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A Solution Methodology for Wasserstein-based Data-Driven Distributionally Robust Problems with Right-Hand-Sided Rectangular-Supported Uncertainty

Autor(es):

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Self presentation



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- Period: 8
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Theoretical background

Stochastic optimization

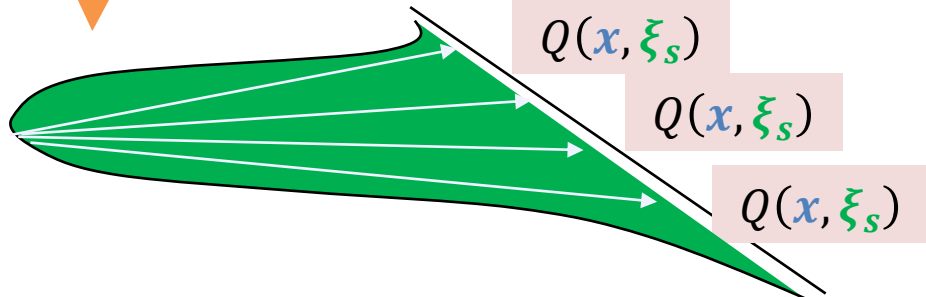
- Two-stage model

$$\min_{\mathbf{x} \in \mathcal{X}} \mathbf{c}^T \mathbf{x} + \mathbb{E}[Q(\mathbf{x}, \xi)]$$

Uncertainty realization

The data-generating probability distribution is known

x

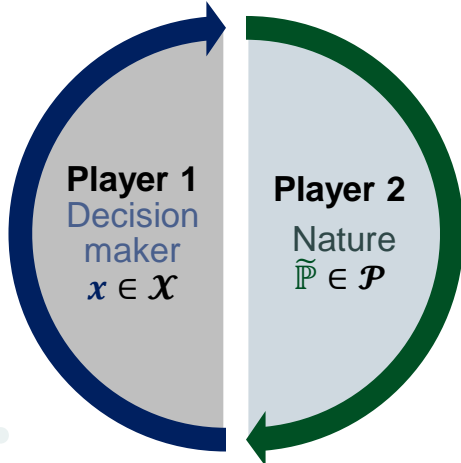


Distributionally robust optimization

- DRO two-stage model

$$\min_{x \in \mathcal{X}} \mathbf{c}^\top \mathbf{x} + \sup_{\tilde{\mathbb{P}} \in \mathcal{P}} \mathbb{E}^{\tilde{\mathbb{P}}} [Q(\mathbf{x}, \xi)]$$

Game theory interpretation



Ambiguity set:

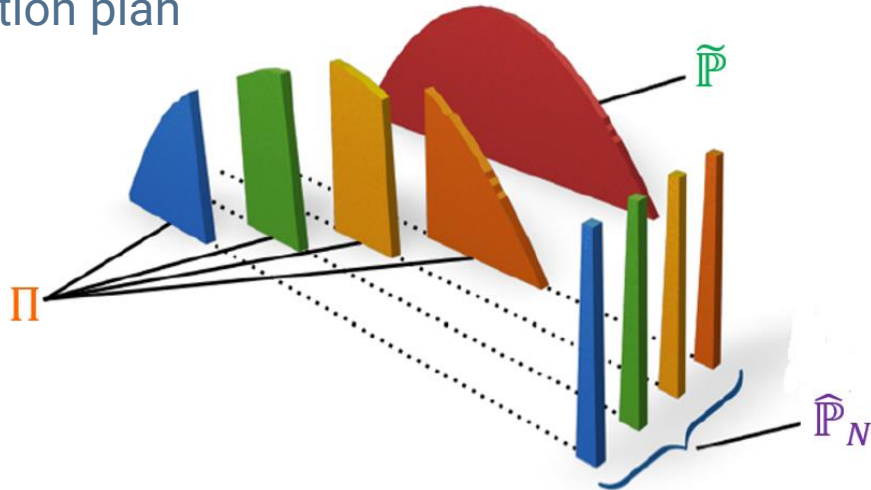
- Moment-based
- Discrepancy-based

Wasserstein ball

- Wasserstein distance

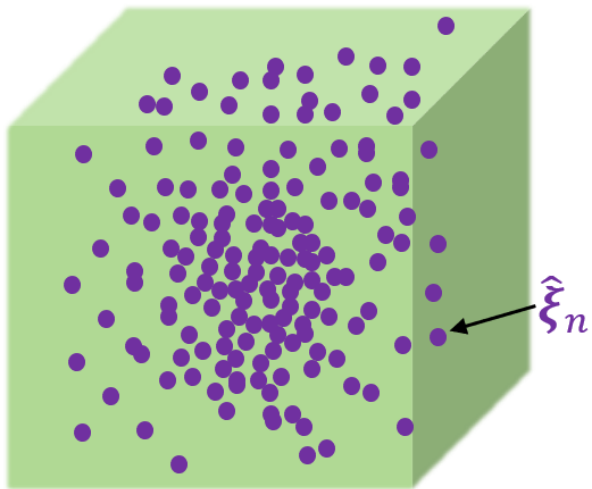
$$d_w(\tilde{\mathbb{P}}, \hat{\mathbb{P}}_N) = \inf \left\{ \int_{\mathbb{E}^2} \|\xi - \hat{\xi}_n\| d\Pi(\xi, \hat{\xi}_n) : \begin{array}{l} \Pi \text{ is a joint distribution of } \xi \text{ and } \hat{\xi}_n \\ \text{with marginals } \tilde{\mathbb{P}} \text{ and } \hat{\mathbb{P}}_N, \text{ respectively} \end{array} \right\}$$

- Transportation plan



Wasserstein ball

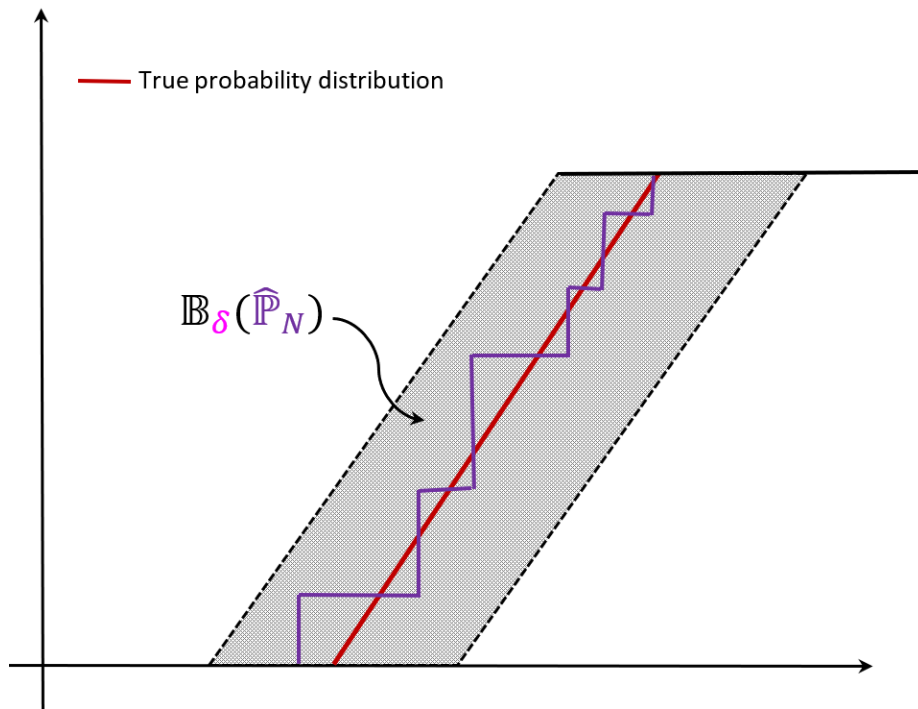
$$\Xi = \times_{i=1}^{d_\xi} [a_i, b_i]$$



Reference distribution

$$\hat{\mathbb{P}}_N = \frac{1}{N} \sum_{n=1}^N \delta_{\hat{\xi}_n}$$

$$\mathbb{B}_\delta d_w(\hat{\mathbb{P}}_N) = \{\tilde{\mathbb{P}} \in \mathcal{M}(\Xi) : d_w(\tilde{\mathbb{P}}, \hat{\mathbb{P}}_N) \leq \delta\}$$





Reformulation of the problem

Equivalent semi-infinite optimization problem

- Parametric linear programs with RHS uncertainty

$$Q(\mathbf{x}, \boldsymbol{\xi}) = \min_{\mathbf{y} \geq 0} \mathbf{q}^\top \mathbf{y}$$
$$s.t. \quad \mathbf{W}\mathbf{y} = \mathbf{H}(\mathbf{x})\boldsymbol{\xi} + \mathbf{r}(\mathbf{x})$$

$$\min_{\mathbf{x} \in \mathcal{X}} \mathbf{c}^\top \mathbf{x} + \sup_{\tilde{\mathbb{P}} \in \mathcal{P}} \mathbb{E}^{\tilde{\mathbb{P}}} [Q(\mathbf{x}, \boldsymbol{\xi})]$$

$$\min_{\mathbf{x}, \lambda, s_n} \mathbf{c}^\top \mathbf{x} + \lambda \delta + \frac{1}{N} \sum_{n=1}^N s_n$$
$$s.t. \quad Q(\mathbf{x}, \boldsymbol{\xi}) - \lambda \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}_n\| \leq s_n, \quad \forall n \leq N, \forall \boldsymbol{\xi} \in \Xi,$$
$$\lambda \geq 0,$$
$$\mathbf{x} \in \mathcal{X}$$

Finite extensive equivalent formulation

Semi-infinite constraint

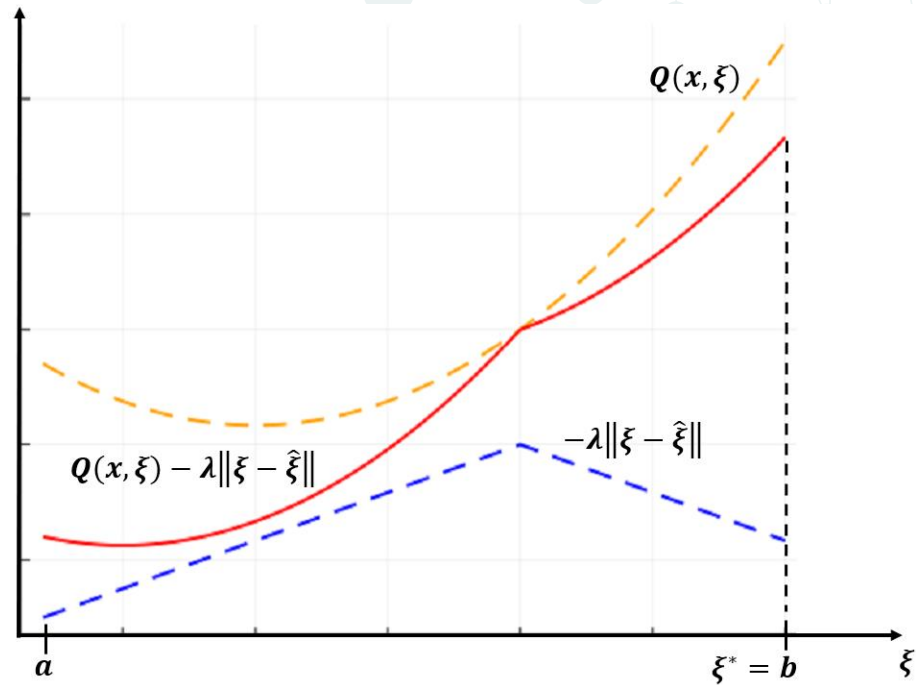
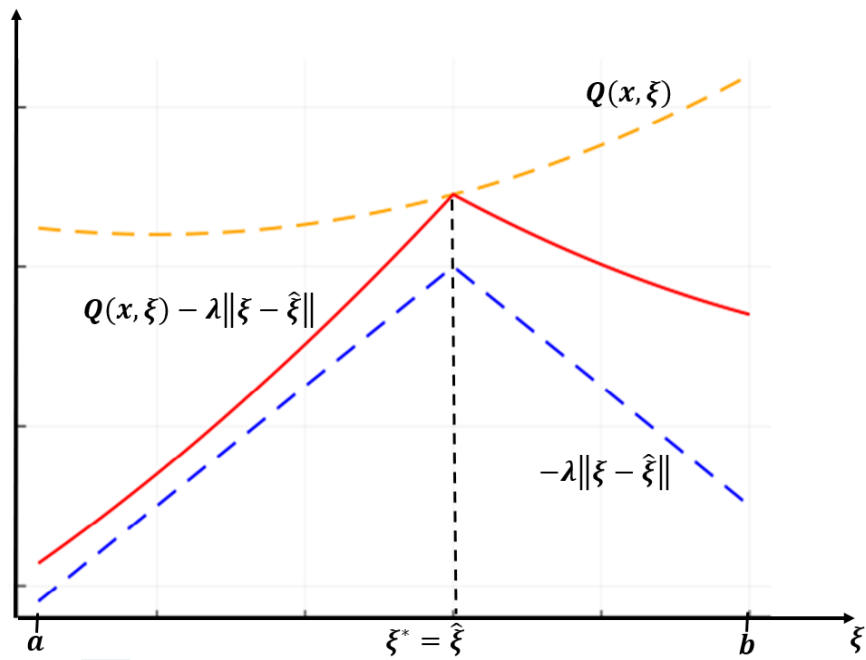
$$Q(\mathbf{x}, \xi) - \lambda \|\xi - \hat{\xi}_n\|_1 \leq s_n, \forall n \leq N, \forall \xi \in \Xi$$

$$\sup_{\xi \in \Xi} \left(Q(\mathbf{x}, \xi) - \lambda \|\xi - \hat{\xi}_n\|_1 \right) \leq s_n, \forall n \leq N$$

Oracle problem

$$\begin{aligned} \min_{\theta, \alpha, \xi} \quad & \theta^\top r(\mathbf{x}) + [H^\top(\mathbf{x})\theta]^\top \xi - \lambda \mathbf{1}^\top \alpha \\ \text{s. t.} \quad & \alpha \geq \xi - \hat{\xi}_n, \\ & \alpha \geq \hat{\xi}_n - \xi, \\ & W^\top \theta \leq \mathbf{q}, \\ & \xi \in \Xi \end{aligned}$$

Solution of the oracle problem



Finite extensive equivalent formulation

Proposition

There exists an optimal solution ξ^* for the oracle problem such that $\xi_i^* \in \{(\hat{\xi}_n)_i, a_i, b_i\}$ for all $i = 1, \dots, d_\xi$

- Optimal solution of the oracle problem

$$\hat{\mathcal{L}}_n = \times_{i=1}^{d_\xi} \{a_i, (\hat{\xi}_n)_i, b_i\} = \{\xi_\ell^*\}_{\ell \in \mathcal{L}_n}, \text{ onde } \mathcal{L}_n = \{1, \dots, |\hat{\mathcal{L}}_n|\}$$

Finite extensive equivalent formulation

Proposed reformulation

$$\begin{aligned} \min_{\mathbf{x}, \lambda, s_n} \quad & \mathbf{c}^\top \mathbf{x} + \lambda \delta + \frac{1}{N} \sum_{n=1}^N s_n \\ \text{s. t.} \quad & Q(\mathbf{x}, \xi_\ell^*) - \lambda \|\xi_\ell^* - \hat{\xi}_n\|_1 \leq s_n, \quad \forall \ell \in \mathcal{L}_n, \forall n \leq N \\ & \lambda \geq 0, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

Existing reformulations

Mehrotra:
Computational
intractable ($N \cdot 3^{d_\xi}$)

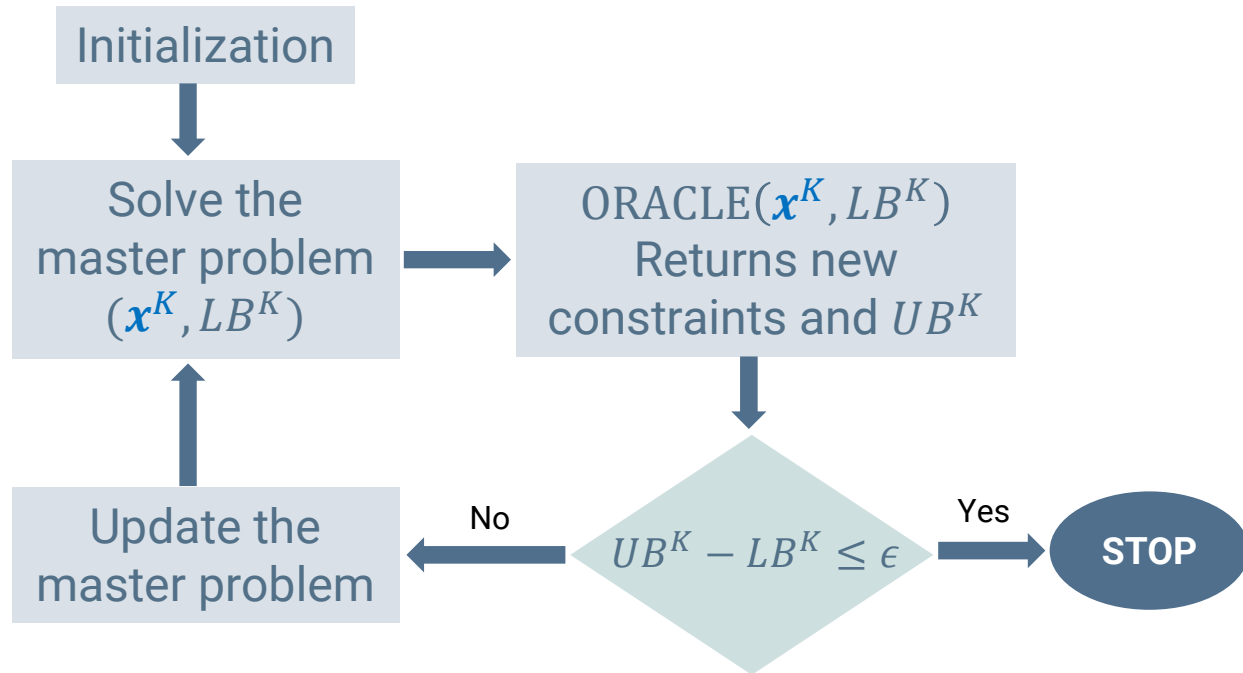
Kuhn:
The resulting PL has
generically
exponential size



Solution
methodology

Decomposition method

- Iterative procedure that converges to the optimal solution of the problem



Column and constraint generation

- Master problem is a linear programming relaxation of the original problem

$$\mathcal{L}_n^K \subseteq \mathcal{L}_n$$

Original problem

Relaxed problem

$$\begin{aligned} \min_{\mathbf{x}, \lambda, s_n} \quad & \mathbf{c}^\top \mathbf{x} + \lambda \delta + \frac{1}{N} \sum_{n=1}^N s_n \\ \text{s. t.} \quad & Q(\mathbf{x}, \xi_\ell^*) \\ & -\lambda \|\xi_\ell^* - \hat{\xi}_n\|_1 \leq s_n, \quad \forall \ell \in \mathcal{L}_n, \forall n \leq N \\ & \lambda \geq 0, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{x}, \lambda, s_n} \quad & \mathbf{c}^\top \mathbf{x} + \lambda \delta + \frac{1}{N} \sum_{n=1}^N s_n \\ \text{s. t.} \quad & q^\top \mathbf{y}_\ell - \lambda \|\xi_\ell^* - \hat{\xi}_n\|_1 \leq s_n, \quad \forall \ell \in \mathcal{L}_n^K, \forall n \leq N \\ & \mathbf{W} \mathbf{y}_\ell = \mathbf{H}(\mathbf{x}) \xi_\ell^* + \mathbf{r}(\mathbf{x}), \quad \forall \ell \in \mathcal{L}_n^K, \forall n \leq N \\ & \lambda \geq 0, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

C&CG algorithm

Initialization

Solve the master problem
(\mathbf{x}^K, LB^K)

$$\xi^* \in \text{Argmin} \begin{cases} \theta^T r(\mathbf{x}^K) + [H^T(\mathbf{x}^K)\theta]^T \xi \\ -\lambda \mathbf{1}^T \alpha \end{cases} \quad \left| \begin{array}{l} \alpha \geq \xi - \hat{\xi}_n, \\ \alpha \geq \hat{\xi}_n - \xi, \\ W^T \theta \leq \mathbf{q}, \\ \xi \in \Xi \end{array} \right.$$

$$\mathcal{L}_n^{K+1} = \mathcal{L}_n^K \cup \{\xi^*\}$$

$$\mathbf{q}^T \mathbf{y}_\ell - \lambda \|\xi_\ell^* - \hat{\xi}_n\|_1 \leq s_n, \quad \forall \ell \in \mathcal{L}_n^{K+1}, \forall n \leq N$$

$$W \mathbf{y}_\ell = H(\mathbf{x}) \xi_\ell^* + \mathbf{r}(\mathbf{x}), \quad \forall \ell \in \mathcal{L}_n^{K+1}, \forall n \leq N$$

No

$$UB^K - LB^K \leq \epsilon$$

Yes

STOP

$$UB^K \leftarrow \min \left\{ UB^{K-1}, \mathbf{c}^T \mathbf{x}^K + \lambda^K \delta + \frac{1}{N} \sum_{n=1}^N \rho(\mathbf{x}^K, \lambda^K) \right\}$$

Bender's multi-cut

- Master problem for the Bender's multi-cut algorithm

$$(\boldsymbol{\theta}_n^k, \boldsymbol{\xi}_n^k) \in \text{Argmin}\{\text{ORACLE}(\boldsymbol{x}^k, \lambda^k)\}$$

Original problem

$$\begin{aligned} \min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda \delta + \frac{1}{N} \sum_{n=1}^N s_n \\ \text{s. t.} \quad & \boldsymbol{\theta}_d^\top \boldsymbol{r}(\boldsymbol{x}) + [\boldsymbol{H}^\top(\boldsymbol{x}) \boldsymbol{\theta}_d]^\top \boldsymbol{\xi}_\ell^* \quad \forall (d, \ell) \\ & -\lambda \|\boldsymbol{\xi}_\ell^* - \hat{\boldsymbol{\xi}}_n\|_1 \leq s_n, \quad \in \mathcal{D} \times \mathcal{L}_n, \forall n \leq N \\ & \lambda \geq 0, \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

Relaxed problem

$$\begin{aligned} \min_{\boldsymbol{x}, \lambda, s_n} \quad & \boldsymbol{c}^\top \boldsymbol{x} + \lambda \delta + \frac{1}{N} \sum_{n=1}^N s_n \\ \text{s. t.} \quad & [\boldsymbol{\theta}_n^k]^\top \boldsymbol{r}(\boldsymbol{x}) + [\boldsymbol{H}^\top(\boldsymbol{x}) \boldsymbol{\theta}_n^k]^\top \boldsymbol{\xi}_n^k \quad \forall k \leq K, \forall n \leq N \\ & -\lambda \|\boldsymbol{\xi}_n^k - \hat{\boldsymbol{\xi}}_n\|_1 \leq s_n, \\ & \lambda \geq 0, \\ & \boldsymbol{x} \in \mathcal{X} \end{aligned}$$

Bender's single-cut

- Master problem for the Bender's single-cut algorithm

$$\mathcal{L} = \bigcup_{n=1}^N \mathcal{L}_n$$

Original problem

$$\begin{aligned} \min_{\mathbf{x}, \lambda, s_n} \quad & \mathbf{c}^\top \mathbf{x} + \lambda \delta + \beta \\ \text{s. t.} \quad & \frac{1}{N} \sum_{n=1}^N \left(\boldsymbol{\theta}_d^\top \mathbf{r}(\mathbf{x}) + [\mathbf{H}^\top(\mathbf{x}) \boldsymbol{\theta}_d]^\top \boldsymbol{\xi}_\ell^* \right) \quad \forall (d, \ell) \in \mathcal{D} \times \mathcal{L}, \\ & \lambda \geq 0, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$

Relaxed problem

$$\begin{aligned} \min_{\mathbf{x}, \lambda, s_n} \quad & \mathbf{c}^\top \mathbf{x} + \lambda \delta + \beta \\ \text{s. t.} \quad & \frac{1}{N} \sum_{n=1}^N \left([\boldsymbol{\theta}_n^k]^\top \mathbf{r}(\mathbf{x}) + [\mathbf{H}^\top(\mathbf{x}) \boldsymbol{\theta}_n^k]^\top \boldsymbol{\xi}_n^k \right) \quad \forall k \leq K, \\ & \lambda \geq 0, \\ & \mathbf{x} \in \mathcal{X} \end{aligned}$$



Results

Numerical experiment

- We test our proposed solution methodology with the two-stage stochastic (single-bus) unit commitment problem
 - First-stage decision:
 - Commit decisions
 - Second-stage decisions:
 - Economic dispatch
- Uncertainty:
 - Uncertain net electricity load

Load – Uncertain renewable generation



Thermal generators



Results

- Illustrative example of a system with 5 thermal generators
- System with 14 and 54 thermal generators

Iteration <i>K</i>	C&CG	Benders multi-cut	Benders single-cut
	<i>UB-LB</i>	<i>UB-LB</i>	<i>UB-LB</i>
1	1273245	1404389	1404388
2	0	147243	282048
3		135559	135559
4		8471	48166
5		8064	19806
6		10656	19806
7		1670	19490
8		1274	16295
9		5548	16295
10		756	16295
11		1278	16295
12		2946	16295
13		760	15564
14		799	15564
15		863	15041
16		799	14310
17		0	13777
40			9142
80			3857
160			2218
165			0
Time (CPUs)	3394	26622	148127

Method	Syst-14	Syst-54
	Time (CPUs)	Time (CPUs)
C&CG	15365	24225
Bender's multi-cut	27833	-
Bender's single-cut	38379	-



References

Some references



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