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MAXIMUM PRINCIPLE AND APPLICATIONS

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Abstract

In this work, we put forward a brief introduction to local second order elliptic operators, based on the classical literature or modern approaches to it, such as [1] [2]. My own master thesis was also used to supply some results. Our object of study are operators that in a sense behave like the Laplacian operator and some of its variants. We present a number of elementary properties and establish an Alexandroff-Bakelman-Pucci estimate. As an application, we examine symmetry results for solutions of elliptical problems.

Keywords: Classical Solutions, Maximum Principle, ABP estimate,Local Operators

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Resumo

Neste trabalho, damos uma breve introdução a teoria linear de operadores elípticos de segunda ordem, baseada na literatura clássica disponível além de trabalhos modernos, tais como [1] [2]. Minha dissertação de mestrado também foi utilizada como base para alguns resultados. Nosso objeto de estudo são operadores que em algum sentido se comportam como o operador laplaciano. Apresentamos uma série de resultados fundamentais para a teoria e demonstramos a estimativa Alexandroff-Bakelman-Pucci. Como aplicação, examinamos resultados de simetria para soluções de problemas elípticos.

Palavras-chave: Soluções Clássicas, Princípio do Máximo, Estimativa ABP,Operadores Locais

Summary

1 Introduction

A partial differential equation is a relation between an unknown function of two or more variables and some of its partial derivatives.In order to be more specific, let $\Omega \subset \mathbb{R}^{\ltimes}$ be an open subset, fix k a positive integer.

Definition 1.1. An expression of the form:

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0$$

is called a k^{th}-order partial differential equation, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and

$$u : \Omega \to \mathbb{R}$$

is the unknown.

Such equation is solved if we can find all functions u such that u verifies it, possibly while also satisfying some auxiliary boundary conditions on a subsetΓ of $\partial\Omega$

Classification of PDE's

Depending on the linearity of the functional F we may classify our PDE's as:

1. Linear:

   Definition 1.2. A PDE is called linear if the operator is of the form:

   $$F = \sum_{|\alpha| \le k} a_\alpha(x) D^\alpha$$

2. Semi linear

   Definition 1.3. A PDE is called semi linear if the operator is of the form

   $$F = \sum_{|\alpha| = k} a_\alpha(x) D^\alpha + a_0(D^{k-1}, \ldots, Du, u, x)$$

   Quasilinear

   Definition 1.4. A PDE is called quasilinear if the operator is of the form

   $$F = \sum_{|\alpha| = k} a_\alpha(D^{k-1}, \ldots, Du, u, x) D^\alpha + a_0(D^{k-1}, \ldots, Du, u, x)$$

   Fully Nonlinear

   Definition 1.5. A PDE is called fully nonlinear if it depends nonlinearly upon the highest order derivative

Example 1.1.

1. Linear

   (a) Laplace's Equation

   $$\Delta u = 0$$

(b) Diffusion Equation

$$u_t - \Delta u = 0$$

(c) Wave Equation

$$u_{tt} - \Delta u = 0$$

2. Semi linear
3. Quasilinear
4. Fully nonlinear

(a) Monge-Ampere Equation

$$Det(D^2 u) = 0$$

2 Elliptical Equations

On this chapter we are going to define what it means for a PDE to be elliptic and work through the classic approach

a Some Definitions

From now on we are mostly concerned about second order PDE's.

Definition 2.1. We say that F is (degenerate) elliptic if

$$F(x, r, p, X) \geq F(x, r, p, Y)$$

for all $x \in \Omega, r \in \mathbb{R}, p \in \mathbb{R}^n, X, Y \in S^n$ provided $X \geq Y$.

Definition 2.2. We say that F is uniformly elliptic if there exists $\lambda, \Lambda > 0$ such that for every $N \in S^n$ for every $M \in \mathbb{R}^{n^2}$

$$\lambda||N|| \leq F(M+N) - F(M) \leq \Lambda||N||$$

Example 2.1. *Consider a second order linear differential operator of the form:*

$$Lu = a_{ij}(x)\partial_i\partial_j u + b_i(x)\partial_i u + c(x)u$$

Where a_{ij} is a real symmetric matrix with eigenvalues between λ, Λ and Einstein summation convention is used. Then L is $(\lambda, n\Lambda)$ elliptic

Remark 2.1. The notion of elipcity depends on the domain.Consider for example the operator as above with the matrix $a_{ij}(x)$ given by:

$$A(x) = \begin{bmatrix} e^{-|x|} & 0 \\ 0 & 1 \end{bmatrix}$$

From the previous example we get that L is uniformly elliptic on any bounded domain Ω but only elliptic on the whole space

b Maximum Principle

One of the principal tools from the analysis of elliptic equations is the so called maximum principle.

In this section we are going to develop the classic maximum principle and some extensions. For now, we are mostly interested in classical solutions ,i.e, we suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$

Consider a non divergence second order linear differential operator

$$Lu = a_{ij}(x)\partial_i\partial_j u + b_i(x)\partial_i u + c(x)u$$

With bounded measurable coefficients such that $|a_{ij}|, |b_i|, |c| \leq \Lambda,$ and $\lambda I \leq A(x) \leq \Lambda I$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, if c = 0 and $Lu \geq 0$ in Ω then* $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$

First let's consider the case that $Lu > 0$ in Ω and suppose that $\max_{\bar{\Omega}} u > \max_{\partial\Omega} u$ if that were the case then there would exist some $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$ and as so we would have $\nabla u(x_0) = 0$ and $D^2 u(x_0) \leq 0$ and therefore $Lu(x_0) \leq 0$ which contradicts the fact that $Lu > 0$ in Ω

To consider the case where $Lu \geq 0$ we are going to consider a perturbation of u which converges uniformly to u. Define $u_\epsilon(x) = u(x) + \epsilon e^{\alpha x_1}$ with α to be chosen later

Evaluating Lu_ϵ we obtain

$$Lu_\epsilon = Lu + L(e^{\alpha x_1}) = Lu + (a_{11}\alpha + b_1)\alpha e^{\alpha x_1} > Lu + e^{\alpha x_1}\alpha(\lambda\alpha - \Lambda) > 0$$

Once we take $\alpha > \frac{\Lambda}{\lambda}$. Applying what we have already proved to u_ϵ we get that:

$$\max_{\bar{\Omega}} u_\epsilon = \max_{\partial\Omega} u_\epsilon \leq \max_{\partial\Omega} u + \epsilon \max_{\partial\Omega} e^{\alpha x_1}$$

Taking the limit as epsilon goes to zero implies the result □

Corollary 2.1. *The preceding theorem guarantees the unicity of solution to the Dirichlet problem:*

$$Lu = f \qquad \text{in } \Omega \tag{1a}$$
$$u = g \qquad \text{in } \partial\Omega \tag{1b}$$

Whenever $c(x) \leq 0$ in Ω

Proof. Let u, v be classical solutions of (13a) and (13b). Define w= u-v, as such w satisfy the following boundary condition problem.

$$Lw = 0 \qquad \text{in } \Omega$$
$$w = 0 \qquad \text{in } \partial\Omega$$

Applying the previous theorem to w and to -w we obtain $\max_{\bar{\Omega}} w = \max_{\partial\Omega} w = 0$ and $min_{\bar{\Omega}} u = min_{\partial\Omega} u = 0$ which implies that w is identically null, therefore we have $u = v$ in Ω □

Let's discuss a little bit about the restrictions of the result.In what follows we present some counter-examples when we remove some hypothesis.

Example 2.2. *In the case we assume Ω unbounded we can consider the Dirichlet problem defined for $\Omega = \{z = (x, y) \in \mathbb{R}^2, |z| > 1\}$ and $Lu = \Delta u$:*

$$Lu = 0 \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{in } \partial\Omega$$

Note that trivially $u(x, y) = 0$ is a solution and $u(x, y) = \log(\sqrt{x^2 + y^2})$ is also a solution.

Example 2.3. *If we assume that the maximum principle holds assuming c(x) positive without further conditions we are led to contradiction once we consider the problem:*

$$\begin{aligned} Lu &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{in } \partial\Omega \end{aligned}$$

Where $Lu = \Delta u + 2u$ *and* $\Omega = \{(x, y) \in \mathbb{R}^2, (x, y) \in [0, 1]^2\}$.
Note that once again we end up with two solutions to the problem, namely $u(x, y) = 0$ *and* $u(x, y) = \sin(x)\sin(y)$

The next result, also known as Hop lemma or Hopf-Oleinik theorem, states that the solution of a uniformly elliptic equation cannot vanish on the boundary where a extremum is attained.

Theorem 2.2. *Let* $u \in C^2(B_R) \cap C(\bar{B_R})$ *satisfying*

$$\begin{aligned} Lu &\leq 0 \qquad \text{in } B_R \\ u &> 0 \qquad \text{in } B_R \\ u(x_0) &= 0 \qquad x_0 \in \partial B_R \end{aligned}$$

Then for every direction $\xi \in \mathbb{R}^n$ *such that* $(\xi, \nu) > 0$ *we have*

$$\liminf_{t \to 0^+} \frac{u(x_0 + t\xi) - u(x_0)}{t} > 0$$

where ν *is the radial normal vector pointing inward to* ∂B_R *at* x_0

Proof. The idea behind the proof is to construct a radial function ϕ which will play the role of a barrier from below for u.

First of all, we may without loss of generality suppose that $c \leq 0$

In fact, if

$$a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu \leq 0$$

Using that u is positive in B_R and summing $-c^+u$ on both sides we have

$$a_{ij}\partial_i\partial_j u + b_i\partial_i u - c^-u \leq -c^+u \leq 0$$

Therefore, we may start the demonstration assuming $c \leq 0$ if not consider the modified operator $\bar{L}u = a_{ij}\partial_i\partial_j u + b_i\partial_i u - c^-u$

Once surpassed those frivolities define

$$\phi(x) = \epsilon(|x|^{-\alpha} - R^{-\alpha})$$

where $\epsilon(\alpha)$ is such that $\epsilon((\frac{R}{2})^{-\alpha} - R^{-\alpha}) < \min_{|x|=\frac{R}{2}} u > 0$ and α is still to be chosen.

It's easy to check that ϕ is radial and for $\frac{R}{2} < r < R$:

$$L(\frac{\phi}{\epsilon}) \geq \alpha r^{-\alpha-2}(\lambda(\alpha+1) - (n-1)\Lambda) - \Lambda| - \alpha r^{-\alpha-1}| + c(r^{-\alpha} - R^{-\alpha}) \geq$$
$$\alpha r^{-\alpha-2}(\lambda(\alpha+1) - (n-1)\Lambda - \Lambda r) + cr^{-\alpha} \geq$$

The above expression is positive once we choose α big enough. Therefore ϕ satisfies:

$$\begin{aligned} &L\phi \geq 0 \geq Lu \qquad \text{in } B_R \setminus B_{\frac{R}{2}} \\ &\phi(R) = 0 = u(x_0) = \min_{|x|=R} u \\ &\phi(\frac{R}{2}) \leq \min_{|x|=\frac{R}{2}} u \\ &\phi'(R) < 0 \end{aligned}$$

Taking t small enough in order that $x_0 + t\xi \in B_R \setminus B_{\frac{R}{2}}$ and using the maximum principle to obtain $\phi \leq u$ in $B_R \setminus B_{\frac{R}{2}}$ we obtain

$$\liminf_{t\to 0^+} \frac{u(x_0+t\xi) - u(x_0)}{t} \geq \liminf_{t\to 0^+} \frac{\phi(x_0+t\xi) - \phi(x_0)}{t} = \nabla\phi(x_0)\cdot\xi =$$
$$\frac{\phi(R)}{R} x_0 \cdot \xi = -\phi(R)(\nu, \xi) > 0 \quad \square$$

Theorem 2.3. ***Strong Maximum Principle:*** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying*

$$\begin{aligned} Lu \leq 0 \qquad &\textit{in } \Omega \\ u \geq 0 \ &\textit{on } \partial\Omega \end{aligned}$$

Then either $u \equiv 0$ or u is strictly positive.

Proof. Let $\Omega_0 = \{x \in \Omega \mid u(x) = 0\}$ and $\Omega^+ = \{x \in \Omega \mid u(x) > 0\}$. Suppose that both sets are nonempty, otherwise there is nothing to be proven. Take $z \in \Omega^+$ such that $d(z, \Omega_0) < d(\partial\Omega)$. Consider a ball of radius R such that $B_R(z)$ is entirely contained in Ω^+ and $\partial B_R(z) \cap \partial\Omega_0$ is nonempty. Let y be a point in this intercession, clearly y is a minimum point therefore $\nabla u(y) = 0$ which contradicts, Hopf lemma applied to $B_R(z)$. $\square$

Corollary 2.2. *Let $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying*

$$\begin{aligned} Lu \leq Lv \qquad &\textit{in } \Omega \\ u \geq v \qquad &\textit{on } \partial B_R \end{aligned}$$

Then either u and v are identical or $u > v$ in Ω.

Proof. Just apply the strong maximum principle to the function $w := u - v$. $\square$

Up to this point we have shown results which characterize an operator satisfying the maximum principle depending on the sign of the coefficients. Now we present another way to prove the validity of the maximum principle.

Theorem 2.4. *If there exists a strictly positive supersolution ψ of $Lu = 0$, or in other words, $\psi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that*

$$\begin{aligned} L\psi \leq 0 \qquad & \text{in } \Omega \\ u > 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

Then L satisfies the maximum principle in Ω.

Proof. Suppose that u is a $C^2(\Omega) \cap C(\bar{\Omega})$ function satisfying,

$$\begin{aligned} Lu \leq 0 \qquad & \text{in } \Omega \\ u \geq 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

we want to conclude that u is a positive function.

The idea here in this proof is to construct an auxiliary function v depending on u and ψsuch that v will satisfy an elliptic equation with the maximum principle propriety. Define$v := \frac{u}{\psi}$ note that v is well defined since ψ is strictly positive.

From the definition of v it is clear that v satisfies the equation:

$$\begin{aligned} A_{ij} v_i j + (b + \frac{A\nabla\psi}{\psi}) \cdot \nabla v + \frac{L\psi}{\psi} \leq 0 \qquad & \text{in } \Omega \\ u > 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

Note that the 0-order term is negative hence we may apply the maximum principle for $v = u\psi$, obtain that v is a positive function and hence since ψ is positive we conclude that so is u. □

3 A priori estimates

Corollary 3.1. *Let Ω be a bounded set and L a second order (λ, Λ) uniformly elliptic operator with bounded coefficients and $c(x) \leq 0$. There exists a constant $C(n, \lambda, \Lambda, diam\Omega) \geq 0$ such that if $Lu = f$ then*

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) + \sup_{x\in\Omega} f(x) \tag{12}$$

Proof. Let $M := \sup_{x\in\Omega} f(x), K := \sup_{x\in\partial\Omega} u(x)$, define $\xi(x) = K + M\Psi(x) = K + M(\exp(\alpha d_0) - \exp(\alpha x_1))$. It follows from the definition that ξ satisfies:

$$L\xi = -M \leq -f^- \leq f \leq Lu \qquad \text{in } \Omega \tag{13a}$$

$$\xi \geq K \geq u^+ \geq= g \qquad \text{on } \partial\Omega \tag{13b}$$

Therefore, the result is a direct conclusion from the maximum principle.

□

The above inequality may be stated in a more general form which is known in the literature as the Alexandrov-Bakelman-Pucci estimate, or ABP estimate for short. This will enable us to move from measure theory estimates to pointwise estimates. Later in this monograph, such an estimate will be an essential element in the proof of the Harnack Inequality. This gives us information over the growth of solutions.

In what follows we introduce the definition of concave envelope for a class of functions $u : \mathbb{R}^d \to \mathbb{R}$. First, we put forward the definition of affine function.

Definition 3.1. A function of the form

$$\ell(x) = a + b \cdot x,$$

with $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$ is called affine.

Remark 3.1. There is also the natural extension of affine functions, namely, vectorial affine maps, which are of the form:

$$\ell(x) = a + b \cdot x,$$

where $a \in \mathbb{R}^d, b \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$. In this case, ℓ also describes a plane in the ambient space.

Definition 3.2. Let $u : \mathbb{R}^d \to \mathbb{R}$ be such that

$$u(x) \leq 0\,,\ x \in \bar{\Omega}$$

The concave envelope of u in Ω at x is denoted by $\Gamma(x)$ and defined as follows:

$$\Gamma(x) = \left\{\ \min\{\ell(x),\, \ell \text{ affine } \mid \ell(x) \geq u^+(x) \text{ in } \Omega\}\right.$$

Next we establish some proprieties of the function Γ.

Lemma 3.1. *Let $\Gamma : \mathbb{R}^d \to \mathbb{R}$ be as in Definition 3.2, Γ is concave.*

Proof. The proof follows from the fact that the minimum of concave functions is concave. Γ is the minimum of affine functions. Recalling that every affine function is concave, we conclude that Γ is concave. $\square$

As a corollary of the previous lemma, we prove the continuity of Γ in Ω as a direct consequence of Theorem A.2.

Given a function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the conditions in Definition 3.2, one may consider the concave envelope Γ as on above and define the contact set of u as follows.

Definition 3.3. If u satisfies the conditions of Definition 3.2, we define the contact set of u, denoted by Σ_u as:

$$\Sigma_u := \{y \in \Omega \,|\, u(y) = \Gamma(y)\}.$$

Remark 3.2. Unless we are dealing simultaneously with more than one concave envelope, we will not use the subindex in order to preserve the clarity of the notation.

Lemma 3.2. *The set $\Sigma_u = \{y \in \Omega | u = \Gamma\}$ is closed.*

Proof. Since u is lower semicontinuous and Γ is continuous, $u - \Gamma$ is lower semicontinuous and therefore $\{u = \Gamma\}$ is the preimage of a closed set by a lower semicontinuous function. The result follows from classic consideration in Analysis. $\square$

Remark 3.3. The superdifferential of Γ is always nonempty in Ω since Γ is concave.

Remark 3.4. If u is differentiable the superdifferential of Γ coincides with Du. On the other hand if Γ is differentiable $D\Gamma$ coincides with the superdifferential of the function Γ.

Lemma 3.3. *Let $u, \Gamma : \mathbb{R}^d \to \mathbb{R}$ be as in Definition 3.2, then*

$$\left(\frac{\sup u}{diam\Omega}\right)^d |B_1| \leq |\nabla u(\Sigma_u)|$$

Proof. Consider a conic function of the form $w(x) := a \cdot \left(1 - \frac{|x|}{R}\right)$ it is clear that $|\nabla w(\Omega)| = |B_a/R|$. Let u be as in Definition 3.2, take $a = \sup_\Omega u > 0$ and $R = diam(\Omega)$. Notice that if you take an hyperplane coming down from infinity that touches the graph of w at its vertex, then it also touches the graph of u in an interior point. Therefore we obtain:

$$\left(\frac{\sup u}{diam\Omega}\right)^d |B_1| \leq |\nabla w(\Omega)| \leq |\nabla u(\Sigma_u)|$$

$\square$

Theorem 3.1. *Let Ω be a bounded set, and L a second order (λ, Λ) uniformly elliptic operator with coefficients in $L^n(\Omega)$ and $c(x) \leq 0$. There is a constant $C(n, \lambda, \Lambda, diam\Omega) \geq 0$ such that, if $Lu = f \in \mathcal{L}^n$ then:*

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) + C||f||_{\mathcal{L}^n(\Omega)} \tag{14}$$

Proof. We will restrict our attention to the case $b \equiv 0$, the interested reader may consult on classical sources for how to contour this difficulty. Notice that we may suppose that u is a negative function on the border of Ω by considering the auxiliary function $v := u - \sup_{\partial} \Omega u$. Hence, from the previous lemma we obtain:

$$\left(\frac{\sup u}{diam\Omega}\right)^d |B_1| \leq |\nabla u(\Sigma_u)|.$$

One may rewrite the last term as

$$|\nabla u(\Sigma_u)| = \int_{\nabla u(\Sigma_u)} 1dx = \int_{\Sigma_u} |D^2u(x)|dx$$

But notice that on the set Σ_u u is a concave function, hence D^2u is a negative matrix. Consider the generalized inequality of arithmetic and geometric means, which say that for given two d-dimensional non negative matrices:

$$det(AB) \leq \left(\frac{Tr(AB)}{d}\right)^d$$

We may apply the above inequality to the matrix $B = -D^2u$ and obtain:

$$|\nabla u(\Sigma_u)| \leq \int_{\Sigma_u} \left|\frac{-Tr(AD^2u(x))dx}{detA}\right|$$

which imply the result.

□

Theorem 3.2 (Maximum Principle for small domain)**.** *Let L be as in the theorem before, additionally suppose that $||b(x)||_{\mathcal{L}^n(\Omega)}, ||c^+(x)||_{\mathcal{L}^\infty(\Omega)} < B$, then given $d_0 > 0$ there is $\delta(n, \lambda, B, d_0)$ such that if $diam\Omega < d_0$, and $|\Omega| < \delta$, L satisfies the maximum principle in Ω.*

Proof. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$, satisfying:

$$Lu \geq 0 \qquad \text{em } \Omega \tag{15a}$$
$$u \leq 0 \qquad \text{em } \partial\Omega \tag{15b}$$

we aim to show that $u \leq 0$ in Ω. For that, consider the auxiliary operator given by:

$$\overline{L}[u] = a_ij(x)u_{x_ix_j}(x) + b_i(x)u_{x_i}(x) - c^-(x)u(x) \geq -c^+(x)u^+(x) =: f.$$

Note that the 0-order term of this operator is negative, hence we may use the ABP estimate to obtain:

$$\sup_{x\in\Omega} u(x) \leq \sup_{x\in\partial\Omega} u(x) + C||c^+u^+||_{\mathcal{L}^n(\Omega)} \leq CB|\Omega|^{1/n} \sup_{x\in\Omega} u^+(x) \tag{16}$$

Choosing a suffiently small $\delta > 0$ such that $CB\delta^{1/n} < 1/2$ we obtain:

$$\sup_{x\in\Omega} u(x) \leq \frac{1}{2} \sup_{x\in\Omega} u^+(x)$$

Therefore $u \leq 0$ in Ω as desired. $\square$

4 Moving Planes Method

With the above result avaliable, we may introduce a common tool in the study of partial differential equations known as moving planes, to expose the utility of this technique we will demonstrate the following result.

Theorem 4.1. *Let Ω be an open,bounded, convex and symmetric in the x_1 direction,furthermore suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ is a lipschitz function and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfying:*

$$\Delta u + f(u) = 0 \qquad \textit{em } \Omega \tag{17a}$$
$$u > 0 \qquad \textit{em } \Omega \tag{17b}$$
$$u = 0 \qquad \textit{em } \partial\Omega \tag{17c}$$

Hence $u(x_1, x_2, \dots, x_n) = u(-x_1, x_2, \dots, x_n) \forall x = (x_1, x_2, \dots, x_n) \in \Omega$ e $u_{x_1} < 0, \forall x$ such that $x_1 > 0$

Remark 4.1. The above result is still valid if we change the laplacian for another operator satisfying the maximum principle that is symmetric with respect to the axis $x_1 = 0$.

Proof. We will start defining the following quantities:

- $T_\lambda = \{x \in \Omega | x_1 = \lambda\}$
- $x_\lambda = (2\lambda - x_1, x_2, ..., x_n)$
- $\Sigma_\lambda = \{x \in \Omega | x_1 = \lambda\}$
- $w_\lambda(x) = u(x_\lambda) - u(x)$ for $x \in \Sigma_\lambda$

This way, it is clear that:

$$\Delta w_\lambda + f(u_\lambda) - f(u) = 0 \tag{18}$$

defining $c_\lambda(x)$ as:

$$c_\lambda(x) = \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)} \qquad , u_\lambda(x) \neq qu(x) \tag{19a}$$
$$c_\lambda(x) = 0 \qquad , u_\lambda(x) = u(x) \tag{19b}$$

Hence, we may write that w_λ satisfies the Dirichlet problem:

$$\Delta w_\lambda(x) + c(x) w_\lambda(x) = 0 \qquad \text{in } \Sigma_\lambda \tag{20a}$$
$$w_\lambda \geq 0 \qquad \text{on } \partial\Sigma_\lambda \tag{20b}$$

As f is Lipschitz we obtain c_λ is uniformely bounded therefore the previous estiamtes are avaliable in our case.

We will split the argument in two parts, first let $R := \sup\{\lambda \in \mathbb{R} | T_\lambda \neq \emptyset\}$. We will show that for every value of λ sufficiently close to R, $w_\lambda > 0$. Following we will consider the case $w_\lambda > 0$ for $\lambda > 0$, and we will show the existence of $\overline{\lambda} < \lambda$ such $w_{\overline{\lambda}} > 0$. This is enough to obtain the result by a symmetry argument which will be later presented.

In order to show the existence of such λ, let $\delta > 0$ as in Theorem 3.2 and choose λ such that $|\Sigma_\lambda| < \delta$, it follows ftom the maximum principle for small domains that L satisfies the maximum principle in Σ_λ. Therefore, either $w_\lambda > 0$ or $w_\lambda \equiv 0$. As $w_\lambda > 0$ on $\partial\Sigma_\lambda \cap \partial\Omega$, it follows that $w_\lambda > 0$ in Σ_λ, hence the first part of the argument is concluded.

To prove the second part of the argument, consider $\lambda^* := \inf\{\lambda \in (0,R) | w_\lambda > 0$ em Σ_λ. From the previous argument we know that λ^* is well defined since this set is non empty. We want to show that $\lambda^* = 0$, since due to the symmetry of the problem we could repeat the same argument starting at the other side and obtain $w_0 \equiv 0$ as desired. Suppose for the sake of contradiction $\lambda^* > 0$, from continuity we in fact have $w_{\lambda^*} \geq 0$ in Σ_{λ^*}. As before, either $w_{\lambda^*} > 0$ or $w_{\lambda^*} \equiv 0$, analogously we conclude that $w_{\lambda^*} > 0$.

Let $K \subset \Sigma_{\lambda^*}$ a compact set such that $|\Sigma_{\lambda^*} \setminus K| < \delta/2$. Hence, due to continuity there is $\varepsilon_0 > 0$ such that $\lambda^* > \varepsilon_0$ satisfying:

$$w_\lambda(x) \geq \frac{\inf_K w_{\lambda^*}}{2} > 0 \qquad \forall \lambda \in (\lambda^* - \varepsilon_0, \lambda) \tag{21a}$$

$$|\Sigma_{\lambda^*} \setminus K| < \delta \tag{21b}$$

From the maximum principle for small domains it follows that $w_\lambda > 0$ wich contradict the definition of λ^*. In this manner we obtain $w_0 \equiv 0$, applying the Höpf lemma to u_λ at points on $T_\lambda \cap \partial\Sigma_\lambda$ we obtain:

$$0 < \frac{\partial w_\lambda}{\partial \nu} = -2\frac{\partial u}{\partial x_1}(\lambda, x_2, ..., x_n). \tag{22}$$

from where we conclude $u_{x_i}(\lambda, x_2, ..., x_n) < 0$ □

a Application:

Theorem 4.2. *Let* $\Omega = B_R(0)$ *, suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is a lipschitz function and* $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ *satisfying:*

$$\Delta u + f(u) = 0 \qquad \textit{in } \Omega \tag{23a}$$

$$u > 0 \qquad \textit{in } \Omega \tag{23b}$$

$$u = 0 \qquad \textit{on } \partial\Omega \tag{23c}$$

Then $u(x_1, x_2, \ldots, x_n) = u(r)$ *e* $u'(r) < 0, \forall r \in (0, R)$.

Proof. This result is a corollary from the previous one. Note that the ball is an open, bounded, convex and symmetric in all directions. Applying the previous result for each direction we obtain that u is radial and the sign og the derivative is a direct consequence fromthe chain rule. □

A Appendix

With the finality of guiding the reader through the theory developed in the following chapters, this chapter will present some basic facts of analysis which are not commonly taught at the more mainstream analysis courses, but are nonetheless necessary for the correct understatement of what follows. The following results were in part found and the interested reader may look for deeper results and applications at [3].

Definition A.1. A subset $C \subset \mathbb{R}^d$ is said to be convex if $(1-\lambda)x + \lambda y \in C$ whenever $x, y \in C$ and $\lambda \in [0,1]$

Definition A.2. Let $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$. The set

$$\{(x,\mu)\,|\,x \in, \mu \in \mathbb{R}, \mu \geq f(x)\}$$

is called the epigraph of f and is denoted by $epi(f)$.

Definition A.3. A function $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is said to be convex if $epi(f)$ is a convex set.

Remark A.1. A convex function must have a convex domain. On the other hand, a locally convex function does not have such a restriction and as such has important applications in regularity theory.

Definition A.4. A function $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is said to be concave if $-f$ is a convex function.

Lemma A.1. *Every convex function $f : S \subset \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ can be extended to a convex function $\bar{f} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$.*

Proof. Define $\bar{f} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ as

$$\bar{f}(x) = \begin{cases} f(x) & x \in S \\ +\infty & x \notin S \end{cases}$$

Let $(x,\mu),(y,\nu) \in \mathbb{R}^{d+1} \cap epi(\bar{f})$, therefore $(1-t)(x,\mu) + t(y,\nu) \in epi(f) \subset epi(\bar{f})$ for every t between 0 and 1. As a direct consequence $epi(\bar{f})$ is convex and convexity of $\bar{f}$ follows.

□

In light of the previous result, the following lemmas will be stated assuming the function f is defined over the whole space $\mathbb{R}^d$.

Lemma A.2. *A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if*

$$f((1-\lambda)x + \lambda y) < (1-\lambda)\alpha + \lambda\beta \qquad \forall \lambda \in (0,1)$$

whenever $f(x) < \alpha, f(y) < \beta$.

Proof. Suppose f is convex. Since $\alpha > f(x)$ and $\beta > f(y)$, the pairs (x, α) and (y, β) are in $epi(f)$. Since f is convex, the epigraph of f is a convex set, which implies:

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) < (1-\lambda)\alpha + \lambda\beta \qquad \lambda \in (0,1).$$

on the other hand, suppose the inequality is true. Let $(x, \alpha), (y, \beta) \in epi(f)$ and $\lambda \in (0,1)$. Then$f((1-\lambda)x + \lambda y) < (1-\lambda)\alpha + \lambda\beta$ which implies that the epigraph of f is convex and therefore the function f is convex. □

Corollary A.1 (Jensen's Inequality)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Then*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \qquad \lambda \in [0,1].$$

Proof. It follows from Lemma A.2 taking the infimum over α, β such that $f(x) < \alpha, f(y) < \beta$. □

Definition A.5. Let $C \subset \mathbb{R}^d$ be a convex set, and suppose $f : C \to \mathbb{R}$ is concave. A vector $p \in \mathbb{R}^d$ is a supergradient of the function f at the point $x \in \mathbb{R}^d$ if for every $y \in \mathbb{R}^d$

$$f(y) \leq f(x) + p \cdot (y - x).$$

Analogously, if f is a convex function, we say that $p \in \mathbb{R}^d$ is a subgradient of f at $x \in \mathbb{R}^d$ if

$$f(y) \geq f(x) + p \cdot (y - x).$$

In both cases we denote the set of all supergradients and subgradients of f at the point x as $\partial f(x)$

Definition A.6. A convex function satisfying the assumption A.5 is said to be superdifferentiable at a point $x \in \mathbb{R}^d$ if $\partial f(x)$ is non-void. In the same manner we define that a concave function f as in A.5 is subdifferentiable at x.

Theorem A.1. *A concave function on a convex set in $\mathbb{R}^d$ is superdifferentiable at each interior point.*

Proof. Let f be a concave function defined on a convex set $C \subset \mathbb{R}^d$, and let x be an interior point of C. Consider the strict subgraph of f,S, as:

$$S := \{(y, \alpha) \in C \times \mathbb{R} : \alpha < f(y)\}$$

It follows from the concavity of f that S is a convex set. Also clear is the fact that the pair $(x, f(x))$ does not belong to the set S. By the Separating Hyperplane Theorem we obtain a nonzero pair $(p, \lambda) \in \mathbb{R}^d \times \mathbb{R}$ such that:

$$p \cdot x + \lambda f(x) \geq p \cdot y + \lambda\alpha, \tag{24}$$

where the inequality above holds for every $y \in C, \alpha < f(y)$. It follows from letting α tend to infinity that λ must be a non-negative number. We proceed to conclude a stronger fact, namely, that λ is indeed strictly positive. Suppose, in order to obtain a contradiction, that $\lambda = 0$. Since x is an interior point, for some $\epsilon > 0$ the ball $B_\epsilon(x)$ is contained in C. Considering points of the form $y = x \pm \epsilon z$, with $z \in B_1$, in 24 we obtain:

$$\begin{cases} 0 \geq & p \cdot z \\ 0 \geq & -p \cdot z \end{cases} \tag{25}$$

We conclude that p must be zero, which contradicts the fact that (p, λ) is nonzero, therefore λ is strictly positive.

Since λ is strictly positive, dividing the whole expression in 24 by λ we obtain:

$$f(x) + (y - x) \cdot (\frac{-p}{\lambda}) \geq \alpha$$

The result follows from letting α tend to $f(y)$ and noticing that$-\frac{p}{\lambda} \in \partial f(x)$. □

Theorem A.2. *Let A be an open convex subset of a finite dimensional vector space over $\mathbb{R}$, let $f : A \to \mathbb{R}$ be a bounded convex function. Then f is continuous on A.*

Proof. Let A and f be as in the theorem, let $x \in A$ an arbitrary point. Consider P the parallelepiped centered at x lying completely inside A, such parallelepiped exists since A is open. Let $y \in \partial P$, for $\lambda \in [0, 1]$, convexity of f implies

$$f((1 - \lambda)x + \lambda y) \leq f(x) + \lambda[f(y) - f(x)]. \tag{26}$$

Also, for $\alpha \in [0, 1/2]$, it follows that

$$\begin{aligned} f(x) &= f\left((1-\alpha)\left[\frac{(1-2\alpha)x}{1-\alpha} + \frac{\alpha y}{1-\alpha}\right] + \alpha(2x - y)\right) \\ &\leq (1-\alpha) f(\frac{(1-2\alpha)x}{1-\alpha} + \frac{\alpha y}{1-\alpha}) + \alpha f(2x - y) \end{aligned}$$

Choosing λ as $\frac{\alpha}{1-\alpha}$ we obtain

$$(1 + \lambda) f(x) \leq f((1 - \lambda)x + \lambda y) + \lambda f(2x - y). \tag{27}$$

Using the two enumerated inequalities we obtain

$$-(\lambda f(2x - y) - f(x)) \leq f(x + \lambda(y - x)) - f(x) \leq \lambda(f(y) - f(x)).$$

Since both $y, 2x - y$ are in ∂P, the above inequality implies that a strict maximum of f cannot be attained in the interior, another conclusion of the previous inequality is that for any vector $z \in P_\lambda := \{x + \lambda(y - x) : y \in \partial P\}$, is true that

$$|f(z) - f(x)| \leq \lambda |\sup_{y \in \partial P} f(y) - f(x)|$$

□

B Abbreviations

B_r:	*The open ball centered at the origin with radius* r.
$B_r(x_0)$:	*The open ball centered at the point*(x_0) *with radius* r.
$\mathbb{S}(d)$:	*The space of* dxd *real symmetric matrices.*

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