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A Solution Methodology for Wasserstein-based Data-Driven Distributionally Robust Problems with Right-Hand-Sided Rectangular-Supported Uncertainty

Autor(es):

Carlos Andrés Gamboa Rodríguez

PUC
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Organizadores: Fernanda Baião / Soraida Aguilar

Layout da Capa: Aline Magalhães dos Santos

Self presentation

- Name: Carlos Andrés Gamboa Rodríguez
- PhD Student
- Period: 8
- Advisor: Davi Valladão
- Co-Advisor: Alexandre Street

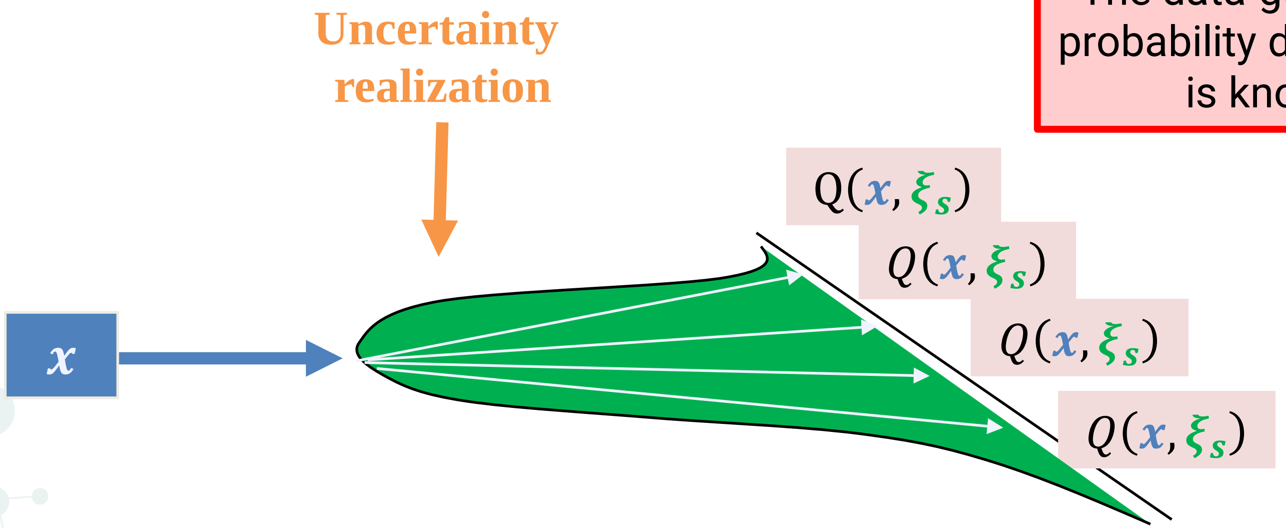
Theoretical background

Stochastic optimization

- Two-stage model

$$\min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{c}^\top \boldsymbol{x} + \mathbb{E}[Q(\boldsymbol{x}, \xi)]$$

The data-generating probability distribution is known

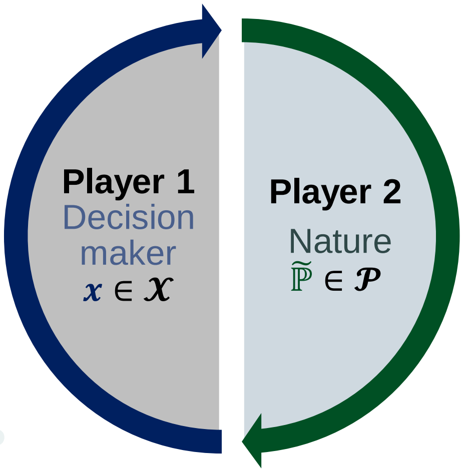


Distributionally robust optimization

- DRO two-stage model

$$\min_{\boldsymbol{x}\in\mathcal{X}} \boldsymbol{c}^\top \boldsymbol{x} + \sup_{\tilde{\mathbb{P}}\in\mathcal{P}} \mathbb{E}^{\tilde{\mathbb{P}}}[Q(\boldsymbol{x},\boldsymbol{\xi})]$$

Game theory interpretation



Ambiguity set:

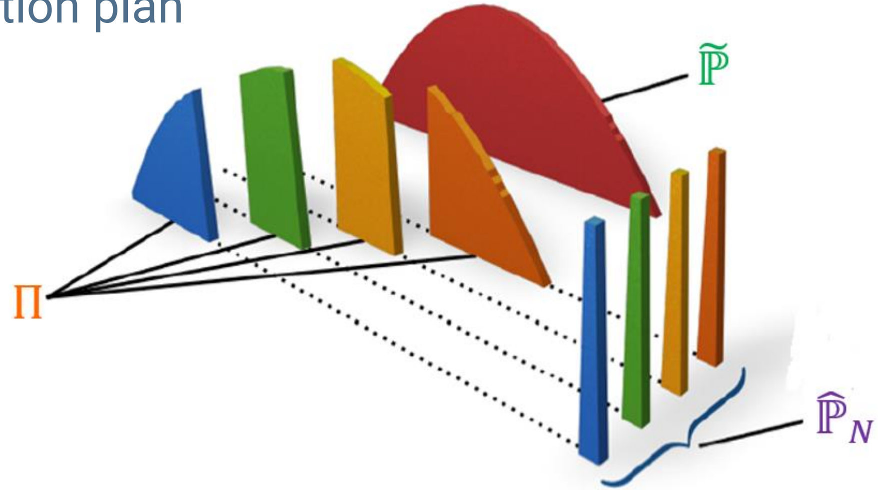
- Moment-based
- Discrepancy-based

Wasserstein ball

- Wasserstein distance

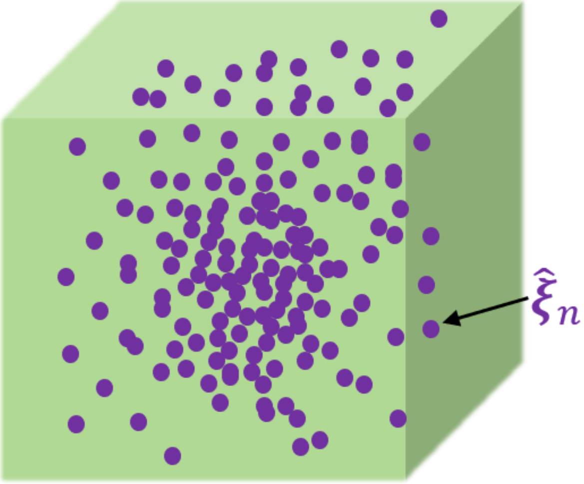
$$d_w(\widetilde{\mathbb{P}}, \widehat{\mathbb{P}}_N) = \inf\left\{\int_{\Xi^2} \|\xi - \hat{\xi}_n\| d\Pi(\xi, \hat{\xi}_n) : \begin{array}{c} \Pi \textit{ is a joint distribution of } \xi \textit{ and } \hat{\xi}_n \\ \textit{with marginals } \widetilde{\mathbb{P}} \textit{ and } \widehat{\mathbb{P}}_N, \textit{respectively} \end{array}\right\}$$

- Transportation plan



Wasserstein ball

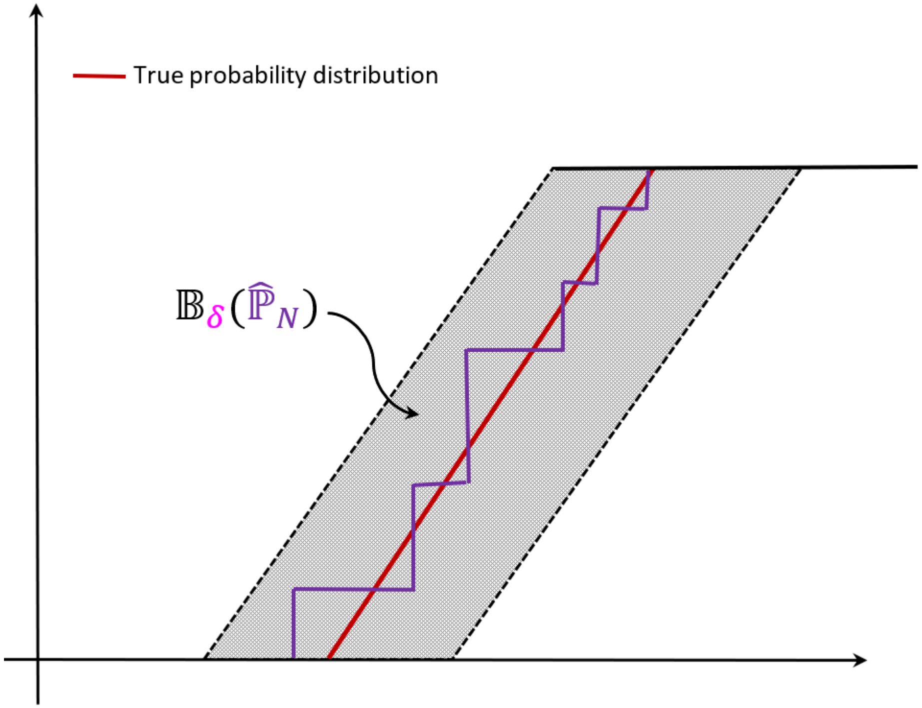
$$\Xi = \times_{i=1}^{d_\xi} [a_i, b_i]$$



Reference distribution

$$\widehat{\mathbb{P}}_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{\hat{\xi}_n}$$

$$\mathbb{B}_\delta d_w(\widehat{\mathbb{P}}_N) = \{\widetilde{\mathbb{P}} \in \mathcal{M}(\Xi): d_w(\widetilde{\mathbb{P}}, \widehat{\mathbb{P}}_N) \leq \delta\}$$



Reformulation of the problem

Equivalent semi-infinite optimization problem

- Parametric linear programs with RHS uncertainty

$$Q(\boldsymbol{x}, \xi) = \min_{\boldsymbol{y} \geq 0} \ \boldsymbol{q}^\top \boldsymbol{y}$$
$$\text{s.t.} \quad \boldsymbol{W}\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{x})\xi + \boldsymbol{r}(\boldsymbol{x})$$

$$\min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{c}^\top \boldsymbol{x} + \sup_{\widetilde{\mathbb{P}} \in \mathcal{P}} \mathbb{E}^{\widetilde{\mathbb{P}}}[Q(\boldsymbol{x}, \xi)]$$

⟺

$$\begin{aligned} \min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \frac{1}{N}\sum_{\boldsymbol{n}=\boldsymbol{1}}^{\boldsymbol{N}} s_n \\ \text{s.t.} \quad & Q(\boldsymbol{x}, \xi) - \lambda\|\xi - \hat{\xi}_n\| \leq s_n, \quad \forall n \leq N, \forall \xi \in \Xi, \\ & \lambda \geq 0, \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

Finite extensive equivalent formulation

Semi-infinite constraint

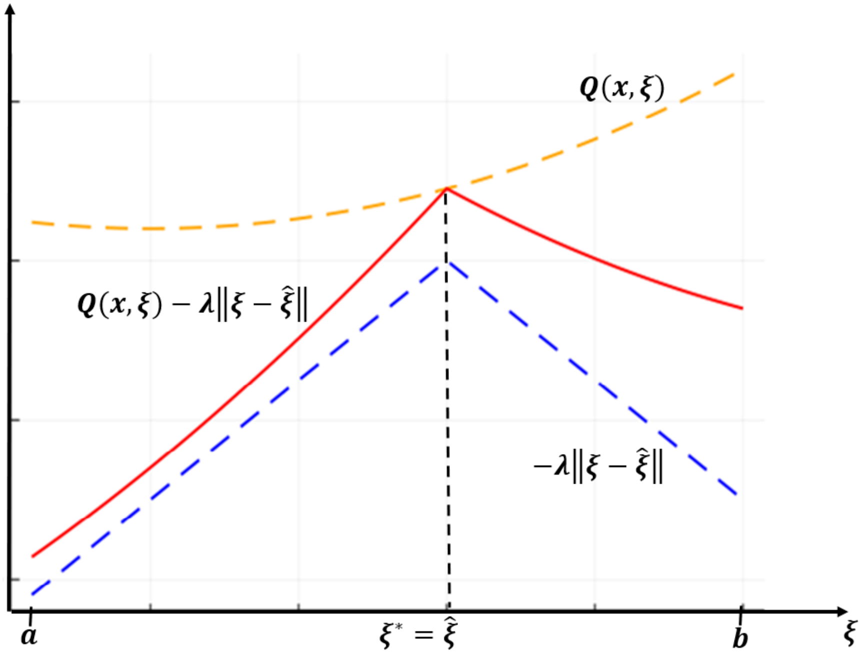
$$Q(\boldsymbol{x}, \xi) - \lambda \left\| \xi - \hat{\boldsymbol{\xi}}_{\boldsymbol{n}} \right\|_1 \leq s_n, \forall n \leq N, \forall \xi \in \Xi$$

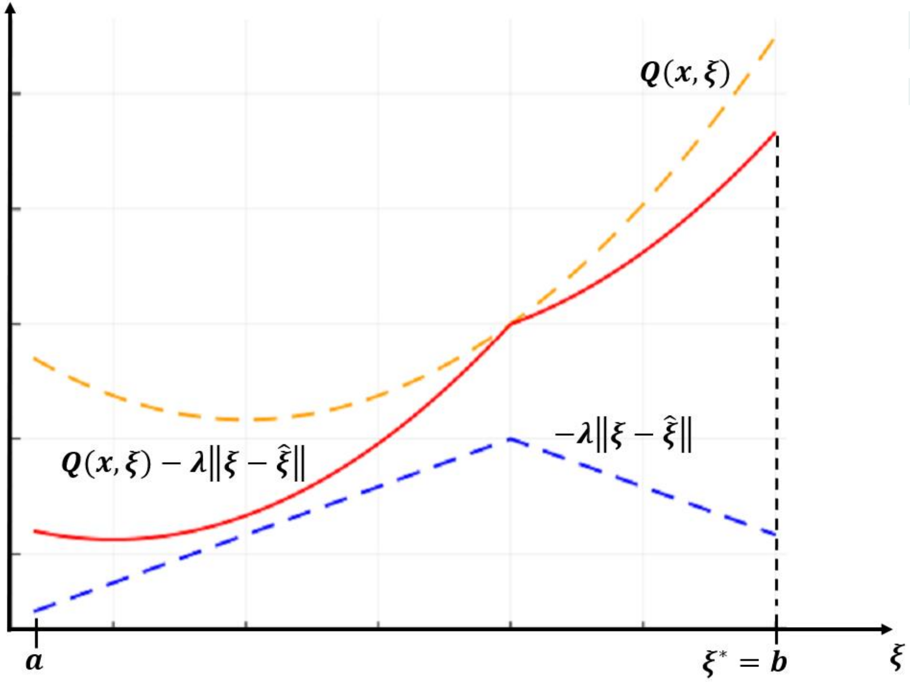
$$\sup_{\xi \in \Xi} \left(Q(\boldsymbol{x}, \xi) - \lambda \left\| \xi - \hat{\xi}_n \right\|_1 \right) \leq s_n, \forall n \leq N$$

Oracle problem

$$\begin{aligned} \min_{\boldsymbol{\theta}, \boldsymbol{\alpha}, \xi} \quad & \boldsymbol{\theta}^\top \boldsymbol{r}(\boldsymbol{x}) + [\boldsymbol{H}^\top(\boldsymbol{x})\boldsymbol{\theta}]^\top \xi - \lambda \mathbf{1}^\top \boldsymbol{\alpha} \\ s.t. \quad & \boldsymbol{\alpha} \geq \xi - \hat{\xi}_n, \\ & \boldsymbol{\alpha} \geq \hat{\xi}_n - \xi, \\ & \boldsymbol{W}^\top \boldsymbol{\theta} \leq \boldsymbol{q}, \\ & \xi \in \Xi \end{aligned}$$

Solution of the oracle problem





Finite extensive equivalent formulation

Proposition

There exists an optimal solution ξ^* for the oracle problem such that $\xi_i^* \in \left\{\left(\hat{\boldsymbol{\xi}}_{\boldsymbol{n}}\right)_i, a_i, b_i\right\}$ for all $i = 1, \dots, d_\xi$

- Optimal solution of the oracle problem

$$\hat{\Xi}_n = \times_{i=1}^{d_\xi} \left\{a_i, \left(\hat{\boldsymbol{\xi}}_{\boldsymbol{n}}\right)_i, b_i\right\} = \{\boldsymbol{\xi}_\ell^*\}_{\ell \in \mathcal{L}_n}, \text{onde } \mathcal{L}_n = \left\{1, \dots, \left|\hat{\Xi}_n\right|\right\}$$

Finite extensive equivalent formulation

Proposed reformulation

$$\min_{\boldsymbol{x}, \lambda, s_n} \quad \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \frac{1}{N}\sum_{\boldsymbol{n}=\mathbf{1}}^{\boldsymbol{N}} s_n$$

$$s.t. \quad Q(\boldsymbol{x}, \boldsymbol{\xi}_\ell^*) - \lambda\left\|\boldsymbol{\xi}_\ell^* - \hat{\boldsymbol{\xi}}_n\right\|_1 \le s_n, \quad \forall \ell \in \mathcal{L}_n, \forall n \le N$$

$$\lambda \ge 0,$$

$$\boldsymbol{x} \in \mathcal{X}$$

Existing reformulations

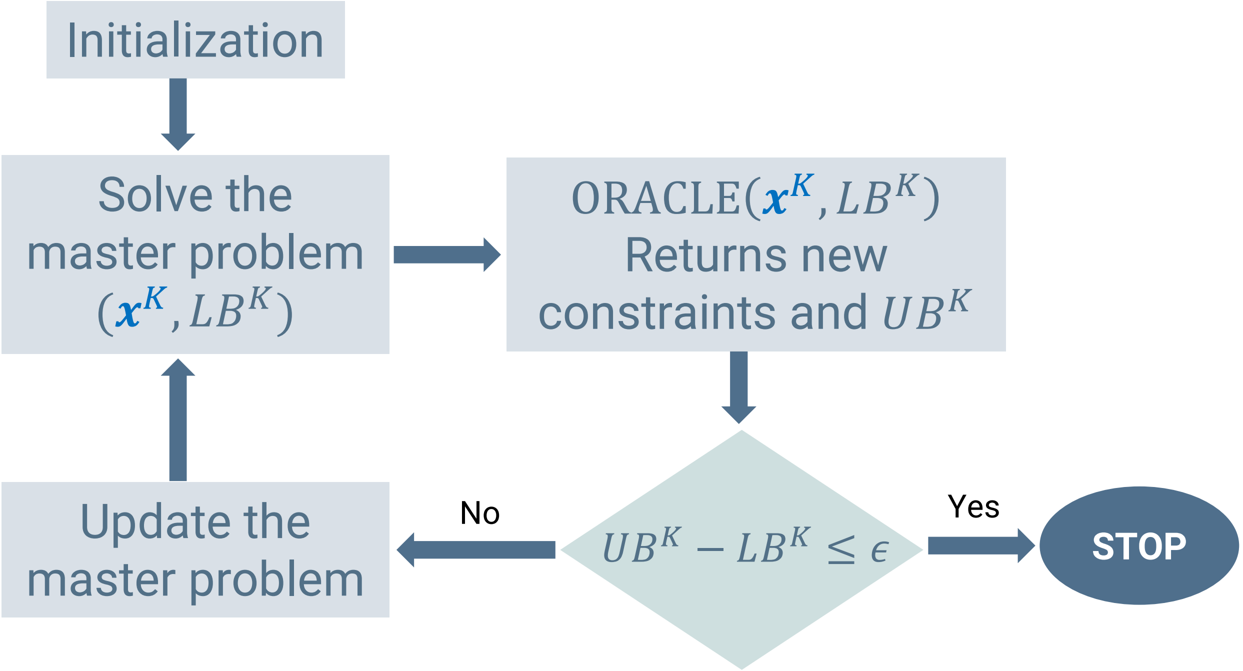
Mehrotra: Computational intractable $(N \cdot 3^{d_\xi})$

Kuhn: The resulting PL has generically exponential size

Solution methodology

Decomposition method

- Iterative procedure that converges to the optimal solution of the problem



Column and constraint generation

- Master problem is a linear programming relaxation of the original problem

$$\mathcal{L}_n^K \subseteq \mathcal{L}_n$$

Original problem

$$\begin{aligned} \min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \frac{1}{N}\sum_{n=1}^{N} s_n \\ s.t. \quad & Q(\boldsymbol{x}, \xi_\ell^*) - \lambda\|\xi_\ell^* - \hat{\xi}_n\|_1 \le s_n, \quad \forall \ell \in \mathcal{L}_n, \forall n \le N \\ & \lambda \ge 0, \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

Relaxed problem

$$\begin{aligned} \min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \frac{1}{N}\sum_{n=1}^{N} s_n \\ s.t. \quad & q^\top \boldsymbol{y}_\ell - \lambda\|\xi_\ell^* - \hat{\xi}_n\|_1 \le s_n, \quad \forall \ell \in \mathcal{L}_n^K, \forall n \le N \\ & \boldsymbol{W}\boldsymbol{y}_\ell = \boldsymbol{H}(\boldsymbol{x})\xi_\ell^* + \boldsymbol{r}(\boldsymbol{x}), \quad \forall \ell \in \mathcal{L}_n^K, \forall n \le N \\ & \lambda \ge 0, \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

C&CG algorithm

Initialization

Solve the master problem $(\boldsymbol{x}^K, LB^K)$

$$\xi^* \in \operatorname{Argmin}\left\{ \begin{array}{c} \boldsymbol{\theta}^\top \boldsymbol{r}(\boldsymbol{x}^K) + [\boldsymbol{H}^\top(\boldsymbol{x}^K)\boldsymbol{\theta}]^\top \xi \\ -\lambda \mathbf{1}^\top \boldsymbol{\alpha} \end{array} \;\middle|\; \begin{array}{c} \boldsymbol{\alpha} \geq \xi - \hat{\xi}_n, \\ \boldsymbol{\alpha} \geq \hat{\xi}_n - \xi, \\ \boldsymbol{W}^\top \boldsymbol{\theta} \leq \boldsymbol{q}, \\ \xi \in \Xi \end{array} \right\}$$

$$UB^K - LB^K \leq \epsilon$$

Yes → STOP

$$UB^K \leftarrow min\left\{ UB^{K-1}, \boldsymbol{c}^\top \boldsymbol{x}^K + \lambda^K \delta + \frac{1}{N}\sum_{n=1}^{N} \rho(\boldsymbol{x}^K, \lambda^K) \right\}$$

No →

$$\mathcal{L}_n^{K+1} = \mathcal{L}_n^K \cup \{\xi^*\}$$

$$\mathbf{q}^\top \boldsymbol{y}_\ell - \lambda \|\xi_\ell^* - \hat{\xi}_n\|_1 \leq s_n, \quad \forall \ell \in \mathcal{L}_n^{K+1}, \forall n \leq N$$

$$\boldsymbol{W}\boldsymbol{y}_\ell = \boldsymbol{H}(\boldsymbol{x})\xi_\ell^* + \boldsymbol{r}(\boldsymbol{x}), \quad \forall \ell \in \mathcal{L}_n^{K+1}, \forall n \leq N$$

Bender's multi-cut

- Master problem for the Bender's multi-cut algorithm

$$(\boldsymbol{\theta}_n^k, \boldsymbol{\xi}_n^k) \in \text{Argmin}\{\text{ORACLE}(\boldsymbol{x}^k, \lambda^k)\}$$

Original problem

$$\begin{aligned} \min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \frac{1}{N}\sum_{n=1}^{N} s_n \\ s.t. \quad & \boldsymbol{\theta}_d^\top \boldsymbol{r}(\boldsymbol{x}) + [\boldsymbol{H}^\top(\boldsymbol{x})\boldsymbol{\theta}_d]^\top \xi_\ell^* - \lambda \left\| \xi_\ell^* - \hat{\xi}_n \right\|_1 \le s_n, \quad \forall (d, \ell) \in \mathcal{D} \times \mathcal{L}_n, \forall n \le N \\ & \lambda \ge 0, \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

Relaxed problem

$$\begin{aligned} \min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \frac{1}{N}\sum_{n=1}^{N} s_n \\ s.t. \quad & [\boldsymbol{\theta}_n^k]^\top \boldsymbol{r}(\boldsymbol{x}) + \left[\boldsymbol{H}^\top(\boldsymbol{x})\boldsymbol{\theta}_n^k\right]^\top \xi_n^k - \lambda \left\| \xi_n^k - \hat{\xi}_n \right\|_1 \le s_n, \quad \forall k \le K, \forall n \le N \\ & \lambda \ge 0, \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

Bender's single-cut

- Master problem for the Bender's single-cut algorithm

$$\mathcal{L} = \bigcup_{n=1}^{N} \mathcal{L}_n$$

Original problem

$$\begin{aligned}
\min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \beta \\
s.t. \quad & \frac{1}{N}\sum_{n=1}^{N} \left(\boldsymbol{\theta}_d^\top \boldsymbol{r}(\boldsymbol{x}) + [\boldsymbol{H}^\top(\boldsymbol{x})\boldsymbol{\theta}_d]^\top \boldsymbol{\xi}_\ell^* \right. \qquad \forall (d,\ell) \in \mathcal{D} \times \mathcal{L}, \\
& \lambda \geq 0, \\
& \boldsymbol{x} \in \mathcal{X}
\end{aligned}$$

Relaxed problem

$$\begin{aligned}
\min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda\delta + \beta \\
s.t. \quad & \frac{1}{N}\sum_{n=1}^{N} \left([\boldsymbol{\theta}_n^k]^\top \boldsymbol{r}(\boldsymbol{x}) + \left[\boldsymbol{H}^\top(\boldsymbol{x})\boldsymbol{\theta}_n^k\right]^\top \boldsymbol{\xi}_n^k \right. \qquad \forall k \leq K, \\
& \lambda \geq 0, \\
& \boldsymbol{x} \in \mathcal{X}
\end{aligned}$$

Results

Numerical experiment

- We test our proposed solution methodology with the two-stage stochastic (single-bus) unit commitment problem

- First-stage decision:
  - Commit decisions
- Second-stage decisions:
  - Economic dispatch

Thermal generators

- Uncertainty:
  - Uncertain net electricity load

Load – Uncertain renewable generation

Results

- Illustrative example of a system with 5 thermal generators

Iteration	C&CG	Benders multi-cut	Benders single-cut
K	$UB-LB$	$UB-LB$	$UB-LB$
1	1273245	1404389	1404388
2	0	147243	282048
3		135559	135559
4		8471	48166
5		8064	19806
6		10656	19806
7		1670	19490
8		1274	16295
9		5548	16295
10		756	16295
11		1278	16295
12		2946	16295
13		760	15564
14		799	15564
15		863	15041
16		799	14310
17		0	13777
40			9142
80			3857
160			2218
165			0
Time (CPUs)	3394	26622	148127

- System with 14 and 54 thermal generators

Method	Syst-14	Syst-54
	Time (CPUs)	Time (CPUs)
C&CG	15365	24225
Bender's multi-cut	27833	-
Bender's single-cut	38379	-

References

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