Pontifícia Universidade Católica
do Rio de Janeiro

Pêdra Daricléa Santos Andrade

Towards a regularity theory for fully nonlinear models

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Edgard Almeida Pimentel

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To my mother Raimunda, with love.

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Abstract

Andrade, Pêdra Daricléa Santos; Pimentel, Edgard (Advisor). **Towards a regularity theory for fully nonlinear models**. Rio de Janeiro, 2020. 76p. Tese de doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

In this thesis, we examine fully nonlinear problems in two distinct contexts. The first part of our work focuses on fully nonlinear mean-field games. In this context, we examine gains of regularity, the existence of solutions, relaxation results, and particular aspects of a one-dimensional problem. The second half of the thesis concerns a (sharp) regularity theory for fully nonlinear equations degenerating with respect to the gradient of the solutions. The fundamental question underlying both topics regards the effects of ellipticity on the intrinsic properties of solutions to nonlinear equations. To be more precise, in the case of mean-field game systems, ellipticity seems to be magnified through the coupling structure. On the other hand, in the degenerate setting, ellipticity collapses, giving rise to intricate regularity phenomena. Our analysis is preceded by some context on both topics.

Keywords

Regularity theory; Mean-field games; Degenerate elliptic equations; Approximation methods; Existence; Viscosity.

Resumo

Andrade, Pêdra Daricléa Santos; Pimentel, Edgard. **Teoria de regularidade para modelos completamente não-lineares**. Rio de Janeiro, 2020. 76p. Tese de Doutorado – Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Neste trabalho examinamos equações completamente não-lineares em dois contextos distintos. A princípio, estudamos jogos de campo médio completamente não-lineares. Aqui, examinamos ganhos de regularidade para as soluções do problema, existência de soluções, resultados de relaxação e aspectos particulares de um example explícito. A segunda metade da tese dedica-se à regularidade ótima das soluções de um modelo completamente não-linear que degenera-se com respeito ao gradiente das soluções. A pergunta fundamental subjacente a ambos os tópicos diz respeito aos efeitos da elipticidade sobre propriedades intrínsecas das soluções de equações não-lineares. Mais precisamente, no caso dos jogos de campo médio, a elipticidade parece magnificada pelos efeitos do acoplamento, enquanto no caso dos problemas degenerados, esta quantidade colapsa em sub-regiões do domínio, dando origem a delicados fenômenos. Nossa análise inclui um breve contexto da inserção do trabalho.

Palavras-chave

Teoria de regularidade; Jogos de Campo Médio; Equações elípticas degeneradas; Métodos de aproximação; Existência; Viscosidade.

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1
Introduction

This thesis comprises two classes of developments. It includes results on fully nonlinear mean-field games as well as on the regularity theory for degenerate fully nonlinear equations. We present our findings in two parts.

The first part of the thesis investigates fully nonlinear mean-field games (MFG, for short) in close connection with a minimization problem. The latter is driven by a Hessian-dependent Lagrangian. We consider the system

$$\begin{cases} F(D^2u) \;=\; m^{1/(p-1)} & \text{in} \quad B_1 \\ \left(F_{ij}(D^2u)\, m\right)_{x_ix_j} \;=\; 0 & \text{in} \quad B_1, \end{cases} \tag{1.1}$$

where F is a fully nonlinear (λ, Λ)-elliptic operator and $p \geq 2$. In (1.1), $F_{ij}(M)$ stands for the derivative of $F(M)$ with respect to the entry m_{ij} of the matrix $M = (m_{ij})_{i,j=1}^d$. We refer to (1.1) as *mean-field game* for the system has an *adjoint structure*. That is to say that the second equation is the formal adjoint of the linearization of the first. In this system, the unknown is a pair (u, m) in a suitable functional space.

The MFG system in (1.1) is obtained as the first compact variation of

$$I[u] \;:=\; \int_{B_1} \left[F(D^2u)\right]^p \,\mathbf{d}x,$$

where I is defined over an appropriate class of functions. Therefore, we can understand the system in (1.1) as the Euler-Lagrange equation associated with the minimization problem.

$$\int_{B_1} \left[F(D^2u)\right]^p \,\mathbf{d}x \quad \longrightarrow \quad \min\,. \tag{1.2}$$

Mean-field game theory is a mathematical framework that aims at examining situations of strategic interaction involving a very large numbers of players. Unfolding at the intersection of analysis of partial differential equations and stochastic methods, this class of problems have attracted the attention of several researchers working on various topics with spillovers on a number of areas; see for instance [67], [68], [25], [59], [33], [34].

Our first result concerns gains of regularity for the solutions to (1.1) in Sobolev spaces. More precisely, we have the following result. Let $u \in \mathcal{C}(B_1)$ be a viscosity solution of

$$F(D^2u) = \mu^{\frac{1}{p-1}} \quad \text{in} \quad B_1,$$

where $F = F(M)$ is uniformly elliptic and convex with respect to M, $\mu \in L^1(B_1)$ is *an arbitrary* non-negative function and $p-1 \geq d$. It follows that $u \in W^{2,p-1}_{loc}(B_1)$, with appropriate estimates. See [22], [49] and [23, Chapter 7]. We also refer the reader to [72], where the convexity assumption on the operator is slightly weakened. However, if we consider a solution (u, μ) to (1.1), gains of regularity are produced. Our first result reads as follows:

Theorem 1 (Improved regularity in Sobolev spaces) *Let* (u, μ) *be a weak solution to* (1.1) *and* F *as above. Then,* $u \in W^{2,\frac{d(p-1)}{d-1}}_{loc}(B_1)$. *In addition, there exists* $C > 0$ *such that*

$$\|u\|_{W^{2,\frac{d(p-1)}{d-1}}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|\mu\|^{\frac{1}{p-1}}_{L^1(B_1)}\right).$$

As a consequence, $u \in \mathcal{C}^{1,\alpha}_{loc}(B_1)$ for every

$$\alpha \in \left(0, 1 - \frac{d-1}{p-1}\right).$$

In particular, the MFG coupling yields improved regularity also in Hölder spaces. Our second result concerns the existence of minimizers for (1.2). Then, a further argument yields the existence of solutions to the associated MFG system. It is the content of the next theorem.

Theorem 2 (Existence of solutions) *Suppose Assumption 2 and Assumption 3, to be detailed later, are in force. Then, there exists a minimizer* $u^* \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ *for* (1.2) *in the space* $W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$. *In addition, such a minimizer yields a solution* (u^*, m^*) *to the fully nonlinear mean-field game system* (1.1).

A further layer of analysis regards the case of non-convex operators F. This is done by applying relaxation arguments to produce information on the minimum of the associated energy. To this end, we consider the relaxed problem

$$\bar{I}[u] := \int_{B_1} \left[\Gamma_F(D^2u)\right]^p \mathbf{d}x \longrightarrow \min \tag{1.3}$$

where Γ_F is the convex envelope of F.

Theorem 3 (A relaxation result) *Suppose Assumption 3, to be detailed later, is in force. Then there exists* $(u_n)_{n\in\mathbb{N}} \subset W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ *such that*

$$I[u_n] \longrightarrow \bar{I}[u^*]$$

and

$$u_n \rightharpoonup u^* \qquad in \qquad W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$$

as $n \to \infty$*, where* $u^* \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ *is the minimizer of* (1.3).

We close the first part of this thesis with a one-dimensional toy-model unveiling distinctive properties of the problem (1.1)-(3.2). We have the following:

Theorem 4 (Explicit solutions) *Let* $F : (0,1) \to \mathbb{R}$ *be given by*

$$F(z) := (1 + z^p)^{\frac{1}{p}}, \tag{1.4}$$

where $p \geq 2$ *is a fixed integer. Then*

1. *A solution* (u, m) *to the associated mean-field game system is given by*

$$u(x) = \frac{A\left((p-1)x - B\right)^{2+\frac{1}{p-1}}}{p(2p-1)} + Cx + D$$

   and

$$m(x) = \left(1 + \left[A\left((p-1)x - B\right)^{\frac{1}{p-1}}\right]^p\right)^{\frac{1}{p}},$$

   where A*,* B*,* C*, and* D *are real constants.*

2. *A minimizing solution* (u^*, m^*) *comprises an affine mapping and a uniform distribution; i.e.,*

$$u^*(x) = A^*x + B^*$$

   and

$$m^*(x) = 1,$$

   where A^* *and* B^* *are real constants.*

Transitioning seamlessly from the first to the second part of this thesis might require some further explanation. The underlying structure lacing up both parts of this work is the notion of ellipticity, as well as its consequences on the associated diffusion processes. In the concrete case of MFG, ellipticity seems to be magnified as an effect of the coupling.

The opposite scenario is a setting where the ellipticity is jeopardized. This is the topic of the second part of our work. Here, we examine solutions to fully degenerate nonlinear elliptic equations degenerating with respect to the gradient, as a modulus of continuity. To be more precise, we consider equations of the form

$$\mathscr{F}(Du, D^2u) \; = \; f(x) \quad \text{in } B_1. \tag{1.5}$$

The source term $f(x)$ is assumed continuous and bounded function and the nonlinear operator $\mathscr{F}\colon \mathbb{R}^d \times \mathcal{S}(d) \to \mathbb{R}$ is degenerate elliptic, with law of degeneracy σ satisfying $\sigma(0) = 0$ and $\sigma(t) > 0$ for every $t > 0$. This means $\mathscr{F}(\vec{p}, M) = \sigma(|\vec{p}|)F(M)$, for an operator $F\colon \mathcal{S}(d) \to \mathbb{R}$, representing the *diffusion agent* of the model and σ is a modulus of continuity, otherwise called the *law of degeneracy* of the equation; precise definitions will be given later.

The emphasis here lies on the fact that along the subregion $\{Du = 0\}$, the equation provides no information on the process. Our first result in this setting reads as follows.

We investigate minimal conditions on the degree of degeneracy σ under which viscosity solutions are of class $\mathcal{C}^1$. In fact, such an improved regularity result is achieved under the (sharp) condition that σ has a Dini continuous inverse σ^{-1}.

Theorem 5 (Differentiability of solutions) *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to*

$$\mathscr{F}(Du, D^2u) = f \quad \textit{in} \quad B_1, \tag{1.6}$$

where $\mathscr{F}(\vec{p}, M) = \sigma(|\vec{p}|)F(M)$. Suppose Assumptions 1, 4, 6 and 5, to be detailed later, hold true. Then $u \in \mathcal{C}^1_{loc}(B_1)$ and there exists a modulus of continuity $\omega : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that

$$\sup_{x \in B_r(x_0)} |Du(x) - Du(x_0)| \leq \omega(r),$$

for every $x_0 \in B_{1/4}$ and $0 < r < 1/4$. In addition, ω depends on $d, \lambda, \Lambda, \sigma$, $\|u\|_{L^\infty(B_1)}$ and $\|f\|_{L^\infty(B_1)}$.

The remainder of this thesis is organized as follows: the next chapter details the main assumptions under which we work and gathers a few preliminary results used throughout this work. The third chapter presents our analysis of fully nonlinear mean-field games. In a fourth chapter, we examine degenerate fully nonlinear equations and produce a regularity theory for the viscosity solutions.

2
Preliminary material and main assumptions

In this chapter we collect important information used in this work. Those include elementary notions and definitions as well as preliminary results. We start detailing our assumptions.

2.1
Main assumptions

Our findings depend on conditions imposed on the elliptic operator F, as well as on the data of the problem. This section puts forward an account of our main assumptions. We start by imposing an ellipticity condition on the operator F. In what follows, $\mathcal{S}(d)$ denotes the space of real symmetric matrices; it will be identified with $\mathbb{R}^{\frac{d(d+1)}{2}}$, whenever convenient.

Assumption 1 (Uniform ellipticity) *We suppose the operator $F : \mathcal{S}(d) \to \mathbb{R}$ to be (λ, Λ)-uniformly elliptic. That is, there exist $0 < \lambda \leq \Lambda$ such that*

$$\lambda \|N\| \leq F(M + N) - F(M) \leq \Lambda \|N\|, \tag{2.1}$$

for every M, $N \in \mathcal{S}(d)$ with $N \geq 0$. In addition, we suppose $F(0) = 0$.

By taking $M \equiv 0$ in Assumption 1, it follows that

$$\lambda \|N\| \leq F(N) \leq \Lambda \|N\|,$$

for every $N \geq 0$. Therefore, uniform ellipticity implies that F satisfies a coercivity condition over non-negative matrices. A classical example of uniform elliptic operators are the *Pucci's extremal operators*. When studying MFG systems, we also impose a convexity condition on F.

Assumption 2 (Convexity of the operator F) *We suppose the operator $F = F(M)$ to be convex with respect to M.*

In certain cases, we must impose further conditions on the growth regime of the operator F. Namely, extend (2.1) to symmetric matrices $N \in \mathcal{S}(d)$. This is the content of the next assumption.

Assumption 3 (Growth Condition) *We suppose that F satisfies*

$$\lambda \|M\| \leq F(M) \leq \Lambda \|M\|,$$

for every $M \in \mathcal{S}(d)$ and $F(0) = 0$.

Consequential on Assumption 3 is a coercivity condition for F in the entire space $\mathcal{S}(d)$. To compare Assumption 1 and Assumption 3 amounts to observe a change in the ellipticity cone of the operator. See Figure 2.1.

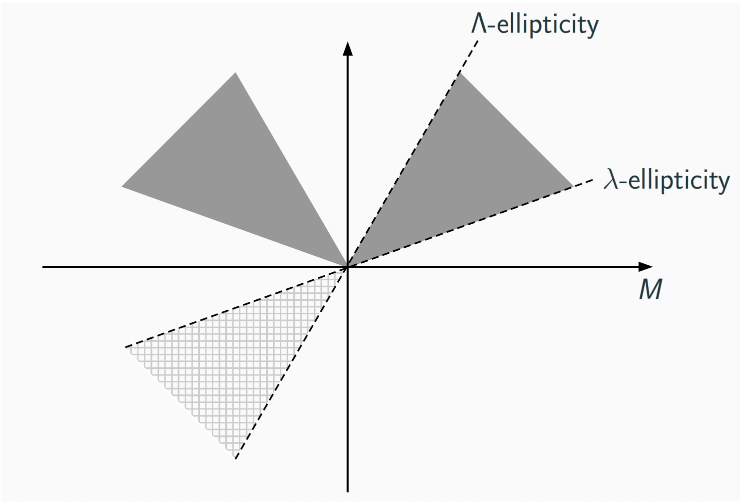

Figure 2.1: The (λ, Λ)-ellipticity condition, as in Assumption 1, confines the image of the operator to the area between the lines $\lambda\|\cdot\|$ and $\Lambda\|\cdot\|$. However, Assumption 3 confines the image of the operator to the region between those lines *within the upper half-plane* of $\mathcal{S}(d) \times \mathbb{R}$.

In what follows, we produce an example of operators that satisfies the growth condition, namely:

Example 1 *Let F be a (λ, Λ)-elliptic operator and $\varepsilon > 0$ a parameter. Consider F_ε given by*

$$F_\varepsilon(M) := |F(M)| + \varepsilon|M|;$$

it is clear that F_ε satisfies the growth condition in the former assumption.

Next, we present the assumptions used the second part of this work. Namely, in the context of degenerate fully nonlinear equations. In this case, we consider a nonlinear operator

$$\mathscr{F}(\vec{p}, M) = \sigma(|\vec{p}|) F(M), \tag{2.2}$$

we call σ its *degeneracy law* and F its *diffusion agent*. This latter nomenclature is justified by the ellipticity condition of F.

Assumption 4 (Law of degeneracy) *We suppose that the degeneracy law $\sigma : [0, +\infty) \to [0, +\infty)$ is a modulus of continuity such that $\sigma(0) = 0$. Moreover, we suppose $\sigma(1) \geq 1$.*

We recall by modulus of continuity we simply mean an increasing function f defined over an interval of $\mathbb{R}_0^+ := (0,+\infty)$ into $\mathbb{R}_0^+$ such that $\lim_{t\to 0} f(t) = 0$. Also, notice that the condition $\sigma(1) \geq 1$ is a mere normalization.

Our next assumption concerns the minimal, sharp condition on the modulus of continuity σ as to ensure the switching from Hölder-continuity to the differentiability of the viscosity solutions to (1.6). It concerns the Dini-continuity of σ^{-1}. Given its importance to our main theorem in this part, below we introduce the formal definition of Dini condition:

Definition 1 (Dini condition) *A modulus of continuity ω is said to satisfy the Dini condition if*

$$\int_0^\tau \frac{\omega(t)}{t}\mathbf{d}t \, < \, +\infty, \tag{2.3}$$

for some $\tau > 0$.

We proceed with an assumption.

Assumption 5 *We suppose $\sigma\colon [0,+\infty) \to [0,+\infty)$ is a modulus of continuity for which its inverse, $\sigma^{-1} : \sigma([0,+\infty)) \to [0,+\infty)$ satisfies the Dini condition* (2.3).

The Dini condition plays an important role in mathematical analysis, notably in harmonic analysis and its applications to the theory of PDEs. Recall a function $f\colon X \to Y$ defined over a metric space (X, d_X) into another metric space (Y, d_Y) is said to be Dini continuous if:

$$d_Y\left(f(x_1), f(x_2)\right) \leq \omega_f\left(d_X(x_1,x_2)\right),$$

for a modulus of continuity ω_f satisfying the Dini condition (2.3). For the sake of precision, it is convenient to define the modulus of continuity of f as

$$\omega_f(t) = \sup_{d_X(x_1,x_2)\leq t} d_Y\left(f(x_1), f(x_2)\right).$$

Obviously any Hölder continuous function h is Dini continuous, as its modulus of continuity is given by $\omega_h(d) = Cd^\alpha$ and

$$\int_0^1 \frac{\omega_h(t)}{t}\mathbf{d}t \, = \, C\alpha^{-1},$$

which is finite.

There are however many important examples of Dini continuous functions that are not Hölder continuous. The classical family of examples is given by:

$$\phi_\alpha(t) = \left(\frac{1}{1-\ln t}\right)^\alpha, \tag{2.4}$$

for any $\alpha > 1$. Further examples of Dini continuous functions can be crafted through generalized power series. Let $(\gamma_k)_{k\in\mathbb{N}} \in c_0$ be a sequence of positive numbers converging to zero and $(a_k)_{k\in\mathbb{N}} \in \ell_1$ be sequence of positive numbers. Define

$$\omega(t) = \sum_{j=1}^{\infty} a_j t^{\gamma_j}.$$

Assume for some $t_\star > 0$ the series is convergent at $t = t_\star$ and that,

$$\sum_{j=1}^{\infty} a_j \frac{\tau^{\gamma_j}}{\gamma_j} < \infty,$$

for some $0 < \tau < t_\star$. Then $\omega(t)$, defined over $(0, t_\star)$ verifies the Dini condition. For instance, $\omega(t) = \sum\limits_{j=1}^{\infty} \frac{\sqrt[j]{t}}{2^j}$ satisfies the Dini condition. Notice that all examples built up through this method fail to be ε–Hölder continuous for all $0 < \varepsilon < 1$.

Similarly, there are a plethora of Dini moduli of continuity that verify $\phi_\alpha(t) = \mathrm{o}(\omega(t))$ for all $\alpha > 1$, where ϕ_α are the standard examples from (2.4). For instance,

$$\tilde{\phi}(t) := \sum_{n=1}^{\infty} a_n \left(\frac{1}{1-\ln t}\right)^{1+\frac{1}{n}},$$

where $a_n = \frac{1}{2^n b_n}$, for $b_n := \int_0^1 \frac{1}{t}\left(\frac{1}{1-\ln t}\right)^{1+\frac{1}{n}} \mathbf{d}t < +\infty$.

Dini condition can also be characterized in terms of the summability of ω evaluated along geometric sequences. That is, a modulus of continuity ω satisfies the Dini condition (2.3) if, and only if,

$$\sum_{n=1}^{\infty} \omega(\tau \cdot \theta^n) \; < \; \infty, \tag{2.5}$$

for every $\theta \in (0,1)$. Indeed, by elementar partition argument, there exist points $\xi_i \in [\tau\theta^i, \tau\theta^{i-1}]$ such that:

$$(1-\theta)\sum_{i=1}^{\infty} \omega(\xi_i) \leq \int_0^{\tau} \frac{\omega(t)}{t}\mathbf{d}t \leq \frac{1-\theta}{\theta}\sum_{i=1}^{\infty} \omega(\xi_i). \tag{2.6}$$

We resort to the characterization in (2.5) further in our arguments.

Finally, we present our assumption concerning the source term f.

Assumption 6 (Continuity of the source term) We suppose the source term $f \in L^\infty(B_1)$ is a continuous function.

In the next section we collect a number of definitions and auxiliary results used throughout this thesis.

2.2 Preliminary notions and results

We start introducing a notion used in the first part of this work. When equipping (1.1) with boundary conditions in the sense of Sobolev, we make use of an affine subspace of $W^{2,p}(B_1)$.

Definition 2 (Sobolev spaces $W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$) *Given $g \in W^{2,p}(B_1)$, we say that $u \in W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$ if $u \in W^{2,p}(B_1)$ and $u - g \in W_0^{1,p}(B_1)$.*

Once this definition is available, to prescribe $u = g$ on ∂B_1 in the Sobolev sense means to consider $u \in W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$. Instead of satisfying the boundary condition in the classical sense, we only require the difference between u and the data to be an element of $W_0^{1,p}(B_1)$. Next, on account of completeness, we proceed by recalling the notion of viscosity solution used in this thesis.

Definition 3 (Viscosity solution) *We say that $u \in \mathcal{C}(B_1)$ is a viscosity subsolution to*

$$G(D^2u, Du, u, x) = g(x) \quad \text{in} \quad B_1 \tag{2.7}$$

if, for every $x_0 \in B_1$ and $\varphi \in \mathcal{C}^2(B_1)$ such that $u - \varphi$ has a local maximum at x_0, we have

$$G(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq g(x_0) \quad \text{in} \quad B_1.$$

Conversely, we say that $u \in \mathcal{C}(B_1)$ is a viscosity supersolution to (2.7) if, $x_0 \in B_1$ and $\varphi \in \mathcal{C}^2(B_1)$ such that $u - \varphi$ has a local minimum at x_0, we have

$$G(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \geq g(x_0) \quad \text{in} \quad B_1.$$

In case u is a viscosity subsolution and supersolution to (2.7), we say u is a viscosity solutions to the equation.

For the notion of viscosity solution, and important related facts on this topic, see [23] and[43]; for the foundations of the L^p-viscosity theory, we refer the reader to [24].

To properly explore the connection between (1.2) and (1.1), we introduce the definition of *solution to the mean-field games system* in (1.1).

Definition 4 (Solution to the MFG problem) *A pair (u, m) is said to be a weak solution to (1.1) if the following conditions are satisfied:*

1. *$u \in \mathcal{C}(B_1)$ and $m \in L^1(B_1)$ satisfies $m \geq 0$ in B_1;*

2. *u is a viscosity solution to*

$$F(D^2u) \;=\; m^{\frac{1}{p-1}} \qquad in \qquad B_1;$$

3. *m satisfies*

$$\int_{B_1} \Big(F_{ij}(D^2u)\, m\Big)\, \phi_{x_i x_j}\, \boldsymbol{dx} \;=\; 0,$$

for every $\phi \in \mathcal{C}^\infty_c(B_1)$.

In general, a solution to (1.1) is a critical point for the functional (1.2). Nevertheless, up to this point, we have not enough information to ensure that solutions to the MFG system are indeed minimizers of the functional $I[u]$. To distinguish solutions minimizing this functional, we introduce a definition.

Definition 5 (Minimizing solution) *We say that* (u^*, m^*) *is a minimizing solution if it solves* (1.1) *and satisfies*

$$I[u^*] \;\leq\; I[u]$$

for every $u \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$.

Part of our analysis concerns the study of relaxation methods. In this realm, a key ingredient is the convex envelope of F. In what follows, we define this object.

Definition 6 (Convex envelope) *Let* $F : \mathcal{S}(d) \to \mathbb{R}$ *be a fully nonlinear operator satisfying Assumption 1. The convex envelope of* F *is the operator* $\Gamma_F : \mathcal{S}(d) \to \mathbb{R}$ *defined by*

$$\Gamma_F(M) \;:=\; \sup\left\{G(M)\,|\, G \;\leq\; F \;\; and \;\; G \; is \; convex\right\}.$$

We are interested in gains of regularity produced by the MFG coupling *vis-a-vis* the equations in (1.1) taken in isolation. To pursue this direction, we rely on the gains of integrability for the second equation in (1.1). This equation is also referred to as *double-divergence equation.*

Lemma 1 (Gains of integrability) *Let* $m \in L^1(B_1)$ *be a non-negative weak solution to the second equation in* (1.1). *Suppose Assumption 1 is in force. Then,* $m \in L^{\frac{d}{d-1}}_{loc}(B_1)$ *and there exists a (universal) constant* $C > 0$ *such that*

$$\|m\|_{L^{\frac{d}{d-1}}(B_{1/2})} \;\leq\; C.$$

For a proof of Lemma 1 we refer the reader to [51]. See also [20] and the references therein. The importance of this result lies in the fact that an integrable solution to the double-divergence equation has higher integrability, depending explicitly on the dimension d.

Because our analysis is heavily based on the direct method in the calculus of variations, part of our arguments are related to the weakly lower semi-continuity of the functional $I[u]$. An important ingredient in the study of this property is the Mazur Theorem. Before we state the theorem, we recall a definition.

Definition 7 (Convex hull set) *Given $X \subset \mathbb{R}^d$, its convex hull is the smallest convex set containing X. It is denoted by $\operatorname{co}(X)$. In case X is the set of values assumed by a sequence $(x_n)_{n\in\mathbb{N}}$, we have*

$$\operatorname{co}[(x_n)_{n\in\mathbb{N}}] := \left\{ \sum_{n\in N} x_n\alpha_n \;\; N \textit{ finite}, \;\; x_n \in X \;\forall n, \;\; \alpha_n \geq 0 \;\forall n, \;\; \textit{and} \;\; \sum_{n\in N} \alpha_n = 1 \right\}.$$

Lemma 2 (Mazur Theorem) *Let X be a linear space and $\ell : X \to \mathbb{R}^+$ be a norm defined on X. If $(x_n)_{n\in\mathbb{N}} \subset X$ is such that*

$$x_n \rightharpoonup x \qquad in \qquad X,$$

there exists a sequence $(y_m)_{m\in\mathbb{N}} \subset \operatorname{co}[(x_n)_{n\in\mathbb{N}}]$ satisfying:

1. *for every $m \in \mathbb{N}$ there exists $M \in \mathbb{N}$ and $(\alpha_m^n)_{n=1}^M$ with*

$$\alpha_m^n > 0 \qquad \sum_{n=1}^{M} \alpha_m^n = 1$$

and

$$y_m = \sum_{n=1}^{M} \alpha_m^n x_n;$$

2. *in addition, we have*

$$\ell\,(y_m - x) \longrightarrow 0 \qquad as \qquad m \to \infty.$$

For a proof of this result, we refer the reader to [80, p. 120, Theorem 2]. See also [74] and [44]. A basic result used in our argument is the Poincaré inequality. Since we need a variant tailored for functions failing compact support, we include it here for the sake of completeness. In the context of functions $u \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$, we make use of a Poincaré's inequality depending intrinsically on g. This is the content of the next lemma.

Lemma 3 (Poincaré's inequality) *Let $u \in W^{1,p}_g(B_1)$ and $C_p > 0$ be the Poincaré's constant associated with $L^p(B_1)$. Then, for every $C < C_p$ there exists $C_1(C, C_p) > 0$ and $C_2 \geq 0$ such that*

$$\int_{B_1} |Du|^p \boldsymbol{dx} - C \int_{B_1} |u|^p \boldsymbol{dx} + C_2 \geq C_1 \left(\int_{B_1} |Du|^p \boldsymbol{dx} + \int_{B_1} |u|^p \boldsymbol{dx} \right).$$

For the proof of Lemma 3, it suffices to apply the compactly supported version of the Poincaré's Inequality to the function $\tilde{u} := u - g$. For the detailed argument, we refer the reader to [45, Lemma 2.7, p. 22].

The remainder of this section collects elements pertaining to the analysis of degenerate fully nonlinear equations.

Next, we introduce auxiliary results capable of ensuring compactness of the solutions for a variant of (1.5). The next proposition is the so-called Crandall-Ishii-Lions Lemma.

Proposition 1 (Crandall-Ishii-Lions Lemma) *Let $\Omega \subset B_1$, $u \in \mathcal{C}(B_1)$ and $\psi \in \mathcal{C}^2(\Omega \times \Omega)$. Let $G : \mathcal{S}(d) \times \mathbb{R}^d \times \mathbb{R} \times B_1 \to \mathbb{R}$ be degenerate elliptic. Set*

$$w(x, y) := u(x) - u(y) \quad \text{for} \quad (x, y) \in \Omega \times \Omega.$$

If the function $w - \psi$ attains its maximum at $(\bar{x}, \bar{y}) \in \Omega \times \Omega$, then for each $\varepsilon > 0$, there exist $X, Y \in \mathcal{S}(d)$ such that

$$G(X, D_x\psi(\bar{x}, \bar{y}), \bar{x}) \leq 0 \leq G(Y, D_y\psi(\bar{x}, \bar{y}), \bar{y}). \tag{2.8}$$

In addition, we have

$$-\left(\frac{1}{\varepsilon} + \|A\| \right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2, \tag{2.9}$$

where $A = D^2\psi(\bar{x}, \bar{y})$.

For a proof of this proposition, we refer the reader to [43, Theorem 3.2]. We close this section with a lemma that plays a key role in our analysis in this part.

Lemma 4 (Modulating ℓ_1 sequences) *Given any sequence of summable numbers $(a_j)_{j\in\mathbb{N}} \in \ell_1$ and $\varepsilon, \delta > 0$, there is a sequence $(c_j)_{j\in\mathbb{N}} \in c_0$, satisfying*

$$\max_{j\in\mathbb{N}} |c_j| \leq \varepsilon^{-1}$$

such that

$$(b_j)_{j\in\mathbb{N}} := \left(\frac{a_j}{c_j} \right)_{j\in\mathbb{N}} \in \ell_1$$

and

$$\varepsilon\left(1-\frac{\delta}{2}\right)\|(a_j)\|_{\ell_1} \leq \|(b_j)\|_{\ell_1} \leq \varepsilon\,(1+\delta)\,\|a_j\|_{\ell_1}.$$

Proof. Let $\delta > 0$. Starting off with the hypothesis $(a_j)_{j\in\mathbb{N}} \in \ell_1$, let n_1 be an integer such that

$$\sum_{k=n_1}^{\infty} |a_k| < \frac{\delta\,\|(a_j)\|_{\ell_1}}{2}.$$

In what follows, let $n_2 > n_1$ be such that

$$\sum_{k=n_2}^{\infty} |a_k| < \frac{\delta\,\|(a_j)\|_{\ell_1}}{2^3};$$

and, in general, let $n_j > n_{j-1}$ be such that

$$\sum_{k=n_j}^{\infty} |a_k| < \frac{\delta\,\|(a_j)\|_{\ell_1}}{2^{2j-1}}$$

for all j. Next we construct the sequence of positive numbers c_j as follows:

$$\begin{aligned} c_1 = \cdots = c_{n_2-1} &= \frac{1}{\varepsilon}, \\ c_{n_2} = \cdots = c_{n_3-1} &= \frac{1}{2\varepsilon}, \\ c_{n_3} = \cdots = c_{n_4-1} &= \frac{1}{2^2\varepsilon}, \\ &\vdots \\ c_{n_j} = \cdots = c_{n_{j+1}-1} &= \frac{1}{2^{j-1}\varepsilon} \end{aligned}$$

and so on. Thus, by the very construction, $(c_j)_{j\in\mathbb{N}} \in c_0$ and

$$\max_{j\in\mathbb{N}} |c_j| \leq \varepsilon^{-1}.$$

Next we estimate, for all $j \geq 1$:

$$\begin{aligned} \sum_{k=n_j}^{n_{j+1}-1} \left|\frac{a_k}{c_k}\right| &= \sum_{k=n_j}^{n_{j+1}-1} \left|\frac{a_k}{1/2^{j-1}\varepsilon}\right| \\ &< 2^{j-1}\varepsilon \sum_{k=n_j}^{n_{j+1}-1} |a_k| \\ &< 2^{j-1}\varepsilon \frac{\delta\,\|(a_j)\|_{\ell_1}}{2^{2j-1}} \\ &= \frac{\varepsilon\delta\,\|(a_j)\|_{\ell_1}}{2^j}. \end{aligned}$$

Hence

$$\sum_{k=1}^{\infty}\left|\frac{a_k}{c_k}\right| \leq \sum_{k=1}^{n_1-1}\left|\frac{a_k}{c_k}\right| + \sum_{k=n_1}^{\infty}\left|\frac{a_k}{c_k}\right|$$
$$= \varepsilon\|a_j\|_{\ell_1} + \varepsilon\delta\left\|(a_j)\right\|_{\ell_1}$$
$$= \varepsilon\left(1+\delta\right)\|a_j\|_{\ell_1}$$

On the other hand, since

$$\sum_{k=1}^{n_1-1}|a_k| + \sum_{k=n_1}^{\infty}|a_k| = \|a_j\|_{\ell_1}$$

and

$$\sum_{k=n_1}^{\infty}|a_k| < \frac{\delta\left\|(a_j)\right\|_{\ell_1}}{2}$$

we have

$$\sum_{k=1}^{n_1-1}|a_k| > \left\|(a_j)\right\|_{\ell_1} - \frac{\delta\left\|(a_j)\right\|_{\ell_1}}{2}$$

Therefore, we obtain

$$\sum_{k=1}^{\infty}\left|\frac{a_k}{c_k}\right| \geq \sum_{k=1}^{n_1-1}\left|\frac{a_k}{c_k}\right|$$
$$= \varepsilon\sum_{k=1}^{n_1-1}|a_k|$$
$$> \varepsilon\left(\left\|(a_j)\right\|_{\ell_1} - \frac{\delta\left\|(a_j)\right\|_{\ell_1}}{2}\right)$$
$$= \varepsilon\left\|(a_j)\right\|_{\ell_1}\left(1-\frac{\delta}{2}\right),$$

and the lemma is finally proven. ■

Remark 1 *Note, in general, one is not allowed to let* $\delta \to 0$ *as the sequence* $(c_j)_{j\in\mathbb{N}}$ *depends itself upon* δ*.*

In the next chapter we detail the analysis of fully nonlinear mean-field games systems.

3
Fully nonlinear mean-field games

In this chapter we present our findings concerning fully nonlinear MFG systems. For ease of presentation, we recall our system of interest in the sequel. We study

$$\begin{cases} F(D^2u) \;=\; m^{1/(p-1)} & \text{in} \quad B_1 \\ (F_{ij}(D^2u)\,m)_{x_ix_j} \;=\; 0 & \text{in} \quad B_1, \end{cases} \tag{3.1}$$

We begin by formally relating (3.1) with the problem

$$\int_{B_1} \left[F(D^2u)\right]^p \mathbf{d}x \;\longrightarrow\; \min\,. \tag{3.2}$$

The natural connection between these structures follows from the first compact variation of the functional in (3.2). Typically, we aim at characterizing the critical points of this mapping in terms of (3.1). This is the content of the next proposition.

Proposition 2 (Euler-Lagrange equation) *The system in* (3.1) *is the Euler-Lagrange equation associated with* (3.2)*.*

Proof. We open the proof by noticing the functional $I[u]$ is well defined in $W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$. Indeed, from Assumption 3, we infer that

$$\left|\int_{B_1} \left[F(D^2u)\right]^p \mathbf{d}x\right| \;\leq\; \int_{B_1} \left|F(D^2u)\right|^p \mathbf{d}x \;\leq\; C\Lambda^p \left\|D^2u\right\|^p_{L^p(B_1)}.$$

Therefore, if we confine the minimization problem to functions in $W^{2,p}(B_1)$, the functional is well defined.

To formally derive its Euler-Lagrange equation, we consider its first compact variation. Let $\varphi \in \mathcal{C}^\infty_0(B_1)$ and define $i : [0,1] \to \mathbb{R}$ as

$$i(\varepsilon) \;:=\; \int_{B_1} \left[F(D^2u^* + \varepsilon D^2\varphi)\right]^p \mathbf{d}x,$$

where $u^* \in W^{2,p}(B_1)$ is a minimizer for $I[u]$; then $\varepsilon = 0$ in minimum point for the real function i. Hence

$$i^{'}(\varepsilon)|_{\varepsilon=0} = 0.$$

We compute

$$\begin{aligned}\frac{d}{d\varepsilon}i(\varepsilon) &= \frac{d}{d\varepsilon}\int_{B_1}\left[F(D^2u+\varepsilon D^2\varphi)\right]^p \,\mathbf{d}x\\ &= \int_{B_1}\frac{\partial}{\partial\varepsilon}\left[F(D^2u+\varepsilon D^2\varphi)\right]^p \,\mathbf{d}x\\ &= p\int_{B_1}\left[F(D^2u+\varepsilon D^2\varphi)\right]^{p-1}\cdot F_{ij}(D^2u+\varepsilon D^2\varphi)\cdot D_{x_ix_j}\varphi\,\mathbf{d}x.\end{aligned}$$

By evaluating $i'(\varepsilon)$ at $\varepsilon = 0$, we recover

$$\int_{B_1}\left[F(D^2u)\right]^{p-1}\cdot F_{ij}(D^2u)\cdot D_{x_ix_j}\varphi\,\mathbf{d}x = 0;$$

integrating by parts twice, we get

$$\int_{B_1}\left(\left[F(D^2u)\right]^{p-1}\cdot F_{ij}(D^2u)\right)_{x_ix_j}\varphi\,\mathbf{d}x = 0, \tag{3.3}$$

for every $\varphi \in \mathcal{C}_0^\infty(B_1)$. Consequential on (3.3) is the fact that

$$\left(\left[F(D^2u)\right]^{p-1}\cdot F_{ij}(D^2u)\right)_{x_ix_j} = 0 \quad \text{in} \quad B_1.$$

By setting $m := [F(D^2u)]^{p-1}$, we recover the fully nonlinear MFG in (3.1). ■

Although we have chosen to present our results in the context of the toy-model (3.1), our techniques account for more general formulations, i.e., including lower order terms. The prototypical gradient-dependence we could include is of the form

$$H(p,x) \sim A(x)\left(1 + |p|^2\right)^{\frac{q}{2}} + V(x), \tag{3.4}$$

where $A \in L^\infty(B_1)$ and V is a merely bounded and measurable potential.

3.1 Some context on MFG systems

The equations comprising (3.1) are a fully nonlinear PDE and an elliptic equation in the double-divergence form. Interesting on their very own merits and leading to foundational developments in the profession, those equations have been largely studied in the course of the last fifty years. Clearly, to put together a meaningful list of references on their regard is out of the scope of this thesis. Therefore, we refrain from mentioning any further work than the monographs [20] and [23].

The fully nonlinear equation in (3.1) can be regarded as the Hamilton-Jacobi associated with an (stochastic) optimal control problem.

Of particular interest is the nonlinear dependence on the Hessian of the value function. Notice also the introduction of the associated density in the underlying cost functional, through the mapping $z \mapsto z^{\frac{1}{p-1}}$. Finally, it is worthy noticing that the choice for this dependence is two-fold. First, it appears naturally in the variational derivation of (3.1). Moreover, it describes a cost functional that penalizes crowds.

The second equation in (3.1) is a Fokker-Planck whose coefficients depend on the value function u. In fact, it describes the distribution associated with the stochastic process with infinitesimal generator

$$F_{ij}(D^2u)\frac{\partial^2}{\partial x_i \partial x_j}.$$

This description is useful in framing the fully nonlinear model (3.1) into the context of the toy-models appearing in the MFG literature.

The mean-field games theory was introduced in [67, 68, 69] as a mathematical framework to model scenarios of strategic interactions involving a (very) large number of players. Its mathematical formulation is completely described by the so-called *master equation*. Under additional assumptions, on the independence of the underlying stochastic process, models simplify substantially. Here, the work-horse of the theory is the coupling of a Hamilton-Jacobi and a Fokker-Planck equation. In this context, several authors advanced the topic in a variety of directions. The existence (and uniqueness) of solutions – both in the elliptic and parabolic settings – is the object of [9, 26, 28, 29, 30, 27, 57, 58, 56, 39, 40, 35, 71], whereas numerical developments are reported in [1, 5, 3, 4], among others. Applications of MFG theory to life and social sciences can be found, for instance in [2, 62, 53]. Finally, the analysis of the master equation has been advanced in [12, 71, 31, 13, 32, 38, 54]. See also the monographs [25, 11, 59].

Since MFG theory embeds in the context of (stochastic) optimal control problems, it is relevant to make sure that the formulation in (3.1) is in line with this framework. In this regard, we notice that Bellman operators are typically Lipschitz-continuous, but fail to be $\mathcal{C}^1$-regular. Nevertheless, to make sense of the second equation in (3.1), the Lipschitz regularity implied by the (λ, Λ)-uniform ellipticity is enough.

A distinctive feature associated with mean-field games systems concerns gains of regularity for the solutions, *vis-a-vis* the same equations taken isolated. We investigate the occurrence of a similar phenomenon in the case of (3.1). In fact, we prove that a solution (u, m) is such that D^2u has better integrability than in the case where u solves $F(D^2u) = \mu$ with $\mu \in L^1(B_1)$ taken arbitrarily.

For values of $p \geq 2$ in a suitable range, this result yields a counterpart in Hölder spaces.

Although substantial advances have been produced by the profession, fully nonlinear formulations are yet to be addressed in the literature. In fact, in [52] the authors put forward a well-posedness analysis based on monotonicity properties and the Minty-Browder machinery; see [50, Chapter 5]. Under certain regularity assumptions on F, together with monotonicity conditions, they claim that solutions to (3.1) would be available; see [52, Section 7.1].

Our findings advance the MFG theory by establishing the existence of weak solutions to (3.1) with no regularity assumptions on F. Indeed, we work under ellipticity and ellipticity-like conditions on the operator governing the MFG system. This is the content of Theorem 2.

Since the analysis of (3.1) interweaves with properties of (3.2), we proceed with some context on the latter. An important aspect concerning the functional in (3.2) is its dependence on the Hessian of the argument function u. The most elementary example is the case $F(M) := Tr(M)$, which leads to the functional governing the equilibrium of thin plates

$$I_\Delta[u] := \int_{B_1} \left|Tr(D^2u)\right|^2 \mathbf{d}x; \tag{3.5}$$

the Euler-Lagrange equation associated with this functional is the plate equation

$$\Delta\Delta u = 0 \quad \text{in} \quad B_1,$$

also known as biharmonic operator.

Functionals of the form (3.5) are relevant in the context of conformally invariant energies. In fact, (3.5) is conformally invariant in dimension four. This fact suggests the development of a regularity program for biharmonic mappings in line with the theory available in the harmonic setting. In [37] the authors develop this theory for biharmonic mappings from (domains in) $\mathbb{R}^d$ into m-dimensional spheres $\mathbb{S}^m$. See also [36].

More general classes of Hessian-dependent functionals can be found in various contexts. First, they represent an important strategy in by-passing the lack of convexity in minimization problems. Consider, for example, the non-convex functional

$$J[u] := \int_{B_1} \left(|Du|^2 - 1\right)^2 \mathbf{d}x. \tag{3.6}$$

An alternative to regularize J is to consider

$$J_\varepsilon[u] := \int_{B_1} \left(|Du|^2 - 1\right)^2 + \varepsilon^2|D^2u|^2\mathbf{d}x; \tag{3.7}$$

since J_ε is convex with respect to the terms of higher order, it is possible to investigate the existence of a minimizer u_ε. Ideally, information on (3.6) would be recovered through (3.7), by taking the singular limit $\varepsilon \to 0$. In some cases, such a limit entails further complexities; these are known as *microstructures*. We mention that J_ε is referred to as Aviles-Giga functional; see [8] and [7]; see also [63].

Functionals depending on the Hessian of a given function also appear in the context of energy-driven pattern formation and nonlinear elasticity, in the study of the mechanics of solids. One example regards the study of wrinkles appearing in a twisted ribbon [64]. In this setting, the energy functional depends on the thickness h of the ribbon, which is regarded as a parameter, as in

$$\int_{B_1} |M(u,v)|^2 \, + \, h^2 |B(u,v)|^2 \mathbf{d}x,$$

where M and B are symmetric tensors accounting for the stretching and bending energies of the system, respectively. Although M depends on u and v only through lower order terms, B depends on v through its Hessian, namely

$$B(u,v) \, \sim \, \|D^2 u\| \, + \, C.$$

The case of interest is the limit $h \to 0$. A further instance where higher order functionals appear in the context of solid mechanics concerns the formation of blister patterns in thin films on compliant substrates [10]. As before, the functional depends on lower order terms and a small (convex) perturbation driven by the Hessian of the minimizers.

In this literature, it is relevant to obtain *matching* upper and lower bounds for the functional. It means that both upper and lower bounds scale accordingly with respect to small parameters. In the case of [64], for instance, the small parameter is the thickness of the ribbon, h. For recent developments in this literature we refer the reader to [79], [41], [42] and the references therein.

When examining (3.2), our focus is two-fold. Firstly, we work under the convexity of F and prove the existence of a minimizer, Then we notice that such critical point is indeed a weak (distributional) solution for the associated Euler-Lagrange equation. This fact produces the existence of solutions to (3.1).

Our second approach to (3.2) drops the convexity of the operator. Here, a relaxation argument meets an ellipticity pass-through mechanism for the convex envelope of F. This mechanism closely relates to coercivity. As a consequence, we are able to characterize the minimum of the energy governed by the fully nonlinear operator.

To complete the analysis, we study a class of unidimensional problems admitting explicit solutions. Our endeavours here are inspired by the analysis in [55]. With this respect, we unveil interesting aspects of the problem. For example, we notice that minimizing solutions pair affine functions with uniform distributions, regardless of the growth regime of the functional; see Theorem 9.

3.2 Gains of regularity through MFG couplings

In this section we explore the gains of regularity yielded by the coupling (3.1). This analysis is motivated by findings in the literature of mean-field games regarding the smoothness of solutions to MFG systems. In that context, the work-horse of the theory is the coupling of a Hamilton-Jacobi with a Fokker-Planck equation.

When taken in isolation, those equations are solvable in the weak sense in regularity classes strictly below those required by classical solutions. However, in the presence of a suitable MFG coupling, smooth solutions are available. Motivated by the toy-models studied in the literature, in what follows we investigate gains of regularity for the solutions to fully nonlinear mean-field games. We focus on estimates in Sobolev spaces.

Now, let $u \in \mathcal{C}(B_1)$ be a viscosity solution of

$$F(D^2u) = \mu^{\frac{1}{p-1}} \quad \text{in} \quad B_1,$$

where F satisfies Assumption 1-Assumption 2, $\mu \in L^1(B_1)$ is a non-negative function and $p-1 \geq d$. In this context, it is known that $u \in W^{2,p-1}_{loc}(B_1)$, with appropriate estimates. We refer to reader [21], [22] and [72]. However if (u, m) solves (3.1), this result can be improved. Next we detail the proof of Theorem 1, restated here as a courtesy to the reader.

Theorem 6 (Restatement of Theorem 1) *Let (u, μ) be a weak solution to (3.1). Suppose Assumptions 1 and 2 are in force. Then, $u \in W^{2,\frac{d(p-1)}{d-1}}_{loc}(B_1)$. In addition, there exists a universal constant $C > 0$ such that*

$$\|u\|_{W^{2,\frac{d(p-1)}{d-1}}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|\mu\|^{\frac{1}{p-1}}_{L^1(B_1)}\right).$$

Proof. The result follows by combining standard arguments in $W^{2,p}$-regularity theory with Lemma 1. Indeed, under Assumption 1-Assumption 2, it remains to verify that the right-hand side of the first equation in (3.1) is uniformly bounded in $L^{\frac{d(p-1)}{d-1}}(B_1)$.

Because of Lemma 1, we have $\mu \in L^{\frac{d}{d-1}}(B_1)$. Therefore, there exists $C > 0$ such that

$$\int_{B_1} \left|\mu^{\frac{1}{p-1}}\right|^{\frac{d(p-1)}{d-1}} \mathbf{d}x \leq C$$

and the result follows. ■

The conclusion of Theorem 6 is that the MFG structure in (3.1) entails improved regularity levels for u in Sobolev spaces; see Figure 3.1.

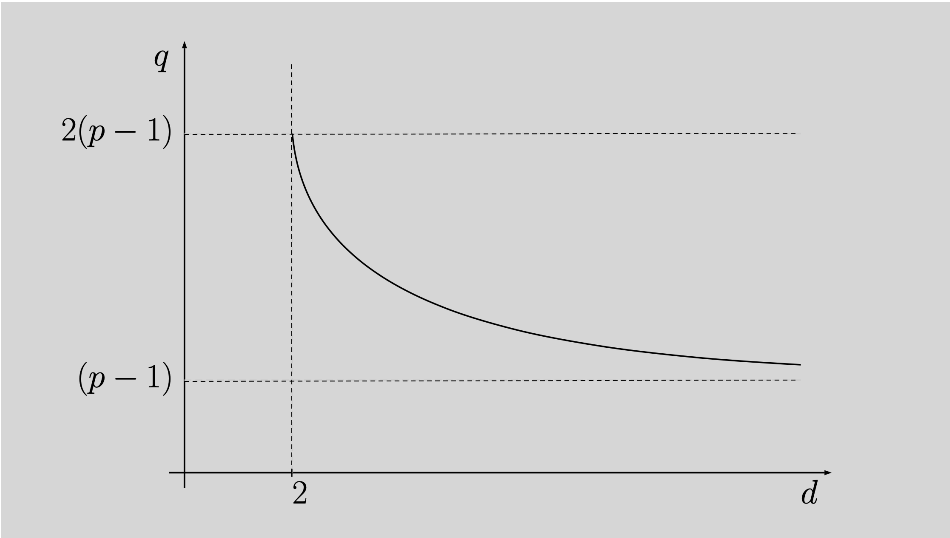

Figure 3.1: Gains of regularity. When equipped with a non-negative right-hand side $\mu \in L^1(B_1)$, the equation $F = \mu^{1/(p-1)}$ has solutions in $W^{2,q}_{loc}(B_1)$, for $d - \varepsilon < q \leq p - 1$. Nevertheless, if we consider a pair (u, μ), solutions to (3.1), the regularity of the solutions to $F = \mu^{1/(p-1)}$ improves; in that case, the range for $d - \varepsilon < q < d(p-1)/(d-1)$ increases by a dimension-dependent factor.

A straightforward consequence of this fact concerns regularity in Hölder spaces. It is put forward in the next corollary.

Corollary 1 (Improved regularity in Hölder spaces) *Let* (u, μ) *be a weak solution to* (3.1) *and suppose Assumption 1-Assumption 2 are in force. Suppose further that* $p - 1 > d$*. Then,* $u \in \mathcal{C}^{1,\alpha^*}_{loc}(B_1)$*, for*

$$\alpha^* := 1 - \frac{d - 1}{p - 1}.$$

In addition, there exists $C > 0$ *such that*

$$\|u\|_{\mathcal{C}^{1,\alpha^*}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|\mu\|^{\frac{1}{p-1}}_{L^1(B_1)}\right).$$

Proof. The proof is immediate and follows from standard Sobolev Embedding Theorems. ■

Remark 2 It is clear that the results in this section hold in the context of an operator $F = F(M, x)$ with variable coefficients, provided it satisfies an oscillation estimate with respect to its fixed-coefficients counterpart. For instance, we can consider the oscillation measure introduced in [22]. Namely, the function $\beta_0 : B_1 \to \mathbb{R}$ given by

$$\beta_0(x) := \sup_{M \in \mathcal{S}(d)} \frac{|F(M,x) - F(M,0)|}{1 + \|M\|}.$$

Appropriate smallness-regimes imposed on β_0 would unlock similar developments for variable coefficients operators. We refer the reader to [23]. In addition, the convexity assumption on F might be weakened. Indeed, we only require a limiting profile with $\mathcal{C}^{1,1}$-estimates to be available; such a limiting configuration can be given by the fixed-coefficients counterpart of $F(M, x)$ or its *recession operator*. See [72] and [75].

Remark 3 (Non-monotone couplings) We notice the gains of regularity discussed here would be available also under the non-monotone coupling term $z \mapsto -z^{\frac{1}{p-1}}$, instead of the monotone one in (3.1).

In the next section we make use of variational techniques to investigate the existence of solutions to (3.1).

3.3 Existence of solutions

Here we investigate the existence of minimizers to (3.2) as well as the existence of solutions to (3.1). We work both under Assumption 1 and Assumption 3. In the context of the former, we prove the existence of minimizers for (3.2) in the class of convex functions $\mathcal{A}$ defined as

$$\mathcal{A} := \left\{ u \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1), \;\; u \;\text{ convex} \right\},$$

where $g \in W^{2,p}(B_1)$ is convex. When working under Assumption 3 we establish the existence of a minimizer for (3.2) in $W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$. In addition, we prove that such a minimizer is a solution to the associated Euler-Lagrange equation, which leads to the existence of a solution to (3.1).

Proposition 3 (Existence of minimizers in $\mathcal{A}$) *Suppose Assumptions 1 and 2 are in force. Then, there exists $u^* \in \mathcal{A}$ such that*

$$I[u^*] \leq I[u] \qquad \text{for every} \qquad u \in \mathcal{A}.$$

The proof of Proposition 3 relies on the weak lower semicontinuity of the functional I in the class $\mathcal{A}$. This is the content of the next proposition.

Proposition 4 *Suppose Assumptions 1 and 2 hold true. Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{A}$ be such that*

$$D^2 u_n \rightharpoonup D^2 u_\infty \quad \textit{in} \quad L^p(B_1, \mathbb{R}^{d^2}).$$

Then,

$$\liminf_{n\to\infty} I[u_n] \,\geq\, I[u_\infty].$$

Proof. We start by establishing the strong lower-semicontinuity of the functional I. Then, we resort to Lemma 2 to obtain the weak lower-semicontinuity.

Step 1 - Because $(u_n)_{n\in\mathbb{N}} \subset \mathcal{A}$, we have $D^2u_n(x) \geq 0$ almost everywhere in B_1. Therefore, Assumption 1 yields

$$F(D^2u_n(x)) \,\geq\, \lambda\|D^2u_n(x)\| \,\geq\, 0, \qquad \text{a.e.}\, x \,\in\, B_1. \tag{3.8}$$

Now, suppose $D^2u_n \to D^2u_\infty$ strongly in $L^p(B_1)$; through a subsequence, if necessary. We infer that $D^2u_n \to D^2u_\infty$ almost everywhere in B_1. Hence, Fatou's Lemma builds upon the (Lipschitz) continuity of F to imply

$$\int_{B_1} \left[F(D^2u_\infty(x)\right]^p \mathbf{d}x \,\leq\, \liminf_{n\to\infty} \int_{B_1} \left[F(D^2u_n(x)\right]^p \mathbf{d}x.$$

Step 2 - To prove the weak lower-semicontinuity we make use of Lemma 2. Notice that

$$0 \,\leq\, \liminf_{n\to\infty} I[u_n] \,<\, \infty.$$

For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for $n > n_\varepsilon$, we have

$$I[u_n] \,\leq\, \liminf_{n\to\infty} I[u_n] \,+\, \varepsilon. \tag{3.9}$$

For $\varepsilon > 0$ fixed, Lemma 2 ensures the existence of $(D^2v_m)_{m\in\mathbb{N}} \subset \operatorname{co}(D^2u_n)_{n\in\mathbb{N}}$ such that $D^2v_m \to D^2u_\infty$ in $L^p(B_1)$. Moreover, for every $m \in \mathbb{N}$, one can find $M > n_\varepsilon$ and points $\alpha_m := (\alpha_m^{n_\varepsilon}, \ldots, \alpha_m^M)$ in the $(M - n_\varepsilon + 1)$-dimensional simplex such that

$$D^2v_m \,=\, \sum_{i=n_\varepsilon}^{M} \alpha_m^i D^2u_i. \tag{3.10}$$

The convexity of $[F(\,\cdot\,)]^p$ combined with the strong lower-semicontinuity and (3.9) leads to

$$I[u_\infty] \,\leq\, I[v_m] \,\leq\, \sum_{i=n_\varepsilon}^{M} \alpha_m^i I[u_i] \,\leq\, \liminf_{n\to\infty} I[u_n] \,+\, \varepsilon.$$

Since $\varepsilon > 0$ was taken arbitrarily, the proof is complete. ■

Once the weak lower-semicontinuity of the functional I has been established, we prove the existence of a minimizer to (3.2) in the class $\mathcal{A}$.

Proof of Proposition 3. We start by setting

$$\underline{m} := \inf_{u \in \mathcal{A}} I[u].$$

Clearly, $\underline{m} > 0$, since g is not trivial. On the other hand, $\underline{m} \leq I[g] < +\infty$. Hence, $0 < \underline{m} < \infty$. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence. There exists $N \in \mathbb{N}$ such that

$$I[u_n] \leq \underline{m} + 1 \qquad \text{for every} \qquad n > N.$$

Therefore,

$$\|D^2 u_n\|_{L^p(B_1)} \leq \frac{1}{\lambda^p} \int_{B_1} \left[F(D^2 u_n)\right]^p \mathbf{d}x \leq C(\underline{m}, \lambda, p), \tag{3.11}$$

for every $n > N$. As a result, we infer that $(D^2 u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^p(B_1)$. Because of Lemma 3, we conclude that $(u_n)_{n \in \mathbb{N}}$ is uniformly bounded in $W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$. Therefore, there exists $u_\infty \in \mathcal{A}$ such that $u_n \rightharpoonup u_\infty$ in $W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$.

Proposition 4 implies that

$$I[u_\infty] \leq \liminf_{n \to \infty} I[u_n] = \underline{m};$$

we set $u^* \equiv u_\infty$ and the proof is complete. ■

Remark 4 If we replace Assumption 1 with Assumption 3 the conclusion of the Proposition 3 changes. In fact, under Assumption 3, the inequality in (3.8) holds true for every $u \in W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$. As a result, we obtain that I is weakly lower semicontinuous over $W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$. Moreover, Assumption 3 yields (3.11) for every minimizing sequence $(u_n)_{n \in \mathbb{N}} \subset W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$.

The conclusion is that under Assumption 2 and Assumption 3, there exists $u^* \in W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$ such that

$$I[u^*] \leq I[u] \qquad \text{for every} \qquad u \in W^{2,p}(B_1) \cap W_g^{1,p}(B_1).$$

That is, problem (3.2) admits a minimizer in $W^{2,p}(B_1) \cap W_g^{1,p}(B_1)$.

In the sequel, the discussion in Remark 4 builds upon standard methods in calculus of variations to produce information on the Euler-Lagrange equation (3.1). Ultimately, it leads to the existence of solutions to (3.1). As before, we restate Theorem 2 in what follows.

Theorem 7 (Restatement of Theorem 2) *Suppose Assumptions 2 and 3 hold true. Then, there exists a minimizer* $u^* \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ *for* (3.2) *in the space* $W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$*. In addition, such a minimizer yields a solution* (u^*, m^*)*, solutions to the fully nonlinear mean-field game system* (3.1).

Proof. We start by noticing that, because F is $\mathcal{L}$-Lipschitz-continuous in $\mathcal{S}(d)$ and $F(0) = 0$, we infer that

$$|F(M)|^p \leq \mathcal{L}^p \, |M|^p$$

and

$$|F_{ij}(M)| \leq \mathcal{L},$$

uniformly in M. By combining Proposition 3 with Remark 4, we infer the existence of a minimizer u^* for (3.2) in the Sobolev space $W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$. Therefore, for every $\phi \in \mathcal{C}^\infty_0(B_1)$, we have

$$\begin{aligned}\int_{B_1} F_{ij}(D^2u^*)F(D^2u^*)^{p-1}\phi_{x_ix_j}\mathbf{d}x &\leq \int_{B_1} \left|F_{ij}(D^2u^*)\right| \left|F(D^2u^*)\right|^{p-1} \left|\phi_{x_ix_j}\right| \mathbf{d}x \\ &\leq C\left(1 + \|D^2u^*\|^p_{L^p(B_1)}\right), \end{aligned} \tag{3.12}$$

where $C = C(\lambda, \Lambda, d, p)$. Since $u^* \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$, the weak form of the Euler-Lagrange equation associated with (3.2) is well-defined for every $\phi \in \mathcal{C}^\infty_c(B_1)$.

By setting $m^* := F(D^2u^*)$, we notice that u^* is a viscosity solution to the first equation in (3.1). In addition, Assumption 3 implies that $m^* \geq 0$. Finally, m^* is clearly integrable, since

$$\int_{B_1} m^* \, \mathbf{d}x \leq C(p, d, \Lambda) + \int_{B_1} \left\|D^2u^*(x)\right\|^p \mathbf{d}x \leq C(p, d, \Lambda, g).$$

The function m^* satisfies the second equation in (3.1) by construction.

Since $F(D^2u) \in L^{p-1}_{loc}(B_1)$, we get that $m \in L^1_{loc}(B_1)$. From the growth condition satisfied by F, we have $m \geq 0$. Therefore, the proof is complete. ■

Remark 5 (Uniqueness of solutions) The uniqueness of solutions follows from strict convexity of the operator F. We expect that a condition in line with the Lasry-Lions monotonicity argument would also lead to uniqueness.

Remark 6 (Logarithmic nonlinearities) Under appropriate convexity assumptions on F, it is reasonable to expect that minor modifications of our arguments would account for a problem of the form

$$\tilde{I}[u] := \int_{B_1} e^{F(D^2u)} \, \mathbf{d}x \quad \longrightarrow \quad \min. \tag{3.13}$$

In fact, if $F = F(M)$ is convex, the function $e^{F(M)}$ is also convex. Also, we notice that under Assumption 3

$$\lambda\|M\| \leq e^{\lambda\|M\|} \leq e^{F(M)}.$$

The interest in (3.13) is mostly motivated by its Euler-Lagrange equation. It gives rise to the following MFG system:

$$\begin{cases} F(D^2u) = \ln m & \text{in} \quad B_1 \\ (F_{ij}(D^2u)\, m)_{x_ix_j} = 0 & \text{in} \quad B_1. \end{cases} \tag{3.14}$$

The problem in (3.14) is known as *MFG with logarithmic nonlinearities* and plays an important role in the mean-field games literature, see [56].

Remark 7 (Lower order terms) We notice that Proposition 3 and Theorem 7 can be adapted to include more general operators, namely, depending on lower-order terms. Suppose

$$F : \mathcal{S}(d) \times \mathbb{R}^d \times \mathbb{R} \times B_1 \longrightarrow \mathbb{R}$$

is such that, for every $M \in \mathcal{S}(d)$, $\xi \in \mathbb{R}^d$, $r \in \mathbb{R}$ and $x \in B_1$, we have

$$\lambda\|M\| + \alpha\|\xi\| + \beta|r| \leq F(M,\xi,r,x) \leq \Lambda\|M\| + A\|\xi\| + B|r|,$$

for some $0 < \lambda \leq \Lambda$, $0 < \alpha \leq A$ and $0 < \beta \leq B$. Here, the Euler-Lagrange equation produces the more general MFG system

$$\begin{cases} F(D^2u, Du, u, x) = m^{1/(p-1)} & \text{in} \quad B_1 \\ (F_{ij}\, m)_{x_ix_j} + (F_{ij}\, m)_{x_i} + F_{ij}\, m = 0 & \text{in} \quad B_1. \end{cases}$$

As mentioned in the introduction, we also believe our methods and techniques would extend to operator of the form

$$F(M) + H(p),$$

provided H satisfies (3.4). For the sake of presentation, we refrain from pursuing explicitly those computations. It is clear that (3.4) builds upon our previous computations to produce the required coercivity and weak-lower semicontinuity, with eventual conditions on the exponent q. We finish this remark by observing that different choices of q would lead to equations involving different operators.

In the sequel we investigate the minimizers of (3.2) in the cases where the operator F fails to be convex.

3.4 Analysis in the non-convex setting

In what follows we consider the case of functionals driven by non-convex operators F. As it is well known, one cannot ensure the existence of minimizers in this context. Nonetheless, information about

$$\min_{u \in W^{2,p}(B_1)\cap W^{1,p}_g(B_1)} I[u]$$

still can be obtained through the so-called *relaxation methods*, that is, by the study of the relaxed problem

$$\bar{I}[u] := \int_{B_1} \left[\Gamma_F(D^2u)\right]^p \mathbf{d}x \longrightarrow \min \tag{3.15}$$

Here, we develop this approach and combine it with the discussion in Section 3.3. We start with a proposition.

Proposition 5 (Coercivity of the convex envelope) *Suppose F satisfies Assumption 3. Then, Γ_F satisfies a coercivity condition of the form*

$$\lambda\|M\| \leq \Gamma_F(M),$$

for every $M \in \mathcal{S}(d)$.

Proof. The result follows from the convexity of the norm. Since $\Gamma_F(M)$ is the supremum taken among all the convex functions $G : \mathcal{S}(d) \to \mathbb{R}$ evaluated at M, we must have

$$\lambda\|M\| \leq \sup\left\{G(M)\,|\,G \leq F \text{ and } G \text{ is convex}\right\}.$$

■

Now we are in position to produce a proof of Theorem 3. We restate and prove this result in the sequel.

Theorem 8 (Restatement of Theorem 3) *Suppose Assumption 3 is in force. Then, there exists $(u_n)_{n\in\mathbb{N}} \subset W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ such that*

$$I[u_n] \longrightarrow \bar{I}[u^*]$$

and

$$u_n \rightharpoonup u^* \quad \text{in} \quad W^{2,p}(B_1),$$

as $n \to \infty$, where $u^* \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ is the minimizer of

$$\bar{I}[u] := \int_{B_1} \left[\Gamma_F(D^2u)\right]^p \, \boldsymbol{d}x.$$

Proof. We start by showing that the convex envelope of F inherits the growth regime imposed by Assumption 3. In fact, Proposition 5 yields the required lower bounds. To obtain the upper bound, notice that $\Gamma_F(M)$ is below F; hence

$$\Gamma_F(M) \leq \Lambda \|M\|,$$

for every $M \in \mathcal{S}(d)$. Therefore, Γ_F is a convex mapping satisfying Assumption 3 and falls within the scope of Proposition 3. We conclude that (3.15) has a minimizer u^*. That is, there exists $u^* \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ such that

$$\bar{I}[u^*] \leq \bar{I}[u],$$

for every $u \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$. Finally, we evoke standard relaxation results and the proof is complete. ■

The contribution of Theorem 8 is to provide information on problem (3.2) in the absence of convexity. Although it might fail to have a minimizer, we characterize its infimum in terms of the convex envelope of F. In addition, the element $u^* \in W^{2,p}(B_1) \cap W^{1,p}_g(B_1)$ that produces the infimum of the relaxed problem is the weak limit of a sequence for which the original functional is defined.

3.5 Explicit solutions in the unidimensional setting

In this section we work out in detail an explicit example in dimension $d = 1$. This concrete model allows us to produce explicit solutions with very simple structure and perform some analysis of the various features involved in the problem. Our analysis is motivated by [55].

Here, we consider the open interval $(0, 1)$ and especialize $F = F(z)$ to be given by

$$F(z) := (1 + z^p)^{\frac{1}{p}}, \tag{3.16}$$

where $p \geq 2$ is a fixed integer. Then we have

$$F''(z) = (1 - p)z^{2p-2}(1 + z^p)^{\frac{1-2p}{p}} + (p - 1)z^{p-1}(1 + z^p)^{\frac{1-p}{p}};$$

therefore, $F''(z)$ might be negative depending on the values of z and p. In that case, F fails to be convex with respect to z. In addition, if $p \in 2\mathbb{N} + 1$ is an odd number, the operator also lacks differentiability.

Under (3.16), the functional in (3.2) becomes

$$I[u] \;=\; \int_0^1 1 \,+\, u_{xx}^p(x)\,\mathbf{d}x,$$

whereas (3.1) comprises

$$\left(1 \,+\, u_{xx}^p(x)\right)^{\frac{1}{p}} \;=\; m^{\frac{1}{p-1}} \tag{3.17}$$

and

$$\left(\frac{u_{xx}^{p-1}(x)\,m(x)}{\left(1 \,+\, u_{xx}^p(x)\right)^{\frac{p-1}{p}}}\right)_{xx} \;=\; 0. \tag{3.18}$$

As a consequence, it follows that

$$\left(u_{xx}^{p-1}\right)_{xx} \;=\; 0.$$

Thus, we discover

$$u(x) \;=\; \frac{A\left((p-1)x \,-\, B\right)^{2+\frac{1}{p-1}}}{p(2p \,-\, 1)} \,+\, Cx \,+\, D \tag{3.19}$$

where A, B, C and D are constants. Using (3.17) we find

$$m(x) \;=\; \left(1 \,+\, \left[A\left((p \,-\, 1)x \,-\, B\right)^{\frac{1}{p-1}}\right]^p\right)^{\frac{1}{p}}, \tag{3.20}$$

Now we turn our attention to the minimization problem driven by $I[u]$. We infer from (3.19) that

$$I[u] \;=\; 1 \,+\, \frac{A\left([(p \,-\, 1) \,-\, B]^{\frac{p}{p-1}} \,-\, (-B)^{\frac{p}{p-1}}\right)}{p}$$

Our goal is to characterize A and B in order to minimize $I[u]$. By resorting to the first order conditions, we find that such constants must be chosen in order to satisfy both

$$\left(A^2(-AB)^{\frac{1}{p-1}}\right)^p - \left(A^2(A(-B+p-1))^{\frac{1}{p-1}}\right)^p = 0$$

and

$$A^{2p-1}\left[Bp\left((-AB)^{\frac{1}{p-1}}\right)^p + p(-B+p-1)\left((A(-B+p-1))^{\frac{1}{p-1}}\right)\right]^p = 0.$$

To ensure that both constraints derived from the first order conditions are met, we must have $A \equiv 0$. Therefore, a solution to (3.17)-(3.18) minimizes the associated functional $I[u]$ if it takes the form

$$u(x) := Cx + D \quad \text{and} \quad m(x) \equiv 1.$$

Among the solutions to the mean-field games system, those minimizing the functional comprise an affine mapping u and a uniform distribution m. A remarkable aspect of this toy-model is that minimizing solutions are independent of the power $p \geq 2$.

In the presence of a boundary condition $u(0) = g(0)$ and $u(1) = g(1)$, we have

$$D := g(0) \quad \text{and} \quad C := g(1) - g(0).$$

The previous findings are summarized in Theorem 4. We restate it in what follows:

Theorem 9 (Restatement of Theorem 4) *Let F be given as in* (3.16). *Then*

1. *A solution (u, m) to the associated mean-field games system is given by*

$$u(x) = \frac{A\left((p-1)x - B\right)^{2+\frac{1}{p-1}}}{p(2p-1)} + Cx + D$$

and

$$m(x) = \left(1 + \left[A\left((p-1)x - B\right)^{\frac{1}{p-1}}\right]^p\right)^{\frac{1}{p}},$$

where A, B, C, and D are real constants.

2. *A minimizing solution (u^*, m^*) comprises an affine mapping and a uniform distribution; i.e.,*

$$u^*(x) = A^*x + B^*$$

and

$$m^*(x) = 1,$$

where A^* and B^* are real constants.

Remark 8 In [52], the authors study the well-posedness of mean-field games systems through monotonicity methods. As a remark, they mention the case of fully nonlinear MFG; see [52, Section 7.1]. In that paper the authors suppose the operators to be convex and of class $\mathcal{C}^\infty$ with respect to the Hessian of the solutions. Therefore, their analysis does not include the class of examples addressed in the present section. In fact, the operator

$$F(z) := (1 + z^p)^{\frac{1}{p}}$$

fails to be convex for odd values of $p \in 2\mathbb{N} + 1$. In addition, $z \mapsto F(z)$ is not smooth. This fact reinforces the importance of explicit examples, such as the one in (3.17)-(3.18), accompanying results stated in more general settings.

4
Degenerate Fully Nonlinear Equations

In this chapter we examine the (sharp) regularity for the solutions to degenerate nonlinear elliptic equations of the form

$$\mathscr{F}(Du, D^2u) \,=\, f(x) \quad \text{in } B_1. \tag{4.1}$$

Here the source term $f(x)$ is continuous and bounded function and the nonlinear operator $\mathscr{F}\colon \mathbb{R}^d \times \mathcal{S}(d) \to \mathbb{R}$ is degenerate elliptic, with law of degeneracy σ, for which its inverse σ^{-1} satisfies the Dini condition. This means $\mathscr{F}(\vec{p}, M) = \sigma(|\vec{p}|)F(M)$, where $F\colon \mathcal{S}(d) \to \mathbb{R}$, representing the *diffusion agent* of the model is uniformly elliptic operator and σ is a modulus of continuity, otherwise called the *law of degeneracy* of the equation. Our main regularity result states that solutions to (4.1) are locally $\mathcal{C}^1$-regular, with the appropriate estimates.

The rationale of our arguments finds itself under the scope of the set of methods usually referred to as *geometric tangential analysis*; see, for example, [76], [73] and [77]. In a nutshell, this class of techniques aims at relating a given problem of interest with an auxiliary, more developed one; ultimately, the geometric structure relating both allows us to transmit information from the latter to the former. This approach is very much inspired by trailblazing ideas developed in [22]. See also [23].

In the concrete case of our work, we explore the connection between (4.1) and the homogeneous, uniformly elliptic, problems driven by F. That is,

$$F(D^2u) \,=\, 0 \qquad \text{in } \; B_1. \tag{4.2}$$

Of particular interest, is the fact that solutions to (4.2) are known to be $\mathcal{C}^{1,\alpha}$-regular, for some $\alpha \in (0,1)$ universal, though unknown. This is the content of the Krylov-Safonov theory [65], [66]. We proceed with some context on the topic.

4.1
Some context on degenerate fully nonlinear equations

The model in (4.1) accounts for a nonlinear diffusion whose ellipticity collapses as the gradient of the solution vanishes. The degeneracy behavior is encoded by the modulus of continuity σ. Our analysis accommodates important examples, accounting for distinct degeneracy-profiles, as the assumptions under which we work are flexible. For example, we mention the log-Lipschitz modulus of continuity, given by

$$\sigma_{\text{log-Lip}}(t) \;:=\; t\left(\ln\left(\frac{1}{t}\right) \,+\, 1\right),$$

and its α-variant

$$\sigma_{\alpha\text{-log-Lip}}(t) \;:=\; t\left(\ln\left(\frac{1}{t}\right) \,+\, 1\right)^{-\alpha},$$

for $\alpha > 0$.

Diffusion processes whose ellipticity is affected by a gradient-dependent term are of fundamental relevance in analysis of partial differential equations. A paramount example – in the variational setting – is the p-Poisson equation

$$\operatorname{div}\left(\mathbf{a}(Du)\right) \;=\; f \qquad \text{in} \;\; B_1,$$

where $\mathbf{a}(\vec{v}) \sim |\vec{v}|^{p-2}\,\vec{v}$. The prototype of the theory is the classical p-Laplacian, which appears in connection with the optimization problem of the p-Dirichlet integral and accounts for a variety of important models in life and social sciences. As regards the regularity of the solutions to p-Poisson equation, these are known to be $\mathcal{C}^{1,\alpha}$-regular, for some $\alpha \in (0,1)$; see, for instance, [48]. For a detailed account of this class of equations, we refer the reader to [70] and the references therein.

A robust nonlinear potential theory for treating variational problems with gradient degeneracy has been developed as an offspring of the pioneering work of De Giorgi [47] and since then it has been a rich and powerful line of research. In this article, though, we are interested in a non-variational counterpart of the theory, to whom Krylov–Safonov work [66] plays the role of De Giorgi's.

Heuristically, the law of degeneracy σ impairs the diffusibility attributes of the model. The stronger the degeneracy law is, the less efficiently the model diffuses, which in turn affects the smoothing properties of the system. That is, using the natural order for laws of degeneracy:

$$\sigma_1 \prec \sigma_2 \quad \text{provided} \quad \sigma_1(t) = \mathrm{o}\left(\sigma_2(t)\right),$$

one should expect that if $\sigma_1 \prec \sigma_2$ then the class of solutions of equations with σ_2 law of degeneracy should be quantitatively smoother than the corresponding class for σ_1. A universal regularity theory for solutions of such equations is the mathematical manifestation of the diffusibility impairment caused by degeneracy.

Indeed, a classical result obtained independently by Trundiger [78] and Caffarelli [22] asserts that if $\sigma \sim 1$, that is, if the equation is non-degenerate, otherwise termed uniformly elliptic, then solutions are locally of class $\mathcal{C}^{1,\alpha}$. If no condition whatsoever is imposed upon the law of degeneracy σ, then solutions may fail to be differentiable; in this case the best one can expect is local Hölder continuity, see [61]. The goal of this part of the thesis is to examine minimal conditions upon σ as to assure solutions retain $\mathcal{C}^1$–differentiability properties. As it happens in many branches of mathematical analysis, $\mathcal{C}^1$ estimate is indeed conceptually more difficult as it often represents a critical borderline regularity.

The heuristic discussion above conveys that such a condition should somehow prevent $\sigma(\vec{p})$ from approaching 0 too abruptly. Our main result in this part captures this insight in a clear and concise format.

In particular, if σ^{-1} behaves as a Hölder continuous function near the origin, then solutions are in fact locally $\mathcal{C}^{1,\gamma}$, for some $0 < \gamma < 1$. This accounts for non-linear elliptic PDEs with power–like degeneracy laws, $\sigma(|\vec{p}|) = \mathrm{O}\left(|\vec{p}|^M\right)$, as $\vec{p} \to 0$, for some $M > 0$, and thus Theorem 5 extends the results from [60], see also [6].

We conclude the introduction by describing, at large, the strategy followed for proving Theorem 5. Given a point $x_0 \in B_{1/4}$, we want to attain the existence of a tangent hyperplane $\mathscr{H}_{x_0} = \ell_{x_0}^{-1}(0)$ and a modulus of continuity ω such that

$$\sup_{x \in B_\gamma(x_0)} |u(x) - \ell_{x_0}(x)| \leq \gamma \omega(\gamma),$$

for all $0 < \gamma \leq 1/4$. This is achieved by means of a geometric recursive construction. Given a family of laws of degeneracy Σ, define the functional space $\Xi_{\varepsilon,\lambda,\Lambda,\Sigma}$ to be the set of all continuous functions $u \in \mathcal{C}(B_1)$ such that $\|u\|_\infty \leq 1$ and

$$\left|\mathscr{F}(Du, D^2u)\right| < \varepsilon$$

in the viscosity sense, for an operator $\mathscr{F}(\vec{p}, M) = \sigma(\vec{p})F(M)$, with F (λ, Λ)-elliptic and $\sigma \in \Sigma$. Then for some positive $\beta > 0$ there exists a modulus of continuity τ such that

$$\Xi_{\varepsilon,\lambda,\Lambda,\Sigma}\Big|_{B_{1/2}} \subset \mathscr{N}_{\tau(\varepsilon)}\left(\mathcal{C}^{1,\beta}(B_{1/2})\right) \tag{4.3}$$

where $\Xi_{\varepsilon,\lambda,\Lambda,\Sigma}\Big|_{B_{1/2}}$ simply represents the restriction of functions in $\Xi_{\varepsilon,\lambda,\Lambda,\Sigma}$ to $B_{1/2}$ and $\mathscr{N}_{\tau(\varepsilon)}\left(\mathcal{C}^{1,\beta}(B_{1/2})\right)$ is the $\tau(\varepsilon)$—neighborhood of $\mathcal{C}^{1,\beta}(B_{1/2})$ within $L^\infty(B_{1/2})$.

Here comes the first main technical difficulty of the proof. To attain such a pivotal result, one must require a sort of "non-collapsing" property upon the family of laws of degeneracy. Otherwise, if one does not prevent a sequence of laws of degeneracy σ_j to converge to a function σ_∞ which vanishes identically on a non-trivial interval $[0,\delta]$, $\delta > 0$, then any function whose Lipschitz norm is less than δ would belong to the limit set of solutions and (4.3) could not hold true. The concept of non-collapsing moduli of continuity and the approximation scheme will be introduced later.

Once such a result is available, the idea is to iterate it, using supporting hyperplanes of $\mathcal{C}^{1,\beta}(B_{1/2})$ functions that are close enough to a scaled version of the preceding element of the sequence. To put forward such strategy, though, one has to tackle two intrinsic difficulties. The first one is that u subtracted an affine function solves a family of equations parametrized by a non-compact set of parameters, for which one nonetheless has to extract some compactness property. This is attained by classical PDE methods, inherent of the viscosity theory. The second, and most challenging difficulty is that these corresponding PDEs are now ruled by a new family of degeneracy laws, which could be collapsing. The main novelty here is a new algorithm for choosing the normalization in each step, based on a sort of "shoring-up" technique, which effectively prevents collapsing of the resulting degeneracy laws.

When it comes to (4.1), the work-horse of the theory has been the model degeneracy $\sigma(t) = t^\theta$, where $\theta > -1$. It leads to equations of the form

$$|Du|^\theta F(D^2u) = f \qquad \text{in } B_1. \tag{4.4}$$

For this choice of σ, the resulting equation can be regarded as a *nonvariational counterpart* of the p-Poisson model. The general theory of (4.4) has known a number of important developments, covering comparison principles, well-posedness for the Dirichlet problem and maximum principles, including an Aleksandroff-Bakelman-Pucci estimate; see [14], [15], [16], [17], [18], [46].

As regards the regularity of the solutions to (4.4), it was first examined in [60], [19] and [6]. In brief, if $u \in \mathcal{C}(B_1)$ is a viscosity solution to (4.4), then we have $u \in \mathcal{C}^{1,\alpha}_{loc}(B_1)$, with the appropriate estimates. Of particular interest is the optimal regularity discovered in the presence of convex operators F; in this case, solutions are $\mathcal{C}^{1,\frac{1}{1+\theta}}$-regular.

In [61] a new perspective on diffusion processes degenerating through a gradient-dependent term is launched. Instead of prescribing the manner in which ellipticity collapses as the gradient of the solutions vanishes, the authors propose the analysis of an equation *holding only at the points where the gradient is large*. Put differently: instead of prescribing a problem that might degenerate in some subregion of the domain, the model in [61] concerns a diffusion process taking place only where $|Du| \geq \gamma$, for some $\gamma > 0$. They prove that solutions to this class of diffusions are locally Hölder-continuous.

The notion of an equation holding only in a subregion where the gradient of the solutions is above certain quantity bears an intrinsic connection with obstacle and free boundary problems.

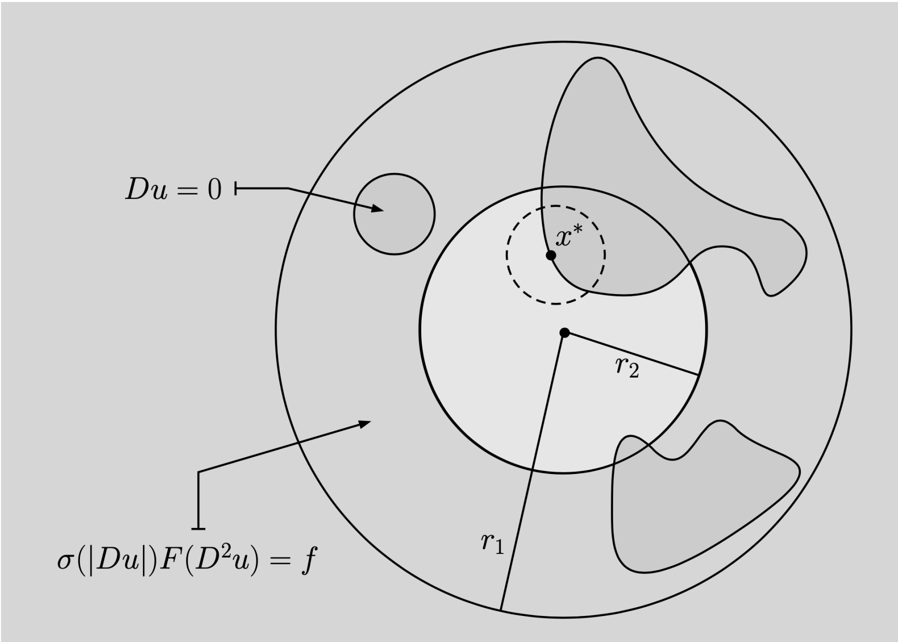

Figure 4.1: By prescribing (4.1) in B_{r_1}, we examine an equation that holds, in fact, in the regions where $|Du| \neq 0$. In the subregions where the gradient vanishes, there is no PDE available. As a consequence, information on the solutions of the problem are not retrieved through usual structures (e.g., degenerate or uniform ellipticity). Away from the interfaces separating both subregions, solutions are known to be regular. The challenge in establishing local regularity results, say, in B_{r_2}, amounts to ensure that at a point $x^* \in B_{r_2} \cap \partial\{Du = 0\}$ it is possible to center a smaller ball within which solutions are regular enough.

In fact, if u is a viscosity solution to (4.1), at the points where its gradient vanishes, the equation falls short in providing information on the diffusion process. An interesting analysis has to do with the *interface* separating the regions $\{Du = 0\}$ and $\{Du \neq 0\}$. Away from this interface, either solutions are locally constant (and therefore locally smooth) or they satisfy a uniformly elliptic equation with bounded left-hand side. Therefore, *away from the interface* a soundly-based regularity theory – such as the Krylov-Safonov Theorem – is available. The critical aspect of the analysis is to understand the behavior of viscosity solutions across $\partial\{Du = 0\}$. Even more important, is the analysis of how the regime-switching across the interface resonates in the regularity theory available for (4.1). See Figure 4.1.

Our findings address those questions. Indeed, in the presence of a degeneracy rate given by a modulus of continuity, solutions cross the interface with a $\mathcal{C}^1$-geometry. The proof of this fact unfolds along three main results.

First, we establish compactness for a variant of (4.1). Then, an approximation result relates solutions to this perturbed equation with $F = 0$. Once this connection is available, an scaling argument builds upon the Dini-continuity of σ^{-1} unlocking an oscillation estimate and ultimately leading to the result – and explicitly characterizing the modulus of continuity of Du. In the sequel we establish compactness for a variant of (4.1).

4.2
Compactness for perturbed PDEs

In this section we produce preliminary levels of compactness for the solutions to a variant of (4.1), by proving that bounded solutions to a family of equations, that are parametrized by vectors of $\mathbb{R}^d$, are uniformly locally Hölder-continuous. This will be attained by means of classical viscosity methods. Part of our arguments involve an scaling argument. As a result, scaled functions satisfy a different equation, i.e.:

$$\mathscr{F}(Du + \xi, D^2u) = f \qquad \text{in } B_1 \tag{4.5}$$

where $\xi \in \mathbb{R}^d$ is *arbitrary* and $\mathscr{F}(\vec{p}, M) = \sigma(|\vec{p}|)F(M)$. The first genuinely challenging instance of our analysis is to produce estimates for the solutions to (4.5) *not depending on* ξ.

Theorem 10 (Hölder-continuity) *Let $u \in \mathcal{C}(B_1)$ be a normalized viscosity solution to* (4.5)*. Suppose Assumptions 1, 4 and 6 are in force. Then, u is locally Hölder-continuous in B_1. In addition, there exists $C > 0$, not depending on $\xi \in \mathbb{R}^d$, such that*

$$\sup_{\substack{x,y \in B_{1/2} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\beta} \leq C,$$

for some $\beta \in (0, 1)$, universal though unknown.

Proof. As commented, the proof follows standard methods in viscosity theory. We will carry all details for completeness.

Let $\omega : \mathbb{R} \to \mathbb{R}$ be defined as $\omega(t) := t - \frac{1}{2}t^2$. For some $0 < r \ll 1$ fixed and a constant $C_0 > 0$ to be determined further in the proof, we prove the existence of L_1, $L_2 > 0$ such that

$$\mathcal{L} := \sup_{x,y \in B_r} \left(u(x) - u(y) - L_1\omega(|x - y|) - L_2(|x - x_0|^2 + |y - x_0|^2)\right) \leq 0,$$

for every $x_0 \in B_{r/2}$. As it is usual when resorting to this class of arguments, we reason through a contradiction argument. That is to say the following: suppose for every $L_1 > 0$ and $L_2 > 0$, there is $x_0 \in B_{r/2}$ for which $\mathcal{L} > 0$. In what follows, we split the proof in several steps.

Step 1 - Consider ψ, $\phi : \overline{B_r} \times \overline{B_r} \to \mathbb{R}$, defined by

$$\psi(x, y) := L_1\omega(|x - y|) + L_2(|x - x_0|^2 + |y - x_0|^2)$$

and

$$\phi(x, y) := u(x) - u(y) - \psi(x, y).$$

Let $(\bar{x}, \bar{y}) \in \overline{B_r} \times \overline{B_r}$ be a maximum point for ϕ. Thus

$$\phi(\bar{x}, \bar{y}) = \mathcal{L} > 0.$$

We therefore conclude

$$\psi(\bar{x}, \bar{y}) < u(\bar{x}) - u(\bar{y}) \leq \operatorname{osc}_{B_1} u \leq 2.$$

It follows that

$$L_1\omega(|\bar{x} - \bar{y}|) + L_2(|\bar{x} - x_0|^2 + |\bar{y} - x_0|^2) \leq 2.$$

As usual, at this point we choose L_2 as to ensure that $\bar{x}$ and $\bar{y}$ are interior points. In fact, if

$$L_2 := \left(\frac{4\sqrt{2}}{r}\right)^2$$

we get

$$|\bar{x} - x_0| \leq \frac{r}{4} \quad \text{and} \quad |\bar{y} - x_0| \leq \frac{r}{4},$$

hence concluding $\bar{x}$, $\bar{y} \in B_r$. Finally, it is straightforward to notice that $\bar{x} \neq \bar{y}$; otherwise, we would have $\mathcal{L} \leq 0$ trivially.

Step 2 - At this point, we resort to the Crandall-Ishii-Lions Lemma, stated in Proposition 1. We proceed by computing $D_x\psi$ and $D_y\psi$ at $(\bar{x}, \bar{y})$. We find

$$D_x\psi(\bar{x}, \bar{y}) = L_1\omega'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}|^{-1}(\bar{x} - \bar{y}) + 2L_2(\bar{x} - x_0),$$

and

$$-D_y\psi(\bar{x}, \bar{y}) = L_1\omega'(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}|^{-1}(\bar{x} - \bar{y}) - 2L_2(\bar{x} - x_0).$$

For ease of presentation, we introduce the following notation:

$$\xi_{\bar{x}} := D_x\psi(\bar{x}, \bar{y}) \quad \text{and} \quad \xi_{\bar{y}} := D_y\psi(\bar{x}, \bar{y}).$$

From Proposition 1 we learn that for every $\varepsilon > 0$, there are matrices $X, Y \in \mathcal{S}(d)$ satisfying the viscosity inequalities

$$\sigma(|\xi_{\bar{x}} + \xi|)F(X) - f(\bar{x}) \leq 0 \leq \sigma(|\xi_{\bar{y}} + \xi|)F(Y) - f(\bar{y}). \tag{4.6}$$

In addition,

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix} + 2L_2 I + \varepsilon A^2, \tag{4.7}$$

where $A := D^2\psi(\bar{x}, \bar{y})$ and

$$Z := L_1\omega''(|\bar{x}-\bar{y}|)\frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2} + L_1\frac{\omega'(|\bar{x}-\bar{y}|)}{|\bar{x}-\bar{y}|}\left(I - \frac{(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2}\right).$$

Step 3 - Next we apply the matrix inequality (4.7) to suitable vectors to recover information on the eigenvalues of $X - Y$. Let $v \in \mathbb{S}^{d-1}$ and consider first $(v, v) \in \mathbb{R}^{2d}$; we obtain

$$\langle (X - Y)v, v\rangle \leq (4L_2 + 2\varepsilon\eta),$$

where $\eta := \|A^2\|$. It is consequential that all eigenvalues of $X - Y$ are bellow $4L_2 + 2\varepsilon\eta$. Furthermore, we apply (4.7) to vectors of the form $(z, -z) \in \mathbb{R}^{2d}$, where

$$z := \frac{\bar{x} - \bar{y}}{|\bar{x} - \bar{y}|};$$

we then get

$$\langle (X - Y)z, z\rangle \leq 4L_1\omega''(|\bar{x} - \bar{y}|) + (4L_2 + 2\varepsilon\eta)|z|^2. \tag{4.8}$$

From the definition of ω, we learn it is twice differentiable, $\omega > 0$ and $\omega'' < 0$. It then follows from (4.8) that at least one eigenvalue of $X - Y$ is bellow $-4L_1 + 4L_2 + 2\varepsilon\eta$. Observe that this quantity will be negative for L_1 sufficiently large. In the sequel, we compute

$$\mathcal{M}^-_{\lambda,\Lambda}(X - Y) \geq 4\lambda L_1 - (\lambda + (d-1)\Lambda)(4L_2 + 2\varepsilon\eta);$$

this inequality builds upon the definition of ellipticity and (4.6) to produce

$$4\lambda L_1 \leq (\lambda + (d-1)\Lambda)(4L_2 + 2\varepsilon\eta) + \frac{f(\bar{x})}{\sigma(|\xi_{\bar{x}} + \xi|)} - \frac{f(\bar{y})}{\sigma(|\xi_{\bar{y}} + \xi|)}. \tag{4.9}$$

Step 4 - At this point we examine two different cases. We start by considering $|\xi| > C_0$, where $C_0 > 0$ is yet to be determined. Estimate the norm of $\xi_{\bar{x}}$ as follows:

$$|\xi_{\bar{x}}| \leq L_1|w'(|\bar{x}-\bar{y}|)| + 2L_2 \leq cL_1, \tag{4.10}$$

for some constant $c > 0$, universal. We choose next $C_0 := 100cL_1$, for L_1 to be fixed later. Since $|\xi_{\bar{x}}| < cL_1$ and $|\xi| > 100cL_1$ it follows that

$$|\xi + \xi_{\bar{x}}| \geq C_0 - \frac{C_0}{100} = \frac{99}{100}C_0;$$

a similar reasoning yields

$$|\xi + \xi_{\bar{y}}| \geq C_0 - \frac{C_0}{100} = \frac{99}{100}C_0;$$

The former inequalities, combined with the fact that σ is nondecreasing yield,

$$\frac{f(\bar{x})}{\sigma(|\xi_{\bar{x}}+\xi|)} \leq \frac{\|f\|_{L^\infty(B_1)}}{\sigma\left(\frac{99C_0}{100}\right)} \leq \|f\|_{L^\infty(B_1)} \tag{4.11}$$

and

$$\frac{-f(\bar{y})}{\sigma(|\xi_{\bar{y}}+\xi|)} \leq \frac{\|f\|_{L^\infty(B_1)}}{\sigma\left(\frac{99C_0}{100}\right)} \leq \|f\|_{L^\infty(B_1)}. \tag{4.12}$$

On their turn, inequalities (4.11) and (4.12) combined with (4.9) yield

$$4\lambda L_1 \leq (\lambda + (d-1)\Lambda)(4L_2 + 2\varepsilon\eta) + 2\|f\|_{L^\infty(B_1)}. \tag{4.13}$$

By choosing $L_1 = L_1(\lambda, \Lambda, d, L_2, r) \gg 1$ sufficiently large, we obtain a contradiction. Consequential on this contradiction is the fact that $\mathcal{L} \leq 0$; hence, we obtain Lipschitz-continuity of the solutions in the case $|\xi| > C_0$.

Step 5 - Consider now the complementary case; i.e., let $|\xi| \leq C_0$, where $C_0 = 100cL_1$ was chosen in the previous step. Define the operator

$$G(x,p,M) := \sigma(|\xi + p|)F(M) - f(x).$$

It follows that $G(x,p,M)$ is uniformly elliptic provided $|p| > \pi C_0$. By using previous regularity results (see, for instance, [61]), we derive Hölder-continuity of the solutions. Gathered with the former step, this fact completes the proof of the theorem. ■

Remark 9 In Theorem 10 we avoid the dependence on $\xi \in \mathbb{R}^d$ by splitting the space into B_{C_0} and $\mathbb{R}^d \setminus B_{C_0}$. In the former case, the Imbert-Silvestre regularity theory implies β-Hölder-continuity for the solutions. However, when analyzing the latter Lipschitz-regularity is available. This is in fact in line with the heuristics associated with the problem: when $|\xi + Du| \gg 1$, the equation is in fact uniformly elliptic.

Once compactness for the solutions of the ξ-perturbed equation is available, we approximate solutions to (4.1) and (4.5) by solutions to $F = 0$. This is our next goal; however before we advance, we need first to introduce a new concept, which is the content of next section.

4.3
Non-collapsing moduli of continuity

In this section we formalize the notion of a family of non-collapsing moduli of continuity.

Definition 8 (Non-collapsing) *A set Γ of moduli of continuity defined over an interval $I \subset \mathbb{R}$ is said to be non-collapsing if for all sequences $(f_n)_{n\in\mathbb{N}} \subset \Gamma$, and all sequences of scalars $(a_n)_{n\in\mathbb{N}} \subset I$, we have*

$$f_n(a_n) \to 0 \quad \textit{implies} \quad a_n \to 0.$$

The former definition plays an important role in the tangential analysis developed in the thesis. In fact, when one tries to connect the prospective regularity theory for $\sigma(|Du|)F(D^2u) = f$ with the one available for $F(D^2h) = 0$, we aim at profiting from a sort of *cancellation* effect, to be understood in the viscosity sense. This is only achievable, however, if one carefully modulates the rate in which $\sigma(t)$ approaches zero, as $t \to 0$. Put differently, we must ensure the degeneracy law is not, itself, degenerate.

Definition 9 *We define the collapsing measure of a family of moduli of continuity Γ defined over an interval $I \subset \mathbb{R}$ as*

$$\mu(\Gamma) := \sup\left\{ s \in I \;\middle|\; \inf_{\sigma\in\Gamma} \sigma(s) = 0 \right\}.$$

For obvious reasons all finite sets of moduli of continuity are non-collapsing, and the interesting environment are infinite sets; for this reason in this section all families of moduli of continuity shall be not finite.

It is not difficult to observe that the measure defined above characterizes non-collapsing sets as follows:

Proposition 6 *Let Γ be a family of moduli of continuity defined over an interval I. The following are equivalent:*

1. *Γ is non-collapsing.*

2. *For all sequences $(f_n)_{n\in\mathbb{N}} \subset \Gamma$ and $a \in I \setminus \{0\}$, $\liminf_{n\to\infty} f_n(a) > 0$.*

3. $\mu(\Gamma) = 0$.

Proof. It is immediate that (2) and (3) are equivalent.
(1) ⇒ (2). Suppose, seeking a contradiction, there was a sequence $(f_n)_{n\in\mathbb{N}}$ and a certain $a > 0$ such that

$$\liminf_{n\to\infty} f_n(a) = 0.$$

Hence, there is a subsequence $(f_{n_k})_{n_k\in\mathbb{N}}$ such that

$$f_{n_k}(a) \to 0,$$

and, since Γ is non-collapsing, we conclude that $a = 0$, which is a contradiction.

(2) ⇒ (1) Let us suppose, for the sake of contradiction, that (1) is not valid. Thus, there would exist $(f_n)_{n\in\mathbb{N}} \subset \Gamma$ and $(a_n)_{n\in\mathbb{N}} \subset I$, with $f_n(a_n) \to 0$ and $a_n \nrightarrow 0$. So, up to a subsequence, there exists a certain $a_0 > 0$ such that

$$a_n \geq a_0 > 0.$$

Since all the functions f_n are non-decreasing, we would have

$$f_n(a_n) \geq f_n(a_0) > 0$$

and, recalling that $f_n(a_n) \to 0$, we would have $f_n(a_0) \to 0$, which contradicts (2). ■

Observe that μ behaves as a kind of "measure of collapse": for non-collapsing sets Γ, we have $\mu(\Gamma) = 0$ and for collapsing sets Γ we have $\mu(\Gamma) > 0$. The higher the value of $\mu(\Gamma)$ the more degenerate the family Γ, otherwise refereed as "more collapsing".

Notice that

$$\mu(\Gamma_1 \cup \Gamma_2) = \max\{\mu(\Gamma_1), \mu(\Gamma_2)\}.$$

However, for infinitely many unions it is possible that $\mu(\Gamma_n) = 0$ for all $n \in \mathbb{N}$ and

$$\mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) = 1.$$

For instance, starting off with a non-collapsing set Γ_1, we can easily consider $\sigma_1, \sigma_2, ...$ such that

$$\mu(\Gamma_1 \cup \{\sigma_j : j \in \mathbb{N}\}) = 1;$$

thus, defining

$$\Gamma_k = \Gamma_1 \cup \{\sigma_1, ..., \sigma_k\}$$

for all $k \geq 2$, we have $\mu(\Gamma_n) = 0$ for all $n \in \mathbb{N}$ and

$$\mu\left(\bigcup_{n=1}^{\infty} \Gamma_n\right) = 1.$$

A plenty of examples of non-collapsing sets of moduli of continuity can be generated by the next propositions:

Proposition 7 *If Γ is a family of moduli of continuity $\sigma\colon I \subset \mathbb{R} \to \mathbb{R}$ such that all $\sigma \in \Gamma$ are increasing and the set*

$$\Gamma^{-1} := \left\{\sigma^{-1} : \sigma \in \Gamma\right\}$$

is equicontinuous, then Γ is non-collapsing.

Proof. If Γ^{-1} is equicontinuous, given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow \sup_{\sigma^{-1}\in\Gamma^{-1}} \left|\sigma^{-1}(x) - \sigma^{-1}(y)\right| < \varepsilon.$$

for all $x, y \in I$. Thus

$$|\sigma(x) - \sigma(y)| < \delta \Rightarrow \sup_{\sigma^{-1}\in\Gamma^{-1}} \left|\sigma^{-1}(\sigma(x)) - \sigma^{-1}(\sigma(y))\right| < \varepsilon,$$

for all $x, y \in [0, 1]$ and all $\sigma \in \Gamma$. Choosing $y = 0$,

$$\sigma(x) < \delta \Rightarrow x < \varepsilon$$

for all $x \in [0, 1]$ and all $\sigma \in \Gamma$, i.e.,

$$x \geq \varepsilon \implies \inf_{\sigma\in\Gamma} \sigma(x) \geq \delta.$$

Hence, Γ is non-collapsing. ∎

Proposition 8 *Let Γ be a family of moduli of continuity and assume*

$$S := \sup_{\omega\in\Gamma} \int_0^1 \frac{\omega^{-1}(t)}{t}\mathbf{d}t < \infty.$$

Then $\mu(\Gamma) = 0$.

Proof. From Proposition 7, if suffices to show

$$\omega_n \in \Gamma,\ \omega_n(a) \to 0 \implies a = 0.$$

Hence, let us suppose, seeking a contradiction, there exist a sequence $\omega_n \in \Gamma$ and a positive $a > 0$ such that $b_n := \omega_n(a) \to 0$. We estimate

$$S \geq \int_0^1 \frac{\omega_n^{-1}(t)}{t}\mathbf{d}t \geq \int_{b_n}^1 \frac{\omega_n^{-1}(t)}{t}\mathbf{d}t \geq a \int_{b_n}^1 \frac{1}{t}\mathbf{d}t \longrightarrow +\infty,$$

as $n \to 0$. We reach a contradiction, and Proposition 8 is proven. ■

Another way of producing a family of non-collapsing moduli of continuity is through a sort of "shoring-up" process.

Definition 10 (Shore-up) *A sequence of moduli of continuity $(\sigma_n)_{n\in\mathbb{N}}$ is said to be shored-up if there exists a sequence of positive numbers $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_n \to 0$ satisfying*

$$\inf_n \sigma_n(\gamma_n) > 0,$$

for every $n \in \mathbb{N}$,

Here is a simple proposition relating the notion of shored-up sequence and non-collapsing moduli of continuity, which is better appreciated through the picture below:

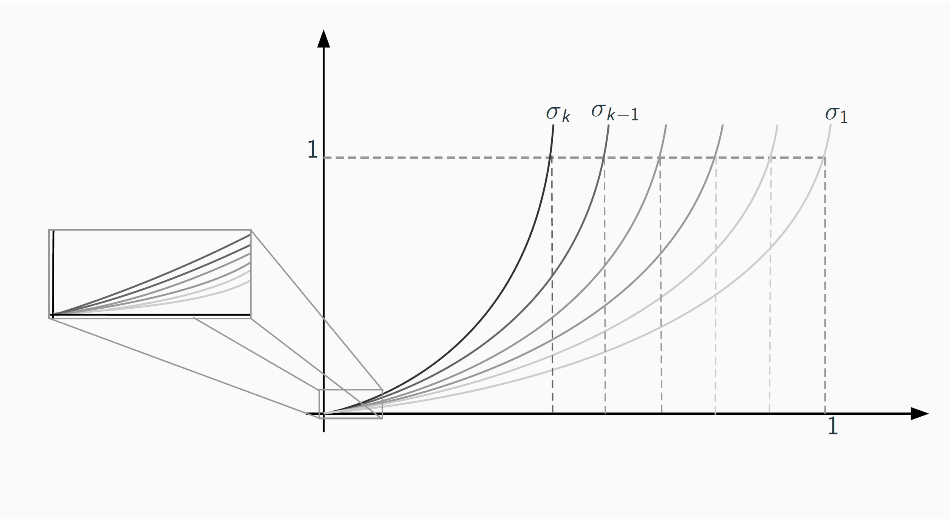

Figure 4.2: The geometric idea of shoring up as to prevent collapsing.

Proposition 9 *If a sequence of moduli of continuity $(\sigma_n)_{n\in\mathbb{N}}$ is shored-up then $\Gamma := \cup_{n\in\mathbb{N}} \{\sigma_n\}$ is non-collapsing.*

Proof. For all $s > 0$, let n_s be an integer such that $\gamma_n < s$ for all $n > n_s$. Since all the functions σ_n are non-decreasing, we have $\sigma_n(\gamma_n) \leq \sigma_n(s)$ for all $n > n_s$. Thus

$$0 < \inf_{n>n_s} \sigma_n(\gamma_n) \leq \inf_{n>n_s} \sigma_n(s).$$

Since $\sigma_1(s) > 0, \ldots, \sigma_{n_s}(s) > 0$, we conclude that

$$\inf_n \sigma_n(s) > 0$$

and $(\sigma_n)_{n\in\mathbb{N}}$ is non collapsing. ■

4.4 Tangential analysis: an approximation result

In this section we establish an approximation result, relating (4.1) and (4.5) with the solutions to the homogeneous, uniformly elliptic, problem $F = 0$. The approximating function whose existence is ensured by the next proposition plays a pivotal role in producing oscillation controls for the solutions to (4.1).

In what follows, we translate Assumption 6 into a smallness condition for the source term f. In fact, throughout this section, we require

$$\|u\|_{L^\infty(B_1)} \leq 1 \qquad \text{and} \qquad \|f\|_{L^\infty(B_1)} < \varepsilon, \tag{4.14}$$

for some $\varepsilon > 0$ yet to be determined. To see the conditions in (4.14) are not restrictive, consider the function

$$v(x) := \frac{u(rx)}{K},$$

for $0 < r \ll 1$ and $K > 0$ to be determined. Notice that v satisfies

$$\overline{\sigma}(|Dv|)\,\overline{F}(D^2 v) = \overline{f} \qquad \text{in } B_1, \tag{4.15}$$

where

$$\overline{\sigma}(t) := \sigma\left(\frac{K}{r}t\right), \qquad \overline{F}(M) := \frac{r^2}{K}F\left(\frac{K}{r^2}M\right)$$

and

$$\overline{f}(x) := \frac{r^2}{K}f(rx).$$

Notice that

$$\overline{\sigma}^{-1}(t) := \frac{r}{K}\sigma^{-1}(t).$$

Indeed,

$$\overline{\sigma}^{-1}(\overline{\sigma}(t)) = \overline{\sigma}^{-1}\left(\sigma\left(\frac{K}{r}t\right)\right) = \frac{r}{K}\sigma^{-1}\left(\sigma\left(\frac{K}{r}t\right)\right) = t.$$

By choosing $r < K$, it follows easily that

$$\int_0^1 \frac{\overline{\sigma}^{-1}(t)}{t}\mathbf{d}t \leq \int_0^1 \frac{\sigma^{-1}(t)}{t}\mathbf{d}t \quad \text{and} \quad \overline{\sigma}(1) = \sigma\left(\frac{K}{r}\right) \geq \sigma(1) \geq 1.$$

Hence, $\overline{\sigma}$ meets Assumptions 5. Clearly, $\overline{F}$ is a (λ, Λ)-elliptic operator. Finally, by setting

$$r := \varepsilon \qquad \text{and} \qquad K := \frac{1}{\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)}}$$

we produce (4.14) and find that (4.15) falls within the same class as (4.1).

Proposition 10 (Approximation Lemma) *Let $\mathfrak{S}$ be a set of non-collapsing moduli of continuity and $u \in \mathcal{C}(B_1)$ be a normalized viscosity solution of an equation of the form*

$$\sigma\left(|Du + \xi|\right) F\left(D^2 u\right) = f \quad in \quad B_1,$$

where $\xi \in \mathbb{R}^d$, $\sigma \in \mathfrak{S}$ satisfies Assumption 4, F satisfies Asumption 1, and f verifies Assumption 6. Given $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta, \lambda, \Lambda, \mathfrak{S}) > 0$ such that if $f \in B_\varepsilon(L^\infty(B_1))$ then we can find a function $h \in B_L\left(\mathcal{C}^{1,\beta}(B_{1/2})\right)$ such that

$$d_{L^\infty(B_{1/2})}(u, h) < \delta,$$

where L and β are universal numbers, in particular independent of $\mathfrak{S}$, δ and ε.

Proof. For ease of presentation we split the proof in five steps.

Step 1 - Suppose the thesis of the lemma fails to hold. Then there exist sequences $(\sigma_j)_{j\in\mathbb{N}}, (\xi_j)_{j\in\mathbb{N}}, (u_j)_{j\in\mathbb{N}}, (F_j)_{j\in\mathbb{N}}, (f_j)_{j\in\mathbb{N}}$ and a number $\delta_0 > 0$ such that, for every $j \in \mathbb{N}$, we have

1. $F_j : \mathcal{S}(d) \to \mathbb{R}$ is a (λ, Λ)-elliptic operator;

2. σ_j is a modulus of continuity satisfying $\sigma_j(0) = 0$ and $\sigma_j(1) \geq 1$. In addition, if $\sigma_j(a_j) \to 0$ then $a_j \to 0$. We observe that such a condition arises naturally under Assumption 5. In fact, if $\sigma_j(a_j) \to 0$, we have

   $$\sigma_j^{-1}(\sigma_j(a_j)) \leq \omega_{\sigma^{-1}}\left(|\sigma_j(a_j)|\right) \longrightarrow 0.$$

   Therefore,,

   $$a_j = \sigma_j^{-1}\left(\sigma_j(a_j)\right) \longrightarrow 0;$$

3. $f_j \in L^\infty(B_1)$ is such that

   $$\|f_j\|_{L^\infty(B_1)} < \frac{1}{j};$$

4.
$$\sigma_j(|Du_j+\xi_j|)F_j(D^2u_j) \;=\; f_j \qquad \text{in} \;\; B_1, \tag{4.16}$$
however,
$$\sup_{x\in B_{1/2}} |u_j(x) \,-\, h(x)| \,\geq\, \delta_0 \tag{4.17}$$
for every $h \in \mathcal{C}^{1,\beta}_{loc}(B_1)$ and every $\beta \in (0,1)$.

Step 2 - Because ellipticity is uniform along the sequence $(F_j)_{j\in\mathbb{N}}$, it follows that $F_j \to F_\infty$ as $j \to \infty$, through a subsequence if necessary. In addition, it follows from Theorem 10 that $(u_j)_{j\in\mathbb{N}}$ converges uniformly to a function u_∞. Our goal is to prove that

$$F_\infty(D^2u_\infty) \;=\; 0 \qquad \text{in} \;\; B_{9/10}.$$

To that end, introduce the second order polynomial $p(x)$, defined as

$$p(x) \;:=\; u_\infty(y) \,+\, b\cdot(x \,-\, y) \,+\, \frac{1}{2}(x \,-\, y)^T M(x \,-\, y);$$

it is clear that $p(y) = u_\infty(y)$; suppose without loss of generality that $p(x) \leq u_\infty(x)$ for $x \in B_{3/4}$. Our goal is to verify that

$$F_\infty(M) \leq 0. \tag{4.18}$$

Step 3 - For $0 < r \ll 1$ fixed, let $(x_j)_{j\in\mathbb{N}}$ be defined by

$$p(x_j) \,-\, u_j(x_j) \;:=\; \max_{x\in B_r}\left(p(x) \,-\, u_j(x_j)\right).$$

We infer from (4.16) that

$$\sigma_j(|b \,+\, \xi_j|)F_j(M) \;\leq\; f_j(x_j).$$

If $(\xi_j)_{j\in\mathbb{N}}$ is an unbounded sequence, consider the (renamed) subsequence satisfying $|\xi_j| > j$, for every $j \in \mathbb{N}$. There exists $j^* \in \mathbb{N}$ such that

$$|b \,+\, \xi_j| \;>\; 1$$

for every $j > j^*$. From Assumption 4 we have

$$F_j(M) \;\leq\; \sigma_j(|b \,+\, \xi_j|)F_j(M) \;\leq\; f_j(x_j),$$

for $j > j^*$. By letting $j \to \infty$, we obtain (4.18). Conversely, if $(\xi_j)_{j\in\mathbb{N}}$ is bounded, at least through a subsequence

$$b + \xi_j \longrightarrow b + \xi^*.$$

If $|b + \xi^*| > 0$, we know $\sigma_j(|b + \xi_j|) \nrightarrow 0$. Hence

$$F_j(M) \leq \frac{f_j(x_j)}{\sigma_j(|b + \xi_j|)} \longrightarrow 0$$

and (4.18) follows. If, on the other hand, $|b + \xi^*| = 0$, we distinguish two cases. The first is $b \equiv 0$ and $\xi_j \to 0$. If there is a subsequence $(\xi_j)_{j\in\mathbb{N}}$ for which $\xi_j \neq 0$, the previous reasoning applies and the argument is complete.

On the opposite, it can be $b = \xi_j = 0$ for every $j \in \mathbb{N}$, sufficiently large. This case is tackled in the next step.

Step 4 - We work under the assumption $b \equiv \xi_j \equiv 0$. Notice that if $\mathrm{Spec}(M) \subset (-\infty, 0]$, ellipticity produces (4.18); in fact

$$F_\infty(M) \leq \lambda \sum_{i=1}^{d} \tau_i \leq 0,$$

where $\{\tau_i,\ i = 1, \ldots, d\}$ are the eigenvalues of M. Hence, we also suppose M has $k > 0$ strictly positive eigenvalues. Let $(e_i)_{i=1}^k$ be the associated eigenvectors and define

$$E := \mathrm{Span}\,\{e_1, e_2, \ldots, e_k\}.$$

Consider the orthogonal sum $\mathbb{R}^d = E \oplus G$ and the orthogonal projection P_E on E. Define the test function

$$\varphi(x) := \kappa \sup_{e\in\mathbb{S}^{d-1}} \langle P_E x, e\rangle + \frac{1}{2} x^T M x.$$

Because $u_j \to u_\infty$ locally uniformly, and $2^{-1}x^T M x$ touches u_∞ at zero, the stability of minimizers implies that φ touches u_j at $x_j^\kappa \in B_r$, for every $0 < \kappa \ll 1$ and $j \gg 1$.

Suppose $x_j^\kappa \in G$. In this case, φ touches u_j at x_j^κ, regardless of the direction $e \in \mathbb{S}^{d-1}$. It follows that

$$\sigma_j\left(|Mx_j^\kappa + \kappa e|\right) F_j(M) \leq f_j(x_j)$$

for every $e \in \mathbb{S}^{d-1}$. By taking supremum with respect to the direction e on both sides of the former inequality, and noticing that

$$\kappa \le \sup_{e \in \mathbb{S}^{d-1}} |Mx_j^\kappa + \kappa e|,$$

we obtain

$$F_j(M) \le \frac{f_j(x_j^\kappa)}{\sigma_j(\kappa)} \longrightarrow 0$$

as $j \to \infty$. To complete the proof we focus on the case $P_E x_j^\kappa \neq 0$. Here

$$\sup_{e \in \mathbb{S}^{d-1}} \left\langle P_E x_j^\kappa, e \right\rangle = |P_e x_j^\kappa|.$$

From the information available for u_j, we obtain

$$\sigma_j\left(\left|Mx_j^\kappa + \kappa \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|}\right|\right) F_j\left(M + \kappa\left(Id + \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|} \otimes \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|}\right)\right) \le f_j(x_j^\kappa).$$

Write x_j^κ as

$$x_j^\kappa, = \sum_{i=1}^d a_i e_i,$$

where $\{e_i,\ i = 1, \ldots, d\}$ are the eigenvectors of M. Hence,

$$Mx_j^\kappa = \sum_{i=1}^k \tau_i a_i e_i + \sum_{i=k+1}^d \tau_i a_i e_i,$$

with $\tau_i > 0$ for $i = 1, \ldots, k$. We then obtain

$$\begin{aligned} \kappa &\le \kappa + \frac{1}{|P_E x_j^\kappa|} \sum_{i=1}^k \tau_i a_i^2 \le \kappa + \frac{1}{|P_E x_j^\kappa|} \left\langle \sum_{i=1}^d \tau_i a_i e_i, \sum_{i=1}^k \tau_i a_i e_i \right\rangle \\ &\le \left\langle Mx_j^\kappa + \kappa \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|}, \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|} \right\rangle \\ &\le \left| Mx_j^\kappa + \kappa \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|} \right|. \end{aligned}$$

Once again we get

$$F_j(M) \le F_j\left(M + \kappa\left(Id + \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|} \otimes \frac{P_E x_j^\kappa}{|P_E x_j^\kappa|}\right)\right) \le \frac{f_j(x_j^\kappa)}{\sigma_j(\kappa)} \longrightarrow 0$$

as $j \to \infty$.

Step 5 - Hence, we conclude that $F_\infty(M) \le 0$ and, therefore, u_∞ is a subsolution to $F_\infty = 0$ in the viscosity sense. To verify that u_∞ is also a supersolution is analogous and we omit the details. Standard results in the regularity theory of viscosity solutions to homogeneous elliptic equations yield $u_\infty \in \mathcal{C}^{1,\beta}_{loc}(B_1)$ for some $\beta \in (0,1)$. By setting $h := u_\infty$ we obtain a contradiction and complete the proof. ■

A fundamental fact about the approximating function h is related to its Taylor expansion centered at an arbitrary point $x_0 \in B_{1/4}$. In fact, we have

$$\sup_{x \in B_r} |h(x) - h(x_0) - Dh(x_0) \cdot (x - x_0)| \leq C r^{1+\beta},$$

for $0 < r \ll 1$.

4.5 Existence of approximating hyperplanes

Let us move forward with the proof of Theorem 5. Hereafter let $L > 0$ and $0 < \beta < 1$ be the universal numbers from Proposition 10. As before, we restate it in what follows.

Theorem 11 (Restatement of Theorem 5) *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to*

$$\mathscr{F}(Du, D^2u) = f \quad \text{in} \quad B_1, \tag{4.19}$$

where $\mathscr{F}(\vec{p}, M) = \sigma(|\vec{p}|)F(M)$. Suppose Assumptions 1, 4, 6 and 5, to be detailed later, hold true. Then $u \in \mathcal{C}^1_{loc}(B_1)$ and there exists a modulus of continuity $\omega : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that

$$\sup_{x \in B_r(x_0)} |Du(x) - Du(x_0)| \leq \omega(r),$$

for every $x_0 \in B_{1/4}$ and $0 < r < 1/4$. In addition, ω depends on $d, \lambda, \Lambda, \sigma$, $\|u\|_{L^\infty(B_1)}$ and $\|f\|_{L^\infty(B_1)}$.

As to ease the presentation, let us define two new moduli of continuity:

$$\gamma(t) := t\sigma(t) \quad \text{and} \quad \omega(t) := \gamma^{-1}(t).$$

Next we make a first choice of constants $0 < r < \mu_1 < 1$, by dividing the analysis in two cases:

Case 1. If $\omega(t) = \mathrm{o}(t^\beta)$, we choose $0 < r < 1/2$ so small that

$$2Lr^\beta = \omega(r) =: \mu_1 > r.$$

This is the most interesting case, for which the degeneracy law is stronger than $t^{\frac{1}{\beta}-1}$.

Case 2. If $t^\beta = \mathrm{O}(\omega(t))$, we fix $0 < \alpha < \beta$ and make $0 < r < 1/2$ so small that

$$2Lr^\beta = r^\alpha =: \mu_1 > r.$$

Notice that, once fixed $0 < \alpha < \beta$, the above choice becomes universal.

In what follows we shall treat both cases concomitantly. Define, hereafter, the ratio

$$0 < \theta := \frac{r}{\mu_1} < 1.$$

Next, under Assumption A5, we know the sequence

$$(a_k)_{k\in\mathbb{N}} := \left(\sigma^{-1}\left(\theta^k\right)\right)_{k\in\mathbb{N}}$$

belongs to ℓ_1. We apply Lemma 4 to the sequence $(a_k)_k$ with, $0 < \delta < \frac{1}{10}$ fixed and $0 < \varepsilon < 1$ chosen in such a way

$$\varepsilon\,(1+\delta) = 1$$

This creates a sequence of positive numbers $(c_k)_k \in c_0$ for which

$$\frac{19}{22}\sum_{i=1}^{\infty}\sigma^{-1}\left(\theta^k\right) \leq \sum_{i=1}^{\infty}\frac{\sigma^{-1}\left(\theta^k\right)}{c_k} \leq \sum_{i=1}^{\infty}\sigma^{-1}\left(\theta^k\right). \tag{4.20}$$

In the sequel, we generate a shored-up sequence of moduli of continuity by the following recursive formula:

$$\begin{aligned}
\sigma_0(t) &= \sigma(t);\\
\sigma_1(t) &= \frac{\mu_1}{r}\sigma(\mu_1 t);\\
\sigma_2(t) &= \frac{\mu_1\mu_2}{r^2}\sigma(\mu_1\mu_2 t);\\
&\vdots\\
\sigma_n(t) &= \frac{\mu_1\mu_2\cdots\mu_n}{r^n}\sigma(\mu_1\mu_2\cdots\mu_n t),
\end{aligned} \tag{4.21}$$

where $\mu_1 > r$ has already been chosen and for $k \geq 2$, the value of μ_k is determined through the following new algorithm:

If

$$\frac{\mu_1^2}{r^2}\sigma\left(\mu_1^2 \cdot c_2\right) \geq 1,$$

then

$$\mu_2 = \mu_1;$$

otherwise

$$\mu_1 < \mu_2 < 1$$

is chosen such that

$$\frac{\mu_1\mu_2}{r^2}\sigma\left((\mu_1\mu_2)\cdot c_2\right) = 1,$$

where c_2 is the 2nd element of the sequence $(c_k)_k \in c_0$ for which (4.20) is verified.

Next we apply the above algorithm recursively, that is: once chosen $r < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$ we decide on the value of μ_{k+1} as:

$$\text{if selecting } \mu_{k+1} = \mu_k \text{ yields } \sigma_{k+1}(c_{k+1}) \geq 1,$$

we set $\mu_{k+1} = \mu_k$. Otherwise, $\mu_{k+1} > \mu_k$ is chosen such that

$$\sigma_{k+1}\left(c_{k+1}\right) = 1,$$

where, as before, c_{k+1} is the $(k+1)$th element of the sequence $(c_k)_k \in c_0$ crafted in Proposition 4, for which (4.20) holds.

Let $\mathfrak{S}$ denote the family of moduli of continuity generated through the described algorithm:

$$\mathfrak{S} := \left\{\sigma_0(t), \sigma_1(t), \cdots, \sigma_n(t), \cdots\right\}.$$

According to Proposition 9, this is a non-collapsing family of moduli of continuity.

The next proposition produces an oscillation control for the difference of the solutions to (4.1) and an affine function.

Proposition 11 *Let $u \in \mathcal{C}(B_1)$ be a normalized viscosity solution to* (4.5)*. Suppose Assumption 1, 4, 6 and 5 are in force. There exists an $\varepsilon > 0$ such that if $\|f\|_{L^\infty(B_1)} < \varepsilon$, then, one can find an affine function $\ell(x)$ and a universal constant $C > 0$ such that*

$$\ell(x) = a + b\cdot x, \qquad \textit{with} \qquad |a| + |b| \leq C$$

and

$$\sup_{x\in B_r} |u(x) - \ell(x)| \leq \mu_1 \cdot r,$$

for constants $r < \mu_1 < 1$.

Proof. From Proposition 10 we infer the existence of $h \in \mathcal{C}^{1,\beta}_{loc}(B_1)$ such that

$$\sup_{x\in B_{9/10}} |u(x) - h(x)| \leq \delta,$$

for some $\delta > 0$, to be set further in the proof. As mentioned before, the regularity of the approximating function h yields

$$\sup_{x \in B_r} |h(x) - h(0) - Dh(0) \cdot x| \leq Lr^{1+\beta}$$

for a universal constant $L > 0$ and every $0 < r \ll 1$. By choosing $a := h(0)$ and $b := Dh(0)$ it is clear that both coefficients are bounded by C. In addition, a straightforward application of the triangular inequality yields

$$\sup_{x \in B_\rho} |u(x) - a - b \cdot x| \leq \delta + \frac{L}{2} r^{1+\beta}.$$

We proceed with the (universal) choices

$$\delta := \frac{\mu_1 \cdot r}{2} \quad \text{and} \quad \mu_1 := Lr^\beta;$$

Choosing δ as above, sets the value of $\epsilon > 0$, through Proposition 10, and the proof is completed. ■

The next proposition extends the statement in Proposition 11 to arbitrarily small radii, in a discrete scale generated by the radius $0 < r \ll 1$. Moving across those discrete scales involve an scaling argument. At this precise point of the argument, scaled solutions fail to satisfy the original equation (4.1). In turn, they satisfy

$$\mathscr{F}_n(Du_n + \xi_n, D^2 u_n) = f_n(x) \qquad \text{in } B_1,$$

where, $\xi_n \in \mathbb{R}^d$ is arbitrary and at each scale the new operator $\mathscr{F}_n(\vec{p}, M)$ has law of degeneracy σ_n and diffusion agent F_n. The switch from (4.1) to (4.5) is justified by the necessity of producing uniform compactness estimates available at this instance of the argument.

Proposition 12 (Oscillation control at discrete scales) *Let $u \in \mathcal{C}(B_1)$ be a normalized viscosity solution to* (4.5)*. Suppose Assumptions 1, 4, 6 and 5 are in force. Then there exists a sequence of real numbers $(\mu_n)_{n \in \mathbb{N}}$ and a sequence of affine functions $(\ell_n)_{n \in \mathbb{N}}$ of the form*

$$\ell_n(x) := A_n + B_n \cdot x$$

satisfying

$$\sup_{x \in B_{r^n}} |u(x) - \ell_n(x)| \leq \left(\prod_{i=1}^n \mu_i \right) r^n, \tag{4.22}$$

$$|A_{n+1} - A_n| \leq C \left(\prod_{i=1}^n \mu_i \right) r^n \tag{4.23}$$

and
$$|B_{n+1} - B_n| \leq C \prod_{i=1}^{n} \mu_i \tag{4.24}$$
for every $n \in \mathbb{N}$.

Proof. We prove the proposition through an induction argument. As before, we proceed in steps.

Step 1 - For μ_1 and $\ell = \ell_1$ as in Proposition 11, consider the auxiliary function

$$u_1(x) := \frac{u(rx) - \ell(rx)}{\mu_1 r}.$$

Notice that u_1 solves

$$\sigma_1\left(\left|Du_1 + \frac{1}{\mu_1}D\ell\right|\right) F_1(D^2u_1) = f_1(x) \qquad \text{in } B_1,$$

where

$$\sigma_1(t) := \frac{\mu_1}{r}\sigma(\mu_1 t),$$

$$F_1(M) := \frac{r}{\mu_1}F\left(\frac{\mu_1}{r}M\right) \qquad \text{and} \qquad f_1(x) := f(rx).$$

The selection through the algorithm preceding 11 ensures that $\sigma_1(1) = 1$. Therefore, u_1 falls within the scope of this result and we infer the existence of an affine function ℓ_1 such that

$$\sup_{x\in B_r} |u_1(x) - \ell_1(x)| \leq \mu_1 \cdot r.$$

At this point, we define u_2 as

$$u_2(x) := \frac{u_1(rx) - \ell_1(rx)}{\mu_2 r},$$

for $r < \mu_1 < \mu_2$ to be chosen. It is clear that u_2 satisfies

$$\sigma_2\left(\left|Du_2 + \frac{1}{\mu_1}D\ell_1\right|\right) F_2(D^2u_2) = f_2(x) \qquad \text{in } B_1,$$

where, as before,

$$\sigma_2(t) = \frac{\mu_1\mu_2}{r^2}\sigma(\mu_1\mu_2 t).$$

The governing diffusion agent for u_2 is given by

$$F_2(M) := \frac{r^2}{\mu_1\mu_2}F\left(\frac{\mu_1\mu_2}{r^2}M\right)$$

and the source term $f_2(x) := f(r^2 x)$. Hence, u_2 meets the requirements of Proposition 11, which ensures the existence of an affine function ℓ_2, with universal bounds, such that

$$\sup_{x \in B_r} |u_2(x) - \ell_2(x)| \leq \mu_1 \cdot r.$$

Proceeding inductively, we notice that

$$u_{k+1}(x) := \frac{u_k(rx) - \ell_k(rx)}{\mu_{k+1} r}$$

solves an equation with degeneracy σ_{k+1}, given by

$$\sigma_{k+1}(t) := \frac{\mu_{k+1}}{r} \sigma_k(\mu_{k+1} t) = \frac{\prod_{i=1}^{k+1} \mu_i}{r^{k+1}} \sigma\left(\prod_{i=1}^{k+1} \mu_i t\right).$$

Recall, $\mu_{k+1} \geq \mu_k$ is determined in such way that either $\mu_{k+1} = \mu_k$ or else

$$\sigma_{k+1}(c_{k+1}) = 1. \tag{4.25}$$

As before, we resort to Proposition 11 to ensure the existence of an affine function ℓ_{k+1} satisfying

$$\sup_{x \in B_r} |u_{k+1}(x) - \ell_{k+1}(x)| \leq \mu_1 \cdot r.$$

Step 2 - Reverting back to the original solution u, we find

$$\sup_{x \in B_{r^k}} |u(x) - \ell_k(x)| \leq \left(\prod_{i=1}^{k} \mu_i\right) r^k,$$

where

$$\begin{aligned} \ell_k(x) &:= \ell_1(x) + \sum_{i=2}^{k} \ell_i\left(r^{-(i-1)} x\right) \left(\prod_{j=1}^{i-1} \mu_j\right) r^{i-1} \\ &= A_k + B_k \cdot x. \end{aligned}$$

In addition, we have

$$|A_{k+1} - A_k| \leq C \left(\prod_{i=1}^{k} \mu_i\right) r^k$$

and

$$|B_{k+1} - B_k| \leq C \left(\prod_{i=1}^{k} \mu_i\right),$$

which completes the proof. ■

4.6 Convergence analysis

In this final section we discuss the convergence of the approximating hyperplanes obtained in Section 4.5. To ensure this fact, we must examine the summability of the series associated with $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$. Such a convergence shall imply a modulus of continuity that takes the form of a sum, associated with the products $\Pi_{i=1}^n \mu_i$, which ultimately yields a proof of Theorem 11.

Proof of Theorem 11.

The algorithm employed to craft the sequence $(\mu_n)_{n\in\mathbb{N}}$ is key in the proof. There are two possibilities:

Either the sequence stabilizes for some $k_0 \geq 2$, that is

$$\mu_{k_0} = \mu_{k_0+1} = \mu_{k_0+2} = \cdots$$

or else for infinitely many k's, there holds $\mu_k < \mu_{k+1}$. And when this happens:

$$\frac{\prod_{i=1}^{k+1}\mu_i}{r^{k+1}}\sigma\left(\left[\prod_{i=1}^{k+1}\mu_i\right]c_{k+1}\right) = 1. \tag{4.26}$$

The former case falls into a classical setting, for which the convergence analysis yields in fact local $\mathcal{C}^{1,\tau}$–regularity of solutions, for some $0 < \tau < \beta$.

Let us now investigate the latter case. Readily from (4.26) one obtains

$$\sigma_{k+1}\left(c_{k+1}\right) = 1, \quad \Longleftrightarrow \quad \frac{\prod_{i=1}^{k+1}\mu_i}{r^{k+1}}\sigma\left(\prod_{i=1}^{k+1}\mu_i \cdot c_{k+1}\right) = 1,$$

which yields

$$\begin{aligned}\prod_{i=1}^{k+1}\mu_i &= \frac{1}{c_{k+1}}\sigma^{-1}\left(\frac{r^{k+1}}{\prod_{i=1}^{k+1}\mu_i}\right)\\ &\leq \frac{\sigma^{-1}\left(\theta^{k+1}\right)}{c_{k+1}}.\end{aligned} \tag{4.27}$$

Estimate (4.20) combined with estimate (4.27) shows the sequence

$$(\tau_k)_{k\in\mathbb{N}} := \left(\prod_{i=1}^{k}\mu_i\right)_{k\in\mathbb{N}}$$

is summable and its ℓ_1 norm is bounded by $\sum\limits_{i=1}^{\infty}\sigma^{-1}(\theta^i)$.

Therefore, it follows from (4.23) and (4.24) that $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are Cauchy sequences. That is, there exist a real number A_∞ and a vector B_∞

such that

$$A_n \longrightarrow A_\infty \qquad \text{and} \qquad B_n \longrightarrow B_\infty.$$

Set $\ell_\infty(x) := A_\infty + B_\infty \cdot x$. Observe also

$$|A_\infty - A_n| \leq C\sum_{i=n}^{\infty} \tau_i r^n \qquad \text{and} \qquad |B_\infty - B_n| \leq C\sum_{i=n}^{\infty} \tau_i.$$

For any $0 < \rho \ll 1$ let $n \in \mathbb{N}$ be such that

$$r^{n+1} < \rho \leq r^n.$$

We then estimate

$$\begin{aligned}\sup_{x\in B_\rho} |u(x) - \ell_\infty(x)| &\leq \sup_{x\in B_{r^n}} |u(x) - \ell_n(x)| + \sup_{x\in B_{r^n}} |\ell_n(x) - \ell_\infty(x)| \\ &\leq C\tau_n r^n + C\left(\sum_{i=n}^{\infty} \tau_i\right) r^n \\ &\leq \frac{1}{r} C \left[\tau_n + \sum_{i=n}^{\infty} \tau_i\right] \rho \\ &\leq \left(C\sum_{i=n}^{\infty} \tau_i\right) \rho.\end{aligned}$$

Finally, set

$$\gamma(t) := C \sum_{i=\lfloor \ln t^{-1} \rfloor}^{\infty} \tau_i,$$

where $\lfloor M \rfloor :=$ the biggest integers that is less than or equal to M. Since $\tau_i \in \ell_1$, $\gamma(t)$ us indeed a modulus of continuity. We have

$$\sup_{x\in B_\rho} |u(x) - u(0) - Du(0) \cdot x| \leq \gamma(\rho)\rho,$$

and the proof of Theorem 11 is finally complete. ■

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A
Notations

In what follows we list some notations used throughout the thesis.

d	stands for dimension of the space.
B_1	is a unity ball.
B_r	represents the ball of radius r.
$\mathcal{C}(B_1)$	denotes the space of continuous functions in B_1.
$\mathcal{C}_c^\infty(B_1)$	is the space of smooth functions with compact support in B_1.
$\mathcal{S}(d)$	is the space of real $d \times d$ symmetric matrices.
F	is a (λ, Λ)-elliptic operator.
$\mathscr{F}$	is the degenerate elliptic operator.
$\mathcal{M}^{\pm}_{\lambda,\Lambda}$	Pucci's extremal operators.
F_{ij}	represents the derivative of $F(M)$ with respect to the entry m_{ij} of $M \in$
(u, m)	denotes a solution to MFG system.
(u^*, m^*)	is a minimizing solution to MFG system.
I	stands for the functional with Hessian-dependent Lagrangian.
$\bar{I}$	is the relaxed functional.
Γ_F	is the convex envelope of F.
$\operatorname{co}(X)$	stands for the convex hull set.
σ	is a modulus of continuity.
$c_0(X)$	is the space of the sequences $(x_j)_{j=1}^{\infty}$ such that $\|x_j\|_X \to 0$.
$\mathscr{H}_{x_0}$	denotes the tangent hyperplane.
Σ	denotes a family of laws of degeneracy.
$\Xi_{\varepsilon,\lambda,\Lambda,\Sigma}$	stands for the set of all continuous functions $u \in \mathcal{C}(\mathcal{B}_\infty)$ such that $\|u\|_\infty$
$\mathscr{N}_{\tau(\varepsilon)}\left(\mathcal{C}^{1,\beta}(B_{1/2})\right)$	is the $\tau(\varepsilon)$—neighborhood of $\mathcal{C}^{1,\beta}(B_{1/2})$.
$\mathfrak{S}$	is a set of non-collapsing moduli of continuity.