{d-1}}\frac{\rho_\epsilon(1-\theta\cdot\theta')}{|1+i(w+k\cdot\theta')/\lambda|^2}\mathrm{d}\theta' \lesssim \frac{1}{C\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}. \tag{2.60}$$

Hence, using estimate (2.60) in (2.59) and integrating in the variables $w$ and

$\theta$ we have that

$$\begin{aligned}
&\left(\int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}} |T_4(w,k,\theta)|^2 \,\mathrm{d}\theta\,\mathrm{d}w\right)^{1/2} \\
&\quad\le \frac{C}{\lambda}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} \rho_\epsilon(1-\theta\cdot\theta')\,|\widehat{G}(w,\bar{k},\theta')|^2\,\mathrm{d}\theta'\,\mathrm{d}\theta\,\mathrm{d}w\right)^{1/2} \\
&\quad\quad\le \frac{C}{\lambda}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}} |\widehat{G}(w,\bar{k},\theta')|^2\,\mathrm{d}\theta'\,\mathrm{d}w\right)^{1/2}.
\end{aligned}$$

Thus, one obtains that the $L^2_{t,\theta}$ norm of $T_4$ is estimated by

$$\|T_4(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} \le \frac{C}{\lambda}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}\left\|\widehat{G}(\cdot,\bar{k},\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}. \tag{2.61}$$

*Conclusion of the proof.* From the decomposition (2.33) and the estimates (2.56), (2.57), (2.58) and (2.61) one concludes that

$$\begin{aligned}
&\|\widehat{u}(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} \\
&\le \left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}\left(\|\widehat{u}(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} + \frac{1}{\lambda}\|\mathcal{K}(\widehat{u})(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}\right) \\
&\quad + \frac{C}{\lambda}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}\|\widehat{u}(t_0,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})} + D\left(\frac{1}{\lambda\epsilon}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}} + \frac{|k|}{\lambda^2}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}}\right. \\
&\quad\quad \left. + \frac{1}{\lambda\epsilon}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{2-q}{q}} + \epsilon^{\frac{s}{2}}\right)\left\|(-\Delta_v)^{s/2}\,\widehat{U_{\mathcal{J}}}(\cdot,k,\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^{d-1})} \\
&\quad\quad\quad + \frac{C}{\lambda}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}\left\|\widehat{G}(\cdot,\bar{k},\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}.
\end{aligned} \tag{2.62}$$

Keep in mind that we are seeking an estimate for large frequencies in the spatial Fourier variable $k \in \mathbb{R}^d$. Set $\epsilon = |k|^{-a}$ and $\lambda = |k|^b$ with numbers $a, b > 0$ to be chosen in the sequel. Since we hope that

$$\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}} \sim \frac{1}{|k|^{s_0}}, \quad \text{for some} \quad s_0 > 0,$$

we can control the term

$$\frac{|k|}{\lambda^2}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{q}} \quad \text{by choosing} \quad \frac{|k|}{\lambda^2} = 1,$$

that is, choosing $b = 1/2$. Recalling that $q \in (1,2)$ one conclude that the leading terms are

$$\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}, \quad \frac{1}{\lambda\sqrt{\epsilon}}\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{2-q}{q}} \quad \text{and} \quad \epsilon^{\frac{s}{2}}.$$

The best option independent of the dimension is choosing $a$ such that

$$\max\left\{\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}, \frac{1}{\lambda\sqrt{\epsilon}}\right\} = \epsilon^{\frac{s}{2}}.$$

In fact,

$$\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}} = \left(|k|^{a/2}\sqrt{|k|^{-1/2}}\right)^{1/2} = \frac{1}{|k|^{\frac{1-2a}{8}}},$$

thus, the best option reduces to find $a$ such that

$$\max\left\{\left(\frac{1}{\sqrt{\epsilon}}\sqrt{\frac{\lambda}{|k|}}\right)^{\frac{1}{2}}, \frac{1}{\lambda\sqrt{\epsilon}}\right\} = \max\left\{\frac{1}{|k|^{\frac{1-2a}{8}}}, \frac{1}{|k|^{\frac{1-a}{2}}}\right\} = \frac{1}{|k|^{\frac{as}{2}}}.$$

The first term in the maximum, being the larger for $|k| \geq 1$, is the constraint. As a consequence, we must have $\frac{1-2a}{8} = \frac{as}{2}$, or, $a = \frac{1/2}{2s+1}$. Computing from (2.62), one concludes that for $|k| \geq 1$,

$$\begin{aligned}\|\widehat{u}(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} \leq \frac{C}{|k|^{\frac{as}{4}}}\Bigg(&\|\widehat{u}(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} + \|\mathcal{K}(\widehat{u})(\cdot,k,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})} \\ &+ \left\|(-\Delta_v)^{s/2}\,\widehat{U_{\mathcal{J}}}(\cdot,k,\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^{d-1})} + \|\widehat{u}(t_0,k,\cdot)\|_{L^2(\mathbb{S}^{d-1})} \\ &+ \frac{\|\widehat{G}(\cdot,\bar{k},\cdot)\|_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}}{|k|^{\frac{as}{4}+\frac{1}{2}}}\Bigg).\end{aligned} \tag{2.63}$$

Take $s_0 = \frac{as}{4}$ and compute using Plancherel theorem

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\left|(-\Delta_x)^{s_0/2}u\right|^2 \mathrm{d}\theta\mathrm{d}x\mathrm{d}t &= \int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\left|\widehat{u}\right|^2|k|^{2s_0}\mathrm{d}\theta\mathrm{d}k\mathrm{d}w \\ &= \left(\int_{|k|\leq 1} + \int_{|k|\geq 1}\right)\int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}}\left|\widehat{u}\right|^2|k|^{2s_0}\mathrm{d}k\mathrm{d}\theta\mathrm{d}w \\ &\leq \|u\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})} + \int_{|k|\geq 1}\int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}}\left|\widehat{u}\right|^2|k|^{2s_0}\mathrm{d}\theta\mathrm{d}w\mathrm{d}k\,.\end{aligned} \tag{2.64}$$

We apply estimate (2.63) in the second term of the right hand side noticing that denoting $A_1 = \{|k| \geq 1, |k_d| \leq 1\}$ and $A_2 = \{|k| \geq 1, |k_d| \geq 1\}$ it follows that

$$\int_{|k|\geq 1}\frac{\|\widehat{G}(\cdot,\bar{k},\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}}{|k|^{2s_0+1}}\mathrm{d}k = \left(\int_{A_1} + \int_{A_2}\right)\frac{\|\widehat{G}(\cdot,\bar{k},\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}}{|k|^{2s_0+1}}\,\mathrm{d}k\,,$$

where, by Plancherel theorem, one has that

$$\int_{A_1} \frac{\|\widehat{G}(\cdot,\bar{k},\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}}{|k|^{2s_0+1}}\,\mathrm{d}k \le \int_{\mathbb{R}^{d-1}} \|\widehat{G}(\cdot,\bar{k},\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}\,\mathrm{d}\bar{k}\left(\int_{|k_d|\le 1}\mathrm{d}k_d\right)$$
$$\le 2\,\|G\|^2_{L^2((t_0,t_1)\times\partial\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\,.$$

Similarly,

$$\begin{aligned}\int_{A_2} \frac{\|\widehat{G}(\cdot,\bar{k},\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}}{|k|^{2s_0+1}}\,\mathrm{d}k &\le \int_{|k_d|\ge 1} \frac{\|\widehat{G}(\cdot,\bar{k},\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}}{|k_d|^{2s_0+1}}\,\mathrm{d}k\\ &\le \int_{\mathbb{R}^{d-1}} \|\widehat{G}(\cdot,\bar{k},\cdot)\|^2_{L^2(\mathbb{R}\times\mathbb{S}^{d-1})}\,\mathrm{d}\bar{k}\left(\int_{|k_d|\ge 1}\frac{\mathrm{d}k_d}{|k_d|^{2s_0+1}}\right)\\ &= \frac{1}{2s_0}\,\|G\|^2_{L^2((t_0,t_1)\times\partial\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\\ &= \frac{1}{2s_0}\,\|\theta_d\, g\|^2_{L^2((t_0,t_1)\times\Gamma^-)}\,.\end{aligned}$$

As a consequence, (2.25) follows from (2.64) and (2.63) and the result is established. ■

**Corollary 2.2.1** *Let $u$ be a solution to* (2.1) *which satisfies the conditions in Theorem 2.2.1. Then, for any $t\in(t_0,t_1)$, we have*

$$\begin{aligned}\left\|(-\Delta_x)^{s_0/2}\,u\right\|_{L^2((t,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} &\le C\left(\frac{1}{\sqrt{t-t_0}}+1\right)\|u\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\\ &\quad + C\left(\left\|(-\Delta_v)^{s/2}\,U_{\mathcal{J}}\right\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{R}^{d-1})} + \|\theta_d\, g\|_{L^2((t_0,t_1)\times\Gamma^-)}\right).\end{aligned} \tag{2.65}$$

*Proof.* Let $\tau\in(t_0,t_1)$ be arbitrary. Then, estimate (2.25) gives that

$$\begin{aligned}\|(-\Delta_x)^{s_0/2}u\|_{L^2((\tau,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} &\le C\Big(\|u(\tau)\|_{L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})} + \|u\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\\ &\quad + \|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{R}^{d-1})} + \|\theta_d\, g\|_{L^2((t_0,t_1)\times\Gamma^-)}\Big)\,.\end{aligned}$$

Taking the time average of the above inequality over $[t_0, t]$ we have

$$\begin{aligned}\left\|(-\Delta_x)^{s_0/2}u\right\|_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} &\leq \frac{1}{t_*-t_0}\int_{t_0}^{t}\left\|(-\Delta_x)^{s_0/2}u\right\|_{L^2((\tau,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})}\mathrm{d}\tau\\ &\leq C\left(\frac{1}{t-t_0}\int_{t_0}^{t}\|u(\tau)\|_{L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\mathrm{d}\tau+\|u\|_{L^2([t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\right)\\ &\quad+C\left(\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|_{L^2([t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}+\|\theta_d g\|_{L^2((t_0,t_1)\times\Gamma^-)}\right)\\ &\leq C\left(\frac{1}{t-t_0}\int_{t_0}^{t_1}\|u(\tau)\|_{L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\mathrm{d}\tau+\|u\|_{L^2([t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\right)\\ &\quad+C\left(\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|_{L^2([t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}+\|\theta_d g\|_{L^2((t_0,t_1)\times\Gamma^-)}\right).\end{aligned}$$

Inequality (2.65) is then obtained by taking square root on both sides of the above inequality. ∎

## 2.3
## A priori estimates and regularity

In this section we implement a classical program in the theory of hypoelliptic equations that consists in proving successive gains of regularity. The interplay of the kinetic variable $\theta$ and the spatial variable was quantified in Theorem 2.2.1 in previous section. It will play an essential role here. Of course, it is understood that solutions are assumed to be sufficiently regular so that all computations are valid.

### 2.3.1
### Regularity estimates up to the boundary.

The mass equation (2.19) with $t' = 0$ gives that

$$m(t) + \Phi_{out}(t) = m(0) + \Phi_{in}(t), \tag{2.66}$$

where, for $t \geq 0$

$$m(t) := \int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u(t,x,\theta)\,\mathrm{d}\theta\,\mathrm{d}x$$

is the mass,

$$\Phi_{out}(t) := \int_0^t\int_{\Gamma^+} u\left(\theta\cdot n(\bar{x})\right)\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau$$

stands for the total mass output in $(0,t)$, and

$$\Phi_{in}(t) := \int_0^t\int_{\Gamma^-} g\left|\theta\cdot n(\bar{x})\right|\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau$$

accounts for the total mass input at $(0,t)$. We assume in the sequel that the total input of mass is finite,

$$m(0) + \sup_{t\geq 0} \Phi_{in}(t) =: c(m_0, \Phi_{in}) < \infty\,.$$

As a consequence, the solutions are assumed with uniform bounded mass

$$\sup_{t\geq 0} m(t) \leq c(m_0, \Phi_{in})\,, \qquad \sup_{t\geq 0} \Phi_{out}(t) \leq c(m_0, \Phi_{in})\,.$$

The following proposition is the analog to [1, Proposition 4.1], see below Proposition A.1.1. We refer to this reference for its proof which is based on Sobolev embedding inequalities and elementary interpolation.

**Proposition 2.3.1** *Let $u \in L^2\big((t_0,t_1); H^s_\theta \cap H^{s_0}_x\big)$, for some $0 < s_0 < s$, be a solution of the RTE (2.1). Then, there exist an explicit $\omega > 1$ and a constant $C(m_0, \Phi_{in}, d, s_0, s) > 0$ such that*

$$\begin{aligned}\int_{t_0}^{t_1} \|u\|^{2\omega}_{L^2_{x,\theta}}\, \mathrm{d}\tau \\ \leq C(m_0, \Phi_{in}, d, s_0, s) \left( \int_{t_0}^{t_1} \int_{\mathbb{R}^d_+} \|u\|^2_{H^s_\theta}\, \mathrm{d}x\, \mathrm{d}\tau + \int_{t_0}^{t_1} \|(-\Delta_x)^{s_0/2}\, u\|^2_{L^2_{x,\theta}}\, \mathrm{d}\tau \right).\end{aligned}$$

*Proof.* By Sobolev embedding, for each $\tau \in (t_0, t)$ we have

$$\int_{\mathbb{R}^d_+} \|u(\tau,x,\cdot)\|^2_{H^2_\theta}\, \mathrm{d}x \geq c_{d,s} \int_{\mathbb{R}^d_+} \left( \int_{\mathbb{S}^{d-1}} u^{p_s}(x,\theta)\, \mathrm{d}\theta \right)^{2/p_s} \mathrm{d}x,$$

$$\|(-\Delta_x)^{s_0/2} u(\tau,x,\cdot)\|^2_{L^2_x} \geq c_{d,s} \left( \int_{\mathbb{S}^{d-1}} u^{q_2}(x,\theta)\, \mathrm{d}\theta \right)^{2/q_2}.$$

where,

$$\frac{1}{q_2} = \frac{1}{2} - \frac{s_0}{d}, \qquad \frac{1}{p_s} = \frac{1}{2} - \frac{s}{d-1}.$$

Notice that $q_2, p_s > 2$. Let

$$\alpha_1 = \frac{2q_2 - 2}{p_s q_2 - 2} \in (0,1), \qquad \alpha_2 = \frac{p_s}{2}\alpha_1 \in (0,1), \qquad r = p_s\alpha_1 + (1-\alpha_1) > 2, \tag{2.67}$$

such that

$$\frac{\alpha_1}{\alpha_2} = \frac{2}{p_s}, \qquad \frac{1-\alpha_1}{1-\alpha_2} = q_2,$$

and

$$r = 2 + \frac{(p_s-2)(q_2-3)}{p_s q_2 - 2} = 3 - \frac{2(p_s + q_2 - 2)}{p_s q_2 - 2} \in (2,3).$$

Then, by Hölder's inequality and Jensen's inequality, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^r \,\mathrm{d}\theta\,\mathrm{d}x\right)^{2/r} &\le \left(\int_{\mathbb{R}^d_+}\left(\int_{\mathbb{S}^{d-1}} u^{p_s}\,\mathrm{d}\theta\right)^{\alpha_1}\left(\int_{\mathbb{S}^{d-1}} u\,\mathrm{d}\theta\right)^{1-\alpha_1}\mathrm{d}x\right)^{2/r}\\
&\le \left(\int_{\mathbb{R}^d_+}\left(\int_{\mathbb{S}^{d-1}} u^{p_s}\,\mathrm{d}\theta\right)^{\frac{\alpha_1}{\alpha_2}}\mathrm{d}x\right)^{2\alpha_2/r}\left(\int_{\mathbb{R}^d_+}\left(\int_{\mathbb{S}^{d-1}} u\,\mathrm{d}\theta\right)^{\frac{1-\alpha_1}{1-\alpha_2}}\mathrm{d}x\right)^{2(1-\alpha_2)/r}\\
&= \left(\int_{\mathbb{R}^d_+}\left(\int_{\mathbb{S}^{d-1}} u^{p_s}\,\mathrm{d}\theta\right)^{2/p_s}\mathrm{d}x\right)^{2\alpha_2/r}\left(\int_{\mathbb{R}^d_+}\left(\int_{\mathbb{S}^{d-1}} u\,\mathrm{d}\theta\right)^{q_2}\mathrm{d}x\right)^{2(1-\alpha_2)/r}\\
&\le \left(\int_{\mathbb{R}^d_+}\left(\int_{\mathbb{S}^{d-1}} u^{p_s}\,\mathrm{d}\theta\right)^{2/p_s}\mathrm{d}x\right)^{2\alpha_2/r}\left(\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^{q_2}\,\mathrm{d}\theta\,\mathrm{d}x\right)^{2(1-\alpha_2)/r}\\
&\le c_{d,s_0,s}\left(\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\right)^{2\alpha_2/r}\left(\left\|(-\Delta_x)^{s_0/2}u\right\|^2_{L^2_{x,\theta}}\,\mathrm{d}x\right)^{(1-\alpha_2)q_2/r}.
\end{aligned}$$

Note that by our choice of (2.67), the parameters safisfy

$$\frac{2\alpha_2}{r}+\frac{(1-\alpha_2)q_2}{r}=1.$$

Thus, if we integrate in time, we have

$$\begin{aligned}
&\int_{t_0}^t\left(\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^r\,\mathrm{d}\theta\,\mathrm{d}x\right)^{2/r}\mathrm{d}\tau\\
&\quad\le c_{d,s_0,s}\int_{t_0}^t\left(\left(\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\right)^{2\alpha_2/r}\left(\left\|(-\Delta_x)^{s_0/2}u\right\|^2_{L^2_{x,\theta}}\,\mathrm{d}x\right)^{(1-\alpha_2)q_2/r}\mathrm{d}\tau\right)\\
&\quad\le c_{d,s_0,s}\left(\int_{t_0}^t\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau\right)^{2\alpha_2/r}\left(\int_{t_0}^t\left\|(-\Delta_x)^{s_0/2}u\right\|^2_{L^2_{x,\theta}}\,\mathrm{d}\tau\right)^{(1-\alpha_2)q_2/r}\mathrm{d}\tau\\
&\quad\le c_{d,s_0,s}\left(\int_{t_0}^t\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau+\int_{t_0}^t\left\|(-\Delta_x)^{s_0/2}u\right\|^2_{L^2_{x,\theta}}\,\mathrm{d}\tau\right).
\end{aligned}\tag{2.68}$$

Let

$$\omega=\frac{2(r-1)}{r}>1,\quad \alpha=\frac{1}{r-1}\in(0,1).\tag{2.69}$$

Hence,

$$r\alpha+(1-\alpha)=2\quad\text{and}\quad\omega=\frac{2}{\alpha r}.$$

Again due to the Hölder's inequality and (2.68), we have that

$$\begin{aligned}
&\int_{t_0}^{t}\left(\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^2 \,\mathrm{d}\theta\,\mathrm{d}x\right)^{\omega}\mathrm{d}\tau\\
&\qquad\le \int_{t_0}^{t}\left(\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^r \,\mathrm{d}\theta\,\mathrm{d}x\right)^{\alpha\omega}\left(\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u \,\mathrm{d}\theta\,\mathrm{d}x\right)^{(1-\alpha)\omega}\mathrm{d}\tau\\
&\qquad\le C(m_0,\Phi_{in})\int_{t_0}^{t}\left(\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u^r \,\mathrm{d}\theta\,\mathrm{d}x\right)^{2/r}\mathrm{d}\tau\\
&\qquad\le C(m_0,\Phi_{in},d,s_0,s)\left(\int_{t_0}^{t}\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau+\int_{t_0}^{t}\|(-\Delta_x)^{s_0/2}\,u\|^2_{L^2_{x,\theta}}\,\mathrm{d}\tau\right),
\end{aligned}$$

where $C(m_0,\Phi_{in},d,s_0,s)=c_{d,s,s_0}\,C^{(1-\alpha)\omega}(m_0,\Phi_{in})$. ∎

**Proposition 2.3.2 (Full energy estimate)** *Let* $u\in\mathcal{C}\Big([t_0,t_1);L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})\Big)$ *be a solution to the RTE* (2.1) *on* $(t_0,t_1)$ *with* $\theta_d\,g\in L^2\Big((t_0,t_1);L^2_{x,\theta}\Big)$. *Then*

$$\sup_{t\in(t_0,t_1)}\|u(t)\|^2_{L^2_{x,\theta}}+\int_{t_0}^{t_1}\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau+\int_{t_0}^{t_1}\|(-\Delta_x)^{s_0/2}u\|^2_{L^2_{x,\theta}}\mathrm{d}\tau \le C\left(\|u(t_0,\cdot,\cdot)\|^2_{L^2_{x,\theta}}+\|\theta_d\,g\|^2_{L^2((t_0,t_1)\times\Gamma^-)}\right),$$

*where the constant depends only on* $C:=C(t_1,d,s,s_0)$ *with* $s_0$ *defined in* (2.25).

*Proof.* The result is a direct consequence of Theorem 2.2.1. Since, we have

$$\begin{aligned}
\int_{t_0}^{t}\|(-\,\Delta_x)^{s_0/2}u\|^2_{L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})}&\le C\Big(\|u(t_0)\|^2_{L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})}+\|u\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1})}\\
&\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|^2_{L^2([t_0,t_1]\times\mathbb{R}^d_+\times\mathcal{S}^{d-1})}+\|\theta_d\,g\|^2_{L^2((t_0,t_1)\times\Gamma^-)}\Big),
\end{aligned}$$

for $s_0=\dfrac{s/4}{2s+1}$. Hence, by the energy estimate (2.22),

$$\begin{aligned}
\sup_{t\in(t_0,t_0+\epsilon_0)}\left(\|u\|^2_{L^2_{x,\theta}}(t)\right)+\int_{t_0}^{t}\int_{\mathbb{S}^{d-1}}\|u\|^2_{H^s_\theta}\mathrm{d}x\,\mathrm{d}\tau+\int_{t_0}^{t}\|(-\Delta_x)^{s_0/2}u\|^2_{L^2_{x,\theta}}\mathrm{d}\tau\\
\le c_{2,1}\|u(t_0,\cdot,\cdot)\|^2_{L^2_{x,\theta}}+c_{2,1}\int_{t_0}^{t}\|u\|^2_{L^2_{x,\theta}}\mathrm{d}\tau+\|\theta_d\,g\|^2_{L^2((t_0,t_1)\times\Gamma^-)},
\end{aligned}$$

where $c_{2,1}$ only depends on $d,s,\delta$ and $\tilde{b}$. By (2.21) we have that

$$\|u\|^2_{L^2_{x,\theta}}(t)\le\|u\|^2_{L^2_{x,\theta}}(t_0)+\|\theta_d\,g\|^2_{L^2((t_0,t_1)\times\Gamma^-)}\quad\text{for any}\quad t\in(t_0,t_0+\epsilon_0).$$

So, we can conclude that,

$$\sup_{t\in(t_0,t_0+\epsilon_0)}\Big(\|u\|^2_{L^2_{x,\theta}}(t)\Big)+\int_{t_0}^{t}\int_{\mathbb{S}^{d-1}}\|u\|^2_{H^s_\theta}\mathrm{d}x\,\mathrm{d}\tau+\int_{t_0}^{t}\|(-\Delta_x)^{s_0/2}u\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\leq c_2\Big(\|u(t_0,\cdot,\cdot)\|^2_{L^2_{x,\theta}}+\|\theta_d\,g\|^2_{L^2([t_0,t_1)\times\mathbb{R}^{d-1}\times\mathbb{S}^{d-1})}\Big).$$

Hence, the proof is complete. ■

**Proposition 2.3.3** *Let* $u\in\mathcal{C}\big((0,T);L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})\big)$ *be a solution of the RTE* (2.1) *on* $(0,T)$ *and assume that*

$$G_T:=\sup_{0<t<T}\Big\{t^{\frac{1}{w-1}}\|\theta_d\,g\|^2_{L^2((t,T)\times\Gamma^-)}\Big\}<\infty,$$

*where* $\omega>1$ *is given in the Proposition 2.3.1. Then,*

$$\|u(t)\|^2_{L^2_{x,\theta}}\leq A\,t^{-\frac{1}{\omega-1}}+\|\theta_d\,g\|^2_{L^2((t/2,T)\times\Gamma^-)}\quad \text{for all}\quad 0<t<T, \tag{2.70}$$

*where*

$$A:=C(m_0,\Phi_{in})\max\Big\{1,G_T\Big\}.$$

*Moreover, for any* $0<t<T$ *,*

$$\int_t^T\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau+\int_t^T\|(-\Delta_x)^{s_0/2}u\|^2_{L^2_{x,\theta}}\mathrm{d}\tau \leq C\Big(A\,t^{-\frac{1}{\omega-1}}+2\|\theta_d\,g\|^2_{L^2((t/2,T)\times\Gamma^-)}\Big). \tag{2.71}$$

*Proof.* Use Proposition 2.3.1 and Proposition 2.3.2, with $t_1=T$ and $t_0=t$, to obtain

$$\int_t^T\|u(\tau)\|^{2\omega}_{L^2_{x,\theta}}\mathrm{d}\tau\leq C\Big(\|u(t)\|^2_{L^2_{x,\theta}}+\|\theta_d\,g\|^2_{L^2((t,T)\times\Gamma^-)}\Big),\quad 0<t\leq T, \tag{2.72}$$

where $C=C(m_0,\Phi_{in})$. Introduce

$$Y(t):=\int_t^T\|u(\tau)\|^{2\omega}_{L^2_{x,\theta}}\mathrm{d}\tau,\quad 0<t<T,$$

then, time differentiation and (2.72) lead to

$$\frac{\mathrm{d}Y}{\mathrm{d}t}(t)+c\,Y^{\omega}(t)\leq\|\theta_d\,g\|^{2w}_{L^2((t,T)\times\Gamma^-)}.$$

where $c:=c(m_0,\Phi_{in},w)$. This is a Bernoulli differential inequality. Therefore, using a standard comparison method with explicit solutions of Bernoulli ODE, one concludes that

$$Y(t)\leq A\,t^{-\frac{1}{w-1}},\qquad 0<t<T,$$

with $A$ as defined in the statement. Again, using Proposition 2.3.2 with $t_0 = \tau$ and $t_1 = T$, it follows that

$$\sup_{s\in(\tau,T)}\left(\|u\|^{2w}_{L^2_{x,\theta}}(s)\right) \leq C\left(\|u(\tau,\cdot,\cdot)\|^{2w}_{L^2_{x,\theta}} + \|\theta_d\, g\|^{2w}_{L^2((\tau,T)\times\Gamma^-)}\right).$$

Thus, integrating in $\tau \in (t,2t) \in (0,T)$ and using the bound on $Y(t)$, we obtain that

$$t\,\|u(2\,t)\|^{2\omega}_{L^2_{x,\theta}} \leq A\,t^{-\frac{1}{w-1}} + t\,\|\theta_d\, g\|^{2\omega}_{L^2((t,T)\times\Gamma^-)}\,,$$

that is,

$$\|u(2\,t)\|^{2}_{L^2_{x,\theta}} \leq A\,t^{-\frac{1}{w-1}} + \|\theta_d\, g\|^{2}_{L^2((t,T)\times\Gamma^-)}\,.$$

This proves estimate (2.75) by renaming $t \to t/2$. Estimate (2.76) follows from both, Proposition 2.3.2 with $t_0 = t$ and $t_1 = T$ that give us

$$\int_t^T\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau + \int_t^T\|(-\Delta_x)^{s_0/2}u\|^2_{L^2_{x,\theta}}\mathrm{d}\tau \leq C\left(\|u(t)\|^2_{L^2_{x,\theta}} + \|\theta_d\, g\|^2_{L^2((t,T)\times\Gamma^-)}\right),$$

and estimate (2.75) used in the right side. ■

**Remark 1** *Recall that for* $0 < t < T$

$$\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|^2_{L^2\left((t,T)\times\mathbb{R}^d_+\times\mathbb{R}^{d-1}\right)} = \int_t^T\int_{\mathbb{R}^d_+}\|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau\,.$$

*Therefore, Proposition 2.3.3 implies,*

$$\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\|^2_{L^2\left((t,T)\times\mathbb{R}^d_+\times\mathbb{R}^{d-1}\right)} \leq C\left(A\,t^{-\frac{1}{\omega-1}} + 2\|\theta_d\, g\|^2_{L^2((t/2,T)\times\Gamma^-)}\right).$$

### 2.3.1.1
### Tangential spatial regularity.

We finish this section studying higher regularity, up to the boundary, for solutions $u(t,x,\theta)$ of the RTE (2.1). We first study the spatial regularity for which the analysis is relatively simple since the differential operator $\partial^\ell_x$, with $\ell = (\ell_1,\cdots,\ell_d)$ a multi-index with nonnegative integer entries, commutes with the RTE operator. Indeed, if $w^\ell(t,x,\theta) = (\partial^\ell_x u)(t,x,\theta)$ then

$$\partial_t w^\ell + \theta\cdot\nabla_x w^\ell = \mathcal{I}(w^\ell) \quad \text{in} \quad (0,T)\times\mathbb{R}^d_+\times\mathbb{S}^{d-1}\,. \tag{2.73}$$

Write $\ell = (\bar{\ell}, \ell_d)$ and take $\ell_d = 0$. Then, $w^{(\bar{\ell},0)}(t,\bar{x},0,\theta) = g^{\bar{\ell}}(t,\bar{x},\theta)$. As a consequence, using Proposition 2.3.2 with $u = w^{(\bar{\ell},0)}$ it follows that

$$\sup_{t\in(t_0,t_1)}\Big(\|w^{(\bar{\ell},0)}(t)\|^2_{L^2_{x,\theta}}\Big) + \int_{t_0}^{t_1}\int_{\mathbb{R}^d_+}\|w^{(\bar{\ell},0)}\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{t_0}^{t_1}\|(-\Delta_x)^{s_0/2}w^{(\bar{\ell},0)}\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\leq C\left(\|w^{(\bar{\ell},0)}(t_0,\cdot,\cdot)\|^2_{L^2_{x,\theta}} + \|\theta_d\, g^{\bar{\ell}}\|^2_{L^2((t_0,t_1)\times\Gamma^-)}\right).$$

Now, classical interpolation gives $\|\partial^{\bar{\ell}}_x\phi\|_{L^2_x} \leq \|(-\Delta_x)^{s_0/2}\partial^{\bar{\ell}}_x\phi\|^{\alpha}_{L^2_x}\|\phi\|^{1-\alpha}_{L^2_x}$ for multi-index $\bar{\ell}$, $s_0 \geq 0$, and $\alpha = \frac{|\bar{\ell}|}{|\bar{\ell}|+s_0}$ g. Then, we have that

$$\|w^{(\bar{\ell},0)}(t)\|_{L^2_{x,\theta}} \leq C_\ell\|(-\Delta_x)^{s_0/2}w^{(\bar{\ell},0)}(t)\|^{\alpha}_{L^2_{x,\theta}}\|u(t)\|^{1-\alpha}_{L^2_{x,\theta}}, \qquad \alpha = \frac{|\bar{\ell}|}{|\bar{\ell}|+s_0}.$$

As a consequence, Proposition 2.3.3 leads to

$$\|w^{(\bar{\ell},0)}(t)\|_{L^2_{x,\theta}} \leq C_{t_0}\|(-\Delta_x)^{s_0/2}w^{(\bar{\ell},0)}(t)\|^{\alpha}_{L^2_{x,\theta}},$$

for a constant $C_{t_0} := C_\ell(t_0, m_0, \Phi_{in}, \|g\|_{L^2_{t,x,\theta}})$. Therefore,

$$\sup_{t\in(t_0,t_1)}\Big(\|w^{(\bar{\ell},0)}(t)\|^2_{L^2_{x,\theta}}\Big) + \tilde{C}_{t_0}\int_{t_0}^{t_1}\|w^{(\bar{\ell},0)}\|^{2/\alpha}_{L^2_{x,\theta}}$$
$$\leq C\left(\|w^{(\bar{\ell},0)}(t_0,\cdot,\cdot)\|^2_{L^2_{x,\theta}} + \|\theta_d\, g^{\bar{\ell}}\|^2_{L^2((t_0,t_1)\times\Gamma^-)}\right).$$

The same argument that proves Proposition 2.3.3 leads to the control

$$\|\partial^{(\bar{\ell},0)}_x u(t)\|_{L^2_{x,\theta}} \leq C\Big(t_0, m_0, \Phi_{in}, \|\theta_d\, g\|_{H^{\bar{\ell}}_{\bar{x}}((t_0/2,T)\times\Gamma^-)}\Big) \quad \text{for all} \quad 0 < t_0 \leq t < T. \tag{2.74}$$

**Proposition 2.3.4 (Tangential spatial regularity estimate)** *Let* $u \in \mathcal{C}\Big([t_0,t_1); L^2(\mathbb{R}^d\times\mathbb{S}^{d-1})\Big)$ *be a solution of the RTE* (2.1) *on* $(0,T)$ *and assume that*

$$\|\theta_d\,\partial^{\bar{\ell}}_{\bar{x}} g\|^2_{L^2((t_0/2,T)\times\Gamma^-)} < \infty, \quad 0 < t_0 < T.$$

*Then,*

$$\|\partial^{(\bar{\ell},0)}_x u(t)\|^2_{L^2_{x,\theta}} \leq C\Big(t_0, m_0, \Phi_{in}, \|\theta_d\, g\|_{H^{\bar{\ell}}_{\bar{x}}((t_0/2,T)\times\Gamma^-)}\Big) \quad \textit{for all} \quad t_0 < t < T. \tag{2.75}$$

*Moreover,*

$$\int_{t_0}^{T}\int_{\mathbb{R}^d_+}\|\partial^{(\bar{\ell},0)}_x u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{t_0}^{T}\|(-\Delta_x)^{s_0/2}\partial^{(\bar{\ell},0)}_x u\|^2_{L^2_{x,\theta}}\mathrm{d}\tau \leq C\Big(t_0, m_0, \Phi_{in}, \|\theta_d\, g\|_{H^{\bar{\ell}}_{\bar{x}}((t_0/2,T)\times\Gamma^-)}\Big). \tag{2.76}$$

**Remark 2** *Note that higher regularity up to the boundary in the normal* *variable $x_d$ is not expected. This can be seen from expression (2.32) by observing that the boundary term $\frac{\widehat{G}(w,\bar{k},\theta)}{\mathrm{i}(w+\theta\cdot k)}$ caps the decay in the Fourier variable $k_d \in (-\infty,\infty)$. In addition, $\theta$-regularity of $u$ is limited at the boundary as well due to the discontinuity on the set $\{\theta_d = 0\}$. Thus, it is unclear how much more regularity the solution enjoys with respect to the one specified in the averaging lemma. Deeper investigation of the optimal regularisation near the boundary is an interesting aspect that will not be addressed in this section. In the next section we will prove, however, that solutions are smooth in all spatial coordinates and time in the interior of the half-space.*

## 2.3.2
## Interior regularity.

In this part of the chapter we prove that solutions are smooth in all variables in the interior of the domain. Take a smooth nonnegative and increasing cut-off function $\phi$ defined as

$$\phi(x_d) := \begin{cases} 0 & \text{if} \quad 0 \le x_d \le 1 \\ 1 & \text{if} \quad x_d \ge 2\,. \end{cases}$$

Set $\phi_\epsilon(x_d) = \phi(x_d/\epsilon)$ for $\epsilon \in (0,1)$. Multiply (2.1) by $\phi_\epsilon$ to obtain

$$\partial_t\left(\phi_\epsilon u\right) + \theta\cdot\nabla_x\left(\phi_\epsilon u\right) = \mathcal{I}\left(\phi_\epsilon u\right) + \theta_d\,\phi_\epsilon'\,u, \quad \text{in} \quad (0,\infty)\times\mathbb{R}^d\times\mathbb{S}^{d-1}\,, \tag{2.77}$$

where, using an extension by zero, we interpret $\phi_\epsilon u$ to be defined in the whole space. As a consequence, $\phi_\epsilon u$ satisfies the RTE (2.1) with a source $F := \theta_d\,\phi_\epsilon'\,u$ for which we can apply the argument presented in [1].

**Proposition 2.3.5 (Interior spatial regularity)** *Let $u \in \mathcal{C}\Big([t_0,t_1); L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})\Big)$ be a solution of the RTE (2.1). Then, for any $l\in\mathbb{N}$ and $\epsilon\in(0,1)$ it follows that*

$$\sup_{t\in(t_0,t_1)}\Big(\|(-\Delta_x)^{ls_0/2}(\phi_\epsilon u)(t)\|^2_{L^2_{x,\theta}}\Big) + \|(-\Delta_x)^{(1+l)s_0/2}(\phi_\epsilon\,u)\|^2_{L^2\left((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1}\right)}$$
$$\le C(t_0,t_1,m_0,\Phi_{in},\phi)\epsilon^{-2(1+ls_0)}\Big(1+\|\theta_d\,g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big). \tag{2.78}$$

*Proof.* We perform induction on the regularity order $l$, proving the result for $l = s_0$ and then moving in steps $ks_0$, with $k = 1,2,\cdots$. For the basis case one notices that the arguments that led to Proposition 2.3.2 can be applied to equation (2.77). In particular, Theorem 2.2.1 is applied in the whole space so that $g = 0$ and with a source $F := \theta_d\, \phi'_\epsilon\, u$[1]. Then,

$$\sup_{t\in(t_0,t_1)}\Big(\|(\phi_\epsilon u)(t)\|^2_{L^2_{x,\theta}}\Big) + \int_{t_0}^{t_1}\int_{\mathbb{R}^d_+}\|\phi_\epsilon u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{t_0}^{t_1}\|(-\Delta_x)^{s_0/2}(\phi_\epsilon u)\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\leq C\left(\|(\phi_\epsilon u)(t_0)\|^2_{L^2_{x,\theta}} + \|\phi_\epsilon u\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} + \|F\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})}\right).$$

The constant depends as $C := C(d,s,s_0)$ with $s_0$ defined in (2.25). In fact, since Theorem 2.2.1 is applied to the whole space we may take $s_0 = \frac{s/4}{2s+1}$. Using Proposition 2.3.3 to control the right side, we obtain that for any $\epsilon > 0$

$$\sup_{t\in(t_0,t_1)}\Big(\|(\phi_\epsilon u)(t)\|^2_{L^2_{x,\theta}}\Big) + \int_{t_0}^{t_1}\|(-\Delta_x)^{s_0/2}(\phi_\epsilon u)\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\leq C(t_0,t_1,m_0,\Phi_{in})\epsilon^{-2}\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,T)\times\Gamma^-)}\Big).$$

In the last inequality we used the fact that $\|\phi'_\epsilon\|_\infty \sim \epsilon^{-1}$. This proves the basis case. Now assume the result to be valid for $l = ks_0$ and arbitrary $\epsilon > 0$, that is

$$\sup_{t\in(t_0,t_1)}\Big(\|(-\Delta_x)^{(k-1)s_0/2}(\phi_\epsilon u)(t)\|^2_{L^2_{x,\theta}}\Big) + \int_{t_0}^{t_1}\|(-\Delta_x)^{ks_0/2}(\phi_\epsilon u)\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\leq C(t_0,t_1,m_0,\Phi_{in})\epsilon^{-2}\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,T)\times\Gamma^-)}\Big),$$

and let us prove it for $l = (k+1)s_0$. To this end differentiate (2.77) by $(-\Delta_x)^{ks_0/2}$ to obtain for $w^{ks_0}_\epsilon := (-\Delta_x)^{ks_0/2}(\phi_\epsilon u)$

$$\partial_t w^{ks_0}_\epsilon + \theta\cdot\nabla_x w^{ks_0}_\epsilon = \mathcal{I}\left(w^{ks_0}_\epsilon\right) + \theta_d\,(-\Delta_x)^{ks_0/2}(\phi'_\epsilon\, u), \quad \text{in} \quad (0,\infty)\times\mathbb{R}^d\times\mathbb{S}^{d-1}. \tag{2.79}$$

Again, we are led to

$$\sup_{t\in(t_0,t_1)}\Big(\|w^{ks_0}_\epsilon(t)\|^2_{L^2_{x,\theta}}\Big) + \int_{t_0}^{t_1}\|(-\Delta_x)^{s_0/2}w^{ks_0}_\epsilon\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\leq C\left(\|w^{ks_0}_\epsilon(t_0)\|^2_{L^2_{x,\theta}} + \|w^{ks_0}_\epsilon\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} + \|F^{ks_0}\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})}\right), \tag{2.80}$$

where $F^{ks_0} = \theta_d\,(-\Delta_x)^{ks_0/2}(\phi'_\epsilon\, u)$. In Proposition A.1.2 we prove that

$$\|F^{ks_0}\|^2_{L^2_x} \leq \epsilon^{-2(1+ks_0)}C_\phi\Big(\|\phi_{\epsilon/2}u\|^2_{L^2_x} + \|(-\Delta_x)^{ks_0/2}(\phi_{\epsilon/2}u)\|^2_{L^2_x}\Big),$$

thus, we can control the latter two term in the right side of (2.80) with the

[1]The source can be treated as the term $\mathcal{K}(u)$, the bounded part of $\mathcal{I}$.

induction hypothesis to obtain that

$$\sup_{t\in(t_0,t_1)}\Big(\|w^{ks_0}_\epsilon(t)\|^2_{L^2_{x,\theta}}\Big)+\int_{t_0}^{t_1}\|(-\Delta_x)^{s_0/2}w^{ks_0}_\epsilon\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\le C\|w^{ks_0}_\epsilon(t_0)\|^2_{L^2_{x,\theta}}+\epsilon^{-2(1+ks_0)}C(t_0,t_1,m_0,\Phi_{in},\phi)\Big(1+\|\theta_d\,g\|^2_{L^2((t_0/2,T)\times\Gamma^-)}\Big)\,.$$

At this point one uses the argument leading to Proposition 2.3.4 to conclude that

$$\sup_{t\in(t_0,t_1)}\Big(\|w^{ks_0}_\epsilon(t)\|^2_{L^2_{x,\theta}}\Big)+\int_{t_0}^{t_1}\|(-\Delta_x)^{s_0/2}w^{ks_0}_\epsilon\|^2_{L^2_{x,\theta}}\mathrm{d}\tau$$
$$\le\epsilon^{-2(1+ks_0)}C(t_0,t_1,m_0,\Phi_{in},\phi)\Big(1+\|\theta_d\,g\|^2_{L^2((t_0/2,T)\times\Gamma^-)}\Big)\,,$$

which proves the inductive step. ■

In order to study the regularity in the variable $\theta\in\mathbb{S}^{d-1}$ we follow the spirit of [1] which consists in studying the regularity in the projected variable $v\in\mathbb{R}^{d-1}$. The main technical difficult lies in the treatments of weights that account for the geometry of the problem. Additionally, derivation in angle is an operation that does not commutes with the equation which complicates the computations, yet, easily solved with a commutator identity of the type given in Proposition A.1.3.

Take the cut-off RTE (2.77) and apply the stereographic projection to it. Recalling the formulas (2.11) and (2.12) it follows that $U^\epsilon_{\mathcal{J}}:=\frac{\phi_\epsilon u_{\mathcal{J}}}{\langle v\rangle^{d-1-2s}}$ satisfies

$$\partial_t U^\epsilon_{\mathcal{J}}+\theta(v)\cdot\nabla_x U^\epsilon_{\mathcal{J}}=-D_0\langle v\rangle^{4s}(-\Delta_v)^sU^\epsilon_{\mathcal{J}}+c_{s,d}\,U^\epsilon_{\mathcal{J}}$$
$$+\frac{\big[\mathcal{I}_h(\phi_\epsilon u)\big]_{\mathcal{J}}}{\langle v\rangle^{d-1-2s}}+\theta_d(v)\phi'_\epsilon\,U_{\mathcal{J}},\quad\text{in}\quad(0,T)\times\mathbb{R}^d\times\mathbb{R}^{d-1}\,.\tag{2.81}$$

Multiply this equation by $(-\Delta_v)^sU^\epsilon_{\mathcal{J}}$ and integrate in all variables. Using in the computation the Cauchy Schwarz inequality inequality in each term as follows

$$\int_v\Big|\theta(v)\cdot\nabla_xU^\epsilon_{\mathcal{J}}\times(-\Delta_v)^sU^\epsilon_{\mathcal{J}}\Big|$$
$$\le\frac{2}{D_0}\int_v\Big|\theta(v)\cdot\nabla_x\frac{U^\epsilon_{\mathcal{J}}}{\langle v\rangle^{2s}}\Big|^2+\frac{D_0}{8}\int_v\Big|\langle v\rangle^{2s}(-\Delta_v)^sU^\epsilon_{\mathcal{J}}\Big|^2$$
$$=C_{d,s}\int_\theta\Big|\theta\cdot\nabla_x(\phi_\epsilon u)\Big|^2+\frac{D_0}{8}\int_v\Big|\langle v\rangle^{2s}(-\Delta_v)^sU^\epsilon_{\mathcal{J}}\Big|^2\,,$$

one concludes that for $0 < t_0 < t < t_1$

$$\Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}(t)\Big\|^2_{L^2_{x,v}} + \frac{D_0}{2}\int_{t_0}^{t}\Big\|\langle v\rangle^{2s}(-\Delta_v)^s U^{\epsilon}_{\mathcal{J}}\Big\|^2_{L^2_{x,v}}\mathrm{d}\tau \le \Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}(t_0)\Big\|^2_{L^2_{x,v}}$$
$$+ C_{d,s}\int_{t_0}^{t}\Big(\|\phi_\epsilon u\|^2_{L^2_{x,\theta}} + \|\mathcal{I}_h(\phi_\epsilon u)\|^2_{L^2_{x,\theta}} + \|\theta\cdot\nabla_x(\phi_\epsilon u)\|^2_{L^2_{x,\theta}} + \|\theta_d\phi'_\epsilon u\|^2_{L^2_{x,\theta}}\Big)\mathrm{d}\tau\,.$$

Recall that $\mathcal{I}_h$ is a bounded operator and given the interior spatial regularity on Proposition 2.3.5 we are led to

$$\Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}(t)\Big\|^2_{L^2_{x,v}} + \frac{D_0}{2}\int_{t_0}^{t}\Big\|\langle v\rangle^{2s}(-\Delta_v)^s U^{\epsilon}_{\mathcal{J}}\Big\|^2_{L^2_{x,v}}\mathrm{d}\tau$$
$$\le \Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}(t_0)\Big\|^2_{L^2_{x,v}} + C_\epsilon(t,t_0,m_0,\Phi_{in},\phi)\Big(1+\|\theta_d\,g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,. \tag{2.82}$$

Now observe that

$$\int_{\mathbb{R}^{d-1}}\Big|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}\Big|^2\mathrm{d}v = \int_{\mathbb{R}^{d-1}}U^{\epsilon}_{\mathcal{J}}\,(-\Delta_v)^s U^{\epsilon}_{\mathcal{J}}\,\mathrm{d}v$$
$$\le \left(\int_{\mathbb{R}^{d-1}}\left|\frac{U^{\epsilon}_{\mathcal{J}}}{\langle v\rangle^{2s}}\right|^2\mathrm{d}v\right)^{\frac12}\left(\int_{\mathbb{R}^{d-1}}\Big|\langle v\rangle^{2s}(-\Delta_v)^s U^{\epsilon}_{\mathcal{J}}\Big|^2\mathrm{d}v\right)^{\frac12},$$

which is equivalent to

$$\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}\|^2_{L^2_v} \le C_d\|\phi_\epsilon u\|_{L^2_\theta}\|\langle v\rangle^{2s}(-\Delta_v)^s U^{\epsilon}_{\mathcal{J}}\|_{L^2_v}\,.$$

This is a spherical analog to the classical Sobolev interpolation in the Euclidean plane. As a consequence, after using Proposition 2.3.5 to control the $L^2_{x,\theta}$-norm of $\phi_\epsilon u$ we can update (2.82) to

$$\Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}(t)\Big\|^2_{L^2_{x,v}} + \frac{C(t_0,m_0,\Phi_{in})}{1+\|g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}}\int_{t_0}^{t}\Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}\Big\|^4_{L^2_{x,v}}\mathrm{d}\tau$$
$$\le \Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}(t_0)\Big\|^2_{L^2_{x,v}} + C_\epsilon(t,t_0,m_0,\Phi_{in},\phi)\Big(1+\|\theta_d\,g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,.$$

From here we can argue as before, for instance as in the proof of Proposition 2.3.3 with $\omega = 2$, to obtain that

$$\sup_{t\in(t_0,t_1)}\Big\|(-\Delta_v)^{s/2}U^{\epsilon}_{\mathcal{J}}(t)\Big\|^2_{L^2_{x,v}} + \int_{t_0}^{t_1}\Big\|\langle v\rangle^{2s}(-\Delta_v)^s U^{\epsilon}_{\mathcal{J}}\Big\|^2_{L^2_{x,v}}\mathrm{d}\tau$$
$$\le C_\epsilon(t_1,t_0,m_0,\Phi_{in},\phi)\Big(1+\|\theta_d\,g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,. \tag{2.83}$$

Before proving interior angular regularisation, we have the tools to prove interior space-time regularisation as shown in the following proposition.

**Proposition 2.3.6 (Space-time interior regularisation)** *Let* $u \in \mathcal{C}\Big([t_0,t_1);L^2(\mathbb{R}^d_+\times\mathbb{S}^{d-1})\Big)$ *be a solution of* (2.1)*. Then, for any* $j_1,j_2\in\mathbb{N}$ *it follows that*

$$\sup_{t\in(2t_0,t_1)}\Big\|(-\Delta_x)^{j_1\frac{s_0}{2}}\partial_t^{j_2}\Big(\phi_\epsilon u\Big)\Big\|^2_{L^2_{x,\theta}}+\int_{t_0}^{t_1}\Big\|(-\Delta_x)^{(1+j_1)\frac{s_0}{2}}\partial_t^{j_2}\Big(\phi_\epsilon u\Big)\Big\|^2_{L^2_{x,\theta}}$$
$$\leq C_\epsilon(t_1,t_0,m_0,\Phi_{in},\phi)\Big(1+\|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,, \tag{2.84}$$

*for any* $0<2\,t_0\leq t_1$*.*

*Proof.* Recall that

$$\partial_t\left(\phi_\epsilon u\right)=\mathcal{I}\left(\phi_\epsilon u\right)+\theta_d\,\phi_\epsilon'\,u-\theta\cdot\nabla_x\left(\phi_\epsilon u\right),\quad \text{in}\quad (0,\infty)\times\mathbb{R}^d\times\mathbb{S}^{d-1}\,.$$

Therefore,

$$\int_{t_0}^{t_1}\Big\|\partial_t\left(\phi_\epsilon u\right)\Big\|^2_{L^2_{x,\theta}}\leq 2\int_{t_0}^{t_1}\Big\|\mathcal{I}\left(\phi_\epsilon u\right)\Big\|^2_{L^2_{x,\theta}}+2\int_{t_0}^{t_1}\Big\|\theta_d\,\phi_\epsilon'\,u-\theta\cdot\nabla_x\left(\phi_\epsilon u\right)\Big\|^2_{L^2_{x,\theta}}\,.$$

Using Proposition 2.3.5 one controls the second term in the right side

$$\int_{t_0}^{t_1}\Big\|\theta_d\,\phi_\epsilon'\,u-\theta\cdot\nabla_x\left(\phi_\epsilon u\right)\Big\|^2_{L^2_{x,\theta}}\mathrm{d}\tau\leq C_\epsilon\Big(1+\|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,.$$

In addition, thanks to formulas (2.11) and (2.12)

$$\Big\|\mathcal{I}(\phi_\epsilon u)\|^2_{L^2_\theta}\sim b(1)\Big\|\langle v\rangle^{2s}(-\Delta_v)^s U^\epsilon_{\mathcal{J}}\Big\|^2_{L^2_v}+\|\phi_\epsilon u\|^2_{L^2_v}\,.$$

Thus, estimate (2.83) leads to the bound

$$\int_{t_0}^{t_1}\Big\|\mathcal{I}(\phi_\epsilon u)\|^2_{L^2_\theta}\mathrm{d}\tau\leq C_\epsilon\Big(1+\|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,,$$

which in turn leads to

$$\int_{t_0}^{t_1}\Big\|\partial_t\left(\phi_\epsilon u\right)\Big\|^2_{L^2_{x,\theta}}\leq C_\epsilon\Big(1+\|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,,\quad \epsilon>0\,. \tag{2.85}$$

Let us prove that $y^j_\epsilon:=\partial^j_t(\phi_\epsilon u)$, with $j=1,2,\cdots$, is smooth in the spatial variable. Time differentiation commutes with the RTE, as a consequence, note that

$$\partial_t y^j_\epsilon+\theta\cdot\nabla_x y^j_\epsilon=\mathcal{I}\left(y^j_\epsilon\right)+\theta_d\,\phi_\epsilon'\,y^j_{\epsilon/2}\,. \tag{2.86}$$

Then, we can invoke Theorem 2.2.1 applied in the whole space, with zero boundary $g=0$, and with a source $F^j_\epsilon:=\theta_d\,\phi'_\epsilon y^j_{\epsilon/2}$. As a consequence,

$$\sup_{t\in(t_0,t_1)} \|y^j_\epsilon\|^2_{L^2_{x,\theta}} + \int_{t_0}^{t_1} \|(-\Delta_x)^{s_0/2} y^j_\epsilon\|^2_{L^2_{x,\theta}} \mathrm{d}\tau \le C\Big( \|y^j_\epsilon(t_0)\|^2_{L^2_{x,\theta}} + \|y^j_\epsilon\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} + \|F^j_\epsilon\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})}\Big)\,, \tag{2.87}$$

where the constant depends as $C := C(d,s,s_0)$ with $s_0$ defined in (2.25).

Since (2.85) is valid for any $\epsilon > 0$, one concludes that for $j = 1$ it holds that

$$\|y^1_\epsilon\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} + \|F^1_\epsilon\|^2_{L^2((t_0,t_1)\times\mathbb{R}^d\times\mathbb{S}^{d-1})} \le C_\epsilon\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,.$$

Consequently,

$$\sup_{t\in(t_0,t_1)} \|y^1_\epsilon\|^2_{L^2_{x,\theta}} + \int_{t_0}^{t_1} \|(-\Delta_x)^{s_0/2} y^1_\epsilon\|^2_{L^2_{x,\theta}} \mathrm{d}\tau \le C\|y^1_\epsilon(t_0)\|^2_{L^2_{x,\theta}} + C_\epsilon\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,,$$

from which the following estimate is readily obtained for, say, $0 < \frac{3}{2}t_0 \le t_1$

$$\sup_{t\in(\frac{3}{2}t_0,t_1)} \|y^1_\epsilon\|^2_{L^2_{x,\theta}} + \int_{t_0}^{t_1} \|(-\Delta_x)^{s_0/2} y^1_\epsilon\|^2_{L^2_{x,\theta}} \mathrm{d}\tau \le C_\epsilon\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,. \tag{2.88}$$

Estimate (2.88) works as the base case for an induction argument along the lines of the proof of Proposition 2.3.5 which proves estimate (2.84) for $j_1 \in \mathbb{N}$ and $j_2 = 1$.

For the general case $j_2 \ge 1$ argue again by induction, using estimate (2.84) for $j_1 \in \mathbb{N}$ and $j_2 = 1$ as the base case and with induction hypothesis given by the same estimate for $j_1 \in \mathbb{N}$ and $j_2 = j$. In order to prove (2.84) for $j+1$, note that thanks to the induction hypothesis $\nabla_x y^j_\epsilon$ and $y^j_{\epsilon/2}$ belong to $L^2_{t,x,\theta}$ with the right control with respect to the boundary $g$. Additionally, all $y^j_\epsilon$ satisfy the same equation (2.86), therefore, we can redo the steps that proved (2.83) to obtain that

$$\sup_{t\in(t_0,t_1)} \Big\|(-\Delta_v)^{s/2} Y^{j,\epsilon}_{\mathcal{J}}(t)\Big\|^2_{L^2_{x,v}} + \int_{t_0}^{t_1} \Big\|\langle v\rangle^{2s}(-\Delta_v)^s Y^{j,\epsilon}_{\mathcal{J}}\Big\|^2_{L^2_{x,v}} \mathrm{d}\tau \le C_\epsilon(t_1,t_0,m_0,\Phi_{in},\phi)\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,,$$

where $Y^{j,\epsilon}_{\mathcal{J}} = \frac{\big[y^j_\epsilon\big]_{\mathcal{J}}}{\langle v\rangle^{d-1-2s}}$. This implies thanks to the argument leading to (2.85) that

$$\int_{t_0}^{t_1} \Big\|y^{j+1}_\epsilon\Big\|^2_{L^2_{x,\theta}} \le C_\epsilon\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,, \quad \epsilon > 0\,.$$

Invoking again Theorem 2.2.1 we can use the energy estimate (2.87) with $j+1$ in place of $j$. From this point we conclude as in the case of $y^1_\epsilon$. ∎

**Proposition 2.3.7 (Interior angular regularity)** *Let $u \in \mathcal{C}\Big([t_0,t_1); L^2(\mathbb{R}^d_+ \times \mathbb{S}^{d-1})\Big)$ be a solution of the RTE (2.1). Assume in addition that the scattering kernel (2.4) satisfies for some integer $N_0 \geq 1$*

$$h(z) = \frac{\tilde{b}(z)}{(1-z)^{1+s}} \in \mathcal{C}^{N_0}\Big([-1,1]\Big). \tag{2.89}$$

*Then, for any $j_1$, $j_2 \in \mathbb{N}$ such that $0 \leq \frac{(j_2-1)s}{2} \leq \lfloor N_0 \rfloor$, and $\epsilon \in (0,1)$ it follows that*

$$\sup_{t\in(t_0,t_1)} \|(-\Delta_x)^{\frac{j_1}{2}}(-\Delta_v)^{j_2\frac{s}{2}}\, U^\epsilon_{\mathcal{J}}\|^2_{L^2_{x,v}} + \int_{t_0}^{t_1} \Big\|\langle v\rangle^{2s}(-\Delta_x)^{\frac{j_1}{2}}(-\Delta_v)^{(1+j_2)\frac{s}{2}}\, U^\epsilon_{\mathcal{J}}\Big\|^2_{L^2_{x,v}} d\tau \\ \leq C_{\epsilon,j_1,j_2}(t_0,t_1,m_0,\Phi_{in},\phi)\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big)\,, \tag{2.90}$$

*where we recall that $U^\epsilon_{\mathcal{J}} = \frac{\phi_\epsilon\, u_{\mathcal{J}}}{\langle v\rangle^{d-1-2s}}$.*

*Proof.* The case $j_1 \in \mathbb{N}$ and $j_2 = 1$ is clear from the argument leading to estimate (2.83) after spatial differentiation of the equation given the smoothness of the spatial variable Proposition 2.3.5. For the general case assume, as inductive hypothesis, that (2.90) for $j_1 \in \mathbb{N}$ and $j_2 = j$. Apply the operator $(-\Delta_x)^{\frac{j_1}{2}}\partial^\kappa_v(1-\Delta_v)^{\frac{\alpha}{2}}$ to the projected RTE (2.81) with $\kappa$ multi-index and $\alpha \in (0,1)$ such that $|\kappa| + \alpha = j\frac{s}{2}$. Then for $V^{j,\epsilon}_{\mathcal{J}} = (-\Delta_x)^{\frac{j_1}{2}}\partial^\kappa_v(1-\Delta_v)^{\frac{\alpha}{2}} U^\epsilon_{\mathcal{J}}$ it holds that

$$\partial_t V^{j,\epsilon}_{\mathcal{J}} = -D_0\langle v\rangle^{4s}(-\Delta_v)^s V^{j,\epsilon}_{\mathcal{J}} + c_{s,d}\, V^{j,\epsilon}_{\mathcal{J}} + F^{j,\epsilon}, \quad \text{in} \quad (0,T)\times\mathbb{R}^d\times\mathbb{R}^{d-1}\,, \tag{2.91}$$

where $F^{j,\epsilon} = \sum_{i=1}^4 F^{j,\epsilon}_i$. Let us estimate each of these sources as we define them starting with the angular fractional Laplacian commutation term

$$F^{j,\epsilon}_1 := \sum_{|m|\leq|\kappa|-1} \begin{pmatrix} \kappa \\ m \end{pmatrix} \Big(\Big(\partial^{\kappa-m}_v\langle v\rangle^{4s}\Big) \times (-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}\partial^m_v(1-\Delta_v)^{\frac{\alpha}{2}} U^\epsilon_{\mathcal{J}} \\ + \mathcal{R}^1_m\Big((-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}\partial^m_v U^\epsilon_{\mathcal{J}}\Big)\Big)\,.$$

In this identity we used first the product rule of classical differentiation to handle the operator $\partial^\kappa_v$ and, then, Proposition A.1.3 to commute the fractional differentiation. The operator $\mathcal{R}^1_m$ satisfies for all $|m|+\alpha \le |\kappa|-1+\alpha = j\frac{s}{2}-1$

$$\Big\|\langle\cdot\rangle^{-2s}\mathcal{R}^1_m\Big((-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m U^\epsilon_{\mathcal{J}}\Big)\Big\|_{L^2_v} \le C\Big\|\langle\cdot\rangle^{2s}(-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m U^\epsilon_{\mathcal{J}}\Big\|_{L^2_v}$$
$$\le C\Big\|\langle\cdot\rangle^{2s}(-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}U^\epsilon_{\mathcal{J}}\Big\|^{1-\frac{|m|+\alpha}{js/2}}_{L^2_v}\Big\|\langle\cdot\rangle^{2s}(-\Delta_v)^s V^{j,\epsilon}_{\mathcal{J}}\Big\|^{\frac{|m|+\alpha}{js/2}}_{L^2_v}.$$

The last inequality follows by interpolation. Furthermore, invoking interpolation again

$$\Big\|\langle\cdot\rangle^{-2s}\Big(\partial_v^{\kappa-m}\langle v\rangle^{4s}\Big)\times(-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m(1-\Delta_v)^{\frac{\alpha}{2}}U^\epsilon_{\mathcal{J}}\Big\|_{L^2_v}$$
$$\le C\Big\|\langle\cdot\rangle^{2s}(-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}U^\epsilon_{\mathcal{J}}\Big\|^{1-\frac{|m|+\alpha}{js/2}}_{L^2_v}\Big\|\langle\cdot\rangle^{2s}(-\Delta_v)^s V^{j,\epsilon}_{\mathcal{J}}\Big\|^{\frac{|m|+\alpha}{js/2}}_{L^2_v}$$
$$+C\Big\|\langle\cdot\rangle^{2s}(-\Delta_v)^s(-\Delta_x)^{\frac{j_1}{2}}U^\epsilon_{\mathcal{J}}\Big\|_{L^2_v}.$$

Similarly, the spatial gradient term is given by

$$F^{j,\epsilon}_2 := (-\Delta_x)^{\frac{j_1}{2}}\partial_v^\kappa\Big(1-\Delta_v\Big)^{\frac{\alpha}{2}}\Big(\theta(v)\cdot\nabla_x U^\epsilon_{\mathcal{J}}\Big)$$
$$=\sum_{|m|\le|\kappa|}\begin{pmatrix}\kappa\\ m\end{pmatrix}\Big(\partial_v^{\kappa-m}\theta(v)\cdot\nabla_x(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m(1-\Delta_v)^{\frac{\alpha}{2}}U^\epsilon_{\mathcal{J}}$$
$$+\mathcal{R}^2_m\Big(\partial_{x_i}(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m U^\epsilon_{\mathcal{J}}\Big)\Big),$$

where Proposition A.1.3 was used for the commutation. Given the explicit form of $\theta_i(v)=\mathcal{J}_i(v)$ presented in (2.7) it follows that

$$\Big\|\langle\cdot\rangle^{-2s}\mathcal{R}^2_m\Big(\partial_{x_i}(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m U^\epsilon_{\mathcal{J}}\Big)\Big\|^2_{L^2_v} \le C\|\langle\cdot\rangle^{-2s}\nabla_x(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m U^\epsilon_{\mathcal{J}}\|^2_{L^2_v}$$
$$\le C\,\|\langle\cdot\rangle^{-2s}\nabla_x(-\Delta_x)^{\frac{j_1}{2}}U^\epsilon_{\mathcal{J}}\|^2_{L^2_v}+C\|\langle\cdot\rangle^{-2s}\nabla_x V^{j,\epsilon}_{\mathcal{J}}\|^2_{L^2_v}.$$

In the same way,

$$\Big\|\langle\cdot\rangle^{-2s}\partial_v^{\kappa-m}\theta(v)\cdot\nabla_x(-\Delta_x)^{\frac{j_1}{2}}\partial_v^m(1-\Delta_v)^{\frac{\alpha}{2}}U^\epsilon_{\mathcal{J}}\Big\|^2_{L^2_v}$$
$$\le C\,\|\langle\cdot\rangle^{-2s}\nabla_x(-\Delta_x)^{\frac{j_1}{2}}U^\epsilon_{\mathcal{J}}\|^2_{L^2_v}+C\|\langle\cdot\rangle^{-2s}\nabla_x V^{j,\epsilon}_{\mathcal{J}}\|^2_{L^2_v}.$$

Repeating previous argument and using Proposition A.1.2, the following estimate for

$$F^{j,\epsilon}_3 := (-\Delta_x)^{\frac{j_1}{2}}\partial_v^\kappa\Big(1-\Delta_v\Big)^{\frac{\alpha}{2}}\Big(\theta_d\,\phi'_\epsilon\,U_{\mathcal{J}}\Big)$$

holds

$$\|\langle\cdot\rangle^{-2s}F^{j,\epsilon}_3\|^2_{L^2_v}\le C_\epsilon\,\|\langle\cdot\rangle^{-2s}(-\Delta_x)^{\frac{j_1}{2}}U^{j,\epsilon/2}_{\mathcal{J}}\|^2_{L^2_v}+C_\epsilon\|\langle\cdot\rangle^{-2s}V^{j,\epsilon/2}_{\mathcal{J}}\|^2_{L^2_v}.$$

Finally, for

$$F_4^{j,\epsilon} = (-\Delta_x)^{\frac{j_1}{2}} \partial_v^{\kappa} \Big(1-\Delta_v\Big)^{\frac{\alpha}{2}} \left( \frac{\big[\mathcal{I}_h(\phi_\epsilon u)\big]_{\mathcal{J}}}{\langle v \rangle^{d-1-2s}} \right),$$

note that

$$h\Big(\theta \cdot \theta'\Big) = h\Big(1 - 2\,\frac{|v-v'|^2}{\langle v\rangle^2 \langle v' \rangle^2}\Big)\,.$$

Therefore,

$$F_4^{j,\epsilon} \sim \frac{\Big[\mathcal{I}_{\tilde h}\Big((-\Delta_x)^{\frac{j_1}{2}}\phi_\epsilon u\Big)\Big]_{\mathcal{J}}}{\langle v\rangle^{d-1-2s}}\,, \quad \text{where} \quad \tilde h = \Big(1-\Delta_v\Big)^{\frac{js/2}{2}} \left(h\Big(1-2\,\frac{|v-v'|^2}{\langle v\rangle^2\langle v'\rangle^2}\Big)\right).$$

As a consequence, since $j\frac{s}{2} \leq N_0$, the estimate holds

$$\|\langle \cdot \rangle^{-2s} F_4^{j,\epsilon}\|^2_{L^2_v} \leq C(\|h\|_{\mathcal{C}^{N_0}}) \Big\|(-\Delta_x)^{\frac{j_1}{2}} u^\epsilon\Big\|^2_{L^2_\theta} \sim \Big\|\langle \cdot \rangle^{-2s}(-\Delta_x)^{\frac{j_1}{2}} U^\epsilon_{\mathcal{J}}\Big\|^2_{L^2_v}\,.$$

Overall, these estimates together with the induction hypothesis lead to the following control valid for any $\delta > 0$

$$\begin{aligned}\int_{t_0}^{t_1} \Big\|\langle \cdot \rangle^{-2s} F^{j,\epsilon}\Big\|^2_{L^2_{x,v}} \mathrm{d}\tau \leq C_\delta\Big(1 + \|\theta_d\, g\|^2_{L^2((t_0/2,t_1)\times\Gamma^-)}\Big) \\ + \delta\int_{t_0}^{t_1} \Big\|\langle \cdot \rangle^{2s}(-\Delta_v)^s V^{j,\epsilon}_{\mathcal{J}}\Big\|^2_{L^2_{x,v}} \mathrm{d}\tau\,.\end{aligned} \tag{2.92}$$

The proof follows from estimate (2.92) multiplying equation (2.91) by $(-\Delta_v)^s V^{j,\epsilon}$ integrating in all variables and choosing $\delta = \frac{D_0}{2}$. ■

## 2.4
## Existence and uniqueness of solution

In this section we prove the well-posedness for the RTE (2.1) using the Lumer–Phillips Theorem. It is tantamount to establish Theorem 1.3.2. The main step consists in describing the domain of the operator

$$\mathcal{L}(u) := \theta \cdot \nabla_x u - \mathcal{I}(u)\,.$$

The central issue is to guarantee the existence of a trace mapping in such domain for which the Green's formula is valid. To this end, consider the Banach space

$$H^s_{\mathcal{L}} = \Big\{ u \in L^2_x \cap H^s_\theta\Big(\mathbb{R}^d_+ \times \mathbb{S}^{d-1}\Big)\,\Big|\, \mathcal{L}(u) \in L^2\Big(\mathbb{R}^d_+ \times \mathbb{S}^{d-1}\Big)\Big\}\,,$$

with norm

$$\|u\|^2_{H^s_{\mathcal{L}}} := \int_{\mathbb{R}^{d-1}_+} \|u\|^2_{H^s_\theta}\,\mathrm{d}x + \|\mathcal{L}u\|^2_{L^2_{x,\theta}}\,.$$

**Proposition 2.4.1 (Local traces $\gamma^{\pm}$)** *Let $K^{\pm}$ be a compact set of*

$$\Gamma^{\pm} = \Big\{(x,\theta) \in \Big\{x \,\Big|\, x_d = 0\Big\} \times \mathbb{S}^{d-1} \,\Big|\, \pm\theta_d < 0\Big\}.$$

*Then, for any $u \in H^1_x \cap H^s_{\mathcal{L}}$ the trace mappings*

$$u \to \gamma_{K^{\pm}}(u) := u\Big|_{K^{\pm}}$$

*satisfy the bound*

$$\|\gamma_{K^{\pm}}(u)\|_{L^2(K^{\pm})} \leq C_{K^{\pm}} \|u\|_{H^s_{\mathcal{L}}}\,. \tag{2.93}$$

*Proof.* Fix $K^+ \subset \Gamma^+$ compact and let $\varphi_{K^+}(\theta) \geq 0$ be a smooth function equal to unity in $K^+$ and compactly supported in $\{\theta_d < 0\}$. For any $w \in H^1_x \cap H^s_{\mathcal{L}}$ it follows that $\mathcal{L}(w) =: f \in L^2_{x,\theta}$ and, since the Green's formula is valid in $H^1_x$,

$$\begin{aligned}\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} f\,\varphi^2_{K^+} w \,\mathrm{d}\theta\,\mathrm{d}x &= \int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{L}(w)\,\varphi^2_{K^+} w\,\mathrm{d}\theta\,\mathrm{d}x \\ &= \int_{\Gamma^+} (\varphi_{K^+} w)^2\,\theta_d\,\mathrm{d}\theta\,\mathrm{d}\bar{x} - \int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{I}(w)\,\varphi^2_{K^+} w\,\mathrm{d}\theta\,\mathrm{d}x\,.\end{aligned}$$

Recalling the weak formulation (2.5), it follows that

$$\begin{aligned}\Big|\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{I}(w)\,\varphi^2_{K^+} w\,\mathrm{d}\theta\,\mathrm{d}x\Big| &\leq \int_{\mathbb{R}^{d-1}_+} \|w\|_{H^s_\theta} \|\varphi^2_{K^+} w\|_{H^s_\theta}\mathrm{d}x \\ &\leq C_{K^+}\int_{\mathbb{R}^{d-1}_+} \|w\|^2_{H^s_\theta}\mathrm{d}x\,.\end{aligned}$$

Also,

$$\begin{aligned}\Big|\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} f\,\varphi^2_{K^+} w\,\mathrm{d}\theta\,\mathrm{d}x\Big| &\leq \int_{\mathbb{R}^d} \|f\|_{H^{-s}_\theta} \|w\,\varphi^2_{K^+}\|_{H^s_\theta}\mathrm{d}x \\ &\leq C_{K^+}\Big(\int_{\mathbb{R}^d_+} \|f\|^2_{H^{-s}_\theta}\mathrm{d}x + \int_{\mathbb{R}^{d-1}} \|w\|^2_{H^s_\theta}\mathrm{d}x\Big)\,.\end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}\mathrm{dist}\Big(\{\theta_d = 0\}, K^+\Big)\int_{K^+} |w|^2\,\mathrm{d}\theta\,\mathrm{d}\bar{x} &\leq \int_{K^+} |w|^2\,|\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x} \\ &\leq \int_{\Gamma^+} (\varphi_{K^+} w)^2\,|\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x} \leq C_{K^+}\|w\|^2_{H^s_{\mathcal{L}}}\,.\end{aligned}$$

This proves the estimate for $\gamma_{K^+}$. For the trace mapping $\gamma_{K^-}$ the proof is similar. ■

Thanks to Proposition 2.4.1, the trace mappings $\gamma^{\pm}(\cdot)$ can be extended by continuity from $H^1_x \cap H^s_{\mathcal{L}}$ to its closure $\overline{H^1_x \cap H^s_{\mathcal{L}}}$ in $H^s_{\mathcal{L}}$. In other words, functions in $\overline{H^1_x \cap H^s_{\mathcal{L}}}$ have well defined traces $\gamma^{\pm}$ in $L^2_{loc}(\Gamma^{\pm})$.

**Theorem 2.4.2** *Let* $H^s_{\Gamma_{loc}} = \left\{ u \in H^s_{\mathcal{L}} \,\middle|\, \gamma^{\pm}(u) \in L^2_{loc}(\Gamma^{\pm}) \right\}$[2]. *Then*

$$\overline{H^1_x \cap H^s_{\mathcal{L}}} = H^s_{\Gamma_{loc}} .$$

*Proof.* We know, due to Proposition 2.4.1, that $\overline{H^1_x \cap H^s_{\mathcal{L}}} \subset H^s_{\Gamma_{loc}}$. Let us prove the opposite inclusion. The strategy consists in proving that the space $H^1_x \cap H^s_{\mathcal{L}} \cap \{\gamma^-(\cdot) = 0\}$ is dense in $H^s_{\mathcal{L}} \cap \{\gamma^-(\cdot) = 0\}$. The rest of the proof is standard using "lifting".

In [4] the following type of approximation problem, in more general domains, was used

$$\begin{cases} \mu u^{\epsilon} + \mathcal{L}_{\epsilon}(u^{\epsilon}) - \epsilon \Delta u^{\epsilon} = f & \text{in } \mathbb{R}^d_+ \times \mathbb{S}^{d-1} , \\ u^{\epsilon} = 0 & \text{on } \Gamma^- , \\ \partial_{x_d} u^{\epsilon} = 0 & \text{on } \Gamma , \end{cases} \tag{2.94}$$

where $\mathcal{L}_{\epsilon}$, for $\epsilon > 0$, is the operator defined as

$$\mathcal{L}_{\epsilon} := \theta \cdot \nabla_x - \mathcal{I}_{\epsilon} ,$$

where $\mathcal{I}_{\epsilon}$ is the scattering operator defined by (2.12) where $(-\Delta_{\theta})^s$ in formula (2.11) is replaced by

$$\left[ (-\Delta^{\epsilon}_{\theta})^s u \right]_{\mathcal{J}} := \langle \cdot \rangle^{d-1+2s} (-\Delta^{\epsilon}_v)^s U_{\mathcal{J}},$$

with

$$(-\Delta^{\epsilon}_v)^s := \frac{1}{1 + \epsilon \langle v \rangle^{2s}} (-\tilde{\Delta}^{\epsilon}_v)^s \frac{1}{1 + \epsilon \langle v \rangle^{2s}} .$$

The operator $(-\tilde{\Delta}^{\epsilon}_v)^s$ is a bounded approximation of the fractional Laplacian defined through its Fourier transform

$$\mathcal{F}_v\{(-\tilde{\Delta}^{\epsilon}_v)^s \varphi\}(\xi) := \frac{|\xi|^{2s}}{1 + \epsilon \langle \xi \rangle^{2s}} \widehat{\varphi}(\xi) , \qquad \xi \in \mathbb{R}^{d-1} .$$

For this approximation it is easy to see that $\|\mathcal{I}_{\epsilon}\|_{\mathcal{B}(L^2_{x,\theta})} \leq C\epsilon^{-3}$, where $\|\cdot\|_{\mathcal{B}}$ is the operator norm.

Given $f \in H^1_x \cap L^2_{\theta}$, problem (2.94) has a unique weak solution $u^{\epsilon} \in H^1_x \cap L^2_{\theta} \cap \{\gamma^-(\cdot) = 0\}$. Furthermore, the normal gradient $\partial_{x_d} u^{\epsilon}$ is forced to vanish in the boundary, then using the Green's formula for $u^{\epsilon}$ and $\partial_{x_d} u^{\epsilon}$ one

[2] In the definition of $H^s_{\Gamma_{loc}}$ is implicit that $\gamma^{\pm}(\cdot)$ are well defined and agree with $L^2_{loc}(\Gamma^{\pm})$ functions.

is led to the energy estimate

$$c\left\|(-\tilde{\Delta}^{\epsilon}_v)^{s/2}\Big(\frac{U^{\epsilon}_{\mathcal{J}}}{1+\epsilon\langle v\rangle^{2s}}\Big)\right\|^2_{L^2_{x,v}} + (\mu - C)\|u^{\epsilon}\|^2_{H^1_x\cap L^2_\theta} \leq \|f\|^2_{H^1_x\cap L^2_\theta}\,.$$

This estimate, with $\mu > C$, is enough to send $\epsilon \to 0$ and conclude that the sequence $\{u^\epsilon\}$ converges, up to a subsequence, weakly in $H^1_x \cap L^2_\theta$ to a weak solution $u$ of the problem

$$\begin{cases} \mu u + \mathcal{L}(u) = f & \text{in } \mathbb{R}^d_+ \times \mathbb{S}^{d-1}\,, \\ u = 0 & \text{on } \Gamma^-\,, \end{cases} \tag{2.95}$$

satisfying the estimate

$$c\left\|(-\Delta_v)^{s/2}U_{\mathcal{J}}\right\|^2_{L^2_{x,v}} + (\mu - C)\|u\|^2_{H^1_x\cap L^2_\theta} \leq \|f\|^2_{H^1_x\cap L^2_\theta}\,.$$

In other words, $u \in H^1_x \cap H^s_{\mathcal{L}} \cap \{\gamma^-(\cdot) = 0\}$.

We are now ready to conclude. Take $\tilde{u} \in H^s_{\mathcal{L}} \cap \{\gamma^-(\cdot) = 0\}$, then $\tilde{u}$ satisfies problem (2.95) with $f$ replaced by some $F \in L^2_{x,\theta}$. Take a sequence $\{f^n\} \subset H^1_x \cap L^2_\theta$ converging strongly to $F$ in $L^2_{x,\theta}$. By the previous argument, there exists a sequence $\{u^n\} \in H^1_x \cap H^s_{\mathcal{L}} \cap \{\gamma^-(\cdot) = 0\}$ solving problem (2.95) with $f$ replaced by $f^n$. Since the Green's formula is valid for such sequence, it is not difficult to conclude that

$$\|u^n - u^m\|_{H^s_{\mathcal{L}}} \leq C\|f^n - f^m\|_{L^2_{x,\theta}}\,.$$

Thus, $\{u^n\}$ is Cauchy and converges strongly to a limit, say $\tilde{w} \in H^s_{\mathcal{L}} \cap \{\gamma^-(\cdot) = 0\}$. Indeed, recall that the trace is continuous from estimate (2.93), consequently, $\tilde{w}$ has a well defined trace with $\gamma^-(\tilde{w}) = 0$. And, of course,

$$\|u^n - \tilde{w}\|_{H^s_{\mathcal{L}}} \leq C\|f^n - F\|_{L^2_{x,\theta}} \to 0 \quad \text{as} \quad n \to 0\,.$$

In this way, $\tilde{w}$ is a weak solution of problem (2.95) with $f$ replaced by $F \in L^2_{x,\theta}$. But, problem (2.95) has a unique weak solution, therefore $\tilde{w} = \tilde{u}$. Consequently,

$$\tilde{u} \in \overline{H^1_x \cap H^s_{\mathcal{L}} \cap \{\gamma^-(\cdot) = 0\}}\,,$$

which proves the result. ■

**Remark 3** *Using the Green's formula, it readily follows that if* $u \in \Big\{w \in H^s_{\mathcal{L}}\,\Big|\,\gamma^-(w) \in L^2\Big(\Gamma^-, |\theta_d|\,d\theta\,d\bar{x}\Big)\Big\}$*, then* $\gamma^+(u) \in L^2\Big(\Gamma^+, |\theta_d|\,d\theta\,d\bar{x}\Big)$*.*

As a corollary, Theorem 2.4.2 gives the Green's identity on compact sets of

$\Gamma^{\pm}$.

**Corollary 2.4.1** *Let $\varphi(x,\theta) \in H^1_x \cap H^s_\theta$ with support in $\{\theta_d < 0\} \cup \{\theta_d > 0\}$. Then for $u \in H^s_{\Gamma_{loc}}$ it follows that*

$$\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{L}(u)\,\varphi\,\mathrm{d}\theta\mathrm{d}x = -\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)\,\varphi\,\mathrm{d}\theta\mathrm{d}x - \int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} u\,\theta\cdot\nabla_x\varphi\,\mathrm{d}\theta\mathrm{d}x \\ - \int_{\Gamma^+} \gamma^+(u)\varphi(\bar{x},\theta)\,\theta_d\,\mathrm{d}\theta\mathrm{d}\bar{x} - \int_{\Gamma^-} \gamma^-(u)\varphi(\bar{x},\theta)\,\theta_d\,\mathrm{d}\theta\mathrm{d}\bar{x}\,.$$

### 2.4.1
### Absorbing boundary conditions.

We consider in this section the RTE with absorbing boundary, that is, the case $g \equiv 0$ where the input boundary intensity is null. Inhomogeneous boundary conditions follow from this case using classical arguments involving "lifting", refer for example to the classical reference [11, Chapter XXI - section 4]. In the case of absorbing boundary conditions, the natural domain of the operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to L^2_{x,\theta}$ is

$$\mathcal{D}(\mathcal{L}) = H^s_{\mathcal{L}} \cap \Big\{\gamma^-(\cdot) = 0\Big\}\,.$$

In $\mathcal{D}(\mathcal{L})$ the Green's identity holds in the whole boundary since the traces belong to $L^2\Big(\Gamma^{\pm}, |\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\Big)$. Therefore, for any $u \in \mathcal{D}(\mathcal{L})$

$$\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{L}(u)\,u\,\mathrm{d}\theta\mathrm{d}x = -\int_{\mathbb{R}^d_+}\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)\,u\,\mathrm{d}\theta\mathrm{d}x - \frac{1}{2}\int_{\Gamma^+} \Big|\gamma^+(u)\Big|^2\theta_d\,\mathrm{d}\theta\mathrm{d}\bar{x} \geq 0\,.$$

We conclude that $\langle \mathcal{L}(u), u\rangle_{L^2_{x,\theta}} \geq 0$ and, consequently, $(-\mathcal{L})$ is a dissipative operator taking values on $L^2_{x,\theta}$.

In order to apply Lumer–Phillips Theorem it suffices to prove that the operator $(-\mathcal{L}) - \mu\, I$ is surjective on $L^2_{x,\theta}$ for some $\mu > 0$. But, this is exactly what we proved when we showed the existence of weak solutions in $\mathcal{D}(\mathcal{L})$ for the problem (2.95) with $f \in L^2_{x,\theta}$. Thus, invoking the Lumer–Phillips theorem in reflexive spaces one proves the following theorem.

**Theorem 2.4.3** *The operator $(-\mathcal{L}) : \mathcal{D}(\mathcal{L}) \to L^2_{x,\theta}$ generates a contraction semigroup.*

Next we recall Theorem 1.3.2, accounting for the case of absorbing boundary conditions, and complete its proof.

**Theorem 2.4.4 (Absorbing boundary)** *Consider* $u_0 \in L^2\Big(\mathbb{R}^d_+ \times \mathbb{S}^{d-1}\Big)$ *and* $f \in L^2\Big((0,T) \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}\Big)$. *Then, the problem*

$$\begin{cases} \partial_t u + \mathcal{L}(u) = f & \textit{in} \quad (0,T) \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}, \\ u = u_0 & \textit{on} \quad \{t=0\} \times \mathbb{R}^d_+ \times \mathbb{S}^{d-1}, \\ u = 0 & \textit{on} \quad (0,T) \times \partial\mathbb{R}^d_+ \times \mathbb{S}^{d-1} \;\; \textit{and} \;\; -(\theta \cdot n(\bar{x})) > 0, \end{cases}$$

*has a unique weak solution*

$$u \in \mathcal{C}\Big([0,T]; L^2(\mathbb{R}^d_+ \times \mathbb{S}^{d-1})\Big) \cap L^2\Big((0,T); L^2_x \cap H^s_\theta(\mathbb{R}^d_+ \times \mathbb{S}^{d-1})\Big)\,.$$

*Its trace satisfies* $\gamma^+(u) \in L^2\Big((0,T) \times \Gamma^+, |\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\Big)$. *Furthermore,* $u \geq 0$ *if* $u_0 \geq 0$ *and* $f \geq 0$.

*Proof.* The statement about existence is direct from Theorem 2.4.3. Uniqueness follows from the fact that solutions with well defined traces in $L^2\Big((0,T) \times \Gamma^\pm, |\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\Big)$ satisfy the Green's formula.

Finally, in order to prove positivity we use estimate (2.22) and Gronwall's lemma to conclude that for any $t \in (0,T)$

$$\begin{aligned} \|u(t)\|^2_{L^2_{x,\theta}} + D_0 \int_0^t \int_{\mathbb{R}^d_+} \|u\|^2_{H^s_\theta}\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_{\Gamma^+} |\gamma^+(u)|^2\,|\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau \\ \leq C(T)\Big(\|u_0\|^2_{L^2_{x,\theta}} + \int_0^t \|f(\tau)\|^2_{L^2_{x,\theta}}\mathrm{d}\tau\Big). \end{aligned}$$

Now, for $s \in (0,1)$ it follows that if $u(x,\cdot) \in H^s_\theta$, then the positive and negative parts of $u$ satisfy $u^\pm \in H^s_\theta$ as well[3]. Then,

$$\int_{\mathbb{S}^{d-1}} \mathcal{I}(u)u^-\mathrm{d}\theta = \int_{\mathbb{S}^{d-1}} \mathcal{I}(u^+)u^-\mathrm{d}\theta + \int_{\mathbb{S}^{d-1}} \mathcal{I}(u^-)u^-\mathrm{d}\theta \leq 0\,.$$

The fact that both terms in the left side are non positive is a direct consequence of the weak formulation (2.5). Applying the Green's formula again, it follows that for $t \in (0,T)$

$$\begin{aligned} \|u^-(t)\|^2_{L^2_{x,\theta}} + \frac{1}{2}\int_0^t \int_{\Gamma^+} |\gamma^+(u^-)|^2\,|\theta_d|\,\mathrm{d}\theta\,\mathrm{d}\bar{x}\,\mathrm{d}\tau \\ \leq \|u_0^-\|^2_{L^2_{x,\theta}} + \int_0^t \int_{\mathbb{R}^d_+} \int_{\mathbb{S}^{d-1}} f\,u^-\mathrm{d}\theta\mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

If $u_0 \geq 0$ and $f \geq 0$, the left side is non positive. One is led to conclude that $u^- \equiv 0$. ■

[3] In fact, one has $\|u^\pm\|_{H^s_\theta} \leq \|u\|_{H^s_\theta}$.

# 3
# Doubly nonlinear degenerate parabolic equations

In this chapter we examine an inhomogeneous doubly nonlinear parabolic equation of the form

$$\partial_t\left(|u|^{\beta-2}u\right) - \mathrm{div}|Du|^{p-2}Du = f \quad \text{in} \quad Q_1, \tag{3.1}$$

where $f \in L^{q,r}(Q_1)$, $p \geq \beta \geq 2$ and $Q_1 := B_1 \times (-1,0]$. As mentioned before, our interest relies on interior regularity. For that reason we do not prescribe boundary conditions or examine the Dirichlet problem.

This part is organised as follows: In Section 3.1, we produce preliminary results and obtain energy estimates; we also obtain a sequential stability of the weak solution to (3.1). In Section 3.2 we establish an approximation result. In Section 3.3, a geometric iteration is established, using the intrinsic scale of the degenerate doubly nonlinear parabolic equation. The proof of Theorem 1.3.3 is the subject of Section 3.4.



## 3.1
## Preliminary compactness

The following proposition ensures the Hölder continuity of the solutions to (3.1) both in time and in space.

**Proposition 3.1.1 (Preliminary compactness of the solutions)** *Let $u$ be a locally bounded weak solution to (3.1) in $Q_1$ with $f \in L^{q,r}(B_1 \times (-1,0])$. Then $u$ is locally of class $C^{0,\alpha}$ in space and of class $C^{0,\alpha/\theta}$ in time, with*

$$\alpha = \min\left\{\alpha_o^-, \frac{(pq-d)r - pq}{q[(p-1)r-(p-\beta)]}\right\},$$

*where $0 < \alpha_0^- \leq 1$ is the optimal Hölder exponent for solutions of the homogeneous equation and $\theta = p - \alpha(p-\beta)$. In addition, there exists $C > 0$, depending only on $\|u\|_{L^\infty(Q_1)}$ and $\|f\|_{L^{q,r}(Q_1)}$, such that*

$$|u(x,t) - u(y,s)| \leq C\left(|x-y|^\alpha + |t-s|^{\frac{\alpha}{\theta}}\right).$$

*where $q$ and $r$ satisfy*

$$\frac{1}{r}+\frac{d}{pq}<1 \qquad \text{and} \qquad \frac{p}{r}+\frac{d}{q}>1.$$

For a proof of this proposition, we refer the reader to [18, Remark 3.5].

By comparing our main result with Proposition 3.1.1 we unveil the quantitative improvement in the modulus of continuity of the solutions arising from the approximation with $p$-caloric equations.

We use approximation methods. As mentioned before, the main tools behind this approach are stability, compactness and scaling properties of the equation. We begin with a proposition concerning the energy estimates of the solution to (3.1).

### 3.1.1
### Energy estimates

Energy estimates play a fundamental role in the regularity theory. The next result is related to this kind of estimates. Notice also that Caccioppoli estimates can be obtained by choosing a special test function in the definition of a weak solution.

**Lemma 3.1.2 (Energy estimates)** *Let $u$ be a weak solution of (3.1). There exists a positive constant $C$, depending only on $d, \beta$ and $p$, such that*

$$\begin{aligned}&\int_0^T\int_{B_1}|Du|^p\phi^p dxdt+\sup_{0<t<T}\int_{B_1}|u|^\beta\phi^p dx\\ &\leq C\left(\int_0^T\int_{B_1}|u|^p|D\phi|^p dxdt+\int_0^T\int_{B_1}|u|^\beta\phi^{p-1}|\phi_t|dxdt+\|f\|_{L^{q,r}(Q_1)}\right),\end{aligned}$$

*for every $\phi\in C_0^\infty(B_1\times(0,T))$ such that $\phi\in[0,1]$.*

*Proof.* Here we resort to the definition of weak solutions in terms of the Steklov averages. First, we choose as a test function $\varphi=u_h\,\phi^p$ so that

$$D\varphi=\phi^p Du_h+pu_h\,\phi^{p-1}D\phi,$$

where $\phi\in C_0^\infty(B_1\times(0,T))$ with $0<\phi<1$. Let $0\leq t_1\leq t\leq t_2\leq T$. Also, recall that

$$\partial_t([|u|^{\beta-2}u]_h)u_h=\left(1-\frac{1}{\beta}\right)\partial_t(|u_h|^\beta).$$

Replacing the above computations into the definition of a weak solution gives

$$\int_{t_1}^{t_2}\int_{B_1}[|Du|]_h^p\,\phi^p dxdt + p\int_{t_1}^{t_2}\int_{B_1}[|Du|^{p-2}Du]_h\cdot D\phi\,\phi^{p-1}udxdt$$
$$-\int_{t_1}^{t_2}\int_{B_1}[|u|^{\beta-2}u]_h\,\partial_t(u_h)\phi^p dxdt - \int_{t_1}^{t_2}\int_{B_1} f\,u_h\,\phi^p\,dxdt = 0.$$

We notice that the time derivative appears. We can manage it by integrating by parts as follows

$$\frac{1}{p}\int_{t_1}^{t_2}\int_{B_1}\partial_t[|u|]_h^\beta\,\phi^p\,dxdt$$
$$= \frac{1}{p\beta}\left[\int_{B_1}|u_h|^\beta\phi^p\,dx\right]_{t_1}^{t_2} - \frac{1}{\beta}\int_{t_1}^{t_2}\int_{B_1}|u_h|^\beta\phi^{p-1}\phi_t\,dxdt.$$

With the above computation the time derivative on $u_h$ has disappeared. Thus, we arrive at

$$\begin{aligned}\int_{t_1}^{t_2}\int_{B_1}[|Du|]_h^p\phi^p dxdt + \int_{B_1}|u_h|^\beta\phi^p dx - \int_{t_1}^{t_2}\int_{B_1} f\,u_h\,\phi^p\,dxdt\\ \le C\int_{t_1}^{t_2}\int_{B_1}[|Du|^{p-2}Du]_h\cdot D\phi\,\phi^{p-1}\,u_h\,dxdt\\ + C\int_{t_1}^{t_2}\int_{B_1}|u_h|^\beta\phi^{p-1}\phi_t\,dxdt.\end{aligned} \tag{3.2}$$

At this time, we can choose $t_1$ and $t_2$ as we need. Set $t_1 = 0$ and $t_2 = \tau$ such that

$$\int_{B_1}|u_h(x,\tau)|^\beta\phi^p(x,\tau)dx \ge \frac{1}{2}\sup_{0<t<T}\int_{B_1}|u_h(x,t)|^\beta\phi^p(x,t)dx,$$

then, we have

$$\left[\int_{B_1}|u_h|^\beta\phi^p dx\right]_{t_1=0}^{t_2} = \int_{B_1}|u_h(x,\tau)|^\beta\phi^p(x,\tau)dx$$
$$\ge \frac{1}{2}\sup_{0<t<T}\int_{B_1}|u_h(x,t)|^\beta\phi^p(x,t)dx$$

On the other hand, we might choose $t_1 = 0$ and $t_2 = T$ and replacing into (3.3) yields

$$\begin{aligned}\int_0^T\int_{B_1}|Du|^p\phi^p dxdt + \sup_{0<t<T}\int_{B_1}|u(x,t)|^\beta\phi^p(x,t)dx\\ \le C(p,\beta)\int_0^T\int_{B_1}[|Du|]_h^{p-1}|D\phi|\,\phi^{p-1}\,|u_h|\,dxdt + \int_0^T\int_{B_1} f\,u_h\,\phi^p\,dxdt\\ + C(p,\beta)\int_0^T\int_{B_1}|u_h|^\beta\phi^{p-1}|\phi_t|\,dxdt.\end{aligned} \tag{3.3}$$

Now, using Young's inequality yields

$$C(p,\beta)\int_0^T\int_{B_1}[|Du|]_h^{p-1}|D\phi|\,\phi^{p-1}\,|u_h|\,dxdt$$
$$\leq C(p,\beta)\int_0^T\int_{B_1}[|Du|]_h^p\phi^p\,dxdt + C(p,\beta)\int_0^T\int_{B_1}|u_h|^p|D\phi|^p\,dxdt.$$

For the last term involving the source term $f$ we proceed as follows. From hypothesis $u_h$ is bounded in $Q_1$ and since $0<\phi<1$ and $p>2$, then $\phi^p\leq\phi<1$, then for $\frac{1}{p_1}+\frac{1}{q}=1$ we have

$$\begin{aligned}
\int_0^T\int_{B_1} f\,u_h\,\phi^p\,dxdt &\leq \int_0^T\int_{B_1} f\,u_h\,\phi\,dxdt\\
&\leq \int_0^T\left(\int_{B_1}|f|^q\,dx\right)^{1/q}\left(\int_{B_1}|u_h|^{p_1}\,dx\right)^{1/p_1}dt\\
&\leq \left(\int_0^T\left|\int_{B_1}|f(x,t)|^q\,dx\right|^{r/q}dt\right)^{1/r}\left(\int_0^T\left|\int_{B_1}|u_h(x,t)|^{p_1}\,dx\right|^{\frac{r}{(r-1)p_1}}dt\right)^{\frac{r-1}{r}}\\
&\leq C\|f\|_{L^{q,r}(Q_1)}
\end{aligned}$$

We pass to the limit in $h\to 0$ and then use the properties of the Steklov average. Hence, the proof is complete. ■

Now we prove an stability result for the weak solutions.

**Proposition 3.1.3 (Sequential stability of weak solutions)** *Let $(\beta_n)_{n\subset\mathbb{N}}\subset\mathbb{R}$ and assume that*

$$(f_n)_{n\subset\mathbb{N}}\subset L^{q,r}_{loc}(Q_1)$$

*and*

$$(u_n)_{n\in\mathbb{N}}\subset L^1_{loc}(Q_1).$$

*Suppose further that $(u_n)_{n\in\mathbb{N}}$ solves*

$$\partial_t\left(|u_n|^{\beta_n-2}u_n\right)-\operatorname{div}|Du_n|^{p-2}Du_n=f_n\quad in\quad Q_1, \tag{3.4}$$

*and*

$$|\beta_n-2|+\|f_n\|_{L^{q,r}(Q_1)}\leq\frac{1}{n}.$$

*If there exists $u_\infty\in L^p_{loc}(0,T;W^{1,p}_{loc}(B_1))\cap L^\infty_{loc}(Q_1)$ such that*

$$\|u_n-u_\infty\|_{L^\infty(Q_{9/10})}\longrightarrow 0,$$

*then $u_\infty$ is a weak solution to*

$$\partial_t u_\infty-\operatorname{div}|Du_\infty|^{p-2}Du_\infty=0\quad in\quad Q_{9/10}. \tag{3.5}$$

*Proof.* Let $\varphi \in \mathcal{C}_0^\infty(Q_1)$. We have

$$\left|\int_0^T \int_{B_{9/10}} -u_\infty \varphi_t + |Du_\infty|^{p-2} Du_\infty \cdot D\varphi \, dxdt\right| \leq \int_0^T \int_{B_1} |\varphi_t| \left||u_n|^{\beta_n - 2} u_n - u_\infty\right| \, dxdt$$
$$+ \int_0^T \int_{B_1} |D\varphi| \left||Du_\infty|^{p-2} Du_\infty - |Du_n|^{p-2} Du_n\right| \, dxdt + \int_0^T \int_{B_1} |f_n \varphi| \, dxdt$$
$$=: \int_0^T \int_{B_1} a_n \, dxdt + \int_0^T \int_{B_1} b_n \, dxdt + \int_0^T \int_{B_1} |f_n \varphi| \, dxdt.$$

Proposition 3.1.1 yields $u_n \in \mathcal{C}^\alpha_{loc}(Q_1)$ and $\|u_n\|_{\mathcal{C}^\alpha(Q_{9/10})} \leq C$. Then

$$\left||u_n|^{\beta_n - 2} u_n - u_\infty\right| \leq |u_n|^{\beta_n - 1} + \|u_\infty\|_{L^\infty(Q_1)} \leq C.$$

Hence, we can conclude that

$$|a_n| \leq C|\varphi_t|.$$

From the hypothesis, we deduce that $|u_n|^{\beta_n - 2} u_n \to u_\infty$ a.e.-$(x,t) \in Q_1$ as $n \to \infty$. Then,

$$\int_0^T \int_{B_1} a_n \, dxdt \to 0, \text{ as } n \to \infty.$$

Notice also that

$$\int_0^T \int_{B_1} |f_n \varphi| \, dxdt \to 0, \text{ as } n \to \infty.$$

For the remaining term, we proceed as follows. Using the energy estimates, we can conclude that $Du_n$ is bounded in $L^p(Q_1)$, for a test function $\phi \in (0,1)$ . Then, there exists a subsequence $Du_n$ that converges weakly to $\zeta$ in $L^p(Q_1)$. The identification $\zeta = Du_\infty$ follows by standard techniques.

Without any loss of generality, we may also assume there is $\xi$ such that

$$|Du_{n_j}|^{p-2} Du_{n_j} \rightharpoonup \xi \text{ in } L^q(B_1 \times (0,T]; \mathbb{R}^n)$$

as $j \to \infty$.

In particular, for $v \in L^p((0,T];W_0^{1,p}(B_1))$ and $p \geq 2$, we have

$$\begin{aligned}
0 &\leq \lim_{j\to\infty} \int_{t_0}^{t_1}\int_{B_1} \langle |Du_{n_j}|^{p-2}Du_{n_j} - |Dv|^{p-2}Dv, Du_{n_j} - Dv\rangle dxdt \\
&= \lim_{j\to\infty} \Big[\int_{t_0}^{t_1} \langle |Du_{n_j}(\cdot,t)|^{p-2}Du_{n_j}(\cdot,t), Du_{n_j}(\cdot,t) - Dv(\cdot,t)\rangle dt \\
&\quad - \int_{t_0}^{t_1}\int_{B_1} |Dv|^{p-2}Dv(Du_{n_j} - Dv)dxdt\Big] \\
&= \int_{t_0}^{t_1} \langle \xi(\cdot,t), Du_\infty(\cdot,t) - Dv(\cdot,t)\rangle dt - \int_{t_0}^{t_1}\int_{B_1} |Dv|^{p-2}Dv(Du_\infty - Dv)dxdt \\
&= \int_{t_0}^{t_1} \langle \xi(\cdot,t) - |Dv(\cdot,t)|^{p-2}Dv(\cdot,t), Du_\infty(\cdot,t) - Dv(\cdot,t)\rangle dt.
\end{aligned}$$

The first inequality of the above computations is a well-known algebraic inequality. For a proof of this result, we refer the reader to [12, Lemma 4.4]

We can now choose $v = u_\infty - \tau\phi$ for $\phi \in C_c^\infty(B_1 \times (0,T])$ and $\tau > 0$ to get $Dv = Du_\infty - \tau D\phi$. Then

$$0 \leq \tau \int_{t_0}^{t_1} \langle \xi(\cdot,t) - |Du_\infty(\cdot,t) - \tau D\phi(\cdot,t)|^{p-2}(Du_\infty(\cdot,t) - \tau D\phi(\cdot,t)), D\phi(\cdot,t)\rangle dt.$$

Cancelling $\tau$ and then sending $\tau \to 0^+$ gives

$$\begin{aligned}
0 &\leq \int_{t_0}^{t_1} \langle \xi(\cdot,t) - |Du_\infty(\cdot,t)|^{p-2}Du_\infty(\cdot,t), D\phi(\cdot,t)\rangle \\
&\leq \liminf_{n_j\to\infty} \int_{t_0}^{t_1} \langle \xi(\cdot,t) - |Du_{n_j}(\cdot,t)|^{p-2}Du_{n_j}(\cdot,t), D\phi(\cdot,t)\rangle = 0.
\end{aligned}$$

As a result

$$\xi = |Du_\infty|^{p-2}Du_\infty. \tag{3.6}$$

In other words

$$\int_0^T\int_{B_1} |Du_n|^{p-2}Du_n\psi \to \int_0^T\int_{B_1} |Du_\infty|^{p-2}Du_\infty\psi.$$

Hence, we conclude that

$$\int_0^T\int_{B_1} b_n\, dxdt = \int_0^T\int_{B_1} |D\varphi| \left||Du_\infty|^{p-2}Du_\infty - |Du_n|^{p-2}Du_n\right| dxdt \to 0,$$

as $n \to \infty$. So, the result is obtained. ■

We end this section by commenting on the scaling properties of the doubly nonlinear parabolic equation.

### 3.1.2
### A word about scaling

There is a one-parameter group of scalings that keeps the equation invariant. It is given by

$$v(x,t) = \frac{u(\rho x, \rho^{\theta} t)}{\rho^{\gamma}},$$

where $\theta = p - \gamma(p-\beta)$. In other words if $u$ solves (3.1), then the function $v$ solves

$$\partial_t \left( |v|^{\beta-2} v \right) - \mathrm{div} |Dv|^{p-2} Dv = \bar{f},$$

where $\bar{f}(x,t) := \frac{f(x,t)}{\rho^{\gamma(p-1)-p}}$. Since,

$$\partial_t \left( |v|^{\beta-2} v \right) = \frac{\partial_t \left( |u|^{\beta-2} u \right)}{\rho^{\gamma(\beta-1)-\theta}} = \frac{\partial_t \left( |u|^{\beta-2} u \right)}{\rho^{\gamma(p-1)-p}},$$

and

$$\mathrm{div} |Dv|^{p-2} Dv = \frac{\mathrm{div} |Du|^{p-2} Du}{\rho^{\gamma(p-1)-p}}.$$

Finally, we always can assume a smallness regime on the $L^{q,r}$-norm of $f$ and also we can require $u$ to be a normalized solution. In fact, if we set

$$w(x,t) := \frac{1}{\|u\|_{L^{\infty}(Q_1)} + \|f\|_{L^{q,r}(Q_1)}/\varepsilon_0} u(x,t).$$

Then, $w$ is such that $\|w\|_{L^{\infty}(Q_1)} \leq 1$ and $\|f\|_{L^{q,r}(Q_1)} \leq \varepsilon_0$.

In the next section, we study an approximation result.

## 3.2
## Approximating the solutions by the parabolic $p$-Laplace equation

We start by studying $\{u = 0\}$. The next proposition is the so-called approximation lemma. It ensures the existence of an auxiliary function $h$ approximating the solutions; the approximating function is more regular than the solutions, which introduces room for improvement at the level of the original problem.

We import regularity from $h$ through its Taylor expansion. For our arguments, it is critical that $h$ vanishes at the points $(x,t) \in S_0(u)$, where the degeneracy set $S_0(u)$ is defined as follows

$$S_0(u) := \{(x,t) \in Q_1 \mid u(x,t) = 0\}.$$

That is, we must guarantee

$$\{(x,t) \in Q_1 \mid u(x,t) = 0\} \subset \{(x,t) \in Q_1 \mid h(x,t) = 0\}.$$

Our first result in this section is a zero level set approximation lemma.

**Proposition 3.2.1 (Zero level-set approximation lemma)** *Let $u$ be a weak solution to (3.1). Suppose A1 is in force. Suppose further that $(x_0,t_0) \in S_0(u)$. Then, given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that if*

$$\beta - 2 + \|f\|_{L^{q,r}(Q_1)} < \varepsilon_0,$$

*one can find $h \in \mathcal{C}^{1,1/2}(Q_1)$ satisfying*

$$\|u - h\|_{L^\infty(Q_{9/10})} \leq \delta,$$

*with $h(x_0,t_0) = 0$.*

*Proof.* We argue by contradiction. Suppose the assertion is false. Then, there exist $\delta_0 > 0$ and sequences $(u_n)_{n\in\mathbb{N}}, (f_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$, such that

$$(\beta_n - 2) + \|f_n\|_{L^{q,r}(Q_1)} \leq \frac{1}{n}$$

and

$$\partial_t\left(|u_n|^{\beta_n-2}u_n\right) - \operatorname{div}|Du_n|^{p-2}Du_n = f_n \quad \text{in} \quad Q_1, \tag{3.7}$$

with

$$\|u_n - h\|_{L^\infty(Q_{9/10})} > \delta_0 \qquad \text{or} \qquad h(x_0,t_0) \neq 0,$$

for all $h \in \mathcal{C}^{1,1/2}(Q_1)$ and $n \in \mathbb{N}$. Proposition 3.1.1 implies that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^{\alpha,\alpha/\theta}(Q_1)$, for some $\alpha \in (0,1)$. Therefore, we can apply Arzela-Ascoli theorem to obtain that $u_n \to u_\infty$ uniformly in $\mathcal{C}^{\alpha_0,\alpha_0/\theta}(Q_{9/10})$ through a subsequence, if necessary, for every $0 < \alpha_0 < \alpha$. Applying Proposition 3.1.3 we can conclude that $u_\infty$ solves

$$(u_\infty)_t - \operatorname{div}|Du_\infty|^{p-2}Du_\infty = 0 \quad \text{in} \quad Q_{9/10}.$$

Standard regularity results available for the solutions of degenerate parabolic PDEs modelled on the homogeneous $p$-Laplace equation ensure that $u_\infty \in \mathcal{C}^{1,1/2}(Q_1)$ (See [13]). In addition, as a consequence of the uniform convergence, $u_\infty(x_0,t_0) = 0$. Now, by taking $h \equiv u_\infty$, we obtain a contradiction and complete the proof. ■

## 3.3
## Improved regularity along singular sets

The next result provides the first step in the iteration process to be implemented. In other words, we start by obtaining a control to the oscillation of the solutions to (3.1) in a ball of radius $0 < \rho \ll 1/2$, to be determined.

**Proposition 3.3.1** *Let $u$ be a weak solution to (3.1). Suppose that A1 is in force. Suppose further that $(x_0, t_0) \in S_0(u)$. Then, given $\gamma$ such that*

$$0 < \gamma < \frac{(pq-d)r - pq}{q[(p-1)r-(p-2)]},$$

*there exists $\varepsilon_0 > 0$ such that, if*

$$(\beta - 2) + \|f\|_{L^{q,r}(Q_1)} < \varepsilon_0,$$

*one can find a constant $0 < \rho \ll 1/2$ for which*

$$\sup_{Q_\rho(x_0,t_0)} |u(x,t)| \leq \rho^\gamma.$$

*Proof.* Take $0 < \delta < 1$, to be chosen later, and apply Proposition 3.2.1. We conclude the existence of $h$ such that

$$\|u - h\|_{L^\infty(Q_1)} \leq \delta,$$

with $h(x_0, t_0) = 0$; in addition, we know that $h \in C^{1,1/2}(Q_1)$. This means that $h$ is locally Lipschitz continuous in space and is locally Hölder continuous in time with exponent $1/2$. Observe also if we choose $0 < \rho \ll 1/2$, then for all $(x,t) \in Q_\rho$, we have

$$\begin{aligned}|h(x,t) - h(x_0,t_0)| \leq& |h(x,t) - h(x_0,t)| + |h(x_0,t) - h(x_0,t_0)| \\ &\leq C_1|x - x_0| + C_2|t - t_0|^{1/2} \\ &\leq C_1\rho + C_2\rho^{\frac{\theta}{2}} \leq C\rho.\end{aligned}$$

Since $\theta > 2$. Then, we can conclude that

$$\sup_{Q_\rho(x_0,t_0)} |h(x,t) - h(x_0,t_0)| \leq C\rho. \tag{3.8}$$

We can also obtain upper bounds for $u$ as follows

$$\begin{aligned}\sup_{Q_\rho(x_0,t_0)} |u(x,t)| &= \sup_{Q_\rho(x_0,t_0)} |u(x,t) - h(x,t) + h(x,t)| \\ &\leq \sup_{Q_\rho(x_0,t_0)} |u(x,t) - h(x,t)| \\ &\quad + \sup_{Q_\rho(x_0,t_0)} |h(x,t) - h(x_0,t_0)| \\ &\leq \delta + C\rho.\end{aligned} \tag{3.9}$$

Now, we make the following (universal) choices

$$\rho := \left(\frac{1}{2C}\right)^{\frac{1}{1-\gamma}} \quad \text{and} \quad \delta := \frac{\rho^\gamma}{2}.$$

With these choices, we obtain

$$\sup_{Q_\rho(x_0,t_0)} |u(x,t)| \leq \rho^\gamma,$$

fixing also $\varepsilon_0 > 0$, through Proposition 3.2.1. The result follows from estimate (3.9). ■

In the sequel we refine Proposition 3.3.1. This is done by producing an oscillation control at discrete scales of the form $(\rho^k)_{k\in\mathbb{N}}$.

**Proposition 3.3.2** *Let $u$ be a normalized weak solution to (3.1). Suppose A1 holds. Suppose further that $(x_0,t_0) \in S_0(u)$. Then, for every $\gamma$ that satisfies*

$$0 < \gamma < \frac{(pq-d)r - pq}{q[(p-1)r - (p-2)]},$$

*there exists $\varepsilon_0 > 0$ such that, if*

$$(\beta \;-\; 2) \;+\; \|f\|_{L^{q,r}(Q_1)} \;<\; \varepsilon_0,$$

*then*

$$\sup_{Q_{\rho^k}(x_0,t_0)} |u(x,t)| \;\leq\; \rho^{k\gamma},$$

*for every $k \in \mathbb{N}$.*

*Proof.* We prove the proposition by induction. Notice that Proposition 3.3.1 accounts for the first step in the induction argument. Suppose the statement holds for $n = k$. We need to verify the case $n = k+1$ also holds. Consider the function $v : Q_1 \to \mathbb{R}$ defined by

$$v(x,t) := \frac{u(\rho^k x, \rho^{k\theta} t)}{\rho^{k\gamma}}.$$

Furthermore, the induction hypothesis guarantees that $\|v\|_{L^\infty(Q_1)} \leq 1$. The scaling properties of the solutions given in Section 3.1.2 tell us that $v$ solves

$$\partial_t \left(|v|^{\beta-2}v\right) - \mathrm{div}|Dv|^{p-2}Dv = f_k,$$

for

$$f_k(x,t) = \rho^{k[\gamma+p(1-\gamma)]} f(x,t).$$

Moreover, we can estimate

$$\begin{aligned} ||f_k||^r_{L^{q,r}(Q_1)} &= \int_{-1}^{0}\left(\int_{B_1}|f_k(x,t)|^q dx\right)^{\frac{r}{q}} dt \\ &= \int_{-1}^{0}\left(\int_{B_1}\rho^{kq[\gamma+p(1-\gamma)]}|f(\rho^k x,\rho^{k\theta}t)|^q dx\right)^{\frac{r}{q}} dt \\ &= \int_{-1}^{0}\left(\int_{B_{\rho^k}}\rho^{kq[\gamma+p(1-\gamma)]-kd}|f(z,\rho^{k\theta}t)|^q dz\right)^{\frac{r}{q}} dt \\ &= \rho^{[kq(\gamma+p(1-\gamma))-kd]\frac{r}{q}}.\rho^{-k\theta}\int_{-\rho^{k\theta}}^{0}\left(\int_{B_{\rho^k}}|f(z,\tau)|^q dz\right)^{\frac{r}{q}} d\tau. \end{aligned}$$

Hence, we can conclude that

$$||f_k||^r_{L^{q,r}(Q_1)} \leq \rho^{[kq(\gamma+p(1-\gamma))-kd]\frac{r}{q}-k\theta}||f||^r_{L^{q,r}(Q_{\rho^k})}.$$

Now, observe that

$$[kq(\gamma+p(1-\gamma))-kd]\frac{r}{q} - k\theta \geq 0 \iff \gamma \leq \frac{(pq-d)r-pq}{q[(p-1)r-(p-2)]}.$$

In addition,

$$\|f_k\|^r_{L^{q,r}(Q_1)} = \rho^{[kq(\gamma+p(1-\gamma))-kd]\frac{r}{q}-k\theta}\,\|f\|^r_{L^{q,r}(Q_{\rho^k})}.$$

Since $0<\rho \ll 1/2$, we can conclude that

$$\|f_k\|_{L^{q,r}(Q_1)} \leq \|f\|_{L^{q,r}(Q_{\rho^k})} \leq \|f\|_{L^{q,r}(Q_1)} \leq \varepsilon_0. \tag{3.10}$$

We conclude that $v$ satisfies the hypothesis of Proposition 3.3.1. Moreover, $|v(x_0,t_0)| = \frac{|u(x_0,t_0)|}{\rho^{k\gamma}} = 0$, then $(x_0,t_0) \in S_0(v)$. Hence, we obtain

$$\sup_{Q_\rho(x_0,t_0)} |v(x,t)| \leq \rho^\gamma.$$

Using the relation between $v$ with $u$ yields

$$\sup_{Q_\rho(x_0,t_0)} |u(x,t)| \le \rho^\gamma \rho^{k\gamma} \le \rho^{(k+1)\gamma},$$

and the proof is complete. ■

With Proposition 3.3.2 in hand, we produce a discrete-to-continuous argument, extending the oscillation control to any radius $0 < R \ll 1/2$, not necessarily of the form $\rho^k$. This is the aim of the next Proposition.

**Proposition 3.3.3** *Let $u$ be a locally bounded weak solution to (3.1). Suppose further that $(x_0, t_0) \in S_0(u)$. Then, for every $\gamma$ that satisfies*

$$0 < \gamma < \frac{(pq-d)r - pq}{q[(p-1)r - (p-2)]},$$

*there exists $\varepsilon_0 > 0$ such that, if*

$$(\beta - 2) + \|f\|_{L^{q,r}(Q_1)} < \varepsilon_0,$$

*then for each $0 < R < \rho \ll 1/2$, we have*

$$\sup_{Q_R(x_0,t_0)} |u(x,t)| \le CR^\gamma,$$

*where $C > 0$ is a universal constant.*

*Proof.* Define the function $w : Q_1 \to \mathbb{R}$ as follows

$$w(x,t) = \lambda\, u(\lambda^a x, \lambda^b t),$$

with $\lambda > 0$ to be fixed. Then,

$$\partial_t \left(|w|^{\beta-2} w\right) = \lambda^{\beta-1+b}\, \partial_t \left(|u|^{\beta-2} u\right),$$

and

$$\operatorname{div}|Dw|^{p-2}Dw = \lambda^{p(1+a)-1} \operatorname{div}|Du|^{p-2}Du.$$

Now, we choose $b$ such that

$$\beta + b - 1 = p(a+1) - 1.$$

Thus, $w$ satisfies

$$\partial_t \left(|w|^{\beta-2} w\right) - \operatorname{div}|Dw|^{p-2}Dw = g(x,t),$$

where $g$ is such that

$$g(x,t) = \lambda^{p(a+1)-1} f(\lambda^a x, \lambda^{p(a+1)-\beta} t).$$

From this, notice that

$$\begin{aligned}
||g||^r_{L^{q,r}(Q_1)} &= \int_{-1}^{0} \left( \int_{B_1} |g(x,t)|^q dx \right)^{\frac{r}{q}} dt \\
&= \int_{-1}^{0} \left( \int_{B_1} \lambda^{[p(a+1)-1]q} |f(\lambda^a x, \lambda^{p(a+1)-\beta} t)|^q dx \right)^{\frac{r}{q}} dt \\
&= \int_{-1}^{0} \left( \int_{B_{\lambda^a}} \lambda^{[p(a+1)-1]q-ad} \, |f(x, \lambda^{p(a+1)-\beta} t)|^q dx \right)^{\frac{r}{q}} dt \\
&= \lambda^{[q(p(a+1)-1)-ad]\frac{r}{q}} . \lambda^{-p(a+1)+\beta} \int_{-\lambda^{k\theta}}^{0} \left( \int_{B_{\rho^k}} |f(z,\tau)|^q dz \right)^{\frac{r}{q}} d\tau
\end{aligned}$$

Notice also that

$$\|g\|^r_{L^{q,r}(Q_1)} \leq \lambda^{[q(p(a+1)-1)-ad]\frac{r}{q}+\beta-p(a+1)} \|f\|_{L^{q,r}(Q_1)}.$$

Now, choosing $a$ such that

$$[q(p(a+1)-1)-ad]\frac{r}{q} + \beta - p(a+1) > 0,$$

holds. Then, given $\varepsilon_0 > 0$, we can choose $0 < \lambda \ll 1$ such that

$$\sup_{Q_1} |w(x,t)| \leq \lambda \sup_{Q_1} |u(x,t)| \qquad \text{and} \qquad \|g\|_{L^{q,r}(Q_1)} \leq \varepsilon_0,$$

Since, $u(x_0,t_0) \in S_0(u)$, we have also that $w(x_0,t_0) \in S_0(w)$. Then, $w$ and $g$ satisfies the assumptions in Proposition 3.3.2.

Hence, for each $0 < R < \rho$, there exists $k \in \mathbb{N}$ such that

$$\rho^{k+1} \leq R < \rho^k.$$

then we can conclude that

$$\sup_{Q_{\rho^k}(x_0,t_0)} |u(x,t)| \leq \rho^{k\gamma}.$$

Since, $\rho^{k+1} \leq R$, we have

$$\sup_{Q_R(x_0,t_0)} |u(x,t)| \leq \sup_{Q_{\rho^k}(x_0,t_0)} |u(x,t)| \leq \rho^{k\gamma} = \frac{\rho^{(k+1)\gamma}}{\rho^{\gamma}} \leq CR^{\gamma},$$

with $C = \rho^{-\gamma}$. Hence, the result is obtained. ■

## 3.4 Hölder regularity

In this section we recall the main theorem for this chapter of the thesis and detail its proof. We state it as follows:

**Theorem 3.4.1 (Improved regularity)** *Let $u$ be a weak solution to (1.6). Suppose A1 holds. Given*

$$0 < \gamma < \frac{(pq-d)r - pq}{q[(p-1)r-(p-2)]},$$

*there exists $\varepsilon = \varepsilon(d,\gamma)$ such that, if $0 < \beta - 2 < \varepsilon$. Then $u$ is locally of class $C^{0,\gamma}$ in space and of class $C^{0,\gamma/\theta}$ in time where*

$$\theta := p - \gamma(p-\beta).$$

*In addition, there exists $C > 0$, depending only on $\gamma$ and the dimension $d$, for which*

$$\sup_{B_R(x_0)\times(t_0-R^{\theta},t_0)} |u(x,t) - u(x_0,t_0)| \leq C\left(|x-x_0|^{\gamma} + |t-t_0|^{\gamma/\theta}\right)\left(\|u\|_{L^{\infty}(Q_1)} + \|f\|_{L^{q,r}(Q_1)}\right).$$

*Proof.* Without loss of generality, we prove sharp Hölder regularity with $(x_0,t_0)$ being the origin. We need to prove that for a constant $C > 0$ we have

$$\sup_{Q_R(x,t)} |u(x,t) - u(0,0)| \leq CR^{\gamma}.$$

Since $u$ is continuous. We can define $\mu \geq 0$ such that

$$\mu := (4|u(0,0)|)^{1/\gamma}.$$

Now, we take any radius $0 < R < \rho$. We will analyze two possible cases as follows:

**Case radius $\mu \le R$ :** This case is the simplest case. Recall that

$$|u(0,0)| = \frac{1}{4}\mu^\gamma \le \frac{1}{4}R^\gamma.$$

and by using Proposition 3.3.3, we can conclude that

$$\sup_{Q_R(x,t)} |u(x,t) - u(0,0)| \le C_1 R^\gamma + |u(0,0)| \le \left(C_1 + \frac{1}{4}\right) R^\gamma.$$

Now, we prove the remaining case.

**Case radius $0 < r < \mu$ :** This case is more subtle. We begin by defining $w$ by

$$w(x,t) := \frac{u(\mu x, \mu^\theta t)}{\mu^\gamma}.$$

Notice that $|w(0,0)| = \frac{1}{4}$ and $w$ satisfies

$$\partial_t \left(|w|^{\beta-2}w\right) - \mathrm{div}|Dw|^{p-2}Dw = \bar{h},$$

for

$$\bar{h}(x,t) := \mu^{k[\gamma+p(1-\gamma)]} f(\mu x, \mu^\theta t).$$

Now, Propositon 3.3.3 yields

$$\sup_{Q_1(x,t)} |w(x,t)| \le \mu^{-\gamma} \sup_{Q_\mu(x,t)} |u(x,t)| \le \mu^{-\gamma} C_1 \mu^\gamma = C_1,$$

since $|u(0,0)| = \frac{1}{4}\mu^\gamma$. By this uniform estimate and the local $\mathcal{C}^{\alpha'}-$regularity estimates there exists a radius $r_0$, depending only on the data $p, d, q, \beta$ and $r$, such that

$$|w(x,t)| \ge \frac{1}{8}, \qquad \forall (x,t) \in Q_{r_0}.$$

This bound from below away from zero implies that the following function

$$v := |w|^{\beta-2}w,$$

satisfies a uniformly parabolic equation of the form

$$\partial_t v - \mathrm{div}(c(x,t)|Dv|^{p-2}Dv) = h \in L^{q,r}(Q_{r_0}), \tag{3.11}$$

with continuous coefficients $c(x,t)$ satisfying the uniform bounds

$$0 < c_0 < c(x,t) < C_0.$$

Moreover, the function $v$ belongs to $\mathcal{C}^{\alpha'}(Q_{r_0})$ with

$$\alpha' = \frac{(pq-d)r - pq}{q[(p-1)r-(p-2)]} > \gamma.$$

A proof of the above result can be found in [44, Theorem 3.4]. The above result can be rewriten as follows

$$\sup_{Q_R(x,t)} |v(x,t) - v(0,0)| \leq C_2 R^{\alpha'}, \quad \text{for all} \quad 0 < R < r_0/2,$$

that is

$$\sup_{Q_R(x,t)} |w(x,t) - w(0,0)| \leq C^* R^{\alpha'}, \quad 0 < R < r_0/2,$$

for a new constant $C^*$, depending on the lower bound and upper bound on $w$. It is easy to see that the above inequality in terms of $u$ has the following form

$$\sup_{Q_R(x,t)} \left| \frac{u(\mu x, \mu^\theta t) - u(0,0)}{\mu^\gamma} \right| \leq C^* r^{\alpha'}, \quad 0 < R < \frac{r_0}{2}.$$

Since $\gamma < \alpha'$, we can conclude that

$$\sup_{Q_{\mu R}(x,t)} |u(x,t) - u(0,0)| \leq C^*(\mu R)^\gamma, \quad 0 < \mu R < \mu r_0/2,$$

and relabelling

$$\sup_{Q_R(x,t)} |u(x,t) - u(0,0)| \leq C^* R^\gamma, \quad 0 < R < \mu r_0/2.$$

It remains to see what happens when $\mu \frac{r_0}{2} \leq R < \mu$. We argue as follows

$$\sup_{Q_R(x,t)} |u(x,t) - u(0,0)| \leq \sup_{Q_\mu(x,t)} |u(x,t) - u(0,0)| \leq C\mu^\gamma \leq C \left(\frac{2R}{r_0}\right)^\gamma = C' R^\gamma.$$

Taking $C = \max\{C_1 + \frac{1}{4}, C'\}$ from both cases, we obtain the following estimate

$$\sup_{Q_R(x,t)} |u(x,t) - u(0,0)| \leq C\, R^\gamma,$$

for every $0 < R < \rho$. Hence the proof is complete. ■

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# A
# Appendix

## A.1
## Technical lemmas

In this chapter we are going to present several technical results and norm bounds that we need along this work.

**Proposition A.1.1** *Let $s, s' > 0$ and $\varphi \in H^s_\theta \cap H^{s'}_x$. Then, for some $r := r(s,s',d) > 2$ and $\alpha := \alpha(s,s',d) \in (0,1)$ it follows that*

$$\|\varphi\|_{L^r_{x,\theta}} \le C\left(\int_{\mathbb{R}^d} \|\varphi(x,\cdot)\|^2_{H^s_\theta} \mathrm{d}x\right)^{\frac{\alpha}{2}} \left(\int_{\mathbb{S}^{d-1}} \left\|(-\Delta_x)^{s'/2}\varphi(\cdot,\theta)\right\|^2_{L^2_x} \mathrm{d}\theta\right)^{\frac{1-\alpha}{2}}.$$

*for some $C := C(s,s',d)$*

*Proof.* Recall that by (2.18) and Sobolev imbedding we have that

$$\int_{\mathbb{R}^d} \|\varphi(x,\cdot)\|^2_{H^s_\theta} \mathrm{d}x \ge c\int_{\mathbb{R}^d}\left(\int_{\mathbb{S}^{d-1}} \left|\varphi(x,\theta)\right|^p \mathrm{d}\theta\right)^{\frac{2}{p}} \mathrm{d}x\,,$$

$$\int_{\mathbb{S}^{d-1}} \|(-\Delta_x)^{s'/2}\varphi(\cdot,\theta)\|^2_{L^2_x} \mathrm{d}\theta \ge c\int_{\mathbb{S}^{d-1}}\left(\int_{\mathbb{R}^d} \left|\varphi(x,\theta)\right|^q \mathrm{d}x\right)^{\frac{2}{q}} d\theta\,,$$

where $c := c(s,s',d)$ and

$$\frac{1}{q} = \frac{1}{2} - \frac{s'}{d}\,, \qquad \frac{1}{p} = \frac{1}{2} - \frac{s}{d-1}\,.$$

Observe that $q$, $p > 2$. Set

$$\alpha_1 = \frac{q-2}{\frac{p}{2}q-2} \in (0,1)\,, \qquad \alpha_2 = \frac{p}{2}\alpha_1 \in (0,1)\,, \qquad r = p\,\alpha_1 + 2(1-\alpha_1) > 2\,, \tag{A.1}$$

so that

$$\frac{\alpha_1}{\alpha_2} = \frac{2}{p}\,, \qquad \frac{1-\alpha_1}{1-\alpha_2} = \frac{q}{2} > 1\,.$$

Then, using Hölder inequality we have that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\big|\varphi(x,\theta)\big|^{r}\mathrm{d}\theta\,\mathrm{d}x\\
&\quad\le\int_{\mathbb{R}^d}\Big(\int_{\mathbb{S}^{d-1}}\big|\varphi(x,\theta)\big|^{p}\mathrm{d}\theta\Big)^{\alpha_1}\Big(\int_{\mathbb{S}^{d-1}}\big|\varphi(x,\theta)\big|^{2}\mathrm{d}\theta\Big)^{1-\alpha_1}\mathrm{d}x\\
&\quad\le\Bigg(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{S}^{d-1}}\big|\varphi(x,\theta)\big|^{p}\mathrm{d}\theta\Big)^{\frac{\alpha_1}{\alpha_2}}\mathrm{d}x\Bigg)^{\alpha_2}\Bigg(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{S}^{d-1}}\big|\varphi(x,\theta)\big|^{2}\mathrm{d}\theta\Big)^{\frac{1-\alpha_1}{1-\alpha_2}}\mathrm{d}x\Bigg)^{1-\alpha_2}\\
&\quad=\Bigg(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{S}^{d-1}}\big|\varphi(x,\theta)\big|^{p}\mathrm{d}\theta\Big)^{\frac{2}{p}}\mathrm{d}x\Bigg)^{\alpha_2}\Bigg(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{S}^{d-1}}\big|\varphi(x,\theta)\big|^{2}\mathrm{d}\theta\Big)^{\frac{q}{2}}\mathrm{d}x\Bigg)^{1-\alpha_2}\\
&\quad\le C\Big(\int_{\mathbb{R}^d}\|\varphi(x,\cdot)\|^2_{H^s_\theta}\mathrm{d}x\Big)^{\alpha_2}\Bigg(\int_{\mathbb{S}^{d-1}}\Big(\int_{\mathbb{R}^d}\big|\varphi(x,\theta)\big|^{q}\mathrm{d}x\Big)^{\frac{2}{q}}\mathrm{d}\theta\Bigg)^{\frac{q}{2}(1-\alpha_2)}\\
&\quad\le C\Big(\int_{\mathbb{R}^d}\|\varphi(x,\cdot)\|^2_{H^s_\theta}\mathrm{d}x\Big)^{\alpha_2}\Big(\int_{\mathbb{S}^{d-1}}\big\|(-\Delta_x)^{s'/2}\varphi(\cdot,\theta)\big\|^2_{L^2_x}\mathrm{d}\theta\Big)^{\frac{q}{2}(1-\alpha_2)}.
\end{aligned}
$$

We used Minkowski's integral inequality for the second term just after the equality. With the definitions (A.1) it is easy to check that

$$2\alpha_2+q(1-\alpha_2)=r\,,$$

thus, the result follows setting $\alpha=\frac{2\alpha_2}{r}$. ■

**Proposition A.1.2** *The following estimate holds for any $l\ge 0$ and $\epsilon>0$*

$$\|(-\Delta_x)^{l/2}(\phi'_\epsilon u)\|_{L^2_x}\le\epsilon^{-1}C_\phi\|(-\Delta_x)^{l/2}(\phi_{\epsilon/2}u)\|_{L^2_x}+\epsilon^{-1-l}C_\phi\|\phi_{\epsilon/2}u\|_{L^2_x}\,.$$

*Proof.* For $l\in\mathbb{N}$ the result is clear. For a general $l>0$ one writes it as $l=\lfloor l\rfloor+\alpha$ with $\alpha\in(0,1)$. In addition, observe that

$$\phi'_\epsilon u=\epsilon^{-1}\phi'(x/\epsilon)u=\epsilon^{-1}\phi'(x/\epsilon)(\phi_{\epsilon/2}u)\,.$$

Therefore, with the notation $\psi_\epsilon(x)=\phi'(x/\epsilon)$, it follows that

$$
\begin{aligned}
(-\Delta_x)^{\alpha/2}(\phi'_\epsilon u)&=\epsilon^{-1}\psi_\epsilon\,(-\Delta_x)^{\alpha/2}(\phi_{\epsilon/2}u)\\
&\qquad+c_{d,s}\epsilon^{-1}\int_{\mathbb{R}^d}\frac{\psi_\epsilon(x)-\psi_\epsilon(x-y)}{|y|^{d+\alpha}}(\phi_{\epsilon/2}u)(x-y)\mathrm{d}y\,.
\end{aligned}
$$

For the last term in the right it follows that

$$
\begin{aligned}
&\epsilon^{-1}\int_{\mathbb{R}^d}\frac{\psi_\epsilon(x)-\psi_\epsilon(x-y)}{|y|^{d+\alpha}}(\phi_{\epsilon/2}u)(x-y)\mathrm{d}y=\\
&\qquad\epsilon^{-1}\Big(\int_{|y|\le\epsilon}+\int_{|y|>\epsilon}\Big)\frac{\psi_\epsilon(x)-\psi_\epsilon(x-y)}{|y|^{d+\alpha}}(\phi_{\epsilon/2}u)(x-y)\mathrm{d}y\,.
\end{aligned}
$$

For the latter integral one has that

$$\left| \int_{|y|>\epsilon} \frac{\psi_\epsilon(x) - \psi_\epsilon(x-y)}{|y|^{d+\alpha}} (\phi_{\epsilon/2} u)(x-y) \mathrm{d}y \right| \leq 2 \|\phi'\|_\infty \int_{\mathbb{R}^d} \frac{\mathbf{1}_{|y|>\epsilon}}{|y|^{d+\alpha}} \left| (\phi_{\epsilon/2} u)(x-y) \right| \mathrm{d}y \,.$$

Since $\int |y|^{-d-\alpha} \mathbf{1}_{|y|>\epsilon} \sim \epsilon^{-\alpha}$, the $L^2_x$-norm of this term is controlled by $\epsilon^{-\alpha} C_\phi \|\phi_{\epsilon/2} u\|_2$. For the first integral one applies the fact that $\psi_\epsilon$ is Lipschitz with constant $\epsilon^{-1}\|\phi''\|_\infty$. As a consequence,

$$\left| \int_{|y|\leq\epsilon} \frac{\psi_\epsilon(x) - \psi_\epsilon(x-y)}{|y|^{d+\alpha}} (\phi_{\epsilon/2} u)(x-y) \mathrm{d}y \right| \\ \leq \epsilon^{-1} \|\phi''\|_\infty \int_{\mathbb{R}^d} \frac{\mathbf{1}_{|y|>\epsilon}}{|y|^{d+\alpha-1}} \left| (\phi_{\epsilon/2} u)(x-y) \right| \mathrm{d}y \,.$$

Since $\int |y|^{-d-\alpha+1} \mathbf{1}_{|y|\leq\epsilon} \sim \epsilon^{1-\alpha}$, the $L^2_x$-norm of this term is controlled by $\epsilon^{-\alpha} C_\phi \|\phi_{\epsilon/2} u\|_2$. Overall these estimates prove that

$$\|(-\Delta_x)^{\alpha/2}(\phi'_\epsilon u)\|_{L^2_x} \leq \epsilon^{-1} C_\phi \|(-\Delta_x)^{\alpha/2}(\phi_{\epsilon/2} u)\|_{L^2_x} + \epsilon^{-1-\alpha} C_\phi \|\phi_{\epsilon/2} u\|_{L^2_x} \,. \quad \text{(A.2)}$$

For the general case note that $(-\Delta_x)^{l/2} = (-\Delta_x)^{\alpha/2}(-\Delta_x)^{\lfloor l \rfloor/2}$. One can use the product rule to treat the operator $(-\Delta_x)^{\lfloor l \rfloor/2}$, then use estimate (A.2) and interpolation to conclude. ■

**Proposition A.1.3 (Commutator)** *Fix dimension $d \geq 2$, $\alpha \in (0,1)$, and $l, k \geq 0$. Then, for any suitable $\varphi$*

$$\left(1 - \Delta_v\right)^{\alpha/2} \left( \langle v \rangle^{2l} \varphi \right) = \langle v \rangle^{2l} \left(1 - \Delta_v\right)^{\alpha/2} \varphi + \mathcal{R}(\varphi) \,, \quad \text{(A.3)}$$

*where the operator $\mathcal{R}$ satisfies the estimate*

$$\left\| \langle \cdot \rangle^{-2k} \mathcal{R}(\varphi) \right\|_{L^2_v} \leq C_{d,\alpha,l,k} \left\| \langle v \rangle^{\max\{0, 2l-1\} - 2k} \varphi \right\|_{L^2_v} . \quad \text{(A.4)}$$

*Proof.* This type of formulas are common in the literature, thus, we will be brief and leave the computational details to the reader.

Keep in mind the identity $\left(1 - \Delta_v\right)^{\alpha/2} = \left(1 - \Delta_v\right)\left(1 - \Delta_v\right)^{-(1-\alpha/2)}$. Let $\mathcal{B}_{1-\alpha/2}$ be the Bessel kernel associated to $\left(1 - \Delta_v\right)^{-(1-\alpha/2)}$, then

$$\left(1 - \Delta_v\right)^{-(1-\alpha/2)} \left( \langle \cdot \rangle^{2l} \varphi \right) = \langle v \rangle^{2l} \left(1 - \Delta_v\right)^{-(1-\alpha/2)} \varphi \\ + \int_{\mathbb{R}^d} \mathcal{B}_{1-\alpha/2}(v' - v) \left( \langle v' \rangle^{2l} - \langle v \rangle^{2l} \right) \varphi(v') \mathrm{d}v' \,.$$

We define

$$\mathcal{R}_1(\varphi) := (1 - \Delta_v) \int_{\mathbb{R}^d} \mathcal{B}_{1-\alpha/2}(v' - v) \left( \langle v' \rangle^{2l} - \langle v \rangle^{2l} \right) \varphi(v') \mathrm{d}v' \,,$$

and prove that satisfies estimate (A.4). Using Taylor expansion, it follows that

$$\langle v' \rangle^{2l} = \langle v + (v'-v) \rangle^{2l} = \langle v \rangle^{2l} + \nabla \langle v \rangle^{2l} \cdot (v'-v) + (v'-v) \cdot \mathcal{H}(v',v)(v'-v)\,,$$

where the remainder is given by

$$\int_0^1 (1-\tau) \nabla^2 \langle \cdot \rangle^{2l} \Big( \tau v' + (1-\tau) v \Big) \mathrm{d}\tau\,.$$

For the first order term in the Taylor expansion, it holds asymptotically

$$\mathcal{B}_{1-\alpha/2}(v'-v)(v'-v) \sim \Big(1-\Delta_v\Big)^{-1-\frac{1-\alpha}{2}}\,, \quad \text{for} \quad |v| \ll 1\,,$$

and consequently,

$$(1-\Delta_v)\mathcal{B}_{1-\alpha/2}(v'-v)(v'-v) \sim \Big(1-\Delta_v\Big)^{-\frac{1-\alpha}{2}}\,, \quad \text{for} \quad |v| \ll 1\,.$$

Similarly, for the second order term one has the asymptotic behaviour

$$\mathcal{B}_{1-\alpha/2}(v'-v) \times (v'-v) \cdot \mathcal{H}(v',v)(v'-v) \sim \Big(1-\Delta_v\Big)^{-2+\frac{\alpha}{2}}\,, \quad \text{for} \quad |v| \ll 1\,,$$

and therefore, for $|v| \ll 1$ it follows that

$$(1-\Delta_v)\mathcal{B}_{1-\alpha/2}(v'-v) \times (v'-v) \cdot \mathcal{H}(v',v)(v'-v) \sim \Big(1-\Delta_v\Big)^{-1+\frac{\alpha}{2}}\,.$$

Additionally and regarding the weight grow, note that

$$\Big|\nabla \langle v \rangle^{2l}\Big| \sim \langle v \rangle^{2l-1}\,, \qquad \Big|\mathcal{H}(v',v)\Big| \lesssim \langle v' \rangle^{\max\{0,2l-2\}} + \langle v \rangle^{\max\{0,2l-2\}}\,.$$

These observations together with classical asymptotic properties of Bessel kernels, see for instance [22, Section 6.1.2], readily lead to (A.4) for the operator $\mathcal{R}_1$. The rest of the proof follows by the standard product rule of differentiation. So, we conclude the proof. ■