Pontifícia Universidade Católica
do Rio de Janeiro

Raphael de Marreiros Cordeiro Machado

Domino tilings of 3D cylinders and regularity of disks

Dissertação de Mestrado

Thesis presented to the Programa de Pós-graduação em Matemática da PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Matemática.

Advisor: Prof. Nicolau Corção Saldanha

Rio de Janeiro
February 2021

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Bibliographic data

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Domino tilings of 3D cylinders and regularity of disks / Raphael de Marreiros Cordeiro Machado; advisor: Nicolau Corção Saldanha. – Rio de janeiro: PUC-Rio , Departamento de Matemática, 2021.

v., 49 f: il. color. ; 30 cm

Dissertação (mestrado) - Pontifícia Universidade Católica do Rio de Janeiro, Departamento de Matemática.

Inclui bibliografia

1. Matemática – Teses. 2. Coberturas por dominós;. 3. Partições por dímeros;. 4. Discos quadriculados;. 5. Movimentos locais;. 6. Flips.. I. Saldanha, Nicolau. II. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Matemática. III. Título.

CDD: 004

Acknowledgments

To CAPES, CNPq, FAPERJ and PUC-Rio for the support.

To my advisor Professor Nicolau Corção Saldanha for his availability and patience in sharing his knowledge.

To Grupo Trivial de Lógica Matemática do IME-UFF for the several seminars.

To my family and friends.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001

Abstract

de Marreiros Cordeiro Machado, Raphael; Saldanha, Nicolau (Advisor). **Domino tilings of 3D cylinders and regularity of disks**. Rio de Janeiro, 2021. 49p. Dissertação de Mestrado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

In this dissertation we study domino tilings of three-dimensional regions. In particular, we consider the flip connectivity problem for cylinders, i.e, regions of the form $\mathcal{D}\times[0,N]$. A flip is a local move: two adjacent dominoes are removed and placed back in a different position. In two dimensions, two domino tilings of the same contractible region are connected by flips. In dimension 3, the problem is subtler. We present the twist, a flip invariant that associates an integer number with a tiling. For many 3D regions, there exist examples of tilings with the same twist which can not be joined by a sequence of flips. Recent papers prove that for certain disks $\mathcal{D}$, called regular, two tilings of the cylinder $\mathcal{D}\times[0,N]$ with the same twist can be joined by a sequence of flips once we add vertical space to the cylinder. These results are presented and discussed. We then prove regularity or irregularity for new families of quadriculated disks. It turns out that a bottleneck often implies irregularity.

Keywords

Domino tilings; Dimer partitions; Quadriculated disks; Local movements; Flips.

Resumo

de Marreiros Cordeiro Machado, Raphael; Saldanha, Nicolau. **Cobertura por dominós de cilindros 3D e regularidade de discos**. Rio de Janeiro, 2021. 49p. Dissertação de Mestrado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nessa dissertação estudamos coberturas por dominós de regiões tridimensionais. Em particular, consideramos o problema de conectividade por flips de cilindros, ou seja, regiões da forma $\mathcal{D} \times [0, N]$. Um flip é um movimento local: dois dominós adjacentes são removidos e recolocados em outra posição. Em duas dimensões, duas coberturas de uma mesma região contrátil podem ser conectadas por flips. Em dimensão 3, o problema é mais sutil. Apresentamos o twist, um invariante por flips que associa uma cobertura a um número inteiro. Para muitas regiões 3D, existem exemplos de coberturas com o mesmo twist que não podem ser ligadas por uma sequência de flips. Artigos recentes mostram que para muitos discos $\mathcal{D}$, chamados regulares, duas coberturas do cilindro $\mathcal{D} \times [0, N]$ com o mesmo twist podem ser ligadas por flips uma vez que adicionamos espaço vertical ao cilindro. Esses resultados são apresentados e discutidos. Nós então demonstramos a regularidade ou irregularidade de vários discos. Verificamos que um gargalo muitas vezes implica na irregularidade.

Palavras-chave

Coberturas por dominós; Partições por dímeros; Discos quadriculados; Movimentos locais; Flips.

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1
Introduction

The tiling problem discusses whether or not a region can be tessellated by a given set of tiles. Many questions about tilings of two-dimensional regions have been studied, in particular, for regions in a regular lattice of the cartesian plane. For instance, in the square lattice, the *domino tiling*. In that case, a region is a connected finite union of closed unit squares and the set of tiles is a domino, i.e, a 2×1 rectangle.

Conway and Lagarias [1] associated groups to the boundaries of the tiles and the region to be tiled. This technique, called boundary invariants, proved useful for determining whether some regions are tileable. Later, Thurston [8] developed the concept of height functions which gives necessary and sufficient conditions for a simply connected region to be tileable by dominoes. The existence of domino tilings led to the idea of *flips*. A two-dimensional flip consists of a $90°$ rotation of two adjacent parallel dominoes. It follows from [8] that any two tilings of a simply connected region can be joined by a sequence of flips. For non simply connected regions, the idea of *flux* of a tiling is presented in [7]. The flux is a flip invariant such that two tilings can be joined by a sequence of flips if and only if they have the same flux.

The three-dimensional problem of flip connectivity between two tilings is much more subtle. For instance, the space of tilings of a simply connected region is no longer connected by flips. Milet and Saldanha [3] introduced the *twist* of a tiling for some regions. The twist is a flip invariant which, for contractible regions, assumes values in $\mathbb{Z}$. In order to study the twist and the flip connectivity the concept of a *regular* disk is defined in [4]. A non trivial balanced quadriculated disk $\mathcal{D}$ is regular if whenever two tilings $\mathbf{t}_0$ and $\mathbf{t}_1$ of $\mathcal{D} \times [0, N]$ have the same twist then $\mathbf{t}_0$ and $\mathbf{t}_1$ can be connected by a sequence of flips provided that some vertical space is allowed. The first main theorem of [4] shows that the rectangle $\mathcal{D} = [0, L] \times [0, M]$ with LM even is regular if and only if $\min\{L, M\} \geq 3$. In this context, it was conjectured that "large" disks are regular.

In this dissertation we investigate whether or not a quadriculated disk is regular. For instance, we prove that the disks in Figure 1.1 are regular.

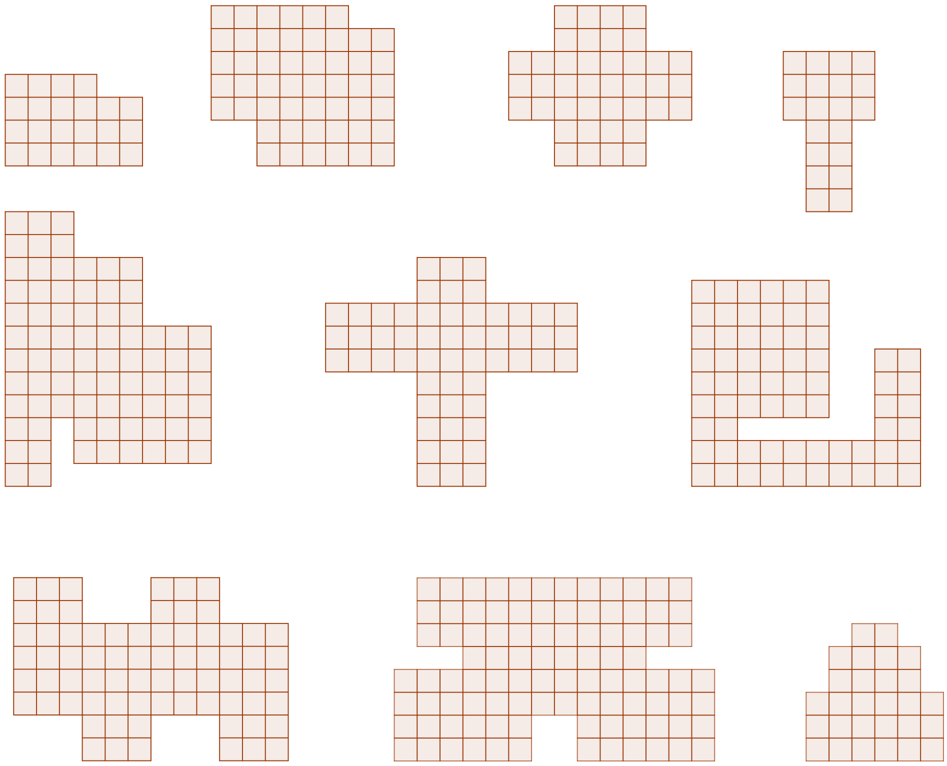

Figure 1.1: Examples of regular disks.

The problem of determining which disks are regular is not trivial. In fact, there are examples of non regular disks which are very similar to some regular disks. For example, consider the Figure 1.2, it contains examples of disks which we prove to be non regular.

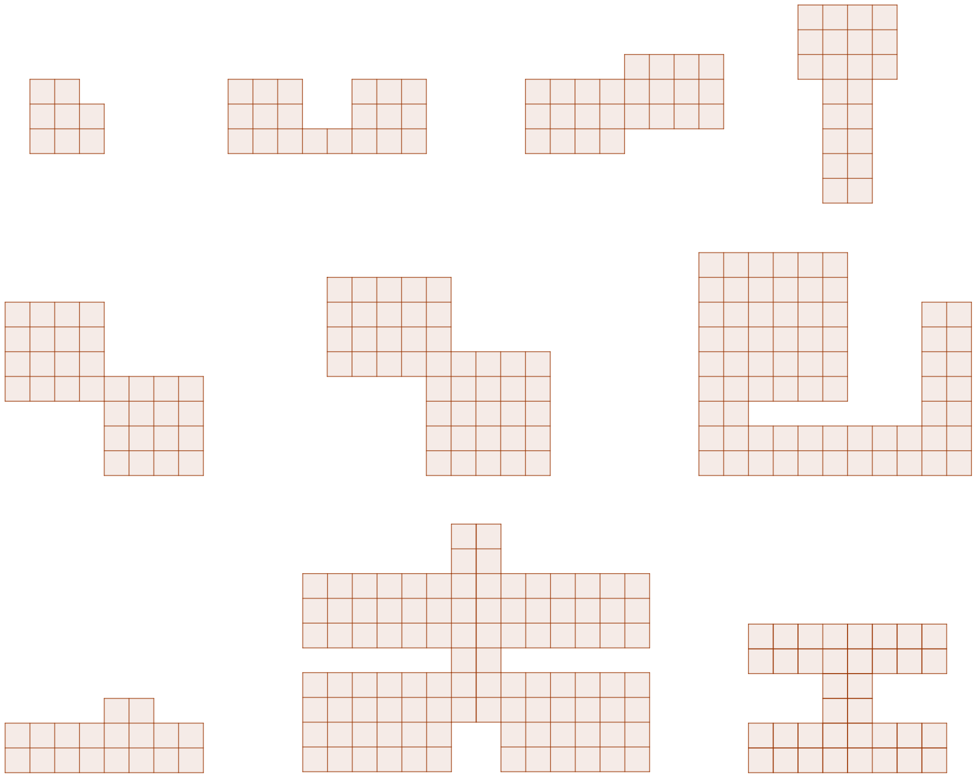

Figure 1.2: Examples of non regular disks.

This text is divided as follows. In Chapter 2 we discuss the well known results about two-dimensional tilings. In Chapter 3 we present the background needed to discuss the three-dimensional problem of flip connectivity. Finally, in Chapter 4 we prove the regularity and the irregularity of many quadriculated disks.

2
Two-dimensional tilings

A *quadriculated region* $\mathcal{D} \subset \mathbb{R}^2$ is a connected finite union of closed unit squares $[a, a+1] \times [b, b+1]$ such that $(a, b) \in \mathbb{Z}^2$. In particular, when $\mathcal{D}$ is connected with simply connected interior we say that $\mathcal{D}$ is a *quadriculated disk*. A *domino* is the union of two adjacent closed unit squares and a *domino tiling* of $\mathcal{D}$ is a covering of $\mathcal{D}$ by dominoes with disjoint interiors. For instance, Figure 2.1 shows two examples of quadriculated regions and tilings.

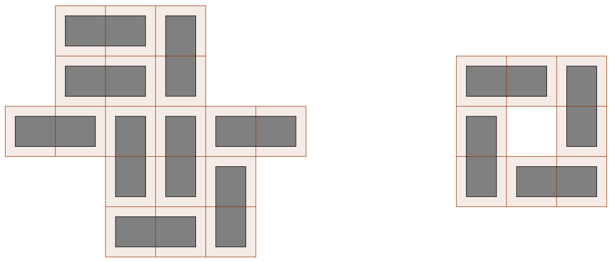

Figure 2.1: Tilings of quadriculated regions.

Initially, a natural problem is determining which regions are *tileable*, i.e, admit a tiling by dominoes. The most basic argument to decide whether a region $\mathcal{D}$ has a tiling is by coloring the unit squares. We say that a unit square $[a, a+1] \times [b, b+1]$ is white (resp. black) if $(-1)^{a+b}$ is equal to $+1$ (resp. -1). Since a domino is formed by a black square and a white square every tileable region $\mathcal{D}$ must have the same number of black and white squares; we say that a quadriculated region with this property is *balanced*. However, the converse does not hold, it is not difficult to construct examples of balanced regions which are not tileable.

Thurston [8] developed a more sophisticated argument than coloring unit squares to study the tiling problem when $\mathcal{D}$ is a quadriculated disk. In fact, he exhibited an algorithm which decides whether or not there exists a tiling for $\mathcal{D}$. This construction is mainly based on the concept of height functions, which associate a three-dimensional object with a tiling.

A problem that we are more interested in, whose solution also is based on height functions, is determining whether two tilings of a region $\mathcal{D}$ can be joined by a sequence of flips. This problem does not have a direct answer; there are regions with tilings that admit no flips. In Figure 2.1 the tiling of the first region admits two flips and the tiling of the second region admits no flips.

First we consider a balanced quadriculated disk and the flip connectivity problem. Then, we present a generalization of the problem for arbitrary regions. Let $\mathcal{V}(\mathcal{D})$ be the set formed by the vertices of the unit squares contained in $\mathcal{D}$. A *height function* is a function $h\colon \mathcal{V}(\mathcal{D}) \to \mathbb{Z}$ which satisfies the following properties:

I. $h(x, y) \mod 4 = \phi(x, y)$ where $\phi\colon \mathcal{V}(\mathcal{D}) \to \mathbb{Z}/4\mathbb{Z}$ is such that $\phi(x, y) = [0]$ if x and y are both even, $\phi(x, y) = [1]$ if x is odd and y is even, $\phi(x, y) = [2]$ if x and y are both odd, $\phi(x, y) = [3]$ if x is even and y is odd.

II. The values of h at adjacent points never differ by more than 3.

III. The values of h at adjacent points contained in the boundary of $\mathcal{D}$ differ by 1.

In Figure 2.2 we exhibit a quadriculated disk and a height function.

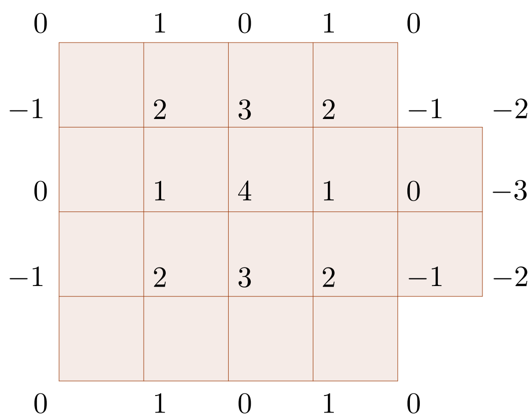

Figure 2.2: Disk with a height function

We have that every height function induces a tiling of $\mathcal{D}$. Indeed, the properties I, II and III ensure that the values of h at adjacent vertices differ either by 1 or 3. Furthermore, each square contained in $\mathcal{D}$ has exactly two vertices such that the values of h differ by 3; the third property implies that these vertices do not belong to the boundary. Then, we obtain a tiling of $\mathcal{D}$ by deleting the edges which contain vertices that differ by 3.

Conversely, every tiling of $\mathcal{D}$ corresponds to a height function $h\colon \mathcal{V}(\mathcal{D}) \to \mathbb{Z}$. Given a tiling, choose an arbitrary vertex v_0 in the boundary of $\mathcal{D}$ and define $h(v_0) = \phi(v_0)$. For an arbitrary vertex $w \in \mathcal{V}(\mathcal{D})$ consider a path, starting at v_0 and ending at w, contained in the boundaries of the dominoes. Suppose that white squares are oriented counterclockwise and black squares are oriented clockwise. Then, $h(w)$ is defined as follows: if we move along a positively (resp. negatively) directed edge the height increases (resp. decreases) by 1. It is not difficult to see that for every cycle based in v_0 this process of adding and subtracting by 1 returns the value $4(w-b)$, where w (resp. b) is the number of white (resp. black) squares enclosed by the cycle. Therefore, the definition of $h(w)$ does not depend on choice of paths, since any two paths contained in the boundaries of dominoes can be concatenate to a cycle which encloses a tiling. Furthermore, notice that h satisfies properties I, II and III. Therefore, h is a well defined height function.

Given two tilings we may assume that their corresponding height functions coincide on the boundary of the quadriculated disk $\mathcal{D}$. Indeed, the choice of basepoint and base value contribute only with a constant in $4\mathbb{Z}$. Furthermore, by construction, two tilings differ by a flip iff their corresponding height functions differ at a unique vertex by ± 4; in that case we say that the height functions are adjacent.

Consider two height functions h_1 and h_2. We say that $h_1 \leq h_2$ if $h_1(v) \leq h_2(v)$ for every $v \in \mathcal{V}(\mathcal{D})$. Furthermore, we say that $h_1 < h_2$ if $h_1 \leq h_2$ and there exists $u \in \mathcal{V}(\mathcal{D})$ such that $h_1(u) < h_2(u)$. We follow [6] and [7] and show that any two tilings of a quadriculated disk can be joined by a sequence of flips.

Lemma 2.1. *Let $h_1 < h_2$ be two height functions coinciding on the boundary of a disk $\mathcal{D}$. Then, there exists a height function h_3, adjacent to h_1, such that $h_1 < h_3 \leq h_2$.*

Proof. Suppose that h_1 has a local minimum $u \in \mathcal{V}(\mathcal{D})$ in the interior of $\mathcal{D}$ such that $h_1(u) < h_2(u)$. Define $h_3(v) = h_1(v)$ if $v \neq u$ and $h_3(u) = h_1(u) + 4$. Therefore, $h_1 < h_3 \leq h_2$. We claim that h_3 is a height function. Then, we must show that h_3 satisfies the properties I,II and III. By construction, h_3 satisfies I. Since u is in the interior of $\mathcal{D}$ it follows that h_3 satisfies III. Since u is a local minimum of h_1, for every adjacent vertex w either $h_1(w) - h_1(u) = 1$ or $h_1(w) - h_1(u) = 3$. Thus, either $h_3(w) - h_3(u) = -3$ or $h_3(w) - h_3(u) = -1$, it follows that h_3 satisfies II. Then, in order to prove the lemma it is sufficient to prove the existence of u.

Let $A \subset \mathcal{V}(\mathcal{D})$ be the subset where $h_2 - h_1$ is maximum. Then, take $u \in A$ such that $h_1(u) \leq h_1(v)$ for every $v \in A$. We claim that u is a local minimum of h_1. Let w be a vertex adjacent to u, there are two cases: either $w \in A$ or $w \notin A$. If $w \in A$, by construction $h_1(u) < h_1(w)$. If $w \notin A$, let $x_1 = h_1(u)$ and $x_2 = h_2(u)$. Since adjacent vertices either differ by 1 or 3, for every $i = 1, 2$, there are only two possible values for $h_i(w)$: y_i and z_i such that $y_i < x_i < z_i$. By the properties of height functions, it is not difficult to see that $z_2 - z_1 = x_2 - x_1 = y_2 - y_1$. Moreover, since $w \notin A$, we have $h_2(w) - h_1(w) < h_2(u) - h_1(u)$. Therefore, $h_1(w) = z_1$ and $h_2(w) = y_2$, the claim then follows. ■

Now, consider two tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ of a quadriculated disk and their corresponding height functions h_1 and h_2, respectively. Notice that $\max(h_1, h_2)$ is a function which satisfies the properties I, II and III. Thus, $\max(h_1, h_2)$ is a height function. Then, by Lemma 2.1 there exists flips joining $\mathbf{t}_1$ and $\mathbf{t}_2$ to the tiling induced by $\max(h_1, h_2)$. Therefore, there exists flips connecting the tilings $\mathbf{t}_1$ and $\mathbf{t}_2$.

In order to study the flip connectivity of tilings for regions which are not a disk, a generalization of the concept of height function was presented in [7]. A straightforward generalization of height functions would not be well defined. A first problem is to define the height function on the boundary if we have a different number of black and white squares contained in the holes of $\mathcal{D}$. A second problem is the possible existence of a vertex v such that all edges going towards and outwards v belong to the boundary of $\mathcal{D}$. This problem is solved by interpreting v as two vertices and defining adjacency such that each vertex belongs to two edges. Figure 2.3 illustrates these two problems.

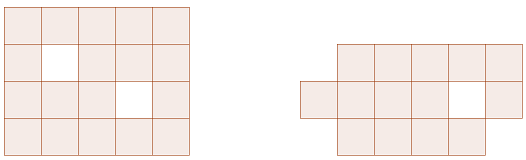

Figure 2.3: Tileable non simply connected regions.

Then, instead of height functions we consider *height sections*, sections of a map with codomain $\mathcal{V}(\mathcal{D})$ and fiber $\mathbb{Z}$. Let $\mathcal{D}^*$ be the set formed by the boundaries of the unit squares which are contained in $\mathcal{D}$. Consider an arbitrary basepoint $v_0 \in \mathcal{V}(\mathcal{D})$ and let $\mathscr{P}$ be the set of all paths in $\mathcal{D}^*$ going from v_0 to some other vertex in $\mathcal{V}(\mathcal{D})$. Given two paths $\gamma_1, \gamma_2 \in \mathscr{P}$ let γ be the cycle formed by the concatenation of γ_1 and γ_2, and let l be the sum of the windings of γ around white squares not contained in $\mathcal{D}$ minus the sum of the windings of γ around black squares not contained in $\mathcal{D}$. Let $\sim$ be the equivalence relation in $\mathscr{P} \times \mathbb{Z}$ such that $(\gamma_1, k_1) \sim (\gamma_2, k_2)$ iff γ_1 and γ_2 have the same endpoint and $k_1 - k_2 = 4l$. Then, the desired sections are obtained from the projection $\pi\colon \mathscr{P} \times \mathbb{Z}/\sim \to \mathcal{V}(\mathcal{D})$ which takes a pair $[(\gamma_1, k_1)]$ to the endpoint of γ_1.

We can define operations on the fibers. Given an element in a fiber and a integer m we define the sum $[(\gamma_1, k_1)] + m = [(\gamma_1, k_1 + m)]$, obtaining another element in the same fiber. Given two elements in the same fiber we define the difference $[(\gamma_1, k_1)] - [(\gamma_2, k_2)] = k_1 - k_2 - 4l$, obtaining an integer. Given two elements in fibers of two adjacent vertices v, w we define the difference $[(\gamma_1, k_1)] - [(\gamma_2, k_2)] = [(\alpha * \gamma_1, k_1)] - [(\gamma_2, k_2)]$, where α is the edge connecting v and w, and $*$ represents the concatenation of two paths. Therefore, the difference of two height sections is a well defined function with domain $\mathcal{V}(\mathcal{D})$ and codomain $4\mathbb{Z}$.

Given a tiling we define its corresponding height section as we did for height functions: by considering paths contained in the boundaries of the dominoes and, as we move along directed edges, we add or subtract 1. The definition of $\mathscr{P} \times \mathbb{Z}/\sim$ implies that this construction does not depend on choices of paths. Furthermore this definition satisfies properties I, II and III.

In Figure 2.4 we show a region with a tiling and its corresponding height section; instead of exhibiting a path and an integer for each vertex, we draw a height section by following a different construction. Consider line segments crossing the squares contained in the holes of $\mathcal{D}$ and fix a basepoint v_0. For each vertex v, consider a path γ from v_0 to v contained in the boundaries of the dominoes such that γ does not cross the line segments. Therefore, we obtain an equivalence class $[(\gamma, k)]$ by following the previous paragraph. It is not difficult to see that this construction does not depend on choice of paths. The figure then shows the integer k at the vertex v. Notice that a different choice of line segments produces a different representation of the same height section.

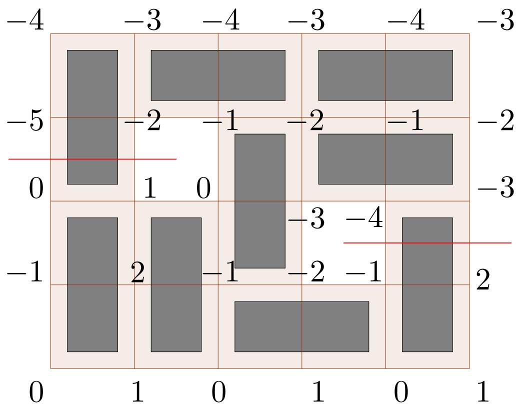

Figure 2.4: Height section of a tiling.

As with height functions the choice of basepoint and base value contributes only with a global constant in $4\mathbb{Z}$. Then, since the values on the boundary differ by 1, once we fix a basepoint and a base value we may assume that any two height sections coincide on the exterior boundary of $\mathcal{D}$. Furthermore, two tilings differ by a flip iff their corresponding height sections differ at a unique vertex by ± 4.

Then, with the analogous interpretation of the concepts of $<$ and adjacency, we have an extension of Lemma 2.1 to height sections.

Lemma 2.2. *Let $h_1 < h_2$ be two height sections coinciding on the boundary of a region $\mathcal{D}$. Then, there exists a height section h_3, adjacent to h_1, such that $h_1 < h_3 \leq h_2$.*

Proof. As with height functions, it is sufficient to prove that h_1 has a local minimum u in $\mathcal{V}(\mathcal{D})$. We have that $h_2 - h_1$ is a well defined function which assumes values in $4\mathbb{Z}$. Therefore, we can consider the subset $A \subset \mathcal{V}(\mathcal{D})$ where $h_2 - h_1$ is maximum. The proof of Lemma 2.1 is based on finding a global minimum of h_1 in A. Unfortunately, we are not able to find a global minimum of a height section, since only elements in adjacent fibers can be compared. However, notice that if h_1 has a local minimum in A, the result follows as with height functions. Suppose by contradiction that there exists no local minimum. Then, for every $v \in A$ there exists $w \in A$ such that w is adjacent to v and $h_1(w) < h_1(v)$. Since A is finite, there exists a cycle $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$ of vertices in A such that $h_1(v_{i+1}) < h_1(v_i)$ for every $i = 0, \ldots, n-1$. Without loss of generality suppose that the cycle is oriented counterclockwise, simple and encloses a minimum area. Clearly this area is greater than 1. Since the area is minimum, any two adjacent vertices in the the cycle must differ by 1, i.e, $h_1(v_i) - h_1(v_{i+1}) = 1$. Since $h_1(v_i) > h_1(v_{i+1})$ and $h_1(v_{i+2}) > h_1(v_{i+1})$ the edges v_i, v_{i+1} and $v_{i+1}v_{i+2}$ form a right angle, otherwise one of the edges would have difference 3. Furthermore, since the cycle encloses a minimum area, $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ can not be the four vertices of a square oriented counterclockwise. Therefore, the cycle can not turn left two consecutive times and this contradicts the fact that the cycle is oriented counterclockwise. ■

Then, as with height functions it follows that any two tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ can be joined by a sequence of flips iff their corresponding height sections coincide on the boundary of $\mathcal{D}$.

Lemma 2.2 (see [7]) has a combinatorial version which allows us to decide whether two tilings are connected by a sequence of flips without computing their corresponding height sections.

A *cut* of $\mathcal{D}$ is a polygonal line contained in $\mathcal{D}^*$ connecting two boundary vertices. The *flux* of a tiling across a cut is defined by counting the number of dominoes crossing the cut where the domino is counted positively (resp. negatively) if its white square is to the left (resp. right) of the cut. Notice that, for a fixed cut, the flux of a tiling is invariant under flips.

Consider a cut which disconnects $\mathcal{D}$ into two regions $\mathcal{D}_1$ and $\mathcal{D}_2$. Suppose $\mathcal{D}_1$ is to the left of the cut and $\mathcal{D}_2$ is to the right of the cut. Then, the flux of any tiling is given by the number of white squares in $\mathcal{D}_1$ minus the number of black squares contained $\mathcal{D}_1$; which is equal to the number of black squares in $\mathcal{D}_2$ minus the number of white squares in $\mathcal{D}_2$. For a cut which does not disconnect $\mathcal{D}$ the flux may admit different values. In Figure 2.5 we show a region, a cut and two tilings with different flux.

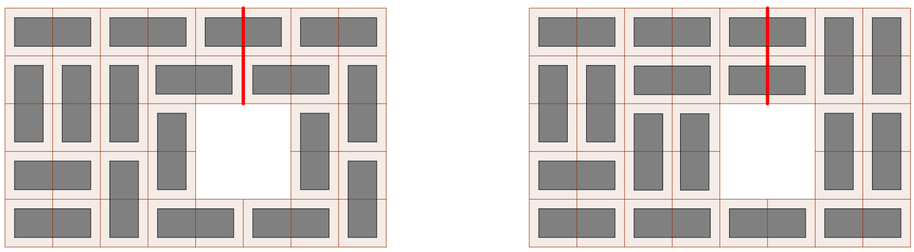

Figure 2.5: The figure shows a cut (in red) and two tilings. The first tiling has flux $+1$ and the second tiling has flux 0.

We then have the following version of Lemma 2.2:

Lemma 2.3. *Let $\mathcal{D}$ be a region with genus n. Consider n disjoint cuts of $\mathcal{D}$ such that every cut does not disconnect $\mathcal{D}$. Then, two tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ can be joined by a sequence of flips if and only if their flux across each of the n cuts are equal.*

Remark 2.4. In the lemma above we assume the existence of the n disjoint cuts not disconnecting $\mathcal{D}$, for a more detailed discussion see [7]. ◇

Proof. Let h_1 and h_2 be the corresponding height sections of $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively. We may assume that h_1 and h_2 coincide on the exterior boundary of $\mathcal{D}$. Consider a cut Γ connecting two boundary points v_0 and v_1 such that v_0 is contained in the exterior boundary of $\mathcal{D}$. It is not difficult to see that $h_1(v_0) - h_1(v_1) - (h_2(v_0) - h_2(v_1)) = 4(f_1 - f_2)$ where f_1 and f_2 are the flux of $\mathbf{t}_1$ and $\mathbf{t}_2$ across Γ, respectively. Then, the flux across the n cuts are equal if and only if h_1 and h_2 coincide on n vertices in the interior boundary of $\mathcal{D}$ and therefore on the whole boundary of $\mathcal{D}$. The result then follows by Lemma 2.2. ■

3
Three-dimensional tilings

In this chapter we present the necessary background about tilings of three-dimensional regions; the material covered is contained in [4].

A *cubiculated region* $\mathcal{R} \subset \mathbb{R}^3$ is a connected finite union of closed unit cubes $[a, a+1] \times [b, b+1] \times [c, c+1]$ such that $(a, b, c) \in \mathbb{Z}^3$. We say that a unit cube is white (resp. black) if $(-1)^{a+b+c}$ is equal to $+1$ (resp. -1). The two-dimensional concepts of domino, domino tiling and flip are naturally extended to three-dimensional regions. A domino is the union of two adjacent unit cubes and a domino tiling of $\mathcal{R}$ is a covering of $\mathcal{R}$ by dominoes with disjoint interiors. Moreover, a flip is local move which consists of a 90° rotation of two adjacent parallel dominoes. Notice that in three-dimensional regions a flip can be performed in three directions. We denote the set of domino tilings of $\mathcal{R}$ by $\mathcal{T}(\mathcal{R})$. Given two tilings $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}(\mathcal{R})$ we write $\mathbf{t}_1 \approx \mathbf{t}_2$ if there exists a sequence of flips joining $\mathbf{t}_1$ and $\mathbf{t}_2$.

We consider a specific family of cubiculated regions. A *cylinder* $\mathcal{R}_N$ is a cubiculated region formed by the cartesian product $\mathcal{D} \times [0, N]$, where $N \in \mathbb{N}$ and $\mathcal{D}$ is a balanced quadriculated disk. For cylinders with N even, we have the *vertical tiling* $\mathbf{t}_{\text{vert},N} \in \mathcal{T}(\mathcal{R}_N)$ which consists of only dominoes of the form $[a, a+1] \times [b, b+1] \times [c, c+2]$.

We draw a tiling $\mathbf{t} \in \mathcal{R}_N$ by floors, i.e, we describe the behavior of $\mathbf{t}$ at each floor $\mathcal{D} \times [k-1, k]$ for every $k = 1, \ldots, N$, as in Figure 3.1. First, we fix the x-axis and the y-axis. The floors are then exhibited in increasing order from the left to the right. Dominoes which are parallel to either the x-axis or the y-axis are represented as planar dominoes. Vertical dominoes, i.e, dominoes parallel to z-axis, are represented by two squares contained in adjacent floors; to avoid confusion the square contained in the highest floor is left unfilled.

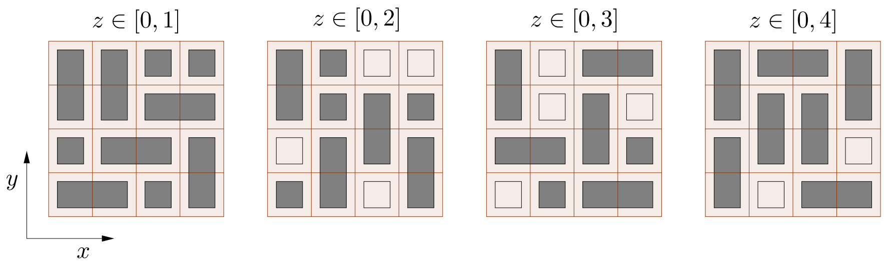

Figure 3.1: A tiling of the region $[0, 4] \times [0, 4] \times [0, 4]$.

A balanced quadriculated disk $\mathcal{D}$ is called *trivial* if either $\mathcal{D}$ is a 2×2 square or every unit square contained in $\mathcal{D}$ is adjacent to at most other two unit squares.

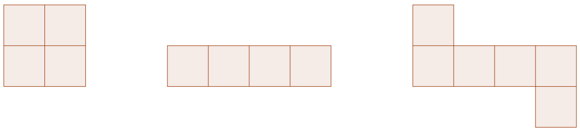

Figure 3.2: Examples of trivial disks.

The following lemma shows that tilings of a cylinder $\mathcal{D} \times [0, N]$ with $\mathcal{D}$ trivial can be joined by a sequence of flips; therefore we usually assume that our disks are not trivial.

Lemma 3.1. *Let $\mathcal{D}$ be a trivial balanced quadriculated disk. Consider two tilings $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_N)$ for some $N \in \mathbb{N}$. Then, $\mathbf{t}_1 \approx \mathbf{t}_2$.*

Proof. If $\mathcal{D}$ is not a 2×2 square then the cylinder $\mathcal{D} \times [0, N]$ can be identified with a planar region. The result then follows by Lemma 2.1.

If $\mathcal{D}$ is a 2×2 square, consider an arbitrary tiling $\mathbf{t}$ of $\mathcal{D} \times [0, N]$. Among the vertical dominoes in $\mathbf{t}$, take d_0 occupying $\mathcal{D} \times [N_0, N_0+2]$ with N_0 minimum. Since $\mathcal{D}$ is a 2×2 square, there exists a vertical domino d_1 in $\mathbf{t}$ adjacent to d. Then, perform a vertical flip obtaining two horizontal dominoes. A repeated argument then shows that $\mathbf{t}$ is equivalent by flips to a tiling of $\mathcal{D} \times [0, N]$ which is formed only by horizontal dominoes. Since $\mathbf{t}$ is arbitrary, the result then follows. ■

In contrast with the two-dimensional case, many simply connected three-dimensional regions admit isolated tilings, i.e, tilings where no flip can be performed. For instance, in Figure 3.3, we exhibit two isolated tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ of the cylinder $[0, 4] \times [0, 4] \times [0, 2]$. Therefore, $\mathbf{t}_1 \not\approx \mathbf{t}_2$.

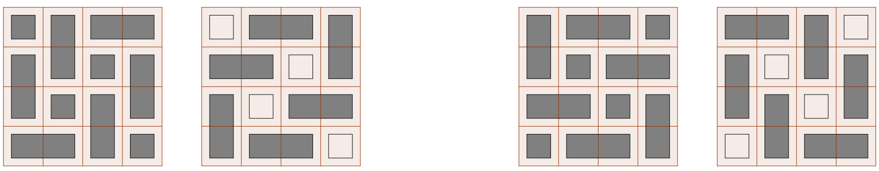

Figure 3.3: Two isolated tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ of $[0, 4] \times [0, 4] \times [0, 2]$.

For a fixed balanced quadriculated disk $\mathcal{D}$, consider $\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_{N_1})$ and $\mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{N_2})$. We now define the concatenation $\mathbf{t}_1 * \mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{N_1+N_2})$ of $\mathbf{t}_1$ and $\mathbf{t}_2$. Translating $\mathbf{t}_2$ by $(0, 0, N_1)$ we obtain a tiling $\tilde{\mathbf{t}_2}$ of $\mathcal{D} \times [N_1, N_1 + N_2]$.

Then, $\mathbf{t}_1 * \mathbf{t}_2$ is defined as the union of $\mathbf{t}_1$ and $\tilde{\mathbf{t}_2}$. Therefore, we draw $\mathbf{t}_1 * \mathbf{t}_2$ by concatenating the drawings of $\mathbf{t}_1$ and $\mathbf{t}_2$.

It turns out that the tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ in Figure 3.3 are such that $\mathbf{t}_1 * \mathbf{t}_{\mathrm{vert},2} \approx \mathbf{t}_2 * \mathbf{t}_{\mathrm{vert},2}$. This example suggests an equivalence relation $\sim$ on tilings. Let $\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_{N_1})$ and $\mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{N_2})$ such that $N_1 \equiv N_2 \pmod 2$. We say that $\mathbf{t}_1 \sim \mathbf{t}_2$ if there exists $M_1, M_2 \in 2\mathbb{N}$ such that $M_1 + N_1 = M_2 + N_2$ and $\mathbf{t}_1 * \mathbf{t}_{\mathrm{vert},M_1} \approx \mathbf{t}_2 * \mathbf{t}_{\mathrm{vert},M_2}$. Notice that if $\mathbf{t}_1 \not\sim \mathbf{t}_2$ then $\mathbf{t}_1 \not\approx \mathbf{t}_2$. Then, in order to study the flip connectivity problem we first study the equivalence relation $\sim$.

3.1 Plugs and corks

Let $\mathcal{D}$ be a balanced quadriculated disk. A *plug* p is a finite union of unit squares contained in $\mathcal{D}$ such that the number of white squares and black squares in p are equal. The number of squares contained in p is denoted by $|p|$. Notice that if p is a plug its complement $\mathcal{D} \smallsetminus \mathrm{int}(p)$ also is a plug and is denoted by p^{-1}. We admit the empty plug $\mathbf{p}_\circ = \emptyset$ and the full plug $\mathbf{p}_\bullet = \mathcal{D}$. The set of plugs in $\mathcal{D}$ is denoted by $\mathcal{P}$.

Let $p_1, p_2 \in \mathcal{P}$ be two plugs and consider two integers N_1, N_2 such that $N_2 > N_1 + 2$. The *cork* $\mathcal{R}_{N_1,N_2;p_1,p_2}$ is defined to be the cubiculated region:

$$\mathcal{R}_{N_1,N_2;p_1,p_2} = (\mathcal{D} \times [N_1, N_2]) \smallsetminus \mathrm{int}((p_1 \times [N_1, N_1 + 1]) \cup (p_2 \times [N_2 - 1, N_2]))$$

In other words, the cork $\mathcal{R}_{N_1,N_2;p_1,p_2}$ is obtained from $\mathcal{D} \times [N_1, N_2]$ by removing the plug p_1 from the $(N_1 + 1)$-th floor and the plug p_2 from the N_2-th floor. Notice that $\mathcal{R}_{0,N;\mathbf{p}_\circ,\mathbf{p}_\circ} = \mathcal{R}_N$. Furthermore, as with cylinders, when $N_2 - N_1$ is even we have the vertical tiling $\mathbf{t}_{\mathrm{vert}} \in \mathcal{T}(\mathcal{R}_{N_1,N_2;p,p})$ formed only by vertical dominoes.

Let $\mathcal{R} = \mathcal{R}_{N_1,N_2;p_1,p_2}$ be a cork and consider a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R})$. We can describe the behavior of $\mathbf{t}$ at each floor $\mathcal{D} \times [N - 1, N]$ by a triple $f_N = (p_{N-1}, f^*_N, p_N)$ for each $N \in \{N_1 + 1, N_1 + 2, \ldots, N_2\}$; f_N is called the *full N-th floor*. The plug $p_N = \mathrm{plug}_N(\mathbf{t}) \in \mathcal{P}$ is formed by the the unit squares $[a, a + 1] \times [b, b + 1]$ such that $[a, a + 1] \times [b, b + 1] \times [N - 1, N + 1]$ is contained in $\mathbf{t}$. Notice that the plugs p_{N-1} and p_N are disjoint. The *reduced N-th floor* f^*_N is formed by the horizontal dominoes of $\mathbf{t}$ which are contained in $\mathcal{D} \times [N - 1, N]$. Therefore, f^*_N is a tiling of $\mathcal{D} \smallsetminus (p_{N-1} \cup p_N)$. By identifying each floor with a triple we can then describe $\mathbf{t}$ by a sequence:

$$\mathbf{t} = (p_{N_1}, f^*_{N_1+1}, p_{N_1+1}, f^*_{N_1+2}, p_{N_1+2}, \ldots, p_{N_2-1}, f^*_{N_2}, p_{N_2}).$$

Example 3.2. Figure 3.4 exhibits a tiling of the cylinder $[0,4]^3$ by describing a sequence of reduced floors and plugs. Notice that the sequence starts and ends with the empty plug.

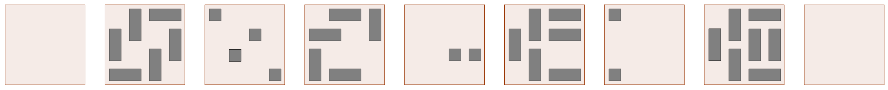

Figure 3.4: A sequence of reduced floors and plugs. $\diamond$

Let $\mathcal{D}$ be a balanced quadriculated disk and $p \in \mathcal{P}$ a plug. A tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{-N,N;p,p})$ is *even* if it is symmetric with respect to the reflection on the xy plane. Therefore, $\mathbf{t}$ is of the form:

$$\mathbf{t} = (p, f_N^*, p_{N_1}, f_{N-1}^*, \ldots, p_1, f_1^*, p_0, f_1^*, p_1, \ldots, f_{N_1}^*, p_{N_1}, f_N^*, p).$$

Notice that by performing one vertical flip for each domino in f_1^* we obtain:

$$\mathbf{t}_1 = (p, f_N^*, p_{N_1}, f_{N-1}^*, \ldots, p_2, f_2^*, p_1, \emptyset, p_1^{-1}, \emptyset, p_1, f_2^*, p_2, \ldots, f_{N_1}^*, p_{N_1}, f_N^*, p).$$

Now, by performing three vertical flips for each domino in f_2 we obtain:

$$\mathbf{t}_2 = (p, f_N^*, p_{N_1}, f_{N-1}^*, \ldots, p_2, \emptyset, p_2^{-1}, \emptyset, p_2, \emptyset, p_2^{-1}, \emptyset, p_2, \ldots, f_{N_1}^*, p_{N_1}, f_N^*, p).$$

A repeated argument proves the following lemma:

Lemma 3.3. *If a tiling* $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{N,N;p,p})$ *is even then* $\mathbf{t} \approx \mathbf{t}_{vert}$.

We have a natural generalization of concatenation of tilings to concatenation of corks. Consider plugs $p_1, p_2, p_3 \in \mathcal{P}$ and tilings $\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_{N_1,N_2;p_1,p_2})$ and $\mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{N_2,N_3;p_2,p_3})$. Then, the concatenation $\mathbf{t}_1 * \mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{N_1,N_3;p_1,p_3})$ is formed by the dominoes of $\mathbf{t}_1$, the dominoes of $\mathbf{t}_2$ and the dominoes $s \times [N_2 - 1, N_2 + 1]$ for each square $s \subset \mathcal{D}$ such that $s \in p_2$. The following lemma allows us to introduce vertical dominoes between any two floors of a tiling.

Lemma 3.4. *Consider a balanced quadriculated disk* $\mathcal{D}$. *Let* $p_1, p_2 \in \mathcal{P}$ *be two plugs and let* $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{N_1,N_2;p_1,p_2})$ *be a tiling. Let* $\mathbf{t}_{vert,p_1} \in \mathcal{T}(\mathcal{R}_{N_1-2,N_1;p_1,p_1})$ *and* $\mathbf{t}_{vert,p_2} \in \mathcal{T}(\mathcal{R}_{N_1,N_1+2;p_2,p_2})$ *be vertical tilings. Consider the concatenations* $\tilde{\mathbf{t}_1} = \mathbf{t}_{vert,p_1} * \mathbf{t} \in \mathcal{T}(\mathcal{R}_{N_1-2,N_2;p_1,p_2})$ *and* $\tilde{\mathbf{t}_2} = \mathbf{t} * \mathbf{t}_{vert,p_2} \in \mathcal{T}(\mathcal{R}_{N_1,N_2+2;p_1,p_2})$. *Then, after a translation by* $(0,0,2)$, $\tilde{\mathbf{t}_1} \approx \tilde{\mathbf{t}_2}$.

Proof. We must move every horizontal domino in $\tilde{\mathbf{t}_1}$ down by two floors. This is performed in increasing order of the z coordinate. Therefore, for each horizontal domino $d \times [k-1,k]$ the region $d \times [k-3,k-1]$ is occupied by two vertical dominoes. The result then follows by performing two flips for each horizontal domino. $\blacksquare$

Lemma 3.5. *Consider a balanced quadriculated disk $\mathcal{D}$ and a plug $p \in \mathcal{P}$. If N is even and $N \geq |p|$ then there exists an even tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{-N,N;p,p})$ such that $plug_0(\mathbf{t}) = \mathbf{p}_\circ$. In particular, $\mathbf{t} \approx \mathbf{t}_{vert}$.*

Proof. We identify $\mathcal{D}$ with a bipartite graph. The vertices are the colored unit squares and two vertices are connected by an edge if and only if their corresponding unit squares are adjacent. Let us fix a spanning tree of $\mathcal{D}$. The distance between two squares is then defined as the distance along the spanning tree.

The proof is by induction on the even integer $|p|$. If $|p| = 0$ then $p = \mathbf{p}_\circ$ and the result follows by taking $\mathbf{t} = \mathbf{t}_{\text{vert}}$. Consider $p \in \mathcal{P}$ such that $|p| \geq 2$. Let l be the minimal distance between a black square and a white square in p. Let s_0 and s_l be the squares which realize this minimum. Consider $\tilde{p} = p \setminus (s_0 \cup s_l)$ so that $|\tilde{p}| = |p| - 2$. For $\tilde{N} = N - 2$, we construct an even tiling $\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_{-N,N;p,p})$ such that $\text{plug}_{-\tilde{N}}(\mathbf{t}_1) = \text{plug}_{\tilde{N}}(\mathbf{t}_1) = \tilde{p}$ and the restriction of $\mathbf{t}_1$ to $\mathcal{R}_{-\tilde{N},\tilde{N};\tilde{p},\tilde{p}}$ is equal to $\mathbf{t}_{\text{vert}} \in \mathcal{T}(\mathcal{R}_{-\tilde{N},\tilde{N};\tilde{p},\tilde{p}})$.

Let $s_0, s_1, \ldots, s_l$ be the squares between s_0 and s_l along the spanning tree. Notice that, since l is minimal, the squares $s_1, \ldots, s_{l-1}$ are not contained in p. Let us denote a horizontal domino $(s_i \cup s_{i+1}) \times [c-1, c]$ by a triple (s_i, s_{i+1}, c); then the horizontal dominoes of $\mathbf{t}_1$ are:

$$
\begin{aligned}
&(s_1, s_2, -N+1), (s_3, s_4, -N+1), \ldots, (s_{l-2}, s_{l-1}, -N+1);\\
&(s_0, s_1, -N+2), (s_2, s_3, -N+2), \ldots (s_{l-1}, s_l, -N+2);\\
&(s_0, s_1, N-1), (s_2, s_3, N-1), \ldots (s_{l-1}, s_l, N-1);\\
&(s_1, s_2, N), (s_3, s_4, N), \ldots, (s_{l-2}, s_{l-1}, N);
\end{aligned}
$$

All other dominoes of $\mathbf{t}_1$ are vertical.

The result then follows by induction. Indeed, by the induction hypothesis $\mathbf{t}_{\text{vert}} \in \mathcal{T}(\mathcal{R}_{-\tilde{N},\tilde{N};\tilde{p},\tilde{p}})$ is equivalent by flips to an even tiling $\mathbf{t}$ such that $\text{plug}_0(\mathbf{t}) = \mathbf{p}_\circ$. Therefore, $\mathbf{t}_1$ is equivalent by flips to a tiling which satisfies the desired properties. ■

Example 3.6. Figure 3.5 below exhibits an example of the tiling $\mathbf{t}_1$ constructed in the proof of Lemma 3.5; we omit some vertical floors of $\mathbf{t}_1$.

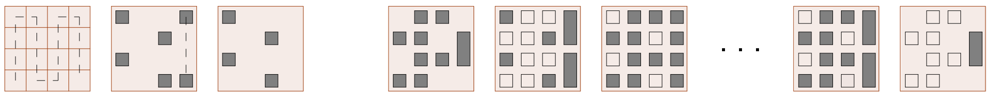

Figure 3.5: The disk $\mathcal{D} = [0,4] \times [0,4]$ with a spanning tree, the plug p with the squares s_0 and s_2, the plug $\tilde{p}$ and the tiling $\mathbf{t}_1$.

$\diamond$

Lemma 3.7. *Let $\mathcal{D}$ be a quadriculated disk and $p \in \mathcal{P}$ be a plug. If $N \geq 2|\mathcal{D}|$ then both corks $\mathcal{R}_{0,N;\mathbf{p}_\circ,p}$ and $\mathcal{R}_{0,N;p,\mathbf{p}_\circ}$ admit tilings.*

Proof. We construct tilings of $\mathcal{R}_{0,N;\mathbf{p}_\circ,p}$; a similar construction shows that there exists tilings of $\mathcal{R}_{0,N;p,\mathbf{p}_\circ}$ Let $N = 2|\mathcal{D}| + k$ for some $k \in \mathbb{N}$. If k is even, by Lemma 3.5, there exists a tiling $\mathbf{t} \in \mathcal{R}_{-N,N;p;p}$ such that $\operatorname{plug}_0(\mathbf{t}) = \mathbf{p}_\circ$. Therefore, by restricting $\mathbf{t}$ to $\mathcal{R}_{0,N}$, we obtain a tiling of $\mathcal{R}_{0,N;\mathbf{p}_\circ,p}$. Suppose k is odd. Then, $k + 1 + |\mathcal{D}|$ is even and, by Lemma 3.5, there exists a tiling $\mathbf{t} \in \mathcal{R}_{0,k+1+|\mathcal{D}|;\mathbf{p}_\circ,p}$. Furthermore, there exists a tiling $\tilde{\mathbf{t}}$ of $\mathcal{R}_{0,|\mathcal{D}|;\mathbf{p}_\circ,\mathbf{p}_\bullet} = \mathcal{R}_{0,|\mathcal{D}|-1,\mathbf{p}_\circ,\mathbf{p}_\circ}$. Therefore, $\tilde{\mathbf{t}} * \mathbf{t}$ is a tiling of $\mathcal{R}_{0,N;\mathbf{p}_\circ,p}$. ■

3.2 Twist

Let $\mathcal{D}$ be a quadriculated disk. The twist of a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_N)$ is defined in [2]. The twist is a flip invariant that associates an integer number $\mathrm{Tw}(\mathbf{t})$ with a tiling; in particular, for a vertical tiling $\mathrm{Tw}(\mathbf{t}_{\mathrm{vert}}) = 0$. Moreover, if $\mathbf{t}$ is a tiling and $\mathbf{t}^{-1}$ is its reflection on the xy plane then $\mathrm{Tw}(\mathbf{t}^{-1}) = -\mathrm{Tw}(\mathbf{t})$. The concatenation of two tilings $\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_{N_1})$ and $\mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{N_2})$ is such that $\mathrm{Tw}(\mathbf{t}_1 * \mathbf{t}_2) = \mathrm{Tw}(\mathbf{t}_1) + \mathrm{Tw}(\mathbf{t}_2)$. Therefore, two tilings equivalent under $\sim$ have the same twist. In this dissertation, we are interested in determining whether two tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ with the same twist are equivalent under $\sim$.

For an arbitrary domino d, let $v(d) \in \{\pm e_1, \pm e_2, \pm e_3\} \subset \mathbb{R}^3$ be the unit vector from the center of the white cube to the center of the black cube of d. Let $u \in \{\pm e_1, \pm e_2\}$ and $X + [0, +\infty)u = \{x + tu \colon x \in X, t \in [0, +\infty)\}$. We define the *$u$-shade* of a subset $X \subset \mathbb{R}^3$ to be the set:

$$\mathcal{S}^u(X) = \operatorname{int}((X + [0, +\infty)u) \smallsetminus X).$$

Given a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R})$ and two dominoes d_1 and d_2 of $\mathbf{t}$, we define the *effect of d_1 on d_2 along u* as $\tau^u(d_1, d_2) \in \{0, \pm\frac{1}{4}\}$ where

$$\tau^u(d_1, d_2) = \begin{cases} \frac{1}{4}\det(v(d_2), v(d_1), u), & d_2 \cap \mathcal{S}^u(d_1) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Since $u \in \{\pm e_1, \pm e_2\}$, notice that $\tau^u(d_1, d_2) = 0$ unless d_1 is a vertical (resp. horizontal) domino and d_2 is a horizontal (resp. vertical) domino such that d_1 and d_2 are contained in $\mathcal{D} \times [N_0, N_0 + 2]$ for some N_0.

The *twist* of $\mathbf{t}$ is defined as the sum:

$$\mathrm{Tw}(\mathbf{t}) = \sum_{d_1, d_2 \in \mathbf{t}} \tau^u(d_1, d_2).$$

The twist is always an integer and it does not depend on the choice of u (see [2], [4]).

Lemma 3.8. *Let $\mathcal{D}$ be a non trivial balanced quadriculated disk and let $a \in \mathbb{Z}$. If $N \geq 4|\mathcal{D}| + 3$ then there exits a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{2|a|N})$ such that $\mathrm{Tw}(\mathbf{t}) = a$.*

Remark 3.9. We do not try to obtain sharp estimates of N. For instance, Figure 3.3 shows two tilings of $[0, 4]^2 \times [0, 2]$ with twist $+2$. ◇

Proof. We construct two tilings $\mathbf{t}_1$ and $\mathbf{t}_2$ of $\mathcal{D} \times [0, N]$ such that $\mathrm{Tw}(\mathbf{t}_1) - \mathrm{Tw}(\mathbf{t}_2) = 1$; therefore $\mathrm{Tw}(\mathbf{t}_1 * \mathbf{t}_2^{-1}) = 1$. The result then follows by considering concatenations and reflections on the xy plane.

Since $\mathcal{D}$ is not trivial, assume without loss of generality that there exists an unit square in $\mathcal{D}$ with adjacent unit squares in the directions e_1 and $\pm e_2$. Moreover, since $\mathcal{D}$ is balanced, there exists at least more two unit squares in $\mathcal{D}$. Therefore, $\mathcal{D}$ contains at least six unit squares; let p be the plug formed by these unit squares. Figure 3.6 shows the floors $2|\mathcal{D}| + 1$, $2|\mathcal{D}| + 2$ and $2|\mathcal{D}| + 3$ of $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively.

Figure 3.6: The floors $2|\mathcal{D}| + 1$, $2|\mathcal{D}| + 2$ and $2|\mathcal{D}| + 3$ of $\mathbf{t}_1$ and $\mathbf{t}_2$. The corks $\mathcal{R}_{2|\mathcal{D}|,2|\mathcal{D}|+2;p,p}$ and $\mathcal{R}_{2|\mathcal{D}|+2,2|\mathcal{D}|+4;p,p}$ are occupied by vertical dominoes.

By Lemma 3.7 we can tile the first $2|\mathcal{D}|$ and the last $N-(2|\mathcal{D}|+3) \geq 2|\mathcal{D}|$ floors of $\mathbf{t}_1$ and $\mathbf{t}_2$ in the same way. Therefore

$$\mathrm{Tw}(\mathbf{t}_1) - \mathrm{Tw}(\mathbf{t}_2) = \sum_{\substack{d_1,d_2 \in \mathbf{t}_1 \\ d_1,d_2 \subset \mathcal{R}_{2|\mathcal{D}|,2|\mathcal{D}|+3}}} \tau^{e_2}(d_1,d_2) \ - \sum_{\substack{d_1,d_2 \in \mathbf{t}_2 \\ d_1,d_2 \subset \mathcal{R}_{2|\mathcal{D}|,2|\mathcal{D}|+3}}} \tau^{e_2}(d_1,d_2) = 1.$$

■

3.3 The domino group and the domino CW-complex

Let $\mathcal{D}$ be a balanced quadriculated disk. Consider the following set:

$$G_{\mathcal{D}} = \left(\bigsqcup_{N \in \mathbb{N}} \mathcal{T}(\mathcal{R}_N) \right) / \sim .$$

We do the usual abuse of notation and instead of thinking about equivalence classes we denote an element of $G_{\mathcal{D}}$ by $\mathbf{t}$. We define a group structure on $G_{\mathcal{D}}$ where the operation is given by the concatenation. The identity element is $\mathbf{t}_{\mathrm{vert},2}$ and the inverse element of a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_N)$ is $\mathbf{t}^{-1} \in \mathcal{T}(\mathcal{R}_N)$ obtained by reflecting $\mathbf{t}$ on the xy plane; by Lemma 3.3 $\mathbf{t} * \mathbf{t}^{-1} \sim \mathbf{t}_{\mathrm{vert},2}$. The group $(G_{\mathcal{D}}, *)$ is called the *domino group*. Furthermore, we define the *even domino group* $G^+_{\mathcal{D}}$ to be the kernel of the homomorphism $G_{\mathcal{D}} \to \mathbb{Z}/(2)$ which takes a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_N)$ to N mod 2. More specifically,

$$G^+_{\mathcal{D}} = \left(\bigsqcup_{N \in \mathbb{N}} \mathcal{T}(\mathcal{R}_{2N}) \right) / \sim .$$

Therefore $G^+_{\mathcal{D}}$ is a normal subgroup of $G_{\mathcal{D}}$ of index 2.

The twist defines a homomorphism $\mathrm{Tw}\colon G^+_{\mathcal{D}} \to \mathbb{Z}$. Notice that if $\mathcal{D}$ is a trivial disk then, by Lemma 3.1, the homomorphism Tw is trivial. However, if $\mathcal{D}$ is a non trivial disk, it follows from Lemma 3.8 that Tw is surjective. We say that a non trivial quadriculated disk $\mathcal{D}$ is *regular* if the map Tw is an isomorphism so that $G_{\mathcal{D}} \simeq \mathbb{Z} \oplus (\mathbb{Z}/(2))$. Therefore, a non trivial disk $\mathcal{D}$ is regular if and only if any two tilings $\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_{N_1})$ and $\mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{N_2})$ with $N_1 \equiv N_2 \pmod 2$ and $\mathrm{Tw}(\mathbf{t}_1) = \mathrm{Tw}(\mathbf{t}_2)$ are such that $\mathbf{t}_1 \sim \mathbf{t}_2$. We focus on these two definitions of regular disks, an equivalent probabilistic definition is presented in [5]. In that sense a disk $\mathcal{D}$ is regular if and only if for any two tilings $\mathbf{t}$ and $\tilde{\mathbf{t}}$ of $\mathcal{D} \times [0, N]$ we have that $\mathrm{Prob}[\mathbf{t} \approx \tilde{\mathbf{t}} | \mathrm{Tw}(\mathbf{t}) = \mathrm{Tw}(\tilde{\mathbf{t}})] \to 1$ as $N \to +\infty$.

When $\mathcal{D}$ is a tileable quadriculated disk which is not regular the domino group $G_{\mathcal{D}}$ is a semidirect product. Indeed, if $\mathcal{D}$ is tileable then there exists

$\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_1)$ with $\mathbf{t}_1 * \mathbf{t}_1 \approx \mathbf{t}_{\mathrm{vert},2}$. Let $H = \{\mathbf{t}_{\mathrm{vert},2}, \mathbf{t}_1\}$ be the subgroup generated by $\mathbf{t}_1$. Let $\mathbf{t} \in G_{\mathcal{D}}$ be a tiling of $\mathcal{D} \times [0, N]$. If N is even then $\mathbf{t} \sim \mathbf{t} * \mathbf{t}_{\mathrm{vert},2}$. If N is odd then $\mathbf{t} \sim \mathbf{t} * \mathbf{t}_{\mathrm{vert},2} \sim (\mathbf{t} * \mathbf{t}_1) * \mathbf{t}_1$. Recall that $G^+_{\mathcal{D}}$ is a normal subgroup of index 2 of $G_{\mathcal{D}}$ and notice that $\phi \colon G^+_{\mathcal{D}} \to G^+_{\mathcal{D}}$ such that $\phi(\mathbf{t}) = \mathbf{t}_1 * \mathbf{t} * \mathbf{t}_1$ is an automorphism with $\phi^2 = Id$. Therefore $G_{\mathcal{D}} \simeq G^+_{\mathcal{D}} \ltimes H$. In the next chapter we present many examples of regular disks and non regular disks.

Given a quadriculated disk $\mathcal{D}$, we identify the domino group $G_{\mathcal{D}}$ with the fundamental group of a 2-complex $\mathcal{C}_{\mathcal{D}}$. The 0-skeleton is formed by the plugs $p \in \mathcal{P}$. Attach an edge between two plugs p_1, p_2 for each valid full floor $f = (p_1, f^*, p_2)$. Therefore, there exists an edge between p_1 and p_2 for each tiling of $\mathcal{D} \smallsetminus (p_1 \cup p_2)$. Furthermore, $f = (p_1, f^*, p_2)$ and $f^{-1} = (p_2, f^*, p_1)$ define two orientations of the same edge. Notice that if p_1 and p_2 are not disjoint then there is no edge between them. In particular, there is a loop based at $\mathbf{p}_\circ$ for each tiling of $\mathcal{D}$ and there are no other loops in $\mathcal{C}_{\mathcal{D}}$. In contrast with the other edges we do not have an orientation for loops, i.e, if we move from $\mathbf{p}_\circ$ to $\mathbf{p}_\circ$ in one move we do not have to specify what orientation was used. Therefore, this construction is not exactly a 1-complex. This fact will be fixed when we attach the 2-cells.

We attach a disk with boundary $f * f$ for each floor $f = (\mathbf{p}_\circ, f^*, \mathbf{p}_\circ)$, therefore the loops f and f^{-1} are now homotopic. The other 2-cells correspond to flips. The horizontal flips are described by bigons. Let $p_1, p_2 \in \mathcal{P}$ be two disjoint plugs and let f_1^*, f_2^* be two tilings of $\mathcal{D} \smallsetminus (p_1 \cup p_2)$ differing by a flip. Attach a bigon in the cycle with vertices p_1 and p_2 and edges $f_1 = (p_1, f_1^*, p_2)$ and $f_2 = (p_1, f_2^*, p_2)$, as in Figure 3.7.

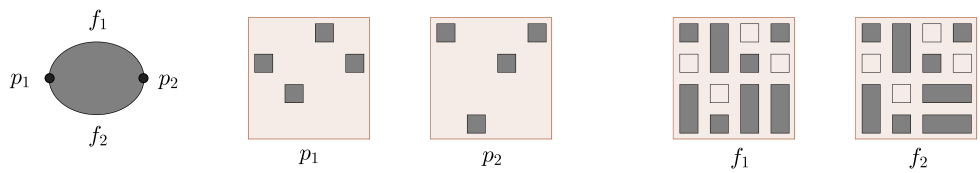

Figure 3.7: Example of a 2-cell defined by a horizontal flip in the disk $\mathcal{D} = [0, 4] \times [0, 4]$.

The vertical flips are described by quadrilaterals. Let $p_1, p_2, \tilde{p_2}, p_3 \in \mathcal{P}$ be plugs such that p_2 and $\tilde{p_2}$ differ by two adjacent unit squares. Let d be the domino formed by the union of these two squares. Suppose that $\tilde{p_2}$, and therefore p_2, is disjoint from both p_1 and p_3. Consider $\tilde{f}_1^*$ and $\tilde{f}_2^*$ tilings of $\mathcal{D} \smallsetminus (p_1 \cup \tilde{p_2})$ and $\mathcal{D} \smallsetminus (\tilde{p_2} \cup p_3)$, respectively. Let f_1 and f_2 be tilings of $\mathcal{D} \smallsetminus (p_1 \cup p_2)$ and $\mathcal{D} \smallsetminus (p_2 \cup p_3)$ obtained from f_1^* and f_2^*, respectively, by adding the domino d. Then, attach a quadrilateral with vertices $p_1, p_2, \tilde{p_2}, p_3$

and edges $f_1 = (p_1, f_1^*, p_2)$, $\tilde{f}_1 = (p_1, \tilde{f}_1^*, \tilde{p}_2)$, $f_2 = (p_2, f_2^*, p_3)$, $\tilde{f}_2 = (\tilde{p}_2, \tilde{f}_2^*, p_3)$, as in Figure 3.8.

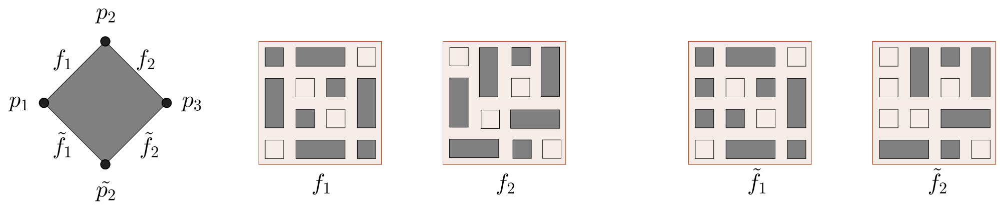

Figure 3.8: Example of a 2-cell defined by a vertical flip in the disk $\mathcal{D} = [0,4] \times [0,4]$.

Given a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_N)$ consider its corresponding sequence $\mathbf{t} = (\mathbf{p}_\circ, f_1^*, p_1, \ldots, p_{N-1}, f_N^*, \mathbf{p}_\circ)$. We then have an identification between $\mathbf{t}$ and a closed path of length N in $\mathcal{C}_\mathcal{D}$. It is not difficult to verify that two tilings are equivalents under $\sim$ if and only if their corresponding paths in $\mathcal{C}_\mathcal{D}$ are homotopic. Therefore, we have that $G_\mathcal{D} \simeq \pi_1(\mathcal{C}_\mathcal{D}, \mathbf{p}_\circ)$.

By the Galois correspondence, the subgroup of index two $G_\mathcal{D}^+$ corresponds to a double cover $\mathcal{C}_\mathcal{D}^+$ of $\mathcal{C}_\mathcal{D}$. The set of vertices of $\mathcal{C}_\mathcal{D}^+$ is the set $\mathcal{P} \times \mathbb{Z}/(2)$ which indicates the plug and the parity of its position. Similarly, the edges of $\mathcal{C}_\mathcal{D}^+$ are floors with parity, i.e, a pair (f, k) where f is a floor and $k \in \mathbb{Z}/(2)$. Notice that $(f, k)^{-1} = (f^{-1}, k+1)$.

3.4 Generators

Recall that, as in the proof of Lemma 3.5, there exists an identification between a quadriculated disk $\mathcal{D}$ and a bipartite graph. We say that a disk $\mathcal{D}$ is *hamiltonian* if its corresponding bipartite graph has a hamiltonian path. Notice that every hamiltonian disk is tileable.

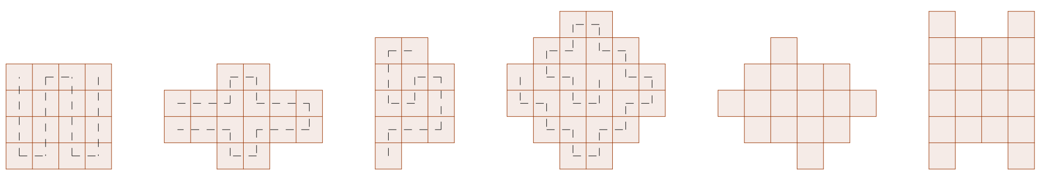

Figure 3.9: The first four disks are hamiltonian. The fifth disk is not hamiltonian and not tileable. The sixth disk is tileable and not hamiltonian.

We have an identification between the domino group $G_\mathcal{D}$ and the fundamental group $\pi_1(\mathcal{C}_\mathcal{D})$ of a 2-complex $\mathcal{C}_\mathcal{D}$. In this section we present a small family of tilings which generates $G_\mathcal{D}$ for a hamiltonian disk $\mathcal{D}$.

Given a hamiltonian quadriculated disk $\mathcal{D}$, consider a fixed hamiltonian path $\gamma_0 = (s_1, \ldots, s_{|\mathcal{D}|})$ where s_i is a unit square contained in $\mathcal{D}$ and s_i and s_{i+1} are adjacent unit squares for each $i = 1, \ldots, |\mathcal{D}|$. We say a that a domino $d \subset \mathcal{D}$ is *contained* in the path γ_0 if there exists i such that $d = s_i \cup s_{i+1}$. Furthermore, we say that a domino $d \subset \mathcal{R}_N$ *respects* γ_0 if its projection $\tilde{d} \subset \mathcal{D}$ is contained in γ_0; in particular, vertical dominoes respect γ_0. In that sense, a tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{N_1,N_2,p_1,p_2})$ respects γ_0 if every domino in $\mathbf{t}$ respects γ_0.

Consider a hamiltonian disk $\mathcal{D}$ with a fixed hamiltonian path γ_0. Suppose without loss of generality that the color of the unit square s_i is $(-1)^i$. Let $d = s_{i_{d,-}} \cup s_{i_{d,+}}$ be a domino not contained in γ_0 so that we may assume $i_{d,-1} + 1 < i_{d,+1}$. Then, we divide γ_0 into three intervals: $I_{d;-1} = \mathbb{Z} \cap [1, i_{d,-1} - 1]$, $I_{d;0} = \mathbb{Z} \cap [i_{d,-} + 1, i_{d,+} - 1]$ and $I_{d;+1} = \mathbb{Z} \cap [i_{d,+} + 1, |\mathcal{D}|]$. Notice that the intervals $I_{d;-1}$ and $I_{d,+1}$ may be empty and the interval $I_{d;0}$ always has even positive cardinality. A plug p is *compatible* with d if there exists no unit square contained in both p and d. For a plug $p \in \mathcal{P}$ compatible with d, define

$$\operatorname{flux}_j(d; p) = \sum_{i \in I_{d;j}, s_i \subset p} (-1)^i \in \mathbb{Z}.$$

In other words, we compute $\operatorname{flux}_j(d; p)$ by adding $+1$ (resp. -1) for each white (resp. black) unit square $s_i \subset p$ with $i \in I_{d;j}$. Notice that, since p is balanced, $\operatorname{flux}_{-1}(d; p) + \operatorname{flux}_0(d; p) + \operatorname{flux}_{+1}(d; p) = 0$. Define the triple $\operatorname{flux}(d; p) = (\operatorname{flux}_{-1}(d; p), \operatorname{flux}_0(d; p), \operatorname{flux}_{+1}(d; p))$. Notice that $\operatorname{flux}(d; p) \in H$ where $H = \{(\phi_{-1}, \phi_0, \phi_{+1}) \in \mathbb{Z}^3 \colon \phi_{-1} + \phi_0 + \phi_{+1} = 0\}$.

For instance, Figure 3.10 below shows a hamiltonian quadriculated disk and a domino d with a compatible plug p. Suppose that the hamiltonian path starts at the top left corner of the square $\mathcal{D} = [0, 4]^2$. Then, by construction, $\operatorname{flux}(d; p) = (-1, 2, -1)$.

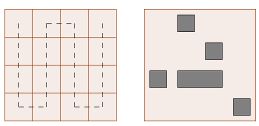

Figure 3.10: A hamiltonian disk with a domino and a compatible plug.

Consider the set

$$\Phi_d = \{(\phi_{-1}, \phi_0, \phi_{+1}) \in H \colon \forall j, \phi_j \in [\phi_j^{\min}, \phi_j^{\max}]\}$$

where $\phi_j^{\min} = -|\{i \in I_{d;j} \colon (-1)^i = -1\}|$ and $\phi_j^{\max} = |\{i \in I_{d;j} \colon (-1)^i = +1\}|$. Notice that there exists a plug $p \in \mathcal{P}$ disjoint from d with $\operatorname{flux}(d; p) = \phi$

if and only if $\phi \in \Phi_d$. A *complete family* of compatible plugs for d is a family $(p_{d,\phi})_{\phi\in\Phi_d}$ such that $\operatorname{flux}(d; p_{d,\phi}) = \phi$ for all $\phi \in \Phi_d$.

For each plug $p \in \mathcal{P}$ and N even let $\mathbf{t}_p \in \mathcal{T}(\mathcal{R}_{0,N;p,\mathbf{p}_\circ})$ be the restriction of the tiling constructed in Lemma 3.5. Notice that all dominoes in $\mathbf{t}_p$ respect γ_0. The following lemma shows that $\mathbf{t}_p$ is well defined.

Lemma 3.10. *Consider a hamiltonian quadriculated disk $\mathcal{D}$ with a fixed path γ_0. Let $p \in \mathcal{P}$ be a plug and let $\mathbf{t}_1 \in \mathcal{T}(\mathcal{R}_{0,N_1;p;\mathbf{p}_\circ})$ and $\mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{0,N_2;p;\mathbf{p}_\circ})$ be two tilings where N_1 and N_2 are both even. If both $\mathbf{t}_1$ and $\mathbf{t}_2$ respect γ_0 then $\mathbf{t}_1 \sim \mathbf{t}_2$.*

Proof. Given an arbitrary tiling $\mathbf{t} \in \mathcal{T}(\mathcal{R}_{0,N;p,\mathbf{p}_\circ})$ respecting γ_0 we construct a planar tiling of a region $\mathcal{D}_p \subset [0, |\mathcal{D}|] \times [0, N]$. The region $\mathcal{D}_p$ is obtained from $[0, |\mathcal{D}|] \times [0, N]$ by removing the unit squares $[i-1, 1] \times [0, 1]$ for which $s_i \subset p$; therefore if $N \geq 2$ the region $\mathcal{D}_p$ is a disk. Let d be a domino in $\mathbf{t}$. Since $\mathbf{t}$ respects γ_0, depending whether d is a vertical domino or a horizontal domino the projection of d on $\mathcal{D}$ is equal to either s_i or $s_i \cup s_{i+1}$. If d is horizontal then $d \subset \mathcal{D} \times [k-1, k]$ and we place a domino $[i-1, i+1] \times [k-1, k]$ in $\mathcal{D}_p$. If d is vertical then $d \subset \mathcal{D} \times [k-1, k+1]$ and we place a domino $[i-1, i] \times [k-1, k+1]$ in $\mathcal{D}_p$. Therefore, we obtain a tiling of $\mathcal{D}_p$. Conversely, given a tiling of $\mathcal{D}_p$ we can construct a tiling of $\mathcal{R}_{0,N;p,\mathbf{p}_\circ}$. Furthermore, a flip in $\mathcal{D}_p$ corresponds to a flip in $\mathcal{R}_{0,N;p,\mathbf{p}_\circ}$. By adding vertical floors we may assume that $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}(\mathcal{R}_{0,N;p,\mathbf{p}_\circ})$. The result then follows by Lemma 2.1. ■

Consider a domino $d \subset \mathcal{D}$ not contained in γ_0. Let $p \in \mathcal{P}$ be a plug compatible with d and consider the plug $\tilde{p} = p \cup d$. Let $\mathbf{t}_p \in \mathcal{T}(\mathcal{R}_{0,N_p;p,\mathbf{p}_\circ})$ and $\mathbf{t}_{\tilde{p}} \in \mathcal{T}(\mathcal{R}_{0,N_{\tilde{p}};\tilde{p},\mathbf{p}_\circ})$. Consider $f = (p, d, \tilde{p}^c)$ and $f_{\text{vert}} = (\tilde{p}^c, \emptyset, \tilde{p})$. We define the tiling $\mathbf{t}_{d;p}$ to be the concatenation

$$\mathbf{t}_{d;p} = \mathbf{t}_p^{-1} * f * f_{\text{vert}} * \mathbf{t}_{\tilde{p}} \in \mathcal{T}(\mathcal{R}_N)$$

where $N = N_p + N_{\tilde{p}} + 2$. Notice that all dominoes contained in $\mathbf{t}_{d;p}$ respect γ_0 except for the domino d contained in the floor f.

Example 3.11. Consider the hamiltonian disk $\mathcal{D}$ and the domino $d \subset \mathcal{D}$ with a compatible plug p, as in the Figure 3.10. The Figure 3.11 below exhibits the tiling $\mathbf{t}_{d;p}$.

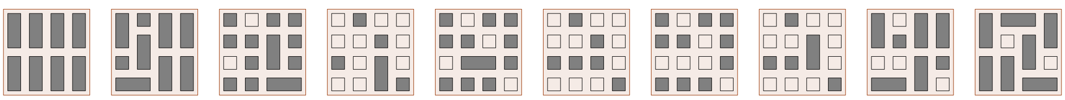

Figure 3.11: The tiling $\mathbf{t}_{d;p}$.

◇

Let $\mathcal{D}$ be a quadriculated disk with a fixed hamiltonian path γ_0. For each domino $d \subset \mathcal{D}$ not contained in γ_0 consider a complete family of compatible plugs $(p_{d,\phi})_{\phi\in\Phi_d}$. The family of tilings $(\mathbf{t}_{d;p_{d,\phi}})$ generates the domino group $G^+_{\mathcal{D}}$ (see [4]). The following lemma shows that this construction is well defined.

Lemma 3.12. *Consider a hamiltonian disk $\mathcal{D}$ with a fixed path γ_0 and a domino $d \subset \mathcal{D}$ not contained in γ_0. Let $p_1, p_2 \in \mathcal{P}$ be two plugs both compatible with d. If flux$(d; p_1) =$ flux$(d; p_2)$ then $\mathbf{t}_{d;p_1} \sim \mathbf{t}_{d;p_2}$.*

Proof. Without loss of generality we may assume that $\mathbf{t}_{d;p_1}, \mathbf{t}_{d;p_2} \in \mathcal{T}(\mathcal{R}_N)$. Let $d = s_{i_{d,-}} \cup s_{i_{d,+}}$ and suppose that $d \subset \mathcal{D} \times [k-1, k]$. Notice that, as in the proof of Lemma 3.10, each tiling $\mathbf{t}_1$ and $\mathbf{t}_2$ corresponds to a planar tiling of a region $\mathcal{D}_p$ obtained from a rectangle by removing two unit squares:

$$\mathcal{D}_p = ([0, |\mathcal{D}|] \times [0, N]) \smallsetminus ((s_{i_{d,-}} \times [k, k-1]) \cup (s_{i_{d,+}} \times [k, k-1])).$$

The hypothesis $\text{flux}(d; p_1) = \text{flux}(d; p_2)$ implies that the two tilings of $\mathcal{D}_p$ have the same flux (in the sense of chapter 2). Therefore, by Lemma 2.3, it follows that $\mathbf{t}_{d;p_1} \approx \mathbf{t}_{d;p_2}$. ■

Consider a non trivial balanced disk $\mathcal{D}$ with a fixed hamiltonian path γ_0. Suppose that $\mathcal{D}$ contains a 2×3 rectangle so that $\mathcal{D}$ is not too small. In order to prove that $\mathcal{D}$ is regular proceed as follows. Construct a tiling $a \in \mathcal{T}(\mathcal{R}_4)$ by taking a tiling of a $2 \times 3 \times 4$ box, as in Figure 3.12, and vertical dominoes outside the box. Notice that $\text{Tw}(a) = +1$.

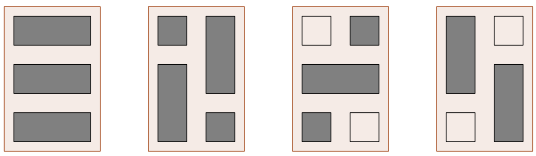

Figure 3.12: Tiling of $2 \times 3 \times 4$ box.

For each domino d_i not contained in γ_0 consider compatible plugs p_j such that the values $\text{flux}(d_i, p_j)$ cover the set Φ_{d_i}. Then, for each tiling $\mathbf{t}_{d_i;p_j}$ compute $\text{Tw}(\mathbf{t}_{d_i;p_j}) = k_{ij}$. The regularity of $\mathcal{D}$ then follows by proving that $\mathbf{t}_{d_i;p_j} \sim a^{k_{ij}}$ for all i, j. Indeed, given a tiling $\mathbf{t} \in G^+_{\mathcal{D}}$ we have $\mathbf{t} \sim a^{\text{Tw}(\mathbf{t})}$. Then, $\text{Tw}\colon G^+_{\mathcal{D}} \to \mathbb{Z}$ is injective and therefore an isomorphism.

4
Disks regularity

In this chapter we discuss the regularity of disks. We say that a unit square $s \subset \mathcal{D}$ (resp. a domino $d \subset \mathcal{D}$) *disconnects* $\mathcal{D}$ if, as a bipartite graph, $\mathcal{D} \smallsetminus s$ (resp. $\mathcal{D} \smallsetminus d$) is not connected. Two disks $\mathcal{D}_1$ and $\mathcal{D}_2$ are *disjoint* if every unit square in $\mathcal{D}_1$ is not adjacent to a unit square in $\mathcal{D}_2$.

4.1
Non regular disks

Theorem 4.1. *Consider a quadriculated disk $\mathcal{D}$. Suppose there exists a domino $d \subset \mathcal{D}$ which disconnects $\mathcal{D}$ such that $\mathcal{D} \smallsetminus d$ is the union of two disjoint balanced quadriculated disks $\mathcal{D}_1, \mathcal{D}_2$ with $|\mathcal{D}_1| = |\mathcal{D}_2|$. Then there exists a surjective homomorphism $\phi\colon G^+_{\mathcal{D}} \to F_2$. In particular, $\mathcal{D}$ is not regular.*

Figure 4.1 below shows two valid floors of a disk $\mathcal{D}$ that satisfies the hypothesis of Theorem 4.1; the proof of Theorem 4.1 is based on the existence of these two floors.

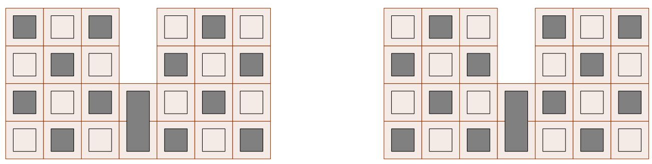

Figure 4.1: Two valid floors of a disk $\mathcal{D}$.

Proof. We construct ϕ by working in $\mathcal{C}^+_{\mathcal{D}}$. We provide a map taking each floor with parity $\mathbf{f} = (f, k) = (p_0, f^*, p_1, k)$ to $\phi(\mathbf{f}) \in \{e, a, a^{-1}, b, b^{-1}\} \subset F_2$. Define $\phi(\mathbf{f}) = e$ unless d is contained in f^* and p_0 marks exactly $\frac{|\mathcal{D}_1|}{2}$ squares in the disk $\mathcal{D}_1$, all of the same color. Therefore all squares in $\mathcal{D} \smallsetminus d$ are marked by either p_0 or p_1, and the marking follows a checkerboard pattern. Then there exist only two floors f_0 and $f_1 = f_0^{-1}$ (and four signed floors) satisfying the conditions above. Set $\mathbf{f_0} = (f_0, 0)$ and $\mathbf{f_1} = (f_1, 0)$ so that $\mathbf{f_0}^{-1} = (f_1, 1)$ and $\mathbf{f_1}^{-1} = (f_0, 1)$.

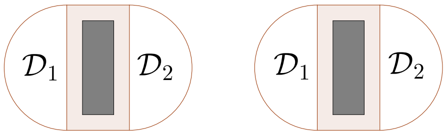

Figure 4.2: p_0 marks $\frac{|\mathcal{D}_1|}{2}$ white or black squares in $\mathcal{D}_1$

Finally, set $\phi(\mathbf{f_0}) = a$, $\phi(\mathbf{f_0}^{-1}) = a^{-1}$, $\phi(\mathbf{f_1}) = b$ and $\phi(\mathbf{f_1}^{-1}) = b^{-1}$. Since neither f_0 nor f_1 are part of the boundary of any 2-cell, ϕ extends to a homomorphism. ■

Theorem 4.2. *Consider a quadriculated disk $\mathcal{D}$. Suppose there exists a 2×2 square $s \subset \mathcal{D}$ such that $\mathcal{D} \smallsetminus s$ is the union of two disjoint balanced quadriculated disks $\mathcal{D}_1, \mathcal{D}_2$ with $|\mathcal{D}_1| = |\mathcal{D}_2|$ such that both $\mathcal{D}_1 \cap s$ and $\mathcal{D}_2 \cap s$ contain adjacent sides of s. Let d_1 and d_2 be the dominoes of s which are adjacent to $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. If $\mathcal{D} \smallsetminus d_1$ and $\mathcal{D} \smallsetminus d_2$ are not connected then there exists a surjective homomorphism $\phi\colon G^+_{\mathcal{D}} \to F_2$. In particular, $\mathcal{D}$ is not regular.*

Figure 4.3 below shows four families of floors of a disk $\mathcal{D}$ that satisfies the hypothesis of Theorem 4.2; the proof of Theorem 4.2 is based on the existence of these four families of floors.

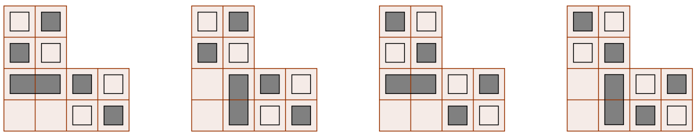

Figure 4.3: Four families of floors of a quadriculated disk $\mathcal{D}$, each family contains three valid floors.

Proof. We construct ϕ by working in $\mathcal{C}^+_{\mathcal{D}}$. We provide a map taking each floor with parity $\mathbf{f} = (f, k) = (p_0, f^*, p_1, k)$ to $\phi(\mathbf{f}) \in \{e, a, a^{-1}, b, b^{-1}\} \subset F_2$. Define $\phi(\mathbf{f}) = e$ unless one of the following conditions hold:

1. d_1 is contained in f^* and p_0 marks exactly $\frac{|\mathcal{D}_1|}{2}$ squares in the disk $\mathcal{D}_1$, all of the same color.

2. d_2 is contained in f^* and p_0 marks exactly $\frac{|\mathcal{D}_2|}{2}$ squares in the disk $\mathcal{D}_2$, all of the same color.

Therefore, there are four classes of floors for which ϕ is non trivial, in each class we have a checkerboard pattern in $\mathcal{D}_1$ or $\mathcal{D}_2$. Let us call these classes 0,1,2,3 in the order shown in the figure below.

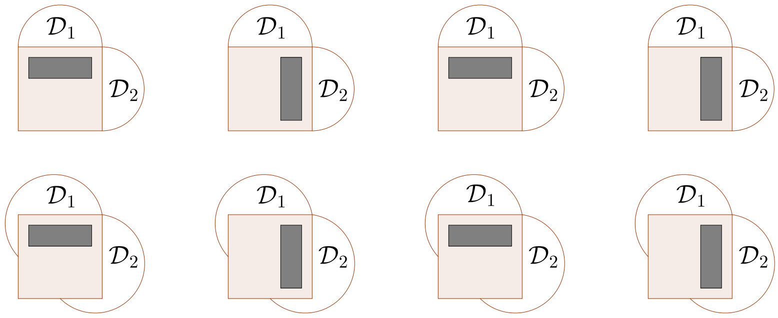

Figure 4.4: Two possible options for $\mathcal{D}$ each option has 4 classes. The plug p_0 marks white squares in $\mathcal{D}_1$ and $\mathcal{D}_2$ on classes 0 and 3, respectively. The plug p_0 marks black squares in $\mathcal{D}_1$ and $\mathcal{D}_2$ on classes 2 and 1, respectively.

If $\mathbf{f} = (f, k)$ is of class $0, 1$ define $\phi(\mathbf{f}) = a$ if $k = 0$ and $\phi(\mathbf{f}) = b^{-1}$ if $k = 1$. If $\mathbf{f} = (f, k)$ is of class $2, 3$ define $\phi(\mathbf{f}) = b$ if $k = 0$ and $\phi(\mathbf{f}) = a^{-1}$ if $k = 1$. A case by case check shows that ϕ extends to a homomorphism since it takes the boundary of any 2-cell to e.

■

Theorem 4.3. *Consider a quadriculated disk $\mathcal{D}$. Suppose there exists a 2×2 square $s \subset \mathcal{D}$ such that $\mathcal{D} \smallsetminus s$ is the union of two disjoint balanced quadriculated disks $\mathcal{D}_1, \mathcal{D}_2$ with $|\mathcal{D}_1| = |\mathcal{D}_2|$ such that both $\mathcal{D}_1 \cap s$ and $\mathcal{D}_2 \cap s$ contain opposite sides of s. Let d_1 and d_2 be the dominoes of s which are adjacent to $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. If $\mathcal{D} \smallsetminus d_1$ and $\mathcal{D} \smallsetminus d_2$ are not connected then there exists a surjective homomorphism $\phi\colon G^+_{\mathcal{D}} \to F_2$. In particular, $\mathcal{D}$ is not regular.*

Figure 4.5 below shows four families of floors of a disk $\mathcal{D}$ that satisfies the hypothesis of Theorem 4.3; the proof of Theorem 4.3 is based on the existence of these four families of floors.

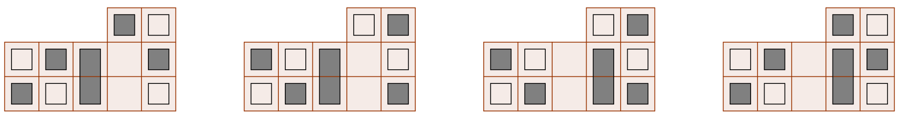

Figure 4.5: Four families of floors of a quadriculated disk $\mathcal{D}$, each family contains three valid floors.

Proof. We construct ϕ by working in $\mathcal{C}^+_{\mathcal{D}}$. We provide a map taking each floor with parity $\mathbf{f} = (f, k) = (p_0, f^*, p_1, k)$ to $\phi(\mathbf{f}) \in \{e, a, a^{-1}, b, b^{-1}\} \subset F_2$. Define $\phi(\mathbf{f}) = e$ unless exactly one of the following conditions hold:

1. d_1 is contained in f^* and p_0 marks exactly $\frac{|\mathcal{D}_1|}{2}$ squares in the disk $\mathcal{D}_1$, all of the same color.

2. d_2 is contained in f^* and p_0 marks exactly $\frac{|\mathcal{D}_2|}{2}$ squares in the disk $\mathcal{D}_2$, all of the same color.

Notice that there are floors satisfying both conditions above:

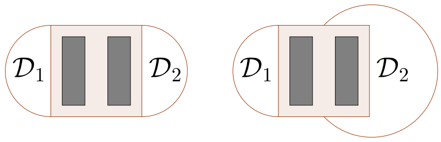

Figure 4.6: Two possible options for $\mathcal{D}$, each option has a floor $\mathbf{f}$ with $d_1, d_2 \subset f^*$ such that p_0 marks $\frac{|\mathcal{D}_1|}{2}$ black squares in $\mathcal{D}_1$ and $\frac{|\mathcal{D}_1|}{2}$ white squares in $\mathcal{D}_2$; by definition $\phi(\mathbf{f}) = e$

There are four classes of floors for which ϕ is non trivial, in each class we have a checkerboard pattern in $\mathcal{D}_1$ or $\mathcal{D}_2$. Let us call these classes 0,1,2,3 in the order shown in the figure below.

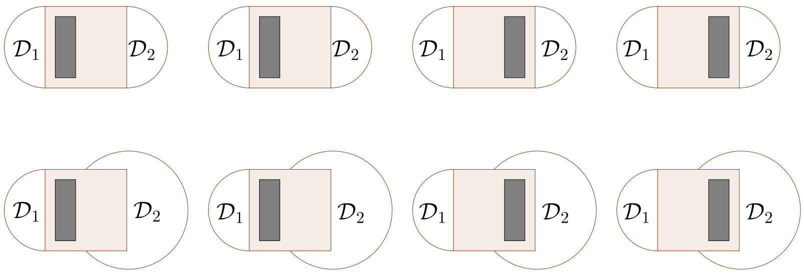

Figure 4.7: Two possible options for $\mathcal{D}$, each option has 4 classes. The plug p_0 marks white squares in $\mathcal{D}_1$ and $\mathcal{D}_2$ on classes 1 and 3, respectively. The plug p_0 marks black squares in $\mathcal{D}_1$ and $\mathcal{D}_2$ on classes 0 and 2, respectively.

If $\mathbf{f} = (f, k)$ is of class j, define $\phi(\mathbf{f})$ as

(j,k)	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$
$\phi(\mathbf{f})$	a	b	a^{-1}	b^{-1}	b^{-1}	a^{-1}	b	a

A case by case check shows that ϕ extends to a homomorphism since it takes the boundary of any 2-cell to e. ■

Example 4.4. Consider the disks below

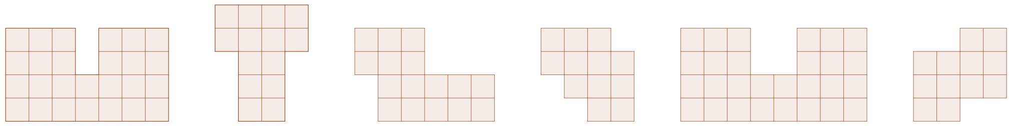

Figure 4.8: Theorem 4.1 shows that the first two disks are not regular. Theorem 4.2 shows that the third and the fourth disk are not regular. Theorem 4.3 shows that the last two disks are not regular.

◇

Consider a tileable quadriculated disk $\mathcal{D}$. Suppose there exists a unit square $s \subset \mathcal{D}$ such that $\mathcal{D} \setminus s$ is the union of two disjoint quadriculated disks $\mathcal{D}_1$ and $\mathcal{D}_2$. For $i = 1, 2$, construct the balanced disks $\tilde{\mathcal{D}}_i = \mathcal{D}_i \cup s \cup r_i$ where r_i is a $1 \times M_i$ rectangle such that the intersection $(\mathcal{D}_i \cup s) \cap r_i$ is an edge of s and $|r_i|$ is sufficiently large, see Figure 4.9.

Let s_i be the square contained in r_i which is adjacent to s. Given a tiling $\mathbf{t}$ of $\mathcal{D} \times [0, N]$, N even, we construct a tiling $\mathbf{t}^i$ of $\tilde{\mathcal{D}}_i \times [0, N]$ for $i = 1, 2$. Set $\mathbf{t}^i|_{\mathcal{D}_i} = \mathbf{t}|_{\mathcal{D}_i}$, i.e, for every domino d in $\mathbf{t}$ which is entirely contained in $\mathcal{D}_i \times [0, N]$ we have $d \subset \mathbf{t}^i$. Moreover, suppose there exists $d = (s \cup \tilde{s}) \times [k-1, k]$ in $\mathbf{t}$ for some $1 \leq k \leq N$ and for some square $\tilde{s}$. If $\tilde{s} \in \mathcal{D}_1$ then let $d \subset \mathbf{t}^1$ and place a domino in $\tilde{\mathcal{D}}_2 \times [0, N]$ occupying $(s \cup s_2) \times [k-1, k]$. If $\tilde{s} \in \mathcal{D}_2$ then let $d \subset \mathbf{t}^2$ and place a domino in $\tilde{\mathcal{D}}_1 \times [0, N]$ occupying $(s \cup s_1) \times [k-1, k]$. Since N is even, the number of horizontal dominoes in $\mathbf{t}$ intersecting $s \times [0, N]$ is even. Then, since r_i is a rectangle, we are left with a region to be tiled which is isomorphic to a simply connected balanced quadriculated disk. However, since $|r_i|$ is sufficiently large, this region has a tiling. Therefore, we have a tiling $\mathbf{t}^i$ of $\tilde{\mathcal{D}}_i \times [0, N]$ for $i = 1, 2$. The following lemma shows that this construction is invariant by flips.

Example 4.5. Figure 4.9 below shows an example of the construction above.

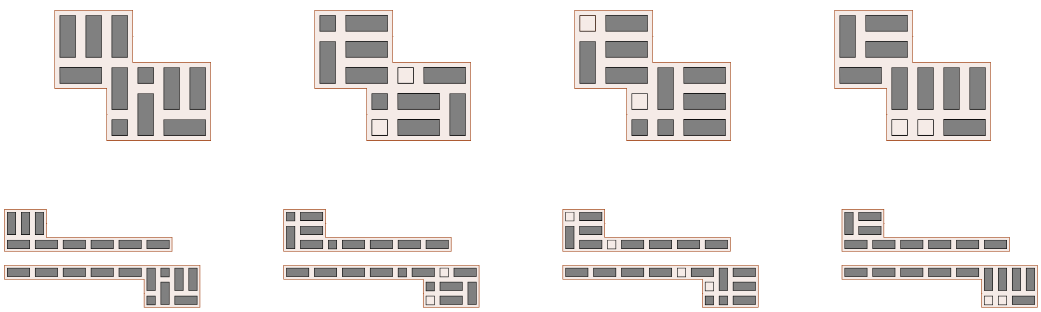

Figure 4.9: Tiling of a disk $\mathcal{D}$ and its extension to $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$.

◇

Lemma 4.6. *Consider a tileable quadriculated disk $\mathcal{D}$. Suppose there exists a square $s \subset \mathcal{D}$ such that $\mathcal{D} \setminus s$ is the union of two disjoint quadriculated disks $\mathcal{D}_1$ and $\mathcal{D}_2$. Consider the disks $\tilde{\mathcal{D}}_1$ and $\tilde{\mathcal{D}}_2$. If $\mathbf{t}_0, \mathbf{t}_1$ are tilings of $\mathcal{D} \times [0, N]$, N even, such that $\mathbf{t}_0 \approx \mathbf{t}_1$ then $\mathbf{t}_0^i \approx \mathbf{t}_1^i$ for $i \in \{1, 2\}$.*

Proof. Suppose without loss of generality that $\mathbf{t}_0$ and $\mathbf{t}_1$ differ by a flip. If the flip is performed on two dominoes d_1, d_2 whose projections $\tilde{d}_1, \tilde{d}_2 \subset \mathcal{D}$ contain s the result follows from the fact that any two tilings of a simply connected quadriculated disk can be joined by a sequence of flips. Otherwise, the result follows by construction. ■

Theorem 4.7. *Consider a tileable quadriculated disk $\mathcal{D}$. Suppose there exists a square $s \subset \mathcal{D}$ such that $\mathcal{D} \setminus s$ is the union of two disjoint quadriculated disks $\mathcal{D}_1$ and $\mathcal{D}_2$. If both $\mathcal{D}_1$ and $\mathcal{D}_2$ contain a 2×3 rectangle then there exists a surjective homomorphism $\phi \colon G^+_{\mathcal{D}} \to \mathbb{Z}^2$. In particular, $\mathcal{D}$ is not regular.*

Proof. By Lemma 4.6 we can construct the quadriculated disks $\tilde{\mathcal{D}}_1, \tilde{\mathcal{D}}_2$ and the homomorphism $\phi \colon G^+_{\mathcal{D}} \to \mathbb{Z}^2$ such that $\phi(\mathbf{t}) = (\mathrm{Tw}(\mathbf{t}^1), \mathrm{Tw}(\mathbf{t}^2))$ is well defined. Since both $\mathcal{D}_1$ and $\mathcal{D}_2$ contain a 2×3 rectangle, given $(t_1, t_2) \in \mathbb{Z}^2$, there exists tilings $\mathbf{t}^1$ of $\tilde{\mathcal{D}}_1 \times [0, N]$ and $\mathbf{t}^2$ of $\tilde{\mathcal{D}}_2 \times [0, N]$ such that $\mathbf{t}^1|_{r_1 \cup s}$ and $\mathbf{t}^2|_{r_2 \cup s}$ are only occupied by vertical dominoes with $\mathrm{Tw}(\mathbf{t}^1) = t_1$ and $\mathrm{Tw}(\mathbf{t}^2) = t_2$. Then $\mathbf{t} = \mathbf{t}^1|_{\mathcal{D}_1} \cup \mathbf{t}^2|_{\mathcal{D}_2}$ is a tiling of $\mathcal{D} \times [0, N]$ such that $\phi(\mathbf{t}) = (t_1, t_2)$. Therefore, ϕ is surjective. ■

Example 4.8. Theorem 4.7 shows that the four disks below are not regular.

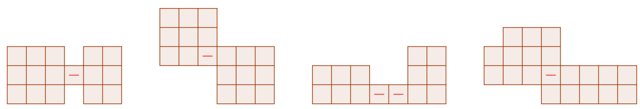

Figure 4.10: Four non regular disks. We draw a red line segment to indicate the squares that disconnect the disk into two disjoint disks containing a 2×3 rectangle.

◇

The theorems in this section show that for some disks $\mathcal{D}$ the existence of a bottleneck (i.e, a domino that disconnects $\mathcal{D}$) implies that $\mathcal{D}$ is not regular. It seems that these results about non regular disks are optimal. In that sense we have the following conjecture:

Conjecture 4.9. *Almost every disk $\mathcal{D}$ that does not admit a bottleneck as in Theorems 4.1, 4.2, 4.3 and 4.7 is regular.*

4.2 Regular disks

In this section we prove the regularity of several quadriculated disks. In fact, given a regular quadriculated disk $\mathcal{D}$ with a hamiltonian cycle γ, we provide a method to construct a new regular disk by gluing together $\mathcal{D}$ and a rectangle $[0, L] \times [0, M]$ with LM even and $L, M \geq 2$. We first prove some technical lemmas.

Lemma 4.10. *Consider a quadriculated disk $\mathcal{D}$ with a hamiltonian cycle $\gamma = (s_1, \dots, s_{|\mathcal{D}|})$. Suppose that $\mathcal{D} \smallsetminus (s_1 \cup s_{|\mathcal{D}|})$ is a balanced disk. Let $p \in \mathcal{P}$ be a plug compatible with a domino $d \subset \mathcal{D}$ which is not contained in γ. If p marks s_1 and $s_{|\mathcal{D}|}$ then $\mathbf{t}_{d;p} \sim \mathbf{t}_{d;p \smallsetminus (s_1 \cup s_{|\mathcal{D}|})}$.*

Proof. Consider the domino $d_1 = s_1 \cup s_{|\mathcal{D}|}$ and let $\mathcal{D}_1 = \mathcal{D} \smallsetminus (s_1 \cup s_{|\mathcal{D}|})$. Then, $\gamma_1 = (s_2, \dots, s_{|\mathcal{D}|-1})$ is a hamiltonian path in $\mathcal{D}_1$. Notice that $p \smallsetminus d_1$ is a well defined plug in $\mathcal{D}_1$. Since $\mathcal{D}_1$ is a balanced disk, we may assume that the tiling $\mathbf{t}_{d;p \smallsetminus d_1} = \mathbf{t}^{-1}_{p \smallsetminus d_1} * f * f_{\text{vert}} * \mathbf{t}_{(p \smallsetminus d_1) \cup d}$ of $\mathcal{D} \times [0, N]$ is such that the column $d_1 \times [0, N]$ is occupied only by vertical dominoes.

By definition, $f = (p \smallsetminus d_1, d, (p \smallsetminus d_1 \cup d)^{-1})$. Let $\mathbf{t}_0$ be the tiling of $\mathcal{D} \times [0, 1]$ which respects γ. Then, by Lemma 3.12, $\mathbf{t}_{d;p \smallsetminus d_1} \sim \mathbf{t}_1$ where $\mathbf{t}_1 = \mathbf{t}_0 * \mathbf{t}_{d;p \smallsetminus d_1} * \mathbf{t}_0$. Let $\tilde{\mathbf{t}_0}$ be the tiling of $\mathcal{D} \times [0, 1]$ such that $\tilde{\mathbf{t}_0}|_{\mathcal{D}_1}$ respects γ_1; therefore $d_1 \in \tilde{\mathbf{t}_0}$. Since any two tilings of a quadriculated disk can be joined by a sequence of flips we have $\mathbf{t}_0 \sim \tilde{\mathbf{t}_0}$. Then, $\mathbf{t}_1 \sim \mathbf{t}_2$ where $\mathbf{t}_2 = \tilde{\mathbf{t}_0} * \mathbf{t}_{d;p \smallsetminus d_1} * \tilde{\mathbf{t}_0}$. Perform flips in $\mathbf{t}_2$ such that all dominoes in $d_1 \times [0, N+2]$ are vertical. These flips modified the floor f of $\mathbf{t}_2$ to $\tilde{f} = (p, d, (p \cup d)^{-1})$. Therefore, by Lemma 3.12, $\mathbf{t}_2 \sim \mathbf{t}_{d;p}$. Thus, $\mathbf{t}_{d;p} \sim \mathbf{t}_{d;p \smallsetminus d_1}$. ∎

Consider a quadriculated disk $\mathcal{D}$ with a hamiltonian cycle $\gamma = (s_1, \dots, s_{|\mathcal{D}|})$. Let $d \subset \mathcal{D}$ be a domino not contained in γ. Suppose there exists adjacent squares s_k, s_l such that $k \in I_{d;0}$ and $l \in I_{d;\pm 1}$. We say that d is *good* if $s_l \notin \{s_1, s_{|\mathcal{D}|}\}$. Notice that if a domino $d \subset \mathcal{D}$ does not disconnect $\mathcal{D}$ then there exists adjacent squares $s_k \in I_{d;0}$ and $s_l \in I_{d;\pm 1}$, but not necessarily $s_l \notin \{s_1, s_{|\mathcal{D}|}\}$. However, if d is good then d does not disconnect $\mathcal{D}$.

Lemma 4.11. *Consider a quadriculated disk $\mathcal{D}$ with a hamiltonian cycle $\gamma = (s_1, \dots, s_{|\mathcal{D}|})$. Suppose that $\mathcal{D} \smallsetminus (s_1 \cup s_{|\mathcal{D}|})$ is a balanced disk. Let $d \subset \mathcal{D} \smallsetminus (s_1 \cup s_{|\mathcal{D}|})$ be a domino not contained in γ with a compatible plug $p \in \mathcal{P}$. If d is good then there exists dominoes $d_1, \dots, d_i \subset \mathcal{D} \smallsetminus (s_1 \cup s_{|\mathcal{D}|})$ with compatible plugs $p_1, \dots, p_i \in \mathcal{P}$ containing s_1 and $s_{|\mathcal{D}|}$ such that $\mathbf{t}_{d;p} \sim \mathbf{t}_{d_1;p_1} * \dots * \mathbf{t}_{d_i;p_i}$ for either $i = 1$ or $i = 3$.*

For example, as in Figure 4.11, consider the disk $\mathcal{D} = [0,3] \times [0,4]$ with the hamiltonian cycle $\gamma = (s_1, \dots, s_{|\mathcal{D}|})$ where $s_1 = [1,2] \times [3,4]$ and $s_{|\mathcal{D}|} = [2,3] \times [3,4]$. Furthermore, Figure 4.11 shows a good domino $d \subset \mathcal{D}$ and two compatible plugs $p_1, p_2 \in \mathcal{P}$.

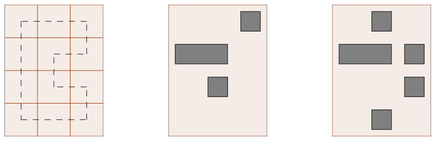

Figure 4.11: The disk $\mathcal{D} = [0,3] \times [0,4]$ with a hamiltonian cycle γ and a good domino d with two compatible plugs p_1 and p_2, respectively.

Consider the plug $\tilde{p_1} = p_1 \cup (s_1 \cup s_2)$. Then, $\tilde{p_1}$ is a plug compatible with d which marks $s_1 \cup s_{|\mathcal{D}|}$ such that $\operatorname{flux}(d; \tilde{p_1}) = \operatorname{flux}(d; p_1)$. Therefore, $\mathbf{t}_{d;p_1} \sim \mathbf{t}_{d;\tilde{p_1}}$.

Let $\mathbf{t}_{d;p_2} = \mathbf{t}_{p_2}^{-1} * f * f_{\mathrm{vert}} * \mathbf{t}_{p_2 \cup d}$ be the tiling of $\mathcal{D} \times [0, N]$. We may assume that, for some $p_0 \in \mathcal{P}$ and for some reduced floor f^*, $f_1 = (p_0, f^*, p_2)$ is the floor below f and $f_2 = (p_2 \cup d, \emptyset, (p_2 \cup d)^{-1})$ is the floor above f_{vert}. Then, as in Figure 4.12, perform three flips in $\mathbf{t}_{d;p_2}$.

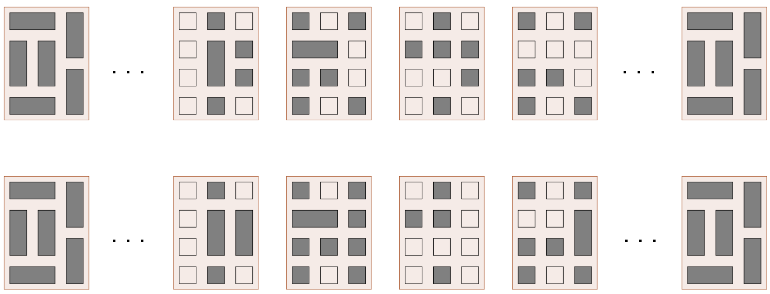

Figure 4.12: The first row shows the tiling $\mathbf{t}_{d;p_2}$, we omit some floors and focus on $f_1 * f * f_{\mathrm{vert}} * f_2$. The second row shows the same region after three flips.

Let $\tilde{f_1}, \tilde{f}, \tilde{f_{\mathrm{vert}}}, \tilde{f_2}$ be the floors obtained after these flips. Then, $\mathbf{t}_{d;p_2}$ is equivalent to a tiling with three dominoes not contained in γ, each in a different floor: $\tilde{f_1}, \tilde{f}, \tilde{f_2}$. Now, as in Lemma 3.4, add a large number of vertical floors between $\tilde{f_1}$ and $\tilde{f}$, and between $\tilde{f_{\mathrm{vert}}}$ and $\tilde{f_2}$. Modify these vertical floors, as in Lemma 3.5 to $\mathbf{t}_{p_2 \smallsetminus d_1} * \mathbf{t}_{p_2 \smallsetminus d_1}^{-1}$ and to $\mathbf{t}_{(p_2 \cup d) \smallsetminus d_1} * \mathbf{t}_{(p_2 \cup d) \smallsetminus d_1}^{-1}$ respectively. Then, $\mathbf{t}_{d;p_2}$ is equivalent to the concatenation of three tilings: $\mathbf{t}_{d;p_2} \sim \mathbf{t}_{d_1;p_0} * \mathbf{t}_{d;p_2 \smallsetminus d_1} * \mathbf{t}_{d_1;(p_2 \cup d) \smallsetminus d_1}$.

By Figure 4.12, it is not difficult to see that we can take a plug $p_3 \in \mathcal{P}$ compatible with $d_1 = [2,3] \times [1,3]$ marking s_1 and $s_{|\mathcal{D}|}$ such that $\operatorname{flux}(d_1; p_0) = \operatorname{flux}(d_1; p_3)$. Moreover, take $p_4 \in \mathcal{P}$ marking s_1 and $s_{|\mathcal{D}|}$, and take p_5 not marking s_1 and $s_{|\mathcal{D}|}$ such that p_4 (resp. p_5) is compatible with d (resp. d_1), $\operatorname{flux}(d; p_4) = \operatorname{flux}(d; p_2 \smallsetminus d_1))$ and $\operatorname{flux}(d_1; p_5) = \operatorname{flux}(d_1; (p_2 \cup d) \smallsetminus d_1)$.

By Lemma 3.12, $\mathbf{t}_{d_1;p_0} \sim \mathbf{t}_{d_1;p_3}$, $\mathbf{t}_{d_1;p_2 \smallsetminus d_1} \sim \mathbf{t}_{d_1;p_4}$ and $\mathbf{t}_{d;(p_2 \cup d) \smallsetminus d_1} \sim \mathbf{t}_{d;p_5}$. Furthermore, by Lemma 4.10, $\mathbf{t}_{d;p_5} \sim \mathbf{t}_{d;p_5 \cup (s_1 \cup s_{|\mathcal{D}|})}$. The result then follows.

Proof of Lemma 4.11. Let $d = s_i \cup s_j$ such that $i < j$ and $\mathrm{flux}(d;p) = \phi$. Consider two cases: $\phi_{-1}\phi_{+1} < 0$ and $\phi_{-1}\phi_{+1} \geq 0$. If $\phi_{-1}\phi_{+1} < 0$ it is not difficult to see that we can take a plug $p_1 \in \mathcal{P}$ compatible with d such that $\mathrm{flux}(d;p_1) = \phi$ and $s_1, s_{|\mathcal{D}|} \in p_1$. Therefore, by Lemma 3.12, $\mathbf{t}_{d;p_1} \sim \mathbf{t}_{d;p}$ and the result follows.

Suppose $\phi_{-1}\phi_{+1} \geq 0$ hence $\phi_0 \leq 0$. We may assume that $|\phi_{-1}|$ and $|\phi_{+1}|$ are maximal. For instance, suppose that $|\phi_{-1}|$ is not maximal. Then, take a plug $p_1 \in \mathcal{P}$ such that $\mathrm{flux}(d;p_1) = \phi$ and either $s_1, s_2, s_{|\mathcal{D}|} \in p_1$ or $s_1, s_{|\mathcal{D}|} \notin p_1$. The desired result then follows by Lemma 3.12 and Lemma 4.10.

Since d is good there exists adjacent squares s_k and s_l such that $d_1 = s_k \cup s_l \subset \mathcal{D} \smallsetminus (s_1 \cup s_{|\mathcal{D}|})$, $k \in I_{d;0}$ and $l \in I_{d;\pm 1}$. Without loss of generality, suppose that $l \in I_{d;+1}$. Since $\phi_0\phi_{+1} \leq 0$ we may assume that either $s_k, s_l \in p$ or $s_k, s_l \notin p$.

Suppose that $s_k, s_l \in p$, the other case is similar. Construct the tiling $\mathbf{t}_{d;p} = \mathbf{t}_p^{-1} * f * f_{\mathrm{vert}} * \mathbf{t}_{p \cup d}$ of $\mathcal{D} \times [0, N]$ for some even N. Let f_1 be the floor below f and f_2 be the floor above f_{vert}. By construction, $f_1 = (\tilde{p}, f^*, p)$ and $f_2 = (p \cup d, \emptyset, (p \cup d)^{-1})$ for some plug $\tilde{p}$ and for some floor f^*. Suppose that f is the M-th floor. Then, the region $d_1 \times [M-1, M+3]$ contained in $f_1 * f * f_{\mathrm{vert}} * f_2$ is occupied by four vertical dominoes. Then, perform three flips such that horizontal dominoes are contained in $d_1 \times [M-1, M]$ and $d_1 \times [M+2, M+3]$. Therefore two vertical dominoes are contained in $d_1 \times [M, M+2]$. Let $\tilde{f}_1, , \tilde{f}, \tilde{f}_{\mathrm{vert}}, \tilde{f}_2$ be the floors obtained after these three flips. More specifically, $\tilde{f}_1 = (\tilde{p},\, f^* \cup d_1,\, p \smallsetminus d_1)$, $\tilde{f} = (p \smallsetminus d_1,\, d,\, (p \cup d)^{-1} \cup d_1)$, $\tilde{f}_{\mathrm{vert}} = ((p \cup d)^{-1} \cup d_1,\, \emptyset,\, (p \cup d) \smallsetminus d_1)$ and $\tilde{f}_2 = ((p \cup d) \smallsetminus d_1, d_1, (p \cup d)^{-1})$. Therefore, $\mathbf{t}_{d;p}$ is equivalent to a tiling $\tilde{\mathbf{t}}_{d;p}$ with three dominoes not respecting γ, each contained in a different floor: $\tilde{f}_1, \tilde{f}, \tilde{f}_2$.

Let us analyze the floors $\tilde{f}_1, \tilde{f}, \tilde{f}_2$ more carefully. In $\tilde{f}$, by construction, $|\mathrm{flux}_{+1}(d; p \smallsetminus d_1)| = |\phi_{+1}| - 1$, and therefore the flux is not maximal. We claim that in $\tilde{f}_1 = (\tilde{p}, f^* \cup d_1, p \smallsetminus d_1)$ the flux $|\mathrm{flux}_{-1}(d_1; \tilde{p})|$ is not maximal. Since $k \in I_{d;0}$ it follows that $I_{d;-1} \subset I_{d_1;-1}$. Recall that ϕ_{-1} is maximal and $\phi_0\phi_{-1} \leq 0$. Then, if $s_{i-1} \notin p$ either $s_i, s_{i-1} \in \tilde{p}$ or s_i and s_{i-1} are parts of horizontal dominoes contained in f^*. Moreover, if $s_{i-1} \in p$ then $s_{i+1} \notin p$, and therefore $i + 1 \neq k$. Then, either $s_i, s_{i+1} \in \tilde{p}$ or s_i and s_{i+1} are parts of horizontal dominoes contained in f^* Thus, in any case, $|\mathrm{flux}_{-1}(d_1; \tilde{p})|$ is not maximal. A similar argument shows that in $\tilde{f}_2$ the flux $|\mathrm{flux}_{-1}(d_1; (p \cup d) \smallsetminus d_1)|$ is not maximal. By adding vertical floors, the proof now follows as in the example $\mathcal{D} = [0, 3] \times [0, 4]$. ■

Lemma 4.12. *Consider a quadriculated disk $\mathcal{D}$ with a hamiltonian cycle $\gamma = (s_1, \ldots, s_{|\mathcal{D}|})$. Suppose there exists a domino $d \subset \mathcal{D}$ such that $\mathcal{D} \smallsetminus d = \mathcal{D}_1 \cup \mathcal{D}_2$ is the union of two disjoint quadriculated disks such that $|\mathcal{D}_1| > |\mathcal{D}_2|$ and $s_1 \cup s_{|\mathcal{D}|} \subset \mathcal{D}_2$. Let $d_0 \subset \mathcal{D}_2$ be a domino which disconnects $\mathcal{D}$. Then, for every plug $p_0 \in \mathcal{P}$ compatible with d_0 there exists a plug $p \in \mathcal{P}$ compatible with d such that $\mathbf{t}_{d_0;p_0} \sim \mathbf{t}_{d;p}$.*

For example, as in Figure 4.13 below, consider the disk $\mathcal{D}$ with the hamiltonian cycle γ.

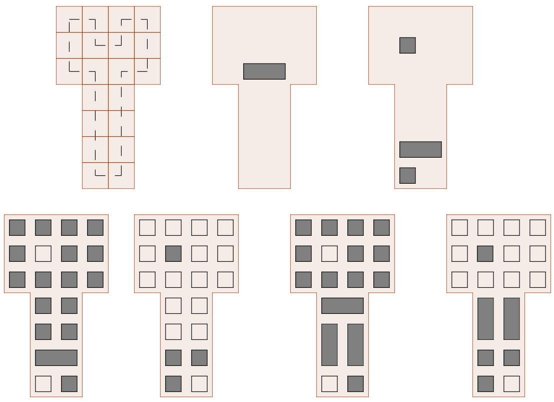

Figure 4.13: The first row shows the disk $\mathcal{D}$, the hamiltonian cycle γ, the domino d and the domino d_0 with the compatible plug p_0. In the second row the first two floors show the region $f * f_{\text{vert}}$ of $\mathbf{t}_{d_0;p_0}$ and the last two floors show the same region after four flips.

Proof. Notice that d_0 and d are not contained in γ, since both dominoes disconnect $\mathcal{D}$. Suppose that $d = s_i \cup s_j$ and $d_0 = s_k \cup s_l$ with $i < j$ and $k < l$. There are three cases: $k \in I_{d;-1}$ and $l \in I_{d+1}$, $k, l \in I_{d;-1}$ and $k, l \in I_{d;+1}$.

Consider the first case. Let $\mathcal{D}_3$ be the region between d_0 and d such that $d_0, d \subset \mathcal{D}_3$. In $\mathcal{D}_3$, γ can be seen as the union of two disjoint paths γ_1 and γ_2. More specifically, $\gamma_1 = (s_k, s_{k+1}, \ldots, s_i)$ and $\gamma_2 = (s_j, s_{j+1}, \ldots, s_l)$. Notice that $\gamma_1 \cup \gamma_2$ is a cycle in $\mathcal{D}_3$ so that $\mathcal{D}_3$ is a balanced quadriculated disk. Therefore, both $|\gamma_1|$ and $|\gamma_2|$ are either even or odd. Consider the case $|\gamma_1|$ and $|\gamma_2|$ both even. Then, by placing dominoes along γ_1 and γ_2, there exists a tiling of $\mathcal{D}_3$ which respects γ, and there exists a tiling of $\mathcal{D}_3 \smallsetminus d_0$ which respects γ except for d. Similarly, if $|\gamma_1|$ and $|\gamma_2|$ are both odd then there exists a tiling of $\mathcal{D}_3$ which respects γ except for d, and there exists a tiling of $\mathcal{D}_3 \smallsetminus d_0$ which respects γ. Suppose that $|\gamma_1|$ and $|\gamma_2|$ are both even, a similar construction holds for the other case. Let $\mathrm{flux}(d_0; p_0) = \phi$. Since $s_1 \cup s_{|\mathcal{D}|} \subset \mathcal{D}_2$ then $|\phi_{-1}| + |\phi_{+1}| < |\mathcal{D}_2|$. Since $|\mathcal{D}_1| > |\mathcal{D}_2|$ we may assume that p_0 does not mark squares in $\mathcal{D}_3$. Construct $\mathbf{t}_{d_0;p_0} = \mathbf{t}_{p_0}^{-1} * f * f_{\text{vert}} * \mathbf{t}_{p_0 \cup d_0}$

where $f = (p_0, d_0, (p_0 \cup d_0)^{-1})$. By construction $f * f_{\text{vert}}|_{\mathcal{D}_3 \smallsetminus d_0}$ is formed only by vertical dominoes. Then, perform vertical flips to obtain two floors of the tiling of $\mathcal{D}_3 \smallsetminus d_0$ which respects γ except for d. Therefore f_{vert} is equivalent to a floor which respects γ except for d and f is equivalent to a floor with a tiling in $\mathcal{D}_3$. This tiling in $\mathcal{D}_3$, by Lemma 2.1, can be connected by flips to the tiling of $\mathcal{D}_3$ that respects γ. Therefore, $\mathbf{t}_{d_0;p_0}$ is equivalent to a tiling that respects γ except for d. The result then follows.

The second and the third cases are similar. Then, without loss of generality, suppose that $k, l \in I_{d;-1}$. Let $\mathcal{D}_3$ be the region determined by $\gamma_1 = (s_1, s_2, \ldots, s_k, s_l, s_{l+1}, \ldots, s_i)$ and $\gamma_2 = (s_j, s_{j+1}, \ldots, s_{|\mathcal{D}|})$. In another words $\mathcal{D}_3$ is equal to $\mathcal{D}_2 \cup d$ minus the squares s_m for $m \in I_{d_0;0}$. Then, since $|I_{d_0;0}|$ is even, $\mathcal{D}_3$ is a balanced quadriculated disk. Therefore, both $|\gamma_1|$ and $|\gamma_2|$ are either even or odd. Suppose that both $|\gamma_1|$ and $|\gamma_2|$ are even, the other case is similar. Notice that the domino d_0 respect γ_1 but does not respect γ. Then, it is useful to divide γ_1 into two paths: $\gamma_1^1 = (s_1, s_2, \ldots, s_k)$ and $\gamma_1^2 = (s_l, s_{l+1}, \ldots, s_i)$. Since γ_1 is even it follows that both $|\gamma_1^1|$ and $|\gamma_1^2|$ are either even or odd. Suppose that both $|\gamma_1^1|$ and $|\gamma_1^2|$ are odd, a similar construction holds for the other case. Therefore, there exists a tiling $\mathbf{t}_1$ of $\mathcal{D}_3$ which respect γ except for the domino d. Furthermore, there exists a tiling $\mathbf{t}_2$ of $\mathcal{D}_3 \smallsetminus d_0$ which respects γ. As in the previous paragraph, we may assume that $\mathbf{t}_{d_0;p_0} = \mathbf{t}_{p_0}^{-1} * f * f_{\text{vert}} * \mathbf{t}_{p_0 \cup d_0}$ is such that $f|_{\mathcal{D}_3} \approx \mathbf{t}_1$ and $f_{\text{vert}}|_{\mathcal{D}_3 \smallsetminus d_0} \approx \mathbf{t}_2$. Then, $\mathbf{t}_{d_0;p_0}$ is equivalent to a tiling respecting γ except for the domino d. Therefore, $\mathbf{t}_{d_0;p_0} \sim \mathbf{t}_{d;p}$ for some plug $p \in \mathcal{P}$ compatible with d. ■

Consider a regular quadriculated disk $\mathcal{D}$ with a hamiltonian cycle γ. Let $d_1, d_2, \ldots, d_k \subset \mathcal{D}$ be the dominoes which disconnect $\mathcal{D}$. Therefore, for every $i = 1, \ldots, k$, we have that $\mathcal{D} \smallsetminus d_i = \mathcal{D}_1^i \cup \mathcal{D}_2^i$ is the union of two disjoint quadriculated disks such that $|\mathcal{D}_1^i| > |\mathcal{D}_2^i|$. We define the *core* $\mathcal{K}_{\mathcal{D}}$ of $\mathcal{D}$ as the intersection $\bigcap_{i=1}^{k} \mathcal{D}_1^i$. In particular, if $\mathcal{D}$ can not be disconnected by removing a domino then $\mathcal{K}_{\mathcal{D}} = \mathcal{D}$. Notice that $\mathcal{K}_{\mathcal{D}}$ is not always a balanced quadriculated disk, as the second disk in Figure 4.14.

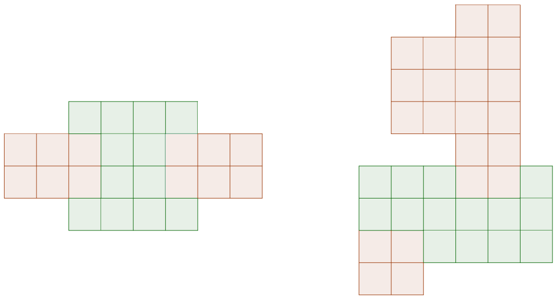

Figure 4.14: Two disks and their cores (highlighted in green).

We say that a quadriculated disk $\mathcal{D}$ is *thin* if $\mathcal{D} = \bigcup_{i=1}^{k} \mathcal{D}_i$ where $\mathcal{D}_i$ is a rectangle with a side of length 2 and $\mathcal{D}_i \bigcap (\bigcup_{j=i+1}^{k} \mathcal{D}_j)$ is a line segment of length 2 for every $i = 1, \ldots, k-1$. Notice that every thin disk has a unique hamiltonian cycle $\gamma = (s_1, \ldots, s_{|\mathcal{D}|})$ and every domino d not contained in γ disconnects $\mathcal{D}$. Furthermore, we may assume that $\mathcal{D} \smallsetminus (s_1 \cup s_{|\mathcal{D}|})$ is a thin disk.

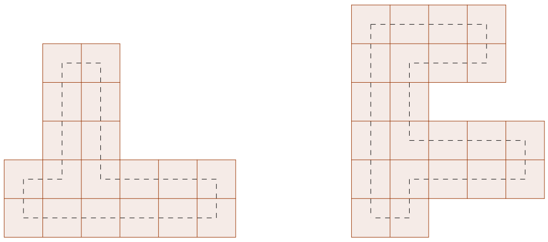

Figure 4.15: Two thin disks.

At this point, a natural question to ask is whether or not thin disks are regular. Unfortunately, this question does not have a direct answer. For instance, Figure 4.16 shows a regular thin disk and a non regular thin disk.

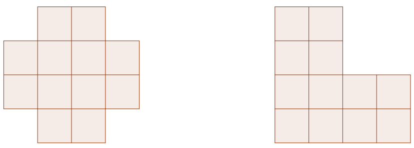

Figure 4.16: A computation shows that the first disk is regular, and Lemma 4.2 shows that the second disk is not regular.

Recall that $\mathcal{D} = [0, L] \times [0, M]$ is regular for LM even such that $\min\{L, M\} \geq 3$. The following two theorems show how to construct new regular disks.

Theorem 4.13. *Consider a regular quadriculated disk $\mathcal{D}$ and a thin disk $\tilde{\mathcal{D}}$ such that $|\tilde{\mathcal{D}}| + 2 < |\mathcal{D}|$. Let $\mathcal{D}_0 = \mathcal{D} \cup \tilde{\mathcal{D}}$ be a quadriculated disk with a hamiltonian cycle $\gamma = (s_1, \ldots, s_{|\mathcal{D}_0|})$. Suppose the intersection $\mathcal{D} \cap \tilde{\mathcal{D}}$ is a line segment of length 2 contained in $\mathcal{K}_{\mathcal{D}}$. Then, $\mathcal{D}_0$ is regular.*

Proof. Let $d_0 \subset \mathcal{D}$ be the domino adjacent to $\tilde{\mathcal{D}}$. Notice that d_0 disconnects $\mathcal{D}_0$. Since γ is a cycle we may assume that $d_1 = s_1 \cup s_{|\mathcal{D}_0|}$ is a domino in $\tilde{\mathcal{D}}$ such that $\tilde{\mathcal{D}} \smallsetminus d_1$ is either empty or a thin disk. Consider a domino $d \subset \mathcal{D}_0$ not contained in γ with a compatible plug $p \in \mathcal{P}$ such that $\operatorname{flux}(d; p) = \phi$. We claim that $\mathbf{t}_{d;p}$ is equivalent to a concatenation of tilings $\mathbf{t}_{\tilde{d}_1, p_1}, \ldots, \mathbf{t}_{\tilde{d}_k, p_k}$ such that p_i contains s_1 and $s_{|\mathcal{D}_0|}$ for every $i = 1, \ldots, k$.

Suppose that $d \subset \mathcal{D}$, $d \neq d_0$ and that d disconnects $\mathcal{D}$. Then, since $\mathcal{D}$ is regular, $\mathcal{D} \smallsetminus d = \mathcal{D}_1 \cup \mathcal{D}_2$ is the union of two disjoint quadriculated disks such that $|\mathcal{D}_1| > |\mathcal{D}_2|$. Therefore, since $\mathcal{K}_{\mathcal{D}} \subset \mathcal{D}_1$ and $s_1 \cup s_{|\mathcal{D}_0|} \subset \tilde{\mathcal{D}}$ it follows that $|I_{d;0}| = |\mathcal{D}_2|$ and $\mathcal{D}_2 \cap \tilde{\mathcal{D}} = \emptyset$. Then, we can take a plug $p_1 \in \mathcal{P}$ compatible with d such that $\operatorname{flux}(d; p_1) = \phi$ and p_1 marks s_1 and $s_{|\mathcal{D}|}$. The claim then follows.

Suppose that $d \subset \mathcal{D}$ and that d does not disconnect $\mathcal{D}$. Since $\tilde{\mathcal{D}}$ is thin it follows that d is good. Then, the claim follows by Lemma 4.11.

Suppose $d \subset \tilde{\mathcal{D}}$. Since $\tilde{\mathcal{D}}$ is thin we have that d disconnects $\tilde{\mathcal{D}}$. By construction, $\mathcal{D}_0 \smallsetminus d_0 = (\mathcal{D} \smallsetminus d_0) \cup \tilde{\mathcal{D}}$ is the disjoint union of two quadriculated disks. Furthermore, by hypothesis, $|\mathcal{D} \smallsetminus d_0| > |\tilde{\mathcal{D}}|$. Then, by Lemma 4.12, $\mathbf{t}_{d;p} \sim \mathbf{t}_{d_0;p_1}$ for some plug $p_1 \in \mathcal{P}$ compatible with d_0. Therefore, we are left with the case $d = d_0$.

Suppose that d_0 occupies the squares s_i, s_j. Let $s_{i\_}$ (resp. $s_{j\_}$) be the unit square adjacent to s_i (resp. s_j) such that $s_{i\_} \cup s_{j\_}$ is a domino contained in $\mathcal{D}$. Since $|\mathcal{D}| - 2 > |\tilde{\mathcal{D}}|$ we can take a plug $p_1 \in \mathcal{P}$ which does not mark the two squares $s_{i\_}, s_{j\_}$ such that $\operatorname{flux}(d_0; p_1) = \phi$. Consider the tiling $\mathbf{t}_{d_0;p_1} = \mathbf{t}_{p_1}^{-1} * f * f_{\text{vert}} * \mathbf{t}_{p_1 \cup d_0}$. As in Figure 4.17, by performing a vertical flip and a horizontal flip, modify the floor f to a floor with a domino in $s_i \cup s_{i\_}$ and a domino in $s_j \cup s_{j\_}$; and modify the floor f_{vert} to a floor with a domino in $s_{i\_} \cup s_{j\_}$. By adding two vertical floors between f and f_{vert} move the domino in $s_j \cup s_{j\_}$ up by two floors. Then, we have a tiling which respects γ_0 except for at most three dominoes, depending whether or not $s_i \cup s_{i\_}$, $s_j \cup s_{j\_}$ and $s_{i\_} \cup s_{j\_}$ are contained in γ_0. Furthermore, each domino which does not respect γ_0 is in a different floor. Now, as in the proof of Lemma 4.11, add a large number of vertical floors around each domino not contained in γ_0. Then, perform flips in each vertical region to obtain $\mathbf{t}_{\tilde{p}}^{-1} * \mathbf{t}_{\tilde{p}}$ for some plug $\tilde{p} \in \mathcal{P}$. Therefore, $\mathbf{t}_{d_0;p_1}$

is equivalent to the concatenation of at most three generators, one for each domino not contained in γ_0. Furthermore, these dominoes are contained in $\mathcal{D}$ and are not equal to d_0. The claim then follows by the previous cases.

Therefore, $\mathbf{t}_{d;p}$ is equivalent to a tiling of $\mathcal{D}_0 \times [0, 2N]$ such that the column $(s_1 \cup s_{|\mathcal{D}_0|}) \times [0, 2N]$ is occupied only by vertical dominoes. Then, the regularity of $\mathcal{D}_0$ follows from the regularity $\mathcal{D}_0 \smallsetminus (s_1 \cup s_{|\mathcal{D}_0|})$. The result then follows by induction on $|\tilde{\mathcal{D}}|$. ■

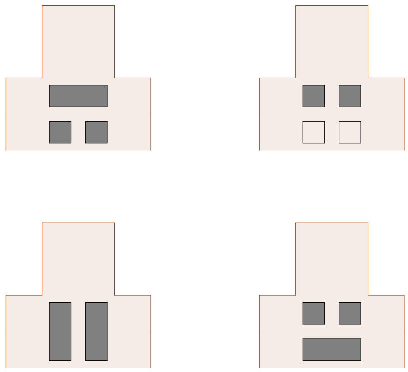

Figure 4.17: The first row shows the region $f * f_{\text{vert}}$ of $\mathbf{t}_{d_0;p_1}$. The second row shows the region $f * f_{\text{vert}}$ after two flips.

Example 4.14. Theorem 4.13 shows that the two disks below are regular.

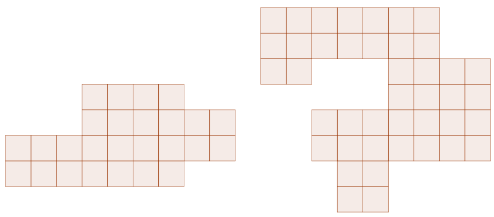

Figure 4.18: Two regular disks.

◇

Theorem 4.15. *Consider a regular quadriculated disk $\mathcal{D}$ with a hamiltonian cycle γ such that $|\mathcal{D}| > 12$. Consider the rectangle $\tilde{\mathcal{D}} = [0, L] \times [0, M]$ where $L \geq 3, M \geq 2$ and LM is even. Let $\mathcal{D}_0 = \mathcal{D} \cup \tilde{\mathcal{D}}$ be a quadriculated disk such that $\mathcal{D} \cap \tilde{\mathcal{D}} = [0, L] \times \{0\} \subset \mathcal{K}_{\mathcal{D}}$. Then, $\mathcal{D}_0$ has a hamiltonian cycle $\tilde{\gamma}$ and is a regular disk.*

Proof. Notice that it is sufficient to consider the cases $M = 2$ and $M = 3$. Indeed, suppose that the lemma holds for $M = 2$ and $M = 3$. Let $\mathcal{D}_2 = [0, L_1] \times [0, 2]$ and $\mathcal{D}_3 = [0, L_2] \times [0, 3]$ for arbitrary numbers $L_1, L_2 \geq 3$

such that $\mathcal{D}_2$ and $\mathcal{D}_3$ are balanced disks. Then, for $i = 2, 3$, $\mathcal{D} \cup \mathcal{D}_i$ is a regular disk with a hamiltomian cycle. Furthermore, notice that $\mathcal{D}_2 \subset \mathcal{K}_{\mathcal{D}\cup\mathcal{D}_2}$ and $\mathcal{D}_3 \subset \mathcal{K}_{\mathcal{D}\cup\mathcal{D}_3}$. Let $M \geq 2$ be an arbitrary number. Then, either $M = 2N$ or $M = 3 + 2(N-1)$ for some $N \geq 1$. The lemma then follows from repeated applications of the cases $M = 2$ and $M = 3$.

Let $\tilde{s_1}, \tilde{s_2}, \tilde{s_3} \subset \mathcal{D}$ be the unit squares in $\mathcal{D}$ which have an edge contained in $[0, 3] \times \{0\}$. Without loss of generality suppose that $\tilde{s_2}$ is adjacent to $\tilde{s_1}$ and $\tilde{s_3}$. Since $\mathcal{D}$ has a hamiltonian cycle, either $\tilde{s_1} \cup \tilde{s_2}$ or $\tilde{s_2} \cup \tilde{s_3}$ is contained in γ. Suppose that $\tilde{s_1} \cup \tilde{s_2}$ is contained in γ, the other case is similar. Then, construct the hamiltonian cycle $\tilde{\gamma} = (s_1, \ldots, s_{|\mathcal{D}_0|})$ in $\mathcal{D}_0$ as the union of γ and the cycle in Figure 4.19. If $M = 2$ let $s_1 = [L-1, L] \times [1, 2]$ and $s_{|\mathcal{D}_0|} = [L-1, L] \times [0, 1]$. If $M = 3$ let $s_1 = [L-2, L-1] \times [0, 1]$ and $s_{|\mathcal{D}_0|} = [L-1, L] \times [0, 1]$.

Figure 4.19: The squares $\tilde{s_1}, \tilde{s_2}, \tilde{s_3}$, the disk $\tilde{\mathcal{D}} = [0, L] \times [0, M]$ with a hamiltonian cycle and the domino $s_1 \cup s_{|\mathcal{D}_0|}$ highlighted in green, for $M = 2$ and $M = 3$, respectively.

Consider $d \subset \mathcal{D}_0$ a domino not contained in $\tilde{\gamma}$ with a compatible plug $p \in \mathcal{P}$ such that $\operatorname{flux}(d; p) = \phi$. We claim that $\mathbf{t}_{d;p}$ is equivalent to a concatenation of tilings $\mathbf{t}_{\tilde{d_1}, p_1}, \ldots, \mathbf{t}_{\tilde{d_k}, p_k}$ such that $d_i \subset \mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ and $s_1, s_{|\mathcal{D}_0|} \in p_i$ for each $i = 1, \ldots, k$.

If d disconnects $\mathcal{D}_0$ then $d \subset \mathcal{D}$ and therefore d disconnects $\mathcal{D}$. Then, as in the proof of Theorem 4.13, take $p_1 \in \mathcal{P}$ compatible with d such that $\operatorname{flux}(d; p_1) = \phi$ and $s_1, s_{|\mathcal{D}_0|} \in p$. If d does not disconnect $\mathcal{D}_0$ then there exists adjacent squares s_k and s_l such that $k \in I_{d;0}$ and $l \in I_{d;\pm 1}$. We consider three cases: $d \subset \mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$, $s_1 \subset d$, $s_{|\mathcal{D}_0|} \subset d$.

Suppose $d \subset \mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$. If d is good the claim then follows by Lemma 4.11. If d is not good then $d_1 = s_k \cup s_l$ is such that $s_l \in \{s_1, s_{|\mathcal{D}_0|}\}$. Notice that an argument similar to the one in the proof of Lemma 4.11 shows that $\mathbf{t}_{d;p} \sim *\mathbf{t}_{d_1;p_1} * \mathbf{t}_{d;p_2} * \mathbf{t}_{d_1;p_3}$ where $s_1, s_{|\mathcal{D}_0|} \in p_2$. Therefore, we are left with the cases $s_1 \subset d$ and $s_{|\mathcal{D}_0|} \subset d$.

Notice that if d is contained in $\mathcal{D}$ and $d \neq \tilde{s_1} \cup \tilde{s_2}$ then d either is good or disconnects $\mathcal{D}$. Indeed, since $d \neq \tilde{s_1} \cup \tilde{s_2}$ then d is not contained in γ. Therefore, if d does not disconnect $\mathcal{D}$ then there exists adjacent squares s_i and s_j such that $i \in I_{d;0}$, $j \in I_{d;\pm 1}$ and $s_i, s_j \subset \mathcal{D}$. Thus, $s_j \notin \{s_1, s_{|\mathcal{D}_0|}\}$ hence d is good.

Suppose that d contains either s_1 or $s_{|\mathcal{D}_0|}$. Then, d is a domino as in Figure 4.20. First, let us consider the case $M = 2$. Therefore, $d = s_{|\mathcal{D}_0|} \cup \tilde{s}$ for some unit square $\tilde{s} \subset \mathcal{D}$. Thus, $\phi_{+1} = 0$ and $\phi_{-1}\phi_0 \leq 0$. We consider two cases: $L > 3$ and $L = 3$

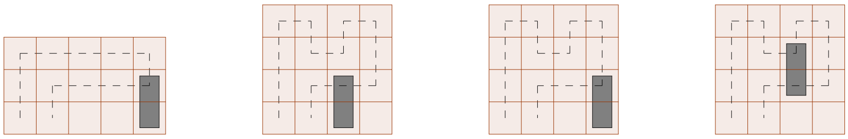

Figure 4.20: The disks $[0, 5] \times [0, 2]$ and $[0, 3] \times [0, 4]$ with the dominoes not respecting $\tilde{\gamma}$ which contain either s_1 or $s_{|\mathcal{D}_0|}$.

If $L > 3$ let $s_u = [0, 1] \times [0, 1]$, $s_v = [1, 2] \times [0, 1]$ and $d_0 = s_u \cup s_v$. Notice that $v \in I_{d;0}$ and $u \in I_{d;-1}$. Since $\phi_{-1}\phi_0 \leq 0$ we may assume that either $s_u, s_v \in p$ or $s_u, s_v \notin p$. Suppose that $s_u, s_v \in p$, the other case is similar. Then, as in the proof of Lemma 4.11, $\mathbf{t}_{d;p} \sim \mathbf{t}_{d_0;p_1} * \mathbf{t}_{d;p\setminus d_0} * \mathbf{t}_{d_0;\tilde{p_1}}$ where p_1 and $\tilde{p_1}$ are plugs compatible with d_0. Furthermore, $\operatorname{flux}_0(d; p \setminus d_0) = |\phi_0| - 1$ is not maximal. Take a plug $\tilde{p} \in \mathcal{P}$ compatible with d which marks $s_{|\mathcal{D}_0|-1}$ and its adjacent square $s \subset \mathcal{D}$ such that $\operatorname{flux}(d; \tilde{p}) = \operatorname{flux}(d; p \setminus d_0)$. Thus, $\mathbf{t}_{d;p\setminus d_0} \sim \mathbf{t}_{d;\tilde{p}}$. By performing two flips in $\mathbf{t}_{d;\tilde{p}}$ and adding some vertical floors we have that $\mathbf{t}_{d;\tilde{p}} \sim \mathbf{t}_{d_2;p_2} * \mathbf{t}_{d_3;p_3}$ where $d_2 = s_{|\mathcal{D}_0|-1} \cup s$ and $d_3 = \tilde{s} \cup s$. Therefore, $\mathbf{t}_{d;p} \sim \mathbf{t}_{d_0;p_1} * \mathbf{t}_{d_2;p_2} * \mathbf{t}_{d_3;p_3} * \mathbf{t}_{d_0;\tilde{p_1}}$. Notice that d_0 and d_2 are good dominoes and $d_3 \subset \mathcal{D} \setminus (\tilde{s_1} \cup \tilde{s_2})$. The claim then follows by the previous cases.

If $L = 3$ let $s \subset \mathcal{D}$ be the 2×2 square that contains $\tilde{s_1}$ and $\tilde{s_2}$. A case by case check shows that there exists a domino $d_0 = s_u \cup s_v \subset s$ such that $d_0 \neq \tilde{s_1} \cup \tilde{s_2}$, $u \in I_{d;0}$ and $v \in I_{d;-1}$. Then, as in the previous paragraph, $\mathbf{t}_{d;p} \sim \mathbf{t}_{d_0;p_1} * \mathbf{t}_{d;\tilde{p}} * \mathbf{t}_{d_0;\tilde{p_1}}$ for some plug $\tilde{p}$ marking $s_{|\mathcal{D}_0|-1}$ and $\tilde{s_2}$. Since $L = 3$ by performing two flips in $\mathbf{t}_{d;\tilde{p}}$ we have that $\mathbf{t}_{d;\tilde{p}} \sim \mathbf{t}_{d_2;p_2}$ where $d_2 = \tilde{s_2} \cup \tilde{s_3}$. Then, $\mathbf{t}_{d;p} \sim \mathbf{t}_{d_0;p_1} * \mathbf{t}_{d_2;p_2} * \mathbf{t}_{d_0;\tilde{p_1}}$. The claim then follows by the previous cases.

A similar argument shows that if $M = 3$ and either s_1 or $s_{|\mathcal{D}_0|}$ is contained in d then $\mathbf{t}_{d;p}$ is equivalent to a concatenation $\mathbf{t}_{d_1;p_1} * \mathbf{t}_{d_2;p_2} * \ldots * \mathbf{t}_{d_i;p_i}$ of tilings such that $d_i \neq \tilde{s_1} \cup \tilde{s_2}$ and either d_i is good or $d_i \subset \mathcal{D}$ for each $i = 1, \ldots, k$.

This case by case analysis proves the claim. Therefore, $\mathbf{t}_{d;p}$ is equivalent to a tiling of $\mathcal{D}_0 \times [0, 2N]$ such that the column $(s_1 \cup s_{|\mathcal{D}_0|}) \times [0, 2N]$ is occupied only by vertical dominoes. Then, the regularity of $\mathcal{D}_0$ follows from the regularity $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$.

Consider $M = 2$, the the regularity of $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ follows by induction on L. Let $L = 3$. By construction $s_1 \cup s_{|\mathcal{D}_0|} = [2, 3] \times [0, 2]$. Then, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is the union of $\mathcal{D}$ and a 2×2 square. Furthermore, $\tilde{\gamma} \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is a hamiltonian cycle in $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$. Thus, by Theorem 4.13, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is regular. Let $L > 3$ be an arbitrary number. Similarly $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is the union of $\mathcal{D}$ and the rectangle $[0, L-1] \times [0, 2]$. Then, by the induction hypothesis, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is regular.

Consider $M = 3$, the regularity of $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ follows by induction on L. Let $L = 4$. By construction $s_1 \cup s_{|\mathcal{D}_0|} = [2, 4] \times [0, 1]$. Then, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is the union of $\mathcal{D}$ and a thin disk. As in Figure 4.21, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ has a hamiltonian cycle. Then, by Theorem 4.13, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is a regular disk.

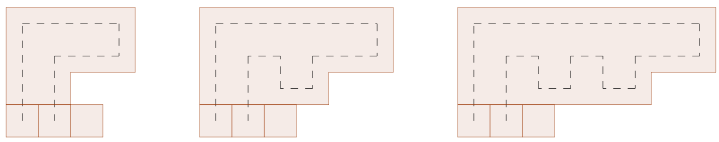

Figure 4.21: The figure shows the squares $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ and $\tilde{\mathcal{D}} \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ with a hamiltonian cycle for $M = 3$ and $L = 4, 6, 8$, respectively.

Let $L > 4$ be an arbitrary even number. Then, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is the union of $\mathcal{D}_1$ and a thin disk, where $\mathcal{D}_1 = \mathcal{D} \cup [0, L-2] \times [0, 3]$. By the induction hypothesis $\mathcal{D}_1$ is a regular disk. By Theorem 4.13, since $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ has a hamiltonian cycle, $\mathcal{D}_0 \setminus (s_1 \cup s_{|\mathcal{D}_0|})$ is a regular disk. ■

Example 4.16. Theorem 4.15 shows that the disks below are regular.

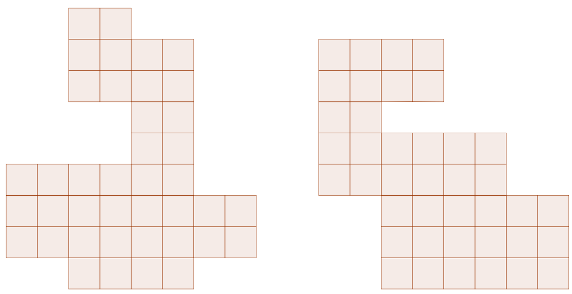

Figure 4.22: Two regular disks.

◇

5
Final remarks

We exhibited many examples of regular disks and non regular disks. However, we did not provide necessary and sufficient conditions for a disk to be regular. We make a few remarks about the results obtained.

The disks that we proved to be non regular can be disconnected by removing either a unit square or a domino. Therefore, our results are consistent with the conjecture that "large" disks are regular. We prove irregularity by constructing surjective homomorphisms from the domino group $G_{\mathcal{D}}$ to either F_2 or $\mathbb{Z}^2$. In several examples, it is not difficult to see that these homomorphisms are not isomorphisms. In that sense, it would be interesting to compute the domino groups and to discuss the differences between the several cases.

We did not exhibit specific families of regular disks. In fact, given a regular disk, we presented two methods to construct a new regular disk. Then, we have the natural question: for which disks the methods presented are not sufficient to prove regularity? Figure 5.1 below shows a quadriculated disk, which seems to be regular, but whose regularity can not be proven by a direct application of Lemma 4.15.

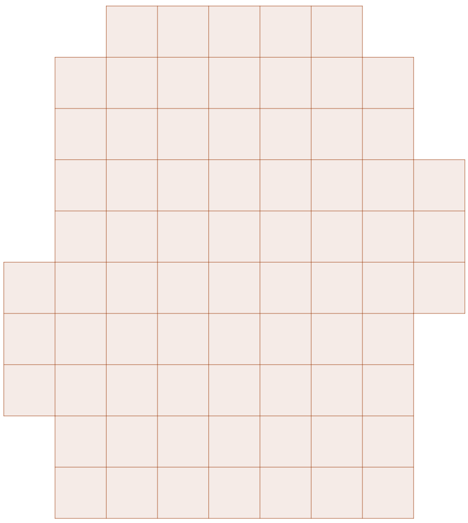

Figure 5.1: Is this a regular disk?

6
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