Pontifícia Universidade Católica
do Rio de Janeiro

Anderson Reis de Vargas

Asymptotic Nets with Constant Affine Mean Curvature

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Marcos Craizer

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Abstract

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Discrete Differential Geometry aims to develop a discrete theory which respects fundamental aspects of smooth theory. With this in mind, some results of smooth theory of Affine Geometry are firstly introduced since their discrete counterparts shall be treated *a posteriori*. The first goal of this work is construct a discrete affine structure for nets $q : \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$ with indefinite Blaschke metric and asymptotic parameters. For this purpose, one defines a conormal vector field ν, which satisfies Lelieuvre's equations and it is associated to a real parameter λ; and an affine normal vector field ξ, which defines the cubic form of the net and makes the structure well defined. This structure allows to study, *e.g.*, ruled surfaces with emphasis on improper affine spheres, which are proved to be equiaffinely congruent to the graph of $z = xy + \varphi(x)$, for some real function φ. Moreover, a definition for singularities is proposed in the case of discrete improper affine spheres from the center-chord construction. Another goal here is to propose a definition for an asymptotic net with constant affine mean curvature (CAMC), in a way that encompasses discrete affine minimal surfaces and discrete affine spheres. Discrete affine minimal surfaces receive a beautiful geometrical characterization directly linked to discrete Lie quadrics. This work is completed with the main result about a discrete version of Cayley surfaces, which are ruled improper affine spheres that can be characterized by the induced connection as: an asymptotic net with CAMC is equiaffinely congruent to a Cayley surface if and only if the cubic form C does not vanish and the affine induced connection is parallel, *i.e.*, $\nabla C \equiv 0$.

Keywords

Discrete Affine Minimal Surfaces; Discrete Affine Spheres; Discrete Ruled Improper Affine Spheres; Discrete Improper Affine Spheres with Singularities; Discrete Cayley Surfaces; Discrete Lie Quadrics.

Resumo

Vargas, Anderson R.; Craizer, Marcos. **Redes assintóticas com curvatura afim média constante**. Rio de Janeiro, 2021. 103p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

A Geometria Diferencial Discreta tem por objetivo desenvolver uma teoria discreta que respeite os aspectos fundamentais da teoria suave. Com isto em mente, são apresentados incialmente resultados da teoria suave da Geometria Afim que terão suas versões discretas tratadas *a posteriori*. O primeiro objetivo deste trabalho é construir uma estrutura afim discreta para as redes assintóticas $q : \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$, com métrica de Blaschke indefinida e parâmetros assintóticos. Com este intuito, são definidos um campo conormal ν, que satisfaz as equações de Lelieuvre e está associado a um parâmetro real λ, e um normal afim ξ que define a forma cúbica da rede e torna a estrutura bem definida. Esta estrutura permite, por exemplo, o estudo das superfícies regradas, com ênfase nas esferas afins impróprias, as quais são congruentes equiafins ao gráfico de $z = xy + \varphi(x)$ para alguma função real φ. Além disso, propõe-se uma definição para as singularidades no caso das esferas afins impróprias discretas a partir da construção centro-corda. Outro objetivo deste trabalho é propor uma definição para as superfícies afins discretas com curvatura afim média constante (CAMC), de forma que englobe as superfícies afins mínimas e as esferas afins. As superfícies afins mínimas discretas recebem uma caracterização geométrica bastante interessane e ligada diretamente às quádricas de Lie discretas. O trabalho se completa com o principal resultado, referente à versão discreta das superfícies de Cayley, esferas afins impróprias regradas caracterizadas a partir da conexão afim induzida: uma rede assintótica com CAMC é congruente equiafim à uma superfície de Cayley se, e somente se, a forma cúbica C é não nula e a conexão afim induzida é paralela, ou seja, $\nabla C \equiv 0$.

Palavras-chave

Superfícies Mínimas Afins Discretas; Esferas Afins Discretas; Esferas Afins Impróprias Regradas Discretas; Esferas Afins Impróprias Discretas com Singularidades; Superfícies de Cayley Discretas; Quádricas de Lie Discretas.

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Il y a deux types de lecteurs: ceux qui se perdent dans ce qu'ils lisent, et ceux qui s'y trouvent.

Paul Valéry.

1
Introduction

Usually, the discretization of parameterized surfaces within discrete differential geometry (DDG) leads to quadrilateral nets, also called quadrilateral meshes. Compared with, *e.g.*, discrete triangulated surfaces, quadrilateral nets do not only discretize smooth surfaces understood as sample sets, but also reflect the combinatorial structure of the parameterizations to be discretized. Quadrilateral nets has been extensively studied and one can focus in those with special geometric properties. One of the most fundamental examples is the discretization of conjugate parameterizations by quadrilateral nets with planar faces, which is not the case in this work, by the way. Discretizing more specific conjugate parameterizations yields planar quadrilateral nets with additional properties, as the discretizations of curvature line parameterizations, that have led to the notions of circular and conical nets, as one can see in Bobenko [3] and Liu at al. [22]. Or the discretizations of Koenigs net and isothermic surfaces, both treated in Bobenko and Suris [5], Doliwa [13] and Müller [29].

Conjugate nets are good models for surfaces with definite metric, but we are interested here in those surfaces with indefinite metric and this leads us to asymptotic nets, wherein instead of planar faces there are planar crosses, that is, the four edges with a common vertex are coplanar. It turns out that asymptotic nets receive this name because it is a natural discretization for surfaces parameterized along asymptotic lines, which is always possible when the Gaussian curvature is negative. Plenty of work was done by using asymptotic parameterizations, as one can see in the long list of references of this thesis [4], [9], [10], [11], [12], [16], [18], [25], [26], [27], [28], only to cite what was relevant for our work.

It is natural to look for applications of asymptotic nets in the context of Computer Aided Geometric Design and Architectural Geometry, for example. The latter is an emerging field of Applied Mathematics, which provides the architecture community with sophisticated geometric knowledge to tackle diverse problems. Focussing also on important aspects such as efficient manufacturing, providing intuitive control of available degrees of freedom, and similar issues, many results in DDG have already been applied in an architectural context, as proposed by Pottmann et al. [33] or mentioned by Käferböck and Pottmann [18]. The first paper is beautifully constructed by the architecture perspective, wherein mathematics appears only as a supporting theory, and it shows stunning images of real projects.

Bobenko and Schief [2, 1999] have given the first consistent definition for discrete affine spheres, both for definite or indefinite metrics. In 2003, Matsuura and Urakawa [26] have done a similar construction in the context of discrete improper affine spheres. Some years later, Craizer, Anciaux and Lewiner [9] introduced a discrete analog of the smooth Weierstrass representation in the indefinite case, giving rise to explicit parameterizations of quadrilateral surfaces in discrete asymptotic coordinates that they called discrete affine minimal surfaces. Their work brought some discrete elements up, like metric, normal affine vector field and cubic form, and these allowed them to have a consistent discrete structure. In the same article they also proposed an interpolation by hyperbolic paraboloids for the skew quadrilaterals, which was extended by Käferböck and Pottman [18, 2013] and was called bilinear patches, but mentioned as discrete Lie quadrics.

After that we give here an affine structure for asymptotic nets with indefinite Blaschke metric and asymptotic parameters that go beyond discrete minimal surfaces. From the conormal vector field, which satisfies Lelieuvre's equations and it is associated to a real positive parameter, we define an affine normal vector field and the cubic form of the net. These elements allow us to set structural equations for the net that added up to the compatibility equations turn all the affine structure consistent.

We also show a 1-parameter family of hyperboloid interpolators for the skew quadrilaterals of the net and how to juxtapose them such that the composed surface has a tangent plane at any point. A hyperboloid is a doubly ruled quadric, *i.e.*, a one-sheeted hyperboloid or a hyperbolic paraboloid from the affine viewpoint. Pieces of those hyperboloids are inserted into the skew quadrilaterals of an asymptotic net such that this interpolator is bounded by the edges of the corresponding quadrilateral. In particular, the edges of a supporting quadrilateral are asymptotic lines of the inserted hyperboloid. This parameterization is very similar to the discretization of curvature line parameterized surfaces by cyclidic nets, which were introduced by Bobenko and Huhnen-Venedey [6].

In their paper Käferböck and Pottman [18] remarked that bilinear patches can be seen as discrete Lie quadrics. In the smooth case one can see that a Lie quadric is intrinsically connected to a surface, since at any point there is a unique Lie quadric, accordingly to Lane [21]. Moreover, a surface has constant affine mean curvature if and only all Lie quadrics also has constant affine mean curvature, and it is minimal if and only if each Lie quadric is a hyperbolic paraboloid.

Note that in the discrete case we have set a 1-parameter family of interpolators at each quadrangle, which allow us to identify all the nets that can have the same parameter for all quadrangles. With this in mind, like Käferböck and Pottman, we called our 1-parameter family of interpolators discrete Lie quadrics and stated when the net has constant affine mean curvature, exactly when all discrete Lie quadrics has the same constant affine mean curvature. That said, we named them nets with CAMC and prove that the definition is well settled.

As examples of nets with CAMC we show both class of discrete affine minimal surfaces and class of affine spheres, which has CAMC in the smooth case. The first class emerges from the work of Craizer, Anciaux and Lewiner [9] as a particular case when the affine mean curvature is zero, which also includes discrete improper affine spheres. Smooth affine spheres were extensively studied and classified, as we can see in Magid and Ryan [24] or Simon [35]. Accordingly to Bobenko and Schief [2] discrete affine spheres has CAMC, so we only have to show that our definition accounts this class too, and fortunately that is done successfully.

A geometric way to characterize smooth affine minimal surfaces was given by Blaschke [1] by stating that a surface is affine minimal if and only if along each asymptotic curve the corresponding second asymptotic directions are parallel to a plane. Käferböck and Pottman [18] proved that a similar characterization can be given in the discrete case, and we gave here another proof to this theorem, which states that a discrete surface is minimal if and only if both assertions are true: the edges of a horizontal (vertical) strip are parallel to a plane and each discrete Lie quadric is a hyperbolic paraboloid.

One class of surfaces that rings a bell when we talk about asymptotic parameters is that of ruled surfaces, since in such a parameterization at least one direction is always a straight line. Martínez and Mílan [25] presented a study about ruled surfaces with flat affine metric, since in authors words, it is the best known class of affine minimal surfaces with indefinite metric. Nomizu and Sasaki [31] dedicate two sections of Chapter 3 to ruled affine spheres and, as an important class, it appears the Cayley surfaces, and they devote an entire section to this class of cubic ruled surfaces. They prove that ruled improper affine spheres are equiaffinely congruent to the graph of $z = xy + \varphi(x)$, for some real function φ and one of the most important case among them is the Cayley Surface, when $\varphi(x) = -\frac{x^3}{3}$. They also prove that Cayley surfaces – the set of all surfaces that are equiaffinely congruent to the Cayley surface – compose one of the six classes of nondegenerate surfaces of $\mathbb{R}^3$ that are homogeneous under equiaffine transformations. Moreover, they prove that a nondegenerate surface is affinely congruent to the Cayley surface if and only if the cubic form C is not zero and is parallel relative to the connection ∇, *i.e.*, $\nabla C \equiv 0$.

We found one reference wherein the author mention discrete Cayley surfaces, namely Matsuura and Urakawa [26], but they do not build up any arguments about their parameterization. So we propose a different one and characterize them in a similar way to the smooth case, *i.e.*, an asymptotic net with CAMC is equiaffinely congruent to a Cayley surface if and only if the cubic form C does not vanish and the affine induced connection is parallel, *i.e.*, $\nabla C \equiv 0$. Note that one of the hypothesis is to be a net with CAMC, which according to our point of view ratifies our definition of discrete surfaces with CAMC. Another important remark is that in a certain way we are defining a discrete connection, although we do not bring this subject up.

Improper affine maps that are not convex can be obtained from a pair of planar curves, as it was made by Craizer [12], and the planar construction can be based on the work of Niethammer et al. [30]. Set by x the mid-point of a chord and by z the area of the planar region bounded by this chord, the two curves and another fixed chord, we show that (x, z) is an improper affine sphere that is called the generalized area distance and we call this approach a centre-chord construction.

A recent paper from last year, but unpublished yet, by Kobayashi and Matsuura [19], shows a construction of improper affine spheres from two curves by using loop groups, which is completely different of our approach. We make use of a discrete centre-chord construction to show that a discrete ruled improper affine sphere is affine congruent to the graph of $z = xy + \varphi(x)$, as in the smooth case. Moreover, we also use this tool to define discrete improper affine spheres with singularities, which is a completely original approach, since discrete singularities make part of a new research subject and suffer from lack of references.

In the smooth case Singularities is a well known field and largely studied by geometers. So we will give here some references that influenced our project. First of all we need to refer to Kokubu at al. [20], since they provide an extensive treatise about singularities, with a general classification of singularities and the criteria for singular points. Moreover, they proved a fundamental relation between the singularities at the surface and the Wigner caustics of the flat fronts. Craizer, Silva and Teixeira [11] studied the singular set of non-convex improper affine maps and they proved that generically it consists of cuspidal edges and swallowtails. Ishikawa and Machida [17] provide the generic classification of singularities of improper affine spheres, surfaces of constant Gaussian curvature and for developable surfaces. This study on improper affine maps was also done by Mílan [27], who added an approach for isolated singularities. All these studies are connected to the Wigner caustics or the midpoint tangent locus (MPTL), which can be found in the work of Giblin [14].

We have only one reference about singularities as a discrete subject, namely Rossman and Yasumoto [34] from 2018, which reveals how new is the subject and how we can make an important contribution to it.

We already said that we have done a discrete centre-chord construction. From it we propose a definition for the discrete midpoint tangent locus, which we call DMPTL, and as in the smooth case it can have cusps. We show that the discrete improper affine sphere created from a pair of curves can have singularities and they are directly linked to the DMPTL associated to these curves. We prove that the edges of the DMPTL provide cuspidal edges at the net, whilst a cusp is associated to a swallowtail vertex at the discrete surface, and the converse is also true. This result reflects exactly the same of the smooth theory, which is an evidence of consistency.

Let us go back a couple of pages and bring here the discussion of discrete Lie quadrics. Note that we interpolated all quadrangles of the net by discrete Lie quadrics such that at each point of the compound surface has a well defined tangent plane that varies continuously when moving from one patch to an adjacent one. Note that two adjacent patches may form a cuspidal edge since the juxtaposition only predicts a coinciding tangent plane. As we show, it is possible to have a swallowtail point in one vertex between two cuspidal edges, without interfere in the interpolation discussion.

All we have accomplished here was mentioned above. So let us now describe the organization of this thesis.

In chapter 2 we present definitions and results from the smooth theory that we judge essential for the subsequent chapters. At first we turn to the affine differential structure, which is followed by a discussion on the definition of indefinite affine spheres. After that we treat the minimal surfaces and give some examples, which is also done for the basic hyperboloid. Then we appeal to the importance of Lie quadrics in the subject of surfaces with constant affine mean curvature. Thereafter we use the centre-chord construction to prove that the generalized area distance is an indefinite improper affine sphere and all indefinite improper affine sphere is the generalized area distance of a pair of curves. Moreover, we define the singularity set and present the relation between the MPTL and the singularities at the surface. The end of the chapter is dedicated to Cayley surfaces and the proof of their characterization.

The third chapter is dedicated to develop the discrete theory of asymptotic nets. We define the discrete conormal and normal vector fields, the coefficients of the cubic form and we give the structural equations, as well as the compatibility ones, finishing it with the proof of the consistency of the affine structure. We also show how to interpolate a quadrangle by a hyperboloid and what comes from the juxtaposition of these interpolators.

Chapter 4 is dedicated to ruled improper affine spheres and singularities at improper affine spheres. We do a discrete centre-chord construction from a pair of discrete curves in the plane and we show that this leads to the discrete generalized area distance, which is proved to be a discrete improper affine sphere. From this result it emerges the equation form for all ruled improper affine spheres. After that we propose a definition for the DMPTL and the main theorem shows a relation between the singularities of the discrete generalized area distance and it: all the edges and cusps of the DMPTL are associated to cuspidal edges and swallowtail points of the net, respectively. We also propose a definition for cuspidal edges and swallowtail points in the case of improper affine spheres, since there is none so far.

The follow chapter shows the definition of nets with constant affine mean curvature (CAMC) from the discrete Lie quadrics, its consistency and how this class of nets encompasses the discrete minimal surfaces and the discrete affine spheres. We show that all discrete affine minimal surfaces are a particular case of CAMC with $H = 0$, how this simplifies all the affine structure of chapter 3, and how this matches with the work of Craizer, Anciaux and Lewiner [10]. Thereon we give another proof to the geometric characterization of discrete minimal surfaces in terms of discrete Lie quadrics. At the end we frame the class of affine spheres after Bobenko and Schief [2] as a class of nets with CAMC.

In chapter 6 we treat the case of discrete Cayley surfaces, which are ruled improper affine spheres. Besides the definition, we give a characterization for such discrete surfaces, which is one of the main results of this thesis.

The last chapter brings the final considerations about the work developed here and the impressions of this humble student for future works.

2
Preliminaries on smooth affine theory

The main goal of this chapter is to show all the smooth theory needed to the development of the discrete theory that compose all the subsequent chapters. Some details will be left out when judged unnecessary, whereas others will be seen in an opposite perspective, since their importance for the main discrete results. In any case, everything here is well described and largely studied in the literature, which can be consulted through out the bibliographic references.

2.1
Affine differential structure

Consider[1] a parameterized smooth surface $q : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, where U is an open subset of the plane and denote

$$L(u,v) = [\, q_u, q_v, q_{uu}],$$
$$M(u,v) = [\, q_u, q_v, q_{uv}],$$
$$N(u,v) = [\, q_u, q_v, q_{vv}].$$

The surface is non-degenerate if $LN - M^2 \neq 0$, and in this case, the Berwald-Blaschke metric is defined by

$$ds^2 = \frac{1}{|LN - M^2|^{1/4}} \left(L\, du^2 + 2M\, du\, dv + N\, dv^2 \right).$$

If $LN - M^2 > 0$, the metric is called *definite* and the surface is locally convex. On the contrary, if $LN - M^2 < 0$, the metric is called *indefinite* and the surface is locally hyperbolic, *i.e.*, the tangent plane crosses the surface.

From now on we shall assume that the affine surface has indefinite metric. Besides that, under a change of coordinates, we may consider $L = N = 0$. Such coordinates are known as *asymptotic*. In this case, it is possible to take $M > 0$, and the metric takes the form $ds^2 = 2\Omega\, du\, dv$, where $\Omega^2 = M$. Without loss of generality, we shall take $\Omega > 0$.

[1]All details of this section can be found in Buchin [7] and Nomizu-Sasaki [31].

Since $L = N = 0$, we know that q_{uu} and q_{vv} live in the tangent space and can be written in terms of the basis $\{q_u, q_v\}$. In addition, with the expression of the transversal field q_{uv}, we obtain the Gauss equations for the q map:

$$\begin{aligned} q_{uu} &= \frac{\Omega_u}{\Omega}\, q_u + \frac{A}{\Omega}\, q_v \\ q_{vv} &= \frac{B}{\Omega}\, q_u + \frac{\Omega_v}{\Omega}\, q_v \\ q_{uv} &= \Omega\, \xi \end{aligned} \tag{2.1}$$

where $A = [q_u, q_{uu}, \xi]$ and $B = [q_v, q_{vv}, \xi]$ are the coefficients of the affine cubic form $A\, du^3 + B\, dv^3$.

ξ is called the *affine normal* vector field. The direction of the affine normal has a simple geometrical meaning in the hyperbolic case: consider an infinitesimal quadrangle composed of asymptotic lines and build two planes, each one being parallel to a pair of opposite edges of the quadrangle, then the affine normal is parallel to the line of intersection of these two planes.

The shape operator in basis $\{q_u, q_v\}$ is given by

$$\begin{aligned} \xi_u &= -H\, q_u + \frac{A_v}{\Omega^2}\, q_v \\ \xi_v &= \frac{B_u}{\Omega^2}\, q_u - H\, q_v. \end{aligned} \tag{2.2}$$

Note that the trace of the shape operator is $-2H$, where H is the affine mean curvature.

By comparing the mixed third derivatives of q we obtain a compatibility equation for the parameters Ω, A, B and H:

$$H = \frac{\Omega_u \Omega_v - \Omega \Omega_{uv} - AB}{\Omega^3}. \tag{2.3}$$

And comparing the mixed two derivatives of ξ, it is obtained two other compatibility equations for the same parameters,

$$\begin{aligned} H_u &= \frac{AB_u}{\Omega^3} - \frac{1}{\Omega}\left(\frac{A_v}{\Omega}\right)_v \\ H_v &= \frac{BA_v}{\Omega^3} - \frac{1}{\Omega}\left(\frac{B_u}{\Omega}\right)_u. \end{aligned} \tag{2.4}$$

Conversely, given Ω, A, B and H satisfying compatibility equations (2.3) and (2.4), there exist a surface $q(u, v)$ according to the equations (2.1).

The vector field $\nu = \frac{1}{\Omega}\, q_u \times q_v$ is called *conormal* and satisfies Lelieuvre's equations

$$\begin{aligned} q_u &= \quad \nu \times \nu_u \\ q_v &= -\nu \times \nu_v \end{aligned} \tag{2.5}$$

It also satisfies $\Delta(\nu) = -H\nu$, where $\Delta(\nu)$ denotes the laplacian of ν with respect to the Blaschke metric. In asymptotic parameters, $\Delta(\nu) = \Omega^{-1}\,\nu_{uv}$. Moreover, the affine normal vector field $\xi = \Omega^{-1}\,q_{uv}$ satisfies $\nu \cdot \xi = 1$.

One last remark is the concept of equiaffine transformations of $\mathbb{R}^3$, defined as a map with determinant 1. The set off all such transformations is denoted by $SL(3,\mathbb{R})$. And the most important, all the quantities in this section are invariant under equiaffine transformations of $\mathbb{R}^3$.

2.2
Indefinite affine spheres

In their paper Bobenko and Schief [2] use purely geometric techniques to discretize affine spheres with definite and indefinite metric, by discussing the relations between affine spheres and their duals. We are interested only in the indefinite case, so let us take a brief look in how they are defined and in their duality relations.

Definition 2.2.1 *A non-degenerate surface in* $\mathbb{R}^3$ *is called an* affine sphere *if all affine normals intersect at a point. If this point is not infinite, it may be chosen as the origin of* $\mathbb{R}^3$ *in a way that* $\xi = H\,q$*, where* H *is the affine mean curvature.*

It is well known that the affine mean curvature of the affine sphere is constant. That said, Bobenko and Schief [2] chose to work only with proper affine spheres, that is, they assume that $H \neq 0$. Moreover, they make a distinction between the cases of a definite Blaschke metric (when the surface is convex or $K > 0$) and of an indefinite Blaschke metric (when the surface is hyperbolic or $K < 0$).

A simple way to obtain indefinite affine spheres is by using the conormal vector field ν, as shown by the following theorem. Its proof can be found in the already mentioned paper [2, p.265].

Theorem 2.2.2 (Duality relations for indefinite affine spheres) *Indefinite affine spheres and their duals are equivalently described by the Lelieuvre's equations*

$$\begin{aligned} q_u &= \nu \times \nu_u, \quad \nu_u = q_u \times q, \\ q_v &= \nu_v \times \nu, \quad \nu_v = q \times q_v, \end{aligned} \tag{2.6}$$

which imply that $q \cdot \nu = 1$.

Since Bobenko and Schief does not pay any attention to improper affine spheres, we think that it is important to have a simpler definition stated. So the definition (2.2.1) can be split in two depending on the value of H, as one can see in Simon [35]:

Definition 2.2.3 *A surface in* $\mathbb{R}^3$ *is a* proper affine sphere *if* $H \neq 0$ *and the position vector with respect to a fixed point (the centre of the affine sphere) is parallel to the direction of the affine normal* ξ*.*

Note that this is the same definition stated by Bobenko and Schief, but with "proper" as an addendum.

Definition 2.2.4 *A surface in* $\mathbb{R}^3$ *is an* improper affine sphere *if the direction of the affine normal* ξ *is constant.*

Observe that this definition with the equations (2.2) allow us to conclude that $H = 0$ in the case of an improper affine sphere.

Proposition 2.2.5 *A surface in* $\mathbb{R}^3$ *is an affine sphere if and only if* $A_v = 0$ *and* $B_u = 0$*, that is,* A *and* B *depend only on* u *and* v*, respectively.*

Proof.
It follows directly from the two last definitions (2.2.3), (2.2.4) and the shape operator (2.2). □

2.3 Affine minimal surfaces

Definition 2.3.1 *A surface is called* affine minimal *if its affine mean curvatures* H *vanishes, or equivalently, if its conormal vector field satisfies the equation* $\nu_{uv} = 0$*.*

The last part of this definition gives a straightforward resolution, since it holds if and only if $\nu(u, v)$ takes the form $\nu(u, v) = \nu^1(u) + \nu^2(v)$, where ν^1 and ν^2 are two real functions of one variable. That said, from Lelieuvre's equation (2.5) it is possible to get an immersion q which is a parameterization in asymptotic coordinates of an affine minimal surface.

Note that by definition (2.2.4), all improper affine spheres are minimal surfaces.

Example 1.
Consider the conormal vector field given by $\nu(u, v) = (u, v, u^2 + v^2)$. Straightforward calculations give us the immersion associated:

$$q(u, v) = \left(u^2 v - \frac{v^3}{3}, uv^2 - \frac{u^3}{3}, -uv\right),$$

hence

$$\Omega(u,v) = u^2 + v^2, \qquad \xi(u,v) = \frac{1}{u^2+v^2}(2u, 2v, -1),$$

$$A(u,v) = \frac{-2v}{u^2+v^2} \qquad \text{and} \qquad B(u,v) = \frac{2u}{u^2+v^2}.$$

Then, by equation (2.3) we have $H = 0$ and the surface is minimal, as expected. See Figure (2.1).

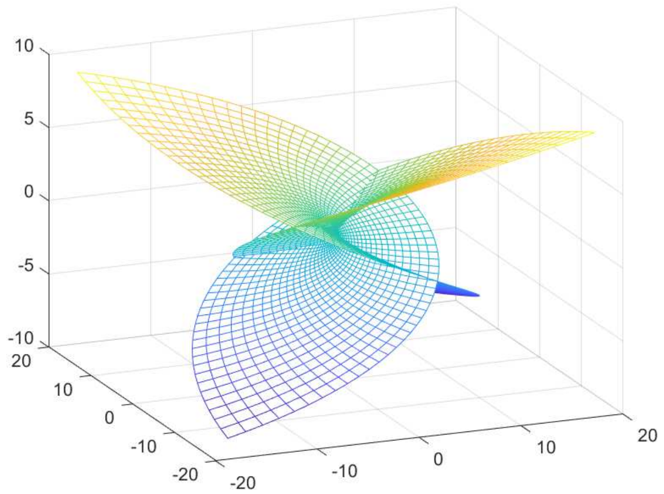


Figure 2.1: First example of a minimal surface.

Example 2.
Consider the conormal vector field given by $\nu(u,v) = \left(-\frac{v^2}{2}, \frac{v-u}{2}, 1\right)$. Straightforward calculations give us the immersion associated:

$$q(u,v) = \left(\frac{u+v}{2}, \frac{v^2}{2}, \frac{uv^2}{4} - \frac{v^3}{12}\right),$$

hence

$$\Omega(u,v) = \frac{v}{2}, \quad \text{and} \quad \xi(u,v) = (0,0,1).$$

Then, we have to take $v > 0$ and the surface is an improper affine sphere. See Figure (2.2).

Besides the definition of an affine minimal surface, Blaschke [1, p.180] gave a geometric characterization for it.

Theorem 2.3.2 *A surface is affine minimal if and only if along each asymptotic curve the corresponding second asymptotic directions are parallel to a plane.*

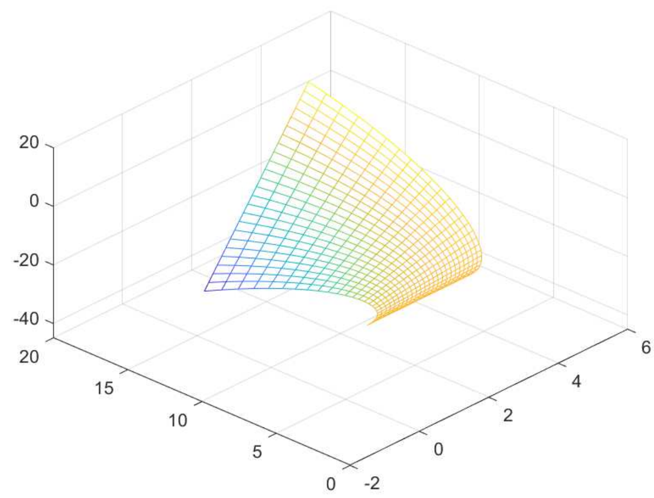


Figure 2.2: Example of a minimal surface that is also an improper affine sphere.

2.4 The basic hyperboloid

Consider the hyperboloid $\mathcal{H}_a$ given by the equation

$$z + az^2 = xy \tag{2.7}$$

parameterized by asymptotic coordinates (u, v),

$$\psi(u, v) = \frac{1}{1 - auv}(u, v, uv). \tag{2.8}$$

The equation (2.8) gives $L = N = 0$ (as expected for asymptotic coordinates) and $M = (1 - auv)^{-4}$. Thus,

$$\Omega(u, v) = \frac{1}{(1 - auv)^2},$$

$$\nu(u, v) = \frac{1}{\Omega}\,\psi_u \times \psi_v = \frac{1}{1 - auv}(-v, -u, 1 + auv), \tag{2.9}$$

$$\xi(u, v) = \frac{1}{\Omega}\psi_{uv} = \frac{1}{1 - auv}(2au, 2av, 1 + auv).$$

Note that

$$\xi - (0, 0, 1) = 2\,a\psi,$$

which means that the hyperboloid $\mathcal{H}_a$ is an *affine sphere* with center at the point $(0, 0, -\frac{1}{2a})$ and affine mean curvature $H = 2a$. Furthermore, according to the conormal vector field, it is easy to see that

$$\nu_{uv} = 2\,a\Omega\nu.$$

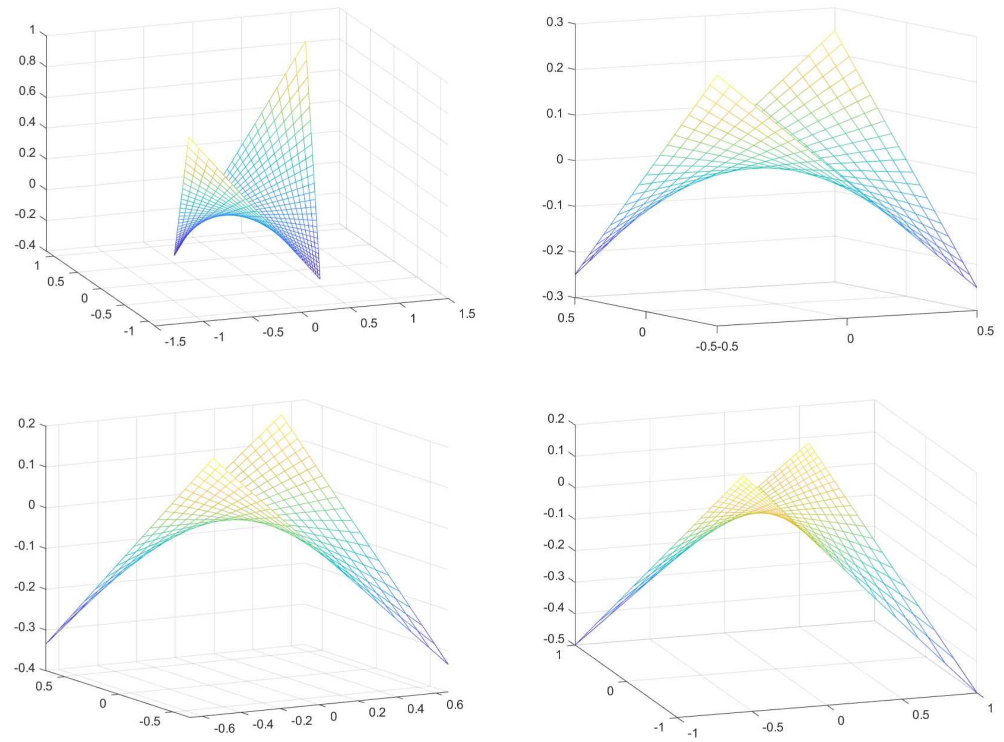


Figure 2.3: Four examples of the basic hyperboloid for different values of a: $a = 0.8$ on the upper left, $a = 0$ on the upper right, $a = -1$ on the lower left and $a = -2$ on the lower right.

2.5 Lie quadrics

Accordingly to Lane [21, p.144], the *quadric of Lie* at the point P_x of the surface S can be defined as follows. Consider two neighbouring points of P_x, P_1 and P_2, on one of the asymptotic curves C through P_x. Let us assume that S is parameterized in asymptotic coordinates $x(u,v)$, that is, we can set $P_1 = x(u+\Delta u, v)$, $P_2 = x(u-\Delta u, v)$ and the curve C is the u-direction. At each of these three points let us draw the tangent of the asymptotic curve in v-direction, *i.e.*, $x_v(u,v)$, $x_v(u+\Delta u, v)$ and $x_v(u-\Delta u, v)$. These three asymptotic tangents determine a quadric surface, and the limit of this quadric when P_1 and P_2 approach P_x, that is, Δu approaches zero, is a quadric surface called *the Lie quadric* at the point P_x of the surface S.

Let us assume that $x(0,0) = (0,0,0)$ and write the expansion

$$x(u,v) = x_u u + x_v v + \frac{1}{2}\left(x_{uu}u^2 + 2x_{uv}uv + x_{vv}v^2\right) + \ldots,$$

where $x_u = x_u(0,0)$, $x_v = x_v(0,0)$, $x_{uu} = x_{uu}(0,0)$, $x_{uv} = x_{uv}(0,0)$ and $x_{vv} = x_{vv}(0,0)$.

Then $X = x(u,0)$, a point in the u-asymptotic line close to $(0,0)$, is

expanded as
$$X = x_u u + \frac{1}{2} x_{uu} u^2 + \dots \tag{2.10}$$
and the v-asymptotic line at this point can be written as
$$X_v = x_v(u,0) = x_v + x_{uv} u + \frac{1}{2} x_{uuv} u^2 + \dots . \tag{2.11}$$

If Y is an arbitrary point in the v-asymptotic line passing through X, then it can be written as a linear combination of X and X_v, $Y = X + kX_v$, for some k, as we can see in next Lemma.

Lemma 2.5.1 *We can write*
$$Y = x_1 x_u + x_2 x_v + x_3 x_{uv},$$
where
$$\begin{aligned} x_1 &= u + \frac{1}{2}[\theta_u + k(\beta\gamma + \theta_{uv})]u^2 + O(3) \\ x_2 &= k + O(2) \\ x_3 &= ku + \frac{1}{2} k \theta_u u^2 + O(3) \end{aligned}$$

Proof.
Lane writes Gauss equations (2.1) as
$$\begin{cases} x_{uu} = \theta_u x_u + \beta x_v \\ x_{vv} = \gamma x_u + \theta_v x_v \end{cases} \tag{2.12}$$
where $\theta = \log(\Omega)$, $\beta = \dfrac{A}{\Omega}$, $\gamma = \dfrac{B}{\Omega}$ and Ω is our old friend from Section 2.1.

Replacing these formulae for x_{uu} and x_{vv} in equations (2.10) and (2.11), we have
$$X = x_u u + \frac{1}{2}\theta_u x_u u^2 + \frac{1}{2}\beta x_v u^2 + \dots$$
and
$$X_v = x_v + x_{uv} u + \frac{1}{2}[(\theta_{uv} + \beta\gamma) x_u + (\beta_v + \beta\theta_v) x_v + \theta_u x_{uv}] u^2 + \dots$$
Comparing the coefficients of x_u, x_v and x_{uv}, we come to the conclusion. $\square$

Proposition 2.5.2 *The local equation of the Lie quadric at a point $x(0,0)$ of a surface S is*
$$x_1 x_2 - x_3 - \frac{1}{2}(\beta\gamma + \theta_{uv}) x_3^2 = 0. \tag{2.13}$$

Proof.
Equation (2.13) follows the above Lemma up to order 3. $\square$

Remark that there is a symmetry regarding to x_1 and x_2, which shows that the Lie quadric remains the same if the two families of asymptotic curves are interchanged. Note also that the basic hyperboloid (2.7) is an example of a Lie quadric.

2.5.1 Affine mean curvature of the Lie quadric

The affine mean curvature H can be written as $H = S - J$, where

$$S = -\frac{1}{\Omega}\left(\log \Omega\right)_{uv} \quad \text{and} \quad J = \frac{AB}{\Omega^3}.$$

Then, for the above Lie quadric wherein $x(0,0) = (0,0,0)$, we have

$$S = -\frac{\theta_{uv}}{\Omega} \quad \text{and} \quad J = \frac{\beta\gamma}{\Omega}$$

and we can write

$$H = -\frac{1}{\Omega}(\beta\gamma + \theta_{uv}).$$

Consider the linear map T setting

$$(1,0,0) \mapsto x_u, \quad (0,1,0) \mapsto x_v, \quad (0,0,1) \mapsto x_{uv}.$$

Then $\det(T) = \Omega^2$. We conclude that the affine mean curvature of the Lie quadric is

$$H(LieQuadric) = -\frac{\beta\gamma + \theta_{uv}}{\sqrt{\Omega^2}} = \frac{H\Omega}{\Omega} = H,$$

the affine mean curvature of the surface at $(0,0,0)$. In other words, a surface has constant affine mean curvature if and only if each Lie quadric of the surface has also constant affine mean curvature, and both curvatures coincide.

2.6 The generalized area distance of a pair of curves

Craizer, Teixeira and Silva [11] have shown that the generalized area distance of a pair of curves is an indefinite improper affine sphere with singularities, and the converse is also true. Moreover, the singularity set of the improper affine sphere corresponds to the area evolute of the pair of curves, also called midpoint parallel tangent locus [14].

In this section we aim to show the definitions and the geometric construction that lead us to those results.

2.6.1
Centre-chord construction

Consider two planar curves $\alpha : I \longrightarrow \mathbb{R}^2$ and $\beta : J \longrightarrow \mathbb{R}^2$, where $I, J \subset \mathbb{R}$. Denote by $f(s,t)$ the midpoint of the chord $\overline{\alpha(s)\beta(t)}$, hence

$$f(s,t) = \frac{1}{2}(\alpha(s) + \beta(t)) = \frac{1}{2}(\alpha_1(s) + \beta_1(t), \alpha_2(s) + \beta_2(t)).$$

Take

$$g(s,t) = \frac{1}{2}(\beta(t) - \alpha(s)) = \frac{1}{2}(\beta_1(t) - \alpha_1(s), \beta_2(t) - \alpha_2(s)),$$

and

$$n(s,t) = g(s,t)^{\perp} = \frac{1}{2}(\alpha_2(s) - \beta_2(t), \beta_1(t) - \alpha_1(s)),$$

where the symbol $\perp$ means anticlockwise rotation of ninety degrees. So n is orthogonal to the chord $\overline{\alpha(s)\beta(t)}$ with half of its length.

Define $z(s,t)$ by the relation $\nabla z = n$, where the gradient is taken with respect to f, which means that z satisfies $z_s = n \cdot f_s$ and $z_t = n \cdot f_t$.

Lemma 2.6.1 *The function z is well defined up to a constant.*

Proof.
Let us assume the existence of z. Then

$$z_s = \frac{1}{4}\left((\alpha_2(s) - \beta_2(t))\alpha_1'(s) + (\beta_1(t) - \alpha_1(s))\alpha_2'(s)\right) = \frac{1}{4}\left[\beta(t) - \alpha(s), \alpha'(s)\right]$$

and

$$z_t = \frac{1}{4}\left((\alpha_2(s) - \beta_2(t))\beta_1'(t) + (\beta_1(t) - \alpha_1(s))\beta_2'(t)\right) = \frac{1}{4}\left[\beta(t) - \alpha(s), \beta'(t)\right].$$

Moreover, z is well defined if and only if $z_{st} = z_{ts}$. But

$$z_{st} = \frac{1}{4}[\beta'(t), \alpha'(s)] = z_{ts},$$

and the Lemma is proved. □

By Green's Theorem we know that the area of a region bounded by a closed curve γ is given by

$$A = \int_{\gamma} F \cdot dr,$$

where $F = \frac{1}{2}(-y, x)$ is a vector field in the plane.

Let us fix a chord L_0 connecting $\alpha(0)$ with $\beta(0)$ and denote by C_0 the line integral along L_0. Now consider γ the path given by L_0, $\beta([0,t])$, $\overline{\beta(t)\alpha(s)}$

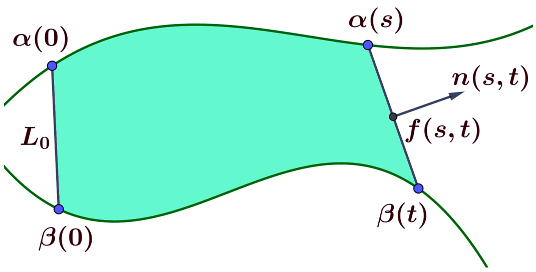


Figure 2.4: Generalized area distance of the pair of curves (α, β).

and $\alpha([s,0])$ as in Fig.(2.4). Then the area $A(s,t)$ of the region bounded by γ is given by

$$2A(s,t) = C_0 + \int_0^t [\beta(t), \beta'(t)]dt + \int_0^s [\alpha'(s), \alpha(s)]ds + [\beta(t), \alpha(s)].$$

Thus

$$2A_s = [\alpha'(s), \alpha(s)] + [\beta(t), \alpha'(s)] = [\alpha'(s), \alpha(s) - \beta(t)]$$

and

$$2A_t = [\beta(t), \beta'(t)] + [\beta'(t), \alpha(s)] = [\beta'(t), \alpha(s) - \beta(t)].$$

Then we conclude that

$$z(s,t) = \frac{A(s,t)}{2} + C,$$

for some constant C.

Definition 2.6.2 *The map $q : I \times J \longrightarrow \mathbb{R}^3$ given by*

$$q(s,t) \longrightarrow (f(s,t), z(s,t)),$$

where f and z are the maps defined above, is called the generalized area distance *of the pair of curves $(\alpha(s), \beta(t))$.*

Proposition 2.6.3 *The generalized area distance map is an indefinite improper affine sphere, and conversely, all indefinite improper affine sphere is the generalized area distance of a pair of planar curves.*

Proof.
Note that

$$q_s = (f_s, z_s) = \frac{1}{2}(\alpha'(s), -[\alpha'(s), y(s,t)]),$$

$$q_t = (f_t, z_t) = \frac{1}{2}(\beta'(t), -[\beta'(t), y(s,t)]),$$

and

$$q_{ss} = \frac{1}{2}(\alpha''(s), -[\alpha''(s), y(s,t)]),$$

$$q_{tt} = \frac{1}{2}(\beta''(t), -[\beta''(t), y(s,t)]).$$

Since the curves are planar, let us suppose that $[\alpha'(s), \beta'(t)] \neq 0$ and write

$$\alpha''(s) = a(s,t)\alpha'(s) + b(s,t)\beta'(t),$$

$$\beta''(t) = c(s,t)\alpha'(s) + d(s,t)\beta'(t),$$

for some scalar functions a, b, c, d. Then we conclude that

$$q_{ss} = a(s,t)q_s + b(s,t)q_t,$$

$$q_{tt} = c(s,t)q_s + d(s,t)q_t,$$

which means that $L = N = 0$, that is, (s,t) are asymptotic coordinates and the metric is indefinite.

From
$$q_{st} = \frac{1}{4}(0, 0, -[\alpha'(s), \beta'(t)])$$
we obtain $M = -\left(\frac{1}{4}[\alpha'(s), \beta'(t)]\right)^2$. The sign of M is associated with the orientation of the basis $\{q_s, q_t, q_{st}\}$, so we can change the roles of s and t to obtain $M > 0$. Then we can assume that and set

$$\Omega(s,t) = \frac{1}{4}|[\alpha'(s), \beta'(t)]|.$$

Thus $\xi = (0, 0, \pm 1)$ and q is an improper affine sphere.

Let us now assume that q is an indefinite improper affine sphere, so we can set $\xi = (0, 0, 1)$ and $q(s,t) = (f(s,t), z(s,t))$ in asymptotic parameters (s,t), where $f(s,t)$ is the projection of q in the plane $\{e_1, e_2\}$.

First of all, we know that $q_{st} = \Omega\,\xi$, hence $f_{st} = 0$ and $z_{st} = \Omega$. Then $f(s,t) = \alpha(s) + \beta(t)$ for some planar curves $\alpha : I \longrightarrow \mathbb{R}^2$ and $\beta : J \longrightarrow \mathbb{R}^2$, and

$$q_s = (f_s, z_s) = (\alpha'(s), z_s),$$

$$q_t = (f_t, z_t) = (\beta'(t), z_t),$$

$$q_{st} = (f_{st}, z_{st}) = (0, 0, \Omega),$$

which means that
$$\Omega^2 = [q_s, q_t, q_{st}] = \Omega[\alpha', \beta']$$

and

$$\Omega(s,t) = [\alpha'(s), \beta'(t)].$$

Let us assume that $[\alpha'(s), \beta'(t)] \neq 0$, since we suppose a non degenerate surface. Since $z_{st} = [\alpha'(s), \beta'(t)]$ (note that here the sign is opposite to z_{st} in Lemma (2.6.1) and this will produce a sign difference in the end results), by integration we get

$$z_s = [\alpha'(s), \beta(t) + u(s)] = [f_s, \beta(t) + u(s)],$$

$$z_t = [\alpha(s) + v(t), \beta'(t)] = [\alpha(s) + v(t), f_t],$$

for some planar curves u and v defined in I and J, respectively.

By assuming that $z_s = [f_s, y]$ and $z_t = [f_t, y]$ (in our construction these two have opposite sign), we conclude that $y(s,t) = \beta(t) + u(s) = -(\alpha(s) + v(t))$, which means that $u(s) = -\alpha(s)$, $v(t) = \beta(t)$ and $y(s,t) = \beta(t) - \alpha(s)$.

And we have that q is the generalized area distance of the pair of curves $(\alpha(s), \beta(t))$

Besides some changes of sign we also chose not to take the midpoint for f, but this does not change the proof, since the relation between z and the area keeps the same. □

Remark.

We did not calculate the conormal vector field ν in the proof of the last Proposition since was not necessary for our purpose, but we can verify that $\nu(s,t) = (-n(s,t), 1)$, which means that it is planar, and the Figure (2.2) is an example of a generalized area distance obtained from the pair of curves $\alpha(u) = (u, 0)$ and $\beta(v) = (v, v^2)$.

2.6.2 Singularity set

We have seen that if q is the generalized area distance of a pair of curves $(\alpha(s), \beta(t))$, then its Blaschke metric is given by $\Omega(s,t) = \frac{1}{4}[\alpha'(s)\beta'(t)]$.

Definition 2.6.4 *The singularity set S of q consists of all pairs (s,t) for which $\Omega = 0$, that is, $[\alpha'(s), \beta'(t)] = 0$.*

Geometrically, the set $f(S)$ consists of all midpoints of chords connecting $\alpha(s)$ and $\beta(t)$ with parallel tangents. Because of it, this set can be called *midpoint parallel tangent locus* (MPTL) or *area evolute.*

Consider $r \longrightarrow \gamma(r) = (s(r), t(r))$ a parametrization for the singular set S and denote by $\eta(r)$ the null direction of df. Then, in order to verify the regularity of $f(S)$ at a point $f(\gamma(r_0)) = f(s_0, t_0)$, we must to check whether

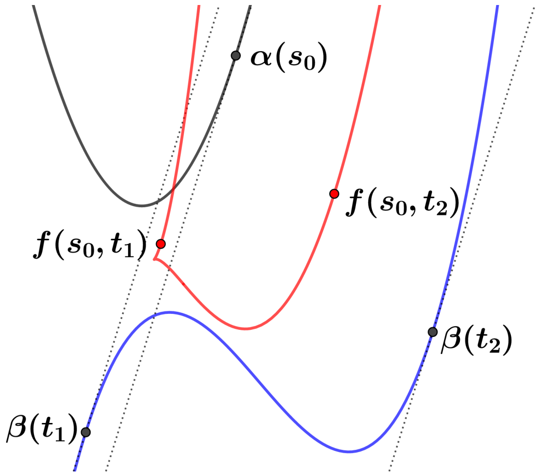


Figure 2.5: Example of the MPTL of the pair of curves $(\alpha(s), \beta(t))$.

$\eta(r_0)$ is parallel to $\gamma'(r_0)$ or not. In other words, whether or not the null vector is

$$df \cdot (\Omega_t, -\Omega_s) = \Omega_t \alpha'(s_0) - \Omega_s \beta'(t_0),$$

where $\Omega_s = [\alpha''(s), \beta'(t)]$ and $\Omega_t = [\alpha'(s), \beta''(t)]$.

Lemma 2.6.5 *The following statements are equivalent:*

1. *The affine tangent vectors of α at s_0 and β at t_0 are opposite, i.e., $b(t_0)\alpha'(s_0) + a(s_0)\beta'(t_o) = 0$.*

2. *The euclidean curvatures k_1 of α at s_0 and k_2 of β at t_0 are equal.*

3. *The null direction η of df is tangent to S, i.e., $\Omega_t\, \alpha'(s_0) - \Omega_s\, \beta'(t_0) = 0$.*

4. *The direction $(b(t_0), a(s_0))$ that vanishes the cubic form is tangent to S.*

Any point $(s,t) \in S$ that does not satisfy these conditions will be called a regular singular point.

The proof of this Lemma appears in Craizer, Teixeira and Silva [11, p.72-73], but it is relevant to say that item (3) will be of better use for us to recognize an irregular singular point.

The next Proposition can be found in Kokubo et al. [20, p.306] and Craizer, Teixeira and Silva [11, p.73].

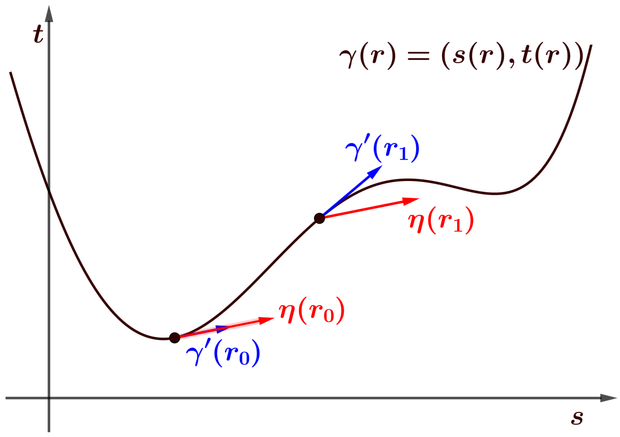


Figure 2.6: Example of a singular set S and the null direction η in two different points. At r_0, γ' and η are parallel, and at r_1 they are not. By Proposition (2.6.6), the first point gives us a swallowtail if $A'(r_0) \neq 0$ and the second one necessarily a cuspidal edge.

Proposition 2.6.6 *Let $\gamma(r_0)$ be a nondegenerate singularity of a front $q : I \times J \longrightarrow \mathbb{R}^3$. Then*

1. *The germ of q at $\gamma(r_0)$ is locally diffeomorphic to a cuspidal edge if and only if $\eta(r_0)$ is not parallel to $\gamma'(r_0)$.*
2. *The germ of q at $\gamma(r_0)$ is locally diffeomorphic to a swallowtail if and only if $\eta(r_0)$ is parallel to $\gamma'(r_0)$ and $A'(r_0) \neq 0$, where*

$$A(r) = [\eta(r), \gamma'(r)].$$

A singularity is called a *front* if the map $(s, t) \longrightarrow (q, \nu)$ is an immersion at (s_0, t_0).

The next results, Lemma and Corollaries, are proved in Craizer, Teixeira and Silva [11, p.73-74]. They intent to describe the nature of the surface at singular points from the behavior of the singular set, that is, the nature of a singular point in the surface – if it is a cuspidal edge or a swallowtail – depends on whether a point in the singular set is regular or not.

Corollary 2.6.7 *Assume that $(s_0, t_0) \in S$ is a regular singular point. Then $f(S)$ is smooth at (s_0, t_0) and the germ of the singularity $q(s_0, t_0)$ is diffeomorphic to a cuspidal edge.*

This Corollary says that a singular point of the surface is a cuspidal edge whenever associated to a regular point of the set $f(S)$. Now consider a singular point (s_0, t_0) associated with the parameter r_0 such that the Lemma (2.6.5) holds. It is possible to assume that, close to (s_0, t_0), $\alpha(s)$

and $\beta(t)$ are parameterized by affine arc-lenght. Define $\lambda(r)$ by the equation $\alpha'(r) + \lambda(r)\beta'(r) = 0$.

Lemma 2.6.8 *The following statements are equivalent:*

1. $\lambda'(r_0) \neq 0$.
2. $k_1'(r_0) \neq k_2'(r_0)$.
3. $A'(r_0) \neq 0$.

Corollary 2.6.9 *Suppose that $(s_0, t_0) \in S$ is not regular and the conditions of Lemma (2.6.8) hold. Then $f(S)$ has an ordinary cusp at (s_0, t_0) and the germ of $q(s_0, t_0)$ is diffeomorphic to a swallowtail.*

In other words, whenever the set $f(S)$ has a cusp the associated singular point at the surface is a swallowtail.

Example:

Consider the pair of curves (α, β) where $\alpha(s) = (s, s^2 + 2)$ and $\beta(t) = \left(t, \frac{t^3 - 6t^2 + 5t}{6}\right)$, for $(s, t) \in \mathbb{R}^2$. These two curves appeared in Figure (2.5), so we will make use of them to understand the last results.

We know that

$$\Omega(s, t) = \frac{1}{4}[\alpha'(s)\beta'(t)] = \frac{1}{4}\left(\frac{3t^2 - 12t + 5}{6} - 2s\right),$$

which means that $\Omega = 0$ if and only if $s = \frac{3t^2 - 12t + 5}{12}$.

Then, the singular set S is

$$S = \left\{(s, t) \in \mathbb{R}^2 \middle/ s = \frac{3t^2 - 12t + 5}{12}\right\},$$

and $f(S)$ is the red curve in Figure (2.5).

From Lemma (2.6.5), we want to know when the null direction η of df is tangent to S, *i.e.*, $\Omega_t\, \alpha'(s_0) - \Omega_s\, \beta'(t_0) = 0$. But, $\Omega_s = -\frac{1}{2}$ and $\Omega_t = \frac{t-2}{4}$. Hence, we are interested in pairs (s, t) such that

$$\frac{t - 2}{4}(1, 2s) + \frac{1}{2}\left(1, \frac{3t^2 - 12t + 5}{6}\right) = (0, 0).$$

And this happens at the point $\left(\frac{5}{12}, 0\right)$. Thus we conclude from Corollary (2.6.9) that $f(S)$ has a cusp at this point and the germ of q is differmorphic to a swallowtail, as one can see in Figure (2.7). On the other hand, Corollary (2.6.7) point out that all other points in $f(S)$ are regular and the germ of q will be diffeomorphic to a cuspidal edge.

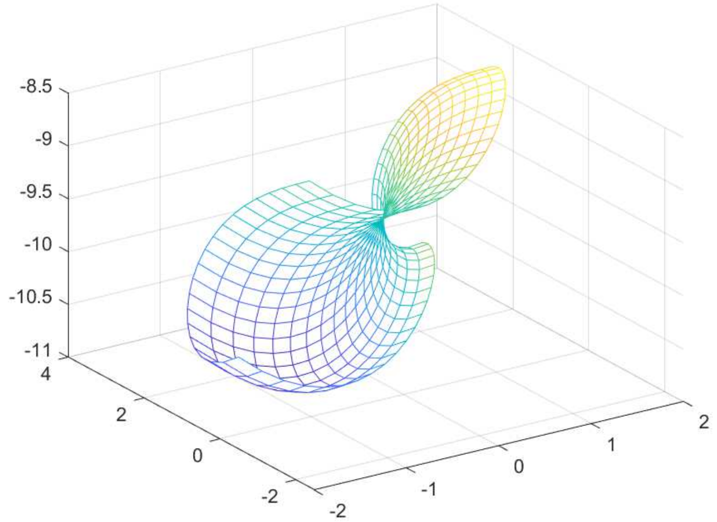


Figure 2.7: Surface $q(s,t) = (f(s,t), -z(s,t))$, where z is the generalized area distance of the pair of curves $\alpha(s) = (s, s^2 + 2)$ and $\beta(t) = \left(t, \frac{t^3-6t^2+5t}{6}\right)$. The swallowtail is quite evident, although the cuspidal edge is not. The sign of z was changed to improve the visualization.

2.7 Characterization of smooth Cayley Surfaces

This section aims to discuss some properties of the so called Cayley Surfaces, since it is a model of equiaffinely homogenous surfaces and of affine spheres with affine metric of constant mean curvature (these being our focus). This study was meticulous done by Nomizu [31] and all the absent proofs in this work can be found there.

Definition 2.7.1 *A surface $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is ruled if it can be written as*

$$f(x, y) = yF(x) + G(x),$$

for some functions $F, G : \mathbb{R} \longrightarrow \mathbb{R}^3$.

Our intention here is to study a surface that is the graph of $z = xy+\varphi(x)$, where $\varphi(x)$ is an arbitrary smooth function defined on $\mathbb{R}$. It is easy to see that this surface is a ruled (see Def.2.7.1) improper affine sphere, since the normal vector ξ is constant.

When we take $\varphi(x) = -\frac{x^3}{3}$, we find one of the most important surface among them, the Cayley Surfaces:

$$z = xy - \frac{x^3}{3}, \tag{2.14}$$

and all its equiaffinely congruent surfaces.

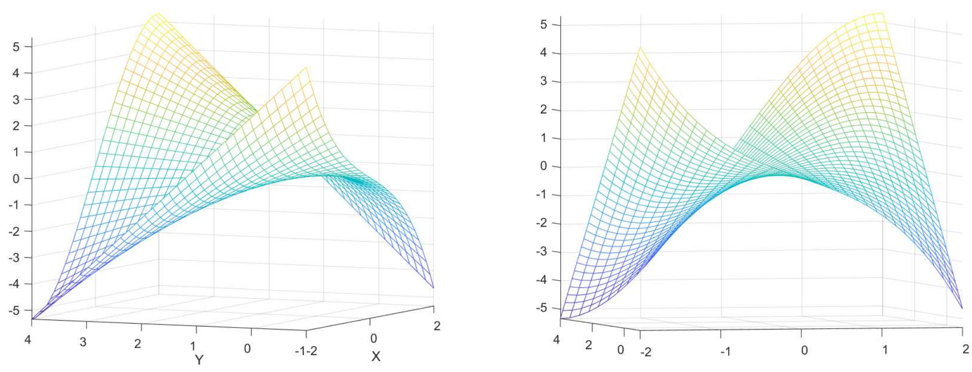


Figure 2.8: A Cayley Surface from two different view points.

Accordingly to Nomizu, this surface is affine homogeneous, that is, there is a Lie subgroup of the group of all equiaffine transformations of $\mathbb{R}^3$ that acts transitively on the surface. This information leads us to the theorem about classification of the affine homogeneous surfaces:

Theorem 2.7.2 *Any nondegenerate surface in $\mathbb{R}^3$ that is homogeneous under equiaffine transformations is a quadric or is affinely congruent to one of the following surfaces:*

$$\begin{array}{cc} (I) & xyz = 1, \\ (II) & (x^2+y^2)z = 1, \\ (III) & x^2(z-y^2)^3 = 1, \\ (IV) & x^2(z-y^2)^3 = -1, \\ (V) & z = xy - \frac{x^3}{3}, \\ (VI) & z = xy + log(x). \end{array}$$

The full proof of this Theorem can be found in Nomizu [31, p.102]. Let us only explain what affine congruence means and give one particular fact used in the demonstration that will be necessary for our further results.

If a nondegenerate surface M^2 in $\mathbb{R}^3$ is equiaffinely homogeneous and the image $\phi(M^2)$ is also a nondegenerate equiaffinely homogeneous surface for any affine transformation ϕ of $\mathbb{R}^3$ onto itself, the two surfaces M^2 and $\phi(M^2)$ are called affinely congruent. This means that the above Theorem is up to affine congruence.

In its proof Nomizu uses the Pick invariant J to show that the first four equations will be found if $J \neq 0$ and, in the other case, if $J = 0$, the last two equations will follow.

Some sections ahead Nomizu treats the local classification of affine spheres whose affine metrics have constant curvature and gives a list that exhausts all such surfaces in the following Theorem:

Theorem 2.7.3 *Let $f : M^2 \longrightarrow \mathbb{R}^3$ be an affine sphere with Blaschke structure. If the affine metric h is flat, then the surface is locally affine congruent to one of the following surfaces:*

$$\begin{array}{ll} (1) & z = x^2 + y^2, \\ (2) & xyz = 1, \\ (3) & (x^2+y^2)z = 1, \\ (4) & z = xy + \varphi(x), \end{array}$$

where φ is an arbitrary function of x.

The Pick invariant is again used in the proof by Nomizu, so that the second and third equations can be found if $J \neq 0$. If $J = 0$, the first equation follows from the definite metric whilst the last one follows from the indefinite one.

Another result due to Nomizu and Sasaki [31] refers to ruled affine spheres and will be of our interest in chapter 4.

Theorem 2.7.4 *If $f : M^2 \longrightarrow \mathbb{R}^3$ is a ruled improper affine sphere, then it is locally of the form (4) in Theorem (2.7.3).*

As we can see the Cayley surface appears explicitly in the Theorem (2.7.2) and implicitly in the Theorem (2.7.3), what is a sign of its importance. In 1989, Nomizu and Pinkall [32] presented a paper in which the characterization of the Cayley surface was made, as we shall see in the next theorem.

Theorem 2.7.5 [Characterization of Cayley surfaces] *Let M^2 be a non-degenerate surface with Blaschke structure in $\mathbb{R}^3$. Then M^2 is affinely congruent to a Cayley surface if and only if the cubic form C is not 0 and it is parallel to ∇, i.e., $\nabla C = 0$.*

Proof.
It is easy to see that the parametrization $\psi(x, y) = (x, y, z(x, y))$ given by the equation (2.14) is not asymptotic, but this problem can be solved by a change of parameters. So we can reparameterize it along the asymptotic lines by setting $x = u$ and $y = v + \frac{u^2}{2}$ and get

$$\phi(u, v) = \left(u, v + \frac{u^2}{2}, uv + \frac{u^3}{6}\right) \tag{2.15}$$

Since $\phi_u = \left(1, u, v + \frac{u^2}{2}\right)$, $\phi_v = (0, 1, u)$, $\phi_{uu} = (0, 1, u)$, $\phi_{uv} = (0, 0, 1)$ and $\phi_{vv} = (0, 0, 0)$, we have $L = [\phi_u, \phi_v, \phi_{uu}] = 0$, $M = [\phi_u, \phi_v, \phi_{uv}] = 1$ and $N = [\phi_u, \phi_v, \phi_{vv}] = 0$, which confirms asymptotic coordinates. Moreover, it says that the metric is constant $\Omega = 1 > 0$ and the affine normal vector field is also constant $\xi = (0, 0, 1)$. It follows that the coefficients of the cubic form are $A = 1$ and $B = 0$ (this means that the surface is ruled), and by formula (2.3) we come to a surface with constant affine mean curvature, since $H = 0$.

Now let us take a look at the cubic form and the covariant derivative ∇C. We know that $C = A\,du^3 + B\,dv^3$ in asymptotic coordinates. Let be $U = \frac{\partial}{\partial u}$ and $V = \frac{\partial}{\partial v}$, so $C(U, U, U) = A$, $C(U, U, V) = C(U, V, V) = 0$ and $C(V, V, V) = B$.

In terms of the covariant derivative ∇ the equations (2.1) lead us to

$$\nabla_U U = \frac{\Omega_u}{\Omega} U + \frac{A}{\Omega} V,$$

$$\nabla_V V = \frac{B}{\Omega} U + \frac{\Omega_v}{\Omega} V,$$
$$\nabla_U V = 0.$$

We want to calculate $\nabla_Z C$ where Z is a direction in the space spanned by $\{U, V\}$, so it is sufficient to calculate $\nabla_U C$ and $\nabla_V C$. With the use of the above covariant derivatives, we have:

$$\begin{aligned}
\nabla_U C(U,U,U) &= \nabla C(U,U,U,U) \\
&= U\,C(U,U,U) - 3C(\nabla_U U, U, U) = A_u - 3A\tfrac{\Omega_u}{\Omega} \\
\nabla_U C(U,U,V) &= \nabla C(U,U,V,U) \\
&= U\,C(V,U,U) - C(\nabla_U V, U, U) - 2C(V, \nabla_U U, U) = 0 \\
\nabla_U C(U,V,V) &= \nabla C(U,V,V,U) \\
&= U\,C(V,V,U) - 2C(\nabla_U V, V, U) - C(V, V, \nabla_U U) = -\tfrac{AB}{\Omega} \\
\nabla_U C(V,V,V) &= \nabla C(V,V,V,U) \\
&= U\,C(V,V,V) - 3C(\nabla_U V, V, V) = B_u \\
\nabla_V C(U,U,U) &= \nabla C(U,U,U,V) \\
&= V\,C(U,U,U) - 3C(\nabla_V U, U, U) = A_v \\
\nabla_V C(U,U,V) &= \nabla C(UU,V,V) \\
&= V\,C(V,U,U) - C(\nabla_V V, U, U) - 2C(V, \nabla_V U, U) = -\tfrac{AB}{\Omega} \\
\nabla_V C(U,V,V) &= \nabla C(U,V,V,V) \\
&= V\,C(U,V,V) - C(\nabla_V U, V, V) - 2C(U, \nabla_V V, V) = 0 \\
\nabla_V C(V,V,V) &= \nabla C(V,V,V,V) \\
&= V\,C(V,V,V) - 3C(\nabla_V V, V, V) = B_v - 3B\tfrac{\Omega_v}{\Omega}
\end{aligned}$$

So we have $\nabla C = 0$ if and only all the eight terms above are equal to zero.

Note that in the Cayley surface we already have that A and H are constant and $B = 0$, which means that $\nabla C = 0$. Conversely, let us suppose that $\nabla C = 0$, that is,

$$\begin{aligned} &(i) \quad A_u = 3A\frac{\Omega_u}{\Omega} \\ &(ii) \quad A_v = B_u = 0 \\ &(iii) \quad B_v = 3B\frac{\Omega_v}{\Omega} \\ &(iv) \quad \frac{AB}{\Omega} = 0 \end{aligned}$$

From (iv) we have that $A = 0$ or $B = 0$. Let us suppose that $B = 0$ and $A \neq 0$.

Let R be the curvature of the surface. Let us prove that $R(X,Y)Z = 0$ for all tangent directions X, Y and Z. Since the tangent space has dimension 2, X, Y and Z are linear combinations of U and V, with $[U,V] = 0$, it is enough to calculate $R(U,V)U$ and $R(V,V)V$, because it is know that $R(U,U)U = R(V,V)U = R(U,U)V = R(V,V)V = 0$, $R(U,V)U = -R(V,U)U$ and $R(U,V)V = -R(V,U)V$. Then from relations (i) to (iv) we have:

$$\begin{aligned} R(U,V)U &= \nabla_V\nabla_U U - \nabla_U\nabla_V U = \nabla_V\left(\frac{\Omega_u}{\Omega}U + \frac{A}{\Omega}V\right) \\ &= \left(\frac{\Omega_u}{\Omega}\right)_v U + \frac{\Omega_u}{\Omega}\nabla_V U + \left(\frac{A}{\Omega}\right)_v V + \frac{A}{\Omega}\nabla_V V \\ &= \left(\frac{\Omega_u}{\Omega}\right)_v U + \frac{A_v\Omega - A\Omega_v}{\Omega^2}V + \frac{AB}{\Omega^2}U + \frac{A\Omega_v}{\Omega^2} \\ &= \left(\frac{\Omega_u}{\Omega}\right)_v U \\ R(V,U)V &= \nabla_U\nabla_V V - \nabla_V\nabla_U V = \nabla_U\left(\frac{B}{\Omega}U + \frac{\Omega_v}{\Omega}V\right) \\ &= \left(\frac{\Omega_v}{\Omega}\right)_u V + \frac{\Omega_v}{\Omega}\nabla_U V = \left(\frac{\Omega_u}{\Omega}\right)_v V \end{aligned}$$

By setting $Z = \alpha U + \beta V$, we get $R(U,V)Z = \left(\frac{\Omega_u}{\Omega}\right)_v (\alpha U - \beta V)$. So from the First Bianchi identity

$$R(U,V)Z + R(V,Z)U + R(Z,U)V = 0$$

in addition to the bilinearity of R we come to $R(V,U)V = 0$, which means that the metric is flat, that is, Ω is constant. Thus the relation (iv) and the formula (2.3) give us $H = 0$ and by equation (2.2) we have $\xi = (0,0,1)$, *i.e.*, the surface is an improper affine sphere.

Let us take $\Omega = 1$. Moreover, since $\Omega_u = 0$, the relation (i) turns to $A_u = 0$. So from now on we have $A_u = A_v = 0$, *i.e.*, A is also constant and w.l.g. let us take $A = 1$.

Let us rewrite the Gauss equations (2.1) with the information we have until now:

$$q_{uu} = q_v, \quad q_{vv} = 0 \quad \text{and} \quad q_{uv} = \xi.$$

From q_{vv} and q_{uv} we get $q_v = u\xi + C$, where C is a constant vector in $\mathbb{R}^3$. By integration on v we come to

$$q = uv\xi + vC + D(u),$$

where $D(u)$ is a vector function of u. From this equation we get

$$q_u = v\xi + D'(u).$$

From q_{uu} we find $q_{uu} = u\xi + C$, which by integration on u it follows that $q_u = \frac{u^2}{2}\xi + uC + E(v)$, where $E(v)$ is a vector function of v. By comparing these two equations for q_u we find

$$D'(u) = \frac{u^2}{2}\xi + uC + P,$$

$$D(u) = \frac{u^3}{6}\xi + \frac{u^2}{2}C + uP + R,$$

where P and R are constant vectors in $\mathbb{R}^3$. We may assume that $R = 0$, and ξ, C and P are linearly independent. Then

$$q = \left(uv + \frac{u^3}{6}\right)\xi + \left(v + \frac{u^2}{2}\right)C + uP.$$

By setting $x = u$, $y = v + \frac{u^2}{2}$ and $z = uv + \frac{u^3}{6}$, we get $z = xy - \frac{x^3}{3}$ and the proof is completed. $\square$

Remark.

Another way to start our proof could be without using the curvature. Note that the relation(iv) shows us that the Pick invariant J is given by $J = \frac{AB}{\Omega^3} = 0$. So the Theorem (2.7.2) assures that the surface is an improper affine sphere (V) or (VI), and the Theorem (2.7.3) also ensures that by surface (4). That said, without loss of generality, we can take $\xi = (0,0,1)$. From that point on the demonstration would follow similarly.

3
Structure of non-degenerate asymptotic nets

3.1
Notation

Given a discrete function $f : D \subset \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$, we denote the discrete partial derivatives with respect to u and v, respectively, by

$$f_1\left(u + \frac{1}{2}, v\right) = f(u+1, v) - f(u, v),$$

$$f_2\left(u, v + \frac{1}{2}\right) = f(u, v+1) - f(u, v).$$

From these equations, the second order partial derivatives follow

$$\begin{aligned}
f_{11}(u,v) &= f(u+1,v) - 2f(u,v) + f(u-1,v)\\
f_{22}(u,v) &= f(u,v+1) - 2f(u,v) + f(u,v-1)\\
f_{12}(u,v) &= f(u+1,v+1) + f(u,v) - f(u+1,v) - f(u,v+1).
\end{aligned}$$

The quadrangle formed by the vertices $f(u,v)$, $f(u+1,v)$, $f(u,v+1)$ and $f(u+1,v+1)$ will be referred to as the quadrangle $\left(u + \frac{1}{2}, v + \frac{1}{2}\right)$.

Given two vectors V_1, $V_2 \in \mathbb{R}^3$, we denote by $V_1 \times V_2$ the cross product and by $V_1 \cdot V_2$ the dot product between them. And given three vectors V_1, V_2, $V_3 \in \mathbb{R}^3$, we denote by $[V_1, V_2, V_3] = (V_1 \times V_2) \cdot V_3$ their determinant.

3.2
Basic structure

A net $q : D \subset \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$ is called asymptotic if the "crosses are planar", *i.e.*, $q_1(u+\frac{1}{2}, v)$, $q_1(u-\frac{1}{2}, v)$, $q_2(u, v+\frac{1}{2})$ and $q_2(u, v-\frac{1}{2})$ are coplanar (see [4], p.66). From this definition, for each quadrangle $\left(u + \frac{1}{2}, v + \frac{1}{2}\right)$, we can define

$$L\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \left[q_1\left(u+\tfrac{1}{2}, v\right), q_2\left(u, v+\tfrac{1}{2}\right), q_1\left(u-\tfrac{1}{2}, v\right)\right] = 0,$$

$$N\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \left[q_1\left(u+\tfrac{1}{2}, v\right), q_2\left(u, v+\tfrac{1}{2}\right), q_2\left(u, v-\tfrac{1}{2}\right)\right] = 0,$$

$$M\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \left[q_1\left(u+\tfrac{1}{2}, v\right), q_2\left(u, v+\tfrac{1}{2}\right), q_2\left(u+1, v+\tfrac{1}{2}\right)\right].$$

Observe that $L = N = 0$, in line with the smooth theory and according with the planar crosses, since geometrically these determinants correspond to volumes. In particular, M is the volume of the tetrahedron of vertices $q(u, v)$, $q(u + 1, v)$, $q(u, v + 1)$ and $q(u + 1, v + 1)$.

We say that the asymptotic net is *non-degenerate* if all its elementary quadrangles are non-planar, which means that $M \neq 0$ in any quadrangle. More than that, M does not change sign so, from now on, we shall assume that $M\left(u + \frac{1}{2}, v + \frac{1}{2}\right) > 0$ for any $(u, v) \in D$. Besides that, like its smooth counterpart, the definition of an asymptotic net implies that all vertices are saddle points, in other words, the plane spanned by $q_1\left(u + \frac{1}{2}, v\right)$, $q_1\left(u - \frac{1}{2}, v\right)$, $q_2\left(u, v + \frac{1}{2}\right)$ and $q_2\left(u, v - \frac{1}{2}\right)$ crosses the surface.

Similarly, define the *affine metric* Ω at a quadrangle $\left(u + \frac{1}{2}, v + \frac{1}{2}\right)$ by

$$\Omega\left(u + \tfrac{1}{2}, v + \tfrac{1}{2}\right) = \sqrt{M\left(u + \tfrac{1}{2}, v + \tfrac{1}{2}\right)}. \tag{3.1}$$

Note that we are taking $\Omega > 0$ for definiteness, in order to agree with the smooth case.

We remark that if $A \in SL(3, \mathbb{R}^3)$, then Aq is also a non-degenerate asymptotic net and its affine metric remains Ω.

In the discrete case (see [4], p.70), the *conormal vector field* ν with respect to an asymptotic net q is a vector-valued map defined at the vertices (u, v) satisfying the discrete Lelieuvre's equations

$$\begin{aligned} \nu(u, v) \times \nu_1\left(u + \tfrac{1}{2}, v\right) &= \quad q_1\left(u + \tfrac{1}{2}, v\right), \\ \nu(u, v) \times \nu_2\left(u, v + \tfrac{1}{2}\right) &= -q_2\left(u, v + \tfrac{1}{2}\right). \end{aligned}$$

Since $\nu_1\left(u + \frac{1}{2}, v\right) = \nu(u + 1, v) - \nu(u, v)$ and $\nu_2\left(u, v + \frac{1}{2}\right) = \nu(u, v + 1) - \nu(u, v)$, we come to simpler discrete Lelieuvre's equations

$$\begin{aligned} \nu(u, v) \times \nu(u + 1, v) &= \quad q_1\left(u + \tfrac{1}{2}, v\right), \\ \nu(u, v) \times \nu(u, v + 1) &= -q_2\left(u, v + \tfrac{1}{2}\right). \end{aligned} \tag{3.2}$$

Let us take the vector field $\nu_\rho(u, v) = \rho\nu(u, v)$, if $u + v$ is even, and $\nu_\rho(u, v) = \rho^{-1}\nu(u, v)$, if $u + v$ is odd, for any constant $\rho > 0$. This operation is known as *black-white re-scaling* and shows us that ν_ρ also satisfies Lelieuvre's equations (3.2), *i.e.*, is a way to obtain another conormal vector field with respect to the q net. Conversely, we can see that any conormal vector field with respect to the asymptotic net q is obtained from ν by a black-white re-scaling.

We can easily see that any co-normal vector field is orthogonal to the planar crosses by

$$\begin{aligned} -q_1(u+\tfrac{1}{2},v)\times q_2(u,v+\tfrac{1}{2}) &= (\nu(u,v)\times\nu(u+1,v))\times(\nu(u,v)\times\nu(u,v+1)) \\ &= [\nu(u,v),\nu(u+1,v),\nu(u,v+1)]\nu(u,v). \end{aligned}$$

Moreover, the conormal vector field is invariant under affine transformations. In fact, for any $A\in SL(3,\mathbb{R}^3)$, $(A^{-1})^t\nu$ is a co-normal vector field for the asymptotic net Aq, since $((A^{-1})^t\nu)\cdot(Aq)=0$.

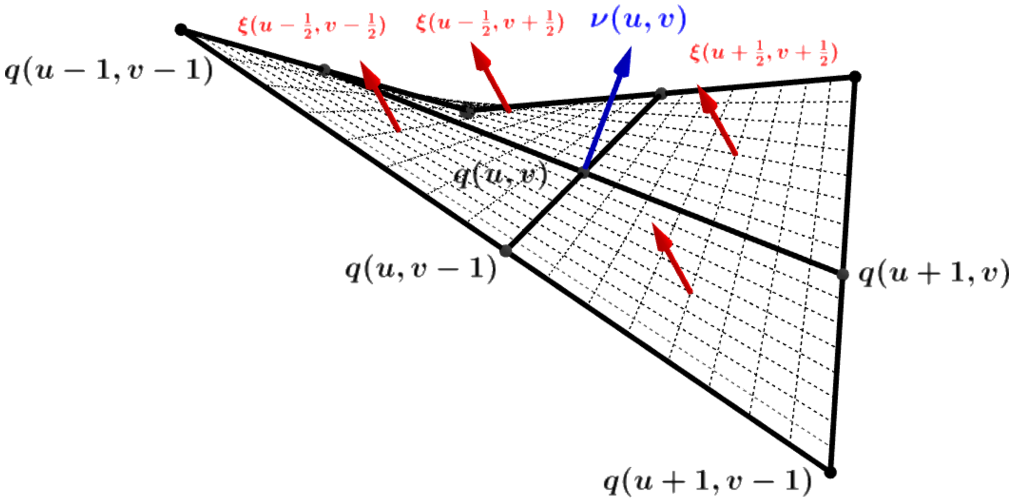


Figure 3.1: Example of a planar cross with the conormal vector at the vertex and the normal vectors at the faces representing ξ_e or ξ_o.

A two-dimensional net $f : D\subset\mathbb{Z}^2\to\mathbb{R}^3$ is called a *Moutard net* if for any quadrangle $(u+\frac{1}{2},v+\frac{1}{2})$, $f(u+1,v)+f(u,v+1)$ is parallel to $f(u,v)+f(u+1,v+1)$.

Proposition 3.2.1 *A vector field $\nu(u,v)$ is the conormal vector field of an asymptotic net $q(u,v)$ if and only if it defines a Moutard net,* i.e.,

$$\lambda^2\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)(\nu(u,v)+\nu(u+1,v+1))=\nu(u,v+1)+\nu(u+1,v), \quad (3.3)$$

for some map $\lambda:(\mathbb{Z}^2)^*\to\mathbb{R}^+$.

Proof.
From Lelieuvre's equations we have

$$q_1\left(u+\tfrac{1}{2},v\right)+q_2\left(u+1,v+\tfrac{1}{2}\right)=(\nu(u,v)+\nu(u+1,v+1))\times\nu(u+1,v)$$

and

$$q_1\left(u+\tfrac{1}{2},v+1\right)+q_2\left(u,v+\tfrac{1}{2}\right)=\nu(u,v+1)\times(\nu(u,v)+\nu(u+1,v+1)).$$

Thus

$$(q_{21}-q_{12})\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=(\nu(u,v)+\nu(u+1,v+1))\times(\nu(u+1,v)+\nu(u+1,v+1)).$$

Now, if ν is a conormal vector field of an asymptotic net, the first member of the above equation is zero, and then ν must satisfy equation (3.3). Reciprocally, if ν satisfies equation (3.3), then the second member is zero, what means $q_{12}=q_{21}$, which implies that, starting from an arbitrary point, it is possible to define an asymptotic net with conormal vector field ν without any ambiguity.

Note that under a black-white re-scaling on ν, the map λ is multiplied by the same positive constant of ν, since

$$\lambda^2(u+\tfrac{1}{2},v+\tfrac{1}{2})(\rho\nu(u,v)+\rho\nu(u+1,v+1))=\rho^{-1}\nu(u,v+1)+\rho^{-1}\nu(u+1,v)$$

or

$$(\rho\lambda)^2(u+\tfrac{1}{2},v+\tfrac{1}{2})(\nu(u,v)+\nu(u+1,v+1))=\nu(u,v+1)+\nu(u+1,v),$$

for some $\rho>0$. □

Lemma 3.2.2 *In terms of conormals, the affine metric is given by*

$$\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=\lambda^{-1}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)[\nu(u,v),\nu(u,v+1),\nu(u+1,v)].$$

One can also write

$$\nu(u,v)=\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\left(q_1(u+\tfrac{1}{2},v)\times q_2(u,v+\tfrac{1}{2})\right).$$

Proof.

$$\Omega^2=\left[q_1\left(u+\tfrac{1}{2},v\right),\,q_2\left(u,v+\tfrac{1}{2}\right),\,q_2\left(u+1,v+\tfrac{1}{2}\right)\right]$$

$$=[\nu(u,v)\times\nu(u+1,v),\,-\nu(u,v)\times\nu(u,v+1),-\nu(u+1,v)\times\nu(u+1,v+1)]$$

$$=[\nu(u,v)\times\nu(u+1,v),\,\nu(u,v)\times\nu(u,v+1),\\ \nu(u+1,v)\times(\nu(u,v)+\nu(u+1,v+1))]$$

$$=\frac{1}{\lambda^2}[\nu(u,v)\times\nu(u+1,v),\,\nu(u,v)\times\nu(u,v+1),\\ \nu(u+1,v)\times(\nu(u+1,v)+\nu(u,v+1))]$$

$$= \frac{1}{\lambda^2}[\nu(u,v),\nu(u+1,v),\nu(u,v+1)]^2$$

Then

$$\Omega = \lambda^{-1}[\nu(u,v),\nu(u,v+1),\nu(u+1,v)].$$

Moreover,

$$\begin{aligned} q_1(u+\tfrac{1}{2},v)\times q_2(u,v+\tfrac{1}{2}) = &\quad (\nu(u,v)\times\nu(u+1,v))\times(\nu(u,v)\times\nu(u,v+1)) \\ = &\quad [\nu(u,v),\nu(u+1,v),\nu(u,v+1)]\nu(u,v) \\ = &\quad \lambda\,\Omega\,\nu(u,v) \end{aligned}$$

and it follows that

$$\nu(u,v) = \frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\left(q_1(u+\tfrac{1}{2},v)\times q_2(u,v+\tfrac{1}{2})\right).$$

□

Remark.
It is possible to obtain $\nu(u,v)$ by considering the other three quadrangles of which $q(u,v)$ is a vertex:

$$\nu(u,v) = \frac{\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}\left(q_1(u-\tfrac{1}{2},v)\times q_2(u,v+\tfrac{1}{2})\right),$$

$$\nu(u,v) = \frac{\lambda^{-1}\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}\left(q_1(u-\tfrac{1}{2},v)\times q_2(u,v-\tfrac{1}{2})\right),$$

$$\nu(u,v) = \frac{\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}\left(q_1(u+\tfrac{1}{2},v)\times q_2(u,v-\tfrac{1}{2})\right).$$

Note that in even quadrangles, *i.e.*, when $u+v$ is even, it appears λ^{-1}, whilst in odd ones it appears only λ.

Another observation about Lemma (3.2.2) refers to the fact that since λ is chosen to be positive, so it will also be positive the basis $\{q_1,q_2,\nu\}$.

3.3 Asymptotic nets and interpolations by hyperboloids

3.3.1 The interpolator hyperboloid

Given four non coplanar points A, B, C and D in $\mathbb{R}^3$, we can interpolate the quadrangle $ABCD$ by the quadric $\mathcal{H}_c = \mathcal{H}(c, A, B, C, D)$ given by the following formulae:

$$\phi(u,v) = A + \frac{u}{1-cuv}(B-A) + \frac{v}{1-cuv}(C-A) \qquad (3.4)$$
$$+\frac{uv}{1-cuv}\left((1-c)D + (1+c)A - (B+C)\right),$$

where $0 \le u \le 1$, $0 \le v \le 1$, $c \in \mathbb{R}$ and $|c| < 1$.

Let us take $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,1,0)$ and $D = \frac{1}{1-c}(1,1,1)$. In this case, the interpolation is given by

$$\phi(u,v) = \frac{u}{1-cuv}(1,0,0) + \frac{v}{1-cuv}(0,1,0) + \frac{uv}{1-cuv}(0,0,1), \qquad (3.5)$$

which coincides with the basic hyperboloid (2.8) by taking $c = a$.

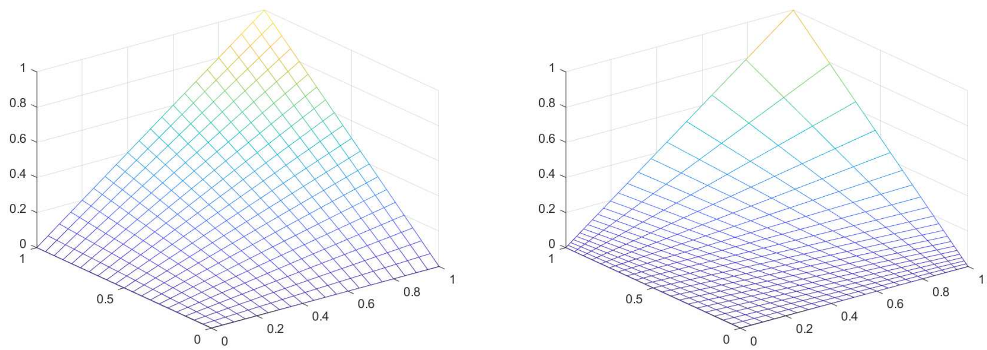


Figure 3.2: Two interpolations for the quadrangle $ABCD$ where $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,1,0)$ and $D = (1,1,1)$ with $c = 0$ on the left and $c = 0.8$ on the right. Note that as c approaches 1 more planar the interpolation is at the origin.

Take a look at the co-normal vectors at the vertices ν_A, ν_B, ν_C and ν_D. From equation (2.9), it follows that $\nu_A = \nu(0,0) = (0,0,1)$, $\nu_B = \nu(1,0) = (0,-1,1)$, $\nu_C = \nu(0,1) = (-1,0,1)$ and $\nu_D = \nu(1,1) = \frac{1}{1-c}(-1,-1,1+c)$.

From Proposition 3.2.1 we write $\lambda^2(\nu_A + \nu_D) = \nu_B + \nu_C$. Then

$$\lambda = \sqrt{1-c}. \qquad (3.6)$$

Now consider the affine map composed by a translation of A with the linear map T that realizes: $(1,0,0) \mapsto B - A$, $(0,1,0) \mapsto C - A$ and $(0,0,1) \mapsto$

$(1-c)(D-A)-(B-A)-(C-A)$. Then the image of the basic hyperboloid given by the equation (3.5) under this affine map is the interpolator hyperboloid given by the equation (3.4). Moreover, we have $det(T) = (1-c)\,\Omega^2$.

Insofar as the affine mean curvature of the basic hyperboloid is $-2c$, we conclude that the affine mean curvature of the interpolator hyperboloid is

$$H_{c,A,B,C,D} = \frac{-2c}{\sqrt{1-c\,\Omega}}. \tag{3.7}$$

Remark.

If we take any quadrangle A, B, C, D of a given basic hyperboloid $\mathcal{H}_a$ and we choose c such that

$$\frac{-2c}{\sqrt{1-c\,\Omega}} = -2a, \tag{3.8}$$

then equation (3.6) allow us to conclude the special relation

$$1-\lambda^2 = a\lambda\Omega. \tag{3.9}$$

3.3.2
Juxtaposition of two interpolator hyperboloids

We have seen how to interpolate a quadrangle by a hyperboloid and the relation between the parameter λ and the curvature of the interpolator. But this extends to all net, so we have to understand what happens in two adjacent quadrangles, in other words, how the juxtaposition of two adjacent hyperboloids interfere in the parameters associated to each quadrangle.

We know that λ and ν are invariant under affine transformations, so we can choose conveniently our two adjacent quadrangles $ABDC$ and $ACFE$ such that $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,1,0)$, $D = (1,1,1)$, $E = (x_1, y_1, 0)$ and $F = (x_2, y_2, x_2)$, for some real numbers x_1, x_2, y_1 and y_2. E and F were chosen in a way that the crosses are planar at A and C. See Figure (3.3).

The interpolations of the quadrangles $ABDC$ and $ACFE$ are given by

$$\phi(u,v) = \frac{u}{1-cuv}(1,0,0) + \frac{v}{1-cuv}(0,1,0) + \frac{uv}{1-cuv}\left((0,0,1) - c(1,1,1)\right)$$

and

$$\begin{aligned}\psi(s,t) = \frac{s}{1-bst}(x_1,y_1,0) + \frac{t}{1-bst}(0,1,0)\\ + \frac{st}{1-bst}\left((1-b)(x_2,y_2,x_2) - (x_1,y_1+1,0)\right).\end{aligned}$$

The two quadrics must be tangent along the common edge AC, *i.e.*, they must have the same tangent plane along it, which means the same normal vector at any point of the edge. This happens when $\psi_s(0,t) \times \psi_t(0,t) =$

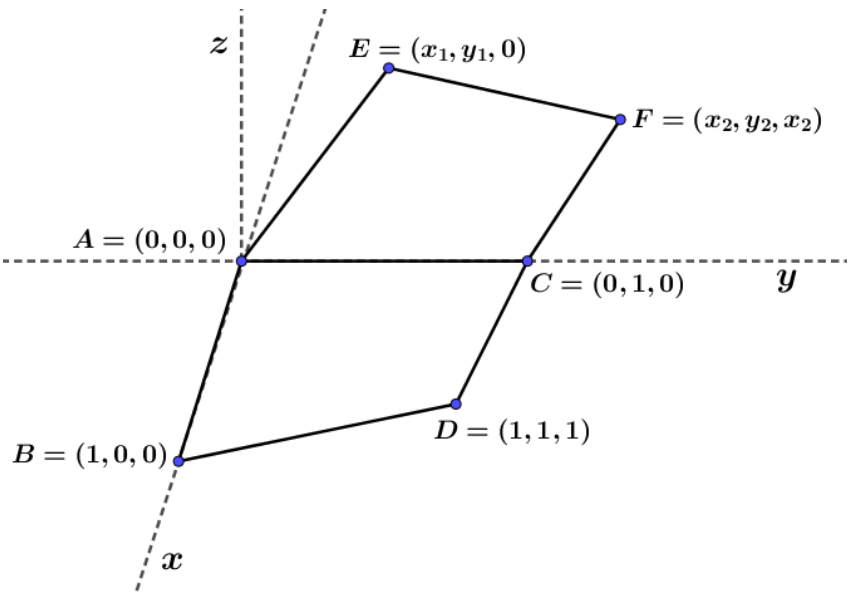


Figure 3.3: Two adjacent quadrangles properly chosen.

$\phi_u(0, v) \times \phi_v(0, v)$ or

$$\frac{(1-c)v}{1-cv} = \frac{x_2(1-b)t}{x_1 - x_1 t + x_2(1-b)t} \tag{3.10}$$

for $0 \leq v \leq 1$ and $0 \leq t \leq 1$.

Besides, the conormal vectors at the vertices A and C must coincide. On the one hand we have $\nu_A = \frac{1}{\sqrt{1-c}}(0, 0, 1)$ and $\nu_C = \sqrt{1-c}\,(-1, 0, 1)$ from ϕ, on the other hand we have $\nu_A = \frac{-x_1}{\sqrt{1-b}\sqrt{x_1 x_2}}(0, 0, 1)$ and $\nu_C = \frac{-x_2\sqrt{1-b}}{\sqrt{x_1 x_2}}(-1, 0, 1)$ from ψ. So they coincide if and only if

$$\frac{1}{\sqrt{1-c}} = \frac{-x_1}{\sqrt{1-b}\sqrt{x_1 x_2}} \quad \Leftrightarrow \quad \frac{1}{1-c} = \frac{x_1}{(1-b)x_2} \tag{3.11}$$

Observe that (3.10) and (3.11) occur if and only if $0 \leq v = t \leq 1$, meaning that the two quadrics must be tangent along the common edge AC with coincident conormal vectors at the vertices if and only if (3.11) is valid.

Let us use the equation (3.3) on the quadrangles $ABDC$ and $AEFC$, *i.e.*, there are parameters λ and μ such that

$$\lambda^2(\nu_A + \nu_D) = \nu_B + \nu_C \quad \text{and} \quad \mu^2(\nu_E + \nu_C) = \nu_A + \nu_F.$$

From ϕ we get $\nu_B = \sqrt{1-c}\,(0, -1, 1)$ and $\nu_D = \frac{1}{\sqrt{1-c}}(-1, -1, 1)$. Then

$$\lambda = \sqrt{1-c}. \tag{3.12}$$

From ψ we have

$$\nu_E = \frac{\psi_s(1,0) \times \psi_t(1,0)}{M^{1/2}} = \frac{\sqrt{1-b}}{\sqrt{x_1 x_2}}(x_2 y_1, -x_1 x_2, x_1 y_2 - x_2 y_1)$$

and

$$\nu_F = \frac{\psi_s(1,1) \times \psi_t(1,1)}{M^{1/2}} = \frac{1}{\sqrt{x_1 x_2}\sqrt{1-b}}(x_2 y_1 - x_2, -x_1 x_2, x_1 y_2 - x_2 y_1 - x_1 + x_2).$$

Hence

$$\mu = \frac{1}{\sqrt{1-b}}. \tag{3.13}$$

From (3.11), (3.12) and (3.13), we conclude that the two quadrics must be tangent along the common edge AC with coincident conormal vectors at the vertices A and C if and only if

$$\lambda\mu = \frac{\sqrt{1-c}}{\sqrt{1-b}} = \frac{\sqrt{x_2}}{\sqrt{x_1}}. \tag{3.14}$$

That said we come to the following result.

Proposition 3.3.1 *In any asymptotic net, if each quadrangle is interpolated by a hyperboloid (3.4) in such a way that two adjacent ones are always tangent along the common edge, it holds*

$$\lambda^2\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right)\lambda^2\left(u-\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \frac{1-c\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}{1-c\left(u-\frac{1}{2}, v+\frac{1}{2}\right)}$$

and

$$\lambda^2\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right)\lambda^2\left(u+\tfrac{1}{2}, v-\tfrac{1}{2}\right) = \frac{1-c\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}{1-c\left(u+\frac{1}{2}, v-\frac{1}{2}\right)},$$

where c is the curvature of the interpolator of the respective quadrangle.

3.4 Discrete hyperboloid

Consider the basic hyperboloid (2.7). Since the asymptotic curves of the smooth hyperboloid are straight lines, sampling it in the domain of asymptotic parameters generate an asymptotic net q. Denote by Δu and Δv the distance between samples in u and v directions, respectively. Straightforward calculations lead us to

$$\nu(u,v) \times \nu(u+\Delta u, v) = q(u+\Delta u, v) - q(u,v) = q_1(u+\tfrac{\Delta u}{2}, v),$$

$$\nu(u, v+\Delta v) \times \nu(u,v) = q(u, v+\Delta v) - q(u,v) = q_2(u, v+\tfrac{\Delta v}{2}).$$

Whereas the discrete Lelieuvre's formulas hold, one can use the samples of the smooth $\nu(u,v)$ as a co-normal vector field of q. We shall call this pair (q,ν) the *discrete hyperboloid*.

From the equation (2.9) whith $a=c$ and the Lemma 3.2.2, it follows that

$$\Omega(u+\tfrac{\Delta u}{2}, v+\tfrac{\Delta v}{2}) = \frac{\Delta u \Delta v}{\sqrt{[1-c(u+\Delta u)(v+\Delta v)][1-cv(u+\Delta u)][1-cu(v+\Delta v)][1-cuv]}}$$

and

$$\lambda(u+\tfrac{\Delta u}{2}, v+\tfrac{\Delta v}{2}) = \sqrt{\frac{[1-cuv][1-c(u+\Delta u)(v+\Delta v)]}{[1-cv(u+\Delta u)][1-cu(v+\Delta v)]}}.$$

From the above formulas, it is possible to verify that

$$1-\lambda^2 = c\lambda\Omega$$

holds at any quadrangle, which is in agreement with the fact that the discrete hyperboloid is a basic one.

3.5 Affine normal vector fields and the cubic form

3.5.1 Affine normal vector field

At each quadrangle $\left(u+\frac{1}{2}, v+\frac{1}{2}\right)$, we define two affine normal vectors ξ_e and ξ_o by

$$\xi_e\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \frac{\lambda\left(u+\frac{1}{2}, v+\frac{1}{2}\right) q_{12}\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)} \tag{3.15}$$

and

$$\xi_o\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \frac{\lambda^{-1}\left(u+\frac{1}{2}, v+\frac{1}{2}\right) q_{12}\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}. \tag{3.16}$$

Observe that ξ_e satisfies $\nu(u,v)\cdot\xi_e\left(u+\frac{1}{2}, v+\frac{1}{2}\right)=1$ for each quadrangle, since from Lemma (3.2.2)

$$\begin{aligned}
&\nu(u,v)\cdot\xi_e\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \\
&= \frac{\lambda^{-1}\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}\left(q_1(u+\tfrac{1}{2}, v)\times q_2(u, v+\tfrac{1}{2})\right)\cdot\frac{\lambda\left(u+\frac{1}{2}, v+\frac{1}{2}\right) q_{12}\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)} \\
&= \frac{1}{\Omega^2\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}\left[q_1\left(u+\tfrac{1}{2}, v\right), q_2\left(u, v+\tfrac{1}{2}\right), q_{12}\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right)\right] = 1
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
\nu(u+1,v+1)\cdot\xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)&=1,\\
\nu(u+1,v)\cdot\xi_o\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)&=1,\\
\nu(u,v+1)\cdot\xi_o\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)&=1.
\end{aligned}
$$

3.5.2 Coefficients of the cubic form

At any vertex we can define the coefficients of the cubic form in a similar way of their smooth counterpart as

$$A(u,v)=\left[q_1\left(u-\tfrac{1}{2},v\right),q_1\left(u+\tfrac{1}{2},v\right),\xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\right]$$

and

$$B(u,v)=\left[q_2\left(u,v-\tfrac{1}{2}\right),q_2\left(u,v+\tfrac{1}{2}\right),\xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\right].$$

By using Lelieuvre's equations and the inner products above, one can verify that

$$
\begin{aligned}
A(u,v)&=\left[q_1\left(u-\tfrac{1}{2},v\right),q_1\left(u+\tfrac{1}{2},v\right),\xi_o\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)\right]\\
&=\left[q_1\left(u-\tfrac{1}{2},v\right),q_1\left(u+\tfrac{1}{2},v\right),\xi_o\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right)\right]\\
&=\left[q_1\left(u-\tfrac{1}{2},v\right),q_1\left(u+\tfrac{1}{2},v\right),\xi_e\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
B(u,v)&=\left[q_2\left(u,v-\tfrac{1}{2}\right),q_2\left(u,v+\tfrac{1}{2}\right),\xi_o\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)\right]\\
&=\left[q_2\left(u,v-\tfrac{1}{2}\right),q_2\left(u,v+\tfrac{1}{2}\right),\xi_o\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right)\right]\\
&=\left[q_2\left(u,v-\tfrac{1}{2}\right),q_2\left(u,v+\tfrac{1}{2}\right),\xi_e\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)\right]
\end{aligned}
$$

3.6 Structural equations

Since at each vertex of the net the crosses are planar, the second derivatives $q_{11}(u,v)$ and $q_{22}(u,v)$ can be written as linear combination of $q_1\left(u+\frac{1}{2},v\right)$ and $q_2\left(u,v+\frac{1}{2}\right)$.

Proposition 3.6.1 *The structural equations of the affine immersion are given by*

$$
\begin{aligned}
q_{11}(u,v)&=\alpha\,q_1\left(u+\tfrac{1}{2},v\right)+\beta\,q_2\left(u,v+\tfrac{1}{2}\right)\\
q_{22}(u,v)&=\gamma\,q_1\left(u+\tfrac{1}{2},v\right)+\delta\,q_2\left(u,v+\tfrac{1}{2}\right)
\end{aligned}
\tag{3.17}
$$

where

$$\alpha=\frac{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)-\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)},$$

$$\beta = \frac{\lambda^{-1}\left(u+\frac{1}{2}, v+\frac{1}{2}\right) A(u,v)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)},$$

$$\gamma = \frac{\lambda^{-1}\left(u+\frac{1}{2}, v+\frac{1}{2}\right) B(u,v)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)},$$

$$\delta = \frac{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right) - \lambda^{-1}\left(u+\frac{1}{2}, v+\frac{1}{2}\right)\lambda^{-1}\left(u+\frac{1}{2}, v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2}, v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}.$$

Proof.
We know that

$$\left[q_1\left(u+\tfrac{1}{2}, v\right), q_2\left(u, v+\tfrac{1}{2}\right), q_{12}\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right)\right] = \Omega^2\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) \quad (3.18)$$

and

$$q_{11} = q_1\left(u+\tfrac{1}{2}, v\right) - q_1\left(u-\tfrac{1}{2}, v\right).$$

Replacing equations (3.17) in (3.18), using (3.15), Lemma (3.2.2) and its remark, we reach the expected conclusion.

It is important to say that $1-\alpha$ and $-\beta$ are the coefficients of the expansion of $q_1(u-\frac{1}{2}, v)$ in the basis $\{q_1(u+\frac{1}{2}, v), q_2(u, v+\frac{1}{2})\}$. ☐

Remark.
Note that the equations (3.17) can be rewritten by considering any combination $\{q_1(u \pm \frac{1}{2}, v), q_2(u, v \pm \frac{1}{2})\}$ as a basis of the plane at (u,v). For example, if we chose $\{q_1(u+\frac{1}{2}, v), q_2(u, v-\frac{1}{2})\}$ then

$$q_{11}(u,v) = \alpha\, q_1\left(u+\tfrac{1}{2}, v\right) + \beta\, q_2\left(u, v-\tfrac{1}{2}\right),$$

where

$$\alpha = \frac{\Omega\left(u+\frac{1}{2}, v-\frac{1}{2}\right) - \lambda^{-1}\left(u+\frac{1}{2}, v-\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2}, v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2}, v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v-\frac{1}{2}\right)}$$

and

$$\beta = \frac{\lambda^{-1}\left(u+\frac{1}{2}, v-\frac{1}{2}\right) A(u,v)}{\Omega\left(u+\frac{1}{2}, v-\frac{1}{2}\right)}.$$

In this case $q_{22}(u,v)$ does not change.

3.7
Gauss equations for the derivatives of ξ

We define the derivatives of ξ in the u-direction as

$$\xi_1^-\left(u, v+\tfrac{1}{2}\right) = \xi_e\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) - \xi_o\left(u-\tfrac{1}{2}, v+\tfrac{1}{2}\right),$$

$$\xi_1^+\left(u, v+\tfrac{1}{2}\right) = \xi_o\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) - \xi_e\left(u-\tfrac{1}{2}, v+\tfrac{1}{2}\right).$$

Note that these vectors are orthogonal to $\nu(u,v)$ and $\nu(u,v+1)$, respectively.

Similarly, in the v-direction we define

$$\xi_2^-\left(u+\tfrac{1}{2}, v\right) = \xi_e\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) - \xi_o\left(u+\tfrac{1}{2}, v-\tfrac{1}{2}\right),$$

$$\xi_2^+\left(u+\tfrac{1}{2}, v\right) = \xi_o\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) - \xi_e\left(u+\tfrac{1}{2}, v-\tfrac{1}{2}\right),$$

and they are orthogonal to $\nu(u,v)$ and $\nu(u+1,v)$, respectively.

From the orthogonality of ξ_1^- and ξ_2^- with $\nu(u,v)$ we can write them in the basis $\{q_1(u+\frac{1}{2},v), q_2(u,v+\frac{1}{2})\}$.

Proposition 3.7.1 *Gauss equations for the derivatives of the normal vector field are given by*

$$\xi_1^-\left(u,v+\tfrac{1}{2}\right) = \frac{\lambda^{-2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)-\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\, q_1\left(u+\tfrac{1}{2},v\right)$$
$$+\frac{\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v+1)-A(u,v)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}\, q_2\left(u,v+\tfrac{1}{2}\right)$$

and

$$\xi_2^-\left(u+\tfrac{1}{2},v\right) = \frac{\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)B(u+1,v)-B(u,v)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}\, q_1\left(u+\tfrac{1}{2},v\right)$$
$$+\frac{\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)-\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\, q_2\left(u,v+\tfrac{1}{2}\right).$$

Proof.
$\xi_1^-\left(u,v+\frac{1}{2}\right)$ can be written in the basis $\{q_1(u+\frac{1}{2},v), q_2(u,v+\frac{1}{2})\}$ as

$$\xi_1^-\left(u,v+\tfrac{1}{2}\right) = \alpha\, q_1\left(u+\tfrac{1}{2},v\right) + \beta\, q_2\left(u,v+\tfrac{1}{2}\right).$$

So

$$\left[\xi_1^-\left(u,v+\tfrac{1}{2}\right), q_2\left(u,v+\tfrac{1}{2}\right), \xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\right] = \alpha\left[q_1\left(u+\tfrac{1}{2},v\right), q_2\left(u,v+\tfrac{1}{2}\right), \xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\right],$$

and then

$$-\left[\xi_o\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right), q_2\left(u,v+\tfrac{1}{2}\right), \xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\right] = \alpha\lambda\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right).$$

Since

$$\lambda^{-1}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right) =$$

$$q_1(u+\tfrac{1}{2},v+1)-q_1(u+\tfrac{1}{2},v),$$

$$\lambda\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)\xi_o\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)= \\ q_1(u-\tfrac{1}{2},v+1)-q_1(u-\tfrac{1}{2},v),$$

$$q_1(u-\tfrac{1}{2},v)=\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_1\left(u+\tfrac{1}{2},v\right) \\ -\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_2\left(u,v+\tfrac{1}{2}\right)$$

and

$$q_1(u-\tfrac{1}{2},v+1)=\frac{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_1\left(u+\tfrac{1}{2},v+1\right) \\ -\frac{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v+1)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_2\left(u,v+\tfrac{1}{2}\right),$$

we conclude that

$$\alpha=\frac{\lambda^{-2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)-\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}.$$

Similarly we have

$$\left[\xi_1^-\left(u,v+\tfrac{1}{2}\right),q_1\left(u+\tfrac{1}{2},v\right),\xi_o\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)\right]= \\ -\beta\lambda\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right).$$

It is only necessary to replace $\xi_o\left(u-\frac{1}{2},v+\frac{1}{2}\right)$ again to conclude that

$$\beta=\frac{\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v+1)-A(u,v)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}.$$

The coefficients of $\xi_2^-\left(u+\frac{1}{2},v\right)$ can be found in the same way. □

Remark.

Note that the above equations can be rewritten by considering any vertex $q(u\pm 1,v)$ or $q(u,v\pm 1)$ and their respective plane by using ξ^+ instead of ξ^- when necessary. For example, by considering the plane determined in the vertex $q(u,v+1)$, the first equation would be written as

$$\xi_1^+\left(u,v+\tfrac{1}{2}\right)=\frac{\lambda^{-2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)-\lambda^{2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_1\left(u+\tfrac{1}{2},v+1\right) \\ +\frac{A(u,v+1)-\lambda^{-2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v)}{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}q_2\left(u,v+\tfrac{1}{2}\right).$$

3.7.1 Compatibility equations

The compatibility equations can be obtained by comparing as q_{112} and q_{121} as the mixed second derivatives of ξ.

Observe that $q_{112}\left(u,v+\frac{1}{2}\right)=q_{11}(u,v+1)-q_{11}(u,v)$. On one hand we have

$$q_{11}(u,v+1)=\left(1-\frac{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\right)q_1\left(u+\tfrac{1}{2},v+1\right)$$
$$+\frac{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v+1)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_2\left(u,v+\tfrac{1}{2}\right)$$

and

$$q_{11}(u,v)=\left(1-\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\right)q_1\left(u+\tfrac{1}{2},v\right)$$
$$+\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v+1)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_2\left(u,v+\tfrac{1}{2}\right).$$

But

$$q_1\left(u+\tfrac{1}{2},v+1\right)=q_1\left(u+\tfrac{1}{2},v\right)+q_{12}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)$$
$$=q_1\left(u+\tfrac{1}{2},v\right)+\lambda^{-1}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\xi_e\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right).$$

So the coefficient of $q_1\left(u+\frac{1}{2},v\right)$ in the expansion of $q_{112}\left(u,v+\frac{1}{2}\right)$ in the basis $\{q_1\left(u+\frac{1}{2},v\right),q_2\left(u,v+\frac{1}{2}\right),\xi_e\left(u+\frac{1}{2},v+\frac{1}{2}\right)\}$ is

$$\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}-\frac{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}.$$

On the other hand we can take $q_{11}(u,v)$ in terms of the quadrangle $\left(u+\frac{1}{2},v-\frac{1}{2}\right)$:

$$q_{11}(u,v)=\left(1-\frac{\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}\right)q_1\left(u+\tfrac{1}{2},v\right)$$
$$+\frac{\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)A(u,v+1)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}q_2\left(u,v-\tfrac{1}{2}\right).$$

We also set

$$q_2\left(u,v-\tfrac{1}{2}\right)=\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)B(u,v+1)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_1\left(u+\tfrac{1}{2},v\right)$$

$$+\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_2\left(u,v+\tfrac{1}{2}\right)$$

and then the coefficient of $q_1\left(u+\frac{1}{2},v\right)$ in the expansion of $q_{112}\left(u,v+\frac{1}{2}\right)$ in the basis $\{q_1\left(u+\frac{1}{2},v\right),q_2\left(u,v+\frac{1}{2}\right),\xi_e\left(u+\frac{1}{2},v+\frac{1}{2}\right)\}$ is now given by

$$\frac{\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}-\frac{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}$$
$$+\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)A(u,v)B(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}.$$

Comparing the two found coefficients we come to the first compatibility equation:

$$\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)} \tag{3.19}$$
$$=\frac{\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}+\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)A(u,v)B(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}$$

Let us now take a look at the second derivatives of the normal vector field ξ. Observe that

$$\xi_1^-\left(u,v+\tfrac{1}{2}\right)-\xi_1^+\left(u,v-\tfrac{1}{2}\right)=\xi_2^-\left(u,v+\tfrac{1}{2}\right)-\xi_2^+\left(u,v-\tfrac{1}{2}\right).$$

From Proposition (3.7.1) and its remark we know that

$$\xi_1^-\left(u,v+\tfrac{1}{2}\right)=\frac{\lambda^{-2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)-\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_1\left(u+\tfrac{1}{2},v\right)$$
$$+\frac{\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v+1)-A(u,v)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}q_2\left(u,v+\tfrac{1}{2}\right),$$

$$\xi_1^+\left(u,v-\tfrac{1}{2}\right)=\frac{\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)-\lambda^{2}\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\lambda^{-1}\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}q_1\left(u+\tfrac{1}{2},v\right)$$
$$+\frac{A(u,v)-\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)A(u,v-1)}{\lambda^{-1}\left(u+\frac{1}{2},v-\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}q_2\left(u,v-\tfrac{1}{2}\right),$$

$$\xi_2^-\left(u+\tfrac{1}{2},v\right)=\frac{\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)B(u,v+1)-B(u,v)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}q_1\left(u+\tfrac{1}{2},v\right)$$

$$+\frac{\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)-\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\, q_2\left(u,v+\tfrac{1}{2}\right)$$

and

$$\xi_2^-\left(u-\tfrac{1}{2},v\right)=\frac{B(u,v)-\lambda^{-2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)B(u-1,v)}{\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}\, q_1\left(u-\tfrac{1}{2},v\right)$$
$$+\frac{\lambda^{-2}\left(u-\frac{1}{2},v-\frac{1}{2}\right)-\lambda^{2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}\, q_2\left(u,v+\tfrac{1}{2}\right).$$

Moreover, Proposition (3.17) gives us

$$q_1(u-\tfrac{1}{2},v)=\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\, q_1\left(u+\tfrac{1}{2},v\right)$$
$$-\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\, q_2\left(u,v+\tfrac{1}{2}\right)$$

and

$$q_2\left(u,v-\tfrac{1}{2}\right)=\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)B(u,v+1)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\, q_1\left(u+\tfrac{1}{2},v\right)$$
$$+\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\, q_2\left(u,v+\tfrac{1}{2}\right).$$

If we compare the coefficients of $q_1\left(u+\frac{1}{2},v\right)$, we obtain the second compatibility equation:

$$\frac{\lambda^{-2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)-\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}-\frac{\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)-\lambda^{2}\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\lambda^{-1}\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)} \tag{3.20}$$
$$+\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)B(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\cdot\frac{A(u,v)-\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)A(u,v-1)}{\lambda^{-1}\left(u+\frac{1}{2},v-\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}$$
$$=\frac{\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)B(u+1,v)-B(u,v)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}$$
$$-\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\cdot\frac{B(u,v)-\lambda^{-2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)B(u-1,v)}{\lambda^{-1}\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}$$

Similarly, the third compatibility equation can be obtained by comparing the coefficients of $q_2\left(u,v+\frac{1}{2}\right)$:

$$
\begin{aligned}
&\frac{\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)-\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)} - \frac{\lambda^{-2}\left(u-\frac{1}{2},v-\frac{1}{2}\right)-\lambda^{2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)} \\
&+\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\cdot\frac{B(u,v)-\lambda^{-2}\left(u-\frac{1}{2},v+\frac{1}{2}\right)B(u-1,v)}{\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)} \\
&= \frac{\lambda^{2}\left(u+\frac{1}{2},v+\frac{1}{2}\right)A(u,v+1)-A(u,v)}{\lambda\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)} \\
&\quad-\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\cdot\frac{A(u,v)-\lambda^{-2}\left(u+\frac{1}{2},v-\frac{1}{2}\right)A(u,v-1)}{\lambda^{-1}\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}
\end{aligned}
\tag{3.21}
$$

Theorem 3.7.2 *Given functions Ω, λ, A and B satisfying the compatibility equations (3.19), (3.20) and (3.21), there exists an asymptotic net q satisfying Gauss equations (3.7.1). Moreover, two asymptotic nets with the same Ω, λ, A and B are affine equivalent.*

Proof.
We begin by choosing four points $q(0,0)$, $q(1,0)$, $q(0,1)$ and $q(1,1)$ satistying

$$[q(1,0)-q(0,0), q(0,1)-q(0,0), q(1,1)-q(1,0)] = \Omega^2\left(\tfrac{1}{2},\tfrac{1}{2}\right).$$

These four points are determined up to an affine transformation of $\mathbb{R}^3$.

From a quadrangle $\left(u-\frac{1}{2}, v-\frac{1}{2}\right)$ it is possible to extend the definition of q to the quadrangles $\left(u+\frac{1}{2}, v-\frac{1}{2}\right)$ and $\left(u-\frac{1}{2}, v+\frac{1}{2}\right)$ by using equations (3.17). From this point one can calculate both normal vectors $\xi_{e(o)}\left(u+\frac{1}{2}, v-\frac{1}{2}\right)$ and $\xi_{e(o)}\left(u-\frac{1}{2}, v+\frac{1}{2}\right)$. Furthermore, they satisfy Gauss equations (3.7.1) and the first compatibility equation determine the coherence of those extensions.

Then one can extend the definition of q to the quadrangle $\left(u+\frac{1}{2}, v+\frac{1}{2}\right)$ in two different ways: from $\left(u+\frac{1}{2}, v-\frac{1}{2}\right)$ and $\left(u-\frac{1}{2}, v+\frac{1}{2}\right)$. We just need to be sure that both extensions lead to the same result, and for this purpose we have to verify that both affine normals $\xi_{e(o)}\left(u+\frac{1}{2}, v+\frac{1}{2}\right)$ are the same, which reduces to verify that $\xi^{\pm}_{12}(u,v) = \xi^{\pm}_{21}(u,v)$. But this is assured by second and third compatibility equations. $\square$

4
Discrete indefinite improper affine spheres

In this chapter one can find two different approaches for improper affine spheres. In the first one, ruled improper affine spheres are the subject and a discrete genuine version of the Theorem (2.7.4) is given, based on a discrete centre-chord construction. In the second one, it is proposed a study about the singularities of discrete indefinite improper affine spheres, with original definitions and criteria to decide the nature of the singularity.

4.1
Ruled nets

Ruled nets are defined in the same way as in smooth case, that is, in at least one of the coordinates direction, u-curves or v-curves are all straight lines. If this happens for both parameters, the net is called double ruled.

Lemma 4.1.1 *Let $q : \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$ be an asymptotic net. Then the net q is ruled if and only if $A(u,v) = 0$ or $B(u,v) = 0$ for all $(u,v) \in \mathbb{Z}^2$.*

Proof.
Note that $B(u,v) = 0, \forall (u,v) \in \mathbb{Z}^2$ if and only if

$$B(u,v) = \left[q_2\left(u, v - \tfrac{1}{2}\right), q_2\left(u, v + \tfrac{1}{2}\right), \xi_+\left(u + \tfrac{1}{2}, v + \tfrac{1}{2}\right)\right] = 0,$$

equivalently

$$\left[q_2\left(u, v - \tfrac{1}{2}\right), q_2\left(u, v + \tfrac{1}{2}\right), q_{12}\left(u + \tfrac{1}{2}, v + \tfrac{1}{2}\right)\right] = 0,$$

which means that $q_2\left(u, v - \frac{1}{2}\right)$ is in the space spanned by $q_2\left(u, v + \frac{1}{2}\right)$ and $q_{12}\left(u + \frac{1}{2}, v + \frac{1}{2}\right)$.

Moreover, by the asymptotic net definition $q_2\left(u, v - \frac{1}{2}\right)$ is also in the space spanned by $q_1\left(u + \frac{1}{2}, v\right)$ and $q_2\left(u, v + \frac{1}{2}\right)$.

Since these two spaces are flat our first assumption is true if and only if $q_2\left(u, v - \frac{1}{2}\right)$ is in the same direction of $q_2\left(u, v + \frac{1}{2}\right)$. Since u and v are arbitrary, for u fixed $q_2\left(u, v + \frac{1}{2}\right)$ are in the same direction for all $v \in \mathbb{Z}$, that is, the net is ruled. □

Remark

Note that $A = 0$ or $B = 0$ determine if u-curves or v-curves are straight lines, respectively. If both A and B are null, then the net q is double ruled.

4.2 Centre-chord construction

We have done the centre-chord construction for the smooth case in Chapter 2. Now we propose the same construction for the discrete case, that is, we start from two discrete planar curves and construct a net.

Consider $\alpha : I \longrightarrow \mathbb{R}^2$ and $\beta : J \longrightarrow \mathbb{R}^2$, where $I, J \subset \mathbb{Z}$; $x(u,v) = \alpha(u) + \beta(v)$ and $y(u,v) = \beta(v) - \alpha(u)$, where $[\alpha_1(u+\frac{1}{2}), \beta_2(v+\frac{1}{2})] \neq 0$.

Define a function $z : I \times J \longrightarrow \mathbb{R}$ such that

$$z_1(u+\tfrac{1}{2}, v) = [x_1(u+\tfrac{1}{2}, v), y(u,v)],$$

$$z_2(u, v+\tfrac{1}{2}) = [x_2(u, v+\tfrac{1}{2}), y(u,v)].$$

Then

$$z_1(u+\tfrac{1}{2}, v) = [\alpha_1(u+\tfrac{1}{2}), y(u,v)],$$
$$z_2(u, v+\tfrac{1}{2}) = [\beta_2(v+\tfrac{1}{2}), y(u,v)].$$

In order to be sure that there is such a function let us calculate and compare the mixed second derivatives of z:

$$z_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) = [\alpha_1(u+\tfrac{1}{2}), \beta_2(v+\tfrac{1}{2})] = z_{21}(u+\tfrac{1}{2}, v+\tfrac{1}{2}).$$

Let us define a net

$$q(u,v) = (x(u,v), z(u,v)).$$

Then

$$\begin{aligned}
q_1(u+\tfrac{1}{2}, v) &= (\alpha_1(u+\tfrac{1}{2}), [\alpha_1(u+\tfrac{1}{2}), y(u,v)]),\\
q_2(u, v+\tfrac{1}{2}) &= (\beta_2(v+\tfrac{1}{2}), [\beta_2(v+\tfrac{1}{2}), y(u,v)]),\\
q_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) &= (0, [\alpha_1(u+\tfrac{1}{2}), \beta_2(v+\tfrac{1}{2})]),
\end{aligned}$$

$$\begin{aligned}
q_{11}(u,v) &= (\alpha_{11}(u), [\alpha_{11}(u), y(u,v)]),\\
q_{22}(u,v) &= (\beta_{22}(v), [\beta_{22}(v), y(u,v)]).
\end{aligned}$$

Since $[\alpha_1(u+\frac{1}{2}), \beta_2(v+\frac{1}{2})] \neq 0$, we can write α_{11} and β_{22} as a linear combination of α_1 and β_2 as

$$\alpha_{11}(u) = a(u,v)\alpha_1(u+\tfrac{1}{2}) + b(u,v)\beta_2(v+\tfrac{1}{2})$$

$$\beta_{22}(v) = c(u,v)\alpha_1(u+\tfrac{1}{2}) + d(u,v)\beta_2(v+\tfrac{1}{2})$$

for some functions a, b, c, d. So we have

$$q_{11}(u,v)(u) = a(u,v)q_1(u+\tfrac{1}{2},v) + b(u,v)q_2(u,v+\tfrac{1}{2})$$

$$q_{22}(u,v)(v) = c(u,v)q_1(u+\tfrac{1}{2},v) + d(u,v)q_2(u,v+\tfrac{1}{2})$$
$$q_{12}(u+\tfrac{1}{2},v+\tfrac{1}{2}) = [\alpha_1(u+\tfrac{1}{2}), \beta_2(v+\tfrac{1}{2})]\xi$$

where $\xi = (0,0,1)$. This means that $q(u,v)$ is asymptotically parameterized and an improper affine sphere.

The cubic form is given by

$$A = [q_1(u-\tfrac{1}{2},v), q_1(u+\tfrac{1}{2},v), \xi] = [\alpha_1(u-\tfrac{1}{2}), \alpha_1(u+\tfrac{1}{2})],$$

$$B = [q_2(u,v-\tfrac{1}{2}), q_2(u,v+\tfrac{1}{2})\xi] = [\beta_2(v-\tfrac{1}{2}), \beta_2(v+\tfrac{1}{2})].$$

Since α and β are planar curves, q is ruled if and only if α or β is a straight line.

4.2.1
Discrete generalized area distance

In the smooth case we have seen that the centre-chord construction leads to the generalized area distance map. The same can be made in the discrete case, as we shall see.

Definition 4.2.1 *The map $q : I \times J \longrightarrow \mathbb{R}^3$ given by*

$$q(u,u) \longrightarrow (x(u,v), z(u,v)),$$

where x and z are the maps defined in the centre-chord construction, is called the discrete generalized area distance *of the pair of discrete planar curves $(\alpha(u), \beta(v))$.*

Note that the area $A(u,v)$ of the region limited by the curves α, β and the segments $\overline{\alpha(0)\beta(0)}$ and $\overline{\alpha(u)\beta(v)}$, can be expressed as a sum of areas of triangles, as shown if Figure (4.1).

Then $A_1(u+\frac{1}{2},v) = A(u+1,v) - A(u,v)$ is equal to the area of the triangle formed by $\alpha_1(u+\frac{1}{2})$ and $\beta(v)$, that is,

$$A_1(u+\tfrac{1}{2},v) = \frac{1}{2}\left[\alpha_1(u+\tfrac{1}{2}), \beta(v)-\alpha(u)\right] = \frac{1}{2}z_1(u+\tfrac{1}{2},v).$$

Similarly,

$$A_2(u,v+\tfrac{1}{2}) = \frac{1}{2}\left[\beta_2(v+\tfrac{1}{2}), \beta(v)-\alpha(u)\right] = \frac{1}{2}z_2(u,v+\tfrac{1}{2}),$$

although this one does not correspond to Figure (4.1).

Then, up to a constant, z is equal to the double area.

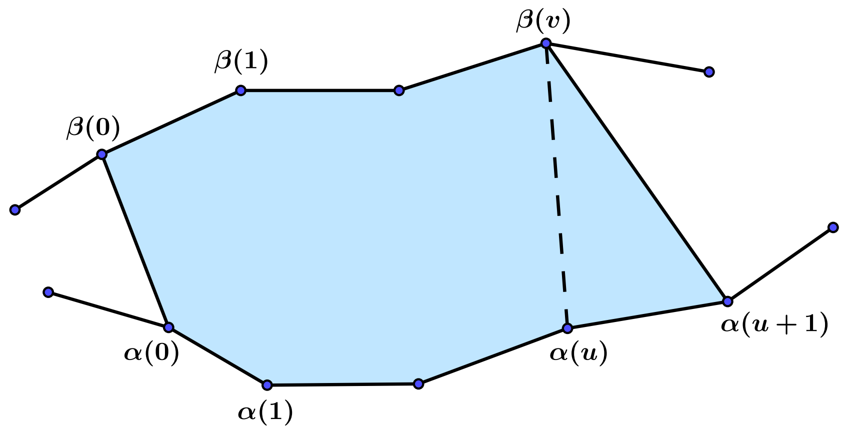


Figure 4.1: Discrete generalized area distance.

Remark.
In the smooth case z is half the area. This did not take place here because in discrete centre-chord construction we chose to let the half out, *i.e.*, we set up $x(u,v) = \alpha(u)+\beta(v)$ and $y(u,v) = \beta(v)-\alpha(u)$, instead of $x(u,v) = \frac{1}{2}(\alpha(u)+\beta(v))$ and $y(u,v) = \frac{1}{2}(\beta(v)-\alpha(u))$. This difference does not change any result of this chapter and we will take the half when it is convenient.

Proposition 4.2.2 *If $q : I \times J \longrightarrow \mathbb{R}^3$ is an improper affine sphere, then it can be parameterized by centre-chord construction, that is,*

$$q(u,v) = (x(u,v), z(u,v)),$$

where $x(u,v) = \alpha(u)+\beta(v)$, $z_1(u+\frac{1}{2},v) = [x_1(u+\frac{1}{2},v), y(u,v)]$, $z_2(u,v+\frac{1}{2}) = [x_2(u,v+\frac{1}{2}), y(u,v)]$ and $y(u,v) = \beta(v)-\alpha(u)$, for some curves $\alpha : I \longrightarrow \mathbb{R}^2$ and $\beta : J \longrightarrow \mathbb{R}^2$, where $I, J \subset \mathbb{Z}$. In another words, the map q is the discrete generalized area distance of the pair of curves (α,β).

Proof.
Since q is an improper affine sphere, let us assume that $\xi = (0,0,1)$ and let us take an asymptotic parameterization

$$q(u,v) = (x(u,v), z(u,v)),$$

where $x(u,v)$ is the projection in the plane $\{(1,0,0),(0,1,0)\}$.

Note that

$$q_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) = \Omega_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2})\,\xi,$$

but we also can write it as

$$q_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) = (x_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}), z_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2})),$$

which means that

$$x_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) = 0$$

and

$$z_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2})) = \Omega_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}).$$

Thus,

$$x(u,v) = \alpha(u) + \beta(v),$$

for some curves $\alpha : I \longrightarrow \mathbb{R}^2$ and $\beta : J \longrightarrow \mathbb{R}^2$, where $I, J \subset \mathbb{Z}$.

Moreover,

$$\begin{aligned} q_1(u+\tfrac{1}{2}, v) &= \left(\alpha_1(u+\tfrac{1}{2}),\, z_1(u+\tfrac{1}{2})\right), \\ q_2(u, v+\tfrac{1}{2}) &= \left(\beta_2(v+\tfrac{1}{2}),\, z_2(v+\tfrac{1}{2})\right), \\ q_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) &= \left(0,\, \Omega_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2})\right). \end{aligned}$$

Hence

$$\Omega^2 = [q_1, q_2, q_{12}] = \Omega\,\left[\alpha_1(u+\tfrac{1}{2}),\, \beta_2(v+\tfrac{1}{2})\right]$$

and

$$\Omega_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) = \left[\alpha_1(u+\tfrac{1}{2}),\, \beta_2(v+\tfrac{1}{2})\right] > 0.$$

Thereafter,

$$z_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2})) = \left[\alpha_1(u+\tfrac{1}{2}),\, \beta_2(v+\tfrac{1}{2})\right],$$

and by discrete integration we have

$$z_1(u+\tfrac{1}{2},v) = \left[\alpha_1(u+\tfrac{1}{2}),\ \beta(v)+f(u)\right],$$
$$z_2(u,v+\tfrac{1}{2}) = \left[\alpha(u)+g(v),\ \beta_2(v+\tfrac{1}{2})\right],$$

for some curves f and g.

If we set

$$z_1(u+\tfrac{1}{2},v) = \left[x_1(u+\tfrac{1}{2}),\ y(u,v)\right],$$
$$z_2(u,v+\tfrac{1}{2}) = \left[x_2(v+\tfrac{1}{2}),\ y(u,v)\right],$$

we find $y(u,v) = \beta(v) - \alpha(u)$ and the proof is completed. □

Theorem 4.2.3 *If q is a discrete ruled improper affine sphere, then it is of the form*

$$z = xy + \varphi(x),$$

for some real function φ.

Proof.
Since the net is as improper affine sphere we can take the above centre-chord parameterization. Furthermore, the ruled hypothesis allow us to take α or β as a straight line. So w.l.g. let us set

$$\alpha(u) = (\alpha^1(u), \alpha^2(u)) \quad \text{and} \quad \beta = (0, v),$$

where $\alpha^1, \alpha^2 : \mathbb{Z} \longrightarrow \mathbb{R}$ and $\alpha_1(u+\frac{1}{2}) \neq 0$.

Then

$$x(u,v) = (\alpha^1(u), \alpha^2(u)+v),$$
$$y(u,v) = (-\alpha^1(u), v-\alpha^2(u)),$$
$$z_1(u+\tfrac{1}{2},v) = v\alpha_1^1(u+\tfrac{1}{2}) - \alpha_1^1(u+\tfrac{1}{2})\alpha^2(u) + \alpha^1(u)\alpha_1^2(u+\tfrac{1}{2}),$$
$$z_2(u,v+\tfrac{1}{2}) = \alpha_1(u+\tfrac{1}{2}).$$

By discrete integration on v we have $z(u,v) = v\alpha_1(u+\frac{1}{2}) + g(u)$ for some real function g. But the first derivative of z on u must agree with the expression of z_1, so

$$g_1(u+\tfrac{1}{2}) = -[\alpha_1(u+\tfrac{1}{2}), \alpha(u)]$$

and

$$q(u,v) = \left(\alpha^1(u),\ \alpha^2(u)+v,\ v\alpha^1(u)+g(u)\right),$$

which means that $z = xy + \varphi(x)$, where $\varphi(x) = g(u) - \alpha^1(u)\alpha^2(u)$. □

4.3
Discrete improper indefinite affine spheres with singularities

Our intention here is to give a proper definition to singularities for discrete improper indefinite affine spheres and how to classify them. From Proposition (4.2.2) they can be parameterized by centre-chord construction, so we will make use of this tool.

Consider two discrete planar curves $\alpha : I \longrightarrow \mathbb{R}^2$ and $\beta : J \longrightarrow \mathbb{R}^2$, where $I, J \subset \mathbb{Z}$, such that:

- For any point $\alpha(u)$ and any triplet $\beta(v-1)$, $\beta(v)$, $\beta(v+1)$, we have that $\beta(v)$ is within the interior of the angle $\beta(v-1)\alpha(u)\beta(v+1)$, supposed less than 180°.
- For any point $\beta(v)$ and any triplet $\alpha(u-1)$, $\alpha(u)$, $\alpha(u+1)$, we have that $\alpha(u)$ is within the interior of the angle $\alpha(u-1)\beta(v)\alpha(u+1)$, supposed less than 180°. See Figure (4.2).

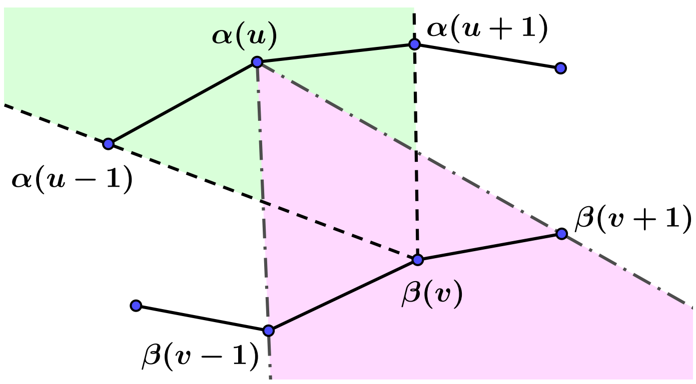


Figure 4.2: Restriction to the pair of planar curves (α, β).

This restriction is made to simplify our first model of singularities, but we think that is something to be explored in future works about the subject.

4.3.1
Singularity set

The singular set defined in the smooth case gives us a set of all midpoints of chords connecting α and β with parallel tangents. So it is essential to set up a definition for parallelism in the context of discrete planar curves α and β. First of all, it is not possible to take parallel derivatives, because it would imply in two parallel edges and this is something that we must avoid, since it would generate a planar quadrangle in the net, which is degenerate by definition. Then we will make a definition of parallelism by a relation between a point of one curve and an edge of the other, as follow.

Definition 4.3.1 *Given a pair of planar curves* (α, β) *we say that:*

- $\beta_2(v+\frac{1}{2})$ *is parallel to* α *at* $\alpha(u)$ *if* $\alpha(u-1)$ *and* $\alpha(u+1)$ *are in the same half-plane determined by the straight line given by* $\alpha(u)+r\beta_2(v+\frac{1}{2})$*, for* $r \in \mathbb{R}$*. See Figure (4.3);*
- *Similarly,* $\alpha_1(u+\frac{1}{2})$ *is parallel to* β *at* $\beta(v)$ *if* $\beta(v-1)$ *and* $\beta(v+1)$ *are in the same half-plane determined by the straight line given by* $\beta(v)+r\alpha_1(u+\frac{1}{2})$*, for* $r \in \mathbb{R}$*.*

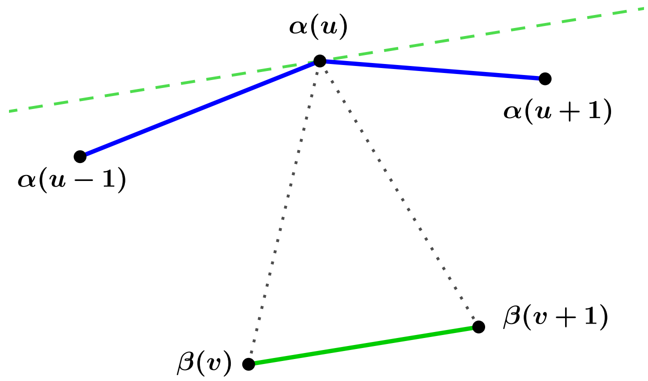


Figure 4.3: Parallelism between $\beta_2(v+\frac{1}{2})$ and $\alpha(u)$. Note that the green dotted line is parallel to the derivative of β (an edge of the curve β) and let $\alpha(u-1)$ and $\alpha(u+1)$ in the same half-plane.

Definition 4.3.2 *The singularity set* S *of* q *consists of all pairs* (u, v) *for which* $\alpha_1(u+\frac{1}{2})$ *is parallel to* β *at* $\beta(v)$ *or* $\beta_2(v+\frac{1}{2})$ *is parallel to* α *at* $\alpha(u)$*, accordingly to Definition (4.3.1).*

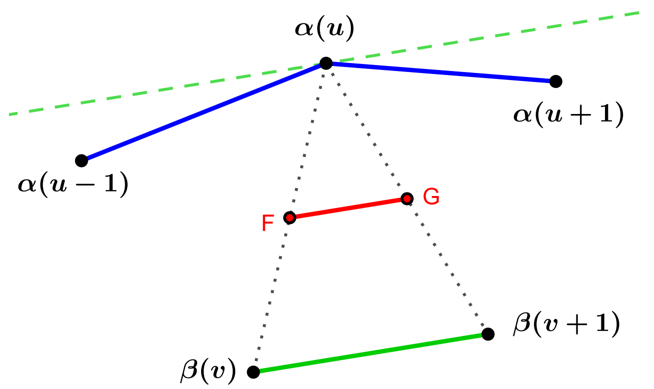


Figure 4.4: The DMPTL curve is formed by the midsegments of the triangles formed in points of tangency. In this example, FG represents an edge of the DMPTL of the pair of curves (α, β).

Geometrically, the set $x(S)$ consists of all midpoints of chords connecting $\alpha(u)$ and $\beta(v)$ where one of two kinds of tangents occurs. This set generate a discrete curve that shall be called *discrete midpoint parallel tangent locus* (DMPTL) or *discrete area evolute* of the pair of curves (α, β). Observe that the parallelism at one point is associated to a triangle formed by the point of one curve and the derivative of the other, as we can see on Figure (4.4). So the DMPTL curve will be formed by the midsegments of each of these triangles.

Let us study the behaviour of the DMPTL curve and how to construct it pass by pass. Suppose that $\alpha_1(u+\frac{1}{2})$ is parallel to $\beta(v)$ for some u and v, then we have formed a triangle and its midsegment is part of the DMPTL. The next pass is to decide what adjacent triangle we should choose and that is going to be clear after next Proposition.

Proposition 4.3.3 *Let A, B and C be three successive points on α; D, E and F successive points on β, such that AB is parallel to E. Then only one of the three follow statements is true:*

- *BC is parallel to E, as in Figure (4.6 - left);*
- *EF is parallel to B, as in Figure (4.6 - right);*
- *DE is parallel to B, as in Figure (4.7).*

Proof.
Let us fix A, B, D, E and F such that the hypothesis keep valid and see what can happens to the point C.

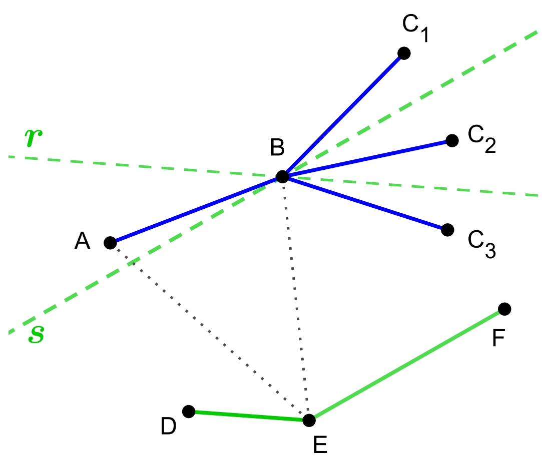


Figure 4.5: The three possible configurations for the vertex C.

Let r and s be straight lines passing by B and parallel to the edges DE and EF, respectively. They divide the plane in four open regions and the point C can be only in three of them. In fact, it can not be at the same region wherein is the point A, because the restriction made to the curves α and β at the beginning of the section. Also, since there is no parallelism between edges of α and β, C can not be either in the straight line r or in s.

From the parallelism between AB and E, one know that A and E are in two different of those regions. Let us call region A that one wherein is the point A, and similarly for region E. The other two will be regions $\overline{A}$ and $\overline{E}$, where the points $\overline{A}$ and $\overline{E}$ are symmetric to A and E with respect to B, respectively.

Suppose that C is in the region $\overline{A}$ (C_2 in Figure 4.5). So neither s nor r let A and C in the same half-plane, which means that the parallel to BC passing by E do that with D and F. Then BC is parallel to E and we get the first item of the Proposition.

Let us know suppose that C is in the region $\overline{E}$ (C_1 in Figure 4.5). So s let A and C in the same half-plane, which means that EF is parallel to B and we get the second item of the Proposition.

Finally, C can be in the region E (C_3 in Figure 4.5). So r let A and C in the same half-plane, which means that DE is parallel to B and we get the third item of the Proposition.

There is no other possibility to the point C and the parallelism is uniquely determined by the chosen region for C, so the proof is completed. □

Figures (4.6) and (4.7) show that in all of three options we have constructed the DMPTL given by MNP composed by two midsegments. Now we can extend this construction for all points of both curves α and β, from the points M and P.

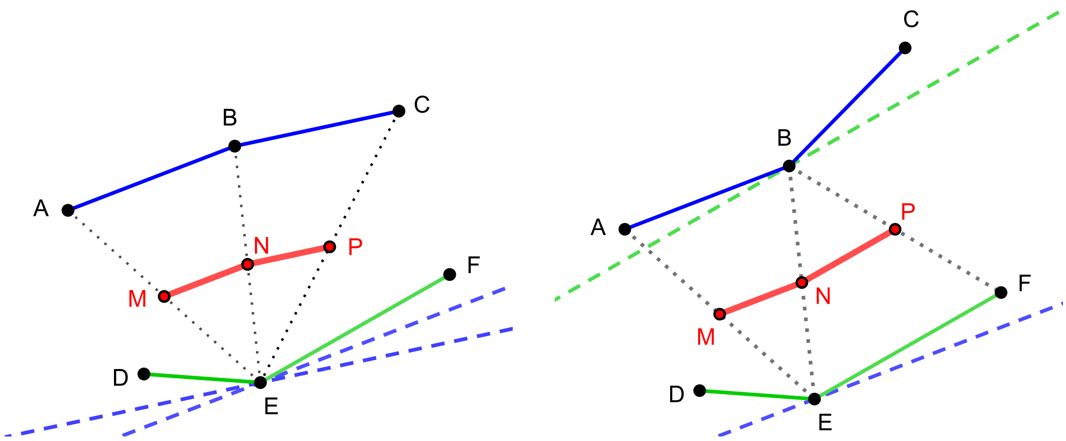


Figure 4.6: On the left, the first possibility for the construction of the DMPTL, where BC is parallel to E. On the right, the second one, where EF is parallel to B.

Observe that in first and second possibilities, the orientation of u or v did not change, whilst in the third one, the orientation of v has changed. This point of orientation change at the curve DMPTL shall be called a *cusp*. From the restriction made on the pair of curves α and β we have that a cusp occurs at N when M and P are in the same half-plane determined by BE, as in Figure (4.7).

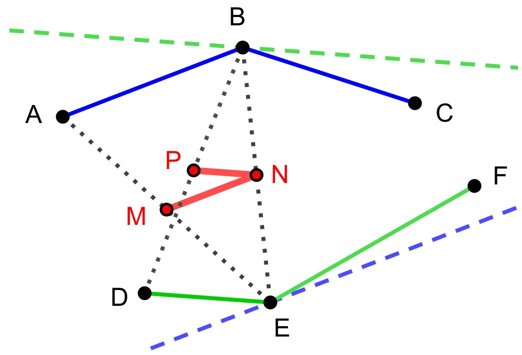
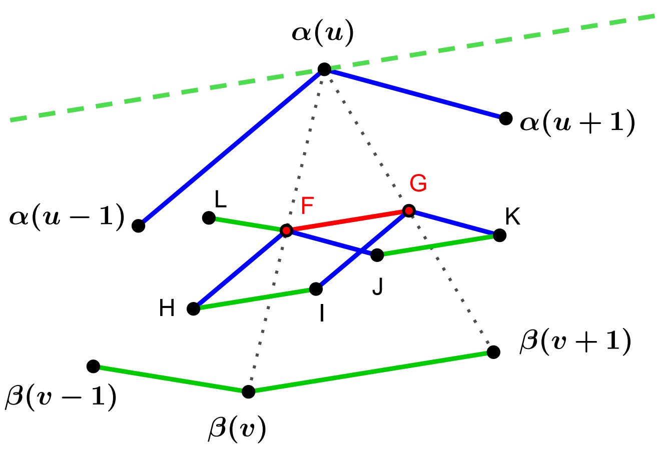


Figure 4.7: Third possibility for the construction of the DMPTL, where DE is parallel to B.

4.3.2
Relation between the DMPTL and the singularities at the net

From the curve DMPTL associated to the pair of curves (α, β) we want to define and classify the singularities in the discrete generalized area distance, *i.e.*, the indefinite improper affine sphere.

Let us begin by the edges of the DMPTL, which will be correspondent to regular points in the smooth case, that is, they will generate the cuspidal edges of the net.



$$F = x(u,v)$$
$$G = x(u,v+1)$$
$$H = x(u-1,v)$$
$$I = x(u-1,v+1)$$
$$J = x(u+1,v)$$
$$K = x(u+1,v+1)$$
$$L = x(u,v-1)$$

Figure 4.8: The edge of the DMPTL let the vertices of the two adjacent quadrilaterals in the same half-plane.

Consider a pair of curves (α, β) such that $\beta_2(v+\frac{1}{2})$ is parallel to $\alpha(u)$, as in Figure (4.8). The segment FG (in red) is an edge of the DMPTL and generate the edge $q_2\left(u, v+\frac{1}{2}\right)$ of the net q. Note that in the plane, the straight line that contains FG let H, I, J and K in the same half-plane, that is, $x(u-1,v)$, $x(u-1,v+1)$, $x(u+1,v)$ and $x(u+1,v+1)$, respectively, because the parallelism with the edges of α (blue segments). This means that the same happens at the net in both star planes[1] at $q(u,v)$ and $q(u,v+1)$, since points x are projections of the net q in the plane of α and β. Then we come to the definition of a cuspidal edge.

Definition 4.3.4 *Let $q : I \times J \subset \mathbb{Z}^2 \longrightarrow R^3$ be an indefinite improper affine sphere. Then the edge $q_1\left(u+\frac{1}{2}, v\right)$ is called a* cuspidal edge *if the straight line $q(u,v) + rq_1\left(u+\frac{1}{2}, v\right)$, with $r \in \mathbb{R}$, in the star plane at $q(u,v)$, let $q(u,v-1)$ and $q(u,v+1)$ in the same half-plane. Similarly, the edge $q_2\left(u, v+\frac{1}{2}\right)$ is a* cuspidal edge *if the straight line $q(u,v) + sq_2\left(u, v+\frac{1}{2}\right)$, with $s \in \mathbb{R}$, in the star plane at $q(u,v)$, let $q(u-1,v)$ and $q(u+1,v)$ in the same half-plane. This can be seen in Figure (4.9).*

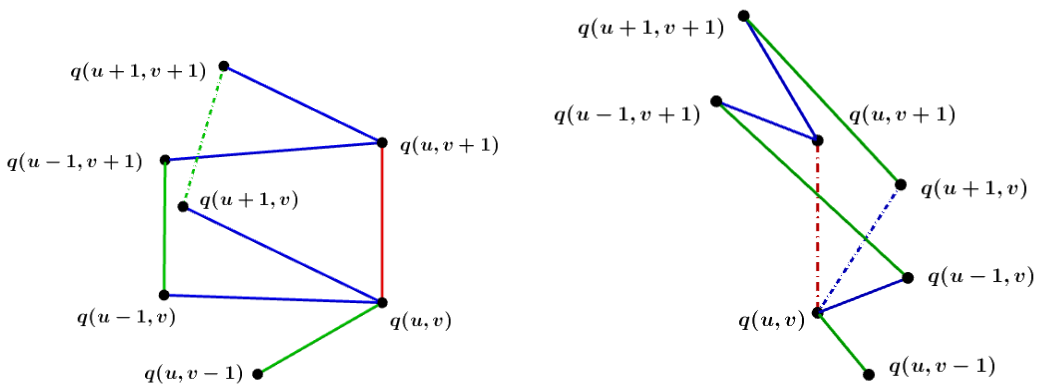


Figure 4.9: A cuspidal edge (in red) in the net q from two different viewpoints. It is easy to see that in the star plane at $q(u,v)$ the pair of vertices $q(u-1,v)$ and $q(u+1,v)$ is in the same half-plane, and the same happens to the pair $q(u-1,v+1)$ and $q(u+1,v+1)$ in the star plane at $q(u,v+1)$.

Proposition 4.3.5 *Let $q : I \times J \subset \mathbb{Z}^2 \longrightarrow R^3$ be an indefinite improper affine sphere. Then the edge $q_1\left(u+\frac{1}{2}, v\right)$ is a cuspidal one if and only if the star plane at $q(u,v)$ let $q(u+1,v-1)$ and $q(u+1,v+1)$ in the same half-space determined by it. Similarly, the edge $q_2\left(u, v+\frac{1}{2}\right)$ is cuspidal if and only if the star plane at $q(u,v)$ let $q(u-1,v+1)$ and $q(u+1,v+1)$ in the same half-space determined by it. See Figure (4.9).*

[1]Since at each vertex the four edges are planar, from now on, the plane that contains these edges will be referred to as the star plane at the vertex.

Proof.
Consider two planes Π_1 and Π_2 that intersect each other at the straight line γ. Then two points A and B are in the same half-plane of Π_1 determined by γ if and only if they are in the same half-space determined by Π_2. Applying this to the star planes at $q(u, v)$ and $q(u + 1, v)$, the proposition follows. □

Before we discuss about a cusp of the DMPTL, let us make some notes about possible configurations for star planes in the net.

Definition 4.3.6 *A star plane at $q(u, v)$ is called* typical *if the four points $q(u+1, v)$, $q(u, v+1)$, $q(u-1, v)$ and $q(u, v-1)$ appear in this order, clockwise or counter clockwise, with respect to $q(u, v)$. And it shall be called* atypical *otherwise. See Figure (4.10).*

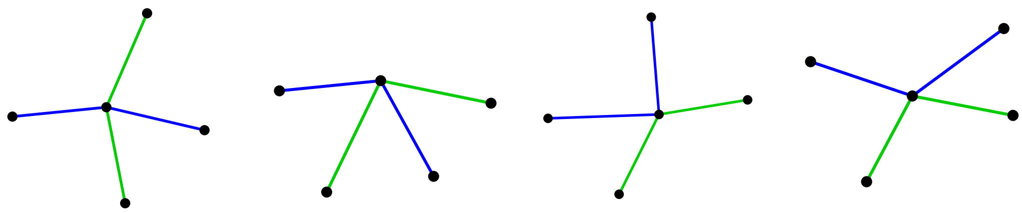


Figure 4.10: Both figures on the left show two different possibilities of a typical star, whilst both on the right show two possible configurations for an atypical one. The colors identify the u and v directions.

A typical star can have two different configurations that are of interest, as shown in Figure (4.10). If there is or not two different directions that form an angle which measure is greater than 180° at $q(u, v)$.

If all pairs of two different directions form angles with less than 180° (first star on Figure 4.10), then there is no cuspidal edges at the star. Otherwise, there is a pair of cuspidal edges and they are those forming the bigger angle (second star on Figure 4.10). In fact, since both of them let the three other vertices at the same half-plane, they are cuspidal edges.

An atypical star has always a pair of cuspidal edges, which can be either in same or different directions. The first case can be seen in Figure (4.10) on the third star, since both green directions let both blue ones in the same half-plane. We could have the inverse situation where both blue directions would let both green ones in the same half-plane, and they would be the pair of cuspidal edges. If these two cases do not happen, then we have the second possibility (last star on Figure 4.10), since one of green directions let both blue ones in the same half-plane and a similar split up happens when one of the blue directions is taken.

We have defined a cusp in the DMPTL by the change of direction and orientation. That lead us to the definition of a swallowtail point in the net.

Definition 4.3.7 *Let $q : I \times J \subset \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$ be an indefinite improper affine sphere. Then the vertex $q(u,v)$ is called a* swallowtail point *if two adjacent edges in different directions are cuspidal and the star plane at $q(u,v)$ is an atypical one. See last star on Figure (4.10).*

Theorem 4.3.8 *Consider two discrete planar curves $\alpha : I \longrightarrow \mathbb{R}^2$ and $\beta : J \longrightarrow \mathbb{R}^2$, where $I, J \subset \mathbb{Z}$, and let $q : I \times J \longrightarrow \mathbb{R}^3$ be the discrete generalized area distance given by*

$$q(u,v) = (x(u,v), z(u,v)),$$

where $x(u,v) = \alpha(u) + \beta(v)$, $y(u,v) = \beta(v) - \alpha(u)$, $z_1(u + \frac{1}{2}, v) = [x_1(u + \frac{1}{2}, v), y(u,v)]$ and $z_2(u, v + \frac{1}{2}) = [x_2(u, v + \frac{1}{2}), y(u,v)]$. Then

(i) An edge of the net q is cuspidal if and only if it is associated to an edge of the planar curve DMPTL.

(ii) A vertex of the net q is a swallowtail point if and only if it is associated to a cusp of the planar curve DMPTL.

Proof.
Observe that item (i) follows directly from the construction of the definition of a cuspidal edge (4.3.4). Then we have to think only about the item (ii).

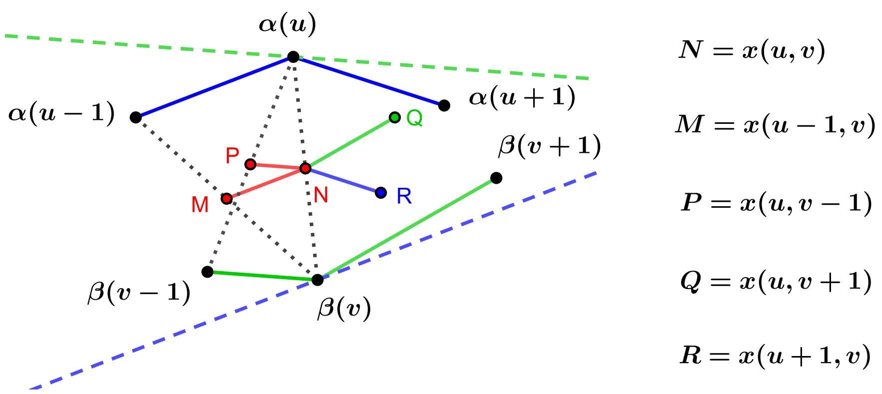


Figure 4.11: A cusp in the DMPTL forms an atypical star.

A cusp $x(u,v)$ in the DMPTL is a point of direction change as one can see in Figure (4.11) and it generates the vertex $q(u,v)$ of the net. Moreover, it connects two edges of the DMPTL that generate two adjacent cuspidal edges in different directions in the star plane at $q(u,v)$. So it is only necessary to verify that this star is an atypical one.

Note that the edge $x(u,v)x(u,v-1)$ let $x(u-1,v)$ and $x(u+1,v)$ in the same half-plane, whilst the edge $x(u,v)x(u-1,v)$ does the same with the points $x(u,v-1)$ and $x(u,v+1)$. That means these five points form an atypical star in the plane that contains the DMPTL and this plane is a projection of the star plane at $q(u,v)$. Thus we come to the expected conclusion. □

Let us remember that a swallowtail point in the smooth case always imply in self-intersection, so we expect that behaviour in discrete case too.

Proposition 4.3.9 *Let $q : I \times J \subset \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$ be an indefinite improper affine sphere. If $q(u,v)$ is a swallowtail point, then there is a pair of quadrangles, with $q(u,v)$ as a vertex, that intersect each other.*

Proof.
Let $q(u,v)$ be a swallowtail point in an indefinite improper affine sphere, with adjacent cuspidal edges $q_1\left(u-\frac{1}{2},v\right)$ and $q_2\left(u,v-\frac{1}{2}\right)$ as shown in Figure (4.12).

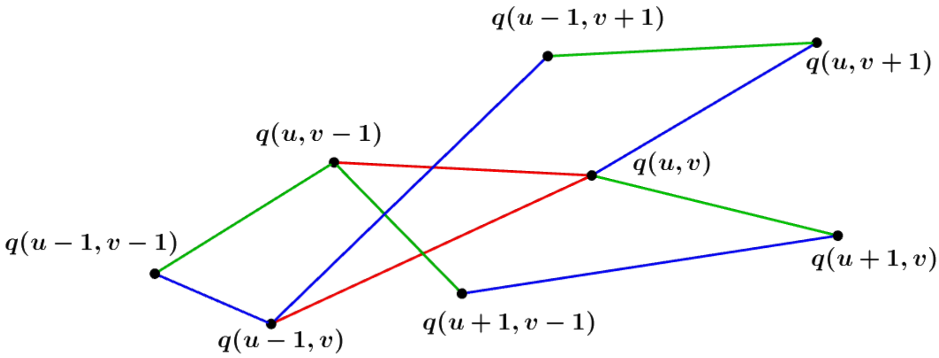


Figure 4.12: Swallowtail point.

Since $q_1\left(u-\frac{1}{2},v\right)$ is a cuspidal edge, by Proposition (4.3.5) the star plane at $q(u,v)$, let us call it Π, leaves $q(u-1,v-1)$ and $q(u-1,v+1)$ in the same half-space. Similarly, $q_2\left(u,v-\frac{1}{2}\right)$ is a cuspidal edge and Π let $q(u-1,v-1)$ and $q(u+1,v-1)$ in the same half-space (see Figure 4.12). Thus we conclude that all three vertices $q(u-1,v-1)$, $q(u-1,v+1)$ and $q(u+1,v-1)$ are in the same half-space determined by Π.

Consider the star planes at $q(u+1,v-1)$ and $q(u-1,v+1)$. They intersect the plane Π in two straight lines, $q(u,v-1)q(u+1,v)$ and $q(u-1,v),q(u,v+1)$, respectively. But these straight lines intersect each other since the star is atypical. Then, the planes must also intersect each other, which means that this happens to que quadrangles $\left(u+\frac{1}{2},v-\frac{1}{2}\right)$ and $\left(u-\frac{1}{2},v+\frac{1}{2}\right)$. □

Remark.
This proof also shows that intersecting quadrangles are either both even or both odd.

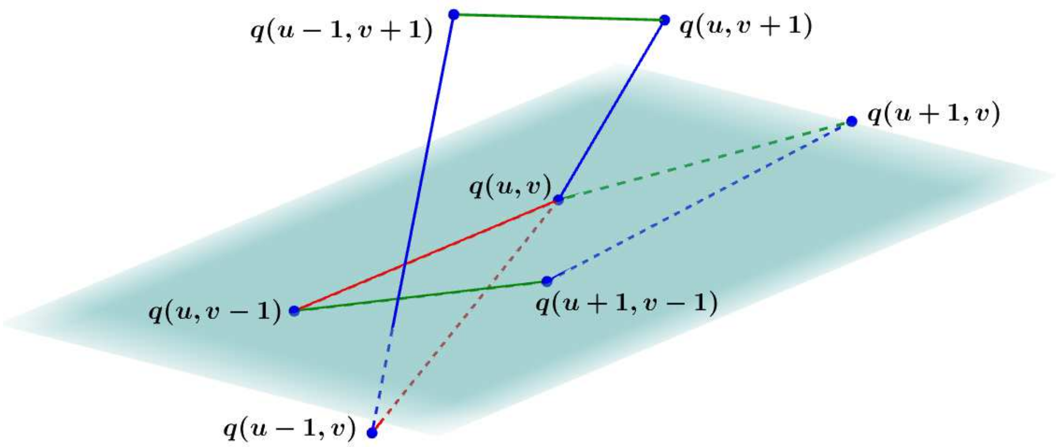


Figure 4.13: A swallowtail at $q(u,v)$ with two adjacent cuspidal edges (in red). Note that blue and green edges are v and u-directions, respectively. The star plane at $q(u,v-1)$ helps to see the intersection between two quadrangles.

After all we can summarize the relation between the planar curve DMPTL given by the pair of curves α and β, and the asymptotic net at a star plane wherein there are two cuspidal edges and eventually one swallowtail point, in three configurations. Let us see each of them separately.

Fisrt configuration: This is the first possibility considered by Proposition (4.3.3). In DMPTL we have a pair of edges in the same direction (Figure 4.14 - left) and in an atypical star a pair of cuspidal edges (Figure 4.14 - right). The full net is shown in Figure (4.16 - left).

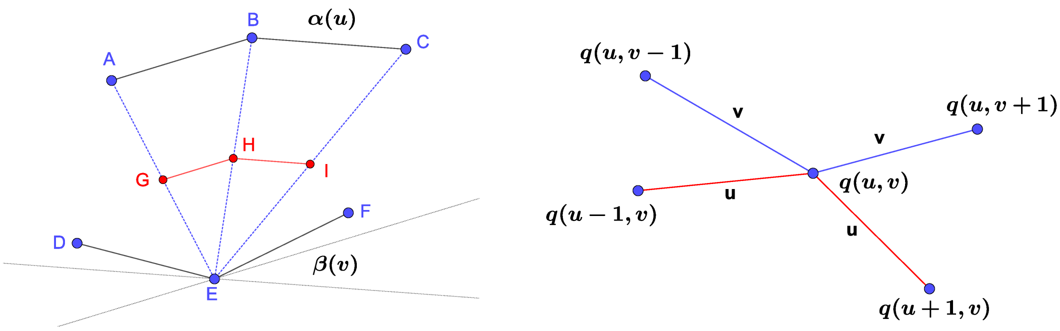


Figure 4.14: A pair of edges in DMPTL (left) that produces a pair of cuspidal edges in the same direction $u-u$ in an atypical star (right), when there is no swallowtail point.

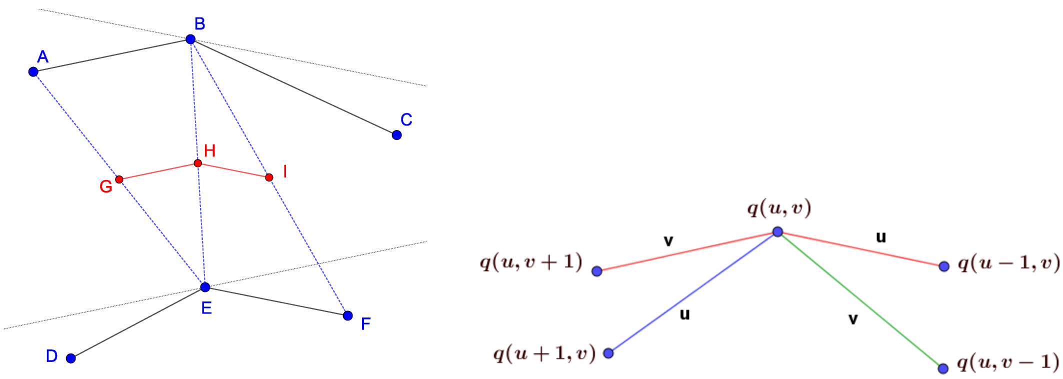


Figure 4.15: A pair of edges in DMPTL (left) that produces a pair of cuspidal edges in different directions $u-v$ in a typical star (right), when there is no swallowtail point.

Second configuration: This represents the second possibility shown in Proposition (4.3.3). One can see the DMPTL with a pair of edges in different directions (Figure 4.15 - left) and a typical star with a pair of cuspidal edges (Figure 4.15 - right). The full net can be seen in Figure (4.16 - right).

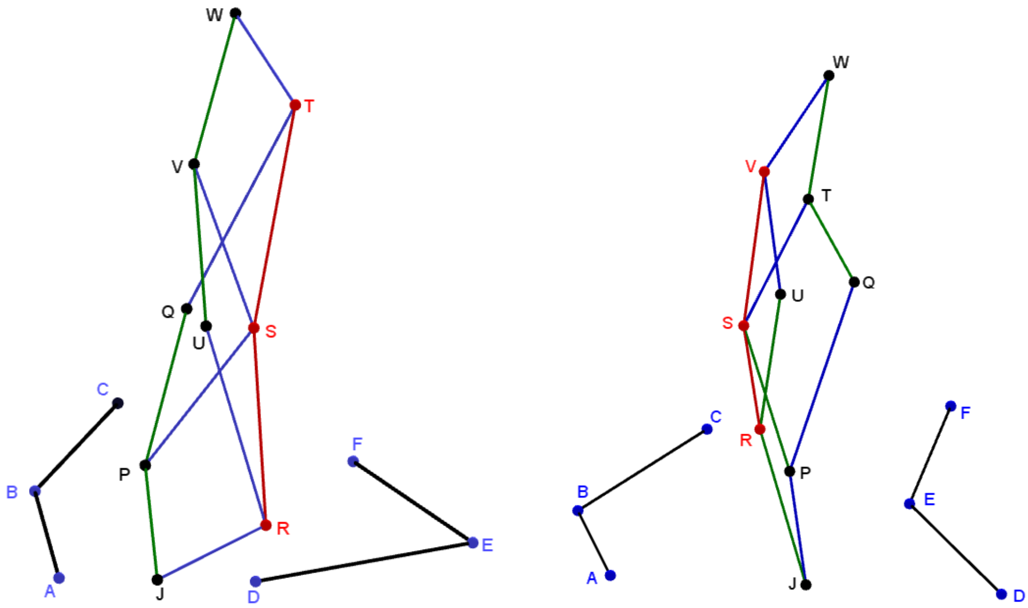


Figure 4.16: Discrete generalized area distance of the pair (α, β) with an atypical (left) and a typical (right) star, both of them with a pair of cuspidal edges with no swallowtail, representing first and second configurations, respectively.

Third configuration: As the other two, it represents the third possibility of Proposition (4.3.3). It is possile to visualize the DMPTL with a pair of edges in different directions (Figure 4.17 - left) and an atypical star where there is a swallowtail point between a pair of cuspidal edges (Figure 4.17 - right). The full net is drawn in Figure (4.18).

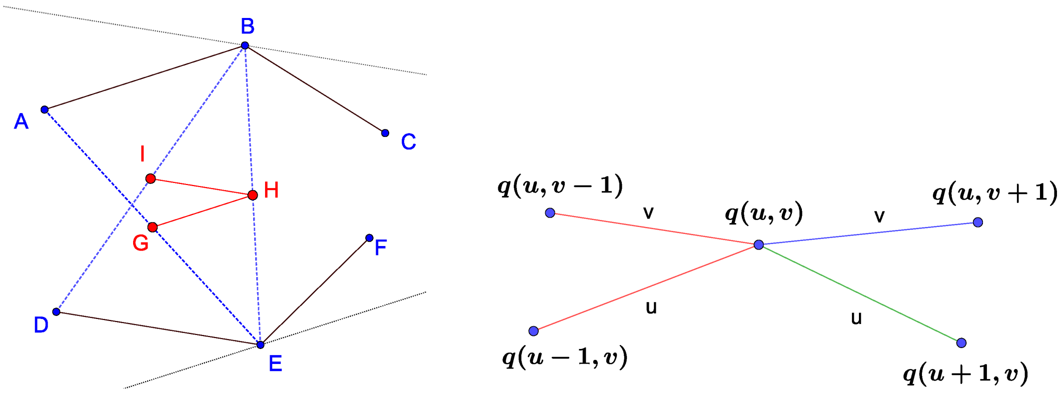


Figure 4.17: A cusp in DMPTL (left) that produces a swallowtail point in an atypical star (right) with a pair of cuspidal edges in different directions $u - v$.

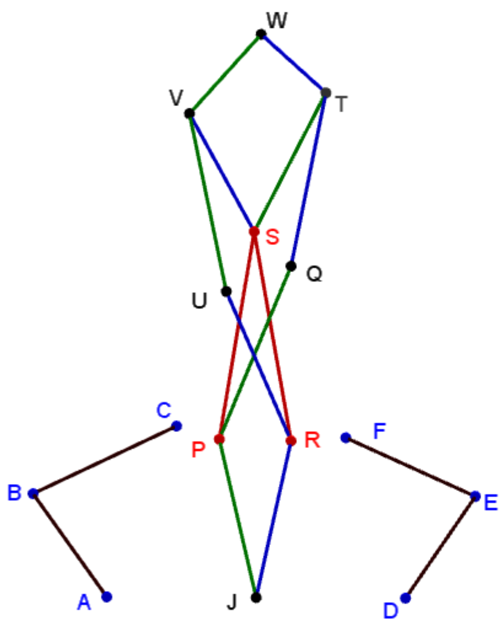


Figure 4.18: Discrete generalized area distance of the pair (α, β) with an atypical star, with a pair of cuspidal edges and a swallowtail point.

4.3.3
Example of a discrete improper affine sphere with singularities

Let us construct here an example of a discrete improper affine sphere with singularities from two curves α and β making use of centre-chord construction.

Consider the curves given by

$$\alpha(u) = \left(u,\, 5 - \frac{(u-2)^2}{8}\right) \quad \text{and} \quad \beta(v) = (v^2 - 2,\, v).$$

We show in Figure (4.19) the MPTL in the smooth case aiming to compare it with the DMPTL of Figure (4.20). Note that both of them are formed by two connected components and only one presents a cusp, which means that both surfaces (smooth and discrete) generated by the pair (α, β) contain two cuspidal curves and a unique swallowtail point.

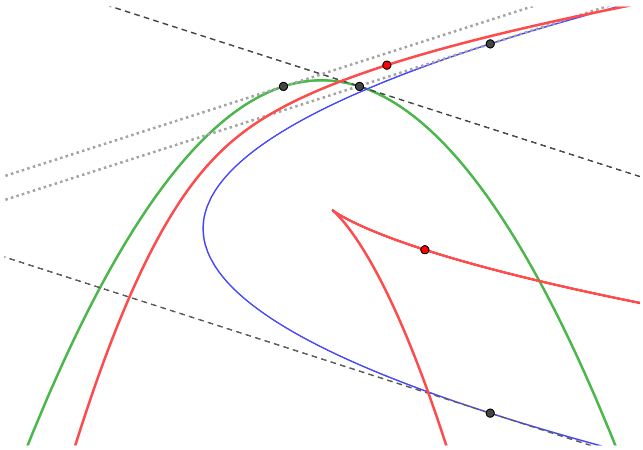


Figure 4.19: MPTL composed by two curves (in red) associated to the pair of curves (α, β) of the example.

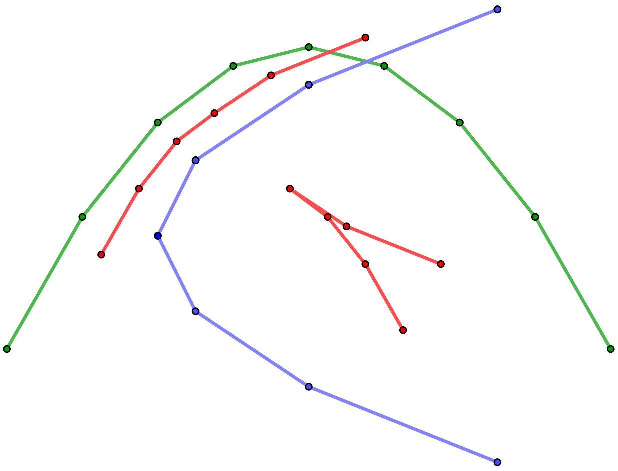


Figure 4.20: DMPTL composed by two discrete curves (in red) associated to the pair of discrete curves (α, β) of the example.

As we can see, Figure (4.21) shows the discrete improper affine sphere constructed from the pair (α, β) and its easy to see one cuspidal curve composed only by cuspidal edges and another one with a swallowtail point, both of them mimic the DMPTL.

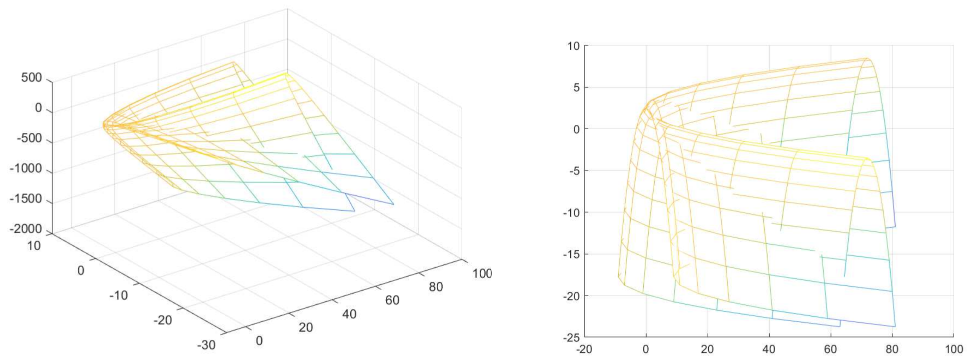


Figure 4.21: A discrete improper affine sphere with singularities and its upper view.

This net is given by $q(u,v) = (x(u,v), z(u,v))$, where

$$x(u,v) = \frac{1}{2}\left(u + v^2 - 2,\, 5 + v - \frac{(u-2)^2}{8}\right),$$

$$z(u,v) = \frac{1}{4}\left(-\frac{15u}{4} - 3v + uv - \frac{uv^2}{8} + \frac{u^2v^2}{8} - \frac{3u(u-1)}{16} - \frac{u(u^2-1)}{12} + \frac{v^2}{2} - 4v(v-1) + \frac{v(v-1)(v-2)}{3}\right).$$

5
Asymptotic nets with constant affine mean curvature

In this chapter we present a definition for nets with constant affine mean curvature from the discussion made in Section 3.2 about hyperboloid interpolators under the perspective of Lie quadrics in the smooth case. After that we show that discrete minimal surfaces and affine spheres have this property, as we expect from the smooth theory.

5.1
Constant affine mean curvature

In the smooth theory the mean curvature is an extrinsic measure that in a way describes how the surface is embedded in a certain space. At a point of the surface, it is the average of the principal curvatures. So if we think in the discrete world, what could the mean curvature be at a point in a net? Especially in an asymptotic net, such a concept does not have any sense at a vertex, since each of them has four coplanar edges. Then we have to see the mean curvature here as a general property of the net.

We discussed in Section 3.2 how to interpolate the quadrilaterals of the net by pieces of hyperboloids in such a way that two adjacent interpolators have the same tangent plane at the common edge. Our intention is to define the idea of affine mean curvature from the hyperboloid interpolators and, as it was said by Käferböck and Pottman [18], these interpolators can be seen as discrete Lie quadrics. So from now on, we will refer to the interpolator a discrete Lie quadric.

We have proved that at each quadrangle there is a relation between the parameter λ, the metric Ω and the affine mean curvature $-2c$ of the discrete Lie quadric, and we can chose c such that

$$1-\lambda^2\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=a\,\lambda\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right). \tag{5.1}$$

This means that the discrete Lie quadric has constant affine mean curvature $-2a$ through all the net and we can state the following definition.

Definition 5.1.1 *An asymptotic net has constant affine mean curvature if each discrete Lie quadric has the same constant affine mean curvature. Such a net shall be referred as an asymptotic net with* CAMC *and will satisfy equation (5.1).*

Note that in each quadrangle there is a 1-parameter family of discrete Lie quadrics, differently from its smooth counterpart, where at each point there is a unique Lie quadric. So we stated here that when we can chose for each quadrangle a Lie quadric with the same constant affine mean curvature, the net shall have constant affine mean curvature.

Lemma 5.1.2 *An asymptotic net can not have* CAMC *for two different values of a.*

Proof.
We already know that equation (5.1) holds for some map $\lambda : (Z^2)^* \longrightarrow \mathbb{R}^+$. Let us suppose that it also holds for $\rho\lambda$, with $\rho > 0$. Then both equations are valid

$$1 - \lambda^2 = a\lambda\Omega \quad \text{and} \quad 1 - \rho^2\lambda^2 = a\rho\lambda\Omega.$$

It follows that

$$1 - \rho^2\lambda^2 = \rho(1 - \lambda^2) \quad \Leftrightarrow \quad (\rho - 1)\left(\rho - \frac{1}{\lambda^2}\right) = 0,$$

which means $\rho = 1$.

Then we conclude that under a black-white re-scaling on λ, the equation (5.1) will not hold anymore. □

This Lemma establishes that asymptotic nets with *CAMC* are well defined and combined with Lemma (3.2.2), we have a fixed conormal vector field for the net, since any black-white re-scaling on ν automatically changes λ.

The first and obvious example of asymptotic nets with *CAMC* is the discrete hyperboloid as seen in Section 3.3. We will see ahead that both classes of affine minimal surfaces and proper affine spheres have *CAMC* too.

5.2 Discrete affine minimal surfaces

Asymptotic nets that are affine minimal were well treated in Craizer, Anciaux and Lewiner [10], and they were followed by Käferböck and Pottmann [18], Huhnen-Venedey and Rörig [16], among others. Let us see the first work here as a particular case of nets with CAMC when λ is identically 1, from the perspective of the theory developed in chapter 3. Moreover, it is given a geometric characterization for these nets, in a different way of that given by Käferböck and Pottmann [18], that agrees with Blaschke characterization in Theorem (2.3.2).

Firstly we need to establish the conditions when a net is minimal. Note that Definition (2.3.1) about smooth minimal surfaces says that $\nu_{uv} = 0$ means minimality. So in the discrete case we shall call *discrete affine minimal surface* a net in which $\nu_{12} = 0$. This is proved by Craizer, Anciaux and Lewiner [10], *i.e.*, they prove that a discrete minimal surface is actually the critical set of a variational area functional, which is equivalent to $\nu_{12} = 0$.

Proposition 5.2.1 *A discrete affine surface is minimal if and only if the map* $\lambda : (\mathbb{Z}^2)^* \to \mathbb{R}^+$ *associated to the conormal vector field* ν *is identically* 1.

Proof.
Equation (3.3) says that

$$\lambda^2\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right)(\nu(u,v)+\nu(u+1,v+1)) = \nu(u,v+1)+\nu(u+1,v).$$

Then

$$\begin{aligned}
\nu_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) &= \nu_2(u+1, v+\tfrac{1}{2}) - \nu_2(u, v+\tfrac{1}{2}) \\
&= \nu(u+1,v+1) - \nu(u+1,v) - \nu(u,v+1) + \nu(u,v) \\
&= (1-\lambda^2)(\nu(u,v)+\nu(u+1,v+1))
\end{aligned}$$

Since the surface is minimal if and only if $\nu_{12} = 0$, thus it happens if and only if $\lambda = 1$. □

Observe that Proposition (5.2.1) and Definition (5.1.1) declare that a discrete minimal surface has CAMC equal to zero, as expected.

Now we can see all the theory developed in chapter 3, with $\lambda = 1$, to be sure that it agrees with the work of Craizer, Anciaux and Lewiner [10]. Let us bring here the important results.

From Lemma (3.2.2) we can write the affine metric only in terms of the conormal vector field.

Lemma 5.2.2 *In terms of conormals, the affine metric is given by*

$$\Omega\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = [\nu(u,v), \nu(u,v+1), \nu(u+1,v)].$$

One can also write

$$\nu(u,v) = \frac{1}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}\left(q_1(u+\tfrac{1}{2},v)\times q_2(u,v+\tfrac{1}{2})\right).$$

This Lemma with the Remarks of Lemma (3.2.2) agrees with Theorem 1.1 from Craizer, Anciaux and Lewiner [10].

The affine normal map becomes simpler in the case where $\lambda = 1$, since $\xi_e = \xi_o$. Then the affine normal vector becomes

$$\xi\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right) = \frac{q_{12}\left(u+\frac{1}{2}, v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)} \tag{5.2}$$

as in the above mentioned reference.

5.2.1 Structural equations for discrete minimal surfaces

The coefficients of the cubic form keep the same, only without difference between ξ_e and ξ_o:

$$A(u,v) = \left[q_1\left(u-\tfrac{1}{2}, v\right), q_1\left(u+\tfrac{1}{2}, v\right), \xi\left(u\pm\tfrac{1}{2}, v\pm\tfrac{1}{2}\right)\right]$$

and

$$B(u,v) = \left[q_2\left(u, v-\tfrac{1}{2}\right), q_2\left(u, v+\tfrac{1}{2}\right), \xi\left(u\pm\tfrac{1}{2}, v\pm\tfrac{1}{2}\right)\right].$$

Then, from Proposition (3.6.1) we can write knew structural equations.

Proposition 5.2.3 *The structural equations of the discrete affine minimal surface are given by*

$$q_{11}(u,v) = \frac{\Omega_1\left(u, v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)} q_1\left(u+\tfrac{1}{2}, v\right) + \frac{A(u,v)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)} q_2\left(u, v+\tfrac{1}{2}\right)$$

$$q_{22}(u,v) = \frac{B(u,v)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)} q_1\left(u+\tfrac{1}{2}, v\right) + \frac{\Omega_2\left(u+\frac{1}{2}, v\right)}{\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right)} q_2\left(u, v+\tfrac{1}{2}\right)$$

Note that these equations are exactly equal to the structural equations (2.1) in the smooth case, only in a discrete version.

Let us see now the Gauss equations for the derivatives of ξ, from Proposition (3.7.1).

Proposition 5.2.4 *Gauss equations for the derivatives of the normal vector field in a discrete minimal surface are given by*

$$\xi_1\left(u,v+\tfrac{1}{2}\right)=\frac{A_2\left(u,v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}\,q_2\left(u,v+\tfrac{1}{2}\right)$$

and

$$\xi_2\left(u+\tfrac{1}{2},v\right)=\frac{B_1\left(u+\frac{1}{2},v\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}\,q_1\left(u+\tfrac{1}{2},v\right)$$

Corollary 5.2.5 *A discrete affine minimal surface is an improper affine sphere if and only if $A=A(u)$ and $B=B(v)$.*

5.2.2 Compatibility equations for discrete minimal surfaces

We are now able to write the three compatibility equations for discrete affine minimal surfaces. From the first of them (3.19) we get:

$$\frac{\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}=\frac{\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}+\frac{A(u,v)B(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}$$

which is equivalent to

$$A(u,v)B(u,v)=\\ \Omega\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right)-\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)$$

Let us make some changes in order to have an equation similar to the smooth one. Note that

$$\Omega_1\left(u,v+\tfrac{1}{2}\right)=\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)-\Omega\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right),$$

$$\Omega_2\left(u+\tfrac{1}{2},v\right)=\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)-\Omega\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right)$$

and

$$\Omega_{12}(u,v)=\Omega(u+\tfrac{1}{2},v+\tfrac{1}{2})-\Omega(u-\tfrac{1}{2},v+\tfrac{1}{2})-\Omega(u+\tfrac{1}{2},v-\tfrac{1}{2})+\Omega(u-\tfrac{1}{2},v-\tfrac{1}{2}).$$

Then the first compatibility equation for a discrete minimal surface can be written as

$$A(u,v)B(u,v)=\Omega_1\left(u,v+\tfrac{1}{2}\right)\Omega_2\left(u+\tfrac{1}{2},v\right)-\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega_{12}(u,v),$$

which is a straight discrete version of the equation (2.3) when $H = 0$.

The second compatibility equation (3.20) for an asymptotic net with $\lambda = 1$ gives us

$$\frac{B(u,v)\,A_2\left(u,v-\frac{1}{2}\right)}{\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)} = \frac{B_1\left(u+\frac{1}{2},v\right)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)} - \frac{B_1\left(u-\frac{1}{2},v\right)}{\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}$$

or

$$B(u,v)\,A_2\left(u,v-\tfrac{1}{2}\right) = \Omega\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)B_1\left(u+\tfrac{1}{2},v\right) - \Omega\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right)B_1\left(u-\tfrac{1}{2},v\right).$$

From the third compatibility equation (3.21) we get

$$\frac{A(u,v)\,B_1\left(u-\frac{1}{2},v\right)}{\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)} = \frac{A_2\left(u,v+\frac{1}{2}\right)}{\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)} - \frac{A_2\left(u,v-\frac{1}{2}\right)}{\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}.$$

or

$$A(u,v)\,B_1\left(u-\tfrac{1}{2},v\right) = \Omega\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)A_2\left(u,v+\tfrac{1}{2}\right) - \Omega\left(u-\tfrac{1}{2},v+\tfrac{1}{2}\right)A_2\left(u,v-\tfrac{1}{2}\right).$$

Note that both second and third compatibility equations are similar to equations (2.4) when $H = 0$. Moreover, all of them appear in the mentioned paper.

5.2.3
Geometric characterization of discrete minimal surfaces

Käferböck and Pottmann [18] deal with the extension of asymptotic nets to smooth surfaces by gluing bilinear patches – discrete Lie quadrics – into the skew quadrilaterals. The crucial difference here is that they obtained smooth surfaces as a piecewise smooth discretization of surfaces parameterized along asymptotic lines, inasmuch we have made a juxtaposition in order to have only coincident tangent planes at the edges, including possible singularities. Our construction is more general, since it applies to all asymptotic nets with CAMC, whilst they are interested just in minimal asymptotic nets.

They proved that the only quadrilateral net which can be extended by discrete Lie quadrics to overall continuously differentiable surfaces are those asymptotic nets in which the edges that join two neighboring net polylines are parallel to a plane, as a discrete counterpart to Blaschke characterization. Moreover, these surfaces can be seen as discrete affine minimal surfaces with negative curvature.

We are going to show here this Blaschke characterization from the fact that our gluing discrete Lie quadrics need to have only coincident planes at the edges. Although this result is not totally original, we think it is relevant from the perspective of the developed theory that has been proposed along this work.

Let us remember some equations that we found in Section 3.2.2 about the juxtaposition of two adjacent discrete Lie quadrics of quadrangles $ABDC$ and $ACFE$ such that $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 1, 0)$, $D = (1, 1, 1)$, $E = (x_1, y_1, 0)$ and $F = (x_2, y_2, x_2)$, for some real numbers x_1, x_2, y_1 and y_2. See Figure (5.2).

We have seen that there is a strict relation between the two parameters associated to a quadrangle, λ (from the conormal vector field) and c (from the discrete Lie quadric), and this relation is different in two adjacent quadrangles, as shown by the following equations

$$\lambda = \sqrt{1-c} \quad \text{and} \quad \mu = \frac{1}{\sqrt{1-b}}, \tag{5.3}$$

where they are associated to the quadrangles $ABDC$ and $ACFE$, respectively.

We also found a relation between all the parameters and the edges

$$\lambda\mu = \frac{\sqrt{1-a}}{\sqrt{1-b}} = \frac{\sqrt{x_2}}{\sqrt{x_1}}. \tag{5.4}$$

After Proposition 5.2.1, the surface is minimal if and only if $\lambda = \mu = 1$. So from relations (5.3) and (5.4), the surface is minimal if and only if $x_1 = x_2$ and $a = b = 0$. That said we have almost proved the follow theorem.

Theorem 5.2.6 [Geometric characterization of discrete minimal surfaces]
Let $q : \mathbb{Z}^2 \to \mathbb{R}^3$ be an asymptotic net and $\mathcal{L}_{c\left(u+\frac{1}{2},v+\frac{1}{2}\right)}$ the discrete Lie quadric for the quadrangle $\left(u+\frac{1}{2}, v+\frac{1}{2}\right)$. The surface represented by q is affine minimal if and only if the follow conditions are valid:

(i) The edges of a vertical strip, i.e., *$q_1\left(u+\frac{1}{2}, v\right)$ with u fixed, are parallel to one plane. The same property is satisfied by the edges of a horizontal strip $q_2\left(u, v+\frac{1}{2}\right)$, with v fixed,.*

(ii) Each discrete Lie quadric $\mathcal{L}_{c\left(u+\frac{1}{2},v+\frac{1}{2}\right)}$ is a hyperbolic paraboloid, i.e., *for all u and v, $c\left(u+\frac{1}{2}, v+\frac{1}{2}\right) = 0$.*

Proof
At any vertex of an asymptotic net the four adjacent edges are coplanar, consequently

$$q_2\left(u, v+\tfrac{1}{2}\right) = x_1\, q_2\left(u, v-\tfrac{1}{2}\right) + y_1\, q_1\left(u+\tfrac{1}{2}, v\right),$$

for some real constants x_1 and y_1. But $q(u+1, v+1) - q(u,v)$ is not in the same plane, so we can write it in terms of the basis $\left\{q_2\left(u, v-\tfrac{1}{2}\right), q_1\left(u+\tfrac{1}{2}, v\right), \nu(u,v)\right\}$ with real coefficients x_2, y_2 and z:

$$q(u+1, v+1) - q(u,v) = x_2\, q_2\left(u, v-\tfrac{1}{2}\right) + y_2\, q_1\left(u+\tfrac{1}{2}, v\right) + z\,\nu(u,v)$$

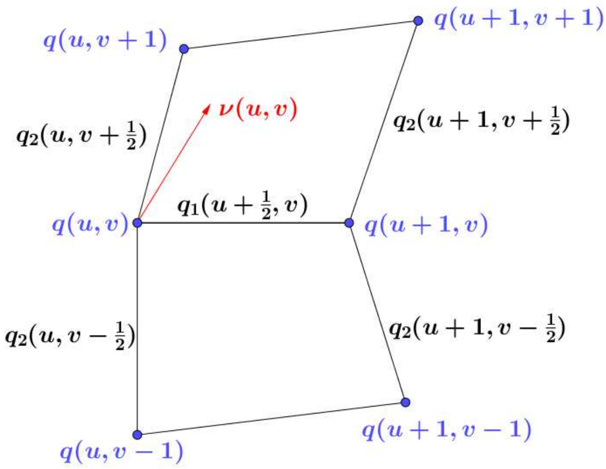


Figure 5.1: Two adjacent quadrangles of a vertical strip in an asymptotic net.

We can apply an affine transformation over the net such that Figure 5.1 is transformed in Figure 5.2, *i.e.*,

$$\begin{array}{ll} q(u,v) \mapsto A = (0,0,0) & q(u+1,v) \mapsto C = (0,1,0) \\ q(u,v-1) \mapsto B = (1,0,0) & q(u+1,v-1) \mapsto D = (1,1,1) \\ q(u,v+1) \mapsto E = (x_1,y_1,0) & q(u+1,v+1) \mapsto F = (x_2,y_2,x_2) \end{array}$$

From this point we can turn to the preceding calculations and translate those conclusions for the q net:

(i) In the cartesian coordinates, $x_1 = x_2$ means that the edges of a vertical strip are parallel to the plane $x = 0$. So in the q net the edges of a vertical strip $q_1\left(u+\tfrac{1}{2}, v\right)$, with u fixed, are parallel to the plane spanned by $q_1\left(u+\tfrac{1}{2}, v\right)$ and $\nu(u,v)$. And we come to the same conclusion for the horizontal strips.

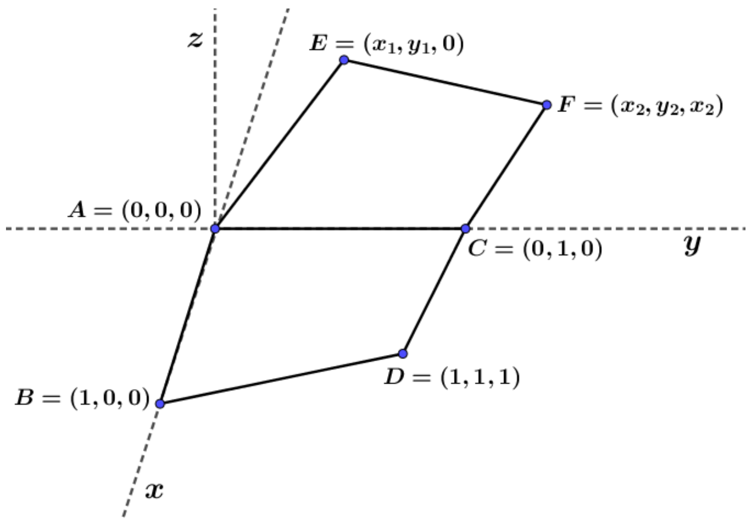


Figure 5.2: A convenient choice of quadrangles of an asymptotic net in cartesian coordinates.

(ii) From $c = 0$ we know that the discrete Lie quadric is a hyperbolic paraboloid meaning that each quadrangle is interpolated by such a surface. □

5.2.4
Examples of discrete minimal surfaces

Example 1.
Consider the conormal vector field given by $\nu(u, v) = (u, v, u^2 + v^2)$, the same that was taken in Example 1 of a smooth minimal surface in Section 2.3.

Note that

$$\nu(u, v) + \nu(u + 1, v + 1) = \Big(2u + 1, 2v + 1, u^2 + v^2 + (u + 1)^2 + (v + 1)^2\Big)$$

$$= \nu(u, v + 1) + \nu(u + 1, v),$$

which means that λ is identically 1 and the asymptotic net q associated to ν is minimal.

By a discrete integration we get

$$q(u, v) = \left(u^2 v - \frac{v(v^2 - 1)}{3},\ uv^2 - \frac{u(u^2 - 1)}{3},\ -uv\right)$$

and the net is shown in Figure (5.3 - left).

It is important to remark that the vertices of the discrete surface are not points of the smooth one since the equations that define them are different,

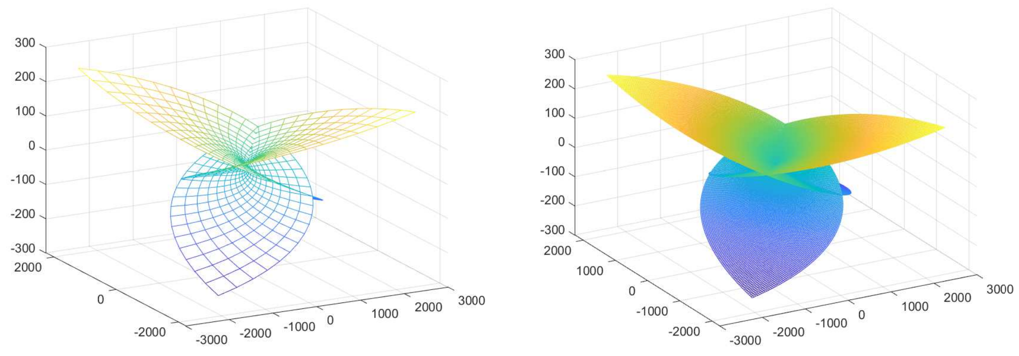


Figure 5.3: Discrete minimal surface of Example 1 on the left and its smooth sister on the right.

but the graphs seem almost the same. Another great difference is the integer domain in the discrete case.

Example 2.

Let us make use of an Exemple of Section 2.3. again, by considering the conormal vector field given by $\nu(u,v) = \left(-\frac{v^2}{2}, \frac{v-u}{2}, 1\right)$. This is a especial one because it was constructed from the centre-chord theory, which also means that it is an improper affine sphere.

Note that

$$\nu(u,v) + \nu(u+1, v+1) = \left(-\frac{v^2}{2} - \frac{(v+1)^2}{2}, v-u, 2\right)$$

$$= \nu(u, v+1) + \nu(u+1, v),$$

which means that λ is identically 1 and the asymptotic net q associated to ν is minimal, as expected.

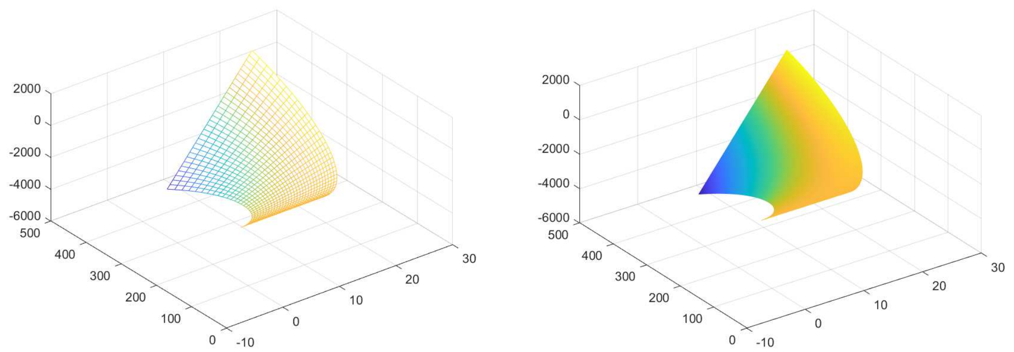


Figure 5.4: Discrete minimal surface of Example 2 on the left and its smooth sister on the right.

By a discrete integration we get

$$q(u,v) = \left(\frac{u+v}{2}, \frac{v^2}{2}, \frac{uv^2}{4} - \frac{v(v^2-1)}{12}\right)$$

and the net is shown in Figure (5.4 - left).

The remark made in Example 1 keeps valid here.

5.3 Discrete proper affine spheres

According to Bobenko and Schief [2], a pair (q, ν) is called a *discrete affine sphere* if q and ν satisfy Lelieuvre's equations (3.2) and its dual:

$$\begin{aligned} \nu(u+1,v) - \nu(u,v) &= 2a\, q(u+1,v) \times q(u,v), \\ \nu(u,v+1) - \nu(u,v) &= 2a\, q(u,v) \times q(u,v+1), \end{aligned} \tag{5.5}$$

These dual relations say in particular that both nets q and ν are asymptotic and Moutard at the same time. It seems important to observe that we are choosing $2a$ instead of 1, as it was done by Bobenko, who normalized it, because this will be proven convenient to our future conclusions.

In the smooth case we have made an observation about the difference between proper and improper affine spheres. The same is true for discrete affine spheres, that is, we shall call *discrete proper affine sphere* when the dual relation (5.5) is satisfied.

Lemma 5.3.1 *The duality relations implies that*

(i) $q(u,v)\cdot\nu(u,v) = q(u+1,v)\cdot\nu(u,v) = q(u,v+1)\cdot\nu(u,v) = \frac{1}{2a}$, *for some constant* $a > 0$.

(ii) $1 + 2a\, q(u+1,v)\cdot\nu(u,v+1) = 1 + 2a\, q(u,v+1)\cdot\nu(u+1,v) = 2\lambda^2\left(u+\frac{1}{2}, v+\frac{1}{2}\right)$.

(iii) $\lambda^2\left(u+\frac{1}{2}, v+\frac{1}{2}\right)(q(u,v) + q(u+1,v+1)) = q(u+1,v) + q(u,v+1)$.

Proof.

(i) From equations (3.2) we have $(q(u+1,v) - q(u,v))\cdot\nu(u,v) = 0$ and $(q(u,v+1) - q(u,v))\cdot\nu(u,v) = 0$. Then

$$q(u,v)\cdot\nu(u,v) = q(u+1,v)\cdot\nu(u,v) = q(u,v+1)\cdot\nu(u,v) = \frac{1}{2a}.$$

(ii) Let us do the inner product of $q(u,v+1)$ with both members of the equation (3.3)

$$\lambda^2\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)(\nu(u,v)+\nu(u+1,v+1))\cdot q(u,v+1)$$
$$=\nu(u,v+1)+\nu(u+1,v)\cdot q(u,v+1)$$

and use (i). Hence

$$\lambda^2\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\left(\frac{1}{2a}+\frac{1}{2a}\right)=\frac{1}{2a}+\nu(u,v+1)\cdot q(u,v+1)$$

then

$$2\lambda^2\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=1+2a\,\nu(u,v+1)\cdot q(u,v+1).$$

Similarly,

$$2\lambda^2\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=1+2a\,\nu(u,v+1)\cdot q(u+1,v).$$

(iii) Since q is also a Moutard net, $q(u,v)+q(u+1,v+1)$ and $q(u+1,v)+q(u,v+1)$ are in the same direction. From (i) and (ii) we conclude that

$$\lambda^2\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)(q(u,v)+q(u+1,v+1))=q(u,v+1)+q(u+1,v). \qquad \square$$

Lemma 5.3.2 *Similarly to Lemma (3.2.2) the net $q(u,v)$ can be writen directly in terms of the conormal vector field as*

$$q(u,v)=-\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)}{2a\,\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\,\nu_1\left(u+\tfrac{1}{2},v\right)\times\nu_2\left(u,v+\tfrac{1}{2}\right).$$

Proof.
From Lemma (3.2.2) we know that

$$\Omega=\lambda^{-1}\left[\nu(u,v),\nu(u,v+1),\nu(u+1,v)\right].$$

The dual equations (5.5) give us
$(\nu(u+1,v)-\nu(u,v))\times(\nu(u,v+1)-\nu(u,v))=$

$$-\,4a^2\left[q(u,v),q(u+1,v),q(u,v+1)\right]q(u,v),$$

and thus
$-[\nu(u,v),\nu(u,v+1),\nu(u+1,v)]=$

$$-\,4a^2\left[q(u,v),q(u+1,v),q(u,v+1)\right]q(u,v)\cdot\nu(u,v),$$

which by Lemma (5.3.1)*(i)* we can write

$$\lambda\Omega=2a\left[q(u,v),q(u+1,v),q(u,v+1)\right].$$

Therefore

$$\nu_1\left(u+\tfrac{1}{2},v\right)\times\nu_2\left(u,v+\tfrac{1}{2}\right) = -2a\lambda\Omega\, q(u,v). \qquad \square$$

This Lemma can be seen as the dual of Lemma (3.2.2) and looking at its Remark, in the same way, it is possible to get the equations with respect to the other three quadrangles:

$$q(u,v) = -\frac{\lambda\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{2a\,\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}\,\nu_1\left(u-\tfrac{1}{2},v\right)\times\nu_2\left(u,v+\tfrac{1}{2}\right),$$

$$q(u,v) = -\frac{\lambda^{-1}\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{2a\,\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}\,\nu_1\left(u-\tfrac{1}{2},v\right)\times\nu_2\left(u,v-\tfrac{1}{2}\right),$$

$$q(u,v) = -\frac{\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)}{2a\,\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}\,\nu_1\left(u+\tfrac{1}{2},v\right)\times\nu_2\left(u,v-\tfrac{1}{2}\right).$$

Proposition 5.3.3 *Any affine sphere has CAMC, i.e., it holds*

$$1-\lambda^2 = a\lambda\Omega.$$

Proof.
From Lemma (5.3.1)*(iii)* it follows that

$$\begin{aligned} q(u+1,v+1) &= \lambda^{-2}\left(q(u,v+1)+q(u+1,v)\right)-q(u,v) \\ &= \lambda^{-2}\left\{(q(u,v+1)-q(u,v))+(q(u+1,v)-q(u,v))\right\} \\ &\quad + 2\lambda^{-2}q(u,v)-q(u,v). \end{aligned}$$

Moreover,

$$q(u+1,v) = (q(u+1,v))-q(u,v))+q(u,v).$$

Then

$$\begin{aligned} \Omega^2 &= [q(u+1,v))-q(u,v), q(u,v+1)-q(u,v), q(u+1,v+1)-q(u+1,v)] \\ &= [q(u+1,v))-q(u,v), q(u,v+1)-q(u,v), 2(\lambda^{-2}-1)q(u,v)] \\ &= \frac{2(1-\lambda^2)}{\lambda^2}[q(u,v), q(u+1,v), q(u,v+1)]. \end{aligned}$$

Since

$$\lambda\Omega = 2a\,[q(u,v), q(u+1,v), q(u,v+1)],$$

we come to

$$1-\lambda^2 = a\lambda\Omega. \qquad \square$$

Remark.
Note that this fact is in according to the smooth counterpart and it is the reason we have taken the constant $2a$ in the formulae (5.5).

5.3.1
Examples of discrete proper affine spheres

Unfortunately it is very difficult to explicit affine spheres by formulas, since they are solutions of PDEs. So the only class of proper affine spheres in asymptotic parameters that we know is the 1-parameter family of discrete hyperboloids (Section 3.3) taken as samples of the basic ones

$$q(u, v) = \frac{1}{1 - auv}(u, v, uv)$$

discussed in Section 2.4 for $a \neq 0$, since when $a = 0$ we have an improper affine sphere.

6
Discrete Cayley surfaces

Section 2.7 was dedicated to the characterization of smooth Cayley surfaces, accordingly to Nomizu and Sasaki [31]. We make in this chapter a discrete approach to these surfaces, with a similar characterization, which gives us a very good example for our previous discussion on discrete surfaces with CAMC.

6.1
Defining discrete Cayley surfaces

We have found only one reference to a discrete Cayley surface in Matsuura and Urakawa [26], wherein it appears as an example of discrete improper affine spheres. However, the given parameterization does not agree with our construction.

Our first step is to find a solution for the discrete version of the Gauss equations of the Cayley surface given in asymptotic coordinates by (2.15). In the smooth case we have

$$\begin{aligned}\phi_{uu} &= (0,1,u) = \phi_v,\\ \phi_{vv} &= (0,0,0),\\ \phi_{uv} &= (0,0,1) = \xi.\end{aligned}$$

Then the discrete version can be written in the following form

$$\begin{aligned}&q_{11}(u,v) = q_2(u, v+\tfrac{1}{2}) = (0,1,u),\\ &q_{22}(u,v) = (0,0,0),\\ &q_{12}(u+\tfrac{1}{2}, v+\tfrac{1}{2}) = (0,0,1).\end{aligned}$$

We shall assume as initial conditions $q(0,0) = (0,0,0)$, $q(0,1) = (0,1,0)$ and $q(1,0) = (1,0,0)$. So the solution shall be

$$q(u,v) = \left(u, v + \frac{u(u-1)}{2}, uv + \frac{u(u^2-1)}{6}\right), \ (u,v) \in \mathbb{Z}^2. \tag{6.1}$$

As we always remark in discrete examples, this is not a sample of the smooth Cayley surface, since the equations are different. But they are quite the same and satisfy the same PDE system.

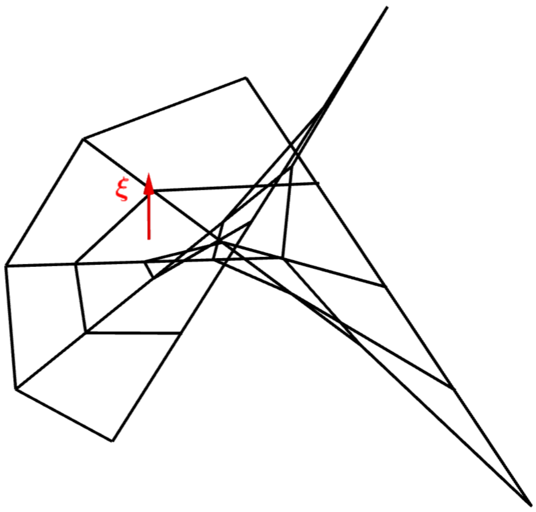


Figure 6.1: Discrete Cayley surface with its affine normal vector.

Thus we conclude that this is the discrete version of Cayley surface and, as in the smooth case, one can easily get $\Omega\left(u+\frac{1}{2}, v+\frac{1}{2}\right) = 1$ and $\xi\left(u+\frac{1}{2}, v+\frac{1}{2}\right) = (0,0,1)$. Moreover $\lambda\left(u+\frac{1}{2}, v+\frac{1}{2}\right) = 1$, which means that the surface is minimal and $H = 0$.

Let us calculate the coefficients A and B:

$$q_1\left(u-\tfrac{1}{2}, v\right) = \left(1, u-1, v+\frac{u(u-1)}{2}\right)$$

$$q_1\left(u+\tfrac{1}{2}, v\right) = \left(1, u, v+\frac{u(u+1)}{2}\right)$$

$$q_2\left(u, v-\tfrac{1}{2}\right) = q_2\left(u, v+\tfrac{1}{2}\right) = (0,1,u)$$

Then

$$A(u,v) = \left[q_1\left(u-\tfrac{1}{2}, v\right), q_1\left(u+\tfrac{1}{2}, v\right), \xi\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right)\right] = 1$$

and

$$B(u,v) = \left[q_2\left(u, v-\tfrac{1}{2}\right), q_2\left(u, v+\tfrac{1}{2}\right), \xi\left(u+\tfrac{1}{2}, v+\tfrac{1}{2}\right)\right] = 0,$$

in line with its smooth counterpart.

6.2 Characterization of discrete Cayley surfaces

Considering Cayley surfaces from equation (6.1) and all structural elements shown above we can give them a good characterization, which is in correspondence with the Theorem (2.7.5), since the equations that appear here are the discrete counterpart for those in the proof of the smooth one.

Theorem 6.2.1 [Characterization of discrete Cayley surfaces] *Let $q : \mathbb{Z}^2 \longrightarrow \mathbb{R}^3$ be an asymptotic net with CAMC. Then the net q is affinely congruent to a discrete Cayley surface if and only if it satisfies the following relations:*

(i) $A_1\left(u+\frac{1}{2},v\right) = 3A(u,v)\left[1-\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\right]$;

(ii) $A_2\left(u,v+\frac{1}{2}\right) = B_1\left(u+\frac{1}{2},v\right) = 0$;

(iii) $B_2\left(u,v+\frac{1}{2}\right) = 3B(u,v)\left[1-\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u+\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}\right]$;

(iv) $\frac{A(u,v)B(u,v)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)} = 0$ *or simply either* $A(u,v) \neq 0$ *or* $B(u,v) \neq 0$.

Proof.
As we have seen the net given by the equation (6.1) satisfies all the four relations in the theorem since A and Ω are constant and $B = 0$. It is only necessary to prove the converse.

Let us suppose that $A(u,v) \neq 0$ and from relation (iv) we have $B(u,v) = 0$, which means that q is ruled by Lemma (4.1.1). Since $B = 0$ the relation (iii) vanishes and does not give any information.

From relation (ii) we get

$$A(u,v+1) = A(u,v), \forall v \in \mathbb{Z},$$

i.e., A is a function of u.

From relation (i) and $A(u,v) \neq 0$ we can write

$$\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)} = f(u,v) \neq 0,$$

for some map $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}$ and $f \neq 0$ since $\lambda \neq 0$ and $\Omega \neq 0$.

Then, from Lemma (3.2.2) and its remark, it follows that

$$\lambda^{-1}(u-\tfrac{1}{2},v+\tfrac{1}{2})\Omega(u-\tfrac{1}{2},v+\tfrac{1}{2})\nu(u,v)=f(u,v)[q_1(u+\tfrac{1}{2},v)\times q_2(u,v+\tfrac{1}{2})],$$

and similarly,

$$\lambda(u+\tfrac{1}{2},v+\tfrac{1}{2})\Omega(u+\tfrac{1}{2},v+\tfrac{1}{2})\nu(u,v+1)=\frac{1}{f(u,v)}[q_1(u+\tfrac{1}{2},v+1)\times q_2(u,v+\tfrac{1}{2})].$$

Hence

$$\lambda^{-1}(u-\tfrac{1}{2},v+\tfrac{1}{2})\lambda(u+\tfrac{1}{2},v+\tfrac{1}{2})\Omega(u-\tfrac{1}{2},v+\tfrac{1}{2})\Omega(u+\tfrac{1}{2},v+\tfrac{1}{2})(\nu(u,v)\times\nu(u,v+1))=$$

$$[q_2(u,v+\tfrac{1}{2}),q_1(u+\tfrac{1}{2},v),q_1(u+\tfrac{1}{2},v+1)]q_2(u,v+\tfrac{1}{2})$$

or equivalently,

$$[q_1(u+\tfrac{1}{2},v),q_2(u,v+\tfrac{1}{2}),q_1(u+\tfrac{1}{2},v+1)]=$$

$$\lambda^{-1}(u-\tfrac{1}{2},v+\tfrac{1}{2})\lambda(u+\tfrac{1}{2},v+\tfrac{1}{2})\Omega(u-\tfrac{1}{2},v+\tfrac{1}{2})\Omega(u+\tfrac{1}{2},v+\tfrac{1}{2}),$$

which means that

$$\Omega(u+\tfrac{1}{2},v+\tfrac{1}{2})=\lambda^{-1}(u-\tfrac{1}{2},v+\tfrac{1}{2})\lambda(u+\tfrac{1}{2},v+\tfrac{1}{2})\Omega(u-\tfrac{1}{2},v+\tfrac{1}{2}). \tag{6.2}$$

Observe that the first compatibility equation (3.19) results in

$$\frac{\lambda^{-1}\left(u+\frac{1}{2},v+\frac{1}{2}\right)\lambda^{-1}\left(u-\frac{1}{2},v+\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v+\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}=\frac{\lambda\left(u+\frac{1}{2},v-\frac{1}{2}\right)\lambda\left(u-\frac{1}{2},v-\frac{1}{2}\right)\Omega\left(u-\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}.$$

So by use of equation (6.2) it becomes

$$\lambda^{-2}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=\lambda^{2}\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right). \tag{6.3}$$

Let us now make use of the fact that the net has CAMC, in other words, there is a constant a such that in every quadrangle the equation (3.9) holds

$$1-\lambda^2=a\lambda\Omega,$$

and we can write

$$\lambda^{-2}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=1+a\lambda^{-1}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)$$

and

$$\lambda^{2}\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)=1-a\lambda\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)\Omega\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right).$$

Since this two expressions are equal, we have

$$a\left(\lambda\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)\Omega\left(u-\tfrac{1}{2},v-\tfrac{1}{2}\right)+\lambda^{-1}\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\right)=0,$$

but this term in parenthesis is always positive, which means that $a=0$ and the net is minimal. Thus we conclude that λ is identically 1.

Thereafter, equation (6.2) states that $\Omega(u+\frac{1}{2},v+\frac{1}{2})=\Omega(u-\frac{1}{2},v+\frac{1}{2})$, in other words, Ω depends only on v, and by replacing these two pieces of information in relation (i), we get

$$A_1(u,v)=0,$$

and therefore A is constant.

Note that $B=0=B(u)$ and $\lambda=1$, so the Corollary (5.2.5) ensures that q is an improper affine sphere, *i.e.*, ξ is constant.

Then the structural equations (3.17) become

$$\begin{aligned}
q_{11}(u,v) &= \frac{A}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_2(u,v+\tfrac{1}{2}),\\
q_{22}(u,v) &= \frac{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)-\Omega\left(u+\frac{1}{2},v-\frac{1}{2}\right)}{\Omega\left(u+\frac{1}{2},v+\frac{1}{2}\right)}q_2(u,v+\tfrac{1}{2}),\\
q_{12}(u,v) &= \Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\xi.
\end{aligned}$$

The last equation can be integrated in u such that (remember that Ω is a function of v and ξ is constant)

$$q_2\left(u,v+\frac{1}{2}\right)=u\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)\xi+C,$$

where C is a constant vector in $\mathbb{R}^3$. Hence

$$q_{22}(u,v)=u\left(\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)-\Omega\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right)\right)\xi.$$

Observe that $q_{22}(u,v)$ has been written in two different ways, as a multiple of q_2 and ξ, but these two vectors are always linearly independent, so we conclude that

$$\Omega\left(u+\tfrac{1}{2},v+\tfrac{1}{2}\right)=\Omega\left(u+\tfrac{1}{2},v-\tfrac{1}{2}\right).$$

which means that Ω is constant.

Without loss of generality, let us suppose that $A=1$, $\Omega=1$ and

$\xi = (0,0,1)$. Then the structural equations can be written as

$$\begin{aligned} q_{11}(u,v) &= q_2(u, v + \tfrac{1}{2}) \\ q_{22}(u,v) &= 0 \\ q_{12}(u,v) &= \xi \end{aligned}$$

Let us take as initial conditions $q(0,0) = (0,0,0)$, $q(0,1) = (0,1,0)$ and $q(1,0) = (1,0,0)$ as above.

From q_{22} and q_{12} we get $q_2(u, v + \frac{1}{2}) = u\xi + C$, where C is a constant vector in $\mathbb{R}^3$. Then $C = q_2(0, \frac{1}{2}) = q(0,1) - q(0,0) = (0,1,0)$ and we can write $q_2(u, v + \frac{1}{2}) = (0,1,u)$.

By discrete integration on v we come to

$$q(u,v) = uv\xi + vC + D(u),$$

where $D(u)$ is a vector function of u.

Hence $D_{11}(u) = q_{11}(u,v) = (0,1,u)$ or

$$D(u-1) + D(u+1) - 2D(u) = (0,1,u).$$

Then,

$$D(u+1) - D(u) = \left(1, 1, \frac{u(u+1)}{2}\right),$$

and we come to

$$D(u) = \left(u, \frac{u(u-1)}{2}, \frac{u(u+1)(u-1)}{6}\right).$$

Thereafter

$$q(u,v) = \left(u, v + \frac{u(u-1)}{2}, uv + \frac{u(u^2-1)}{6}\right),$$

which by equation (6.1) is the discrete Cayley surface. □

7
Final Considerations

After all this work I believe I have made a good contribution to Mathematics research, in particular to Discrete Affine Geometry, a growing field along the last two decades. I have proposed an original definition for discrete singularities for indefinite improper affine spheres, although I did not show as many examples as I would like to. I also proposed the first definition for asymptotic nets with constant affine mean curvature, which was called CAMC. Maybe there is another way to give a definition for that, but what was made here seems quite solid, since it encompasses the set of indefinite affine spheres and the set of minimal surfaces, which agrees with the smooth case. In order to complete this work with a great result, I gave a discrete version for Cayley surfaces and I proved how to characterize them from the fact that they are asymptotic nets with CAMC.

Since the lack of references about discrete singularities, I hope that what was proposed in Chapter 4 will be a kick-off for researchers in this brand new area. For example, in the smooth case a swallowtail point use to be isolated, as we have seen in the example of Subsection 4.3.3, but in several attempts with discrete examples, it seems that sometimes there is a pair of adjacent swallowtail points close to the correspondent one of the smooth case. This can be seen by the construction of the DMPTL curve. Since the discrete improper affine sphere is not a sample of its smooth counterpart, I believe that this happens because it is highly improbable that a discrete swallowtail point overlaps the smooth one. I also have the impression that these pairs disappear insofar the discrete net coincides with the smooth one and the DMPTL converges to the MPTL.

All figures in this work were made in Geogebra and Matlab. The first one is an easy tool for planar constructions and it proved to be a good choice for examples of small nets, mainly when I wanted to bring up a particular characteristic, like the case in Chapter 4 in which I needed to show the possible cases of singularities. The Matlab was used to construct all surfaces, since is an excellent software to do everything, if you have programming skills, what is not my case by the way. That said, I could not construct small nets in Matlab with a good understand of what was going on close to a swallowtail point, mainly when there were several of them. Therefore I believe that someone with this kind of skill will aggregate great data to the study of discrete singularities.

Another observation about discrete singularities is that we have defined them only for indefinite improper affine spheres, since the centre-chord construction proved to be the perfect tool for the task. But what about general asymptotic nets? Is it possible that the definitions work in the same way or at least similarly? Maybe they work for a bigger set of asymptotic nets, like those with CAMC. These are only a couple of questions that can be made, which shows that there is plenty of work to do about the subject from now on.

Lie quadrics have an important role in the characterization of surfaces with constant affine mean curvature, especially in the case of minimal surfaces, so it is natural to extend them to the discrete case, as it was mentioned by Käferböck and Pottman [18] and we stablished in Section 5.1. But one can question if it is not artificial since there is a 1-parameter family of discrete Lie quadrics for each quadrangle of the net. I believe that the characterization for the smooth case by Lie quadrics is a strong argument for the definition we have made and the uniqueness proved in Lemma (5.1.2) states that nets with CAMC were well defined, and we proved that discrete minimal surfaces and affine spheres are examples of them. Are there other classes of nets with CAMC? Is it possible to classify all the nets with CAMC, if we know them? So much to do.

Among all the nets with CAMC we chose to work with discrete Cayley surfaces and I am very happy with the result. But I would like to leave here a little note about one of the hypothesis used in Theorem (6.2.1). In the smooth case, Martinéz and Milán [25] proved that $AB = 0$ implies that the surface is locally ruled, which is very easy to see, but this is completely different in the discrete universe, since the continuity is the argument key used in the proof. I tried very hard to find something similar or a counterexample for asymptotic nets. Although unsuccessful, my intuition says that it is possible to have some conclusion in this point.

Another remark about Cayley characterization is the importance of the CAMC hypothesis. All paths that I tried to follow led me to Rome and in this case Caesar is called CAMC.

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