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Energetic analysis of electromechanical systems

Projeto de Graduação

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Resumo

Análise energética de sistemas eletromecânicos

Sistemas eletromecânicos são sistemas compostos por dois subsistemas de diferentes naturezas, um de origem mecânica e o outro de origem eletromagnética. Devido a interação entre esses dois subsistemas, um termo de acoplamento está sempre presente nas equações que regem a sua dinâmica. Este acoplamento implica em uma influência mútua entre as duas partes, ou seja, a dinâmica do subsistema mecânico influencia a dinâmica do subsistema eletromagnético e vice e versa. O projeto final visa caracterizar e analisar a dinâmica de sistemas eletromecânicos. Como esses sistemas são compostos por dois subsistemas de diferentes naturezas, as energias no sistema também têm origens diferentes: algumas são mecânicas, como energias cinética e potencial, enquanto outras são eletromagnéticas, como energias magnética e elétrica. Para uma descrição adequada da dinâmica de um sistema eletromecânico, não é suficiente descrever cada subsistema separadamente e, portanto, deve-se considerar termos de acoplamento que proporcionam uma interação entre os diferentes tipos de energias presentes no sistema. Como existem diversas aplicações no dia a dia, sistemas eletromecânicos têm atraído cada vez mais a atenção de pesquisadores e por isso as publicações na áreas são cada vez mais frequentes. Porém, muitas são as referências (livros e artigos publicados em revistas) que descrevem a dinâmica desse tipo de sistema de forma errada, não levando em consideração a interação entre os subsistemas mecânico e eletromagnético ou mesmo descrevendo sistemas eletromecânicos somente fazendo uso de parâmetros mecânicos. É necessário descrever a dinâmica desse tipo de sistema com parâmetros de ambas as partes, pois somente com essa interação entre os dois subsistemas o sistema pode ser dito eletromecânico. Outro foco do projeto final é fazer uma análise energética de sistemas eletromecânicos para que essa interação entre os dois subsistemas seja melhor compreendida. Essa análise será possível após a obtenção das equações da dinâmica envolvendo termos de acoplamento entre variáveis mecânicas e eletromagnéticas através do método de Lagrange. Esses termos de acoplamento introduzem nas equações da dinâmica termos giroscópicos e circulatórios (matrizes do tipo G e N, respectivamente) que podem gerar vibrações auto-excitadas. Dessa forma, uma vez obtidas as equações da dinâmica, será feita uma análise de estabilidade dessas equações, uma novidade na literatura.

Palavras–chave

Eletromecânico; Energia; Co-energia; Lagrange; Acomplamento.

Abstract

Energetic analysis of electromechanical systems

Electromechanical systems are systems composed of two subsystems of different natures, one of mechanical origin and the other of electromagnetic origin. Due to the interaction between these two subsystems, a coupling term is always present in the equations that govern its dynamics. This coupling implies a mutual influence between the two parts, that is, the dynamics of the mechanical subsystem influences the dynamics of the electromagnetic subsystem and vice versa. The final project aims to characterize and analyze the dynamics of electromechanical systems. Since these systems are composed of two subsystems of different natures, the energies in the system also have different origins: some are mechanical, as energies kinetic and potential, while others are electromagnetic, like magnetic and electrical energies. For a proper description of the dynamics of an electromechanical system, it is not sufficient to describe each subsystem separately and, therefore, one must consider coupling terms that provide an interaction between the different types of energies present in the system. Once there are several applications in everyday life, electromechanical systems are attracting the attention of researchers and that is why publications in the areas are more and more frequent. However, there are many references (books and published articles) that describe the dynamics of this type of system in a wrong way, without taking into account the interaction between the mechanical and electromagnetic subsystems or even describing electromechanical systems using only mechanical parameters. It is necessary to describe the dynamics of this type of system with parameters of both natures, because only with this interaction between the two subsystems the can system be called electromechanical. In addition, another focus of the final project is to do an energetic analysis of electromechanical systems so that this interaction between the two subsystems is better comprehended. This analysis will be possible after obtaining the equations of the dynamics involving coupling terms between mechanical and electromagnetic variables through the Lagrange method. These coupling terms introduce into the dynamics equations gyroscopic and circular terms (matrices of type G and N, respectively) that can generate self-excited vibrations. This way, once the dynamic equations are obtained, a stability analysis of these equations will be made, a novelty in literature.

Keywords

Electromechanical; Energy; Co-energy; Lagrange; Coupling.

Summary

Figures list

Tables list

1
Introduction

Electromechanical systems are composed by two coupled subsystems, a mechanical and an electromagnetic. They can be found in several applications used in daily life. However, even though they are so common, it is still a challenge to find references correctly describing their dynamics. Several published papers, books and thesis present serious mistakes in the description of the dynamics of electromechanical systems. A common error found in the literature is the neglection of the dynamics of the electromagnetic subsystem and its interactions with the mechanical subsystem (see reference *Cveticanin, L., Zukovic, M., Balthazar, J. M.* from [1]). Without the dynamics of the electromagnetic subsystem, the electromechanical system becomes a purely mechanical system described by mechanical variables. The recent published paper [2] discusses about some of the references with mistakes and shows how the neglection of the electromagnetic subsystem changes the dynamics.

The objective of this final project is to make a step by step of how to describe properly the dynamics of an electromechanical system using the Lagrangian method, also seen in [3; 4; 5; 6]. To accomplish this goal, the report starts by introducing some important concepts in chapter 2 [7]. After that, it was found the necessity to recall electromagnetic fundamentals, explored in chapter 3 [8]. Chapter 4 shows how to obtain the energies and co-energies found in electromagnetic systems, followed by several examples of how to obtain the dynamic equations of a pure electromagnetic system [5] in chapter 5. To start the final project's main topic, chapter 6 introduces some possible coupling terms in electromechanical systems and then the dynamic equations of several examples are going to be deducted in chapter 7.2 [5].

In chapter 8, an electromechanical system called electromagnetic loudspeaker is analyzed. The coupling term that produces the interaction between the mechanical subsystem and the electromagnetic subsystem in this loudspeaker is a transducer, presented in chapter 6. After all the calculations, an energy analysis will be made to compare the different types of energy (kinetic, potential, electric and magnetic) and show their interplay by the results of numerical integrations using the $4^{th} - 5^{th}$ order Runge-Kutta method with the *ode45 MATLAB* function.

Chapter 9 main objective is to show and discuss different types of errors committed while trying to find the dynamic equations of an electromechanical system and why they are critical. To achieve this goal, an electromechanical

system called galvanometer will be used as example. A wrong way of finding this system's dynamics will be shown and compared with the right one described in chapter 7.2.5. The wrong system dynamics does not correctly characterize the mutual influence between the mechanical and electromagnetic subsystems of the galvanometer. A routine was developed using the software *MATLAB* to simulate how the galvanometer system responds over time considering both dynamics (the wrong and the right one). In the simulations, it was used the analytical solution of the wrong dynamics and a numerical approximation of the right dynamics. The numerical approximation of the right dynamics was obtained by the integration of the initial value problem that gives the right system dynamics by the $4^{th}-5^{th}$ order Runge-Kutta method with the *ode45 MATLAB* function. Several phase portrait are shown and commented to make clear the difference between the two situations.

In chapter 10, some conclusions and future projects are presented and the products developed during the final project are listed.

2
Important concepts

2.1
Energy and Co-energy

When a system with only a mechanical domain is analysed, the topic of energy and co-energy is not often mentioned, but to describe electromagnetic and electromechanical systems, this concept is very important. To obtain them, a constitutive law is required and a graphic representation can help in their visualization, once the energy is represented by the area below the line of the constitutive equation and the co-energy, by the area above it. We can then conclude that energy and co-energy are complementary.

For a better understanding of this concept, lets begin with a simple example: a particle with mass m moving in the positive direction of x. Notice that we are dealing with a linear and non-relativistic system. The kinetic energy of the particle can be found through the work done by a force f acting on it.

Knowing that the constitutive equation for Newtonian dynamics is given by $p = m\,v$ (where p is the momentum and v is the velocity of the particle) and that Newton's second law is $f = \frac{dp}{dt}$, we have:

$$f\ dx = \frac{dp}{dt}\ dx = \frac{dx}{dt}\ dp = v\ dp \quad \rightarrow \quad f\ dx = \frac{p}{m}\ dp.$$

By the integration of this equation from 0 to p we get the kinetic energy:

$$T(p) = \frac{p^2}{2m}. \tag{2-1}$$

We can see in figure 2.1 that the co-energy is given by a triangular area, that can be obtained when subtracting the kinetic energy from a general square area of the graphic. Then, we have:

$$T^*(v) = p\,v - T(p) \quad \rightarrow \quad T^*(v) = \frac{m\,v^2}{2}. \tag{2-2}$$

The kinetic energy and co-energy, in this case, have the same area and, therefore, are equal in quantities. This fact explains why the kinetic co-energy is often known just as kinetic energy. However, this is only valid because of the system's linearity. So, if we take in consideration a relativistic situation, for example, this is no longer true.
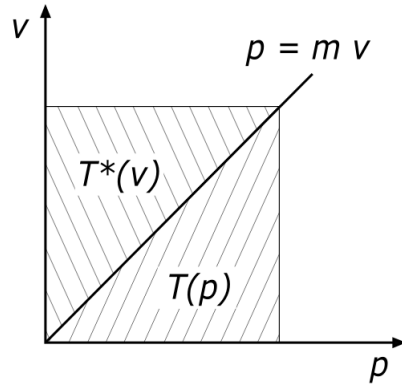
Figura 2.1: Graphic representation of kinetic energy and co-energy.

2.2
Lagrangian and Co-Lagrangian

The Lagrangian is usually used to describe the dynamics of mechanical systems. It is defined as the difference between the stored kinetic co-energy (related to the masses and moments of inertia) and the total potential energy (related to the stiffness elements and the action of gravity).

The same logic used to describe co-energy as the complement of the energy can be used for the Lagrangian. So there is a complement for this equation, that is named Co-Lagrangian. The Co-Lagrangian is described as the opposite of the Lagrangian: it is the difference between the total potential co-energy and the stored kinetic energy.

This concepts can be extended to electromagnetic and mechanical domains. For mechanical systems, the Lagrangian is expressed in terms of displacements and velocities, while the Co-Lagrangian, in terms of momenta and forces. For electromagnetic systems, the Lagrangian is expressed in terms of currents and charge, while the Co-Lagrangian, in terms of voltages and flux linkage.

For a better visualization, the next table shows the energy and co-energy relations in the Lagrangian and Co-Lagrangian:

Domain	Lagrangian	Co-Lagrangian
Mechanical	Kinetic co-energy - Potential energy	Potential co-energy - Kinetic energy
Electromagnetic	Magnetic co-energy - Electric energy	Electric co-energy - Magnetic energy

Tabela 2.1: Energy and co-energy relations in the Lagrangian and Co-Lagrangian.

The notations used here for the Lagrangian and Co-Lagrangian for electromagnetic systems are:

– Z $\rightarrow$ Lagrangian;

– Z^* $\rightarrow$ Co-Lagrangian;

– E_m $\rightarrow$ magnetic energy;

– E_e $\rightarrow$ electric energy;

– E_m^*, E_e^* $\rightarrow$ the respective coenergies.

Also, for the Lagrangian and Co-Lagrangian for mechanical systems, the notations are:

– L $\rightarrow$ Lagrangian;

– L^* $\rightarrow$ Co-Lagrangian;

– T $\rightarrow$ kinetic energy;

– V $\rightarrow$ potential energy;

– T^*, V^* $\rightarrow$ the respective coenergies.

We can then write the Lagrangian and Co-Lagrangian explicitly for both domains:

1. For electromagnetic systems:

$$Z = E_m^* - E_e \ \text{ and } \ Z^* = E_e^* - E_m. \tag{2-3}$$

2. For mechanical systems:

$$L = T^* - V \ \text{ and } \ L^* = V^* - T. \tag{2-4}$$

Where the generic formula to find the dynamics is:

$$\frac{d}{dt}\left(\frac{\partial X}{\partial \dot{z}_i}\right) - \frac{\partial X}{\partial z_i} = \frac{d\delta W}{d\delta z_i}. \tag{2-5}$$

3
Electromagnetic background

Now we find ourselves in need of a electromagnetic background [8] to fully understand how to describe the dynamics of an electromagnetic system. Concepts such as magnetic and electrostatic forces, magnetic and electrostatic fields, potential and flux will be approached and, with these knowledge, the description of charged particles and rigid bodies dynamics will be possible.

3.1
Electrostatic force

When a polarized body find itself close to another body also polarized, they tend to attract or repel each other. For example, if a positively charged body is near a negatively charged body, the free electrons of the second one are attracted to the protons in the first one. However, the opposite occurs when two different bodies are charged equally, that is, two positively charged bodies are repelled when close to each other (the same is valid for two negatively charged). The force that results in this interaction is called Coulomb force of attraction and is given as follow:

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{\|\mathbf{r}\|^2}\hat{r} \ \text{ or } \ \mathbf{F}_{12} = \frac{1}{4\pi\epsilon}\frac{q_1 q_2}{\|\mathbf{r}\|^3}\mathbf{r}. \tag{3-1}$$

Here, ϵ is called electric permittivity, q_1 and q_2 are the charge of the particles in question, measured in Coulombs (C). Also, $\mathbf{r}$ is the position vector between q_1 and q_2, $\|\mathbf{r}\|$ is the distance between them and $\hat{r}$ is the unit vector in the position of $\mathbf{r}$.

It is possible to see that, when the two bodies have different charges, the sign changes and, consequently, the direction of the force. It is also important to notice that $\mathbf{F}_{12} = \mathbf{F}_{21}$.

For a general situation where there are n charged particles:

$$\mathbf{F} = \sum_{j=1}^{n}\mathbf{F}_{ij}. \tag{3-2}$$

Figure 3.1 is a representation of these situations.

3.2
Electric field

An intuitive idea of what the electric field is can be given as follow: it is the electric force per unit charge. Analyzing equation (3.2), it is possible to
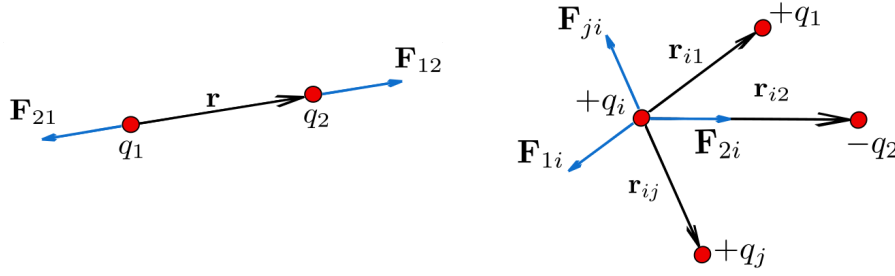
Figura 3.1: Coulomb forces between two or more charged particles.

see that the direction of this field is the direction of the electric force.

$$\mathbf{E}_P = \frac{\mathbf{F}_{Q_q}}{q} = \frac{1}{4\pi\epsilon} \frac{Q}{\|\mathbf{r}\|^2}\hat{r}. \tag{3-3}$$

P represents a point where we can find a positive charge q (called test charge) influenced by another charge Q. So, $\mathbf{E}_P$ is the electric field resultant by Q at point P.
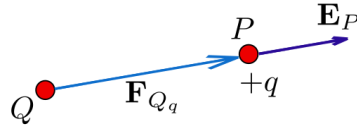
Figura 3.2: Representation of the explanation.

3.3
Electric flux

Taking an electric field $\mathbf{E}$ (variable in time t and position $\mathbf{r}$) and a surface s (also variable in time) inserted in the electric field, the electric flux can be defined as the flow of the electric field through an unit area dA of $s(t)$.

$$\Phi = \iint_{s(t)} \mathbf{E}(\mathbf{r}, t) \, . \, d\mathbf{A}. \tag{3-4}$$

With it we obtain a scalar and its unit is $\frac{Nm^2}{C}$.

For a case where neither the surface or the electric field varies in time, we can have the electric flux for when there is an angle θ between the normal of the surface and the electric field in which it is immersed:

$$\Phi = \mathbf{E} \, . \, \mathbf{A} = E \, A \, cos\theta. \tag{3-5}$$

You can see the case in figure 3.3.

Figura 3.3: Electric flux.

3.4
Electrostatic potential energy

To introduce this concept, take two positive charges $(+q)$. The idea is to bring one of them, in a hypothetical long distance (∞), at a distance R from the other charge, stationary. To accomplish this, we need the work done by the force opposite to the one imposed by the non moving charge 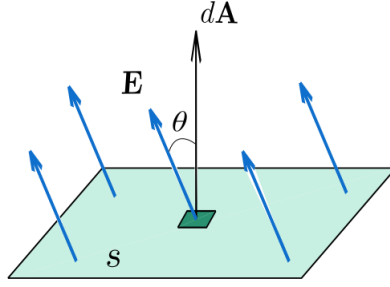at the other one ($\mathbf{F}$ in figure 3.4). It is important to have in mind that this work is equal to the electrostatic potential energy stored in the moving charge we aim.
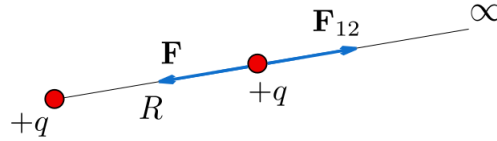
Figura 3.4: System described.

Here, we are talking about a conservative field, so the work done (and, therefore, the potential energy) does not depend on the path taken by the charge. Since $\mathbf{F} = -\mathbf{F}_{12}$, the work in question is given by:

$$W = \int_{\infty}^{R} \mathbf{F} \cdot d\mathbf{r} = \int_{R}^{\infty} \mathbf{F}_{12} \cdot d\mathbf{r}. \tag{3-6}$$

Substituting then $\mathbf{F}_{12}$ (Coulomb's law):

$$W = \frac{1}{4\pi\epsilon} \frac{q_1 q_2}{R}. \tag{3-7}$$

Once more, if the two charges equals in sign, the work done is positive, and if the opposite occurs, the work in negative.

Generalizing the situation for N charges and substituting W by U (potential energy), we have:

$$U = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N(i \neq j)} \frac{1}{4\pi\epsilon} \frac{q_i q_j}{r_{ij}}. \tag{3-8}$$

Notice that the term $\frac{1}{2}$ implies that the potential energy between the two charges is equally distributed.

3.5
Electric potential (voltage)

The relation seen between electrostatic force and electric field is also observed with electrostatic potential energy and electric potential, meaning that the potential energy is the electric potential per unit charge, just like the electric field is the electrostatic force divided by charge. So, at a point P, the voltage (or electric potential) is:

$$V_P = \int_R^\infty \frac{\mathbf{F}_{12}}{q} \cdot d\mathbf{r} = \int_R^\infty \mathbf{E}_P \cdot d\mathbf{r}. \tag{3-9}$$

The voltage is measured in $\frac{J}{C}$, also called volts V.

Substituting $\mathbf{E}_P$, we can get, for a charge Q, the voltage at a distance R at a point P:

$$V_P = \frac{1}{4\pi\epsilon} \frac{Q}{R}. \tag{3-10}$$

We can also obtain the voltage between two points (A and B) separated by a distance R, where $\mathbf{E}$ is the electric field that contains those two points. This integral does not depend of the path between A and B.

$$V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{r}. \tag{3-11}$$

In a closed loop (from point A to point A), the integral iguals zero as follow. This is called Kirchoff's voltage law, very used when dealing with circuits.

$$\oint \mathbf{E} \cdot d\mathbf{r} = 0. \tag{3-12}$$

If we take the differential form of equation (3-11) with respect to r:

$$dV = -\int_B^A E \, dr \ \text{ or } \ E = -\frac{dV}{dr}. \tag{3-13}$$

Generalizing for any set of coordinates:

$$\mathbf{E} = -\nabla V. \tag{3-14}$$

Analyzing it, we can see that a particle positively charge goes naturally from a high voltage to a low voltage and with a negatively charged one, from a low voltage to a high voltage. The unit for the electric field is, therefore, $\frac{V}{m}$.

3.6
Electromagnetic force

Being $\mathbf{B}(\mathbf{r}, t)$ a magnetic field and considering the previous knowledge that $\mathbf{F}(\mathbf{r}, t) = q\mathbf{E}(\mathbf{r}, t)$, the force resultant of both electric and magnetic field in a charge q is:

$$\mathbf{f}(\mathbf{r}, t) = q\,\mathbf{E}(\mathbf{r}, t) + q\,\mathbf{v}_P^O \times \mathbf{B}(\mathbf{r}, t). \qquad (3\text{-}15)$$

Here, $\mathbf{v}_P^O$ is the particle's velocity at point P regarding a inertial point in space O. This equation in called Lorentz law.

To find the magnetic field's direction, is commonly used the right-hand rule, where your thumb is the direction of the velocity vector, the four finger remaining are direction of the magnetic field and the palm of the hand is the direction of the electromagnetic force $\mathbf{F}$.

Since the magnetic force is perpendicular to the charge's velocity, it does not result in any change in the work done by the particle. Also, the magnetic force only acts in moving particles (differently from the electrostatic force, that acts in stationary particles and can change its velocity). Therefore, the magnetic force cannot influence in the kinetic energy, but can change the direction of the particle.

4
Energy

In this chapter, we will focus on introducing some elements that may be present in a circuit [6] (capacitor, inductor, resistor and the source), the energy that each one of them exert in the system and how to calculate it.

4.1
Capacitor

A capacitor is an element of the circuit represented by two plates that can transfer charge from one to another. The capacitance c is a property assigned to the capacitor and it is defined as the capacity to store electric energy. For capacitors, the circuit's current through it equals the variation with time of the charge:

$$i = \dot{q} = \frac{dq}{dt}. \tag{4-1}$$

Since an usual capacitor is approximately linear, the constitutive equation and it's graphic representation are:
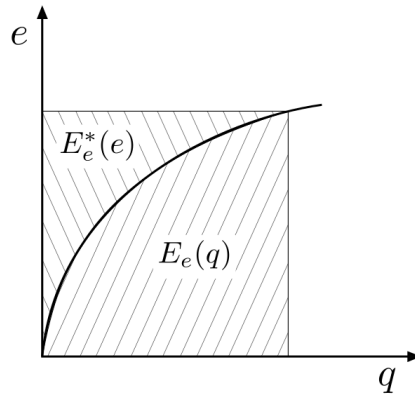
$$q = c\,e. \tag{4-2}$$

Figura 4.1: Capacitor's constitutive law.

Calculating the work done in the capacitor by integrating the power inserted in the circuit, it is possible to obtain the electric energy stored in an ideal capacitor.

$$E_e(q) = \int_0^t e\dot{q}\,dt = \int_0^q e\,dq.$$

So the electrical energy of a circuit is given by:

$$E_e(q) = \frac{q^2}{2c}. \tag{4-3}$$

Where c_{ir} is the mutual capacitance between two capacitors with capacitances c_i and c_r for s numbers of capacitors.

Subtracting the energy from a generic area eq, we have:

$$E_e^*(e) = eq - E_e(q).$$

So the electrical co-energy is:

$$E_e^*(e) = \frac{c\,e^2}{2}. \tag{4-4}$$

Some times you can not disregard the mutual capacitance between two capacitors, so the general equations for the electrical energy and co-energy are:

$$E_e(q) = \frac{1}{2} \sum_{i,r}^{s} \left(\frac{1}{c_{ir}}\, q_i\, q_r \right); \tag{4-5}$$

$$E_e^*(e) = \frac{1}{2} \sum_{i,r}^{s} \left(c_{ir}\, e_i\, e_r \right). \tag{4-6}$$

4.2
Inductor

An inductor is an element of the circuit that stores energy in the form of a magnetic field and is represented as a coil with N spires in the circuit. The inductance l of an inductor can be defined as the ratio between the total flux and the electric current present.

$$l = \frac{N\Phi}{\dot{q}}. \tag{4-7}$$

The voltage between two ends of an inductor equals the derivative in time of the flux linkage:

$$e = \frac{d\lambda}{dt}. \tag{4-8}$$

It is shown in the next figure the constitutive law's graphic for linear inductors. This equation is given by:

$$\lambda = l\,\dot{q}.$$

Here, we can do the same process done with capacitors:

$$E_m(\lambda) = \int_0^t e\dot{q}\, dt = \int_0^\lambda \dot{q}\, d\lambda.$$

So the magnetic energy in a circuit is:

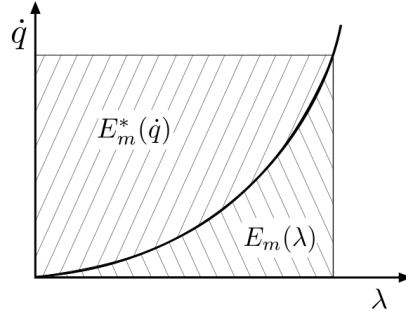$$E_m(\lambda) = \frac{\lambda^2}{2l}. \tag{4-9}$$

Figura 4.2: Inductor's constitutive law.

To obtain the magnetic co-energy, it is sufficient to subtract the magnetic energy from a generic area $\lambda \, \dot{q}$ in the graphic as follow:

$$E_m^*(\dot{q}) = \lambda \, \dot{q} - E_m(\lambda) \quad \rightarrow \quad E_m^*(\dot{q}) = \frac{l \, \dot{q}^2}{2}. \tag{4-10}$$

But not always we can desconsider the mutual inductances between two coils. So the general form for the magnetic co-energy in a circuit for s coils is given by:

$$E_m(\lambda) = \frac{1}{2} \sum_{i,r}^{s} \left(\frac{1}{l_{ir}} \, \lambda_i \, \lambda_r \right); \tag{4-11}$$

$$E_m^*(\dot{q}) = \frac{1}{2} \sum_{i,r}^{s} (l_{ir} \, \dot{q}_i \, \dot{q}_r). \tag{4-12}$$

The need to use energy and co-energy will became clear with the examples.

4.3
Non-conservative elements

Non-conservative elements are described with the concept of virtual work. Two possible non-conservative elements that can be found in a circuit are described next.

4.3.1
Source

The following relations in a pure electrical systems describe the work done by a source:

$$\delta_{f_i} = v_i \, \delta q_i \quad \rightarrow \quad \text{Generalized voltages;}$$
$$\delta_{I_k} = \dot{q}_k \, \delta \lambda_k \quad \rightarrow \quad \text{Generalized currents.}$$

The first relation refers to a situation when there is a voltage source and the second one, when there is a current source providing energy to the circuit.

To determinate the signal to each expression above, we will attribute that it is positive when the element supplies energy to the circuit and negative when the element absorbs energy from it.

4.3.2
Resistor

A resistor is an energy-dissipating element and the next equation gives us an important relation:

$$e = r\dot{q}. \tag{4-13}$$

Note that both e and v are voltages, so it is possible to use the first relation (generalized voltages) to find a virtual work in the charge formulation. We'll create an expression specifically for the resistance dissipation in the system, denoted as D:

$$\delta_d = r_i \, \dot{q}_i \, \delta q_i.$$

Using the second one (generalized currents), we obtain a flux linkage formulation:

$$\delta_d = \dot{q} \, \delta\lambda = \frac{e}{r}\delta\lambda \quad \rightarrow \quad \delta_d = \frac{1}{r} \, \dot{\lambda}\delta\lambda.$$
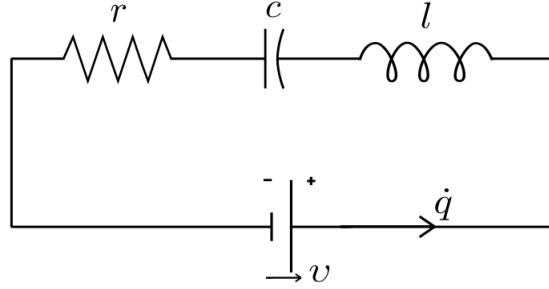
Generalizing, we get:

$$\delta_{d_i} = r_i \, \dot{q}_i \, \delta q_i \quad \text{and} \quad \delta_{d_s} = \frac{1}{r} \, \dot{\lambda}_s\delta\lambda_s. \tag{4-14}$$

It will became clear during the chapter 6 why this two versions of the equation are important when finding the dynamics of a electromechanical system.

5
Electromagnetic systems examples

Example 1: Consider the following circuit, where the capacitor is initially discharged:

To find the dynamics of the system, we will be using the Lagrangian method as follow.

$$E_m^* = \frac{l\dot{q}^2}{2}, \quad E_e = \frac{q^2}{2c} \quad \rightarrow \quad Z = E_m^* - E_e = \frac{l\dot{q}^2}{2} - \frac{q^2}{2c}.$$

Obtaining the virtual work:

$$\delta_f = v\,\delta q, \quad \delta_d = r\,\dot{q}\,\delta q \quad \rightarrow \quad \delta W = \delta_f - \delta_d = v\,\delta q - r\,\dot{q}\,\delta q.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}}\right) - \frac{\partial Z}{\partial q} = \frac{d\delta W}{d\delta q}.$$

$$l\ddot{q} + \frac{q}{c} = v - r\dot{q} \quad \rightarrow \quad l\ddot{q} + \frac{q}{c} + r\dot{q} = v.$$

Example 2: Consider the following circuit, where the capacitor is initially discharged (there is only one inductor in the system, so it is not worth mentioning the existence of a mutual inductance).

To find the dynamics of the system, we will be using the Lagrangian method.

In this case, as the circuit has more than one loop, we need to find the constraint that surrounds the currents (our parameterization). Note that there are two degrees of freedom in the system and the restriction in this case is given by: $q_3 = q_1 + q_2$.

With this we can now obtain the energies necessary for the construction of the Lagrangian function:

$$E_m^* = \frac{l_1 \dot{q}_1{}^2}{2}, \quad E_e = \frac{q_2^2}{2c_1} + \frac{(q_1 + q_2)^2}{2c_2},$$

$$Z = E_m^* - E_e = \frac{l_1 \dot{q}_1{}^2}{2} - \left(\frac{q_2^2}{2c_1} + \frac{(q_1 + q_2)^2}{2c_2} \right).$$

Obtaining the virtual work:

$$\delta_f = v_1 \, \delta q_1 + v_2 \, \delta q_2, \quad \delta_d = r_1 \, \dot{q}_1 \, \delta q_1 + r_2 \, (\dot{q}_1 + \dot{q}_2) \, (\delta q_1 + \delta q_2),$$

$$\delta W = \delta_f - \delta_d = v_1 \, \delta q_1 + v_2 \, \delta q_2 - r_1 \, \dot{q}_1 \, \delta q_1 - r_2 \, (\dot{q}_1 + \dot{q}_2) \, (\delta q_1 + \delta q_2).$$

Finding the dynamics:

$$\frac{d}{dt} \left(\frac{\partial Z}{\partial \dot{q}_i} \right) - \frac{\partial Z}{\partial q_i} = \frac{d \delta W}{d \delta q_i}.$$
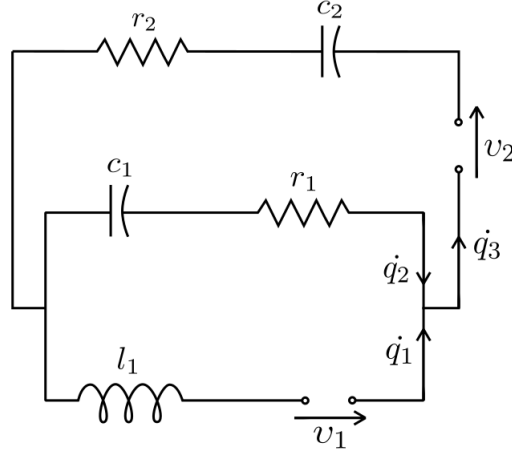
For q_1:

$$\frac{\partial Z}{\partial \dot{q}_1} = l_1 \dot{q}_1, \quad \frac{d}{dt} \left(\frac{\partial Z}{\partial \dot{q}_1} \right) = l_1 \ddot{q}_1, \quad \frac{\partial Z}{\partial q_1} = \frac{-(q_1 + q_2)}{c_2}, \quad \frac{d \delta W}{d \delta q_1} = v_1 + v_2 - r_2 (\dot{q}_1 + \dot{q}_2).$$

$$l_1 \ddot{q}_1 + r_2 (\dot{q}_1 + \dot{q}_2) + \frac{1}{c_2} (q_1 + q_2) = v_1 + v_2.$$

For q_2:

$$\frac{\partial Z}{\partial \dot{q}_2} = 0, \quad \frac{d}{dt} \left(\frac{\partial Z}{\partial \dot{q}_2} \right) = 0, \quad \frac{\partial Z}{\partial q_2} = -\frac{q_2}{c_1} - \frac{(q_1 + q_2)}{c_2}, \quad \frac{d \delta W}{d \delta q_2} = v_2 - r_1 \dot{q}_2 - r_2 (\dot{q}_1 + \dot{q}_2).$$

$$r_2 \dot{q}_1 + (r_1 + r_2) \dot{q}_2 + \frac{q_1}{c_2} + \left(\frac{1}{c_1} + \frac{1}{c_2} \right) q_2 = v_2.$$

Example 3: Take the following circuit. Here, the smaller coil is inside the larger one and they have a time-dependent mutual inductance l_{12}. In addition, the system voltage v_2 also varies over time with the following relation:

$v_2 = v_0 \, sen(wt)$, where v_0 is a constant.
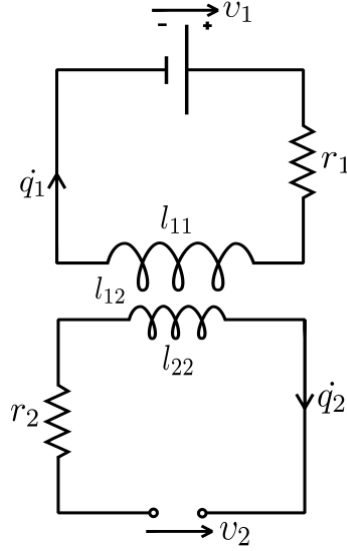
We want to find the system dynamics through the Lagrangian method. Note that in this case we have two independent currents.

$$E_m^* = \frac{1}{2} \left(l_{11}\dot{q}_1{}^2 + 2l_{12}\dot{q}_1\dot{q}_2 + l_{22}\dot{q}_2{}^2 \right), \quad E_e = 0$$

$$Z = E_m^* - E_e = \frac{1}{2} \left(l_{11}\dot{q}_1{}^2 + 2l_{12}\dot{q}_1\dot{q}_2 + l_{22}\dot{q}_2{}^2 \right).$$

Obtaining the virtual work:

$$\delta_f = v_1 \, \delta q_1 + v_2 \, \delta q_2, \quad \delta_d = r_1 \, \dot{q}_1 \, \delta q_1 + r_2 \, \dot{q}_2 \, \delta q_2,$$

$$\delta W = \delta_f - \delta_d = v_1 \, \delta q_1 + v_2 \, \delta q_2 - r_1 \, \dot{q}_1 \, \delta q_1 - r_2 \, \dot{q}_2 \, \delta q_2.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_i} \right) - \frac{\partial Z}{\partial q_i} = \frac{d\delta W}{d\delta q_i}.$$

For q_1:

$$\frac{\partial Z}{\partial \dot{q}_1} = l_{11}\dot{q}_1 + l_{12}\dot{q}_2, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_1} \right) = l_{11}\ddot{q}_1 + l_{12}\ddot{q}_2 + \dot{l}_{12}\dot{q}_2, \quad \frac{\partial Z}{\partial q_1} = 0, \quad \frac{d\delta W}{d\delta q_1} = v_1 - r_1\dot{q}_1.$$

$$l_{11}\ddot{q}_1 + l_{12}\ddot{q}_2 + r_1\dot{q}_1 + \dot{l}_{12}\dot{q}_2 = v_1.$$

For q_2:

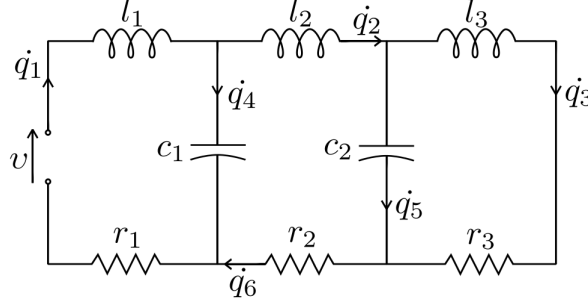$$\frac{\partial Z}{\partial \dot{q}_2} = l_{12}\dot{q}_1 + l_{22}\dot{q}_2, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_2} \right) = l_{12}\ddot{q}_1 + \dot{l}_{12}\dot{q}_1 + l_{22}\ddot{q}_2, \quad \frac{\partial Z}{\partial q_2} = 0, \quad \frac{d\delta W}{d\delta q_2} = v_2 - r_2\dot{q}_2.$$

$$l_{12}\ddot{q}_1 + l_{22}\ddot{q}_2 + \dot{l}_{12}\dot{q}_1 + r_2\dot{q}_2 = v_2.$$

Example 4: Take the following circuit where we disconsider the mutual inductances. To find the dynamics of the system, we will be using the Lagrangian method as follow and considering that the capacitors are initially charged: initial charges of q_{04} to c_1 and q_{05} to c_2.

Notice that this example has three degrees of freedom (six distinct currents, but only three are independent from the others) and the equations that give us the necessary constraints are:

$$q_4 = q_1 - q_2,$$

$$q_5 = q_2 - q_3,$$

$$q_6 = q_1 - q_4 = q_2.$$

With this we can start by finding the energies necessary for the construction of the Lagrangian function:

$$E_m^* = \frac{1}{2}\left(l_1 \dot{q}_1{}^2 + l_2 \dot{q}_2{}^2 + l_3 \dot{q}_3{}^2\right),$$

$$E_e = \frac{(q_4 + q_{04})^2}{2c_2} + \frac{(q_5 + q_{05})^2}{2c_1} = \frac{(q_1 - q_2 + q_{04})^2}{2c_1} + \frac{(q_2 - q_3 + q_{05})^2}{2c_2},$$

$$Z = E_m^* - E_e = \frac{1}{2}\left(l_1 \dot{q}_1{}^2 + l_2 \dot{q}_2{}^2 + l_3 \dot{q}_3{}^2\right) - \frac{(q_1 - q_2 + q_{04})^2}{2c_1} - \frac{(q_2 - q_3 + q_{05})^2}{2c_2}.$$

Obtaining the virtual work:

$$\delta_f = v\,\delta q_1, \quad \delta_d = r_1\,\dot{q}_1\,\delta q_1 + r_2\,\dot{q}_2\,\delta q_2 + r_3\,\dot{q}_3\,\delta q_3,$$

$$\delta W = \delta_f - \delta_d = v\,\delta q_1 - r_1\,\dot{q}_1\,\delta q_1 - r_2\,\dot{q}_2\,\delta q_2 - r_3\,\dot{q}_3\,\delta q_3.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_i}\right) - \frac{\partial Z}{\partial q_i} = \frac{d\delta W}{d\delta q_i}.$$

For q_1:

$$\frac{\partial Z}{\partial \dot{q}_1} = l_1 \dot{q}_1, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_1}\right) = l_1 \ddot{q}_1, \quad \frac{\partial Z}{\partial q_1} = -\frac{(q_1 - q_2 + q_{04})}{c_1}, \quad \frac{d\delta W}{d\delta q_1} = v - r_1 \dot{q}_1.$$

$$l_1 \ddot{q}_1 + r_1 \dot{q}_1 + \frac{q_1}{c_1} - \frac{q_2}{c_1} = v - \frac{q_{04}}{c_1}.$$

For q_2:

$$\frac{\partial Z}{\partial \dot{q}_2} = l_2 \dot{q}_2, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_2}\right) = l_2 \ddot{q}_2, \quad \frac{\partial Z}{\partial q_2} = \frac{(q_1 - q_2 + q_{04})}{c_1} - \frac{(q_2 - q_3 + q_{05})}{c_2}, \quad \frac{d\delta W}{d\delta q_2} = -r_2 \dot{q}_2.$$

$$l_2 \ddot{q}_2 + r_2 \dot{q}_2 - \frac{q_1}{c_1} + \left(\frac{1}{c_1} + \frac{1}{c_2}\right) q_2 - \frac{1}{c_2} q_3 = v + \frac{q_{04}}{c_1} - \frac{q_{05}}{c_5}.$$

For q_3:

$$\frac{\partial Z}{\partial \dot{q}_3} = l_3 \dot{q}_3, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_3}\right) = l_3 \ddot{q}_3, \quad \frac{\partial Z}{\partial q_3} = \frac{(q_2 - q_3 + q_{05})}{c_2}, \quad \frac{d\delta W}{d\delta q_3} = -r_3 \dot{q}_3.$$

$$l_3 \ddot{q}_3 + r_3 \dot{q}_3 - \frac{q_2}{c_2} + \frac{q_3}{c_2} = \frac{q_{05}}{c_2}.$$

Example 5: Lets take the previous example, but now considering the mutual capacitance (c_{12}) between two discharged capacitors, and find the dynamics equations for the system using the Lagrangian method.

$$E_m^* = \frac{1}{2}\left(l_1 \dot{q}_1^{\,2} + l_2 \dot{q}_2^{\,2} + l_3 \dot{q}_3^{\,2}\right),$$

$$E_e = \frac{1}{2}\left(\frac{q_4^2}{c_2} + \frac{q_5^2}{c_1} + \frac{2\,q_4\,q_5}{c_{12}}\right) = \frac{(q_1 - q_2)^2}{2c_1} + \frac{(q_2 - q_3)^2}{2c_2} + \frac{(q_1 - q_2)\,(q_2 - q_3)}{c_{12}},$$

$$Z = E_m^* - E_e = \frac{1}{2}\left(l_1 \dot{q}_1^{\,2} + l_2 \dot{q}_2^{\,2} + l_3 \dot{q}_3^{\,2}\right) - \frac{(q_1 - q_2)^2}{2c_1} - \frac{(q_2 - q_3)^2}{2c_2} - \frac{(q_1 - q_2)\,(q_2 - q_3)}{c_{12}}.$$

Obtaining the virtual work:

$$\delta_f = v\,\delta q_1, \quad \delta_d = r_1\,\dot{q}_1\,\delta q_1 + r_2\,\dot{q}_2\,\delta q_2 + r_3\,\dot{q}_3\,\delta q_3,$$

$$\delta W = \delta_f - \delta_d = v\,\delta q_1 - r_1\,\dot{q}_1\,\delta q_1 - r_2\,\dot{q}_2\,\delta q_2 - r_3\,\dot{q}_3\,\delta q_3.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_i}\right) - \frac{\partial Z}{\partial q_i} = \frac{d\delta W}{d\delta q_i}.$$

For q_1:

$$\frac{\partial Z}{\partial \dot{q}_1} = l_1 \dot{q}_1, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_1}\right) = l_1 \ddot{q}_1, \quad \frac{\partial Z}{\partial q_1} = -\frac{(q_1 - q_2)}{c_1} - \frac{(q_2 - q_3)}{c_{12}}, \quad \frac{d\delta W}{d\delta q_1} = v - r_1 \dot{q}_1.$$

$$l_1\ddot{q}_1 + r_1\dot{q}_1 + \frac{q_1}{c_1} + \left(\frac{1}{c_{12}} - \frac{1}{c_1}\right)q_2 - \frac{q_3}{c_{12}} = v.$$

For q_2:

$$\frac{\partial Z}{\partial \dot{q}_2} = l_2\dot{q}_2, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_2}\right) = l_2\ddot{q}_2, \quad \frac{\partial Z}{\partial q_2} = \frac{(q_1 - q_2)}{c_1} - \frac{(q_2 - q_3)}{c_2} - \frac{(q_1 - 2q_2 + q_3)}{c_{12}}, \quad \frac{d\delta W}{d\delta q_2} = -r_2\dot{q}_2.$$
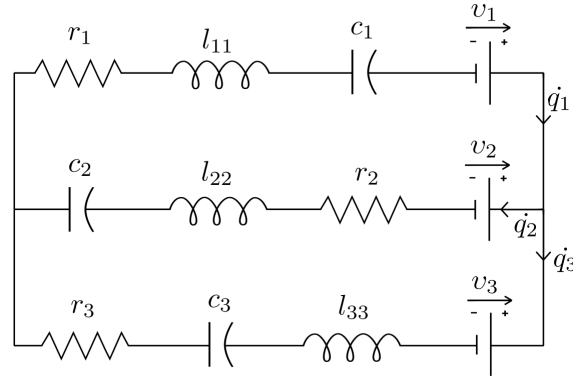
$$l_2\ddot{q}_2 + r_2\dot{q}_2 - \frac{q_1}{c_1} + \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{2}{c_{12}}\right)q_2 + \left(\frac{1}{c_{12}} - \frac{1}{c_2}\right)q_3 = v.$$

For q_3:

$$\frac{\partial Z}{\partial \dot{q}_3} = l_3\dot{q}_3, \quad \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_3}\right) = l_3\ddot{q}_3, \quad \frac{\partial Z}{\partial q_3} = \frac{(q_2 - q_3)}{c_2} + \frac{(q_1 - q_2)}{c_{12}}, \quad \frac{d\delta W}{d\delta q_3} = -r_3\dot{q}_3.$$

$$l_3\ddot{q}_3 + r_3\dot{q}_3 - \frac{q_1}{c_{12}} + \left(\frac{1}{c_{12}} - \frac{1}{c_2}\right)q_2 + \frac{q_3}{c_2} = 0.$$

Example 6: Here, we will be finding the dynamics equations, through the Lagrangian method, of the following circuit:

It is given that the capacitors are initially discharged. We must consider the mutual inductance, where $l_{12} = l_{21}$ is the mutual inductance between the inductor 1 and inductor 2 and so on. We also know that: $q_1 = q_2 + q_3 \quad \rightarrow q_3 = q_1 - q_2$.

There are two degrees of freedom and so:

$$E_m^* = \frac{1}{2}\left(l_{11}\dot{q}_1^{\,2} + l_{22}\dot{q}_2^{\,2} + l_{33}\dot{q}_3^{\,2} + 2l_{12}\dot{q}_1\dot{q}_2 + 2l_{13}\dot{q}_1\dot{q}_3 + 2l_{23}\dot{q}_2\dot{q}_3\right), \quad E_e = \frac{q_1^2}{2c_1} + \frac{q_2^2}{2c_2} + \frac{q_3^2}{2c_3},$$

$$Z = E_m^* - E_e = \frac{1}{2}\left(l_{11}\dot{q}_1^{\,2} + l_{22}\dot{q}_2^{\,2} + l_{33}\dot{q}_3^{\,2} + 2l_{12}\dot{q}_1\dot{q}_2 + 2l_{13}\dot{q}_1\dot{q}_3 + 2l_{23}\dot{q}_2\dot{q}_3\right) - \frac{q_1^2}{2c_1} - \frac{q_2^2}{2c_2} - \frac{q_3^2}{2c_3}.$$

Obtaining the virtual work:

$$\delta_f = v_1\,\delta q_1 + v_2\,\delta q_2 + v_3\,\delta q_3, \quad \delta_d = r_1\,\dot{q}_1\,\delta q_1 + r_2\,\dot{q}_2\,\delta q_2 + r_3\,\dot{q}_3\,\delta q_3,$$

$$\delta W = \delta_f - \delta_d = v_1 \, \delta q_1 + v_2 \, \delta q_2 + v_3 \, \delta q_3 - r_1 \, \dot{q}_1 \, \delta q_1 - r_2 \, \dot{q}_2 \, \delta q_2 - r_3 \, \dot{q}_3 \, \delta q_3.$$

Replacing q_3, we get:

$$Z = E_m^* - E_e = \frac{1}{2}[l_{11}\dot{q}_1{}^2 + l_{22}\dot{q}_2{}^2 + l_{33}(\dot{q}_1 - \dot{q}_2)^2 + 2l_{12}\dot{q}_1\dot{q}_2 + 2l_{13}\dot{q}_1(\dot{q}_1 - \dot{q}_2) +$$

$$+2l_{23}\dot{q}_2(\dot{q}_1 - \dot{q}_2)] - \frac{q_1^2}{2c_1} - \frac{q_2^2}{2c_2} - \frac{(\dot{q}_1 - \dot{q}_2)^2}{2c_3}.$$

$$\delta W = v_1 \, \delta q_1 + v_2 \, \delta q_2 + v_3 \, (\delta q_1 - \delta q_2) - r_1 \, \dot{q}_1 \, \delta q_1 - r_2 \, \dot{q}_2 \, \delta q_2 - r_3 \, (\dot{q}_1 - \dot{q}_2) \, (\delta q_1 - \delta q_2).$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_i}\right) - \frac{\partial Z}{\partial q_i} = \frac{d\delta W}{d\delta q_i}.$$

For q_1:

$$\frac{\partial Z}{\partial \dot{q}_1} = l_{11}\dot{q}_1 + l_{33}(\dot{q}_1 - \dot{q}_2) + l_{12}\dot{q}_2 + 2l_{13}\dot{q}_1 - l_{13}\dot{q}_2 + l_{23}\dot{q}_2,$$

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_1}\right) = (l_{11} + l_{33} + 2l_{13})\ddot{q}_1 + (-l_{33} + l_{12} - l_{13} + l_{23})\ddot{q}_2,$$

$$\frac{\partial Z}{\partial q_1} = -\frac{q_1}{c_1} - \frac{(q_1 - q_2)}{c_3}, \quad \frac{d\delta W}{d\delta q_1} = v_1 + v_3 - [r_1\dot{q}_1 + r_3(\dot{q}_1 - \dot{q}_2)].$$

$$(l_{11} + 2l_{13} + l_{33})\ddot{q}_1 + (l_{12} - l_{13} + l_{23} - l_{33})\ddot{q}_2 + (r_1 + r_3)\dot{q}_1 - r_3\dot{q}_3 + \left(\frac{1}{c_1} + \frac{1}{c_3}\right)q_1 - \frac{1}{c_3}q_2 = v_1 + v_3.$$

For q_2:

$$\frac{\partial Z}{\partial \dot{q}_2} = l_{22}\dot{q}_2 - l_{33}(\dot{q}_1 - \dot{q}_2) + l_{12}\dot{q}_1 - l_{13}\dot{q}_1 - 2l_{23}\dot{q}_2 + l_{23}\dot{q}_1,$$

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_2}\right) = (l_{22} + l_{33} - 2l_{23})\ddot{q}_2 + (-l_{33} + l_{12} - l_{13} + l_{23})\ddot{q}_1,$$

$$\frac{\partial Z}{\partial q_2} = -\frac{q_2}{c_2} + \frac{(q_1 - q_2)}{c_3}, \quad \frac{d\delta W}{d\delta q_2} = v_2 - v_3 - r_2\dot{q}_2 + r_3(\dot{q}_1 - \dot{q}_2).$$

$$(l_{22} - 2l_{23} + l_{33})\ddot{q}_2 + (l_{12} - l_{13} + l_{23} - l_{33})\ddot{q}_1 - r_3\dot{q}_1 + (r_2 + r_3)\dot{q}_2 - \frac{q_1}{c_3} + \left(\frac{1}{c_2} + \frac{1}{c_3}\right)q_2 = v_2 - v_3.$$

Example 7: To find the dynamic equations of the circuit, we will be using the Lagrangian method as follow. The capacitors are initially discharged and the equation of constraint in this case is: $q_2 = q_1 + q_0$, where q_0 is the electric charge transmitted from the current source to the circuit. Therefore,

it is not a variable in the system (but it still varies with time) and the virtual variation is resumed to $\delta q_2 = \delta q_1$.
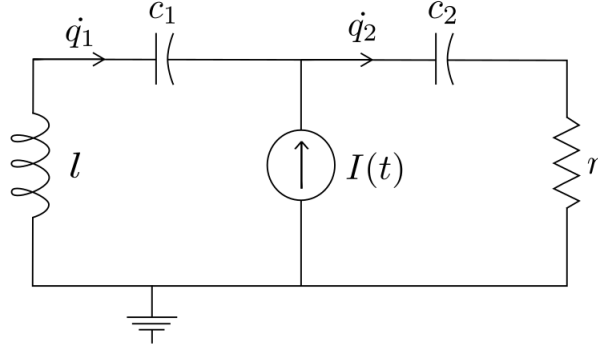
$$E_m^* = \frac{l\dot{q}_1^{\,2}}{2}, \quad E_e = \frac{q_1^2}{2c_1} + \frac{q_2^2}{2c_2} = \frac{q_1^2}{2c_1} + \frac{(q_1 + q_0)^2}{2c_2},$$

$$Z = E_m^* - E_e = \frac{l\dot{q}_1^{\,2}}{2} - \frac{q_1^2}{2c_1} - \frac{(q_1 + q_0)^2}{2c_2}.$$

In this example we are using the charge formulation to find the system's dynamics, so the current source does not contribute to the work done with this parametrization.

$$\delta_d = r\,\dot{q}_2\,\delta q_2 = r\,(\dot{q}_1 + \dot{q}_0)^2\,\delta q_1,$$

$$\delta W = -\delta_d = -r\,(\dot{q}_1 + \dot{q}_0)^2\,\delta q_1.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_1}\right) - \frac{\partial Z}{\partial q_1} = \frac{d\delta W}{d\delta q_1}.$$

$$l\ddot{q}_1 + \frac{q_1}{c_1} + \frac{(q_1 + q_0)}{c_2} = -r\,(\dot{q}_1 + \dot{q}_0),$$

$$l\ddot{q}_1 + r\dot{q}_1 + \left(\frac{1}{c_1} + \frac{1}{c_2}\right)q_1 = -r\dot{q}_0 - \frac{q_0}{c_2}.$$

Example 8: Take the same circuit from the last example. To find the dynamic equations we will be using the Co-Lagrangian function (flux linkage formulation). For a better understanding of the flux linkage formulation process, the next figure shows the coordinates λ_i.

The voltage drops when crossing a capacitor, so the voltage across c_1 is $\left(\dot{\lambda}_2 - \dot{\lambda}_1\right)$ and across c_1, $\left(\dot{\lambda}_3 - \dot{\lambda}_2\right)$.

Now it is possible to write the energies for the formulation of the Co-Lagrangian using the relations found before:

$$E_e^* = \frac{c_1}{2}\left(\dot{\lambda}_2 - \dot{\lambda}_1\right)^2 + \frac{c_2}{2}\left(\dot{\lambda}_3 - \dot{\lambda}_2\right)^2, \quad E_m = \frac{\lambda_1^2}{2l},$$
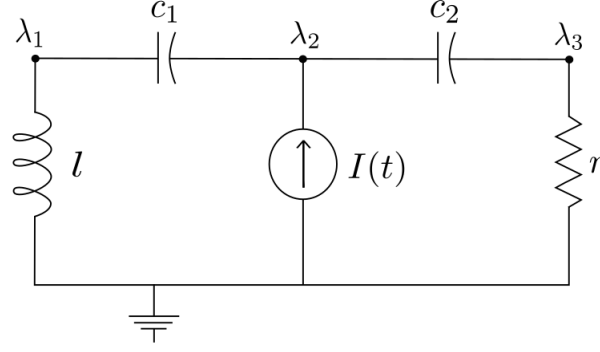
$$Z^* = E_e^* - E_m = \frac{c_1}{2}\left(\dot{\lambda}_2 - \dot{\lambda}_1\right)^2 + \frac{c_2}{2}\left(\dot{\lambda}_3 - \dot{\lambda}_2\right)^2 - \frac{\lambda_1^2}{2l}.$$

Obtaining the virtual work:

$$\delta_f = q_0\,\delta\lambda_2, \quad \delta_d = \frac{\dot{\lambda}_3}{r}\,\delta\lambda_3,$$

$$\delta W = \delta_f - \delta_d = q_0\,\delta\lambda_2 - \frac{\dot{\lambda}_3}{r}\,\delta\lambda_3.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z^*}{\partial\dot{\lambda}_i}\right) - \frac{\partial Z^*}{\partial\lambda_i} = \frac{d\delta W}{d\delta\lambda_i}.$$

For λ_1:

$$c_1\left(\ddot{\lambda}_1 - \ddot{\lambda}_2\right) + \frac{\lambda_1}{l} = 0.$$

For λ_2:

$$-c_1\ddot{\lambda}_1 + (c_1 + c_2)\,\ddot{\lambda}_2 - c_2\ddot{\lambda}_3 = q_0.$$

For λ_3:

$$c_2\left(\ddot{\lambda}_3 - \ddot{\lambda}_2\right) + \frac{\dot{\lambda}_3}{r} = 0.$$

Example 9: Take the next circuit. We will find the dynamic equations through the Lagrangian method. The capacitors are initially discharged and the equation of constraint in this case is: $q_3 = q_2 + q_0$, where q_0 is the electric charge supplied to the circuit from the current source. The virtual variation is resumed to $\delta q_3 = \delta q_2$, once q_0 is not a variable. Notice that q_1 is a independent variable.

$$E_m^* = \frac{l_1\dot{q}_1^{\,2}}{2} + \frac{l_2\,(\dot{q}_2 + \dot{q}_0)^2}{2}, \quad E_e = \frac{q_1^2}{2c_1} + \frac{q_2^2}{2c_2} + \frac{q_3^2}{2c_3} = \frac{q_1^2}{2c_1} + \frac{q_2^2}{2c_2} + \frac{(q_2 + q_0)^2}{2c_3},$$

$$Z = E_m^* - E_e = \frac{l_1\dot{q}_1^{\,2}}{2} + \frac{l_2\,(\dot{q}_2 + \dot{q}_0)^2}{2} - \frac{q_1^2}{2c_1} - \frac{q_2^2}{2c_2} - \frac{(q_2 + q_0)^2}{2c_3}.$$

Just like in example 6, the current source does not contribute to the work done with this parametrization.
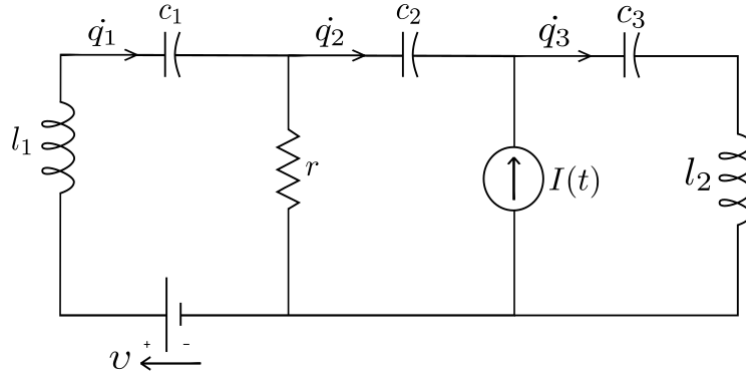
$$\delta_f = v\,\delta q_1, \quad \delta_d = r\,(\dot{q}_1 - \dot{q}_2)\,(\delta q_1 - \delta q_2),$$

$$\delta W = \delta_f - \delta_d = v\,\delta q_1 - r\,(\dot{q}_1 - \dot{q}_2)\,(\delta q_1 - \delta q_2) = [v - r\,(\dot{q}_1 + \dot{q}_2)]\,\delta q_1 + (r\dot{q}_1 - r\dot{q}_2)\,\delta q_2.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{q}_i}\right) - \frac{\partial Z}{\partial q_i} = \frac{d\delta W}{d\delta q_i}.$$
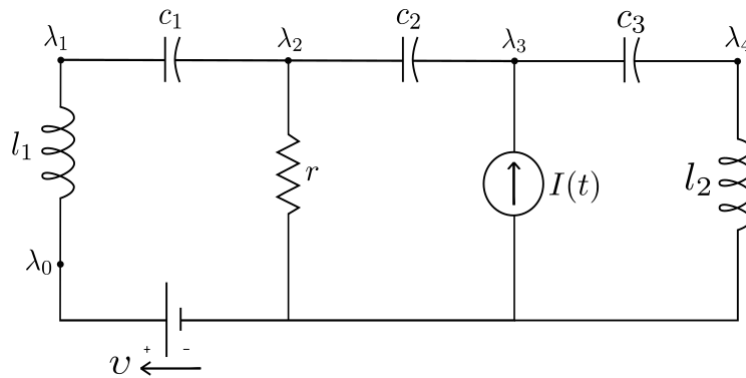
For q_1:

$$l_1\ddot{q}_1 + r\,(\dot{q}_1 + \dot{q}_2) + \frac{q_1}{c_1} = v.$$

For q_2:

$$l_2\ddot{q}_2 + r\,(\dot{q}_1 - \dot{q}_2) + \frac{q_2}{c_2} + \frac{q_2}{c_3} = -l_2\ddot{q}_0 - \frac{q_0}{c_3}.$$

Example 10: Take the same circuit from the last example. We will find the dynamic equations using the Co-Lagrangian function.

Using the relations found before, it is possible to find the dynamics through the flux linkage formulation:

$$E_e^* = \frac{c_1}{2}\left(\dot{\lambda}_2 - \dot{\lambda}_1\right)^2 + \frac{c_2}{2}\left(\dot{\lambda}_3 - \dot{\lambda}_2\right)^2 + \frac{c_3}{2}\left(\dot{\lambda}_4 - \dot{\lambda}_3\right)^2, \quad E_m = \frac{(\lambda_1 - \lambda_0)^2}{2l_1} + \frac{\lambda_4^2}{2l_2},$$

$$Z^* = E_e^* - E_m = \frac{c_1}{2}\left(\dot{\lambda}_2 - \dot{\lambda}_1\right)^2 + \frac{c_2}{2}\left(\dot{\lambda}_3 - \dot{\lambda}_2\right)^2 + \frac{c_3}{2}\left(\dot{\lambda}_4 - \dot{\lambda}_3\right)^2 - \frac{(\lambda_1 - \lambda_0)^2}{2l_1} - \frac{\lambda_4^2}{2l_2}.$$

Notice that the voltage source contribute through the magnetic energy, but not through the virtual work.

$$\delta_f = q_0\,\delta\lambda_3, \quad \delta_d = \frac{\dot{\lambda}_2}{r}\,\delta\lambda_2,$$

$$\delta W = \delta_f - \delta_d = q_0\,\delta\lambda_3 - \frac{\dot{\lambda}_2}{r}\,\delta\lambda_2.$$

Finding the dynamics:

$$\frac{d}{dt}\left(\frac{\partial Z^*}{\partial \dot{\lambda}_i}\right) - \frac{\partial Z^*}{\partial \lambda_i} = \frac{d\delta W}{d\delta\lambda_i}.$$

For λ_1:

$$c_1\left(\ddot{\lambda}_1 - \ddot{\lambda}_2\right) + \frac{(\lambda_1 + \lambda_0)}{l_1} = 0.$$

For λ_2:

$$-c_1\ddot{\lambda}_1 + (c_1 + c_2)\,\ddot{\lambda}_2 - c_2\ddot{\lambda}_3 + \frac{\dot{\lambda}_2}{r} = 0.$$

For λ_3:

$$-c_2\ddot{\lambda}_2 + (c_2 + c_3)\,\ddot{\lambda}_3 - c_3\ddot{\lambda}_4 = q_0.$$

For λ_4:

$$c_3\left(\ddot{\lambda}_4 - \ddot{\lambda}_3\right) + \frac{\lambda_4}{l_2} = 0.$$

6
Coupling term

As said before, electromechanical systems are composed by electromagnetic and mechanical subsystems. Nevertheless, it is not sufficient to combine these two parts to obtain the system's dynamics, once there is an interaction between them and, therefore, must exist a term of connection between the fields to represent these always present energy transference [9]. We will call it coupling field.

These chapter uses energy balance and the concepts of energy and co-energy to find coupling terms for electromechanical systems.

It is important to notice that this interaction exists when an element for connection between the two subsystems is present in the system or when the body in question is charged and immersed in a electric or magnetic field. Next, possible coupling terms are shown.

6.1
Movable-plate capacitor

Here, we already discussed common capacitors: elements from a circuit that store energy between two plates. To obtain a capacitor capable of transforming electrical energy into mechanical energy or vice versa, it is necessary to introduce a new characteristic to this element: one or more plates must be able to move. This characterizes a movable-plate capacitor [5], a conservative energy storing transducer. Figure 6.1 illustrates this case.
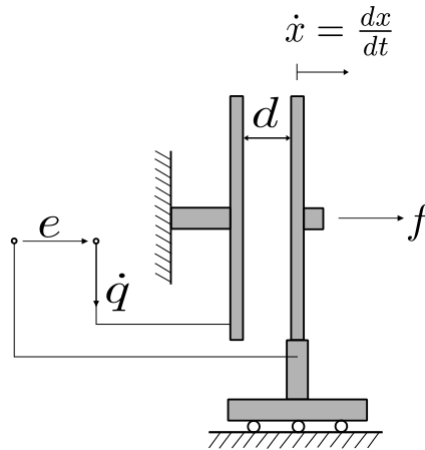

Figura 6.1: Movable-plate capacitor.

Since the plates are moving, the capacitance must be variable. For a distance represented by x in the case where only one plate is moving, we write

$c(x)$ for the variable capacitance. Now, using the energy and co-energy relations in equations (4-3) and (4-4), it is possible to obtain the coupling term for a electromechanical system with a movable-plate capacitor:

$$E_e(x,q) \;=\; \frac{q^2}{2c(x)}, \tag{6-1}$$

$$E_e^*(x,e) \;=\; \frac{c(x)e^2}{2}. \tag{6-2}$$

This variable capacitance is written in terms of the plate's area (A), a constant called electric permittivity (ϵ) and the distance between the plates in natural conditions (d). It will be mentioned and explained in the examples.

Since the capacitor is a circuit's natural element when present, adding a new coupling term would be redundant, so, in this case, this term is simply the electrical energy (with the charge formulation) or the electrical co-energy (with the link flux formulation).

6.2
Movable-core inductor

An inductor with a movable-core [5], also a conservative energy storing transducer, is illustrated in figure 6.2. This element converts magnetic energy into mechanical energy or vice versa.
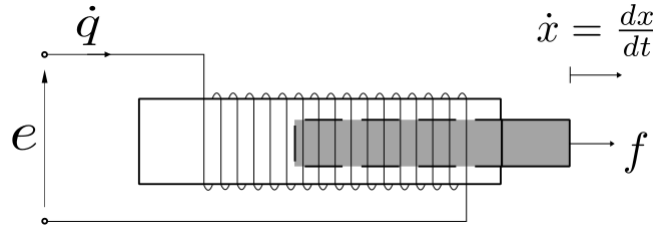
Figura 6.2: Movable-core inductor.

To find the coupling term for a circuit with a movable-core inductor, we can use the same logic as the one used with a movable-plate capacitor. Therefore, using the energy and co-energy relations in equations (4-9) and (4-10), the coupling terms are:

$$E_m(x,\lambda) \;=\; \frac{\lambda^2}{2l(x)}, \tag{6-3}$$

$$E_m^*(x,\dot{q}) \;=\; \frac{l(x)\dot{q}^2}{2}. \tag{6-4}$$

In the examples it will be shown how the inductance varies with x.

The same analyses used for the movable-plate capacitor can be done in this case: it would be redundant to introduce another coupling term in this kind of systems, so it is sufficient to use the magnetic energy (with the flux linkage formulation) or the magnetic co-energy (with the charge formulation).

6.3
Moving-coil transducer

Now we will introduce a new element that can be used to determinate a different type of coupling term. A moving-coil transducer [5] is also an energy transformer, but, differently from the others, this element converts electrical power into mechanical power or vice versa and can not store energy. Figure 6.3 shows how the transducer is composed: a coil that can move vertically is placed in a magnet that yields an uniform magnetic field B. Figure 6.4 is the transducer's symbolic representation that will be used.
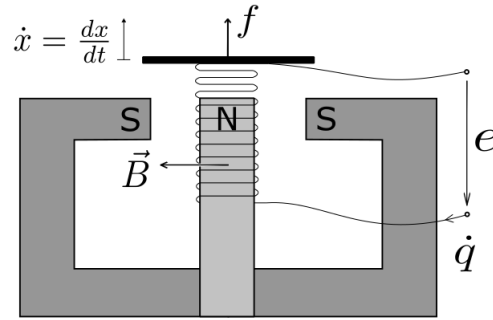
Figura 6.3: Moving-coil transducer.
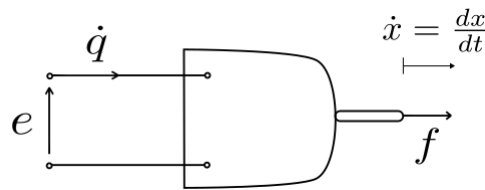
Figura 6.4: A symbolic representation of the moving-coil transducer

It is possible to obtain the constitutive equation for the moving-coil transducer from the two equations: one is equation (3-15) (Lorentz law) obtained before and the other one is given next (called Faraday's law), where f is the force needed to keep the coil in equilibrium, e is the voltage drop in the coil and $dl = r\, d\theta$.

$$de = \dot{\mathbf{x}} \times \mathbf{B}.\mathbf{dl} = -\dot{x}Br d\theta, \tag{6-5}$$

$$\mathbf{f} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}). \tag{6-6}$$

There are two facts to point out now: figure 6.3 also shows that $\mathbf{B}$, $\dot{\mathbf{x}}$ and $\mathbf{dl}$ are orthogonal to each other and that the electrostatic part can be removed when compared to the action of the magnetic one. Taking equation (6-5) and integrating it over θ, we get:

$$e = 2\pi n r B \dot{x}.$$

Since B is uniform, we can use ϱ to refer to the constant term (called the transducer constant) and the last equation can be written as:

$$e = \varrho \dot{x}. \tag{6-7}$$

Also, if its taken in consideration that a large number of particles is forming the current, the force df (taken from equation (6-6)) acting in a dl is:

$$\mathbf{df} = \dot{q}\, \mathbf{dl} \times \mathbf{B}.$$

And, using the same relation as before $(dl = r\, d\theta)$, df can be written as:

$$df = \dot{q} r d\theta B. \tag{6-8}$$

Now integrating equation (6-8) over the part of the conductor that is in contact with B, the magnetic field, we get:

$$f = -\dot{q} 2\pi n r B = -\varrho \dot{q}. \tag{6-9}$$

With equations (6-7) and (6-9), we have the two necessary constitutive equations for the moving-coil transducer.

As said before, the moving-coil transducer is an element that converts mechanical power into electric power, therefore, the total power handed to the transducer is the sum of both constitutive equations:

$$e\dot{q} + f\dot{x} = \varrho \dot{x}\dot{q} - \varrho \dot{q}\dot{x} = 0.$$

It is now explicit the reason why this element does not store energy. We also notice that the magnetic energy is constant, so it's value can be taken as the reference one:

$$U_m = 0. \tag{6-10}$$

To obtain the co-energy correspondent to this element, we can start by writing equation (6-7) in terms of flux linkage formulation, where x_0 is a position reference:

$$\lambda = \varrho(x - x_0).$$

The next relation is known from the energy chapter:

$$U_m^*(x, \dot{q}) = \lambda \dot{q} - U_m(x, \lambda).$$

It is now possible to find the magnetic co-energy substituting λ in the

last relation:
$$U_m^*(x, \dot{q}) = \varrho\dot{q}(x - x_0). \qquad (6\text{-}11)$$

Differently from the other cases, this term should be added to the usual Lagrangian function, once it doesn't appear naturally in the formulation. That's the reason why we will use the notation U to refer to this type of energy.

6.4
DC Motor

A DC motor is a machine that converts electrical energy into mechanical energy by rotating a shaft and generating a torque through a conductor material with a continuous current found inside a magnetic field.

The torque generated by the motor is given by the product of the current with a electromagnetic constant referred to as the strength of the magnetic field (k_e). Therefore:
$$\tau_e = k_e\dot{q}. \qquad (6\text{-}12)$$

Equation (6-12) must be multiplied by an angular displacement α to form an energy and, by doing that, we get the magnetic co-energy used for a DC motor:
$$U_m^*(\alpha, \dot{q}) = k_e\dot{q}\alpha. \qquad (6\text{-}13)$$

Just as in the last case, this term should be added to the Lagrangian function, once it doesn't appear naturally in the Lagrangian formulation.

7
Lagrangian for electromechanical systems

7.1
Formula

Now that it is known possible coupling terms for an electromechanical system, it is necessary to describe the Lagrangian formulation for this case. The Lagrangian function for an electromechanical system [5] is written as:

$$\Gamma = T^* - V + E_m^* - E_e \pm U^*, \qquad (7\text{-}1)$$

where T^* is the kinetic co-energy, V the potential energy, E_m^* the magnetic co-energy and E_e the electric energy. The coupling term U^* can have an electric or magnetic origin and it's signal depends on this fact. If it is transmitted as a magnetic coupling (U_m^*), the signal is positive and if it is transmitted as an electric one (U_e^*), the signal is negative. This is shown in the next equations:

$$\Gamma = T^* - V + (E_m^* + U_m^*) - E_e, \qquad (7\text{-}2)$$

$$\Gamma = T^* - V + E_m^* - (E_e + U_e^*). \qquad (7\text{-}3)$$

Being z_i a generalized coordinate of the system, each differential equation of the electromechanical system dynamics can be found by:

$$\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{z}_i}\right) - \frac{\partial \Gamma}{\partial z_i} = \frac{d\delta W}{d\delta z_i}. \qquad (7\text{-}4)$$

7.2
Examples

7.2.1
Capacitive microphone

Take figure 7.1 showing an electromechanical system in a horizontal position composed by a source v, a resistor, an inductor, a coil spring with natural length x_0, a damper and a movable-plate capacitor [5]. We'll take x (deflection) and q (charge) as our generalized variables. The capacitance in question is initially charged with q_0 and varies with x, the distance from the equilibrium position as shown bellow. Also, the movable plate has mass m. This system is called capacitive microphone.
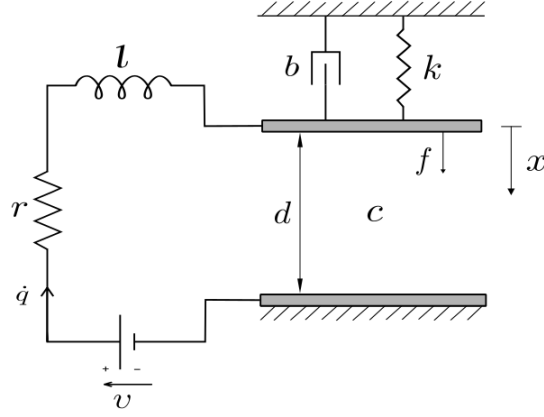
Figura 7.1: Capacitive microphone.

The capacitance of the movable-plate capacitor in this case can be written as:

$$c(x) = \frac{\epsilon A}{d - x}.$$

Where the constants used to describe it are as given in the section about movable-plate capacitors.

To find the dynamics equations using Lagrangian, we will be dividing the system in two: the mechanical and electromagnetic subsystems. The first one depends only of the coordinate x and the second one, of q.

For the mechanical subsystem:

$$T^* = \frac{m\dot{x}^2}{2}, \quad V = \frac{kx^2}{2}.$$

For the electromagnetic subsystem:

$$E_m^* = \frac{l\dot{q}^2}{2}, \quad E_e = \frac{(q_0 + q)^2}{2c(x)} = \frac{(d - x)}{2\epsilon A}(q_0 + q)^2.$$

Notice that, in this case, there is no need to add a term of coupling, once the variable capacitance in the system is already sufficient. Therefore, the Lagrangian for the electromechanical system is:

$$\Gamma = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} + \frac{l\dot{q}^2}{2} - \frac{(d - x)}{2\epsilon A}(q_0 + q)^2.$$

Now it is possible to derive this and obtain the dynamics equations, where the virtual work is:

$$\delta_f = \upsilon\,\delta q + f\delta x, \quad \delta_d = r\,\dot{q}\,\delta q + b\,\dot{x}\,\delta x.$$

$$\delta W = \delta_f - \delta_d = \upsilon\,\delta q + f\delta x - r\,\dot{q}\,\delta q - b\,\dot{x}\,\delta x.$$

For x:

$$\frac{\partial \Gamma}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{x}}\right) = m\ddot{x}, \quad \frac{\partial \Gamma}{\partial x} = -kx + \frac{(q_0 + q)^2}{2\epsilon A}, \quad \frac{d\delta W}{d\delta x} = f - b\dot{x}.$$

For q:

$$\frac{\partial \Gamma}{\partial \dot{q}} = l\dot{q}, \quad \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}}\right) = l\ddot{q}, \quad \frac{\partial \Gamma}{\partial q} = -\frac{(d-x)}{\epsilon A}(q_0 + q), \quad \frac{d\delta W}{d\delta q} = v - r\dot{q}.$$

The dynamic equations of motion for the capacitive michophone are then:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \Gamma}{\partial \dot{x}}\right) - \dfrac{\partial \Gamma}{\partial x} = \dfrac{d\delta W}{d\delta x} & \rightarrow \quad m\ddot{x} + b\dot{x} + kx - \dfrac{(q_0 + q)^2}{2\epsilon A} = f, \\[3mm] \dfrac{d}{dt}\left(\dfrac{\partial \Gamma}{\partial \dot{q}}\right) - \dfrac{\partial \Gamma}{\partial q} = \dfrac{d\delta W}{d\delta q} & \rightarrow \quad l\ddot{q} + r\dot{q} + \dfrac{(d-x)}{\epsilon A}(q_0 + q) = v. \end{cases}$$

7.2.2
Electromagnetic plunger

A plunger of mass m is vertically attracted to an electromagnet when a switch closes the circuit in the figure. Since the gap of the plunger is x_d when the current is zero, the magnetic force generated by the current through the coil can only attract the plunger to a maximum $x = -x_d$. This system is called electromagnetic plunger [5].
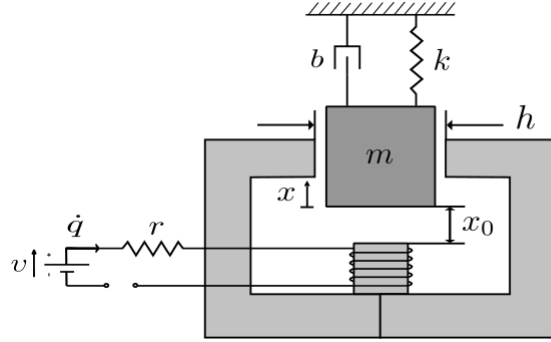
Figura 7.2: Electromagnetic plunger.

The inductance of the electromagnet is variable with x and is given as:

$$l(x) = \frac{l_0}{1 + \frac{(x_d + x)}{h}}.$$

Now it is possible to find the dynamics equations following the steps shown before.

For the mechanical subsystem:

$$T^* = \frac{m\dot{x}^2}{2}, \quad V = \frac{kx^2}{2}.$$

For the electromagnetic subsystem:

$$E_m^* = \frac{l(x)\dot{q}^2}{2}, \quad E_e = 0.$$

Finding the Lagrangian:

$$\Gamma = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} + \frac{l(x)\dot{q}^2}{2}.$$

Obtaining the virtual work:

$$\delta_f = \upsilon\,\delta q, \quad \delta_d = r\,\dot{q}\,\delta q + b\,\dot{x}\,\delta x,$$

$$\delta W = \delta_f - \delta_d = \upsilon\,\delta q - r\,\dot{q}\,\delta q - b\,\dot{x}\,\delta x.$$

For x:

$$\frac{\partial\Gamma}{\partial\dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial\Gamma}{\partial\dot{x}}\right) = m\ddot{x}, \quad \frac{\partial\Gamma}{\partial x} = -kx + l'(x)\frac{\dot{q}^2}{2}, \quad \frac{d\delta W}{d\delta x} = -b\dot{x}.$$

For q:

$$\frac{\partial\Gamma}{\partial\dot{q}} = l\dot{q}, \quad \frac{d}{dt}\left(\frac{\partial\Gamma}{\partial\dot{q}}\right) = \frac{d}{dt}[l(x)\dot{q}], \quad \frac{\partial\Gamma}{\partial q} = 0, \quad \frac{d\delta W}{d\delta q} = \upsilon - r\dot{q}.$$

The dynamic equations of motion for the electromagnetic plunger are then:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial\Gamma}{\partial\dot{x}}\right) - \dfrac{\partial\Gamma}{\partial x} = \dfrac{d\delta W}{d\delta x} \quad \rightarrow \quad m\ddot{x} + b\dot{x} + kx - l'(x)\dfrac{\dot{q}^2}{2} = 0, \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial\Gamma}{\partial\dot{q}}\right) - \dfrac{\partial\Gamma}{\partial q} = \dfrac{d\delta W}{d\delta q} \quad \rightarrow \quad \dfrac{d}{dt}[l(x)\dot{q}] + r\dot{q} = \upsilon. \end{cases}$$

7.2.3
Electromagnetic loudspeaker

Figure 7.3 shows us an electromagnetic loudspeaker [5], a system that generates acoustical energy through electrical energy. The mechanical subsystem is composed by a mass, a spring and a damper, while the electromagnetic part, by a source, resistor and inductor. A moving-coil transducer with constant ϱ is used as a coupling element between the subsystems.
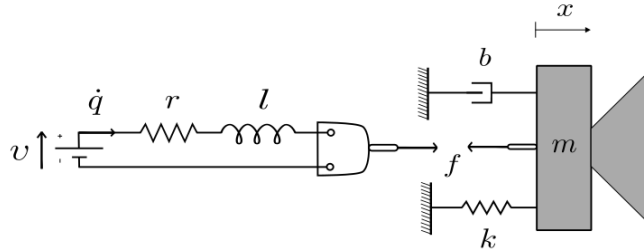

Figura 7.3: Electromagnetic loudspeaker.

This example is only valid in low frequencies situations.

To find the equations that describe the system's dynamics through the Lagrangian, we'll use the coordinates given in the figure.

For the mechanical subsystem:

$$T^* = \frac{m\dot{x}^2}{2}, \quad V = \frac{kx^2}{2}.$$

For the electromagnetic subsystem:

$$E_m^* = \frac{l\dot{q}^2}{2}, \quad E_e = 0.$$

The coupling term is given by:

$$U_m^* = \varrho\dot{q}x.$$

So the Lagrangian is given by:

$$\Gamma = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} + \frac{l\dot{q}^2}{2} + \varrho\dot{q}x.$$

Obtaining the virtual work:

$$\delta_f = v\,\delta q, \quad \delta_d = r\,\dot{q}\,\delta q + b\,\dot{x}\,\delta x,$$

$$\delta W = \delta_f - \delta_d = v\,\delta q - r\,\dot{q}\,\delta q - b\,\dot{x}\,\delta x.$$

For x:

$$\frac{\partial\Gamma}{\partial\dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial\Gamma}{\partial\dot{x}}\right) = m\ddot{x}, \quad \frac{\partial\Gamma}{\partial x} = -kx + \varrho\dot{q}, \quad \frac{d\delta W}{d\delta x} = -b\dot{x}.$$

For q:

$$\frac{\partial\Gamma}{\partial\dot{q}} = l\dot{q} + \varrho x, \quad \frac{d}{dt}\left(\frac{\partial\Gamma}{\partial\dot{q}}\right) = l\ddot{q} + \varrho\dot{x}, \quad \frac{\partial\Gamma}{\partial q} = 0, \quad \frac{d\delta W}{d\delta q} = v - r\dot{q}.$$

The dynamic equations of motion for the electromagnetic loudspeaker are then:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial\Gamma}{\partial\dot{x}}\right) - \dfrac{\partial\Gamma}{\partial x} = \dfrac{d\delta W}{d\delta x} & \rightarrow \quad m\ddot{x} + b\dot{x} + kx - \varrho\dot{q} = 0, \\[2mm] \dfrac{d}{dt}\left(\dfrac{\partial\Gamma}{\partial\dot{q}}\right) - \dfrac{\partial\Gamma}{\partial q} = \dfrac{d\delta W}{d\delta q} & \rightarrow \quad l\ddot{q} + r\dot{q} + \varrho\dot{x} = v. \end{cases} \qquad (7\text{-}5)$$

7.2.4
Scotch-yoke

The system analyzed in this example is composed by a DC motor that moves a cart by using a component called scotch-yoke and in can be seen in figure 7.4 [10; 11; 12]. It is possible to notice that there is a mechanical coupling resultant of the disk's rotational motion, that converts it to a translation motion through the so called scotch-yoke (the motor has a pin that slides to an opening in a plate attached to the card). There is also the electromechanical coupling given by the DC motor.
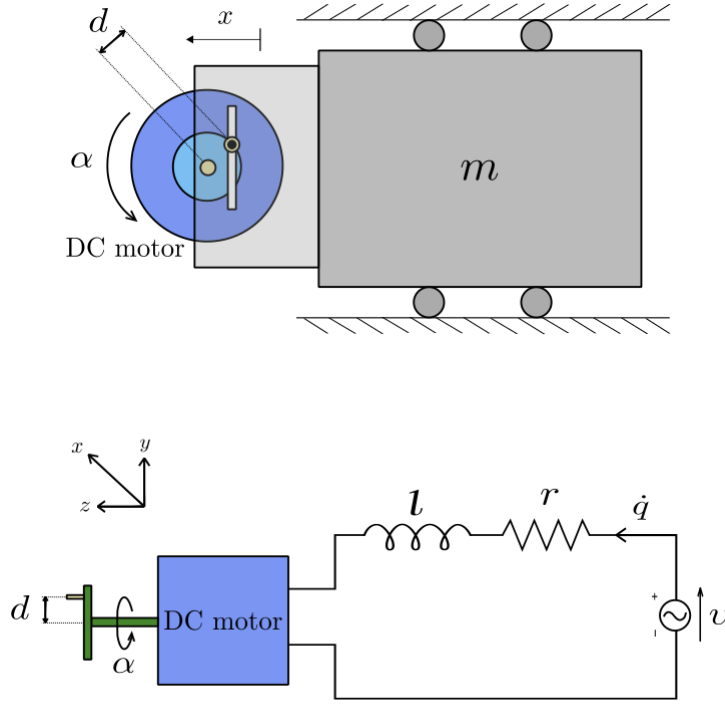
Figura 7.4: Cart-motor system with the scotch-yoke and DC motor.

Being m the cart mass, j_m the moment of inertia of the motor, k_e the strength of the magnetic field (the motor electromagnetic force constant), α the angular displacement of the disk and d the distance from the center of the plate to the pin that gives us the mechanical coupling, find the system's dynamics equations through the Lagrangian.

For the mechanical subsystem we have:

$$T^* = \frac{j_m \dot{\alpha}^2}{2} + \frac{m \dot{x}^2}{2}, \quad V = 0.$$

For the electromagnetic subsystem:

$$E_m^* = \frac{l \dot{q}^2}{2}, \quad E_e = 0.$$

The coupling term is given by:

$$U_m^* = k_e \dot{q} \alpha.$$

However, the scotch-yoke gives us the following relation between x and α:

$$x = d\cos(\alpha) \quad \rightarrow \quad \dot{x} = -d\sin(\alpha)\dot{\alpha}.$$

So the kinetic co-energy can be also written as:

$$T^* = \frac{j_m \dot{\alpha}^2}{2} + \frac{m[-d\,sin(\alpha)\dot{\alpha}]^2}{2}.$$

Now it is possible to find the Lagrangian:

$$\Gamma = \frac{j_m \dot{\alpha}^2}{2} + \frac{m[-d\,sin(\alpha)\dot{\alpha}]^2}{2} + \frac{l\dot{q}^2}{2} + k_e \dot{q}\alpha.$$

Obtaining the virtual work:

$$\delta_f = v\,\delta q, \quad \delta_d = r\,\dot{q}\,\delta q + b\,\dot{\alpha}\,\delta\alpha,$$

$$\delta W = \delta_f - \delta_d = v\,\delta q - r\,\dot{q}\,\delta q - b\,\dot{\alpha}\,\delta\alpha.$$

For α:

$$\frac{\partial \Gamma}{\partial \dot{\alpha}} = j_m \dot{\alpha} + md^2 sin^2(\alpha)\,\dot{\alpha}, \quad \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{\alpha}}\right) = j_m \ddot{\alpha} + md^2 sin^2(\alpha)\,\ddot{\alpha} + 2md^2 sin(\alpha)cos(\alpha)\dot{\alpha}^2,$$

$$\frac{\partial \Gamma}{\partial \alpha} = md^2 sin(\alpha)cos(\alpha)\dot{\alpha}^2 + k_e \dot{q}, \quad \frac{d\delta W}{d\delta\alpha} = -b\dot{\alpha}.$$

For q:

$$\frac{\partial \Gamma}{\partial \dot{q}} = l\dot{q} + k_e \alpha, \quad \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}}\right) = l\ddot{q} + k_e \dot{\alpha}, \quad \frac{\partial \Gamma}{\partial q} = 0, \quad \frac{d\delta W}{d\delta q} = v - r\dot{q}.$$

The dynamic equations of motion for the scotch-yoke are then:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \Gamma}{\partial \dot{\alpha}}\right) - \dfrac{\partial \Gamma}{\partial \alpha} = \dfrac{d\delta W}{d\delta\alpha} \quad \rightarrow \quad [j_m + md^2 sin^2(\alpha)]\ddot{\alpha} + md^2 sin(\alpha)cos(\alpha)\dot{\alpha}^2 + b\dot{\alpha} - k_e \dot{q} = 0, \\[2mm] \dfrac{d}{dt}\left(\dfrac{\partial \Gamma}{\partial \dot{q}}\right) - \dfrac{\partial \Gamma}{\partial q} = \dfrac{d\delta W}{d\delta q} \quad \rightarrow \quad l\ddot{q} + r\dot{q} + k_e \dot{\alpha} = v. \end{cases}$$

7.2.5
Galvanometer

A galvanometer is a tool capable of measuring electric currents with low intensities. It works with a mechanical device that moves through an electromagnetic force produced by the current passing through the circuit in the system. This current going through the moving-coil produces a magnetic field, making a pointer move in a graduate scale, just as shown in figure 7.5. The system is composed by a circuit with inductance l, resistance r, a source v, a permanent magnet, a torsional spring k, a damper b, a fix core with height L, diameter $2d$ and a coil involving the core. The generalized variables in this example are the current $\dot{q}$ and the pointer's angle given by θ. It is not taken into account any capacitance in the system.

Lorentz and Faraday's laws can be determined in a similar way as in the transducer example and are given by:

$$e = \dot{\theta} BdL \quad \text{and} \quad f = \dot{q} BdL. \tag{7-6}$$

The constant BdL can then be represented as γ. With the same steps made in the transducer, the magnetic co-energy is given by:
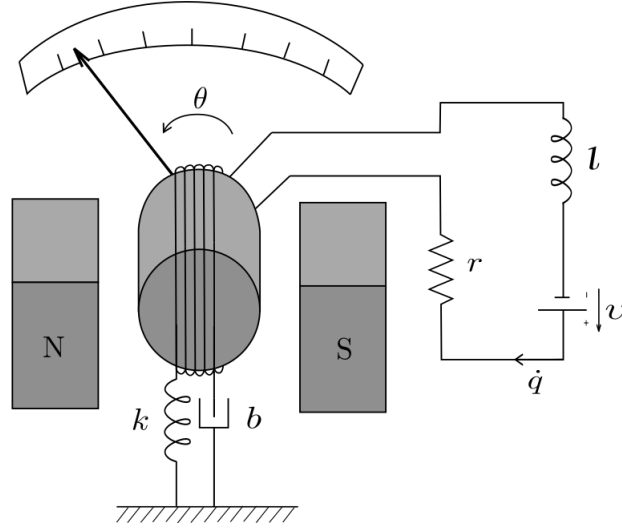
Figura 7.5: Galvanometer.

$$U_m^* = \gamma \dot{q} \theta.$$

The others energies in the system are:

$$T^* = \frac{j\dot{\theta}^2}{2}, \quad V = \frac{k\theta^2}{2}, \quad E_e = 0, \quad E_m^* = \frac{l\dot{q}^2}{2}.$$

The Lagrangian function is then given by:

$$\Gamma = \frac{j\dot{\theta}^2}{2} - \frac{k\theta^2}{2} + \frac{l\dot{q}^2}{2} + \gamma \dot{q} \theta.$$

The virtual work is:

$$\delta_f = v \, \delta q, \quad \delta_d = r \, \dot{q} \, \delta q + b \, \dot{\theta} \, \delta\theta,$$

$$\delta W = \delta_f - \delta_d = v \, \delta q - r \, \dot{q} \, \delta q - b \, \dot{\theta} \, \delta\theta.$$

Now it is possible to find the dynamics equations.

For θ:

$$\frac{\partial \Gamma}{\partial \dot{\theta}} = j\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{\theta}}\right) = j\ddot{\theta}, \quad \frac{\partial \Gamma}{\partial \theta} = -k\theta + \gamma \, \dot{q}, \quad \frac{d\delta W}{d\delta\theta} = -b\dot{\theta}.$$

For q:

$$\frac{\partial \Gamma}{\partial \dot{q}} = l\dot{q} + \gamma\theta, \quad \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}}\right) = l\ddot{q} + \gamma \, \dot{\theta}, \quad \frac{\partial \Gamma}{\partial q} = 0, \quad \frac{d\delta W}{d\delta q} = v - r\dot{q}.$$

The dynamic equations of motion for the galvanometer are then:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \Gamma}{\partial \dot{\theta}}\right) - \dfrac{\partial \Gamma}{\partial \theta} = \dfrac{d\delta W}{d\delta\theta} \quad \rightarrow \quad j\ddot{\theta} + b\dot{\theta} + k\theta - \gamma \, \dot{q} = 0, \\[3mm] \dfrac{d}{dt}\left(\dfrac{\partial \Gamma}{\partial \dot{q}}\right) - \dfrac{\partial \Gamma}{\partial q} = \dfrac{d\delta W}{d\delta q} \quad \rightarrow \quad l\ddot{q} + r\dot{q} + \gamma \, \dot{\theta} = v. \end{cases} \qquad (7\text{-}7)$$

8
Energetic analysis of an electromagnetic loudspeaker

To analyze the interplay between the different types of energy in this system, a routine was developed using the software $MATLAB$ to simulate how the electromagnetic loudspeaker responds during 15 seconds to a situation where the initial conditions are $x(0) = 1, q(0) = 0, \dot{x}(0) = 0$ and $\dot{q}(0) = 0$. To simulate, the initial value problem that gives the system dynamics was integrated by the $4^{th}-5^{th}$ order Runge-Kutta method with the $ode45$ $MATLAB$ function. The time-step used was 0.002 seconds and the parameters were chosen for a better interpretation of the results and they are given by:

$m = 0.15\ kg$	$b = 0\ Ns/m$
$k = 0.10\ N/m$	$\varrho = 0.30\ mT$
$l = 1.00\ H$	$r = 0\ \Omega$
$v = 0\ V$	-

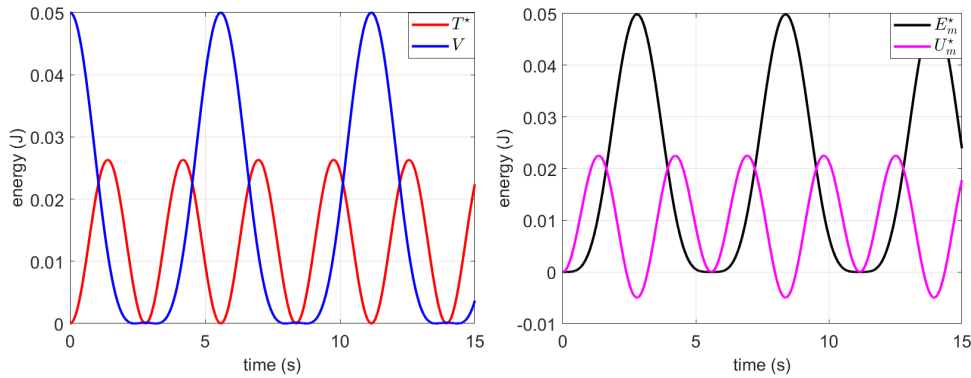Tabela 8.1: Used parameters in the electromagnetic loudspeaker simulation.

Figura 8.1: Graphics showing the different types of energies in the system.

Figure 8.1 compares the kinetic co-energy T^* with the potential energy V and the magnetic co-energy E_m^* with the electric energy E_e. It is possible to notice that the potential energy reaches its maximum and minimum values when the kinetic energy is in its minimum. Something similar, but not equal, occurs this the magnetic co-energy and electric energy: when E_m^* reaches its maximum, E_e is in its minimum (a negative value), but when the magnetic co-energy reaches its minimum, the electric energy is null, reaching a local minimum.

It is also possible to do an energy balance of the system. Using equation (7-5) and multiplying the first equation by $\dot{x}(t)$ and the second one by $\dot{q}(t)$:

$$\begin{cases} m\ddot{x}(t)\dot{x}(t) + b\dot{x}(t)\dot{x}(t) + kx(t)\dot{x}(t) - \varrho\dot{q}(t)\dot{x}(t) = 0, \\ l\ddot{q}(t)\dot{q}(t) + r\dot{q}(t)\dot{q}(t) + \varrho\dot{x}(t)\dot{q}(t) = v\dot{q}(t). \end{cases} \quad (8\text{-}1)$$

Adding the two equations found in (8-1) and making $b = 0, r = 0, v = 0$ to simplify the analysis:

$$\frac{d}{dt}\left(\frac{m\dot{x}(t)^2}{2} + \frac{kx(t)^2}{2} + \frac{l\dot{q}(t)^2}{2}\right) = 0. \quad (8\text{-}2)$$

It is possible to see in equation (8-2) that the coupling term of the system dynamics no longer appears. This happens because the moving-coil transducer is an element that does not store energy and, therefore, does not contribute to the energy balance. Figure 8.2 shows the graphic representation of equation (8-2), with the sum of the different types of energy: mechanical $(T^* + V)$ and electromagnetic (E_m^*). The green line represents this sum and is a constant with a value that depends, in this case, on the parameter k of the system.
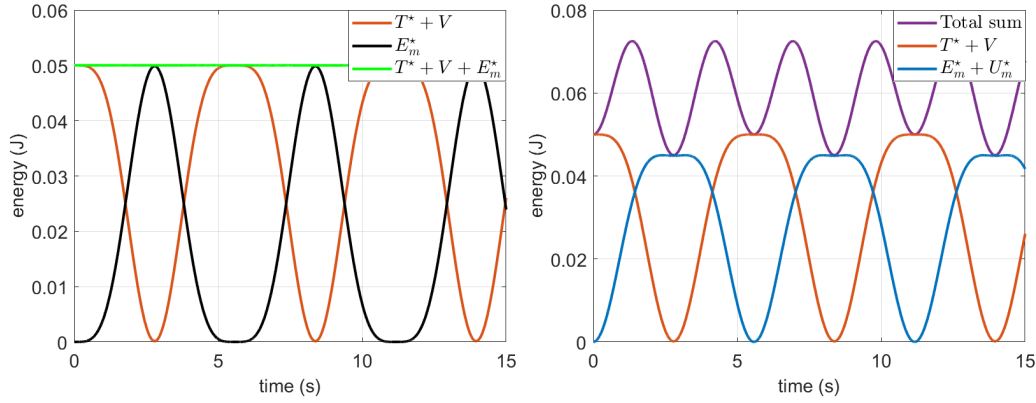

Figura 8.2: Energy balance and different energies sums, respectively.

Figure 8.2 also shows two types of sums: one adding the two energies of mechanical origin and another one adding the two energies of electromagnetic origin. It is also shown the total sum of the energies, that is, the sum of all energies regardless of its origin. This last one does not equal a constant, showing once more that U_m^*, energy passing through the transducer, is not stored in this element, it is only transmitted from one subsystem to another.

After this simple example, it is possible to change one of the parameters so a more accurate analysis can be made. Giving a $v = sin(t)$, the same graphics can be analyzed. The patterns in figures 8.3 and 8.4 are repeated every 50 seconds.

Figure 8.3 shows a different pattern compared to the previous one: now, the potential energy V reaches its maximums when the kinetic energy T^* is in its minimums and vice versa. The relation between E_m^* and U_m^* also changes: their minimums are always close and the same occurs to the maximums. The

fact that U_m^* is, in the most part, negative, shows that this energy is flowing contrary to the one before most of the time.
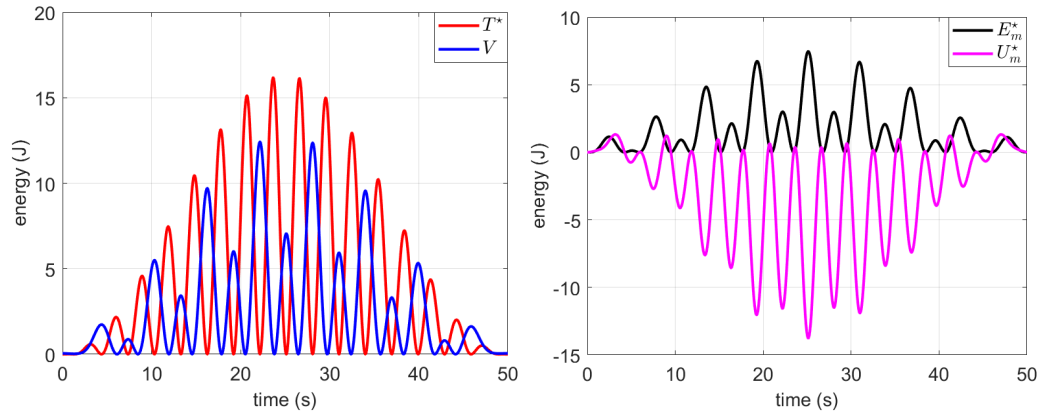

Figura 8.3: Graphics showing the different types of energies in the system.

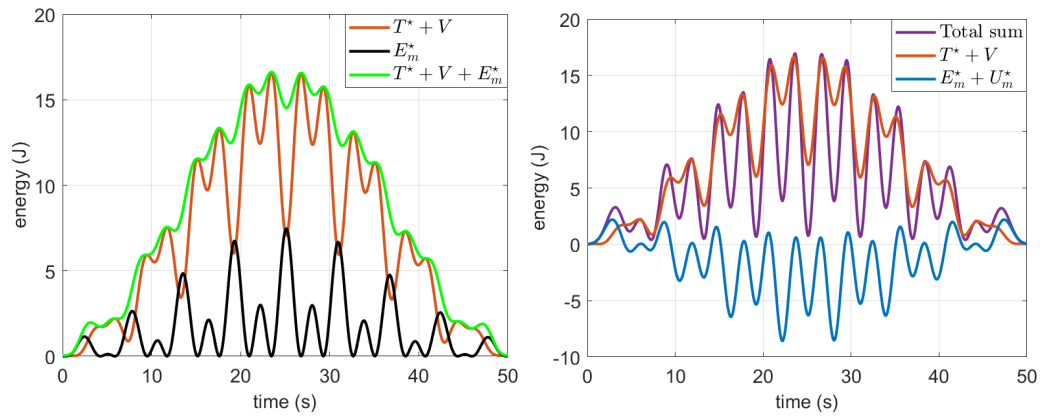The different kinds of energy sums are given in figure 8.4.

Figura 8.4: Energy balance and different energies sums, respectively.

9
Comparison between the right and wrong ways of finding an electromechanical system dynamics

The main goal of this chapter is to exemplify and analyse several mistakes made in literature when trying to obtain an electromechanical system dynamics while making use of an specific example: the galvanometer (seen in figure 7.5).

As said before, electromechanical systems are composed by two coupled subsystems: a mechanical and an electromagnetic. But even though they are so common and can be found in several applications used in daily life, it is still a challenge to find references correctly describing their dynamics. Published papers, books and thesis present numerous mistakes in the description of the dynamics of this type of system. Some common errors found in the literature are: neglecting the dynamics of the electromagnetic subsystem and its interactions with the mechanical subsystem, ignore an inductance present in the electromagnetic subsystem or even assume a functional relation between the forces of interaction between the subsystems.

The next step here is to use the galvanometer to compare the right and wrong ways of finding its dynamics and then analyse the differences. This will serve as an example of why the kind of mistakes mentioned before are critical when describing any electromechanical system.

The right dynamic equations of motion for the galvanometer are given by equation 7-7 and are shown again next:

$$\begin{cases} j\ddot{\theta} + b\dot{\theta} + k\theta - \gamma\,\dot{q} = 0, \\ l\ddot{q} + r\dot{q} + \gamma\,\dot{\theta} = v. \end{cases}$$

At this point, it is interesting to show an example that, at first, may seem reasonable, but after a close look is clear that it is not. Thus, it will be possible to compare this two solutions: the right one (as developed in section 7.2.5) and the wrong one (shown next). For example, one can find the next idea tempting, even though it is **wrong**:

Let L be the length of the coil's wire and $2b$ the fixed core's diameter. Having in mind that the coil's torque is given by:

$$\tau = 2dF, \tag{9-1}$$

where $F = \dfrac{1}{2}BL\dot{q}$, it is possible to obtain the following relation:

$$j\ddot{\theta} = \tau - k\theta.$$

Since the coil is emerged in a magnetic field, equation (7-6) can be combined with the next relation:

$$v - e = r\dot{q}. \tag{9-2}$$

Equation (9-2) does not consider the existence of an capacitance in the system (just as in the right case). However, it is neglecting the existence of an inductance and this is the first mistake made here.

The second mistake comes with the combination of all the equations obtained until this point. By doing this, we find only one equation of motion when there is suppose to be two, once there are two coordinates in the system: θ and $\dot{q}$. But to continue the idea, the equation of motion obtained by doing this is:

$$\ddot{\theta} + \left[\frac{\gamma^2}{rj}\right]\dot{\theta} + \left[\frac{k}{j}\right]\theta = \left[\frac{\gamma}{rj}\right]v,$$

where γ is once again BdL.

We can also say, for a matter of simplification, that:

$$\xi = \frac{\gamma^2}{2r\sqrt{kj}}, \quad \omega_n = \sqrt{\frac{k}{j}}, \quad p = \frac{\gamma}{kr}.$$

The wrong equation of motion for the galvanometer can then be written as:

$$\ddot{\theta} + 2\xi\omega_n\dot{\theta} + \omega_n^2\theta = \omega_n^2 pv. \tag{9-3}$$

Solving equation (9-3) for θ, we get:

$$\theta_{wrong}(t) = vp\left[1 - e^{-\xi\omega_n t}\left(cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}sin(\omega_d t)\right)\right], \tag{9-4}$$

$$\dot{\theta}_{wrong}(t) = vp\left[\xi\omega_n e^{-\xi\omega_n t}\left(cos(\omega_d t) - \frac{\xi\omega_n}{\omega_d}sin(\omega_d t)\right) - e^{-\xi\omega_n t}\left(-\omega_d sin(\omega_d t)\right.\right.$$
$$\left.\left. -\xi\omega_n cos(\omega_d t)\right)\right]. \tag{9-5}$$

With the solution for the wrong way of describing the galvanometer's dynamics and the simulation of the right one, it is possible to compare both with different kinds of graphics. The same initial conditions and parameters given in table 9.1 were used for both cases.

A routine was developed using the software $MATLAB$ to simulate how the galvanometer system responds with time, but now with different propose: show and discuss different types of errors committed while trying to find the dynamic equations of an electromechanical system and why they are critical when trying to describe the dynamics of an electromechanical system. The simulation shows how the galvanometer (figure 7.5) responds during 200 seconds to a situation where the initial conditions are $\theta(0) = 0, q(0) = 0, \dot{\theta}(0) = 0$ and $\dot{q}(0) = 0$.

To simulate, the initial value problem that gives the system dynamics was integrated by the $4^{th}-5^{th}$ order Runge-Kutta method with the *ode45 MATLAB* function. The time-step used was 0.002 seconds and the parameters were chosen for a better interpretation of the results and they are given by:

$j = 0.50\ kg\ m^2$	$b = 0\ Ns/m$
$k = 0.20\ N/m$	$\gamma = 1.00\ m^2\ T$
$l = 10.00\ H$	$r = 10.00\ \Omega$
$v = 1\ V$	-

Tabela 9.1: Used parameters in the galvanometer simulation.

First, lets compare the solutions obtained for $\theta(t)$, given in figure 9.1. The blue curve represents the wrong solution and the red, the right one developed in this report in section 7.2.5. The reader can notice that both solutions tend to the same value (for the given parameters and initial conditions, this value is 0.5 rad) in permanent state, but when it is still in transient state, the damping factors are distinct. The wrong solution is more damped than the right one. This fact results in different times for the system to achieve permanent state (the wrong one achieves this faster).
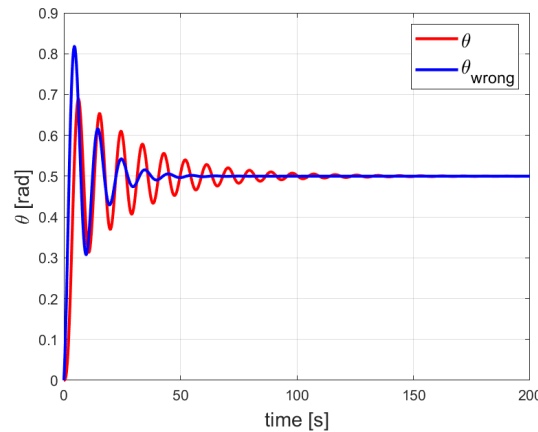

Figura 9.1: Solutions for both the right and wrong way of obtaining the dynamic equations.

Another way of comparing the results is to make phase portraits. Figure 9.2 shows a phase portrait of the $\dot{\theta}$ as function of θ. For both cases the form is similar and tend to the point (0.5, 0), but just as mentioned for figure 9.1, the response of the wrong solution is more damped then the right one, converging more quickly.

There are other interesting phase portraits to analyse. Figure 9.3 shows two distinct phase portrait: the first one is of the $\dot{\theta}$ as function of $\dot{q}$ and the second, of the τ as function of $\dot{\theta}$. Those graphics clearly show that the relations
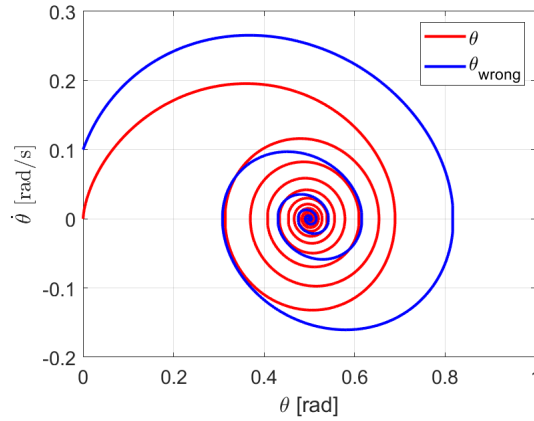
Figura 9.2: Phase portrait of the $\dot{\theta}$ as function of θ.

obtained for the wrong solution can be written as functions (using equations (9-4), (9-5) and (9-1)). This is not correct and, therefore, is the third possible error when obtaning the dynamic equations for an electromechanical system. The relations of $\dot{\theta}$ and $\dot{q}$ just as τ and $\dot{\theta}$ are non-functional (red curves), once they depend on initial conditions. Those relations can only be obtained after solving the dynamic equations for the system's coordenates.
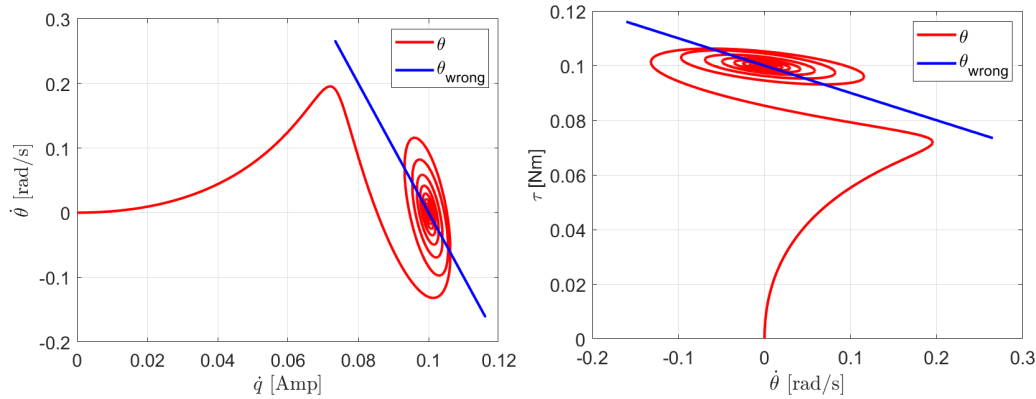

Figura 9.3: Phase portrait of the $\dot{\theta}$ as function of $\dot{q}$ and of the τ as function of $\dot{\theta}$, respectively.

Neglecting the inductance and assuming functional relations for the force of interaction between the subsystems are only two of the many common mistakes made in literature when trying to describe the dynamics of an electromechanical system. This chapter shows that they are indeed critical mistakes.

It is also important to highlight the fact that the galvanometer is a measuring instrument and having the right equations of motion for this kind of electromechanical system is crucial to obtain the precise measurement. Desenvolving incorrectly the dynamics of a measuring device can be very dangerous and/or expensive depending on the applications.

10
Conclusions and future projects

In the final project it was possible to characterize the mutual influence between the two subsystems that compose an electromechanical system and the interplay of the energies between them. It was verified that these coupling terms can have a magnetic origin or an electric origin.

In the analysis of the electromagnetic loudspeaker, it was possible to conclude that the moving-coil transducer is an element that does not store energy in system and, therefore, does not contribute to the energy balance. Which means that the transducer's energy is only transmitted from one subsystem to another. That is, it was also explored in the final project the fact that a coupling element does not necessarily have to store energy, it can only transform it, making the energy transition in this case a little bit different from a pure mechanical system.

Also, possible errors in describing the dynamics of an electromechanical system were commented using the example of an galvanometer, a measuring device. Different types of graphics were made so that it was clear that neglecting the inductance, finding only one equation of motion that uses only mechanical coordinates and assuming functional relations for the interaction between the subsystems of an electromechanical system are mistakes that change completely how the system varies with time.

This project has four products, listed bellow:

– A didactic text explaining how to describe the dynamics of electrome-chanical systems using the Lagrangian method with several examples of different electromechanical systems [13];

– Routines using the software *MATLAB* to solve an electromechanical system's equations of motions and also to make graphics that enable important interpretations about them;

– A congress proceeding paper entitled "Coupling in electromechanical sys-tems"submitted to the Regional Meeting of Applied and Computational Mathematics;

– A paper submitted to the congress XIV Academic Meeting on Compu-tational Modeling, entitled "Electromagnetic loudspeaker: an energetic approach".

There are a lot of possible approaches for a next step in this field of research. For example, the comparison between a pure mechanical system

with coupled equations of motions and an electromechanical system (a system that always has a coupling term) using the circulatory matrix and gyroscopic matrix. After the comparison, a stability analysis can be made. Also, there is a possibility of focusing in piezoelectric materials, a material that transforms deformation into difference of potential and vice versa and, therefore, is an electromechanical system.

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