Pontifícia Universidade Católica
do Rio de Janeiro

Arnaldo João do Nascimento Junior

Essays on Behavioral Finance

Tese de Doutorado

Thesis presented to the Programa de Pós-graduação em Engenharia de Produção of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Engenharia de Produção.

Advisor : Prof. Luiz Eduardo Teixeira Brandão
Co-advisor: Prof. Marcelo Cabus Klotzle

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Abstract

João do Nascimento Junior, Arnaldo; Teixeira Brandão, Luiz Eduardo (Advisor); Cabus Klotzle, Marcelo (Co-Advisor). **Essays on Behavioral Finance**. Rio de Janeiro, 2021. 95p. Tese de Doutorado – Departamento de Engenharia Industrial, Pontifícia Universidade Católica do Rio de Janeiro.

Based on *Cumulative Prospect Theory*, three essays are presented in this thesis. All three works are linked by a deeper understanding of *Probability Weighting Functions* and its connection with decisions in a risk scenario.

The first essay is an empirical work using prospect theory to analyze the narrow framing bias in investment decisions in certain emerging countries: Brazil, China, Russia, Mexico and South Africa. In all cases, we empirically identified the predictive power of prospect theory for stock returns. We also found that the probability weighting function is the most important factor in this predictive power.

The second essay is a theoretical work proposing an axiomatization for the Goldstein-Einhorn weighting function. Since 1987, the well known Goldstein-Einhorn Weighting Function is widely used in many empirical and theoretical papers. Richard Gonzalez and George Wu proposed an axiomatization for it in 1999. The present work analyses their preference condition and finds a bigger family of weighting functions. We provide useful examples and suggest a new preference condition which is necessary and sufficient for Goldstein-Einhorn function. This new preference condition simulates the behavior of people in risky attitudes.

The third essay propose a measure to evaluate the psychological features of attractiveness and discriminability in the context of probability weighting functions. These concepts are important to help us understand how some emotions drive our behavior. We propose measures in absolute and in the relative sense and compare with some particular cases found in the literature. Our findings are consistent with the qualitative understanding widespread in the literature and provide a quantitative analysis for it.

Keywords

Prospect Theory; Probability Weighting Function; Axiomatization; Discriminability; Attractiveness.

Resumo

João do Nascimento Junior, Arnaldo; Teixeira Brandão, Luiz Eduardo; Cabus Klotzle, Marcelo. **Ensaios em Finanças Comportamentais**. Rio de Janeiro, 2021. 95p. Proposta de Tese de Doutorado – Departamento de Engenharia Industrial, Pontifícia Universidade Católica do Rio de Janeiro.

Baseado na *Teoria Cumulativa da Perspectiva*, três ensaios são apresentados nessa tese. Todos os três trabalhos estão conectados pelo entendimento aprofundado da *Função de Ponderação de Probabilidade* e suas conexões cenários de decisão sob risco.

O primeiro ensaio é um trabalho empírico utilizando a teoria da perspectiva para analisar o viés do efeito de enquadramento em decisões de investimentos em certos países emergentes: Brasil, China, Russia, México e África do Sul. Em todos os casos, identificamos empiricamente o poder preditivo da teoria da perspectiva para os retornos dos ativos. Também encontramos que a função de ponderação de probabilidade é o fator mais importante para o poder preditivo.

O segundo ensaio é um trbalho teórico propondo uma axiomatização da função de ponderação de Goldstein-Einhorn. Desde 1987, a conhecida função de ponderação de Goldstein-Einhorn é largamente utilizada em trabalhos em muitos artigos empíricos e teóricos. Richard Gonzalez e George Wu propuseram uma axiomatização para esta função em 1999. O trabalho que apresentamos analisa a condição de preferência dos autores e encontra uma família maior de funções de ponderação. Fornecemos exemplos úteis e sugerimos uma nova condição de preferência que é necessária e suficiente para a função de Goldstein-Einhorn. Esta nova condição de prefer6encia simula o comportamento das pessoas em situações que envolvem atitutes arriscadas.

O terceiro ensaio propõe uma medida para as características psicológicas chamadas de atratividade e discriminabilidade, no contexto das funções de ponderação de probabilidades. Esse conceitos são importantes para nos ajudar a entender como algumas emoções influenciam nosso comportamento. Propomos medidas no sentido absoluto e relativo e as comparamos com alguns exemplos particulares encontrados na literatura. Nossos resultados são consistentes com o entendimento qualitativo encontrado na literatura e fornece um entendimento quantitativo para ele.

Palavras-chave

Teoria da Perspectiva; Função de Distorção de Probabilidade; Axiomatização; Discriminabilidade; Atratividade.

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1
Presentation

This thesis is built on the basis of the *Cumulative Prospect Theory* ([4]). We present three essays that we expect shed light on some important problems in the literature. All works are linked by a deeper study of *Probability Weighting Function* and its connection between psychological bias and economic decisions in a risk scenario.

Prospect Theory was introduced by two Israeli psychologists, Amos Tversky and Daniel Kahneman ([3] and [4]). Based on this work, Daniel Kahneman won the Nobel Prize in economics in 2002.

Roughly speaking, traditional finance operates on the assumption that investors are rational and make decisions with the aim of maximizing their expected utility. Moreover, investors are assumed to be risk averse (concave utility function) and to use objective probabilities. However, this theory was first criticized by [1] and [2] subsequently showed that the theory is contradictory. After that, a series of works identified many anomalies not explained by the traditional theory. Cumulative Prospect Theory is still the most commonly used economic theory to account for the anomalies identified in situations of risk and uncertainty.

Probability weighting functions overstates small probabilities and underestimates moderate and large probabilities. This characteristic is called a *regressive effect*. In particular, the characteristic of overvaluing small probabilities — for both gains and losses — explains the demand for lotteries and insurance.

These weighted perceptions are studied by [58] and [31], in their paper on salience theory. According to the article, "salient payoffs" are results that draw the attention of decision makers. For example, a lottery prize would be salient, as would the value of a stolen car. Similarly, [5] also view the high returns/past losses of a stock as salient.

According to [31], from the psychological point of view, detecting salience is a key mechanism that allows human beings — with their limited cognitive resources — to focus their attention on relevant subsets of the available data. It is thus necessary to find a probability function that can handle the phenomena we have just described.

The first essay is a practical application in the stock market. The second essay helps to create a safe ground for part of this theory and the third essay creates measures to be used in practical applications.

In more detail, the first essay is based on the methodology developed by [5] and expand it to emerging markets, understanding its behavioral differences. We use prospect theory to analyze the relationship between PTV (Prospect Theory Value) and stock returns in the emerging markets, more specifically, Brazil, Mexico, South Africa, Russia and China.

The main result reveals that in the Brazilian market, the PTV of the past distribution of returns predicts a subsequent negative return. In the case of the other emerging countries, we found evidence of different behaviors: China and Russia, as well as Brazil, have a negative relation. On the contrary, Mexico has significant positive relations and South Africa has mixed ones. Another contribution of our study is analyzing the behavior of the most common probability weighting functions found in the literature and their relationship to stock returns in emerging markets.

While in this first essay we analyze different existing weighting functions, in the second essay we propose an axiomatization for Goldstein-Einhorn weighting function ([10]). It is one of the most common weighting function found in experimental and empirical papers in the literature.

Probability weighting function is a key element of the Cumulative Prospect Theory ([4]) and Rank-Dependent Utility Theory ([6]). Since 1979 efforts have been made to build an axiomatization through which these weighting functions can be deduced. Axiomatization is important not only to build the theoretical safe ground but also to understand and predict the behavior of the decision maker. Some examples of such works are [7], [11], [18] and [12].

Richard Gonzalez and George Wu ([18]) suggested a preference condition that is proposed as necessary and sufficient to get the Goldstein-Einhorn Probability Weighting Function ([10]).

The purpose of this second essay is discuss the preference condition proposed by Gonzalez and Wu, and shows that it leads us to a wider set of solutions. We also present a new axiomatization which provide us with the Goldstein-Einhorn function as the unique solution.

Also based on the work of Gonzalez and Wu ([18]), in the third and final essay we proposed formal measures of Attractiveness and Discriminability. Attractiveness represents how attracted an individual an individual is to some risk prospect and Discriminability reflects the ability to perceive changes in probabilities.

These measures are used to gauge the degree of departure from the objective probability, which can be interpreted as a gauge of the departure from the rational behavior. In practical applications, they have been used in a wide range of situations, as described in the introduction of the essay.

We apply our measures in the most common two-parametric families of weighting functions found in the literature: NEO-additive, CRS, Goldstein-Einhorn and Prelec. We perform a sensitivity analysis in each family to understand how a changing in the parameter affects the variance of our measures.

2
Theoretical Framework

In this section, we explain the theoretical basis (*Cumulative Prospect Theory* ([4])) on which we developed all of our three essays.

Daniel Kahneman and Amos Tversky ([3]), in 1979, present the original version of the Prospect Theory. This seminal work contains all essential ideas of the theory but it had some limitations. In 1992, the same authors introduced the Cumulative Prospect Theory (CPT, [4]), which solved the previous problems and it is still the most commonly used economic theory to account for the anomalies identified in situations of risk and uncertainty.

In all of our essays, we used the theoretical framework designed by Cumulative Prospect Theory ([4]). Formally, the CPT is modeled by considering a game with a result (gain or loss) x_i for an associated probability p_i. Moreover, negative indices are used for losses, i.e., $x_{-i} < 0$, and positive indices are used for gains, $x_i > 0$. The game can thus be represented mathematically as:

$$(x_{-m}, p_{-m}; ...; x_{-1}, p_{-1}; x_0, p_0; x_1, p_1; ...; x_n, p_n) \tag{2-1}$$

where, $x_i > x_j$, if $i > j$, $x_0 = 0$ and $\sum_{i=-m}^{n} p_i = 1$. By considering CPT, the value of the game (PTV - Prospect Theory Value) will be given by the following expression:

$$PTV = \sum_{i=-m}^{n} \pi_i v(x_i), \tag{2-2}$$

where,

$$\pi_i = \begin{cases} w^+(p_i + ... + p_n) - w^+(p_{i+1} + ... + p_n), \text{ for } 0 \leq i \leq n \\ w^-(p_{-m} + ... + p_i) - w^-(p_{-m} + ... + p_{i-1}), \text{ for } -m \leq i \leq 0 \end{cases}$$

and

$$v(x) = \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda(-x)^\beta, & x < 0 \end{cases} \tag{2-3}$$

where $\alpha, \beta \in (0, 1)$ and $\lambda > 1$.

The function $v(\cdot)$ is called *Value Function* and the function $w(\cdot)$ is called *Probability Weighting Function*.

The value function is similar to the utility function but it is only applied to the change of wealth (outcome of the game) rather than the total wealth.

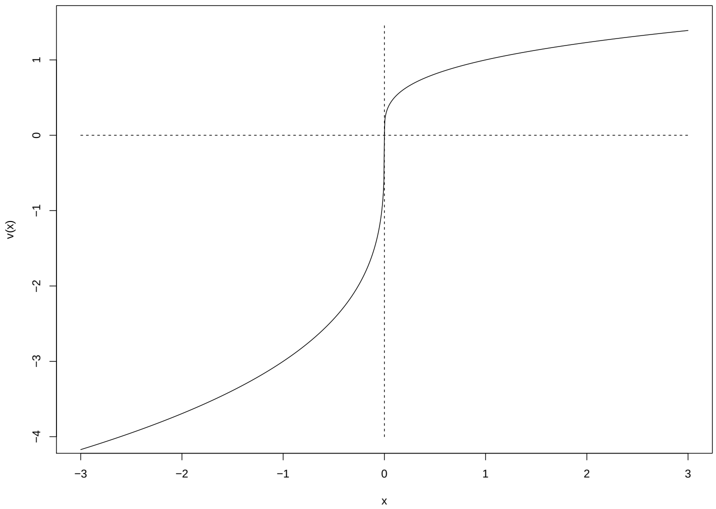


Figure 2.1: Value Function proposed by [4]. The plot uses $\alpha = \beta = 0.3$ and $\lambda = 3$.

This function (Figure 2.1) captures the empirical finding called *loss aversion*. It means that people are more sensitive to losses than to gains of the same magnitude. This phenomenal is represented by $\lambda > 1$ in the expression (2-3). Figure 2.1 shows the shape of the value function and we can also identify another important feature. People are risk averse for gains (concave part) and risk seeking for losses (convex part).

The weighting function w carries a lot of psychological bias and it is the focal point of two essays. Figure 2.2 shows the most common shape (inverse s-shape) of the weighting function found in the experimental works in the literature. Mathematically speaking, it is a function $w : [0,1] \to [0,1]$ which is continuous, strictly increasing and $w(0) = 0$, $w(1) = 1$. Psychologically speaking, it translate the fact that people usually overestimate small probabilities and underestimate large probabilities. This features is called *Regressive Effect*. Overestimate small probabilities contributes to risk seeking behavior for gains and risk aversion for losses. On the other hand, underestimate larger probabilities contributes to risk aversion for gains and risk seeking for losses. For instance, overvaluing small probabilities (for both gains and losses) explains the demand for lotteries and insurance.

Many papers, such as [4], [7], [8], [9], [10], [11] and [36], present different expressions of weighting functions and explore their motivations, properties and consequences.

Through an axiomatic treatment, [11] obtained the Prelec I, Prelec II and Power functions in a mathematically rigorous way. In [36], we found a simpler axiomatization for the family Prelec II. Similarly, [7] also obtained his

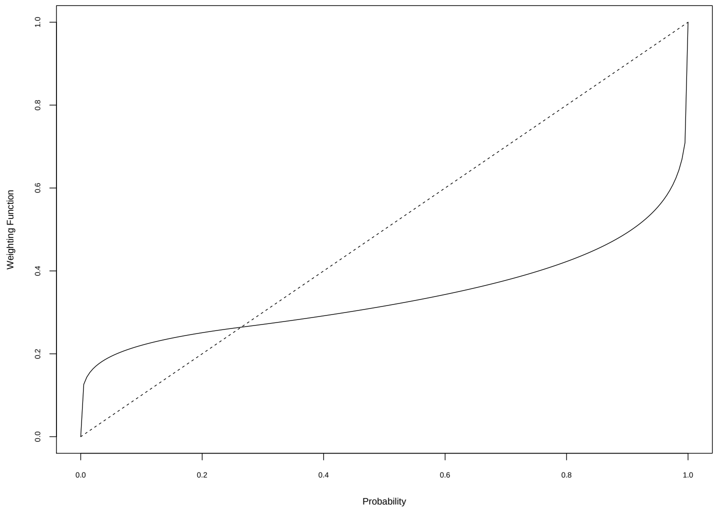


Figure 2.2: Shape of the Probability Weighting Function $w(p)$.

function axiomatically.

Suggested as a function that fits the behaviors identified and the data obtained in their experiments, the explicit version of the Tversky-Kahneman function first appears in [4]. [9] assumes that w follows a log-odds transformation, and [8] suggests a natural generalization (for two parameters) of Tversky-Kahneman and Karmarkar. In [10], the authors obtain their expression by allowing the log-odds transformation suggested by [9] to have a linear coefficient.

In short, the concepts presented in this sections are the building block of Cumulative Prospect Theory and all the three essays in this thesis.

3
Essay I: Prospect Theory and Narrow Framing Bias: Evidence from Emerging Markets

3.1
Introduction

As we pointed out earlier, [3] and [4] introduced Prospect Theory, which is still the most commonly used economic theory to account for the anomalies identified in situations of uncertainty. Along these lines, many studies have applied prospect theory to asset pricing.

Using Cumulative Prospect Theory, [5] show that, in the US market, the investors' mental model (narrow framing) causes them to assume the historical distribution of returns as a good proxy for future returns. The study shows that this investor behavior has an important predictive power for stock returns in markets where most investors are individuals rather than institutional investors who typically use more sophisticated techniques to model future stock prices. More specifically, the authors find empirical evidence that the prospect theory value of the distribution of past returns is negatively correlated with future returns; that is, on average, stocks whose past returns have a high (low) Prospect Theory Value (defined as PTV from this point forward) have a subsequent low (high) return.

The explanation for this phenomenon is that because the PTV represents utility for the investor, high PTVs become attractive and the investor is thus willing to pay a higher premium to buy the stock, overvaluing it and consequently earning a lower return.

Although [5] test their hypothesis also in the international stock market, the result is provided in an aggregated form, without further detail about each country in the sample. Analyzing at an aggregate level, authors focus only on the point where the outcome supports their hypothesis, not saying much about countries that have not behaved as expected.

Furthermore, most studies using Prospect Theory and stock returns, fall on the American and European stock markets. There is a lack of research applying Prospect Theory to understand, in more detail, stock returns in emerging markets.

Our study is based on the methodology developed by [5] and tries to expand it to emerging markets, understanding its behavioral differences. We use prospect theory to analyze the relationship between PTV and stock returns in the emerging markets, more specifically, Brazil, Mexico, South Africa, Russia and China. Considering data from the World Bank, these countries share 22.07% of the world´s GDP, 77.21% of the top ten emerging market´s GDP, and are important representative emerging countries for each continent. An analysis of the narrow framing in emerging countries is the first contribution of our study.

For all countries, we performed additional robustness tests, using different time windows and different probability weighting functions, which is the variable with the most predictive power for stock returns.

Corroborating [5], our main result reveals that in the Brazilian market, the PTV of the past distribution of returns predicts a subsequent negative return. In the case of the other emerging countries, we found evidence of different behaviors: China and Russia, as well as Brazil, have a negative relation. On the contrary, Mexico has significant positive relations and South Africa has mixed ones.

To the best of our knowledge this is the first paper that deals with a) Applying Prospect Theory to analyze the relation of PTV and stock returns in emerging markets; b) Analysing how different probability weighting functions affect the relation between PTV and future stock returns.

The remainder of this article is organized as follows: in Section 3.2, we make a literature review, in Section 3.3, we introduce the prospect theory, with a particular focus on the probability weighting functions. In Section 3.4, we discuss the empirical results in the five emerging countries considered. In Section 3.5, we present the conclusions.

3.2
Literature Review

As we mentioned previously, the main finding of [5] is a negative relation between PTV and the future stock return. The authors take the distribution of monthly returns over the previous five years for stocks traded on the US market and use both a time-series portfolio analysis and the Fama-MacBeth methodology to observe that in a cross-section analysis, the PTV of the stock's historical returns is negatively related to the subsequent return. The authors also show that among the components of the cumulative prospect theory, the probability weighting function contributes most to the predictability of returns.

Our study is consistent with some prior studies that use the same methodology we employ. Relating PTV to idiosyncratic volatility, [19] apply the model of [5] to the South Korean stock market. Unlike the findings of [5] for the US market, the authors find a positive relationship between the PTV and the subsequent returns. The authors argue that this conflict may be due to cultural differences between the two countries. When considering idiosyncratic volatility, the authors find that the negative relationship between volatility and future returns in the Korean market is caused by the stocks' PTV, which is more pronounced in stocks with a negative value.

In [20], the authors apply the model from [5] to US corporate bond returns. Corroborating the results found for the stock market by [5], the authors conclude that there is a negative relationship between PTV and future returns in the fixed income market. Moreover, when considering only the junk bonds market, their study shows that risk aversion is the most critical component in predicting returns in the bond market as a whole and that the probability weighting function has the highest weight.

Another study conducted for the currency market is [21]. Based on the distribution of historical returns in the currency market, they find that currencies with a high (low) PTV have a subsequent low (high) mean return. Similarly, the study also concludes that the probability weighting function is the most crucial element of prospect theory concerning predictability.

Another contribution of [5] is that it confirms that stocks with high PTV are stocks with highly skewed past returns. An intuitive justification presented for this phenomenon is that in looking at the historical distribution of returns, investors identify the positive skewness, leading them to treat the stocks as if they were a type of lottery (Barberis and Huang, 2008) and to thereby view them as attractive. The investor thus overvalues the stock and obtains a subsequent low return.

In [30], the authors use the framing effect and the study by [5] to analyze IPOs in the Chinese market. The authors found that in IPO returns from 2006 to 2012, 30% were negatively skewed, and 70% were positively skewed. In the negatively skewed IPO returns, there was a substantially positive relationship between the skewness and the discounted offer price. However, no relationship was found between the discounted offer price and the positively skewed IPO returns.

3.3 Methodology

We employ the model developed in [5]. Given a specific stock, we take the raw returns from the previous four years (48 months) and order the returns from the most negative to the most positive. If there are m negative returns, then there are $n = 48 - m$ positive returns; the most negative will be named r_{-m}, and the most positive will be r_n.

The distribution of the returns will be equiprobable ($p_i = 1/48$), and thus:

$$(r_{-m}, 1/48; ...; r_{-1}, 1/48; r_1, 1/48; ...; r_n, 1/48).$$

and the PTV can be calculated as:

$$PTV = \sum_{i=-m}^{-1} \left[w^{-} \left(\frac{i+m+1}{48} \right) - w^{-} \left(\frac{i+m}{48} \right) \right] + \sum_{i=1}^{n} \left[w^{+} \left(\frac{n-i+1}{48} \right) - w^{+} \left(\frac{n-i}{48} \right) \right].$$

As noted by [5] and Bordalo et al (2013), the probability weighting function, $w(\cdot)$, is critical for asset pricing.

In [8], [11], [32], [14] we find the most common probability weighting functions used in experimental tests. However there are few works that estimate the parameters of such functions.

Table 3.1 shows the functions that will be used in the empirical tests, with the values of their respective parameters estimated in [8] and [33].

Name	Function	Parameter Value
Tversky-Kahneman (TK)	$w(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{1/\gamma}}$	$\gamma = 0.71$
Goldstein-Einhorn (GE)	$w(p) = \frac{sp^\gamma}{sp^\gamma + (1-p)^\gamma}$	$s = 0.84$ and $\gamma = 0.68$
Prelec I (Pr I)	$w(p) = e^{-(-\ln p)^\theta}$	$\theta = 0.74$
Prelec II (Pr II)	$w(p) = e^{-\beta(-\ln p)^\theta}$	$\theta = 0.534$ and $\beta = 1.083$

Table 3.1: Weighting functions and their respective parameters

3.4 Empirical Results

We have separated the analysis into two parts: First, we analyze the narrow framing bias in the case of Brazil; due to the availability of data, this analysis is performed in more detail. We then analyze the same phenomenon in China, Mexico, Russia and South-Africa.

For Brazil, as in [5], to examine our main hypothesis, i.e., whether a stock's PTV has a negative relationship with its subsequent return, we use two widely known methodologies: the first is portfolio formation analysis, in which we use a time series of monthly stock returns; the second methodology is the Fama-MacBeth regression analysis. In both cases, we perform robustness tests to ratify the evidence of both methodologies.

Concerning the other emerging countries selected, we only perform a portfolio formation analysis by using time series analysis and also perform their respective robustness tests.

3.4.1 Data

For each country, we used the Datastream database to obtain the stock prices in local currency of all stocks, with at least four years of monthly return data, listed on the major exchange in each market. Both, active and delisted stocks were included and we also required that, for each month, the number of stocks with a valid PTV was at least thirty. We also made an adjustment for dividends and stock splits.

In Table 3.2, we have listed the countries chosen and their respective sample periods. The periods are different because, in our separate analysis of the countries, we sought to obtain the largest sample available for each country.

Country	Period
Brazil, South Africa	1990 to 2019
Maxico	1988 to 2019
China	1992 to 2019
Russia	1994 to 2019

Table 3.2: Emerging countries and the sample period

The returns used were the gross returns rather than the excess returns. We made this choice because, in some countries, specially Brazil, reliable data for the risk-free rate data are only available from 2001 onwards, which would narrow our sample, resulting in loss of statistical significance.

To calculate the return, we used the log of the gross return on each stock.

$$r_{t+1} = \ln\left(\frac{P_{t+1}}{P_t}\right) \tag{3-1}$$

Concerning the Fama-MacBeth regressions, we used the same control variable as [5]: *PTV, MKT, Beta, Size, Bm, Mom, Rev, Lt rev, Illiq, Ivol, Max, Min, Skew, Eiskew and Coskew.*

The variables *Beta (β), Size, Bm, Mom, Rev, Lt rev, Illiq* and *Ivol* are variables commonly used in cross-section analysis, as they are known to have a predictive power for returns. Furthermore, the variables *Max, Min, Skew, Eiskew* and *Coskew* are also known in cross-section analyses and account for the skewness aspects of the returns.

In Brazil, reliable data for the risk-free rate, Small Minus Big (SMB) and High Minus Low (HML) data are only available from 2001 onwards, which means that the results of the regressions have little statistical significance. For this reason, the only risk factor generated since 1990 was the market return (MKT). For the latter, we used the log of the gross returns of the Bovespa index.

Due to this limitation, to calculate the Beta, Ivol and Eiskew variables, we will use the regression:

$$r_{it} = \alpha_i + \beta_{it} MKT_t + \epsilon_{it}, \tag{3-2}$$

and Ivol will thus be defined, as in [34]:

$$Ivol_i = \sqrt{Var(\epsilon_{it})}. \tag{3-3}$$

Moreover, Eiskew, as in [35], will be:

$$Eiskew_i = \frac{\frac{1}{n}\sum_{i=1}^{n} \epsilon_{it}^3}{\left(\frac{1}{n}\sum_{i=1}^{n} \epsilon_{it}^2\right)^{3/2}} \tag{3-4}$$

3.4.2 Evidence in the Brazilian Market

3.4.2.1 Time-Series Analysis and Robustness Tests

In this section, we only analyze the Brazilian market; the other countries will be analyzed in Section 3.4.2.4.

The portfolios are constructed by ordering the quantiles obtained from the increasing PTVs. The portfolios determined by the quantiles are numbered from P1 to P8; i.e., P1 corresponds to the portfolio with the lowest PTV, and P8 is the portfolio with the highest PTV.

Starting in January 1990 and ending in August 2019, at the beginning of each month, the quantiles are ordered based on the increasing PTVs. In the subsequent month, for each portfolio, we calculate the mean return, both equal-weighted (EW) and value-weighted (VW). This process gives us a time series of monthly returns for each portfolio. We use this time series to compute the mean return of each quantiles over the entire sample.

Table 3.3 shows the mean gross return of each quantile; the last column is the difference between the mean value of the first quantile (P1) and the mean value of the last quantile (P8). Considering the main hypothesis of our paper, the rightmost column is the most important, as it shows the negative relationship between the PTV and the stock return; i.e., P1 $-$ P8 $>$ 0.

Consistent with [5], the last column in Table 3.3 evidences our hypothesis that stocks with higher PTVs earn a lower subsequent return. The explanation

		P1	P2	P3	P4	P5	P6	P7	P8	P1 − P8
	EW	0.0120	0.0020	0.0044	0.0049	0.0043	0.0013	0.004	-0.0066	**0.0186**
Gross		(2.13)	(0.44)	(0.99)	(1.31)	(1.02)	(0.31)	(0.18)	(-1.67)	(2.70)
return	VW	**0.0277**	**0.0159**	**0.0166**	**0.0187**	**0.0116**	**0.0128**	**0.0093**	0.0018	**0.0258**
		(2.59)	(2.56)	(3.08)	(2.43)	(2.21)	(3.04)	(1.95)	(0.44)	(2.32)

Table 3.3: Portfolio Analysis

for this phenomenon is that because the PTV represents utility for the investor, high PTVs become attractive and the investor is thus willing to pay a higher premium to buy the stock, overvaluing it and consequently earning a lower return.

We found this evidence not only for EW portfolios but also for VW portfolios. In this case, contrary to what occurs in the US market ([5]), in Brazil, in the VW portfolios, the difference between the returns in the extreme portfolios is greater than that in the equal-weighted portfolios, although the negative relationship is preserved in both EW and VW portfolios.

Figure 3.1 graphically shows what takes place in Table 3.3, i.e., the evolution of the mean values of the returns in both the EW and VW portfolios. Note the downward trend in both cases.

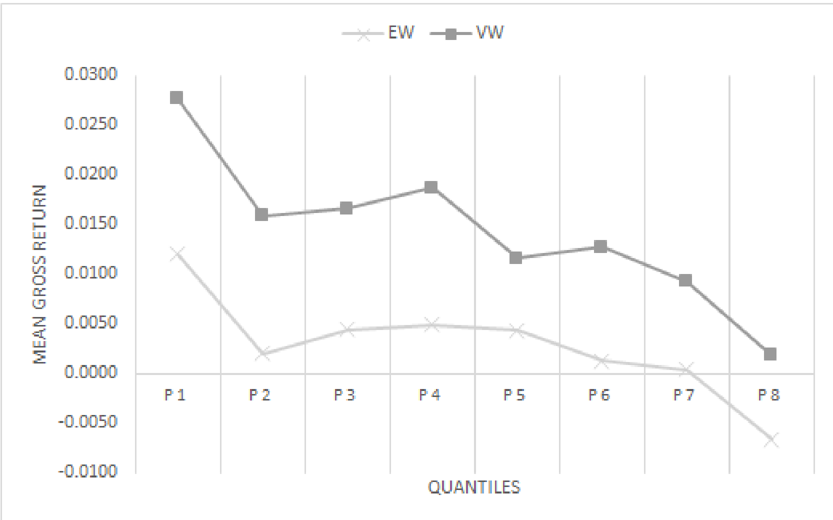


Figure 3.1: Evolution of the portfolios

In this section, we perform three additional tests to evaluate the robustness of our result: first, we reconstructed the portfolios by using different time windows; next, we used different probability weighting functions, as discussed in Section 3.3; and finally, we skipped one month between constructing the portfolios and calculating the return.

Table 3.4 shows the difference in the mean returns between the first (P1) and last (P8) portfolio, with their respective t-statistic values. The columns represent the different probability weighting functions presented in Table 2, and the lines represent the time windows used to calculate the PTV. Panel A presents the values for the equal-weighted portfolios (EW), and Panel B presents the results of the value-weighted portfolios (VW).

Panel A: Equal-weighted portfolio				
	TK	GE	Pr I	Pr II
Past 5 years	0.0035	0.0031	0.0037	0.0055
	(0.67)	(0.53)	(0.71)	(1.10)
Past 4 years	**0.0186**	**0.0136**	**0.0168**	**0.0147**
	(2.70)	(1.82)	(2.34)	(2.16)
Past 3 years	**0.0205**	0.0099	0.0100	0.0107
	(2.35)	(1.23)	(1.28)	(1.42)
Past 2 years	**0.0199**	**0.0154**	**0.0156**	0.0101
	(2.77)	(2.07)	(2.19)	(1.47)

Panel B: Value-weighted portfolio				
Past 5 years	0.0082	**0.0168**	0.0131	0.0032
	(0.76)	(1.70)	(1.59)	(0.41)
Past 4 years	**0.0258**	0.0145	**0.0167**	0.0074
	(2.32)	(1.62)	(2.06)	(0.99)
Past 3 years	**0.0206**	0.0128	**0.0151**	**0.0180**
	(2.10)	(1.52)	(1.83)	(2.11)
Past 2 years	0.0166	0.0129	**0.0184**	**0.0188**
	(1.55)	(1.30)	(1.78)	(1.78)

Table 3.4: Different windows and different weighting functions

A decreasing behavior in the different windows can also be identified in Figure 3.2, where the mean values across the portfolios are only shown for the Tversky-Kahneman (TK) weighting function.

In the next robustness test, as in [5], we skip one month between the portfolio formation and the return calculation. Table 3.5 shows the results of this test. As in Table 3.4, the difference remains positive for most of the functions, although fewer cases have statistical significance. Consistent with [5], we also note that there is a decrease in the magnitude of the portfolio difference. This decline occurs mainly in the EW portfolios.

Once again, in Figure 3.3, for the TK weighting function, we can identify a decreasing behavior when we look at the evolution of the values of the returns from portfolio P1 to portfolio P8.

From the empirical results we have just shown, we have strong evidence that our main hypothesis is true, i.e., Brazilian investors do analyze investment options by looking at the past distribution of returns and thus decide to invest based on the PTV. High PTVs lead investors to pay high premiums for the corresponding stock, thus obtaining lower returns.

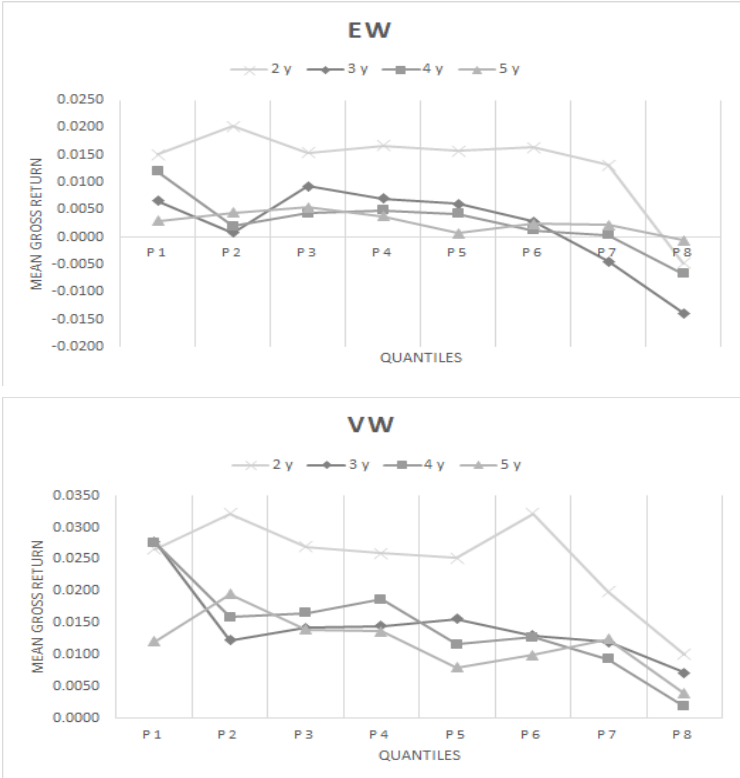


Figure 3.2: Evolution of the returns for the TK function.

Panel A: Equal-weighted portfolio				
	TK	GE	Pr I	Pr II
Past 5 years	0.0025	-0.0013	0.0000	0.0017
	(0.51)	(-0.21)	(0.01)	(0.33)
Past 4 years	0.0141	0.0068	0.0110	0.0079
	(1.57)	(0.72)	(1.20)	(1.01)
Past 3 years	**0.0162**	**0.0062**	0.0051	0.0057
	(1.82)	(1.82)	(0.65)	(0.78)
Past 2 years	**0.0166**	0.0089	0.0096	0.0110
	(2.15)	(1.32)	(1.39)	(0.00)

Panel B: Value-weighted portfolio				
Past 5 years	0.0011	0.0061	0.0075	0.0025
	(0.10)	(0.70)	(0.91)	(0.34)
Past 4 years	**0.0229**	0.0047	0.0046	0.0027
	(2.47)	(0.52)	(0.57)	(0.35)
Past 3 years	0.0165	0.0069	0.0107	0.0107
	(1.55)	(0.82)	(1.42)	(1.30)
Past 2 years	**0.0196**	0.0079	0.0123	.00126
	(1.82)	(0.81)	(1.21)	(1.28)

Table 3.5: Portfolio analysis for different windows, probability weighting functions and a one-month skip

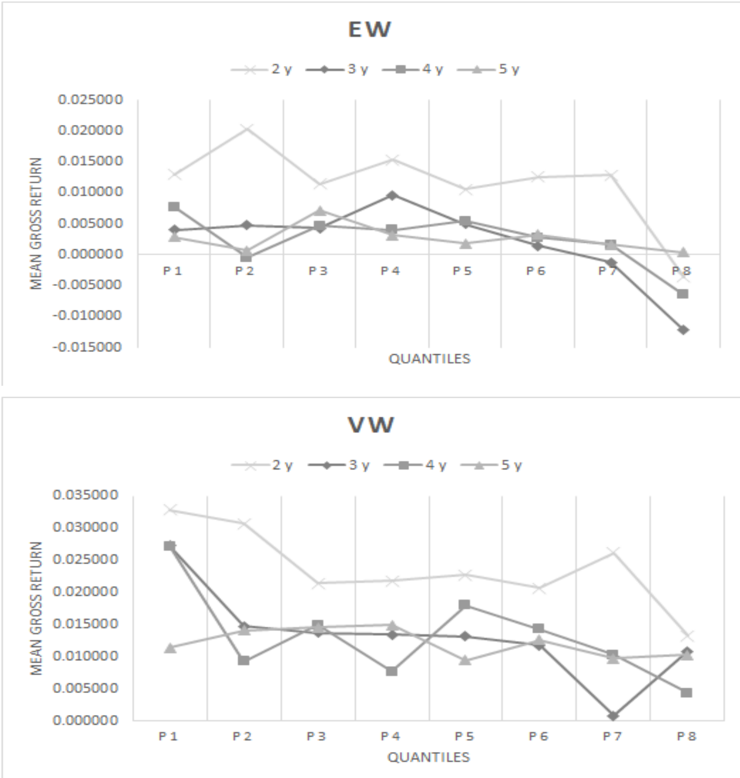


Figure 3.3: Evolution of the returns for the TK function with an one-month skip

3.4.2.2 Fama-MacBeth Methodology and Robustness Tests

In this section, we test our hypothesis through the Fama-MacBeth methodology by using the variables presented in Section 3.4.1.

Fama-MacBeth regression analysis allow us to examine the relation between pairs of variables and control for a large set of other variables when examining the relation of interest.

In general, a Fama-MacBeth regression can be represented by,

$$Y_{i,t} = \alpha_{0,t} + \alpha_{1,t}X1_{i,t} + \alpha_{2,t}X2_{i,t} + \ldots + \epsilon_{i,t} \tag{3-5}$$

where $Y_{i,t}$ is the dependent variable and $X1_{i,t}, X2_{i,t}$, etc, are the dependent variables.

For instance, considering our scenario, we may run a Fama-MacBeth regression where the independent variable is the one-month-ahead raw return r_{t+1} and the dependent variables are beta (β_t), size ($Size_t$) and book-to-market (Bm_t). In this case, the equation (3-5) becomes,

$$r_{i,t+1} = \alpha_{0,t} + \alpha_{1,t}\beta_{i,t} + \alpha_{2,t}Size_{i,t} + \alpha_{3,t}Bm_{i,t} + \epsilon_{i,t+1} \tag{3-6}$$

where the index i represent a specific asset and t is the month.

Table 3.6 presents a summary of those statistical variables. Panel A shows the mean and standard deviation of the variables, and Panel B shows the correlations between them.

Panel A: Mean and Standard Deviations

	PTV	Beta	Size	Bm	Mom	Rev	Illiq	Lt rev	Ivol	Max	Min	Skew	Eiskew	Coskew
Mean	-0.06	0.36	4.71	1.87	0.11	0.01	0.28	1.03	0.01	0.03	0.03	0.07	0.11	-0.29
SD	0.06	0.43	2.88	2.70	0.65	0.02	1.37	30.22	0.01	0.03	0.03	1.07	1.02	2.90

Panel B: Correlations

	PTV	Beta	Size	Bm	Mom	Rev	Illiq	Lt rev	Ivol	Max	Min	Skew	Eiskew	Coskew
PTV	1.00													
Beta	-0.61	1.00												
Size	-0.08	0.39	1.00											
Bm	0.34	-0.26	-0.13	1.00										
Mom	0.19	0.02	0.07	0.17	1.00									
Rev	0.56	-0.07	0.13	0.06	0.30	1.00								
Illiq	0.08	-0.14	-0.18	0.12	-0.03	-0.01	1.00							
Lt Rev	0.09	-0.02	0.00	0.05	0.44	0.03	0.01	1.00						
Ivol	-0.79	0.58	0.08	-0.25	-0.07	-0.03	-0.13	-0.07	1.00					
Max	-0.75	0.60	0.10	-0.25	-0.04	0.05	-0.13	-0.06	0.93	1.00				
Min	-0.82	0.56	0.07	-0.23	-0.08	-0.07	-0.13	-0.07	0.93	0.89	1.00			
Skew	0.32	-0.04	-0.13	-0.08	0.10	0.39	0.23	0.05	0.14	0.24	0.03	1.00		
Eiskew	0.31	-0.02	-0.10	-0.10	0.10	0.37	0.21	0.05	0.11	0.22	0.01	0.98	1.00	
Coskew	-0.02	0.05	0.01	0.08	-0.02	0.03	0.05	0.02	0.08	0.10	0.07	0.13	0.05	1.00

Table 3.6: Summary of the control variables

In line with the US market, we can observe that among the shares traded in Brazil, the PTV variable is positively correlated with past performance measures (Rev, Mom and Lt rev), negatively correlated with the past volatility measure (Ivol), and positively correlated with past skewness (Skew). Moreover, contrary to results in the US market, in our case, companies with high PTV tend to be value companies with lower market capitalization.

The result obtained in Section 3.4.2.1 is now tested by using Fama-MacBeth regressions. Table 3.7 shows the mean of the time-series coefficients of the independent variables. We divided these variables into two groups: variables that account for the skewness of the returns (Max, Min, Skew, Eiskew, Coskew; columns(6) – (9)) and variables that are not related to the skewness (columns (1) – (5)). For each coefficient estimate, we also calculated the associated t-statistic by using the Newey-West correction (window L = 12) for the robust standard error.

The results of Table 3.7 show that in most cases (regressions 2, 3, 4, 7 and 8), the PTV has an important predictive power for the returns. Moreover, the relationship is negative in all cases, reinforcing the result found in Section 3.4.2.1. Furthermore, when we control for the liquidity variable (Illiq), we obtain a significant increase in the magnitude of the PTV coefficient. Contrary to what occurs in the US market, an even greater increase occurs when we control for the skewness variables Skew and Eiskew. As in Section 3.4.2.1, we perform additional tests by using the Fama-MacBeth methodology to

	Control					Skew Control			
	1	2	3	4	5	6	7	8	9
PTV	-0.047	**-0.061**	**-0.057**	**-0.174**	-0.080	-0.078	**-0.440**	**-0.349**	-0.074
	(-1.58)	(-2.14)	(-1.82)	(-3.23)	(-0.94)	(-1.07)	(-3.32)	(-3.66)	(-1.03)
Beta		0.010	0.014	0.005	0.041	0.046	0.050	0.042	0.044
		(0.58)	(0.71)	(0.20)	(0.87)	(0.99)	(0.98)	(0.92)	(0.98)
Size		0.002	0.001	**0.004**	**0.003**	0.002	**0.003**	**0.003**	0.001
		(0.69)	(0.50)	(5.26)	(2.16)	(1.43)	(2.09)	(2.35)	(1.24)
Bm		**0.000**	**0.000**	**0.003**	**0.003**	**0.002**	**0.001**	**0.001**	**0.002**
		(1.95)	(2.07)	(3.87)	(3.84)	(3.69)	(3.64)	(3.64)	(3.69)
Mom		-0.009	-0.008	-0.004	0.020	0.030	0.040	0.035	0.029
		(-1.03)	(-0.72)	(-0.72)	(0.63)	(0.98)	(1.20)	(1.17)	(0.96)
Rev			-0.032	**-0.088**	**-0.155**	**-0.126**	**-0.124**	**-0.128**	**-0.122**
			(-1.38)	(-7.82)	(-2.25)	(-1.75)	(-1.77)	(-1.79)	(-1.70)
Illiq				-3.065	-2.433	-2.399	-1.970	-1.869	-2.461
				(-1.16)	(-0.89)	(-0.91)	(-0.88)	(-0.87)	(-0.95)
Lt Rev					-0.007	-0.003	0.003	0.002	-0.002
					(-3.65)	(-1.71)	(1.31)	(0.85)	(-0.94)
Ivol					-0.821	-0.719	-0.912	-0.811	-0.664
					(-1.07)	(-1.00)	-(1.18)	(-1.16)	(-1.00)
Max						**1.793**	**1.700**	**1.743**	**1.772**
						(48.11)	(27.45)	(56.65)	(45.21)
Min						**-1.777**	**-1.743**	**-1.749**	**-1.754**
						(-28.37)	(-40.12)	(-39.84)	(-28.37)
Skew							**0.019**		
							(4.06)		
Eiskew								**0.015**	
								(5.12)	
Coskew									-0.001
									(-0.47)

Table 3.7: Fama-MacBeth regression analysis

support our result. First, we run the regressions in different windows for the distributions of the returns, and then we use the different probability weighting functions from Table 3.1. To simplify the analysis and understanding, the robustness tests contain only PTVs, as they are of the most interest to our study.

Table 3.8 shows the PTVs for the different probability weighting functions for our 4-year window. The first line is thus the same as that shown in Table 3.7, and the others represent the same regressions, with only changes in the weighting functions.

Note that the PTVs are negative in all cases, which confirms our result from Section 3.4.2.1.

Table 3.9 is similar to Table 3.8, differing only in the size of the window, which in this case is 5 years. Here, we can also note that virtually all the PTVs are negative, except in some regressions of the GE function. However, the cases where positive values occur have a significantly lower statistical significance than do the other cases.

Considering the portfolio analysis (time series) and the regression analy-

	Control					Skew Control			
	1	2	3	4	5	6	7	8	9
TK	-0.047	**-0.061**	**-0.057**	**-0.174**	-0.080	-0.078	**-0.440**	**-0.349**	-0.074
	(-1.58)	(-2.14)	(-1.82)	(-3.23)	(-0.94)	(-1.07)	(-3.32)	(-3.66)	(-1.03)
GE	-0.052	-0.036	-0.026	**-0.376**	-0.266	-0.232	-0.320	-0.298	-0.226
	(-1.66)	(-0.94)	(-0.63)	(-1.71)	(-1.38)	(-1.04)	(-1.29)	(-1.28)	(-1.05)
Pr I	**-0.087**	-0.065	-0.053	**-0.211**	-0.195	-0.185	-0.338	-0.310	-0.165
	(-1.77)	(-1.14)	(-0.87)	(-2.34)	(-1.00)	(-0.93)	(-1.43)	(-1.43)	(-0.92)
Pr II	**-0.082**	-0.035	-0.024	-0.143	-0.176	-0.149	-0.394	-0.316	-0.144
	(-1.74)	(-0.67)	(-0.45)	(-1.63)	(-1.05)	(-0.97)	(-1.54)	(-1.55)	(-0.98)

Table 3.8: Fama-MacBeth regression for different weighting functions and a 4-year window

	Control					Skew Control			
	1	2	3	4	5	6	7	8	9
TK	-0.038	**-0.067**	**-0.073**	**-0.242**	-0.101	-0.104	**-0.503**	**-0.330**	-0.096
	(-1.26)	(-3.11)	(-3.44)	(-2.03)	(-0.81)	(-0.87)	(-2.26)	(-2.79)	(-0.82)
GE	-0.033	**-0.042**	**-0.047**	0.713	0.820	0.795	0.643	0.788	0.804
	(-1.36)	(-1.79)	(-2.07)	(0.91)	(1.07)	(1.04)	(0.91)	(1.00)	(1.04)
Pr I	-0.068	**-0.084**	**-0.091**	-3.219	-2.989	-2.987	-3.059	-3.031	-2.973
	(-1.58)	(-2.28)	(-2.53)	(-1.06)	(-1.00)	(-1.00)	(-1.03)	(-1.02)	(-1.00)
Pr II	-0.070	-0.049	**-0.059**	-0.372	-0.209	-0.230	-0.335	-0.123	-0.231
	(-1.69)	(-1.46)	(-1.78)	(-1.20)	(-0.69)	(-0.75)	(-1.13)	(-0.55)	(-0.77)

Table 3.9: Fama-MacBeth regression for different weighting functions and a 5-year window

sis, we thus come to similar conclusions: PTV has a negative relationship with stock returns. Moreover, we can see that the predictive power is greater when we consider the TK weighting function.

3.4.2.3
Representativeness of the Variables

As in [5], we also examined which characteristic most influences the predictive power of TK. To this end, each column in Table 3.10 corresponds to a different Fama-MacBeth regression with a different TK component.

The prospect variables we consider are the following: loss aversion (LA is represented by λ), probability weighting (PW is represented by γ and δ), concavity/convexity (CC is represented by α) and their combination, namely, LACC, PAPW, CCPW. For example, in column (7), the prospect variable is PTV, and it is, therefore, the same regression as in column (6) of Table 3.7. In column (1), the prospect variable is only the LA component, and in this regression, we use $\lambda = 2.25$ and use $(\alpha, \gamma, \delta) = (1, 1, 1)$ in place of $(\alpha, \gamma, \delta) = (0.88; 0.71; 0.71)$. Similarly, in column (2), we consider only the PW components and then perform the regression in which $(\gamma, \delta) = (0.71; 0.71)$ and $(\alpha, \lambda) = (1, 1)$. In column (3), we have $(\alpha, \gamma, \delta, \lambda) = (0.88; 1; 1; 1)$;

in column (4), $(\alpha, \gamma, \delta, \lambda) = (0.88; 1; 1; 2.25)$; in column (5), $(\alpha, \gamma, \delta, \lambda) = (1; 0.71; 0.71; 2.25)$ and in column (6), $(\alpha, \gamma, \delta, \lambda) = (0.88; 0.71; 0.71; 1)$.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	LA	PW	CC	LACC	LAPW	CCPW	PTV
PTV	**-0.121**	**-0.235**	**-0.175**	**-0.179**	**-0.091**	**-0.258**	**-0.078**
	(-0.72)	**(-1.91)**	(-0.82)	(-0.93)	**(-1.39)**	**(-1.87)**	**(-1.07)**
Beta	0.046	0.037	0.057	0.037	0.047	0.040	0.046
	(1.00)	(0.92)	(0.99)	(0.94)	(0.98)	(0.93)	(0.99)
Size	0.001	0.002	0.002	0.002	0.002	0.002	0.002
	(1.03)	(2.00)	(1.68)	(1.60)	(1.21)	(2.54)	(1.43)
Bm	0.002	0.002	0.002	0.002	0.002	0.002	0.002
	(3.70)	(3.71)	(3.71)	(3.67)	(3.67)	(3.72)	(3.69)
Mom	0.045	-0.014	-0.014	0.042	0.031	-0.015	0.030
	(0.99)	(-0.81)	(-0.80)	(1.01)	(1.00)	(-0.76)	(0.98)
Rev	-0.143	-0.030	0.004	-0.127	-0.130	-0.031	-0.126
	(-1.56)	(-1.17)	(0.08)	(-1.66)	(-1.71)	(-1.06)	(-1.75)
Illiq	-3.043	-1.762	-2.306	-2.958	-2.447	-1.744	-2.399
	(-1.12)	(-0.62)	(-0.82)	(-1.09)	(-0.93)	(-0.61)	(-0.91)
Lt rev	-0.004	0.000	-0.001	-0.004	-0.002	0.001	-0.003
	(-1.72)	(0.07)	(-0.54)	(-1.51)	(-1.47)	(0.48)	(-1.71)
Ivol	-0.990	0.309	0.199	-0.852	-0.763	0.371	-0.719
	(-1.01)	(0.99)	(0.78)	(-1.02)	(-1.00)	(0.99)	(-1.00)
Max	1.785	1.802	1.787	1.778	1.792	1.801	1.793
	(43.86)	(45.50)	(44.59)	(41.11)	(48.11)	(45.32)	(48.11)
Min	-1.775	-1.776	-1.760	-1.775	-1.777	-1.775	-1.777
	(-28.37)	(-29.12)	(-34.62)	(-28.15)	(-28.56)	(-29.19)	(-28.37)

Table 3.10: Fama-MacBeth regression for different prospect variables

The results in Table 3.10 suggest that the PW function is most responsible for the predictive power of the PTV variable. In the four cases where the weighting function appears (columns (2), (5), (6) and (7)), the significance of the result increases. This evidence is related to that found in the robustness tests in Section 3.4.2.2, where changing the PW function substantially changes the significance of the PTV variable, although the relationship between the PTV and the return remains negative.

3.4.2.4
Evidence in Emerging Markets

In the previous sections, we found strong evidence for the Brazilian market that PTV and stock returns are negatively related, i.e., stocks with a high PTV have a subsequent low return.

In this section, we use a portfolio analysis to study the same phenomenon for several emerging countries. We selected one emerging country per continent:

Russia (Europe), China (Asia), Mexico (North America) and South Africa (Africa). The sample periods for the returns are from January 1994 to August 2019, from January 1992 to August 2019, from January 1988 to August 2019 and from January 1990 to August 2019, respectively.

We carried out our study by analyzing only EW portfolios, since in this case, as noted by [5], the individual investor has a greater weight in the results. Moreover, weighted portfolios showed no conclusive results in most cases.

The results suggest different behaviors for the countries under analysis. In China and Russia, as in Brazil, there is a negative relationship between the PTV and the subsequent return. Contrary to the results we found for Brazil, China and Russia, as well as the results in [5] for the US market, we found a positive relationship between the PTV and the subsequent returns in Mexico. In South Africa, the only country for which we analyzed EW and VW portfolios, we found a positive relationship in EW portfolios and a negative relationship in VW portfolios.

These different results reinforce the arguments of [29], [28] and [19]: different investor behaviors may be due to cultural and socioeconomic factors.

We will now perform a more detailed analysis of each country.

As in Brazil and the United States, China and Russia also present the same narrow framing bias. The negative relationship between the PTV and the subsequent return is shown in Table 3.11 for China and in Table 3.12 for Russia. In both tables, the results presented denote the values of the P1 − P8 portfolios, with the value of the respective t-statistic shown in parentheses. Panel A corresponds to the returns constructed in the month subsequent to the portfolio formation, and Panel B shows the result with a one-month skip. As in the case of Brazil, we tested a number of scenarios by using the different weighting functions presented in Table 3.1 and different windows for the distribution of past returns. We also skipped one month from the portfolio formation to the return calculation.

In the case of both China and Russia, in terms of statistical significance, the best performance is reflected by the TK weighting function and a three-year window.

Another important point is that similar to the case in Brazil, in both China and Russia, we also noticed a decrease in the values of the returns when we skipped one month. This finding was reflected with greater statistical significance in certain situations, such as in the TK function in both Table 3.11 and Table 3.12 and in the 3-year window in Table 3.11 and the 2- and 3-year windows in Table 3.12.

In the case of Mexico, on average, we were able to identify a significantly

Panel A: subsequent month				
	5 years	4 years	3 years	2 years
TK	**0.0082**	**0.0082**	**0.0114**	**0.0082**
	(2.30)	(2.25)	(2.79)	(1.97)
GE	0.0016	0.0029	**0.0082**	0.0060
	(1.65)	(1.57)	(2.16)	(1.49)
Pr I	0.0057	**0.0063**	**0.0098**	0.0066
	(1.59)	(1.71)	(2.60)	(1.63)
Pr II	0.0035	0.0054	**0.0093**	**0.0062**
	(1.00)	(1.56)	(2.64)	(1.77)
Panel B: one-month skip				
TK	**0.0066**	**0.0061**	**0.0081**	0.0046
	(1.95)	(1.78)	(1.89)	(1.22)
GE	0.0036	0.0035	0.0054	0.0024
	(1.08)	(0.94)	(1.43)	(0.66)
Pr I	0.0045	0.0048	**0.0074**	0.0020
	(1.30)	(1.28)	(1.85)	(0.57)
Pr II	0.0018	0.0037	0.0060	0.0035
	(0.54)	(1.09)	(1.65)	(1.06)

Table 3.11: Portfolio analysis for China

Panel A: subsequent month				
	5 years	4 years	3 years	2 years
TK	**0.0109**	**0.0118**	**0.0212**	**0.0290**
	(2.18)	(2.06)	(3.80)	(3.66)
GE	0.0085	0.0067	**0.0165**	**0.0207**
	(1.32)	(0.72)	(2.29)	(2.42)
Pr I	0.0077	0.0070	**0.0174**	**0.0210**
	(1.22)	(1.18)	(2.48)	(2.62)
Pr II	0.0055	0.0040	**0.0145**	**0.0156**
	(0.84)	(0.68)	(1.93)	(1.80)
Panel B: one-month skip				
TK	0.0083	**0.0099**	**0.0154**	**0.0246**
	(1.61)	(1.90)	(2.77)	(3.13)
GE	0.0043	0.0043	**0.0121**	**0.0170**
	(0.74)	(0.65)	(1.79)	(2.03)
Pr I	0.0057	0.0039	**0.0122**	**0.0164**
	(0.98)	(0.66)	(1.78)	(2.08)
Pr II	0.0065	0.0030	0.0059	0.0110
	(1.03)	(0.48)	(0.85)	(1.38)

Table 3.12: Portfolio analysis for Russia

positive relationship between the PTV and the returns. In Table 3.13, we show the difference in mean values between P1 and P8 for 5-, 4-, 3-, and 2-year windows and the weighting functions presented in Table 3.1. Panel A presents the return calculated for the month subsequent to the portfolio formation, and Panel B presents the results for the one-month skip in the return calculation.

In all the cases tested, we can observe a significant positive relationship between PTV and stock returns. In other words, high (low) PTVs correspond to high (low) returns. Moreover, in terms of magnitude, the returns are virtually unchanged in all the windows of the first three weighting functions (TK, GE, Pr I). Only in the Pr II function is there a slight decline.

Panel A: subsequent month				
	5 years	4 years	3 years	2 years
TK	**-0.0110**	**-0.0159**	**-0.0157**	**-0.0119**
	(-2.14)	(-3.47)	(-3.04)	(-2.22)
GE	**-0.0154**	**-0.0158**	**-0.0157**	**-0.0120**
	(-2.78)	(-2.94)	(-2.55)	(-2.19)
Pr I	**-0.0152**	**-0.0169**	**-0.0188**	**-0.0144**
	(-2.70)	(-3.06)	(-3.11)	(-2.69)
Pr II	**-0.0115**	**-0.0138**	**-0.0145**	**-0.0106**
	(-2.03)	(-2.54)	(-2.37)	(-1.88)
Panel B: one-month skip				
TK	**-0.0111**	**-0.0156**	**-0.0145**	**-0.0115**
	(-2.28)	(-3.39)	(-2.73)	(-2.26)
GE	**-0.0154**	**-0.0165**	**-0.0156**	**-0.0127**
	(-2.79)	(-3.18)	(-2.54)	(-2.28)
Pr I	**-0.0151**	**-0.0161**	**-0.0169**	**-0.0139**
	(-2.85)	(-3.18)	(-2.77)	(-2.61)
Pr II	**-0.0116**	**-0.0130**	**-0.0134**	**-0.0098**
	(-2.04)	(-2.46)	(-2.27)	(-1.79)

Table 3.13: Portfolio analysis for Mexico

In the case of South Africa, we analyzed not only the case of equal-weighted portfolios (EW) but also value-weighted portfolios (VW). We chose to show both cases because the results suggest a mixed behavior. While in the EW case, we have strong evidence of a positive relationship between PTV and stock returns, in the VW case, the behavior is reversed.

In the VW case, when we use the TK weighting function, the results are significant in both the different windows and in the one-month skip for constructing the return. For the other weighting functions, some of the statistical significance was lost, but the behavior remains.

For the weighting functions from Table 3.1, Table 3.14 shows the difference in the mean values between P1 and P8 and the respective t-values for 5-, 4-, 3- and 2-year windows. Panel A shows the results for EW portfolios, and Panel B shows the results for VW portfolios.

Panel A: subsequent month				
	5 years	4 years	3 years	2 years
TK	**-0.0049**	-0.0043	-0.0026	**-0.0071**
	(-2.04)	(-1.56)	(-0.78)	(-1.98)
GE	**-0.0075**	**-0.0079**	-0.0049	**-0.0099**
	(-2.72)	(-2.72)	-1.32	(-2.56)
Pr I	**-0.0072**	**-0.0075**	-0.0051	**-0.0096**
	(-2.51)	(-2.63)	-1.39	(-2.56)
Pr II	**-0.0063**	**-0.0070**	-0.0051	**-0.0093**
	(-2.04)	(-2.22)	-1.29	(-2.41)

Panel B: one-month skip				
TK	**0.0167**	**0.0138**	**0.0147**	**0.0143**
	(2.30)	(2.09)	(1.79)	(1.70)
GE	**0.0104**	0.0058	0.0093	0.0000
	(1.79)	(1.00)	(1.26)	(0.00)
Pr I	0.0049	0.0013	0.0026	0.0015
	(0.83)	(0.24)	(0.40)	(0.24)
Pr II	0.0009	-0.0013	-0.0046	-0.0053
	(0.15)	(-0.25)	(-0.88)	(-0.97)

Table 3.14: Portfolio analysis for South Africa

Across the portfolios, considering the TK weighting function, Figure 3.4 shows the evolution of the mean value of the returns. The first graph corresponds to the EW case, and the second graph corresponds to the VW case.

Table 3.15 shows the same analysis as before, but this time, we skip one month between the portfolio formation and the return calculation. In Panel A, we can observe the same behavior for the EW portfolios, with great statistical significance. In Panel B, we observe the same decreasing behavior among the portfolios, but we lose the statistical significance.

3.5 Conclusion

Narrow framing is a psychological bias that causes individuals to treat situations of uncertainty differently when they are dealing with gains or losses. Consistent with [5], in our case, investors analyze past gains and losses at the level of the stock to calculate the PTV. In Brazil, we used both a portfolio analysis (time series) and a Fama-MacBeth analysis to show a negative relationship between the PTV and the subsequent returns. In other words,

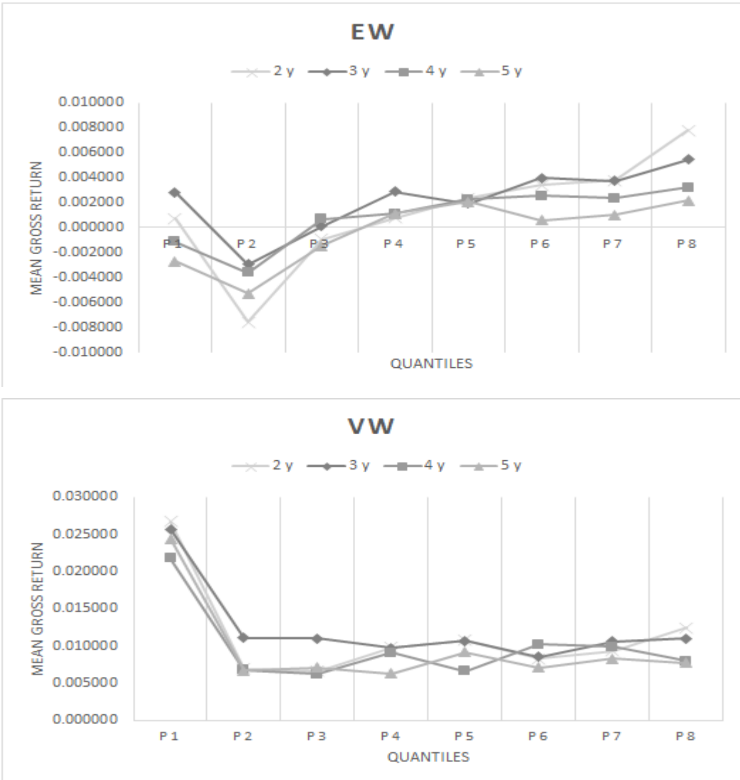


Figure 3.4: Evolution of the mean value of the returns, considering the Tversky-Kahneman (TK) weighting function for South Africa.

Panel A: subsequent month				
	5 years	4 years	3 years	2 years
TK	**-0.0083**	**-0.0075**	**-0.0065**	**-0.0108**
	(-3.56)	(-2.76)	(-1.87)	(-3.08)
GE	**-0.0100**	**-0.0093**	**-0.0080**	**-0.0141**
	(-3.62)	(-2.92)	(-2.05)	(-3.61)
Pr I	**-0.0103**	**-0.0098**	**-0.0093**	**-0.0136**
	(-3.59)	(-3.41)	(-2.43)	(-3.46)
Pr II	**-0.0079**	**-0.0093**	**-0.0091**	**-0.0143**
	(-2.64)	(-3.08)	(-2.26)	(-3.60)
Panel B: one-month skip				
TK	0.0114	0.0113	0.0142	0.0108
	(1.68)	(1.59)	(1.50)	(1.34)
GE	0.0061	0.0047	0.0090	-0.0004
	(0.84)	(0.79)	(1.39)	(-0.05)
Pr I	0.0016	0.0009	0.0034	-0.0003
	(0.26)	(0.16)	(0.55)	(-0.04)
Pr II	0.0007	0.0001	-0.0026	-0.0082
	(0.11)	(0.02)	(-0.46)	(-1.59)

Table 3.15: Portfolio analysis for South Africa with a one-month skip

corroborating the result of [5] for the US market, on average, stocks with high (low) PTVs have a subsequent low (high) return. For other emerging countries, i.e., China, Russia, Mexico and South Africa, we used only portfolio analysis and found results similar to those for Brazil for the first two countries. In the case of Mexico, we found a significantly positive relationship between PTV and returns, and for South Africa, we found a positive relationship for EW portfolios and a negative relationship for VW portfolios. In all cases studied, the result was persistent when we used different probability weighting functions, different time windows, and when a month is skipped. Future studies could deepen the mathematical analysis of the relationships in this model by producing empirical results. Studies have been performed in experimental markets, but there has not yet been any intersection of these studies with stock markets and stock pricing.

4
Essay II: An Axiomatization of the Goldstein-Einhorn Weighting Functions

4.1
Introduction

In this essay we propose an axiomatization for the *Goldstein-Einhorn Probability Weighting Functions* ([10]). It was introduced by [10], adding a linear coefficient in the formulation of [9] and its functional form is given by,

$$w(p) = \frac{ap^b}{ap^b + (1-p)^b}, \quad p \in [0,1]. \tag{4-1}$$

Since then, Goldstein-Einhorn weighting function has been used across a wide spectrum. In an experimental study, [18] tested many weighting functions and found that the two-parameter weighting functions suggested by [10] and [11] performed very well, modeling two important psychological properties, discriminability and attractiveness. Recently, [16] designed a simulation and choice experiment to discriminate among weighting functions. Again, [10] and [11] were the best-fitting models.

In [38], the author used (4-1) to study the violation of branch independence. In [41], the author studied binary gambles and three other kinds of violations: violations of complementary symmetry, violations of consequence monotonicity and of first order stochastic dominance. Violation of coalescing and stochastic dominance was studied by [40].

The most common weighting functions, including (4-1), was applied in [14] to explain important phenomenons like the equity premium puzzle, the long-shot bias in betting markets, and households' under-diversification and their willingness to buy small-scale insurance at exorbitant prices. Using Goldstein-Einhorn weighting function, [27] modeled the distribution of risk taking types in three different experimental data sets, two Swiss and one Chinese.

Considering the theoretical scenario, [17] provides an alternative way to obtain (4-1) based on the concept of indifference prices.

Regarding axiomatization, it is important to build a theoretical safe ground for the theory. In our case, it is important to comprehend what kind of behavior are inferred by such weighting function.

Our axiomatization is represented by a preference condition, which is designed to understand how people make decisions in a risk or uncertain scenario.

Roughly speaking, if one uses a specific weighting function in an empirical work, it means that group of people under consideration have the behavior represented by the associated preference condition. On the other hand, if one designs an experiment to identify the preference condition (risk behavior) of a group, then axiomatization tell what is the weighting function has to be used.

For instance, considering a rare event like a pandemic scenario, a company may consider releasing a job loss insurance. The company needs to understand the level of attractiveness of this product. The first step to do that is to perform an experiment to identify the preference condition and then the weighting function that most fit this group of people. The next step and the whole decision process will be explained in the third essay. Doing this, we will connect both essays (second and third) with a practical application.

Starting in 1979, efforts have been made to build an axiomatization through which these weighting functions can be deduced. An example of such work is [7], which presented the theory of disappointment and proposed the functional form

$$w(p) = \frac{p}{1+(1-p)\eta}.$$

In [11], the author used the common ratio effect to propose,

$$w(p) = e^{(-(-\ln(p))^{\alpha})}$$

and its extension

$$w(p) = e^{(-\beta(-\ln(p))^{\alpha})}.$$

Although it was not the focus of their paper, [18] suggested a preference condition that was necessary and sufficient to get the Goldstein-Einhorn weighting function (equation (4-1)). They formally posed in the appendix as Theorem 1. Since then, the theorem has been reported in several works, [13], [14], [15], [16] and [17].

The purpose of this essay is to discuss the preference condition they proposed and show that it leads us to a wider set of solutions. We present two propositions that found the solution under two different perspectives. Modifying their approach, we present a new axiomatization for the Goldstein-Einhorn functions.

Using these propositions, we study some instructive examples that are helpful not only to understand our solution but also to get insights about psychological characteristics of *discriminability* and *attractiveness*. Those features are very well analyzed by [18].

The remainder of this essay is organized as follows. In Section 4.2, we discuss the preference condition proposed by [18]. In Section 4.3, we present the solutions and provide some examples and characterizations. In Section 4.4 we suggest a new axiomatization. In Section 4.5 we give the conclusions.

4.2
Comments on Gonzalez and Wu (1999)

In this section we discuss the preference condition proposed in [18]. It is important to note that the authors developed an experimental work, so the analysis of the preference condition was not the focus of their paper.

The discussion is carried out under Cumulative Prospect Theory (CPT) and Rank-Dependent Utility Theory (RDU). Generally, the two theories are different but following [18], we use positive outcomes (gains) and then these theories coincide. Analogous conditions can be written for the case of negative outcomes (losses).

Let G denote the set of non-negative two-outcome gambles, $g = (X, p; Y, 1-p)$, where one gets the outcome $X \geq 0$ with probability p or $Y \geq 0$ with probability $1-p$. A preference relation $\precsim$ is assumed over G. In addition, $\prec$ denotes strict preference and $\sim$ denotes indifference.

We represent the preference relation over G by a mapping, U, from G into the set of real numbers, $\mathbb{R}$, such that for all $g_1, g_2 \in G$,

$$g_1 \precsim g_2 \Leftrightarrow U(g_1) \leq U(g_2).$$

Under CPT (or RDU), the preference relation, G, is represented by

$$U(X, p; Y, 1-p) = u(X)w(p) + u(Y)(1 - w(p)) \tag{4-2}$$

where the value function $u : [0, \infty[\to \mathbb{R}$ is continuous and strictly increasing, and the weighting function $w : [0, 1] \to [0, 1]$ is continuous, strictly increasing, $w(0) = 0$ and $w(1) = 1$ (for more details, [4], [39], [18]).

In our framework, because of relations like (4-10), it is convenient to work with w in the open interval $]0, 1[$. Keeping all characteristics of w, instead of emphasizing the assumptions $w(0) = 0$ and $w(1) = 1$, we will assume that the weighting function $w :]0, 1[\to]0, 1[$ is continuous, strictly increasing and onto. [11] and [36] also assume this hypothesis.

We provide the preference condition in [18] as follow,

Preference Condition 4.1 *Suppose the structure of CPT for two-outcome gambles with value function $u : [0, \infty[\to \mathbb{R}$, continuous and strictly increasing, and weighting function w $:]0,1[\to]0,1[$, continuous, strictly increasing and onto. The preference condition is said to hold if, for all non-negatives outcomes $X > X'$ and $Y'' > Y' > Y$, and positive probabilities $\{p, q\}$, the following implication holds*

$$\begin{aligned}&(X, p; Y, 1-p) \sim (X', p; Y', 1-p), (X, p; Y', 1-p) \sim (X', p; Y'', 1-p)\\&\text{and } (X, q; Y, 1-q) \sim (X', q; Y'', 1-q)\end{aligned} \tag{4-3}$$

imply

$$\text{(4-3) holds for } p, q \text{ replaced by } \frac{tp}{1-p+tp} \text{ and } \frac{tq}{1-q+tq}, t > 0. \tag{4-4}$$

As the authors pointed out, the intuition behind the need of specifying the condition in terms of $\overline{p} = \frac{tp}{1-p+tp}$ and $\overline{q} = \frac{tq}{1-q+tq}$ is that the odds ratio of each pair of probabilities are identical, i.e.,

$$\frac{p/(1-p)}{q/(1-q)} = \frac{\overline{p}/(1-\overline{p})}{\overline{q}/(1-\overline{q})}.$$

This odds ratio (constant c in Proposition 4.3) will play an important role in the solution of Problem 4.2.

Following [18] and using (4-2), the three indifference relations in (4-3) are equivalent to,

$$\frac{w(p)}{1-w(p)} = \frac{u(Y') - u(Y)}{u(X) - u(X')}, \tag{4-5}$$

$$\frac{w(p)}{1-w(p)} = \frac{u(Y'') - u(Y')}{u(X) - u(X')}, \tag{4-6}$$

and

$$\frac{w(q)}{1-w(q)} = \frac{u(Y'') - u(Y)}{u(X) - u(X')}. \tag{4-7}$$

The value function u being continuous and strictly increasing, and $X > X'$ and $Y'' > Y' > Y$, the right hand side of (4-5), (4-6), (4-7) are well defined and positive. In addition, we just have to ensure that $Y' = u^{-1}\left(\frac{u(Y)+u(Y'')}{2}\right)$ to make (4-5) and (4-6) hold together. The bijective property of w on $]0,1[$ ensures the existence of positive (p, q) in (4-5) and (4-7).

Adding (4-5) and (4-6) side by side and comparing the result with (4-7) we get

$$2\frac{w(p)}{1-w(p)} = \frac{w(q)}{1-w(q)}. \tag{4-8}$$

So (4-3) conveys the relation (4-8).

By the same considerations (4-4) implies

$$2\frac{w\left(\frac{tp}{1-p+tp}\right)}{1-w\left(\frac{tp}{1-p+tp}\right)} = \frac{w\left(\frac{tq}{1-q+tq}\right)}{1-w\left(\frac{tq}{1-q+tq}\right)}. \tag{4-9}$$

Let $f:]0,\infty[\to]0,\infty[$ be defined by

$$f\left(\frac{p}{1-p}\right) = \frac{w(p)}{1-w(p)}. \tag{4-10}$$

Combining the equations (4-8), (4-9) and (4-10), Preference Condition 4.1 is translated into the implication $2f(x) = f(y) \Rightarrow 2f(tx) = f(ty)$.

In addition, our hypothesis of the weighting function w being continuous, strictly increasing and onto is transferred to f by (4-10). That motivates the following problem,

Problem 4.2 *Find all continuous, strictly increasing and onto functions* $f:]0,\infty[\to]0,\infty[$ *satisfying*

$$2f(x) = f(y) \quad \textit{implies} \quad 2f(tx) = f(ty), \quad \forall t > 0. \tag{4-11}$$

It will be treated in Section 4.3.

The proof presented by Gonzalez and Wu (1999) is based on the solution of the functional equation (induced by)

$$f(x) + f(y) = f(z) \quad \textit{implies} \quad f(tx) + f(ty) = f(tz), \quad \forall t > 0. \tag{4-12}$$

This functional equation under some continuity conditions has $f(x) = ax^b$ as the only solution. Later we will revisit this functional equation and provide a bit more information.

Note that if we set $x = y$ in (4-12) we get (4-11). Roughly speaking, the variables in functional equations can be thought as constrains of the solution. In this sense, the solution of the functional equation (4-12) is more restrictive than that of (4-11).

4.3 Solution and analysis of Problem 4.2

In this section we will solve Problem 4.2 as an independent problem. We will establish and prove a series of propositions that not only solve the problem but also help us to construct some instructive particular solutions.

Our first result links Problem 4.2 to a particular case of the very well known Schröder functional equation.

Proposition 4.3 *Let* $f:]0,\infty[\to]0,\infty[$ *be a continuous, strictly increasing and onto function. Then the implication*

$$2f(x) = f(y) \quad \textit{implies} \quad 2f(tx) = f(ty), \quad \forall t > 0 \tag{4-13}$$

holds if and only if, for some constant $c \in]0,1[$*, f satisfies (a Schröder functional equation)*

$$f(cz) = \frac{1}{2} f(z), \quad \forall z > 0. \tag{4-14}$$

The proof is in the Appendix A. Following the proof, we show that the constant c in equation (4-14) is equal to x/y which is the odds ratio mentioned in the motivation of the preference condition suggested by Gonzalez and Wu (1999).

Theorem 2.10 in Kuczma (1968) states that the the equation (4-14) has a continuous solution depending on an arbitrary function. Tracing back the construction, we have the following result.

Proposition 4.4 *(i) Let* $f_0 : [c,1] \to]0,\infty[$ *be any strictly increasing continuous function whose values at the two end points satisfy the relation* $f_0(c) = \frac{1}{2} f_0(1)$*. Then it has a unique extension,* f*, on* $]0,\infty[$ *satisfying* (4-14)*. Moreover,* f *is continuous and strictly increasing.*

(ii) $f(z) = 2^{-k} f_0(c^{-k} z)$*, for* $z \in]c^{k+1}, c^k]$ *and* $k \in \mathbb{Z}$*.*

(iii) $f(z) > 0$ *for all* $z \in]0,\infty[$*.*

(iv) $\lim_{z\to 0} f(z) = 0$ *and* $\lim_{z\to\infty} f(z) = \infty$*.*

It means that, once f_0 is defined on $[c,1]$, the function f defined by (ii) is the unique extension satisfying (4-14). In view of $\lim_{z\to 0} f(z) = 0$, if we postulate $f(0) = 0$, we get a further continuous extension which satisfies (4-14) on the extended interval $[0,\infty[$.

Next proposition (Proposition 4.5) could be considered as a more elegant form of the solution of (4-14), in the sense that it clearly includes the Goldstein-Einhorn family as the special case of a constant ϕ, that it brings out more hidden parameters - the Fourier coefficients. The drawback is that not all periodic ϕ yields strictly increasing f.

Proposition 4.5 *Let strictly increasing and continuous* $f :]0,\infty[\to]0,\infty[$ *satisfy* (4-14)

$$f(cz) = \frac{1}{2} f(z), \quad \forall z > 0,$$

where $0 < c < 1$ *is a fixed constant. Let* $b > 0$ *be given. Let* $u := \log_2 z$ *and define* ϕ *by*

$$f(z) = \phi(\log_2 z) z^b, \quad \forall z > 0. \tag{4-15}$$

Then (4-14) *is translated into*

$$\phi(\log_2 c + u) = \frac{1}{2c^b}\phi(u), \quad \forall u \in \mathbb{R} \tag{4-16}$$

(which is a Schröder equation in additive form). Conversely, if ϕ satisfies (4-16), *and f is defined by* (4-15) *then f satisfies* (4-14). *The continuity of f corresponds to that of ϕ, but the strict monotonicity of f does not transfer to ϕ. Clearly, by* (4-16),

$$\phi \quad \textit{is periodic with } \log_2 c \textit{ as a period iff} \quad 2c^b = 1, \ \textit{i.e.} \quad b = -(\log_2 c)^{-1}. \tag{4-17}$$

For the periodic ϕ mentioned in (4-17), we take $\Omega := -\log_2 c = b^{-1}$. A further discussion of periodic solution and Fourier Series is left to the Appendix A.

4.3.1 Examples

In the previous section, Proposition 4.4 and Proposition 4.5 presented us not only the solution of Problem 4.2 but also two ways to build examples for the solutions.

In this section, we build two instructive examples. They are helpful in demonstrating the construction of the general solution, and in getting insights about psychological characteristics of the resulting weighting functions. The first example uses the periodic case of ϕ based on Proposition 4.5. The second example uses Proposition 4.4 to build a piecewise linear solution which is easy to analyse and give us many insights about the mathematical features of the solutions.

4.3.1.1 Using Periodic ϕ

The example is based on Proposition 4.5. It takes the form (4-15) and the periodic case (4-17) with $c = 1/2$ and $b = 1$. That leads to the family

$$f(z) = \left[\sin^2(\pi k \log_2 z) + 20\right] z, \quad z \in]0, \infty[. \tag{4-18}$$

It potentially carries some particular solutions of Schroder's equation ($c = 1/2$)

$$f\left(\frac{z}{2}\right) = \frac{1}{2} f(z), \quad z \in]0, \infty[\tag{4-19}$$

which are strictly increasing and continuous. The continuity is clear, we just have to examine its growth.

Consider the case $k = 1$ in (4-18),

$$f(z) = \left[\sin^2(\pi \log_2 z) + 20\right] z, \quad z \in]0, \infty[. \tag{4-20}$$

Let us check if f is increasing. Consider its derivative

$$f'(z) = \frac{\pi}{\ln 2} \sin(2\pi \log_2 z) + \sin^2(\pi \log_2 z) + 20. \tag{4-21}$$

Since $\sin(\cdot)$ falls in $[-1, 1]$, it is easy to check that $f'(z) > 0$ for all $z \in]0, \infty[$. This confirms that with $k = 1$, f is indeed increasing.

Observe that

- f has a unique continuous extension to the closed interval $[0, \infty[$ by taking the definition $f(0) = 0$. It is then increasing and Schröder's equation holds on the extended interval;
- Looking at the expression (4-21), we see that f is many times differentiable on the open interval. However, $f'(z)$ has no limit as z tends to 0 (from the right). So (extended) f is not in the class C^1 on the closed interval $[0, \infty[$.

Now let us analyse the case of $k = 2$ in (4-18). In parallel with the previous case of $k = 1$, this is what we can say.

The function

$$f(z) = \left[\sin^2(2\pi \log_2 z) + 20\right] z, \quad z \in]0, \infty[\tag{4-22}$$

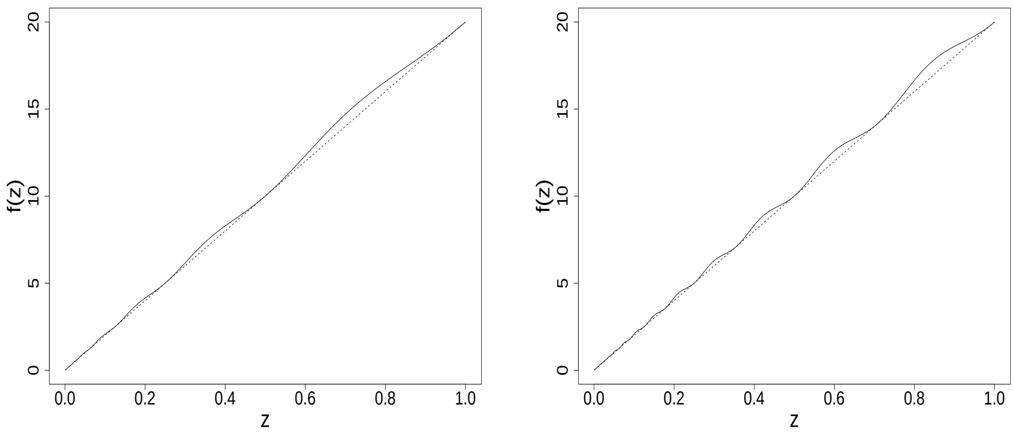
is strictly increasing because its derivative

$$f'(z) = \frac{2\pi}{\ln 2} \sin(4\pi \log_2 z) + \sin^2(2\pi \log_2 z) + 20$$

is checked positive on the open interval by similar arguments (shown for the case of $k = 1$). This confirms that (4-22) is also a (strictly) increasing solution of (4-19).

As we can see in Figure 4.1, the function (4-18) is a perturbation of the function $f(z) = 20z$ and the size of perturbation is given by the period controlled by k. For bigger k (say $k = 6$) we do not get an increasing f and the respective weighting function w is not increasing.

Figure 4.2 shows the corresponding probability weighting function w given by (4-20) and (4-22) through (4-10). It shows overoptimistic individuals once the weighting functions appears to be concave everywhere. This is the



(a) Graph of the function (4-20) (b) Graph of the function (4-22)

Figure 4.1: Plotting of the function ((4-18)) for $k = 1$ and $k = 2$. It can be seen as a perturbation of $f(z) = 20z$ (dashed line).

case of some subjects found in the experiments designed by Gonzalez and Wu (1999).

4.3.1.2
Piecewise linear solution

The example is based on Proposition 4.4. A simple f which is easy, though not differentiable, is our starting point. We will assume $c = 1/3$, and that the initial function f_0 on $[1/3, 1]$ is linear. So the resulting function f is piecewise linear. The disadvantage being non-differentiable at powers of c, and the advantage is that it is so simple. It is easy to visualize if the piecewise linear function is concave or convex without differentiability.

The initial function

$$f_0(z) = 3z + 1, \quad z \in [1/3, 1]$$

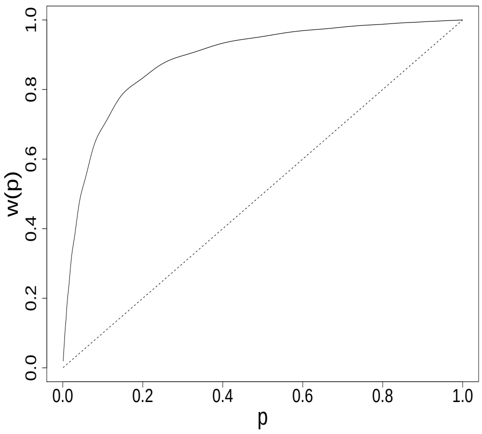
is strictly increasing, continuous, and $f_0(1/3) = \frac{1}{2} f_0(1)$ is satisfied.

Following part (ii) of Proposition 4.4, the unique extension f is given by

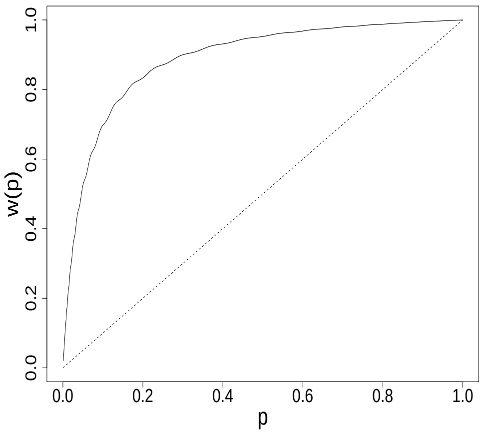
$$f(z) = 2^{-k}(3^{k+1}z + 1), \quad z \in]3^{-(k+1)}, 3^{-k}], \quad k \in \mathbb{Z}. \tag{4-23}$$

It is strictly increasing and continuous on $]0, \infty[$. Furthermore, it is piecewise linear, concave and not differentiable at 3^{-k} for every integer value of k as we can visualize in Figure 4.3.

The associated weighting function w given by (4-10) has graph illustrated in Figure 4.4. It suggests that the weighting function w is continuous, increasing, regressive and piecewise concave-convex. We prove that in the Appendix



(a) Graph of $w(p)$ derived from (4-20)



(b) Graph of $w(p)$ derived from (4-22)

Figure 4.2: Probability Weighting Functions derived from (4-18) for $k = 1$ and $k = 2$. The dashed line is the Identity Function.

A. Furthermore w is not differentiable at the points $\frac{3^{-k}}{1+3^{-k}}$.

The term piecewise concave-convex is an abbreviation for the fact that the weighting function shown in the Figure 4.4 is concave on the connected region covered by the intervals with $k > -1$, and is convex on each interval with $k \leq -1$. The piecewise convex part can be seen when we plot $w(p)$ for $k \in \{-4, -3, -2, -1\}$ in Figure 4.5. It is not convex around the points $\frac{3^{-k}}{1+3^{-k}}$.

Now, let us analyse the general case of a piecewise linear function. For general c, the linear initial function such that $f_0(c) = \frac{1}{2} f_0(1)$ is given by

$$f_0(z) = a(z + 1 - 2c), \quad z \in [c, 1], \quad a > 0.$$

Its extension is

$$f(z) = a[(2c)^{-k} z + 2^{-k}(1 - 2c)], \quad z \in]c^{k+1}, c^k]. \tag{4-24}$$

The family given by (4-24) gives us a lot of information about the behavior of the probability weighting functions. We can see how the parameters a and c influences the inverse s-shape, the fixed point, the attractiveness and the discriminability of the weighting functions. For more details about those important characteristics see for example Wu and Gonzalez (1996), Gonzalez and Wu (1999), Diecidue et al (2009), Abdellaoui et al (2010), Webb and Zank (2011).

From (4-24), if $c < 1/2$ then f is concave and if $c > 1/2$ it is convex. It influences the concave-convex form of w. More precisely, if $c < 1/2$ then w is piecewise concave-convex and if $c > 1/2$ then w is piecewise convex-concave. In addition, the values of a primarily determine the value of the fixed point

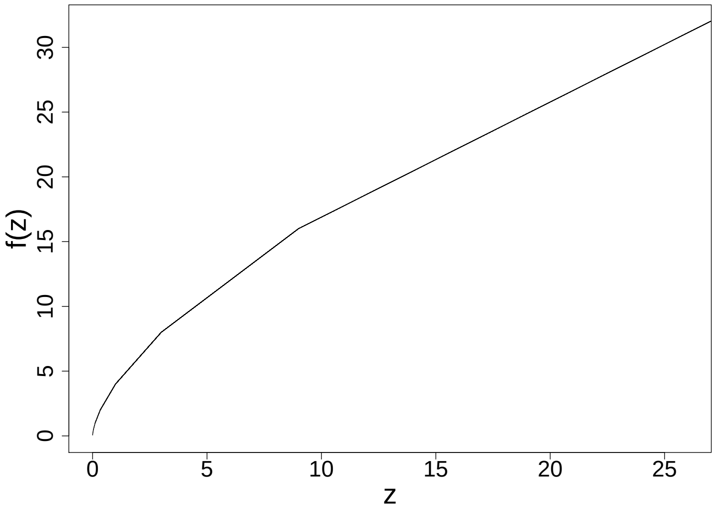


Figure 4.3: Piecewise Linear Function given by (4-23)

of w. Figure 4.6 shows that conclusions. In the Appendix A, we provide a mathematical proof of these claims.

Furthermore there is plenty of empirical works suggesting that in most of the cases, the weighting function is concave-convex, so $c < 1/2$ gets more attention. Regards to the attractiveness and discriminability, the value of a primarily takes care of the attractiveness and the value of c deals with the discriminability. See Figure 4.7.

4.3.2 Condition on ϕ for strictly increasing solution f

Relation (4-15) connects f and ϕ. It is clear that continuity of f correspond to that of ϕ, but the strictly monotinicity of f does not transfer to ϕ. In this section we establish a necessary and sufficient condition for ϕ such that f is strictly increasing.

If f is differentiable we can propose Proposition 4.6.

Proposition 4.6 *f is strictly increasing on $]0, \infty[$ if and only if $f' \geq 0$ on $]0, \infty[$ and $f' > 0$ on a dense subset of $]0, \infty[$.*

A more compact relation can be found if we consider the variable $u = \log_2 z$. Because $\frac{du}{dz} > 0$, it follows from the Chain Rule that

$$\frac{df}{dz} > 0 \Leftrightarrow \frac{df}{du} > 0 \quad \text{and} \quad \frac{df}{dz} = 0 \Leftrightarrow \frac{df}{du} = 0. \tag{4-25}$$

We already know that $f(z) > 0$ ($\forall z > 0$) and thus $\phi(u) > 0$ ($\forall u \in \mathbb{R}$). Since

$$\frac{df}{du} = \phi'(u)2^{bu} + (b \ln 2)\phi(u)2^{bu},$$

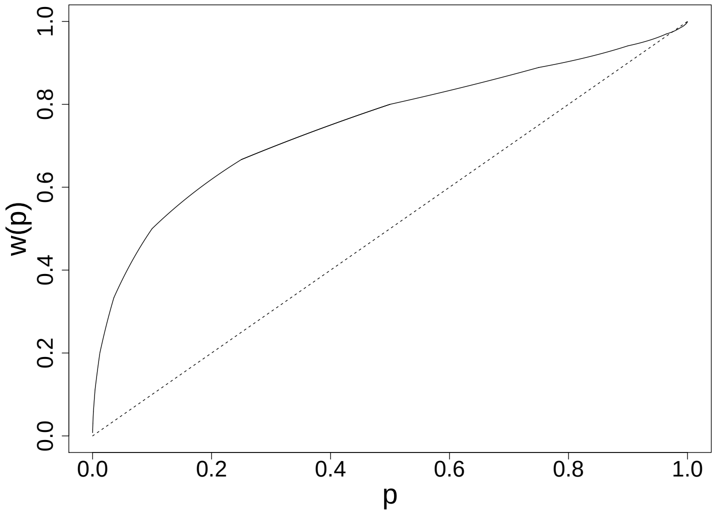


Figure 4.4: Piecewise Weighting Function associated to the Piecewise Linear Function. The dashed line is the Identity Function.

we have

$$\frac{df}{du} > 0 \Leftrightarrow \frac{\phi'(u)}{\phi(u)} > -b\ln 2 \quad \text{and} \quad \frac{df}{du} = 0 \Leftrightarrow \frac{\phi'(u)}{\phi(u)} = -b\ln 2. \tag{4-26}$$

Introducing the function

$$\Phi(u) := \log_2[\phi(u)] \tag{4-27}$$

we rewrite (4-26) in a more compact form

$$\frac{df}{du} > 0 \Leftrightarrow \ \Phi'(u) > -b \quad \text{and} \quad \frac{df}{du} = 0 \Leftrightarrow \Phi'(u) = -b. \tag{4-28}$$

In view of (4-25), (4-28) and Proposition 4.6, we arrive at the following necessary and sufficient condition on ϕ (in the case of a differentiable f).

Proposition 4.7 *f is strictly increasing on $]0,\infty[$ if and only if $\Phi'(u) \geq -b$ ($u \in \mathbb{R}$) and $\Phi'(u) > -b$ on a dense subset of $\mathbb{R}$.*

If f is not differentiable, the best we can do is the discrete formulation of the relation (4-28).

Proposition 4.8 *Suppose that $z_1 > z_0$ then $u_1 = \log_2 z_1 > u_0 = \log_2 z_0$. In this case, f is strictly increasing if and only if*

$$\frac{\Phi(u_1) - \Phi(u_0)}{u_1 - u_0} > -b.$$

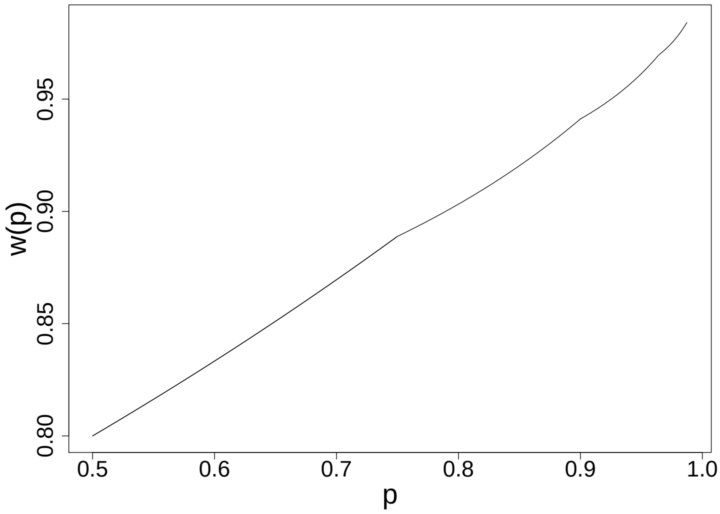


Figure 4.5: Graph of w from $k = -1$ to $k = -4$. It shows that w is piecewise convex, i.e., it is convex by parts but it is not convex around the points $\frac{3^{-k}}{1+3^{-k}}$.

The proof of Proposition 4.8 is in the Appendix A.

4.4 New Preference Condition

As mentioned in the Section 4.2, [18] referred to the following result.

Proposition 4.9 *Let* $f :]0, \infty[\to]0, \infty[$ *be a continuous, strictly increasing and onto function. If the implication*

$$f(x) + f(y) = f(z) \quad \textit{implies} \quad f(tx) + f(ty) = f(tz), \quad \forall t > 0 \tag{4-29}$$

holds then $f(z) = az^b$ *(for some* $a > 0$, $b > 0$*).*

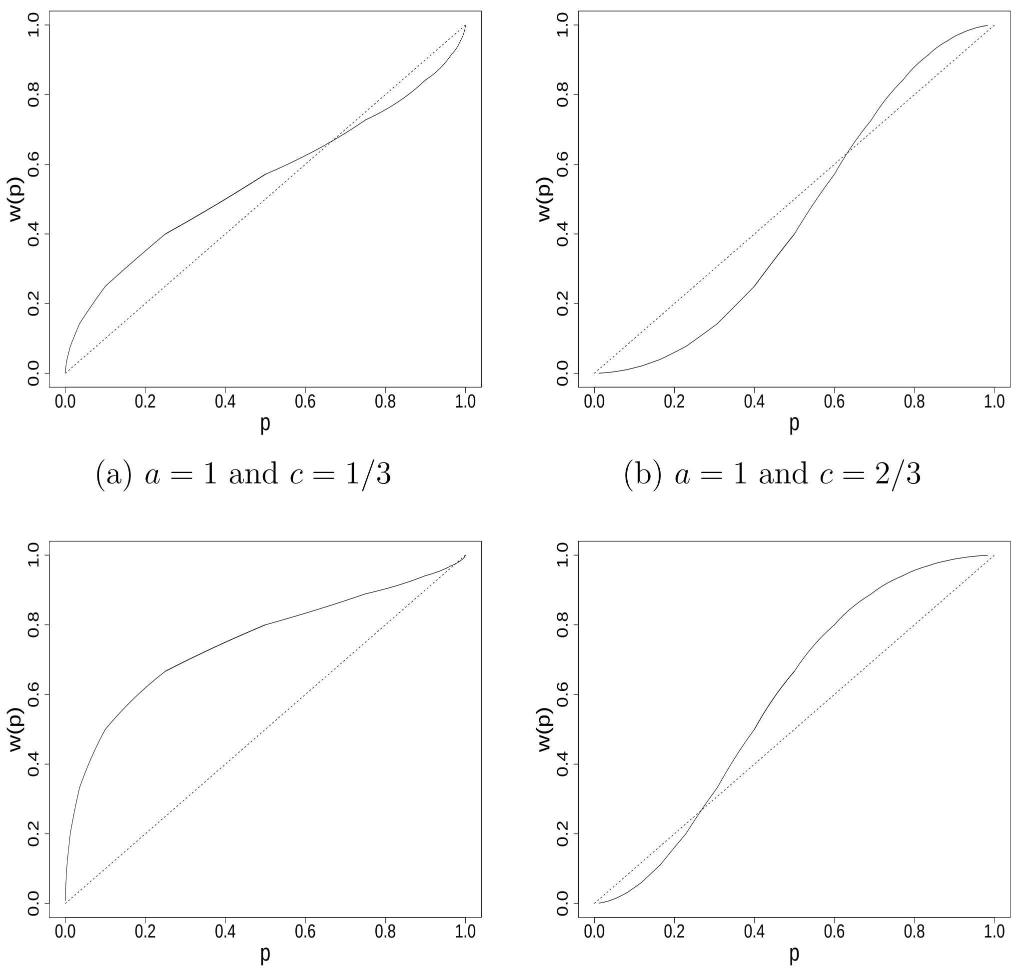
The proof of Proposition 4.9 is in the Appendix A. Since

$$a\left(\frac{p}{1-p}\right)^b = \frac{w(p)}{1-w(p)} \Leftrightarrow w(p) = \frac{ap^b}{ap^b + (1-p)^b},$$

the *Goldstein-Einhron Probability Weighting Functions* correspond to $f(z) = az^b$ $(a > 0, b > 0)$.

In parallel with the relation between the Problem 4.2 and Preference Condition 4.1, we propose Preference Condition 4.10 to tie in with (4-29).

Preference Condition 4.10 *Suppose the structure of CPT for two-outcome gambles with continuous and strictly increasing value function* $u : [0, \infty[\to \mathbb{R}$, *and weighting function* $w :]0, 1[\to]0, 1[$ *which is continuous, strictly increasing and onto. The preference condition is said to hold if, for all non-negative*



(a) $a = 1$ and $c = 1/3$ (b) $a = 1$ and $c = 2/3$

(c) $a = 3$ and $c = 1/3$ (d) $a = 3$ and $c = 2/3$

Figure 4.6: Piecewise Weighting Function derived from 4-24 for different values of a and c. For $c < 1/2$ ((a) and (b)), w is piecewise concave-convex and if $c > 1/2$ ((c) and (d)) w is piecewise-convex-concave. Furthermore, a primarily determine the fixed point and the attractiveness of w

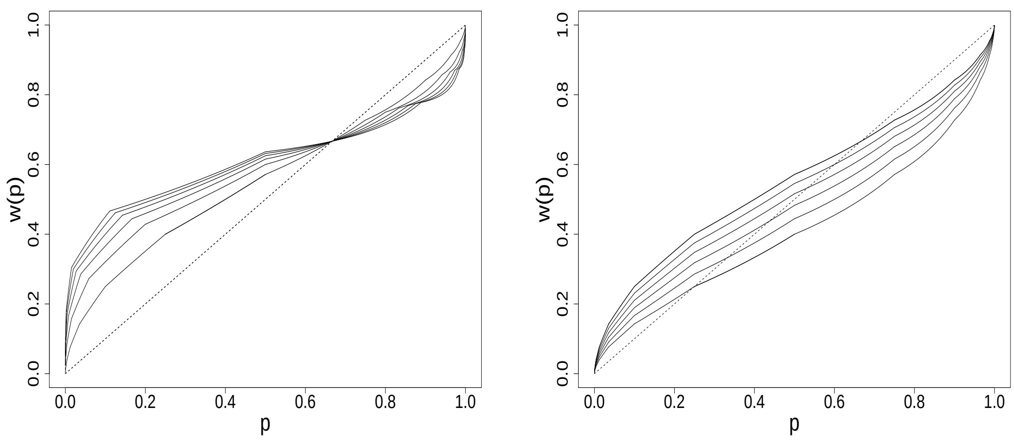
outcomes $X > X'$, $Y'' > Y' > Y$, *and positive probabilities* $\{p, q\}$, *the following implication holds*

$$(X, p; Y, 1-p) \sim (X', p; Y', 1-p), (X, s; Y', 1-s) \sim (X', s; Y'', 1-s)$$
$$\textit{and } (X, q; Y, 1-q) \sim (X', q; Y'', 1-q) \tag{4-30}$$

imply

$$\text{(4-30) } \textit{holds for } p, s, q \textit{ replaced by } \frac{tp}{1-p+tp}, \frac{ts}{1-s+ts}, \frac{tq}{1-q+tq}, t > 0. \tag{4-31}$$

Analogous to the procedure that leads to Problem 4.2 from Preference



(a) $a = 1$ and $c \in \{1/3, 1/4, ..., 1/8\}$ (b) $a \in \{.5, .6, .7, ..., 1\}$ and $c = 1/3$

Figure 4.7: Discriminability and Attractiveness of the Weighting Function w, associated with the general case (4-24). The value of a primarily deals with the Attractiveness of w ((a)) and the value of c controls the Discriminability of w ((b)).

Condition 4.1, Preference Condition 4.10 translates into (4-29) of Proposition 4.9 as indicated below.

The three indifferences in (4-30) are equivalent to

$$\frac{w(p)}{1-w(p)} = \frac{u(Y') - u(Y)}{u(X) - u(X')}, \tag{4-32}$$

$$\frac{w(s)}{1-w(s)} = \frac{u(Y'') - u(Y')}{u(X) - u(X')}, \tag{4-33}$$

$$\frac{w(q)}{1-w(q)} = \frac{u(Y'') - u(Y)}{u(X) - u(X')}. \tag{4-34}$$

The value function u being continuous and strictly increasing, for any chosen $X > X'$ and $Y'' > Y' > Y$, the right hand side of (4-32) to (4-34) are well defined and positive. The assumption that w maps $]0,1[$ bijectively onto $]0,1[$ guarantees the existence of (p, s, q) that meets (4-32), (4-33) and (4-34).

Letting $x = \frac{p}{1-p}$, $y = \frac{s}{1-s}$ and $z = \frac{q}{1-q}$, the above three equations yield $f(x)+f(y) = f(z)$. The three indifference are retained under the substitutions mentioned in (4-31) means that $f(tx) + f(ty) = f(tz)$ follows.

It was mentioned earlier that our hypothesis of the weighting function w being continuous, strictly increasing and onto is transferred to f by (4-10). That assumption on f was entered in Proposition 4.9. Without that, the implication may fail to induce a functional equation.

Using Preference Condition 4.10 and Proposition 4.9, we arrive at the following theorem.

Theorem 4.11 *For non-negative two-outcome gambles under CPT, Preference Condition 4.10 is necessary and sufficient for w be the Goldstein-Einhorn Probability Weighting Functions.*

Theorem 4.11 is the main result of this essay. He build the equivalence relation between the preference condition (risk behavior) and the Goldstein-Einhorn weighting function. As we said in the introduction of this essay, Theorem 4.11 tells us what kind of behavior are inferred by such weighting function.

4.5 Conclusion

Analysing the preference condition presented in [18] (Preference Condition 4.1), and its implied Problem 4.2, we found a large family of weighting functions where the *Goldstein-Einhorn Probability Weighting Functions* constitute a particular sub-family.

We presented two propositions (Proposition 4.4 and Proposition 4.5) that solved Problem 4.2 under two different perspectives. From Proposition 4.4 we built the piecewise linear example that showed us how the concavity of a solution f influences the concavity/convexity of the associated weighting function. This two-parameter family includes the common inverse s-shaped functions found in the literature, not to mention that the larger family of solutions come with far more than two parameters.

Proposition 4.5 clearly encompasses the Goldstein-Einhorn family as a special case (constant function ϕ) and, using Fourier series, it brings out more hidden parameters.

Finally, we provided a new preference condition (Preference Condition 4.10) which is necessary and sufficient to obtain the Goldstein-Einhorn Probability Weighting Functions.

5
Essay III: Measuring Attractiveness and Discriminability

5.1
Introduction

This essay focuses on formal measures of *attractiveness* and *discriminability*. These measurements can be interpreted as a gauge of the departure from the rational behavior.

Our essay is based on [18], which analyze the shape of a probability weighting function in terms of two psychological aspects, attractiveness and discriminability. Attractiveness represents how attracted an individual is to some risk prospect. For instance, individuals who have a financial background might be more attracted (or optimistic) when betting on the prospect involving stock prices than health outcomes.

Discriminability reflects the ability to perceive changes in probabilities. For example, imagine that Vanessa has two markets close to her home. In market A she has a 2% chance of contracting a virus and in market B she has a 1% chance. She easily perceives that A is twice as risky as B and then chooses B. On the other hand, if the same difference occurs near the middle of probabilities, say a 53% for A and a 52% for B, probably she indistinctly chooses A or B.

In many works in the literature, attractiveness and discriminability are linked to optimism/pessimism and likelihood insensitivity, respectively. The more attractive a prospect is, the more optimistic the individual is. In addition, if we have a low ability to discriminate probabilities in some range then we are insensible to changes in probabilities in that range.

In [44], the authors define indexes of pessimism and insensitivity and apply it to an experiment to understand ambiguity attitudes in natural events (The French Stock Index, the temperature in Paris and the temperature in a random country). Based on the results they found behavioral evidences that people are more prudent, invest less, and take out more insurance for unknown probabilities than for known probabilities. In addition, people will be more open to both insurance and long shots, and updating of probabilities after receipt of new information will affect people less for Paris temperature and

The French Stock Index.

Another study, [49], used these indexes to understand the relation between risk attitude and athletic success. They compared risk preference of the players of the Dutch men's field hockey team with a sample of recreational hockey players. They found that professional players were more optimistic than the recreational players about the probability of gain. For larger losses, the professionals were also more optimistic than the recreational players. Regarding to the sensibility to changes in probabilities, professionals were less sensitive for gains and losses. They concluded that optimism, which is usually understood as a bias, may be associated with better outcomes. It contradict the common notion in decision theory that behavioral bias lead to suboptimal outcomes.

The authors in [51], found that individuals with high index of insensitivity are less likely to own stocks. The authors also performed robustness test to controlling for education, financial assets, income, age, family structure, risk aversion, trust, and financial literacy. [53] studied the effect of learning information on people's attitudes toward ambiguity. In terms of decision weights the results indicated that there was significant likelihood insensitivity, but little pessimism. Subjects moved in the direction of expected utility as more information about the historical performance of the stocks became available.

In [18], attractiveness and discriminability are defined in a relative way. It means that these definitions compare the attitudes of two individuals through their respective weighting functions. We propose a definition in an absolute sense where a relative definition becomes natural. The induced definition of relative attractiveness is the same as the one given by [18]. When comparing the definition of relative discriminability, our definition is less stringent.

Finally, [42] proposed a descriptive model with a two-parameter weighting function (CRS family) to understand how people make decisions when the utility function is time-dependent. In this model, one of these parameters depends on the time at which a prospect is resolved. The time parameter is responsible to the sensitivity toward changes in probabilities which is related to the concept of discriminability.

In many works, the authors used a simpler family (called NEO-additive) to suggest indexes to measure the psychological aspects we are dealing with in this paper. The reason to use this family is because it is mathematically easy to work with and its parameters are easy to interpret. For instance, [44] used these indexes to quantitatively analyze Ellsberg-type events and many natural events. [51] developed an experimental work for measuring ambiguity attitudes in economic decisions and applied it in a large representative sample of the population. [53] studied the effect of learning information on people's attitude on the New York Stock Exchange. [52] also suggested indexes that fit all popular ambiguity theories. Our general approach, when applied to the NEO-additive case, finds essentially the same result reported in these works.

Our essay propose definition for attractiveness and discriminability in an absolute and relative sense. It expands the work developed in [18]. Based on these definition, we propose measures for both psychological concepts. In addition, we apply these measures in the most common two-parametric families of weighting functions found in the literature: NEO-additive, CRS, Goldstein-Einhorn and Prelec. We perform a sensitivity analysis in each family to understand how a changing in the parameter affects the variance of our measures. It is important to comprehend which parameter may be taken as an index for attractiveness and discriminability.

For instance, going back to our practical application of the decision in releasing a job loss insurance (section 4.1), we already know how to use the second essay to identify which weighting function to use in the group of people under consideration. Once we know the weighting function and its parameter values, we may use the measures proposed in this third essay to evaluate the level of attractiveness of the group and decide (depending on the rule of decision) whether the product is attractive enough to be released.

This paper is structured as follows. In Section 5.2 we define the psychological concepts of attractiveness and discriminability in an absolute sense and propose measures for it. In Section 5.3 we show the induced measures for these concepts in a relative sense. Section 5.4 contains applications of the measures to the most common weighting functions and in Section 5.5 we summarize the results.

5.2 Attractiveness and Discriminability

In this section we define attractiveness and discriminability in an absolute sense. By "absolute" we mean a definition for the individual himself and not the comparison of two of them as in [18].

We shall assume that a weighting function w is defined on $[0,1]$, is continuous and strictly increasing, with $w(0) = 0$ and $w(1) = 1$ (Figure 2.2). An exception to that is the NEO-additive weighting functions which will be dealt with separately.

Let p be the probability of a prospect X and $w(p)$ be the perceived probability of an individual of the same prospect X.

Definition 5.1 *The (absolute) Attractiveness of an individual, identified by w, to the chance domain of the prospect X at p, is $w(p) - p$.*

We could define the attractiveness at p as $w(p)$ but we want to use the objective probability, p, as a reference. It means that the individual's attractiveness at p is zero when he/she does not distort the objective probability by $w(p)$.

Starting with Definition 5.1, we introduce some set-based measures. Let μ stands for the Lebesgue measure. Measurements in terms of μ are needed when dealing with discontinuous weighting functions, e.g. NEO-additive.

The *Absolute Attractiveness* of an individual on a μ-measurable subset $S \subseteq [0,1]$ is

$$AA_S(w) = \int_S [w(p) - p] d\mu. \tag{5-1}$$

In the special case of $S = [q_1, q_2] \subseteq [0,1]$, and when the individual is implicit, we abbreviate the notation and write

$$AA_{[q_1,q_2]} = \int_{q_1}^{q_2} [w(p) - p] dp = \int_{q_1}^{q_2} w(p) dp - \frac{1}{2}(q_2^2 - q_1^2). \tag{5-2}$$

For global $[q_1, q_2] = [0,1]$ we write

$$AA = \int_0^1 [w(p) - p] dp = \int_0^1 w(p) dp - \frac{1}{2}. \tag{5-3}$$

AA, *Absolute Attractiveness*, is a measure of the "size of the attractiveness". This is a signed measure, as the value may be negative.

Next, we define (absolute) discriminability and also propose a natural measure for it.

Definition 5.2 *The Absolute Discriminability of an individual, identified by w, on a measurable subset $S \subseteq [0,1]$, is*

$$AD_S(w) = \mu(w(S)) - \mu(S). \tag{5-4}$$

In the special case of $S = [q_1, q_2]$ (and for w which is continuous and strictly increasing) we have

$$AD_{[q_1,q_2]}(w) = (w(q_2) - w(q_1)) - (q_2 - q_1). \tag{5-5}$$

For w which is strictly increasing and continuous on $[0,1]$, $w([q_1, q_2]) = [w(q_1), w(q_2)]$. So $\mu(w([q_1, q_2])) = w(q_2) - w(q_1)$.

The interval $[w(q_1), w(q_2)]$ is the individual's perceived interval in lieu of the true interval $[q_1, q_2]$. The room to discriminate, or to work with, is $\mu(w([q_1, q_2]))$. Absolute discriminability on an interval, $AD_{[q_1,q_2]}(w)$, is the change in room to discriminate, $\mu(w([q_1, q_2])) - \mu([q_1, q_2])$.

When w is implicit, we write $AD_{[q_1,q_2]}$.

We shall define a related measure globally (on the interval $[0,1]$). Before giving its precise definition, we present the definition of total variation of a function.

Definition 5.3 *Let I be a finite closed interval and $d : I \to \mathbb{R}$ be a function on I. Consider the collection Π of ordered list of points $a_1 \leq a_2 \leq \ldots \leq a_{N+1} \in I$, where N is an arbitrary natural number. The Total Variation of d on I is given by*

$$TV(d) = \sup \left\{ \sum_{i=1}^{N} |d(a_{i+1}) - d(a_i)| : (a_1, ..., a_{N+1}) \in \Pi \right\}. \tag{5-6}$$

If the total variation is finite then d is called a function of *bounded variation*. It is clear that if d is of bounded variation then its restriction to any closed sub-interval J is also of bounded variation and $TV(d|_J) \leq TV(d)$. A fundamental characterization is that a function has bounded variation if, and only if, it can be written as the difference of two non-decreasing functions g and h, say $d = g - h$. In fact, with $I = [0,1]$, we may define g and h by

$$\begin{aligned} 2g(p) &= TV(d|_{[0,p]}) + d(p), \\ 2h(p) &= TV(d|_{[0,p]}) - d(p) \end{aligned} \tag{5-7}$$

for each $p \in [0,1]$. Then g and h are non-decreasing and $d = g - h$.

The functions d we shall consider are the differences of weighting functions w_1 and w_2. The weighting functions are increasing on $[0,1]$ and thus $d = w_1 - w_2$ is a function of bounded variation.

Weighting functions we are considering will have the boundary property $w(0) = 0$ and $w(1) = 1$. With that assumption, $d = w_1 - w_2$ has the property

$$d(0) = d(1) = 0. \tag{5-8}$$

Taking first $p = 0$ and then $p = 1$ in the definitions (5-7) of g and h, we get

$$g(0) = h(0) = 0, \quad g(1) = h(1) = \frac{TV(d)}{2}. \tag{5-9}$$

To motivate our global definition, pick any point q_3 between q_1 and q_2. We can always see the interval $[q_1, q_2]$ as the union of two intervals $[q_1, q_3]$ and $[q_3, q_2]$. It is already clear the behavioral meaning of the measure, $AD_{[q_1,q_3]}$ on the interval $[q_1, q_3]$. In parallel, It is also clear the behavioral meaning of the measure, $AD_{[q_3,q_2]}$ on the interval $[q_3, q_2]$. Observe that the sum of $AD_{[q_1,q_3]}$ and $AD_{[q_3,q_2]}$ is equal to $AD_{[q_1,q_2]}$, which is the absolute discriminability on $[q_1, q_2]$.

Therefore, a consistent extension of the measure for this local discriminability over a collection of disjoint intervals, each with its own value of local discriminability, is the sum of the measures on each interval. It follows that a reasonable global definition for absolute discriminability on $[0,1]$ is either equal to the sum of the measures on all (disjoint) intervals where d, $d(p) := w(p) - p$, is increasing, or equal to the sum of the measures of all disjoint intervals where d is decreasing, which ever is larger in magnitude. That larger magnitude is taken to be our definition of *Maximum Absolute Discriminability* (MAD). Of the two mentioned sums, one is equal to $h(1) - h(0)$, and the other is $g(1) - g(0)$. In fact, due to (5-8), the two amounts are equal, and is half the total variation of d, as shown in (5-9).

The above discussion suits weighting functions w which are strictly increasing and continuous on $[0,1]$, with $w(0) = 0$, $w(1) = 1$.

For such weighting functions *Maximum Absolute Discriminability* and total variation have the simple relationship

$$MAD = \frac{TV(d)}{2}, \quad \text{where} \quad d(p) = w(p) - p. \tag{5-10}$$

5.3 Relative Attractiveness and Relative Discriminability

In this section, based on the definitions of the last section, we lay the definitions for *Relative Attractiveness* and *Relative Discriminability.* We can interpret both definitions based on the difference function:

$$d(p) = w_1(p) - w_2(p), \quad p \in [0,1] \tag{5-11}$$

where w_1 and w_2 represent the weighting functions of two individuals.

5.3.1 Relative Attractiveness

Definition 5.4 *Individual 1, (identified with w_1), finds the prospect X more attractive than individual 2 if for all $p \in [0,1]$ $w_1(p) \geq w_2(p)$, with at least one strict inequality.*

This definition is taken from [18]. In terms of the difference function $d = w_1 - w_2$, Definition 5.4 asserts that individual 1 finds X more attractive than individual 2 if d is a non-negative function, and $d \neq 0$. For the ease of use, however, we shall allow $d = 0$ and that "more" is not necessarily in a strict sense.

As an illustration we shall examine the function d of various pairs of weighting functions coming from the Goldstein-Einhorn collection

$$w(p) = \frac{ap^b}{ap^b + (1-p)^b}. \tag{5-12}$$

The functions w_1 and w_2 are represented by different pairs of parameters, (a_1, b_1) and (a_2, b_2).

Figure 5.1 presents how the value of a influences the difference function, $d(p)$. In order to see that, we fixed $b_1 = b_2 = 0.5$, $a_2 = 1.5$ and made a_1 vary on $]1.5, 3[$. We have chosen seven values of a_1, which gives us seven curves. The greater the value of a_1, the higher the curve. In every case, $d(p) > 0$ for all value of p on $]0, 1[$ and thus w_1 is more attractive than w_2.

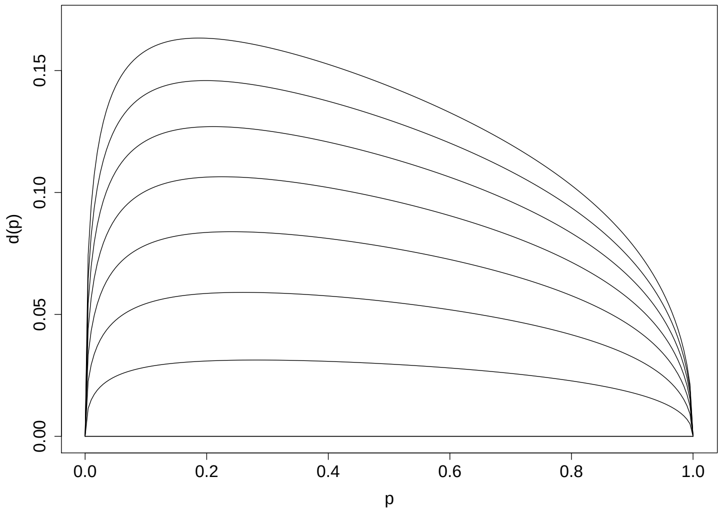


Figure 5.1: Plotting the function $d(p)$ for the Goldstein-Einhorn collection. We have fixed value of b and a_1 varying above reference value $a_2 = 1.5$ ($b_1 = b_2 = 0.5$ and $3 > a_1 > a_2 = 1.5$)

In Figure 5.2, we present how the value of b impacts the difference function. When $b_1 \neq b_2$, we have an interval where $d(p)$ is positive and another interval where it is negative. So, neither w_1 is more attractive than w_2 nor w_2 is more attractive than w_1.

Therefore, to realize relative attractiveness, we must have $b_1 = b_2$. Doing $b = b_1 = b_2$ and varying b on $]0, 1[$, Figure 5.3 shows that the parameter b also influences the relative attractiveness. A lower value of b is represented by a more square graph.

Analysing Figure 5.1, if we take $a_1 = 2$ or $a_1 = 2.5$ we will have that w_1 is more attractive than w_2. However, we see different levels of attractiveness. Looking at the function $d(a_1, a_2; p)$, we have that $d(2, 0.5; p) > d(2.5, 0.5; p)$ for every probability $p \neq 0, 1$. To gauge the different levels we propose a measure of *Relative Attractiveness*,

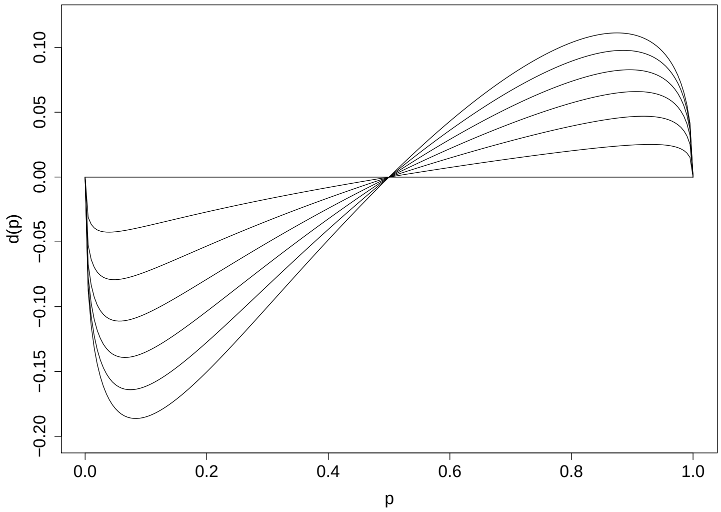


Figure 5.2: Plotting the function $d(p)$ for the Goldstein-Einhorn collection. The figure presents fixed value of a and b_1 varying above reference value $b_2 = 0.5$ ($a_1 = a_2 = 1.5$ and $1 > b_1 \geq b_2 = 0.5$).

$$RA = \int_0^1 [w_1(p) - w_2(p)]dp = \int_0^1 d(p)dp. \tag{5-13}$$

The expression (5-13) is the natural relative measure coming from *Absolute Attractiveness.* In fact, it is reasonable to assume that the relative attractiveness is equal to the difference in the absolute attractiveness of individuals 1 and 2. In other words,

$$RA = AA(w_1) - AA(w_2). \tag{5-14}$$

5.3.2 Relative Discriminability

Based on Definition 5-5 we define *Relative Discriminability.*

Definition 5.5 *Let $w(p)$ be an individual's subjective probability. Then $[w(q_1), w(q_2)]$ is his/her subjective working interval in lieu of the true interval $[q_1, q_2]$. If the length of $[w_1(q_1), w_1(q_2)]$ is greater than $[w_2(q_1), w_2(q_2)]$, then individual 1 exhibits greater discriminability than individual 2 over the interval $[q_1, q_2]$.*

The intuitive meaning is that individual 1 is allocating a more spacious interval (referring to $[w_1(q_1), w_1(q_2)]$) to handle $[q_1, q_2]$ than that of individual 2.

A natural (directed) measure based on Definition 5.5 is

$$RD(w_1, w_2; [q_1, q_2]) = (w_1(q_2) - w_1(q_1)) - (w_2(q_2) - w_2(q_1)) = d(q_2) - d(q_1). \tag{5-15}$$

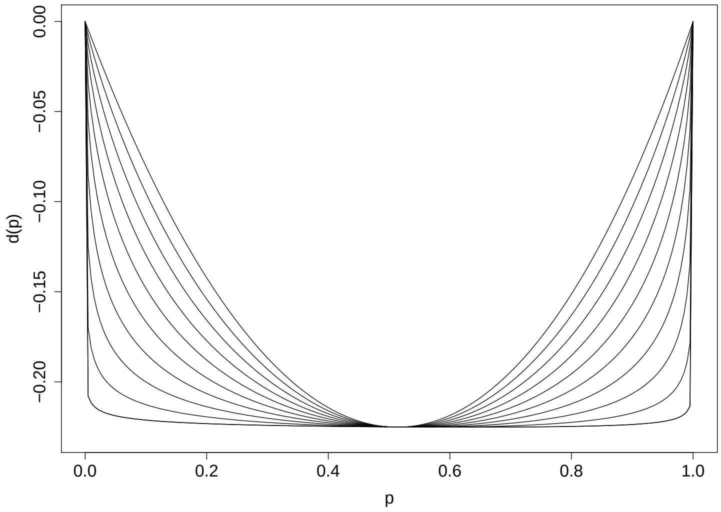


Figure 5.3: Influence of b on the relative elevation. $a_1 = 0.6, a_2 = 1.5$ and $0 < b_1 = b_2 < 1$

Our measure for relative discriminability (expression (5-15)) is the natural measure coming from the absolute discriminability. In other words,

$$RD(w_1, w_2; [q_1, q_2]) = AD_{[q_1,q_2]}(w_1) - AD_{[q_1,q_2]}(w_2). \tag{5-16}$$

We shall refer to $RD(w_1, w_2; [q_1, q_2])$ as the *Relative Discriminability from w_1 to w_2 on* $[q_1, q_2]$. The Definition 5.5 allows us to compare discriminability between any two individuals (or their w) over any interval $[q_1, q_2]$. Under this definition, we always have either w_1 exhibits greater discriminability than w_2 on $[q_1, q_2]$, or w_2 exhibits greater discriminability than w_1 on $[q_1, q_2]$, or w_1 exhibits equal discriminability as w_2 on $[q_1, q_2]$.

In this sense, this definition is more general than the definition given by [18].

In [18], w_1 exhibit greater discriminability than w_2 on $[q_1, q_2]$ when d is strictly increasing on that interval. Furthermore, a natural measure for it is also,

$$GW_{[q_1,q_2]} = d(q_2) - d(q_1). \tag{5-17}$$

The expression (5-17) represents the length of the image of the interval $[q_1, q_2]$ under the strictly increasing continuous map d. Speaking in another way, when w_1 exhibits greater discriminability than w_2 on interval $[q_1, q_2]$, $GW_{[q_1,q_2]}$ is a sensible measure of by how much it is greater.

Note that in Definition 5.5, we do not assume that $d(p)$ is an increasing function on $[q_1, q_2]$. If d is increasing on $[q_1, q_2]$, since $(w_1(q_2) - w_1(q_1)) - (w_2(q_2) - w_2(q_1)) = d(q_2) - d(q_1) > 0$, we get $(w_1(q_2) - w_1(q_1)) > (w_2(q_2) - w_2(q_1))$ as a consequence, and so individual 1 exhibits greater discriminability than individual 2 over the interval $[q_1, q_2]$ under our Definition 5.5.

It is worth noting that distinction between both definitions of relative discriminability. In fact w_1 exhibits greater discriminability than w_2 on interval $[q_1, q_2]$ in the Gonzalez and Wu sense if and only if w_1 exhibits greater discriminability than w_2 on every subinterval of $[q_1, q_2]$ in the new sense laid in Definition 5.5. Despite this difference in meanings, the measures we use to gauge the levels are the same, i.e. (5-15) and (5-17) have the same values.

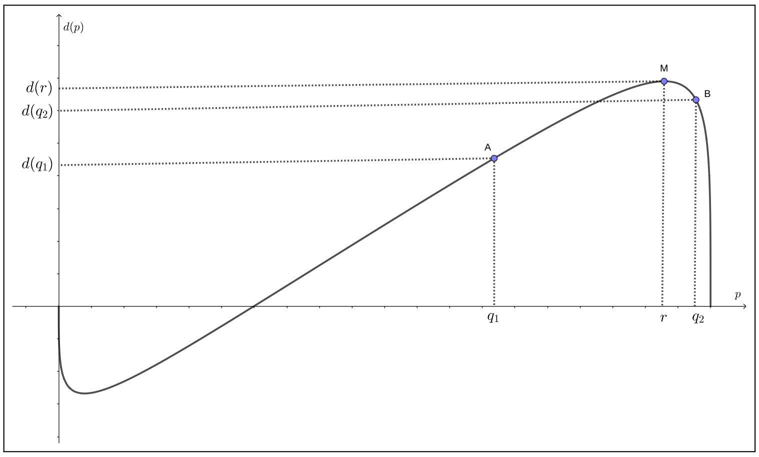


Figure 5.4: Relation between extended and original definitions of Relative Discriminability

In Figure 5.4, under Definition 5.5 w_1 exhibits more discriminability than w_2 on $[q_1, q_2]$. On the other hand, under definition in [18], neither weighting function discriminates more than the other. However, considering M as a local maximum, in [18], w_1 discriminates more than w_2 on $[q_1, r]$ and the opposite happens on $[r, q_2]$. Therefore, we can write

$$RD_{[q_1,q_2]} = [d(r) - d(q_1)] - [d(r) - d(q_2)] = GW_{[q_1,r]} - GW_{[r,q_2]} \tag{5-18}$$

where $GW_{[q_1,r]}$ measures how much w_1 discriminates more than w_2 on $[q_1, r]$ and $GW_{[r,q_2]}$ measures how much w_2 discriminates more than w_1 on $[r, q_2]$. In this particular case, we have $RD_{[q_1,q_2]} > 0$ and, under Definition 5.5, it means that the discrimination of w_1 relative to w_2 on $[q_1, r]$ overcomes the discrimination of w_2 relative to w_1 on $[r, q_2]$. So, if we consider the entire interval $[q_1, q_2]$, it is reasonable to say that w_1 discriminates more than w_2.

Therefore, $RD_{[q_1,q_2]}$ measures the balance between the two measurements coming from equation (5-17). Thus, $RD_{[q_1,q_2]}$ makes good and versatile sense.

We can formulate (5-15) at a higher level of abstraction (e.g. (5-4)). Replace $[q_1, q_2]$ by any reasonable subsets S of $[0, 1]$, and replace the concept of length of intervals by a broader measure, for example, the classical Lebesgue measure μ, say, which measures the "area" of S. Then define a *(directed) measure of relative discriminability from* w_1 *to* w_1 *on* S by

$$RD(w_1, w_2; S) := \mu(w_1(S)) - \mu(w_2(S)). \tag{5-19}$$

In terms of absolute discriminability, $RD(w_1, w_2; S) = AD_S(w_1) - AD_S(w_2)$.

The function $RD(w_1, w_2; S)$ is additive in S. It means we can compute that for the (disjoint) intervals, then sum them up. It also means person 1 is allocating more area to handle the union than person 2, and the extra area is equal to that sum. More area means more working space to discriminate.

In general measure theory, some signed measure, say m, can be decomposed into the difference of two unsigned (non-negative) measures ν_1 and ν_2, known as the positive and negative parts of m. The norm of m is sometime defined by $\nu_1 + \nu_2$ on the global underlying space. Our $m(S) := RD(w_1, w_2; S)$ is a signed measure and admits such decomposition. The norm is tied to the total variation of $w_1 - w_2$.

5.4 Application

In this section we apply our measures to the most common two-parametric families of weighting functions found in the literature: NEO-addtive ([12], [44], [51] and [54]), CRS ([15] and [12]), Goldstein-Einhorn ([10]), Prelec ([11] and [36]).

The first objective of this section is to compare our measures with some particular measures found in the literature. For instance, [46] proposed indexes of optimism (which we can relate with attractiveness) and indexes of likelihood insensitivity (which is related to discriminability) for Goldstein-Einhorn and Prelec weighting functions. [44], [51] and [54] proposed, and used in experimental works, both indexes for NEO-additive weighting functions. [12] suggested an index of relative optimism (which is related to attractiveness) and an index of relative sensitivity (which is related to discriminability) for CRS.

The second objective is to give a broader understanding (local and global) of the impact of each parameter of the weighting function on the psychological factors studied in this paper. This analysis is important to understand which parameter drives the decision of people under risk. This aim is qualitatively well attended in the literature (for example, [18], [13] [12], [15], [27]). We use the term "qualitatively" because the parameters are interpreted on the basis of intuition, experience and playing with graphs, as we can see in the Figure 5.5. However, we are unaware of any work proposing formal and general measures to understand quantitatively their impact.

Figure 5.5 qualitatively shows how the parameters a and b influence the attractiveness and discriminability of the Goldstein-Einhorn weighting functions (expression 5-24). We can intuitively agree that when we fix the value of b and increase the value of the a, the graph becomes higher or more "elevated" (other terminology used in the literature). In other words, an individual having a higher (more elevated) curve is more attracted to a prospect X than an individual with a lower (less elevated) curve. More attraction means more risk seeking for gains and risk aversion for losses. In [50], probabilistic risk seeking is called optimism and probabilistic risk aversion is called pessimism.

On the other hand, if we fix a and decrease the value of b the graph becomes closer to the graph of a step-function. In the literature, this is sometimes related to the intuitive idea of "curvature" of the graph. Furthermore, we can also intuitively identify that when we only change the value of a we have a greater modification in the elevation than in the curvature of the graph. The opposite happens when we fix a and change the parameter b.

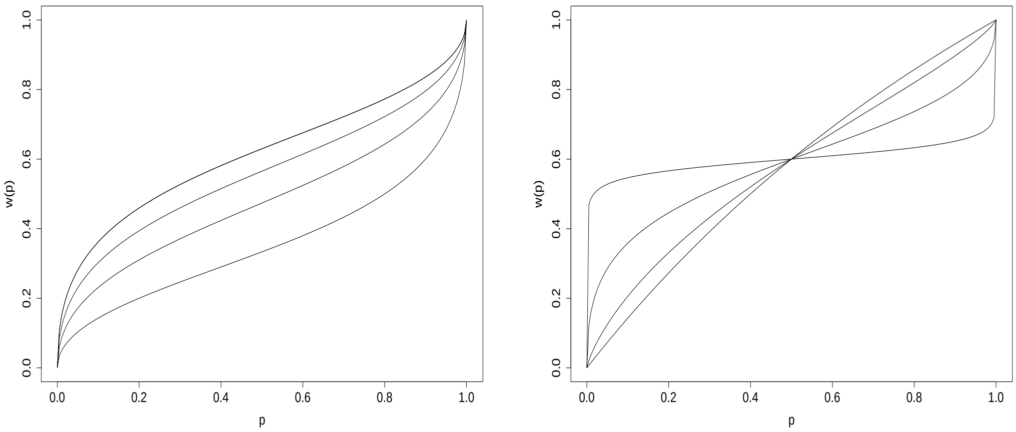
As mentioned in [18], [12], a completely independent influence of each parameter is impossible because the endpoints are attached ($w(0) = 0$ and $w(1) = 1$), Figure 5.5.

The measures proposed in this paper and the sensitivity analysis carried out in this section quantified the impacts of the parameters on each psychological characteristic and also how much this impact is. In particular, our local measures may help us understand how people behave in dealing with almost certain events (probability close to 1) or rare events (probability close to 0). In some weighting families, these behaviors are different from what happens in the global and middle-range of probabilities.

For the NEO-additive family we can directly calculate and interpret the impacts of the parameters. However, for the other families, it is not easy or possible to direct study these impacts. In that case we performed a sensitivity analysis.

Our sensitivity analysis uses the variance-based method described in [55] and [56]. Briefly speaking, the variance-based method (or Sobol's method) provides us reliable information about how the variance of the output depends on the different combinations of values for random inputs. It is a standard approach in sensitivity analysis.

Following the outline of [57], the two-parametric weighting function families, $w(\cdot)$, depends on parameters a and b. Furthermore, let a family of weighting functions be presented with two parameters a and b. Let Y stands for any of our measures. Then Y is a function of a and b, $Y = Y(a, b)$. The impact index of a on Y is defined by



(a) Value of a changing the Attractiveness

(b) Value of b changing the Discriminability

Figure 5.5: Goldstein-Einhorn weighting functions with different values of a and b.

$$I_a(Y) = \frac{\mathrm{Var}_a(E_b(Y|a))}{\mathrm{Var}(Y)} \tag{5-20}$$

where, $E_b(Y|a)$ is the conditional expectation of Y, taken over b. $\mathrm{Var}_a(\cdot)$ is the variance over a and $\mathrm{Var}(Y)$ is the unconditional variance of Y. A similar expression is built for I_b.

Intuitively speaking, if $I_a = 0.7$ and $I_b = 0.2$, it means that 70% of the variance in Y is caused by the variance in a, 20% is caused by the variance in b an 10% is due to interactions between a and b.

The computational implementation follows [45]. Each input parameter follows a uniform distribution and their range of values will be specified later.

To perform one simulation, we generate 32 values for a and 32 values for b. It give us 1024 combination of the pair (a, b) to calculate I_a and I_b, using (5-20). We then perform 400 simulations and analyze the results.

For local measures, we analyze three scenarios: rare event, middle-probability event and almost certain event. We assume probabilities lower than 1% for rare event, probabilities between 10% and 90% for middle-probability event and probabilities greater than 99% for almost certain event.

5.4.1 NEO-additive weighting functions

The weighting functions called NEO-additive have the form

$$w(p) = \begin{cases} 0, & p = 0 \\ a + bp, & 0 < p < 1 \\ 1, & p = 1 \end{cases} \tag{5-21}$$

where $0 \leq a < 1$ and $0 < b \leq 1 - a$.

So far, our discussion was carried out through weighting functions which were strictly increasing and continuous on $[0, 1]$, with $w(0) = 0$ and $w(1) = 1$. However, as mentioned in Section 5.2, this is not the case for NEO-additive function (expression (5-21)). When $a \neq 0$ or $b \neq 1 - a$, the function (5-21) becomes discontinuous at 0 or 1.

In this discontinuous case, when 0 or 1 is in $[q_1, q_2]$, $\mu(w([q_1, q_2])) \neq w(q_2) - w(q_1)$ and the use of $w(q_2) - w(q_1)$ in (5-5) is not appropriate. Because of its friendly expression, we can calculate our absolute measures and compare them with what is found in the literature. Table 5.1 shows the result.

Literature	**Essay III**
Index of Attr. $= \frac{2a+b}{2}$	$AA = \frac{2a+b}{2} - \frac{1}{2}$
Index of Disc. $= 1 - b$	$MAD = 1 - b$

Table 5.1: Comparing the indexes suggested by literature with our measures.

The calculation of AA is straightforward. For MAD, when $(a, b) = (0, 1)$, w is continuous on $[0, 1]$ and $MAD = TV(d)/2 = 0$. When $(a, b) \neq (0, 1)$, the function $d(p)$ is strictly decreasing on $]0, 1[$ and there is no proper interval where d is strictly increasing. It means that MAD is equal to $\mu(d(]0, 1[))$, or more precisely, $MAD = a - (a + b - 1) = 1 - b$. On the other side, $TV(d)$ also takes $\mu(d(]0, 1[))$, but it also captures the magnitude of both jumps (discontinuities), at $p = 0$ and $p = 1$, which are a and $-(a + b - 1)$, respectively. It means that $MAD = TV(d)/2$ also works for NEO-additive.

We can see that the global measure of absolute attractiveness (AA) depends on both parameters (a and b). It is essentially the same value suggested by [44], [51] and [54] (which is equal to $(2a + b)/2$). As for the indexes, linear transformations (normalizations) do not affect them. Both concepts are connected by the idea of elevation and the difference in the measurement comes from the fact that we used the objective probability, p, as a reference. In other words, in equation (5-3) we used $w(p) - p$ instead of only $w(p)$.

[44], [51] and [54] also suggested $1 - b$ as a index of likelihood sensitivity, which we can relate with our measure of discriminability. Both terminologies are connected by the idea of diminishing sensitivity of probabilities ([3]). This value is the same, in magnitude, we found in $|AD_{[0,1]}|$ and MAD (Table 5.1).

It is important to notice that suggesting $1-b$ as an index of sensibility, come from a qualitative understanding as showed in Figure 5.5. On the other hand, index we found in this paper, $MAD = 1-b$, comes from a formal definition of discriminability. It gives a more solid ground for future empirical works involving these psychological aspects.

In talking about the impact of each parameter, all measures are affine with respect to a and b. It means we can write them as $Y(a,b) = ak_a + bk_b + k$.

By the symmetry of a and b, if we choose the same distribution for them, we conclude that

$$I_a \geq I_b \Leftrightarrow |k_a| \geq |k_b|. \tag{5-22}$$

In the Appendix B, we did the calculations considering a and b equipped with the uniform distribution.

Looking at Table 5.1, we find that a has more impact on the variance of AA. For MAD, the parameter b is the only one that impacts their variance.

5.4.2
CRS weighting functions

Now we consider the CRS weighting functions,

$$w(p) = \begin{cases} a^{1-b}p^b, & 0 \leq p \leq a \\ 1-(1-a)^{1-b}(1-p)^b, & a < p \leq 1 \end{cases} \tag{5-23}$$

with $0 \leq a \leq 1$ and $b > 0$. We get the empirically founded inverse s-shape if $0 < a < 1$ and $b < 1$, and exhibit, less frequently found, s-shape if $0 < a < 1$ and $b > 1$.

Table 5.2 shows the results of AA and MAD. The calculation of AA is straightforward. The calculation of MAD is done case by case (Appendix B).

Literature	**Essay III**
Index of Attr. $= a$	$AA = \left(\frac{1-b}{1+b}\right)\left(a - \frac{1}{2}\right)$
Index of Disc. $= 1-b$	$MAD = \begin{cases} 0, & b = 1 \\ \|1-b\|b^{b/(1-b)}, & b \neq 1 \end{cases}$

Table 5.2: Comparing the indexes suggested by literature with our measures.

We compare our measures with those found in [12] (Table 5.2). Inspired by the measure of relative risk aversion in the case of power utility functions ([47], [48]), the authors found $1-b$ as the index of relative sensitivity which is related to discriminability of the weighting function. Both terminologies are connected by the idea of curvature of the weighting function. The authors also consider the whole interval of probabilities, so we can compare it with MAD. MAD also depends only on b but it has different value and behavior, as we can see in Table 5.2 and Figure 5.6. For $b < 1$, $1 - b$ and MAD are strictly decreasing and we have the same qualitative understanding. It means, a greater value of b indicates lower discriminability in both measures. However, for $b > 1$, MAD is strictly increasing and it gives us a different understanding. While $1 - b$ indicates less discriminability when b increases, MAD presents greater discriminability. The behavior of MAD better represents what happens in the graph of w. In this case ($b > 1$), w is first convex and then concave.

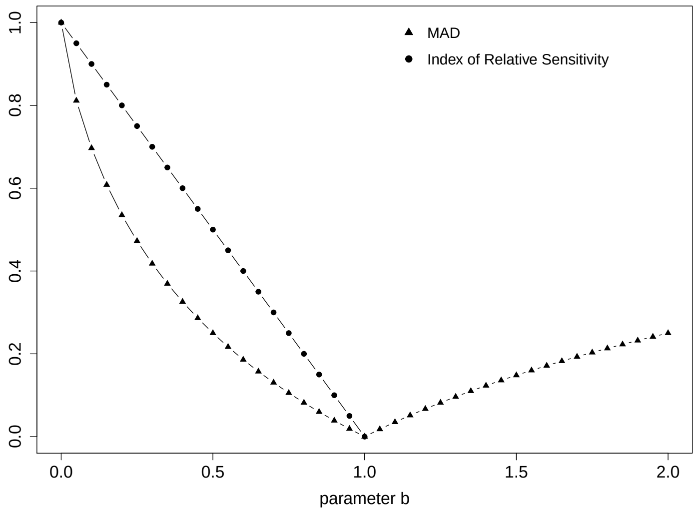


Figure 5.6: CRS function: Graph of MAD and Index of Relative Sensitivity

The authors also suggest a as an index of relative optimism, which we can associate with attractiveness. However, our value for AA found in Table 5.2 depends on a and b. Here we have an important point to distinguish, as pointed out in [12], the value of b is taken as a measure to compare the relative optimism of two individuals. It means that, if (a_1, b_1) are the parameters of the expression 5-23 for the individual 1 and (a_2, b_2) are the parameters for individual 2, then for $a_1 > a_2$ and $b_1 = b_2$ we say that individual 1 exhibits more relative optimism than individual 2. In addition, if we set $a_1 = 0.4$, $a_2 = 0.6$ and $b_1 = b_2 = b$, by varying the value of b we can see that individual 2 exhibits more relative optimism than individual 1, Figure 5.7. However, it presents different degrees of relative optimism. So, we conclude that the parameter b also impacts the relative optimism.

In short, the index of relative optimism, a, suggested in [12] is good enough to say who exhibits more optimism but it doesn't say by how much. On the other hand, our Definition 5-3 tells that a is a good measure to compare

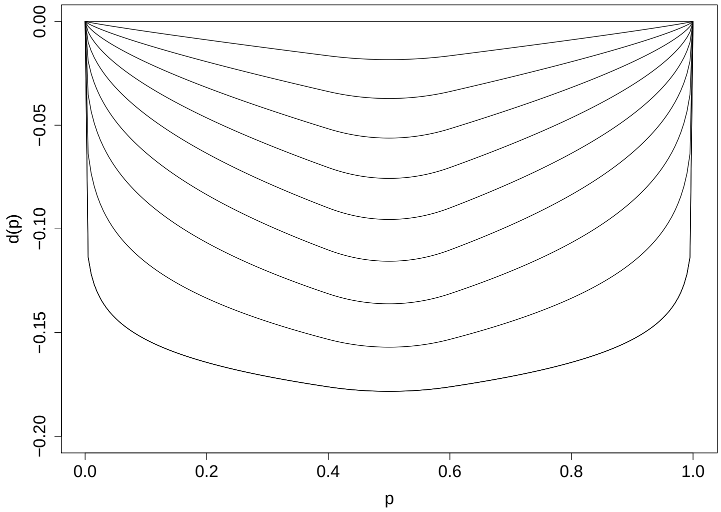


Figure 5.7: CRS function: Influence of b on the Relative Elevation ($a_1 = 0.4$, $a_2 = 0.6$ and $0 < b_1 = b_2 < 1$)

attractiveness (or optimism) and the value of AA in Table 5.2 tells us by how much that difference is.

Speaking of the impact of parameters, the parameter b is the only one that impacts the variance of MAD. In AA, the influence of each parameter is not clear, so, it is necessary to calculate I_a and I_b and perform the sensitivity analysis. We do the same analysis for the local measure, $AA_{[q_1,q_2]}$.

To perform our 400 simulations, the parameters a and b take values drawn from a uniform distribution in $]0,1[$. These range covers the most usual shapes of the CRS family.

Figure 5.8 presents the result. The horizontal axis represents the impact index difference, $I_a - I_b$. This difference varies from -1 to 1 and when $I_a - I_b > 0$ it means that the parameter a has more impact on the variance of AA and the opposite happens if $I_a - I_b < 0$. On the other hand, if $I_a - I_b \approx 0$ the impact of a and b are about the same.

The vertical axis represents the density result of our simulations for $AA_{[q_1,q_2]}$ (equation (5-2)) in four intervals: $[0,1]$, $[0,0.01]$, $[0.1,0.9]$ and $[0.99,1]$. As explained earlier, these intervals represent how people make decisions in a global event ($0 < p < 1$), rare events ($p < 0.01$), middle-probability events ($0.1 < p < 0.9$) and almost certain events ($p > 0.99$). For the other families we will omit the expression "Density on" completely, leaving only the intervals in the charts.

On the global and the middle-range, $[0,1]$ and $[0.1,0.9]$, the parameter a has more impact on the variance of AA. This result fits the idea found in the literature ([12]). However, for rare and almost certain events the parameter

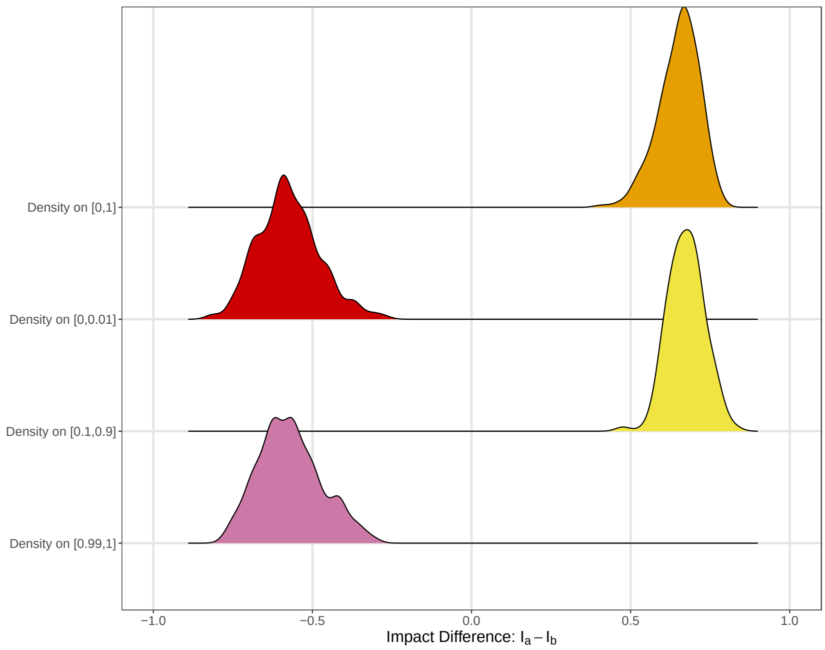


Figure 5.8: CRS family: Impact of a and b on the variance of Absolute Attractiveness ($AA_{[q_1,q_2]}$) in four scenarios: Global ($p \in [0,1]$), Rare events ($p \in [0,0.01]$), Middle-probability events ($p \in [0.1,0.9]$) and Almost Certain events ($p \in [0.99,1]$).

b has more impact. This is a surprising result because most studies take parameter a a good proxy to understand the absolute attractiveness.

Regarding Absolute Discriminability, Figure 5.9 shows us that the impact of each parameter. The MAD result only confirm what we found in the Table 5.2, that is parameter b fully impacts its variance. In the other three scenarios, parameter b also has more impact on the variance of $AD_{[q_1,q_2]}$. However, for probabilities close to 0 ($[0,0.01]$) or close to 1 ($[0.99,1]$) the impact is not as pronounced as in MAD and $AD_{[0.1,0.9]}$. This result confirms what is qualitatively found in the literature, parameter b has more impact in all cases.

5.4.3
Goldstein-Einhorn and Prelec weighting functions

In this section, we will analyze the Goldstein-Einhorn weighting functions,

$$w(p) = \frac{ap^b}{ap^b + (1-p)^b} \tag{5-24}$$

and the Prelec weighting function,

$$w(p) = e^{-a(-\ln p)^b}, \tag{5-25}$$

with $a > 0$ and $0 < b < 1$, in both cases.

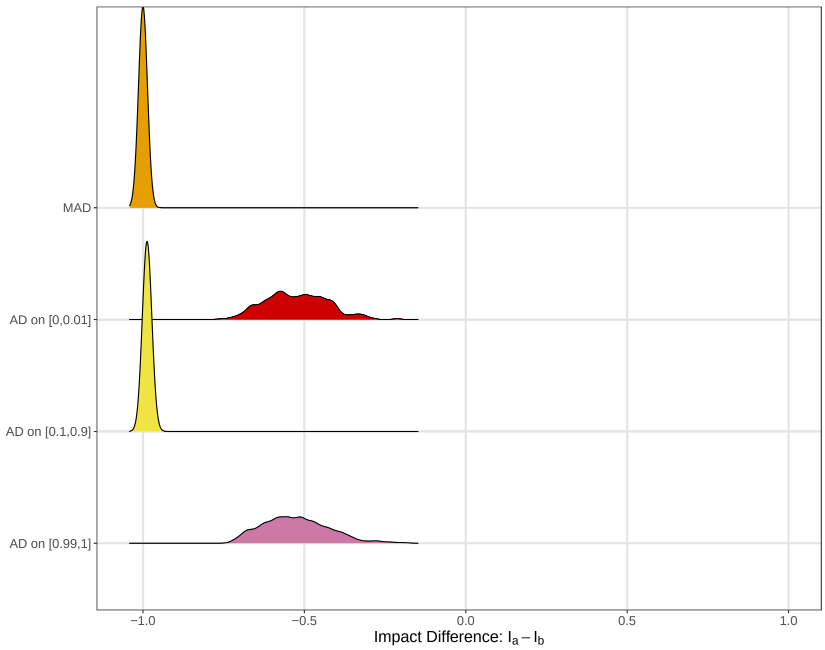


Figure 5.9: CRS family: Impact of a and b on the variance of Maximum Absolute Discriminability (MAD) and on the variance of Absolute Discriminability ($AD_{[q_1,q_2]}$) for Rare events, Middle-probability events and Almost Certain events.

We will compare our absolute measures to the index of optimism (connected to attractiveness) and the index of likelihood sensitivity (connected to discriminability) suggested by [46] and found in many other works in the literature. In both families, parameters a is suggested as the index of optimism and b as the index of likelihood sensitivity. His choices were motivated by an analysis similar to the one we did in Figure 5.5. Doing the same sensitivity analysis, we will find which parameter has more impact on the variance of attractiveness and discriminability. In addition, expressions (5-3) and (5-5) tell us by how much this impact is.

Considering the functions (5-24) and (5-25), we cannot directly study the impact of our measures. As we did in Section 5.4.2, we perform a sensitivity analysis calculating the indexes I_a and I_b.

In the current analysis, the parameter a falls on $]0, 10]$ and b on $]0, 1[$. We believe these variations cover most of the usual shapes of these families of weighting functions.

The first analysis is measuring the impact of a and b on the variance of Absolute Attractiveness (AA) for the Goldstein-Einhorn family. Figure 5.10 shows that in the global and middle-probability cases the variance of $AA_{[q_1,q_2]}$ is almost fully influenced by the parameter a. For rare events, parameter b has more impact on the variance of $AA_{[q_1,q_2]}$ and for almost certain events we do not have a clear cut of the parameter impact. Comparing to the literature, the result for rare and almost certain events are new, since the common understanding is that the parameter a dominates the impact on attractiveness.

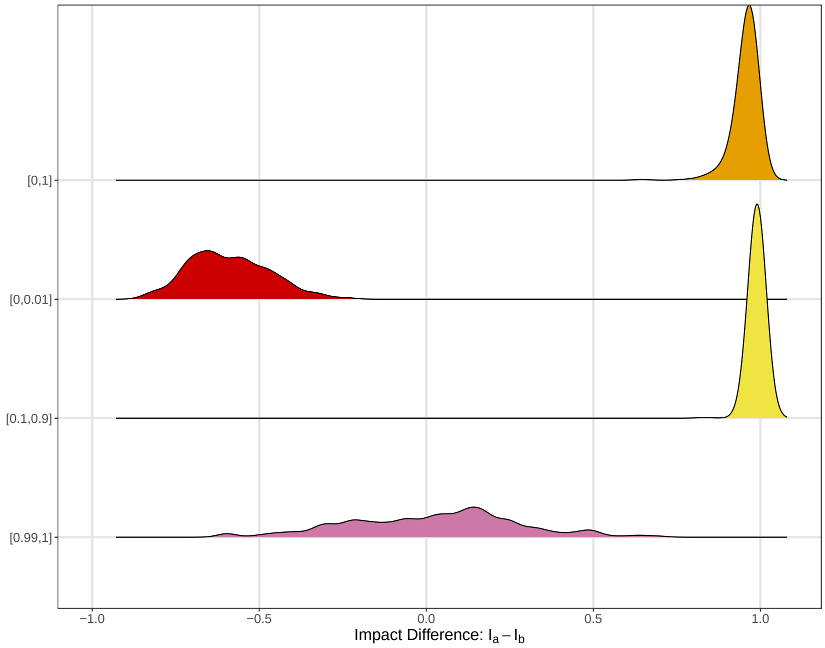


Figure 5.10: Goldstein-Einhron family: Impact of a and b on the variance of Absolute Attractiveness ($AA_{[q_1,q_2]}$) in four scenarios: Global ($p \in [0,1]$), Rare events ($p \in [0, 0.01]$), Middle-probability events ($p \in [0.1, 0.9]$) and Almost Certain events ($p \in [0.99, 1]$).

With regard to the variance of Absolute Discriminability (MAD and $AD_{[q_1,q_2]}$), Figure 5.11 shows that parameter b has more impact in almost all cases, except on $[0.99, 1]$, where the impacts are about the same.

In the case of Prelec family, for Absolute Attractiveness, Figure 5.12 presents interesting result when compared to Goldstein-Einhron family. We find the same result for global interval ($[0,1]$) and middle-probability case ($[0.1, 0.9]$), that is parameter a has more impact. However, in rare events the result is opposite, parameter b has more impact and for almost certain events we don't have a clear cut for Goldstein-Einhorn family and the parameter b has more impact for the Prelec family.

Figure 5.13 presents the result of simulations for Absolute Discriminability in Prelec family. In the case $[0,1]$ and $[0.1, 0.9]$ we confirm the qualitative finding in the literature, parameter b has more influence. However, there is a new finding for MAD where the impact of a and b are similar and in rare events, where parameter a has more impact on the variance of the local absolute discriminability.

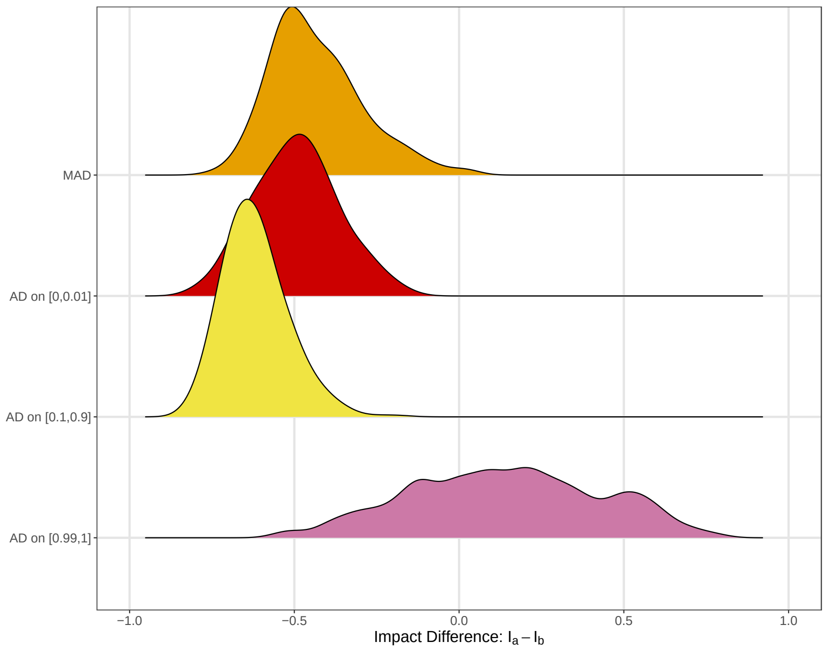


Figure 5.11: Goldstein-Einhron family: Impact of a and b on the variance of Maximum Absolute Discriminability (MAD) and on the variance of Absolute Discriminability ($AD_{[q_1,q_2]}$) for Rare events, Middle-probability events and Almost Certain events.

Regarding to all analyzes in this section, in most cases, we can identify quantitatively what is suggested qualitatively in the literature. That is, the parameter a has more impact on the attractiveness and parameter b has more impact on the discriminability. However, in rare and almost certain events, new results have emerged. It suggests that there is more thinking behind these measures that our eyes can identify.

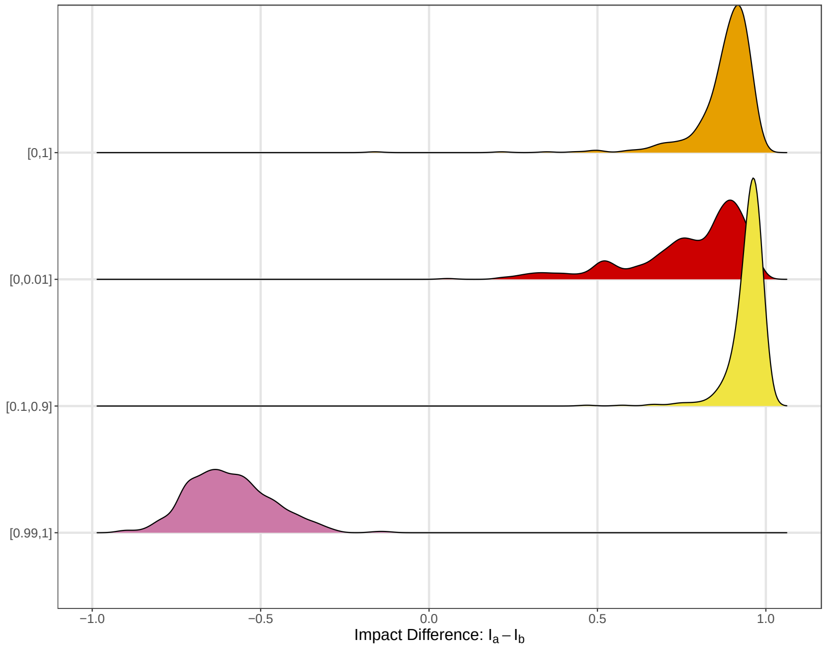


Figure 5.12: Prelec family: Impact of a and b on the variance of Absolute Attractiveness ($AA_{[q_1,q_2]}$) in four scenarios: Global ($p \in [0, 1]$), Rare events ($p \in [0, 0.01]$), Middle-probability events ($p \in [0.1, 0.9]$) and Almost Certain events ($p \in [0.99, 1]$).

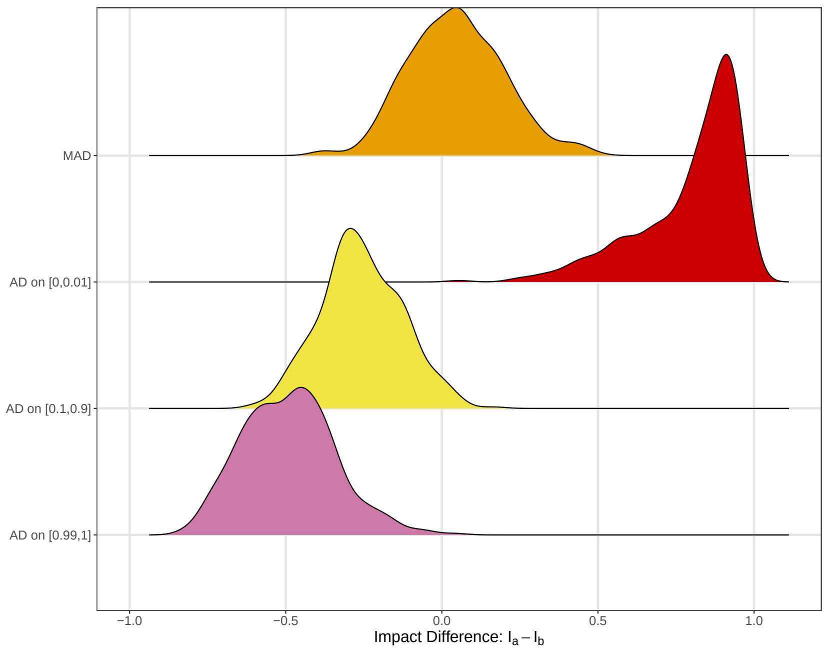


Figure 5.13: Prelec family: Impact of a and b on the variance of Maximum Absolute Discriminability (MAD) and on the variance of Absolute Discriminability ($AD_{[q_1,q_2]}$) for Rare events, Middle-probability events and Almost Certain events.

5.5 Conclusion

In this work, we proposed absolute and relative measures for psychological features called Attractiveness and Discriminability.

In an absolute sense, we suggest definitions for Attractiveness and Discriminability and propose measures for it. These definitions allowed us to establish the respective relative measures in a natural way and also compare it with the those found in [18]. We found that our relative definition for attractiveness in equivalent to [18] but our relative discriminability is an extension to the one found in the same paper.

Additionally, we proposed local and global measures for both psychological features which help us to compared our measures with some particular cases suggested in the literature. We found that our measures confirm the intuitions found in many cases in the literature and shed light on new findings in rare and almost certain events.

6
Conclusions and Futures studies

This thesis was built on the basis of the *Cumulative Prospect Theory* ([4]). We presented three essays that were linked by a deeper study of *Probability Weighting Function* and its connection between psychological bias and economic decisions in a risk scenario.

In the first essay we used Cumulative Prospect Theory and Narrow Framing bias to understand the relation between PTV and subsequent stock return. In Brazil, we found a negative relationship between the PTV and the subsequent returns. In other words, stocks with high (low) PTVs have a subsequent low (high) return. For other emerging countries, i.e., China, Russia, Mexico and South Africa, we found results similar to those for Brazil for the first two countries. In the case of Mexico, we found a significantly positive relationship between PTV and returns, and for South Africa, we found a positive relationship for EW portfolios and a negative relationship for VW portfolios. In addition, probability weighting function was the variable most responsible for the predictive power of the PTV. Then we performed our analysis using the most common weighting function found in the literature and our result persisted.

Future studies could deepen the mathematical analysis of the relationships in this model by producing empirical results. Another interesting path to be taken is to use cultural factors to explain these different relations.

While in this first essay we analyze different existing weighting functions,, in the second essay, we analyzed the preference condition presented in [18] (Preference Condition 4.1), and its implied Problem 4.2. We found a larger family of weighting functions where the Goldstein-Einhorn Probability Weighting Functions constitute a particular sub-family. Our main result is the new preference condition (Preference Condition 4.10) which is necessary and sufficient to obtain the Goldstein-Einhorn Probability Weighting Functions.

A future step for this study is to experimentally test the new preference condition provided.

Also based on the work of Gonzalez and Wu ([18]), in the last essay, we proposed absolute and relative measures for psychological features called Attractiveness and Discriminability.

In an absolute sense, we suggest definitions for Attractiveness and Discriminability and propose measures for it. These definitions allowed us to establish the respective relative measures in a natural way and also compare it with the those found in [18]. We found that our relative definition for attractiveness in equivalent to [18] but our relative discriminability is an extension to the one found in the same paper.

Additionally, we proposed local and global measures for both psychological features which help us to compared our measures with some particular cases suggested in the literature. We found that our measures confirm the intuitions found in many cases in the literature and shed light on new findings in rare and almost certain events.

The natural path to follow from this study is unifying definitions and terminology used in the literature. Attractiveness and Discriminability are connected to Pessimism and Likelihood Insensibility by the elevation and curvature of the probability weighting function. However, each pair of terminologies are based on different definitions. Therefore, a deepen analysis of these definitions is very desirable to find a safe ground for the literature.

A
Appendix

In this section, we present the proofs of all the results mentioned in the Chapter 4, which are Propositions 4.3 – 4.9. Additionally, in the example of the Piecewise Weighting function coming from (4-24), we provide the proofs that w is continuous, strictly increasing, onto and regressive.

A.1
Proof of the Proposition 4.3.

Suppose that the implication holds for f. Since f is continuous, strictly increasing and onto so f^{-1} exists and is continuous too. Let $y > 0$ be given. Then there exists a unique x, say $x = \ell(y)$, such that $2f(x) = f(y)$. In fact, in terms of f^{-1}, $\ell(x) = f^{-1}(\frac{1}{2}f(y))$. Since f is positive valued, $f(x) < 2f(x) = f(y)$. Since f strictly increasing, this implies $x < y$. So

$$0 < \ell(y) < y, \quad \forall y > 0. \tag{A-1}$$

By the description of ℓ we have

$$2f(x) = f(y) \quad \text{iff} \quad x = \ell(y). \tag{A-2}$$

Applying the above definition to the implication (4-13) side by side, it becomes

$$x = \ell(y) \quad \text{implies} \quad tx = \ell(ty).$$

Replacing x by $\ell(y)$ on the right we get the equation

$$t\ell(y) = \ell(ty) \quad (\forall t > 0, y > 0). \tag{A-3}$$

Fixing in (A-3) $y = 1$ we immediately get

$$\ell(t) = ct \quad (\forall t > 0)$$

where $c := \ell(1)$ is a positive constant less than 1 by (A-1). Putting that back into (A-2), the latter becomes

$$2f(x) = f(y) \quad \text{iff} \quad x = cy$$

and so the implication (4-13) becomes

$$x = cy \quad \text{implies} \quad 2f(tx) = f(ty) \quad (\forall t > 0, y > 0).$$

So, the equality on the right side holds for $x = cy$, that is

$$2f(tcy) = f(ty) \quad (\forall t > 0, y > 0).$$

Renaming ty as z we arrive at (4-14). Conversely, it is easy to check that if f is a strictly increasing continuous function satisfying the Schröder equation (4-14) then the implication (4-13) holds because it is apparent that $2f(x) = f(y)$ if and only if $x = cy$.

A.2
Proof of the Proposition 4.4.

It is just the translation of Theorem 2.10 in Kuczma (1968).

A.3
Proof of the Proposition 4.5

Suppose that f satisfies (4-14),

$$f(cz) = \frac{1}{2}f(z), \quad z > 0.$$

Dividing both sides by $c^b z^b$ and defining $\phi(\log_2 z) = \frac{f(z)}{z^b}$ we obtain (4-16). The reciprocal is just a substitution.

Generally speaking, a continuous function ϕ having Ω as a period (positive by convention) admits a trigonometric (Fourier) series representation

$$\phi(x) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty} a_j \cos\left(\frac{2\pi jx}{\Omega}\right) + \sum_{j=1}^{\infty} b_j \sin\left(\frac{2\pi jx}{\Omega}\right),$$

where

$$a_j = \frac{2}{\Omega}\int_0^{\Omega} \phi(s)\cos\left(\frac{2\pi js}{\Omega}\right) ds,$$
$$b_j = \frac{2}{\Omega}\int_0^{\Omega} \phi(s)\sin\left(\frac{2\pi js}{\Omega}\right) ds.$$

The coefficients a_j and b_j are parameters identifying ϕ. In fitting data with functions, we are taking approximations to ϕ and use only finite partial sums. The approximation is thus by (finite) trigonometric polynomials which are differentiable functions (where curvature has a clearer meaning).

A.4
Piecewise Weighting Function coming from (4-24)

$$f(z) = a[(2c)^{-k}z + 2^{-k}(1-2c)], \quad z \in]c^{k+1}, c^k]. \tag{A-4}$$

We recall from (4-10) that

$$w(p) = \frac{f\left(\frac{p}{1-p}\right)}{1 + f\left(\frac{p}{1-p}\right)}. \tag{A-5}$$

A.4.1
Proof that $w(p)$ is continuous, strictly increasing and onto

f given by (4-24) and the mapping $p \mapsto \frac{p}{1-p}$ are continuous, strictly increasing and onto $]0, \infty[$. The asserted properties of w follow easily from (A-5).

A.4.2
Finding the fixed points of f and w

Let $z = \frac{p}{1-p}$. By (4-10), there is a one-to-one correspondence between fixed points of f and w because $w(p^*) = p^*$ if and only if $f(z^*) = z^*$.

Consider first the case of $c = 1/2$. Then $f(z) = az$. It has no fixed point when $a \neq 1$. Every point is a fixed point when $a = 1$. From now on, $c \neq 1/2$ is assumed in this proof.

Start searching for fixed points of (A-4). For each $z \in]c^{k+1}, c^k]$, write

$$z = c^{k+1}\mu,$$

where $1 < \mu \leq \frac{1}{c}$.

Solving the equation for fixed points z^* of f,

$$\begin{aligned}
&f(z) = z \Leftrightarrow \\
&a[(2c)^{-k}z + 2^{-k}(1-2c)] = z \Leftrightarrow \\
&a[(2c)^{-k}c^{k+1}\mu + 2^{-k}(1-2c)] = c^{k+1}\mu \Leftrightarrow \\
&a[c^{-k}c^{k+1}\mu + (1-2c)] = 2^k c^{k+1}\mu \Leftrightarrow \\
&a[c\mu + (1-2c)] = 2^k c^{k+1}\mu \Leftrightarrow \\
&(ac - 2^k c^{k+1})\mu = -a(1-2c).
\end{aligned} \tag{A-6}$$

We have assumed that $c \neq 1/2$. So, by (A-6), $f(z) = z$ is equivalent to

$$ac - 2^k c^{k+1} \neq 0 \quad \text{and} \quad \mu = \frac{-a(1-2c)}{ac - 2^k c^{k+1}}.$$

Next condition to meet is $1 < \mu \leq 1/c$, i.e,

$$1 < \frac{-a(1-2c)}{ac - 2^k c^{k+1}} \leq \frac{1}{c}$$

which is equivalent to

$$c < \frac{-a(1-2c)}{a - 2^k c^k} \leq 1 \tag{A-7}$$

and we seek k satisfying that inequalities. Momentarily let

$$d := 2c.$$

Then $0 < d < 2$ and (as $c \neq 1/2$ has been assumed) $d \neq 1$. Rewrite (A-7) as

$$d < \frac{2a(1-d)}{d^k - a} \leq 2. \tag{A-8}$$

Separating the cases of $c < 1/2$ and $c > 1/2$ is necessary to control the signs.

Case $c < 1/2$. Then $0 < d < 1$. Seek k such that (A-8) holds.

Equivalently,

$$\frac{1}{2} \leq \frac{d^k - a}{2a(1-d)} < \frac{1}{d}.$$

$$\frac{2a(1-d)}{2} \leq d^k - a < \frac{2a(1-d)}{d}.$$

$$a(2-d) \leq d^k < d^{-1}a(2-d).$$

$$\log_d(a(2-d)) - 1 < k \leq \log_d(a(2-d)) \tag{A-9}$$

The length of the half-open interval $]\log_d(a(2-d)) - 1, \log_d(a(2-d))]$ is equal to 1. There is exactly one integer which is in this interval, say k^*. So (A-9) has a unique soluton

$$k^* = \lfloor \log_d(a(2-d)) \rfloor,$$

the greatest integer less than or equal to $\log_d(a(2-d))$.

We interrupt the discussion with an illustration. For $a = 3$ and $c = 1/3$, inequality (A-9) has only one integer solution, which is $k^* = -4$ and then $z^* \approx 39.26$. We check that with Figure A.1.

Below is for the case of $c > 1/2$. We just repeat the above, along with a few simple adjustments.

Case $c > 1/2$. Then $1 < d < 2$. Seek k such that (A-8) holds.

Equivalently,

$$\frac{1}{2} \leq \frac{d^k - a}{2a(1-d)} < \frac{1}{d}.$$

$$\frac{2a(1-d)}{2} \geq d^k - a > \frac{2a(1-d)}{d}.$$

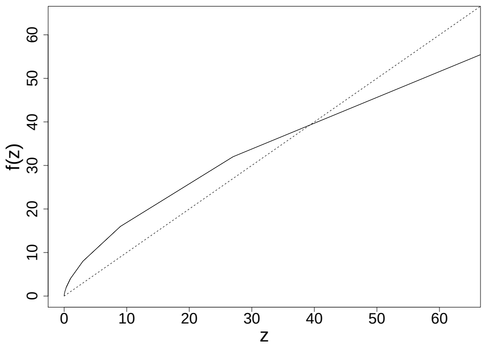


Figure A.1: Fixed point (intersection) of the Piecewise Linear Function with $a = 3$, $c = 1/3$. $k^* = -4$, $z^* \approx 39.26$.

$$d^{-1}a(2-d) < d^k \leq a(2-d)$$

$$\log_d(a(2-d)) - 1 < k \leq \log_d(a(2-d)). \quad \text{(A-10)}$$

So the unique solution for (A-10) is

$$k^* = \lfloor \log_d(a(2-d)) \rfloor.$$

Summarizing, in both cases ($0 < d < 1$ or $1 < d < 2$) f has a unique fixed point z^*. It is contained in the interval $]c^{k^*+1}, c^{k^*}]$, where $k^* = \lfloor \log_d(a(2-d)) \rfloor$. Furthermore, $z^* = c^{k^*+1}\mu$, $\mu = \frac{-a(1-2c)}{ac-2^k c^{k+1}}$ and $1 < \mu \leq \frac{1}{c}$.

A.4.3
Proof that w is regressive

Let z^* be the unique fixed point of f. By (4-10), the assertion is equivalent to $f(z) > z$, for $z \in]0, z^*[$ and $f(z) < z$, for $z \in]z^*, \infty[$.

f being a continuous function, we need only sample one point on each side.

Case of $c < 1/2$. Recall that $k = k^*$, the **largest** integer such that

$$a(2-2c) \leq (2c)^{k^*} \quad \text{(A-11)}$$

and consequently

$$a(2-2c) > (2c)^{k^*+1}. \quad \text{(A-12)}$$

A sample point z which is on the left side of $z^* \in]c^{k^*+1}, c^{k^*}]$ is $z = c^{k^*+1}$. We have to prove that $f(c^{k^*+1}) > c^{k^*+1}$. Evaluate f at that point we get

$$\begin{aligned} f(c^{k^*+1}) &= a[(2c)^{-k^*}c^{k^*+1} + 2^{-k^*}(1-2c)] \\ &= a[2^{-k^*}c + 2^{-k^*}(1-2c)] = a2^{-k^*}(1-c) \\ &= 2^{-(k^*+1)}[a(2-2c)] \end{aligned}$$

and by (A-12),

$$f(c^{k^*+1}) > 2^{-(k^*+1)}(2c)^{k^*+1} = c^{k^*+1}.$$

We claim that it is possible to extend the result to the interval $]0, z^*[$. It means $f(z) > z$ on $]0, z^*[$. We shall prove this claim later.

Next we inspect what happens to points on the right of z^*.

A sample point z which is on the right side of $z^* \in]c^{k^*+1}, c^{k^*}]$ is $z = c^{k^*-1}$ which belongs to $]c^{k^*}, c^{k^*-1}]$. Evaluate f at that point we get

$$\begin{aligned} f(c^{k^*-1}) &= a[(2c)^{-(k^*-1)}c^{k^*-1} + 2^{-(k^*-1)}(1-2c)] \\ &= a[2^{-k^*+1} + 2^{-k^*+1}(1-2c)] = 2^{-k^*+1}[a(2-2c)] \end{aligned}$$

and by (A-11),

$$f(c^{k^*-1}) \leq 2^{-k^*+1}(2c)^{k^*} = 2c^{k^*} = (2c)c^{k^*-1} < c^{k^*-1}.$$

We can extend the result to the interval $]z^*, \infty[$.

Case of $c > 1/2$. The determination whether $f(z) < z$ and/or $f(z) > z$ on the two sides of z^* can be tracked in a similar manner.

Now we prove the claim of the extension by continuity.

Take z_1, z_2 on $]0, z^*[$. If $f(z_1) > z_1$ and $f(z_2) < z_2$, since f is continuous, then by the Intermediate Value Theorem, there exists a point z_3, lying between z_1 and z_2, which is a fixed point of f. Therefore, once we know that f has only one fixed point z^*, then on $]0, z^*[$, or on any interval where f has no fixed point, we cannot have both $f(z_1) > z_1$ and $f(z_2) < z_2$ for some pair z_1 and z_2. In other words, we either have $f(z) < z$ for all z, or $f(z) > z$ for all z on $]0, z^*[$. That is what we meant when we said, we need only sample one point to make a decision.

A.4.4
Convexity of w

Let $f :]0, \infty[\to]0, \infty[$ defined by (A-4) and $w :]0, 1[\to]0, 1[$ given by

$$w(p) = \frac{f\left(\frac{p}{1-p}\right)}{1 + f\left(\frac{p}{1-p}\right)}. \tag{A-13}$$

We proceed to find the first and the second derivative of w on each interval $\left]\frac{c^{k+1}}{1+c^{k+1}}, \frac{c^k}{1+c^k}\right]$. Using (A-13) and after some calculations we get,

$$w'(p) = \frac{a(2c)^{-k}}{[(a(2c)^{-k} - a2^{-k}(1-2c) - 1)p + 1 + a2^{-k}(1-2c)]^2} \tag{A-14}$$

and

$$w''(p) = \frac{2a(2c)^{-2k}[ac^k(1-2c) + (2c)^k - a]}{[(a(2c)^{-k} - a2^{-k}(1-2c) - 1)p + 1 + a2^{-k}(1-2c)]^3}. \tag{A-15}$$

Let us look at the function

$$h(k) = ac^k(1-2c) + (2c)^k - a$$

which appears in the numerator of (A-15). When $h > 0$, w will be convex and when $h < 0$, w will be concave. We divide it in two cases:

1) Convexity on the interval $\left]\frac{c^{k+1}}{1+c^{k+1}}, \frac{c^k}{1+c^k}\right]$*:*

First case $c < 1/2$**:** In this case, $1 - 2c > 0$ and h is strictly decreasing. Furthermore, $k \to -\infty$ implies $h \to \infty$ and $k \to \infty$ implies $h \to -a < 0$. It means that there is a greatest integer k^{**} such that h is non-negative. It means, $h(k) \geq 0$ on $]-\infty, k^{**}]$ and $h(k) < 0$ on $]k^{**}, \infty[$.

Therefore, there is $p^{**} \in]0,1[$ such that w is a piecewise concave function on $]0, p^{**}[$ and piecewise convex on $]p^{**}, 1[$.

Second case $c > 1/2$**:** In this case, $1-2c < 0$ and h is strictly increasing. Furthermore, $k \to -\infty$ implies $h \to -\infty$ and $k \to \infty$ implies $h \to \infty$. It means that there is a smallest integer k^{**} such that h is non-negative. It means, $h(k) < 0$ on $]-\infty, k^{**}[$ and $h(k) \geq 0$ on $[k^{**}, \infty[$.

Therefore, w is a piecewise convex function on $]0, p^{**}[$ and piecewise concave on $]p^{**}, 1[$.

Now, let us analyse what happens around each point $c^k/(1+c^k)$. We will denote $[w']^-$ the derivative of w at $c^k/(1+c^k)$ on the interval $]\frac{c^{k+1}}{1+c^{k+1}}, \frac{c^k}{1+c^k}]$. In the same way, $[w']^+$ is the limit derivative of w at $c^k/(1+c^k)$ on the interval $\left]\frac{c^k}{1+c^k}, \frac{c^{k-1}}{1+c^{k-1}}\right]$. Using (A-14) at the point $c^k/(1+c^k)$, we get,

$$[w']^- = \frac{a(2c)^{-k}(1+c^k)^2}{[a2^{-k+1}(1-c)+1]^2}$$

and

$$[w']^+ = \frac{a(2c)^{-k+1}(1+c^k)^2}{[a2^{-k+1}(1-c)+1]^2}.$$

Thus,

$$\frac{[w']^+}{[w']^-} = 2c. \tag{A-16}$$

2) Convexity on an open interval which includes the point $\frac{c^k}{1+c^k}$*:*

The ratio (A-16) tells us the relation between the tangent lines on each side of the point $c^k/(1+c^k)$.

First case $c < 1/2$**:** The slope of the tangent line from the left ($[w']^-$) is greater than the slope of the tangent line from the right ($[w']^+$). It means that if w is concave on an interval I having $c^k/(1+c^k)$ as the right end point and is also concave on an interval J having $c^k/(1+c^k)$ as the left end point, then w is concave on the union $I \cup J$. This implies the concavity of w on the union of all adjacent intervals $]\frac{c^{k+1}}{1+c^{k+1}}, \frac{c^k}{1+c^k}]$ where w is concave piece by piece.

Second case $c > 1/2$**:** The slope of the tangent line from the left ($[w']^-$) is less than the slope of the tangent line from the right ($[w']^+$). This implies the convexity of w on the union of all adjacent intervals $]\frac{c^{k+1}}{1+c^{k+1}}, \frac{c^k}{1+c^k}]$ where w is convex piece by piece.

Putting together 1) and 2), we have two cases:

a) For $c < 1/2$: there is p^{**} such that w is concave on $]0, p^{**}[$ and piecewise convex on $]p^{**}, 1[$;

b) For $c > 1/2$: there is p^{**} such that w is convex on $]0, p^{**}[$ and piecewise concave on $]p^{**}, 1[$.

A.5
Proof of the Proposition 4.6

The Mean Value Theorem states that, if f is continuous on $[x_1, x_2]$ and differentiable on $]x_1, x_2[$, where $x_1 < x_2$. Then there exist $y \in]x_1, x_2[$ such that

$$f'(y) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{A-17}$$

Suppose that $f' \geq 0$ and that $f' > 0$ on a dense subset of $]0, \infty[$. From $f' \geq 0$ and (A-17) we get that $f(x_2) \geq f(x_1)$ for all $x_2 > x_1$. So f is increasing. If there were $x_1 < x_2$ such that $f(x_1) = f(x_2)$ then the increasing f is constant on $[x_1, x_2]$. So $f'(y) = 0$ for all $y \in]x_1, x_2[$. But that contradicts the assumption that $f' > 0$ on a dense subset of $]0, \infty[$. This proves that f is strictly increasing. Conversely, if f is strictly increasing then, $f' \geq 0$ comes from the definition of the derivative. Additionally, for any $x_1 < x_2$, we have $f(x_1) < f(x_2)$ and so the equation (A-17) implies $f'(y) > 0$ for some $y \in]x_1, x_2[$. Thus, $f' > 0$ on a dense subset.

A.6
Proof of the Proposition 4.8

From the equation (4-15),

$$f(z) = \phi(\log_2 z)z^b \Leftrightarrow \log_2[f(z)] = \log_2[\phi(\log_2 z)] + b\log_2 z. \tag{A-18}$$

It is known that f and $\log_2(\cdot)$ are strictly increasing functions. So, for $z_1 > z_0$ we get from (A-18),

$$\begin{aligned} &\log_2[\phi(\log_2 z_1)] + b\log_2 z_1 > \log_2[\phi(\log_2 z_0)] + b\log_2 z_0 \\ &\Leftrightarrow \frac{\log_2[\phi(\log_2 z_1)] - \log_2[\phi(\log_2 z_0)]}{\log_2 z_1 - \log_2 z_0} > -b \end{aligned}$$

defining $u := \log_2 z$, and (4-27), that $\Phi(u) := \log_2 \phi(u)$, we get what we want.

A.7
Proof of the Proposition 4.9

f is continuous, strictly increasing and onto, so f^{-1} exists. Take any pair (x, y) of positive real numbers and define $z = f^{-1}[f(x) + f(y)]$. It means that $f(z) = f(x) + f(y)$ and so $f(tz) = f(tx) + f(ty)$ follows from the implication. The latter means $tz = f^{-1}[f(tx)+f(ty)]$. Therefore, $f^{-1}[f(tx)+f(ty)] = tz = tf^{-1}[f(x) + f(y)]$ for all $t > 0$.

The pair (x, y) being arbitrary, we arrive at

$$f^{-1}[f(tx) + f(ty)] = tf^{-1}[f(x) + f(y)], \quad \forall t, x, y > 0.$$

This last functional equation is solved in Luce(2001) (Proposition 1). The solutions are $f(z) = az^b$ with constants $a > 0, b > 0$.

B
Appendix

In this section, we present the proofs of all the results mentioned in the Chapter 5.

B.1
Index impact for the NEO-Additive weighting funtions

Let A and B be the random variables which take values a and b. The joint density function is $g(a, b)$ and the marginal density function of A is $g_A(a)$.

Therefore, we have that

$$E_b(x|a) = \int_0^{1-a} x \cdot \frac{g(a,b)}{g_A(a)} db \tag{B-1}$$

where $x \in \{a, b\}$ and

$$g_A(a) = \int_0^{1-a} g(a,b) db. \tag{B-2}$$

Remembering that $E_b(\cdot)$ is a linear function,

$$E_b(Y|a) = E_b((ak_a + bk_b + k)|a) = k_a E_b(a|a) + k_b E_b(b|a) + k \tag{B-3}$$

and using the expression (B-1) we get $E_b(a|a) = a$ and then

$$E_b(Y|a) = ak_a + k + k_b E_b(b|a). \tag{B-4}$$

Now we have to take the variance of $E_b(Y|a)$ over a. In (B-4), the term k is a constant, so

$$Var_a(E_b(Y|a)) = Var_a(ak_a + k_b E_b(b|a))$$

and then,

$$Var_a(E_b(Y|a)) = (k_a)^2 Var_a(a) + (k_b)^2 Var_a(E_b(b|a)) + 2k_a k_b Cov_a(a, E_b(b|a)). \tag{B-5}$$

By symmetry, interchanging a and b yields,

$$Var_b(E_a(Y|b)) = (k_b)^2 Var_b(b) + (k_a)^2 Var_b(E_a(a|b)) + 2k_a k_b Cov_b(b, E_a(a|b)). \tag{B-6}$$

If A is equipped with the uniform distribution on $[0, 1[$ ($A \sim U(0, 1)$) and B with uniform distribution on $]0, 1-a]$, then $g_A(a) = 1$ and $g(a, b) = 1/(1-a)$.

In this case, using (B-1) we get

$$E_b(b|a) = \frac{1-a}{2}, \tag{B-7}$$

and

$$E_b(Y|a) = ak_a + \frac{1-a}{2}k_b + k. \tag{B-8}$$

Since $A \sim U(0,1)$ then $Var(a) = 1/12$, $Var(E_b(b|a)) = Var((1-a)/2) = 1/4 \cdot 12$ and $Cov_a(a, E_b(b|a)) = Cov_a(a, (1-a)/2) = -1/2 \cdot 12$.

From (B-5) we get,

$$Var_a(E_b(Y|a)) = \frac{1}{12}(k_a)^2 + \frac{1}{4 \cdot 12}(k_b)^2 - \frac{1}{12}k_a k_b. \tag{B-9}$$

By symmetry,

$$Var_b(E_a(Y|b)) = \frac{1}{12}(k_b)^2 + \frac{1}{4 \cdot 12}(k_a)^2 - \frac{1}{12}k_a k_b. \tag{B-10}$$

Hence $I_a \geq I_b$ iff,

$$Var_a(E_b(Y|a)) - Var_b(E_a(Y|b)) \geq 0 \Leftrightarrow \frac{1}{16}[(k_a)^2 - (k_b)^2] \geq 0 \Leftrightarrow |k_a| \geq |k_b|. \tag{B-11}$$

B.2
Measures for the CRS weighting funtions

Calculation of MAD for the CRS weighting function family.

Case (i) $b = 1$. In that case, $d = 0$ and so there is no interval where d is strictly increasing/decreasing. Hence $MAD = 0$.

Case (ii) $0 < a < 1$ and $b \neq 1$.

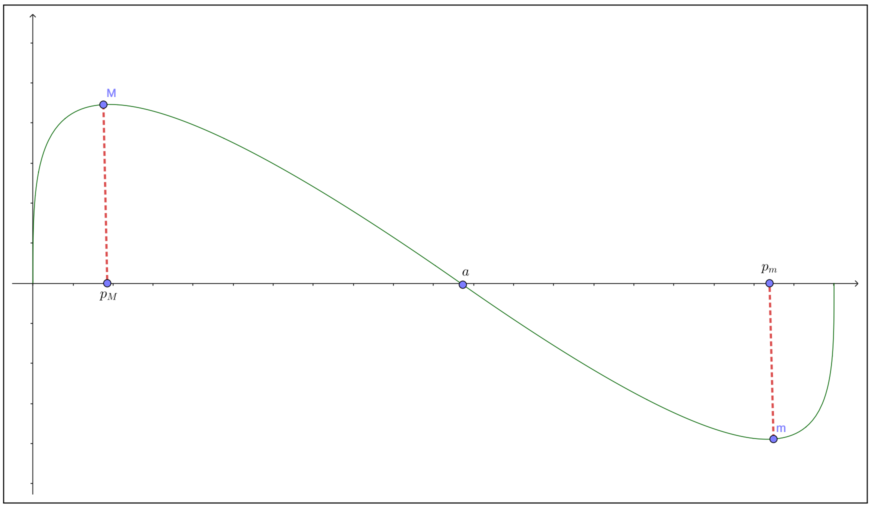


Figure B.1: Graph of $d(p) = w(p) - p$ when $b < 1$

Figure B.1 shows the graph of $d(p) = w(p) - p$, when $b < 1$. We reach the local maximum (point "M", $p_M \in [0, a]$) before the local minimum (point "m",$p_m \in]a, 1]$). (The opposite happens when $b > 1$ where p_m comes first). By the expression (5-23) we can calculate the coordinates of "M" and "m" by using the derivative of d on $]0,1[$.

$$d'(p) = \begin{cases} ba^{1-b}p^{b-1} - 1, & 0 < p \leq a \\ b(1-a)^{1-b}(1-p)^{b-1} - 1, & a < p < 1. \end{cases} \tag{B-12}$$

For $b < 1$, $p_M \in [0, a]$ and by $d'(p_M) = 0$ we find $p_M = ab^{\frac{1}{1-b}}$. On the other hand, if $b > 1$ then $p_M \in [a, 1]$ and we find $p_M = 1 - (1-a)b^{\frac{1}{1-b}}$.

Similarly, for $b < 1$, $p_m \in [a, 1]$ and $p_m = 1 - (1-a)b^{\frac{1}{1-b}}$. For $b > 1$, $p_m \in [0, a]$ and we find $p_m = ab^{\frac{1}{1-b}}$.

So, in both situaltions ($b < 1$ or $b > 1$), we have

$$MAD = \frac{TV(d)}{2} = d(p_M) - d(p_m) = |1-b|b^{\frac{b}{1-b}}. \tag{B-13}$$

Case (iii) $a = 0$ and $b \neq 1$.

$$w(p) = 1 - (1-p)^b, \quad p \in [0, 1] \tag{B-14}$$

and

$$d'(p) = b(1-p)^{b-1} - 1, \quad p \in]0, 1[. \tag{B-15}$$

The point p^* were $d'(p^*) = 0$ is given by $p^* = 1 - b^{1/(1-b)}$. If $b < 1$ it gives a local minimum and if $b > 1$ it gives a local maximum. Therefore, $MAD = TV(d)/2 = |d(p^*)| = |1-b|b^{b/(1-b)}$.

Case (iv) $a = 1$ and $b \neq 1$.

$$w(p) = p^b, \quad p \in [0, 1] \tag{B-16}$$

and

$$d'(p) = bp^{b-1} - 1, \quad p \in]0, 1[. \tag{B-17}$$

The point p^* where $d'(p^*) = 0$ is given by $p^* = b^{1/(1-b)}$. If $b < 1$ it gives a local maximum and if $b > 1$ it gives a local minimum. Therefore, $MAD = TV(d)/2 = |d(p^*)| = |1-b|b^{b/(1-b)}$.

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