Pontifícia Universidade Católica
do Rio de Janeiro

Makson Sales Santos

Mostly regularity theory: interfaces and free boundaries

Tese de Doutorado

Thesis presented to the Programa de Pós–graduação em Matemática of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor : Prof. Edgard Almeida Pimentel
Co-advisor: Prof. Eduardo Vasconcelos Oliveira Teixeira

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To my mother Ilma (in memoriam), with love.

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Abstract

Santos, Makson Sales; Pimentel, Edgard (Advisor); Teixeira, Eduardo (Co-Advisor). **Mostly regularity theory: interfaces and free boundaries**. Rio de Janeiro, 2020. 71p. Tese de doutorado – Departamento de Matemática, Pontifíccia Universidade Católica do Rio de Janeiro.

This thesis focuses on two classes of problems. Firstly, we examine fully nonlinear equation degenerating as a power of the gradient. The interface along which ellipticity collapses introduces substantial difficulties in the analysis and affects the regularity of the solutions. Through methods in harmonic analysis and measure theory we produce a geometric analysis of the problem, which leads to estimates in Sobolev spaces. Furthermore, our findings set an important open problem in the literature, namely: the H lder-continuity for the gradient of solutions in the presence of unbounded source terms. The second part of the thesis focuses on a free transmission problem driven by fully nonlinear operators. On this topic, our results include the optimal regularity of the solutions and an analysis of the associated free boundary.

Keywords

Regularity; Degenerate Equations; Free Boundary; Approximation Methods; Viscosity.

Resumo

Santos, Makson Sales; Pimentel, Edgard; Teixeira, Eduardo. **Teoria de regularidade: interfaces e fronteiras livres** . Rio de Janeiro, 2020. 71p. Tese de Doutorado – Departamento de Matemática, Pontifíccia Universidade Católica do Rio de Janeiro.

Nesta tese estudamos duas classes de problemas. A primeira delas diz respeito a uma equação completamente não-linear que degenera como uma potência do gradiente. A presença desta interface afeta a elipticidade do sistema e produz redução da regularidade. Combinando técnicas da análise harmônica com métodos da teoria da medida, desenvolvemos uma análise tangencial que produz resultados de regularidade para as soluções em espaços de Sobolev. Como consequência, nossos resultados implicam estimativas em espaços de Hölder para o gradiente das soluções, desconhecidas na literatura no caso de termos de fonte não-limitados. A segunda parte trata de um problema de transmissão livre, governado por operadores completamente não-lineares. Neste caso, obtemos regularidade ótima para as soluções, assim como informações sobre a fronteira livre associada.

Palavras-chave

Regularidade; Equações Degeneradas; Fronteira Livre; Métodos de Aproximação; Viscosidade.

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1
Introduction

This thesis reports regularity results for models governed by fully nonlinear partial differential equations (PDE); it comprises two classes of developments. Firstly, we focus on fully nonlinear elliptic equations degenerating as a power of the gradient. In this setting, we establish results on the regularity of the solutions in Sobolev spaces. In the second part of the thesis, we study a free transmission problem.

The fundamental question underlying both topics is the relevance of (non-physical) free boundaries for the regularity of the structures associated with a given equation. In the case of diffusions degenerating as a power of the gradient, the set $\{Du = 0\}$ is of paramount importance. In fact, since ellipticity vanishes at those points, this set is responsible for entailing upper bounds on the regularity of the solutions. On the other hand, when studying free transmission problems, the discontinuity of the diffusion is subject to the geometry of the set $\partial\{u > 0\} \cup \partial\{u < 0\}$.

In the first part of this work, we consider a degenerate fully nonlinear problem of the form

$$|Du|^q F(D^2u) = f(x) \ \text{ in } \ B_1, \tag{1.1}$$

where $q > 0$, $f \in \mathcal{C}(B_1) \cap L^p(B_1)$, with $p > d$ and $F : S(d) \to \mathbb{R}$ is a uniformly elliptic operator. We establish new interior estimates, in fractional Sobolev spaces $W^{\sigma,p(q+1)}_{loc}(B_1)$ where

$$\sigma = \left(\frac{q+2}{q+1}\right)^- .$$

We argue through geometric tangential methods; see, for instance, [59], [52], and [60]. See also [21].

Our goal is to study *Sobolev* regularity results for solutions of (1.1). Notice C^2-estimates are not expected, even in the case of F convex; in fact were f bounded, $C^{1,\frac{1}{1+q}}$-regularity would be sharp in this case, see [2]. Therefore, estimates for the solutions in Sobolev spaces are of pivotal importance. The main idea in [21], is to obtain a suitable decay rate of the set of points where the solution can not be touched by a paraboloid either from above or from bellow.

In the degenerate setting, however, one needs to craft new barriers as to properly gauge the geometric complexity of the graph of u. We introduce the notion of $\mathcal{C}^{1,\alpha}$ -ones which comprises functions of the form:

$$\psi(x) = \ell(x) \pm \frac{M}{2}|x-\zeta|^{1+\alpha},$$

where M is a positive constant and $\ell(\,\cdot\,)$ is an affine function. We name these functions $C^{1,\alpha}$*-cones of opening* M *and vertex* ζ. In this context we extend the Sobolev-estimates in [21] to a degenerate fully nonlinear setting. This is the content of our first main result.

Theorem 1 (Regularity in Sobolev spaces) *Let* $u \in \mathcal{C}(B_1)$ *be a viscosity solution to* (1.1). *Suppose that A1-A3, to be determinate later, hold true. Then* $u \in W^{\sigma,p(q+1)}_{loc}(B_1)$, *for every*

$$\sigma < 1 + \frac{1}{q+1}$$

In addition, there exists a positive constant $C > 0$ *such that*

$$\|u\|_{W^{\sigma,p(q+1)}(\bar{B}_{1/2})} \leq C.$$

Note the q-degeneracy prevents the analysis from accessing information on the Hessian of the solutions. Instead, we control the integrability of a fractional derivative of order $\sigma < 2$. An important consequence of the former result is the Hölder regularity of the gradient for the solutions to (1.1) in the presence of source terms in $L^p(B_1)$, for $p > d$. We state it what follows as a theorem.

Theorem 2 (Almost-sharp $C^{1,\beta}$-regularity) *Let* $u \in \mathcal{C}(B_1)$ *be a normalized viscosity solution of* (1.1). *Suppose A1-A3, to be detailed further, are in force. Then* $u \in \mathcal{C}^{1,\beta}_{loc}(B_1)$ *for all*

$$\beta < \frac{1}{q+1}\left(q+2-\frac{d}{p}\right) - 1.$$

The almost-optimality of the exponent β in Theorem 2 follows from the geometry of (1.1) – encoded in its scaling properties – combined with an explicit example of the form $v(x) \sim |x|^{\frac{1}{q+1}\left(q+2-\frac{d}{p}\right)-1}$.

The second part of this thesis concerns a fully nonlinear free transmission problem of the form

$$F_1(D^2u)\chi_{\{u>0\}} + F_2(D^2u)\chi_{\{u<0\}} = 1 \qquad \text{in} \qquad \Omega^+(u) \cup \Omega^-(u), \tag{1.2}$$

where F_1, $F_2 : \mathcal{S}(d) \to \mathbb{R}$ are (λ, Λ)-elliptic operators, $\Omega^+(u) := \{u > 0\}$ and $\Omega^-(u) := \{u < 0\}$. We prove optimal regularity results for the strong solutions to (1.2) and examine the associated free boundary. In particular, we prove that solutions are locally of class $\mathcal{C}^{1,1}$ and establish non-degeneracy of the free interface. The latter result unlocks the analysis of global solutions. Finally, we focus on the set of non-degenerate points of the free boundary and show that it is, locally, the graph of a $\mathcal{C}^{1,1}$ function.

We study $W^{2,d}$-strong solutions to (1.2). Inspired by ideas firstly put forward in [34], we notice that a $W^{2,d}$-solution to (1.2) solves

$$G(D^2u) = g \qquad \text{in} \qquad B_1$$

in the L^d-sense, where $g \in L^\infty(B_1)$. If we suppose either F_1 or F_2 to be convex, we obtain Hessian-regularity in BMO-spaces. See [23] and [53].

In addition, by requiring both operators to be convex and supposing they are positively homogeneous of degree one, we produce quadratic growth for the solutions. It follows from a dyadic analysis combined with the maximum principle. The argument relies on a scaling strategy, using the L^∞-norms of the solutions as a normalization factor. This machinery was introduced in [24] in the context of an obstacle problem driven by the Laplacian. In [42] the authors take this perspective to the fully nonlinear setting and develop a fairly complete analysis of the obstacle problem governed by fully nonlinear equations. We also refer the reader to [41].

The quadratic-growth results developed in [24] and [42] rely on a smallness condition on the density of the region where solutions are negative.

A further scaling argument – depending on the square of the distance to the free boundary – is capable of relating B_1 with each connected component associated with the transmission problem. This fact extrapolates regularity information for u; namely, we prove that strong solutions to (1.2) are of class $\mathcal{C}^{1,1}$ in $B_{1/2}$. This is the content of our next result.

Theorem 3 (Regularity of the solutions) *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (1.2)*. Suppose A4-A7, to be detailed further, hold true. Suppose further that $V_r(x,u) < C_0$ for every $x \in \partial(\Omega^+(u) \cup \Omega^-(u)) \cap B_{1/2}$, for some $C_0 > 0$. Then, $u \in \mathcal{C}^{1,1}_{loc}(B_1)$ and there exists a universal constant $C > 0$ such that*

$$\left\| D^2u \right\|_{L^\infty(B_{1/2})} \leq C.$$

After examining the regularity of the solutions, we turn our attention to the free transmission interface. We start our analysis with a non-degeneracy result. Very much based on the maximum principle, it follows along the same lines put forward in [42] and [34]. The non-degeneracy property combines with Theorem 3 to control quadratically the growth of the solutions from above and from below.

A further consequence of non-degeneracy concerns global solutions to (1.2); this is the content of our second main result.

Theorem 4 (Characterization of global solutions) *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (1.2) *in $\mathbb{R}^d$. Suppose A4-A8, to be detailed further, are in force. Suppose further there exists $\varepsilon_0 > 0$ such that*

$$\frac{MD\left((B_1 \setminus \Omega) \cap B_r(x)\right)}{r} > \varepsilon_0,$$

where $0 < r \ll 1$ and $x \in \partial\Omega$. Then u is a half-space solution. That is, up to a rotation,

$$u(x) = \frac{\gamma[(x_1)_+]^2}{2} + C,$$

where $C \in \mathbb{R}$ and $\gamma \in (1/\Lambda, 1/\lambda)$ is such that either $F_1(\gamma e_1 \otimes e_1) = 1$ or $F_2(\gamma e_1 \otimes e_1) = 1$.

Finally we examine the regularity of the free boundary. Our analysis focuses on the *non-degenerate points*, see [40]. We consider the set

$$\mathcal{N}(u) := \left\{ x \in B_1 \mid u(x) = 0 \text{ and } \limsup_{z \to x} \frac{|u(z)|}{|x - z|} > 0 \right\},$$

and examine its geometric properties. We prove the following:

Theorem 5 (Regularity of the free boundary) *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (1.2)*. Then $\mathcal{N}(u)$ is, locally, a graph of class $\mathcal{C}^{1,1}$. In addition, there exists a universal constant $C > 0$ such that for all $z \in \mathcal{N}(u)$, we have*

$$|\nu_x - \nu_y| \leq C|x - y|$$

for every x, $y \in B_r(z) \cap \Sigma(u)$ and every $0 < r \ll 1$.

The findings reported in Theorem 5 are related to recent developments concerning nodal sets for broken quasilinear equations; see [40]. We mention that a complete result on the regularity of the free boundary as well as the analysis of the singular set are not included here; see, for instance, [42, 54].

The remainder of this thesis is organized as follows. Chapter 2 presents some results and sets the notation used throughout the thesis. In Section 2.1, we introduce the main assumptions under which we work. Section 2.2, recalls some definitions and collects important tools used in both the first and the second parts of this thesis. Chapter 3 accounts to our analysis of degenerate fully nonlinear equation. In the first section, we briefly present an overview on degenerate fully nonlinear equations. The remaining sections detail the proof of Theorem 1. Chapter 4 reports our findings on the regularity theory for the free transmission problem driven by fully nonlinear operators.

2
Preliminary material and main assumptions

In this chapter we collect elementary notions, auxiliary results and the hypotheses under which we work.

2.1
Main assumptions

In what follows, we detail the main hypotheses used in the first part of this work.

A 1 (Uniform ellipticity) *The operator $F : S(d) \longrightarrow \mathbb{R}$ is (λ, Λ)-uniformly elliptic. That is, for $0 < \lambda \leq \Lambda$, it holds*

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\|,$$

for every M, $N \in \mathcal{S}(d)$, $N \geq 0$.

The next assumption concerns the regularity of F. As in [21], in order to establish Theorem 1, we need F to satisfy a $C^{1,1}$-estimate in the homogeneous setting. Hence, we assume the following:

A 2 ($C^{1,1}$-estimates) *We suppose that $F = 0$ has $C^{1,1}$-interior estimates; i.e., there exists a constant $C > 0$ such that if h is a solution to $F(D^2h) = 0$ in B_1 then*

$$\|h\|_{C^{1,1}(\overline{B}_{1/2})} \leq C.$$

Finally, we impose integrability conditions on the source term f.

A 3 (Integrability of the source term f) *The source term $f : B_1 \to \mathbb{R}$ is such that $f \in \mathcal{C}(B_1) \cap L^p(B_1)$, with $p > d$. In addition, there exists a constant $C > 0$ for which*

$$\|f\|_{L^p(B_1)} \leq C.$$

In the sequel, we present the assumptions used in the second part of this work.

A 4 (Uniform ellipticity for F_i) *For $i = 1, 2$, we suppose the operator $F_i : \mathcal{S}(d) \to \mathbb{R}$ to be (λ, Λ)-uniformly elliptic. That is, for $0 < \lambda \leq \Lambda$, it holds*

$$\lambda\|N\| \leq F_i(M + N) - F_i(M) \leq \Lambda\|N\|,$$

for every M, $N \in \mathcal{S}(d)$, $N \geq 0$, and $i = 1, 2$. We also suppose that $F_i(0) = 0$.

When deriving an elliptic equation satisfied by the strong solutions to (1.2) we suppose the operators F_1 and F_2 are *comparable*. This is the content of the next assumption.

A 5 (Comparable diffusions) *We suppose the operators F_1 and F_2 are comparable in the L^∞-topology. I.e., there exists $C > 0$ such that*

$$\sup_{M\in\mathcal{S}(d)} |F_1(M) - F_2(M)| \leq C.$$

The former assumption is instrumental in proving that u solves an elliptic equation *with right-hand side in L^∞*. We stress that A5 does not require F_1 and F_2 to be close to each other; i.e., the constant $C > 0$ in the assumption does not satisfy a smallness regime.

The next assumption concerns homogeneity of degree 1. It plays a major role in the regularity of the solutions. The argument towards quadratic growth in [24] uses the linearity of the Laplacian operator. In [42] the authors notice that in the fully nonlinear case the condition that parallels linearity is the homogeneity of degree 1.

A 6 (Homogeneity of degree one) *We suppose F_i to be homogeneous of degree one for $i = 1, 2$; that is, for every $\tau \in \mathbb{R}$ and $M \in \mathcal{S}(d)$, we have*

$$F_i(\tau M) = \tau F_i(M),$$

for $i = 1, 2$.

Our next assumption concerns the convexity of the operators F_i.

A 7 (Convexity of the operator F_i) *We suppose the operator $F_i : \mathcal{S}(d) \to \mathbb{R}$ to be convex, for $i = 1, 2$.*

The next assumption is required in the study of non-degeneracy and to characterize global solutions.

A 8 *We suppose $\{Du \neq 0\} \subset \Omega(u) = \Omega^+(u) \cup \Omega^-(u)$.*

In the next section we collect a number of definitions and auxiliary results used throughout this thesis.

2.2
Preliminary notions and results

We start gathering some notation and preliminaries regarding the analysis of degenerate fully nonlinear elliptic equations.

For $r > 0$ and $x_0 \in \mathbb{R}^d$, $B_r(x_0)$ denotes the open ball of radius r centered at x_0, whereas B_r denotes $B_r(0)$. Similarly, $Q_r(x)$ stands for the open cube with side r and center x_0, i.e.,

$$Q_r(x_0) := \left\{ x \in \mathbb{R}^d : |x - x_0|_\infty < \frac{r}{2} \right\},$$

where $|x|_\infty := \max\{|x_1|, \dots |x_d|\}$.

For $M \in S(d)$ we define the Pucci extremal operators to be

$$\mathcal{M}^+(M) := \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} (-\operatorname{Tr}(AM))$$

and

$$\mathcal{M}^-(M) := \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} (-\operatorname{Tr}(AM)),$$

where $\mathcal{A}_{\lambda,\Lambda} := \{A \in S(d) : \lambda I \leq A \leq \Lambda I\}$. It is important to note that $\mathcal{M}^+(M) = -\mathcal{M}^-(-M)$. With this definition we can rewrite the uniformly elliptic of an operator F as

$$\mathcal{M}^-(N) \leq F(M+N) - F(M) \leq \mathcal{M}^+(N),$$

for any $M, N \in S(d)$. In the sequel we introduce the definition of viscosity solution.

Definition 1 (Viscosity solution) *Let $F \in \mathcal{C}(\mathcal{S}(d) \times \mathbb{R}^d \times \mathbb{R} \times B_1, \mathbb{R})$ be a uniformly elliptic operator. We say that $u \in \mathcal{C}(B_1)$ is a viscosity subsolution to $F = 0$ if for every $x_0 \in B_1$ and every $\phi \in \mathcal{C}^2(B_1)$ such that $u - \phi$ attains a local maximum at x_0, we have*

$$F(D^2\phi(x_0), D\phi(x_0), u(x_0), x_0) \leq 0.$$

We say that $u \in \mathcal{C}(B_1)$ is a viscosity supersolution to $F = 0$ if for every $x_0 \in B_1$ and every $\phi \in \mathcal{C}^2(B_1)$ such that $u - \phi$ attains a local minimum at x_0, we have

$$F(D^2\phi(x_0), D\phi(x_0), u(x_0), x_0) \geq 0.$$

If $u \in \mathcal{C}(B_1)$ is a viscosity sub and a supersolution to $F = 0$ we say u is a viscosity solution to $F = 0$ in B_1.

For $g \in L^1_{loc}(\mathbb{R}^d)$ the maximal function of g is defined by

$$m(g)(x) := \sup_{r>0} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |g(y)| dy.$$

Recall that the maximal operator satisfies

$$|\{x \in \mathbb{R}^d : m(g)(x) \geq t\}| \leq \frac{C}{t} \|g\|_{L^1(\mathbb{R}^d)}, \;\; \forall\, t > 0. \tag{2.1}$$

See [21] for further details. Next, we detail the definition of convex envelope and contact set for a continuous function defined on a domain $\Omega \subset \mathbb{R}^d$.

Definition 2 *Let $\Omega \subset \mathbb{R}^d$ be an open set and $v \in \mathcal{C}(\Omega)$. The convex envelope of v in Ω is defined by*

$$\Gamma(v)(x) := \sup_L \{L(x); L \;\leq\; v \; in \, \Omega, \; L \; is \; affine\}.$$

The contact set of v is given by

$$\{x \in \Omega \; ; \; v(x) = \Gamma(v)(x)\}.$$

The next lemma ensures the existence of a barrier function suitable for our analysis of degenerate fully nonlinear problems. See [36] for details.

Lemma 1 *Given $\varepsilon_0 > 0$ there exits a smooth function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$, such that*

1. *$\varphi \geq 0$ in $\mathbb{R}^d \backslash B_{2\sqrt{d}}$;*
2. *$\varphi \leq -2$ in Q_3;*
3. *$\varphi \geq M_b$ in $\mathbb{R}^d$;*
4. *$|D\varphi| \leq \varepsilon_0$ in $\mathbb{R}^d$;*
5. *$\mathcal{M}^-(D^2\varphi) + C\xi \geq 0$ in $\mathbb{R}^d$,*

where $\xi : \mathbb{R}^d \longrightarrow [0, 1]$ is a continuous function with support in $\overline{Q}_1$, C is a positive universal constant and M_b is a positive constant.

We proceed with the definition of $\mathcal{C}^{1,\frac{1}{1+q}}$-cone; these functions play the role of the paraboloids in the uniformly elliptic case.

Definition 3 ($\mathcal{C}^{1,\frac{1}{1+q}}$-cone of opening M and vertex x_0) *We say that ψ is a convex $C^{1,\alpha}$-cone of opening M and vertex x_0 if*

$$\psi(x) \;=\; L(x) \;+\; \frac{M}{2}|x \;-\; x_0|^{1+\alpha},$$

where M is a positive constant, and $L(x)$ is an affine function. Similarly, ψ is a concave $C^{1,\alpha}$-cone of opening M and vertex x_0 if

$$\psi(x) \,=\, L(x) \,-\, \frac{M}{2}|x \,-\, x_0|^{1\,+\,\alpha},,$$

where M is a positive constant, and $L(x)$ is an affine function.

The sets collecting those points that can be touched by a $\mathcal{C}^{1,\alpha}$-cone of certain opening play a pivotal role in our analysis, since their measure yields information on the integrability of the solutions.

Definition 4 *Let $O \subset \Omega$ be an open subset, $0 < \tau_0 < \frac{diam(O)}{5}$ and $M > 0$. We define*

$$\underline{G}_M(u, O) = \underline{G}_M(O)$$

as the set of all points $x_0 \in O$ such that there exists a concave $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 satisfying the following two properties:

1. *$u(x_0) = \psi(x_0)$;*
2. *$u(x) > \psi(x)$ for all $x \in B_{\tau_0}(x_0)$.*

Likewise we define:

$$\overline{G}_M(u, O) = \overline{G}_M(O)$$

as the set of all points $x_0 \in O$ such that there exists a convex $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 satisfying the following two properties:

1. *$u(x_0) = \psi(x_0)$;*
2. *$u(x) < \psi(x)$ for all $x \in B_{\tau_0}(x_0)$.*

Finally

$$G_M(O) = \underline{G}_M(O) \cap \overline{G}_M(O).$$

Next, we note a monotonicity property related to the sets G_M. Let $M_1 > M_2$ and take $x_1 \in \underline{G}_{M_2}(u, O)$. By definition, there exits a concave $C^{1,\frac{1}{q+1}}$-cone of the form

$$\psi(x) = L(x) - \frac{M_2}{2}|x \,-\, x_1|^{1+\frac{1}{q+1}},$$

where L is an affine function, such that $\psi(x_1) = u(x_1)$ and $\psi(x) < u(x)$, for all $x \in B_{\frac{diam(O)}{10}}(x_1)$. Hence,

$$\begin{array}{rcl} u(x) & > & L(x) - \frac{M_2}{2}|x - x_1|^{1+\frac{1}{q+1}} \\ & > & L(x) - \frac{M_1}{2}|x - x_1|^{1+\frac{1}{q+1}} \\ & = & \tilde{\psi}(x), \end{array}$$

for all $x \in B_{\frac{diam(O)}{10}}(x_1)$. Notice that,

$$u(x_1) = \psi(x_1) = \tilde{\psi}(x_1).$$

We conclude that $x_1 \in \underline{G}_{M_1}(u, O)$. It follows that

$$\underline{G}_{M_2}(u, O) \subset \underline{G}_{M_1}(u, O).$$

Similarly, we have

$$\overline{G}_{M_2}(u, O) \subset \overline{G}_{M_1}(u, O).$$

Therefore,

$$G_{M_2}(u, O) \subset G_{M_1}(u, O).$$

Definition 5 *Let $O \subset \Omega$ be an open subset, $0 < \tau_0 < \frac{diam(O)}{5}$ and $M > 0$. We define*

$$\underline{A}_M(u, O) = \underline{A}_M(O) = O \backslash \underline{G}_M(u, O).$$

Similarly

$$\overline{A}_M(u, O) = \overline{A}_M(O) = O \backslash \overline{G}_M(u, O).$$

Finally

$$A_M(u, O) = A_M(O) = O \backslash G_M(u, O).$$

Next we define the $\mathcal{C}^{1,\frac{1}{q+1}}$-aperture function. This structure is directly related to the integrability of solutions to (1.1).

Definition 6 ($\mathcal{C}^{1,\frac{1}{q+1}}$-aperture function) *For $x \in B_{1/2}$ we define*

$$\theta(x) := \theta(u, B_{1/2})(x) = \inf\{M : x \in G_M(B_{1/2})\} \in [0, \infty].$$

Lemma 2 *Let $g : \Omega \to \mathbb{R}$ be a nonnegative and measurable function. Define $\mu_g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as*

$$\mu_g(t) = |\{x \in \Omega : g(x) > t\}|, \quad t > 0.$$

Let $\eta > 0$ and $M > 1$ be constants. Then, for $0 < p < \infty$,

$$g \in L^p(\Omega) \Longleftrightarrow \sum_{k \geq 1} M^{pk} \mu_g(\eta M^k) = S < \infty$$

and

$$C^{-1} S \leq \|g\|^p_{L^p(\Omega)} \leq C(|\Omega| + S),$$

where $C > 0$ is a constant depending only on η, M and p.

The function μ_g defined in Lemma 2 is known as *distribution function of g*. Next, we recall a corollary of the Calderón-Zygmund decomposition. See [22, Lemma 4.2]. Let Q_1 be the unit cube and split it into 2^d cubes of half side. Then, split each one of these 2^d cubes and iterate the process. The cubes obtained in this way are called dyadic cubes.

If Q is a dyadic cube different from Q_1, we say that $\tilde{Q}$ is the predecessor of Q if the latter is one of the 2^d cubes obtained by dividing $\tilde{Q}$.

Lemma 3 (Calderón-Zygmund decomposition) *Let $A \subset B \subset Q_1$ be measurable sets and $0 < \delta < 1$ such that*

(a) $|A| \leq \delta$;

(b) if Q is a dyadic cube such that $|A \cap Q| > \delta |Q|$, then $\tilde{Q} \subset B$, where $\tilde{Q}$ is the predecessor of Q.

Then

$$|A| \leq \delta |B|.$$

Now, we are capable of stating a connection between the distribution function μ_θ and the measure of the sets A_M. In fact,

$$\mu_\theta(t) \leq |A_t(B_{1/2})|;$$

therefore, our interest relies on the summability of

$$\sum_{k \leq 1} M^{p(q+1)k} |A_{M^k}(B_{1/2})|$$

and on the pass-through mechanism transmitting information from *theta* to the solutions u.

In the sequel we state an Aleksandroff-Bakelman-Pucci estimate designed for degenerate equations. We refer the reader to [36] for its proof.

Proposition 1 *Let $G : \Omega \times \mathbb{R}^d \backslash B_{M_F} \times S(d)$, for some $M_F \geq 0$. Suppose that G is continuous and (degenerate) elliptic, i.e. for all $x \in \Omega$, $p \in \mathbb{R}^d$ and $M, N \in S(d)$,*

$$M \leq N \;\Rightarrow\; G(x,p,N) \leq G(x,p,M). \tag{2.2}$$

In addition, suppose that G satisfies the following condition:

$$\begin{cases} |p| \geq M_F \\ G(x,p,M) \geq 0. \end{cases} \quad \Rightarrow \quad \mathcal{M}^+(M) + \gamma(x)|p| + g(x) \geq 0. \tag{2.3}$$

If u is a viscosity supersolution to

$$G(x, Du, D^2u) = 0 \quad in \;\; B_r,$$

then

$$\sup_{B_r} u^- \leq \sup_{\partial B_r} u^- + Cr\left(M_F + \left(\int_{B_r \cap \{u+M_\partial = \Gamma(u)\}} (g^+)^d\right)^{1/d}\right),$$

where $M_\partial = sup_{\partial B_r} u^-$, $\Gamma(u)$ is the convex envelope of $\min(u+M_\partial, 0)$ extended by 0 on B_{2r} and C is a positive constant (only) depending on $\|\gamma\|_{L^d(B_r)}$, d.

The importance of an ABP-type of estimate to our argument relies on the switch pointwise-to-measure control. Once we produce a lower bound for the measure of the contact set of solutions, the geometry of $\mathcal{C}^{1,\alpha}$-cones relates such lower bound with the aperture function θ.

In what follows, we define the fractional Sobolev spaces used in this thesis. We refer to [32, Chapter 2] for further details.

Definition 7 *Let $s \in (0,1)$. For any $p \in [1, +\infty)$ we define $W^{s,p}(\Omega)$ as follows*

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^P(\Omega \times \Omega) \right\}.$$

The norm in $W^{s,p}(\Omega)$ is given by

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dxdy \right)^{\frac{1}{p}},$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} dxdy \right)^{\frac{1}{p}}$$

is the so-called Gagliardo seminorm of u.

Definition 8 *If $s > 1$, we write $s = m+\gamma$, where m is a integer and $\gamma \in (0,1)$. In this case we define,*

$$W^{s,p}(\Omega) := \{u \in W^{m,p}(\Omega) : D^\alpha u \in W^{\gamma,p}(\Omega) \text{ for any } \alpha \text{ s.t. } |\alpha| = m\}.$$

The norm in $W^{s,p}(\Omega)$, when $s > 1$, is defined by

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\gamma,p}(\Omega)} \right)^{\frac{1}{p}}.$$

When $p = 2$, we write $W^{s,2}(\Omega) := H^s(\Omega)$. The space $W^{s,p}_0(\Omega)$ consists of all functions $u \in W^{s,p}(\mathbb{R}^d)$ such that $u = 0$ in $\mathbb{R}^d \setminus \Omega$. In addition, $W^{-s,p}(\Omega)$ denotes the dual space of $W^{s,p}(\Omega)$.

Next, we collect some facts concerning the Fourier transform and its relationship with the fractional Laplacian operator. The Schwartz space will be denoted by $\mathcal{S}$. We refer the reader to [32, Chapter 3]. Standard density arguments allow us to work in less regular spaces such as $L^2(\mathbb{R}^d)$.

Definition 9 (Fourier transform) *Let $u \in \mathcal{S}$. The Fourier transform of u is defined by*

$$\mathcal{F}u(\zeta) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\zeta\cdot x} u(x)dx.$$

It is well known that $\mathcal{F}$ is a isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. See for instance [33, Section 4.3.1, Theorem 1].

Next, we define

Definition 10 *Let $s \in (0,1)$. Then, for any $u \in \mathcal{S}$, we define the fractional Laplacian operator as*

$$(-\Delta)^s u(x) := -\frac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+2s}} dy$$

The interaction between the Fourier transform and the fractional Laplacian operator is a well established fact and we state as follows.

Proposition 2 *Let $s \in (0,1)$ and let $(-\Delta)^s : \mathcal{S} \to L^2(\mathbb{R}^d)$ be the fractional Laplacian operator. Then, for any $u \in \mathcal{S}$,*

$$(-\Delta)^s u = \mathcal{F}^{-1}(|x|^{2s}(\mathcal{F}u)) \text{ for all } x \in \mathbb{R}^d.$$

Proof. See [32, Proposition 3.3]. ■

Definition 11 *For $s \in \mathbb{R}$ we define*

$$\bar{H}^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |x|^{2s}) |\mathcal{F}u(x)|^2 dx < \infty \right\}.$$

Proposition 3 *Let $s \in (0,1)$. Then the fractional Sobolev space $H^s(\mathbb{R}^d)$ coincides with $\bar{H}^s(\mathbb{R}^d)$. In particular, for any $u \in H^s(\mathbb{R}^d)$*

$$[u]^2_{H^s(\mathbb{R}^d)} = 2C(n,s)^{-1} \int_{\mathbb{R}^d} |x|^{2s} |\mathcal{F}u(x)|^2 dx.$$

Proof. See [32, Proposition 3.4]. ■

The next result concerns *local* regularity for the fractional Laplacian equation of order s. See [8, Theorem 1.4] for details.

Theorem 1 *Let $u \in W_0^{s,2}(\bar{\Omega})$ be the unique weak solution to*

$$\begin{cases} (-\Delta)^s u &= f \ \ in \ \Omega \\ u &= 0 \ \ on \ \mathbb{R}^d \setminus \Omega, \end{cases} \tag{2.4}$$

where $\Omega \subset \mathbb{R}^d$ is an arbitrary bounded open set and $s \in (0,1)$. Suppose $f \in W^{-s,2}(\bar{\Omega})$. If $f \in L^p(\Omega)$ with $1 < p < \infty$, then $u \in W^{2s,p}_{loc}(\Omega)$.

The remainder of this chapter fixes some notation and collect elements regarding the second part of the thesis, namely, the fully nonlinear free transmission problem.

We denote by $\Omega^+(u)$ the subset of the unit ball where $u > 0$, whereas $\Omega^-(u)$ stands for the set where $u < 0$. That is,

$$\Omega^+(u) := \{x \in B_1 \mid u(x) > 0\} \quad \text{and} \quad \Omega^-(u) := \{x \in B_1 \mid u(x) < 0\}.$$

When referring to the set where $u \neq 0$ it is convenient to use the notation $\Omega(u) := \Omega^+(u) \cup \Omega^-(u)$. With $\partial\Omega(u)$ we denote the union of the topological boundaries of Ω^+ and Ω^-. I.e.,

$$\partial\Omega(u) := \left(\partial\Omega^+(u) \cup \partial\Omega^-(u)\right) \cap B_1.$$

Also, we denote with $\Sigma(u)$ the set where u vanishes:

$$\Sigma(u) = \{x \in B_1 \mid u(x) = 0\}.$$

Next, we define strong solutions for the free transmission problem

Definition 12 *Let $d < 2p$ and $f \in L^p_{loc}(\Omega)$. A function $u \in W^{2,d}(B_1)$ is a $W^{2,d}$-strong subsolution (respectively, supersolution) of*

$$F(D^2u) = f \ \ in \ \Omega,$$

if

$$F(D^2u) \geq f \ a.e. \ in \ \Omega,$$

(respectively $F(D^2u) \leq f$ a.e. in Ω,). Moreover, u is $W^{2,d}$-strong solution of

$$F(D^2u) = f \ \ in \ \Omega$$

if it is both an $W^{2,d}$-strong subsolution and an $W^{2,d}$-strong supersolution.

Moreover, we recall the notion of L^p-viscosity solution.

Definition 13 *Let $d < 2p$ and $f \in L^p_{loc}(\Omega)$. A function $u \in \mathcal{C}(\Omega)$ is an L^p-viscosity subsolution (supersolution) of*

$$F(D^2u) = f \ \ in \ \Omega,$$

if for all $\phi \in W^{2,p}_{loc}(B_1)$, and point $\hat{x} \in \Omega$ at which $u - \varphi$ has a local maximum (respectively minimum) one has

$$ess \liminf_{x \to \hat{x}} (F(D^2\varphi) - f(x)) \geq 0$$

(respectively $ess \limsup_{x \to \hat{x}} (F(D^2\varphi) - f(x)) \leq 0$). Moreover, u is an L^p-viscosity solution of

$$F(D^2u) = f \ \ in \ \Omega$$

if it is both an L^p-viscosity subsolution and an L^p-viscosity supersolution.

Finally, we introduce the set of non-degenerate points, denoted by $\mathcal{N}(u)$ and defined as

$$\mathcal{N}(u) := \left\{ z \in \Sigma(u) : \limsup_{x \to z} \frac{|u(x)|}{|x - z|} > 0 \right\}.$$

A further condition imposed on the problem regards the subregion $\Omega^+(u)$. To prove quadratic growth of the solutions through the set of methods used in the paper, we consider the quantity

$$V_r(x^*, u) := \frac{\mathrm{vol}\,(B_r(x^*) \cap \Omega^-(u))}{r^d}. \tag{2.5}$$

For ease of notation, $V_r(0,u) =: V_r(u)$. The analysis leading to quadratic growth away from the free boundary supposes $V_r(x^*,u) < C_0$ for some smalls constant $C_0 > 0$ depending on the data of the problem.

We close this section by introducing the notion of thickness. For any set A, we denote by $\mathrm{MD}(A)$ the smallest possible distance between two parallel hyperplanes containing A. We define the thickness of Σ in $B_r(x)$ as

$$\delta_r(u,x) := \frac{\mathrm{MD}(\Sigma(u) \cap B_r(x))}{r}.$$

The thickness δ_r satisfies some elementary properties which we list below. We refer to [49, Chapter 5] for more details.

Proposition 4 *The thickness δ_r satisfies the following properties:*

1. $\delta_1(u_r,0) = \delta_r(u,x)$, *where* $u_r(y) = u(x+ry)/r^2$;
2. *For polynomial global solutions* $P_2 = \Sigma_j a_j x_j^2$, *with* a_j *such that* $G(D^2P_2) = 1$ *we have* $\delta_r(P_2,0) = 0$;
3. *If* u_r *converges to some function* u_0 *then* $\limsup_{r\to 0} \delta_r(u,x_0) \leq \delta_1(u_0,0)$.

The next chapter details our analysis of degenerate fully nonlinear diffusions and puts forward the proof of Theorem 1.

3
Degenerate fully nonlinear equations

In this chapter we detail the proofs of Theorems 1 and 2. The next section provides some context on fully nonlinear equations of the form

$$|Du|^q \, F(D^2u) \; = \; f \qquad \text{in} \;\; B_1. \tag{3.1}$$

3.1
Some context on degenerate fully nonlinear problems

Since the developments in [21] were reported, fully nonlinear equations of the form

$$F(x, \, D^2u) \; = \; f(x) \qquad \text{in } B_1,$$

have been studied by many authors. As a consequence, several results were established concerning the properties of solutions to fully nonlinear operators. More recently, this geometric ideas have been developed into a set of methods and techniques known as *geometric tangential analysis*. See, for instance [3],[4],[61], [63], [62] just to cite a few. We also refer to the surveys [52],[64].

Contrasting the fully nonlinear elliptic case, in the equation (3.1) the diffusion degenerates along an a priori unknown set of critical points $\{|Du| = 0\}$. The degeneracy of the equation is controlled by the exponent q. The larger the value of q more degenerate the equation is. Consequently the smoothing effects on the diffusion become less efficient.

The study of equations of type (3.1) is the subject of a series of papers. In [9] the authors establish fundamental results for equations of the form

$$F(x, Du, D^2u) - g(x, u) \geq (\leq) \;\; 0.$$

Under certain conditions on the operator F, modelled after the p-Laplace operator, they were able to prove a comparison result for subsolutions and supersolutions of $F(x, Du, D^2u) = b(u)$. In addition, they also prove a Liouville type estimate for solutions of

$$-F(x, Du, D^2u) \geq h(x)u^q$$

in $\mathbb{R}^d$.

The study of eigenvalues is the subject of [10] and [11]. Based on [7] the authors extend the definition of the principal eigenvalue. They work with operators of the form

$$G(x, u, Du, D^2u) := F(x, Du, D^2u) + b(x) \cdot Du|Du|^{\alpha} + c(x)|u|^{\alpha}u,$$

where the operator F can be seen as a non-variational extension of the p-Laplacian. For a similar condition see [39]. They prove the existence of non-negative solutions to

$$\begin{cases} G(x, u, Du, D^2u) + \lambda u^{1+\alpha} &= f \;\; \Omega \\ u &= 0 \;\; \partial\Omega \end{cases}$$

and

$$\begin{cases} G(x, \phi, D\phi, D^2\phi) + \overline{\lambda}\phi^{1+\alpha} &= f \;\; \Omega \\ \phi &= 0 \;\; \partial\Omega, \end{cases}$$

where $\lambda < \overline{\lambda}$ and ϕ is of class C^{1-}, i.e., ϕ is α-Hölder continuous for every $\alpha \in (0, 1)$. In addition ϕ is assumed to be locally Lipschitz. The concept of eigenvalue for fully nonlinear operators was generalized in [12]. In that paper, instead of C^2 regularity, the boundary just need to satisfy the uniform exterior cone condition. In addition, they also prove C^{α}-regularity and a maximum principle.

The Aleksandroff-Bakelman-Pucci (ABP) estimate is the object of [36]. In that paper the authors prove the ABP estimate for a class of fully nonlinear elliptic equations of the form

$$F(x, u, Du, D^2u) = 0, \tag{3.2}$$

which can be either degenerate or singular. In addition, they explain how to extend their results to the singular case and for equations of the form

$$F_0(Du, D^2u) + b(x) \cdot Du|Du|^{\alpha}cu|u|^{\alpha} + f_0(x) = 0, \quad x \in \Omega, \tag{3.3}$$

where F is positively homogeneous of order $\alpha \in (-1, 1)$. See [29] for similar estimates and existence results.

In [36] the authors also prove the Harnack inequality for positive solutions of (3.2) in either the singular or the degenerate case. They work under the additional assumption that the operator F is strictly elliptic when the gradient is large. In [30] the authors prove a Harnack inequality to solutions of (3.3), in the singular case, under similar conditions of [36]. See also [13] for similar results in unbounded domains. More recently, Imbert and Silvestre in [38] proved the Harnack inequality for a equation in terms of the Pucci extremal operators, that is more general than particular equations; in fact, their formulation addresses equations holding in regions *where the gradient is large*. In addition, the authors derivate a L^{ε}-estimate for the solutions.

As a central topic in analysis, the regularity theory for fully nonlinear degenerate/singular equations has been studied by many authors in recent years. In [30] the authors work in the singular setting and established C^{α}-estimates for solutions to (3.3). In [36], the authors extend the result in [30] to the degenerate case. In [38], the authors prove Hölder-estimates for solutions of a general equation (involving the Pucci extremal operators) where the gradient is large. Hölder regularity of the gradient of the solutions is the subject of [37]. In that paper the authors proved that solutions to (3.1) are locally of class $C^{1,\alpha}$, for some $0 < \alpha < 1$, provided f is continuous and bounded and $q \geq 0$. The optimal regularity of solutions to (3.1) is given in [2]. Under the additional assumption that F is concave, the authors establish that solutions to (3.1) are of class $\mathcal{C}^{1,\frac{1}{1+q}}$ and this regularity is optimal.

A variable-exponent version of (3.1) is considered in [19]. The authors obtain $\mathcal{C}^{1,\alpha}$-estimates for solutions of

$$|Du|^{q(x)}F(D^2u) = f(x) \quad \text{in} \quad B_1,$$

where q is bounded from below and $\alpha \in (0,1)$ depends on the the L^{∞}-norms of the positive and negative parts of q.

The next section presents a preliminary integrability estimate for the aperture function θ.

3.2 Preliminary estimates for the $\mathcal{C}^{1,\alpha}$-aperture function

In this section we produce an L^{δ}-estimate for the aperture function θ, introduced in Definition 6. This first level of integrability stems from the uniform ellipticity of the operator F and the integrability of the source term f; see A1 and A3. Such estimate is to be refined in a further step of the argument, where geometric arguments build upon A2. To be more precise, we prove the next proposition:

Proposition 5 (L^{δ}-estimate for the aperture function) *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to* (1.1). *Suppose A1 and A3 are in force. Then $\theta \in L^{\delta}(B_1)$, for some $0 < \delta \ll 1$, and there exists a universal constant $C > 0$ such that*

$$\|\theta\|_{L^{\delta}(B_{1/2})} \leq C.$$

The proof of Proposition 5 follows from a few lemmas, exploring the measure of the sets A_M and G_M. The δ-integrability of the aperture function relates to a δ-decay rate for the measure of suitable sets. We start by framing the problem in the context of a bounded domain $\Omega \subset \mathbb{R}^d$ containing balls of d-dependent radii. Furthermore, since we work under A3, scaling arguments allow us to suppose

$$\|f\|_{L^d(B_1)} \leq \delta_0, \tag{3.4}$$

for arbitrary values of $\delta > 0$. We suppose such bounds are available throughout the chapter.

Lemma 4 *Let $u \in \mathcal{C}(B_{6\sqrt{d}})$ be a viscosity solution to* (3.1) *in $B_{6\sqrt{d}}$. Suppose A1 and A3 are in force. Suppose further that Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Then*

$$|\underline{G}_M(u, \Omega) \cap Q_1| \geq 1 - \sigma,$$

where $0 < \sigma < 1$ is a universal constants and $M > 1$ is such that $M = M(\lambda, \Lambda, d, q)$.

Proof. We starting by noticing that $\overline{Q}_1 \subset \overline{Q}_3 \subset B_{2\sqrt{d}}$. Consider the barrier function φ whose existence is ensured by Lemma 1 and set $w := u + 1 + 2\varphi$ in $\overline{B}_{2\sqrt{d}}$. For this choice of w, we have

$$w \geq 0 \quad \text{on} \quad \partial B_{2\sqrt{d}} \qquad \text{and} \qquad \inf_{x \in Q_3} w(x) \leq -2.$$

In addition, w solves

$$G(x, Dw, D^2 w) = 0,$$

where

$$G(x, p, M) := |p - D\varphi|^q F(M - 2D^2\varphi) - f(x).$$

At this point we resort to the ABP estimate as to produce a pointwise-to-measure control. By setting $\gamma \equiv 0$ in Proposition 1, we notice G satisfies (2.2) and (2.3), with

$$g(x) = C\xi(x) + |f(x)|.$$

Hence, by applying Proposition 1 to w we obtain

$$\frac{1}{2c(d)} \leq M_F + |\{w = \Gamma(w)\} \cap Q_1|,$$

provided δ_0 is taken sufficiently small in (3.4). Then, for M_F small enough we have

$$|\{w = \Gamma(w)\} \cap Q_1| \geq 1 - \sigma,$$

with $0 < \sigma < 1$.

Now, we extrapolate information along the contact set tot $\underline{G}_M$. To that end, we show that $(\{w = \Gamma(w)\} \cap Q_1) \subset (\underline{G}_M(\Omega) \cap Q_1)$, for some $M > 1$. Let $x_0 \in \{w = \Gamma(w) \cap Q_1\}$. By the definition of convex envelope, there exists an affine function L such that $L < 0$ on $\partial B_{2\sqrt{d}}$. Recall that $\Gamma(w) < -w^- \leq 0$ in $B_{4\sqrt{d}}$. Hence

$$L \leq \Gamma(w) \leq w = u + 1 + 2\varphi \quad \text{in} \quad B_{2\sqrt{d}},$$

with equalities at x_0. Since $\|D^2\varphi\| \leq C$ in $B_{2\sqrt{d}}$, where C is a universal constant, it follows that there exists a concave $C^{1,\frac{1}{1+q}}$-cone of opening M and vertex x_0

$$\psi(x) = a - \frac{M}{2}|x - x_0|^{1+\frac{1}{1+q}},$$

where $M = M(\lambda, \Lambda, d, q) > 1$ and a is a real number ensuring that

$$\psi \leq L - 1 - 2\varphi \leq u \qquad \text{in} \qquad B_{2\sqrt{d}}, \tag{3.5}$$

with equalities at x_0. Since $L < 0$ and $\varphi \geq 0$ on $\partial B_{2\sqrt{d}}$ we have that $\psi \leq -1$ on $\partial B_{2\sqrt{d}}$. In addition, $\|u\|_{L^\infty}(\Omega) \leq 1$ implies that $\psi(x_0) = u(x_0) \geq -1$. Now, since $x_0 \in B_{2\sqrt{d}}$ and $\{x \in \mathbb{R}^d : \psi(x) \geq -1\}$ is convex, we get $\psi < -1$ in $\mathbb{R}^d \backslash \Omega$.

As a consequence, we find that $\psi \leq u$ in $\mathbb{R}^d \backslash \Omega$. From this and (3.5) we obtain $\psi \leq u$ in Ω; because $\psi(x_0) = u(x_0)$ we get $x_0 \in \underline{G}_M(u, \Omega) \cap Q_1$ and complete the proof. ■

The next result connects the existence of an element $x_1 \in \underline{G}_1(u, \Omega) \cap Q_3$ with the measure of $\underline{G}_M(u, \Omega) \cap Q_1$.

Lemma 5 *Let $u \in \mathcal{C}(B_{6\sqrt{d}})$ be a viscosity solution to* (3.1) *in $B_{6\sqrt{d}}$. Suppose A1 and A3 are in force and Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Suppose further that*

$$\underline{G}_1(u, \Omega) \cap Q_3 \neq \emptyset.$$

Then

$$|\underline{G}_M(u, \Omega) \cap Q_1| \geq 1 - \sigma,$$

where $0 < \sigma < 1$ is a universal constant and $1 < M = M(\lambda, \Lambda, d, q)$.

Proof. Let $x_1 \in \underline{G}_1(u, \Omega) \cap Q_3$ and observe that

$$Q_1 \subset Q_3 \subset B_{3/2\sqrt{d}} \subset B_{4\sqrt{d}}(x_1) \subset B_{6\sqrt{d}} \subset \Omega.$$

From the definition of $C^{1,\frac{1}{1+q}}$-cone of opening 1, we infer the existence of

$$\psi(x) = L(x) - \frac{1}{2}|x - x_1|^{1+\frac{1}{1+q}},$$

where $L(x)$ is an affine function, touching u from bellow at x_1. Hence,

$$\psi_1(x) \leq v(x),$$

where

$$\psi_1(x) := 1 - \frac{1}{16d}|x - x_1|^{1+\frac{1}{1+q}},$$

$$v(x) := L_1(x) + \frac{u(x)}{8d},$$

and

$$L_1(x) := 1 - \frac{L(x)}{8d}.$$

As before, we build an auxiliary function as to resort to the ABP estimate in Proposition 1. Consider φ as in Lemma 1 and define $w(x) := v(x) + \varphi(x)$ in $B_{4\sqrt{d}}(x_1)$. We have that w solves

$$G(x, Dw, D^2w) = 0,$$

where

$$G(x,p,M) := |p - (D\varphi - l_1)|^q \frac{1}{8d} F(8dM - 8dD^2\varphi) - \frac{1}{(8d)^{q+1}} f(x).$$

Once again we take $\gamma \equiv 0$ to conclude G satisfies (2.2) and (2.3), for

$$g(x) = c(d)\xi(x) + |f(x)|.$$

By definition, we have that $w \geq 0$ on $\partial B_{4\sqrt{d}}(x_1)$. In addition, $\inf_{Q_3} w \leq -1$. We apply Proposition 1 to w in $B_{4\sqrt{d}}(x_1)$ and, as in the proof of Lemma 4, specialize the choice of constants as to obtain

$$|\{w = \Gamma(w)\} \cap Q_1| \leq 1 - \sigma,$$

with $0 < \sigma < 1$.

It remains to prove that $(\{w = \Gamma(w)\} \cap Q_1) \subset (\underline{G}_M(u, \Omega) \cap Q_1)$, for some $M > 1$. Let $x_2 \in \{w = \Gamma(w)\} \cap Q_1$. There exists an affine function L_2 such that

$$L_2 \leq \Gamma(w) \leq v + \varphi \;\text{ in }\; B_{4\sqrt{d}}(x_1),$$

with equalities at x_2. Thus, there exists a concave $C^{1,\frac{1}{1+q}}$-cone of opening M_0

$$\psi_2(x) = \widetilde{L}(x) - \frac{M_0}{2}|x - x_2|^{1+\frac{1}{1+q}},$$

where $M_0 = M_0(\lambda, \Lambda, d, q) > 1$ and $\widetilde{L}(x)$ is an affine function, such that

$$\psi_2 \le L_2 - \varphi \le v \quad \text{in} \quad B_{4\sqrt{d}}(x_1), \tag{3.6}$$

with equalities at x_2. Since $L_2 < 0$ on $\partial B_{4\sqrt{d}}(x_1)$, it follows that $\psi \le \psi_1$ on $\partial B_{4\sqrt{d}}(x_1)$. In addition, $x_2 \in Q_1 \subset B_{4\sqrt{d}}(x_1)$ and $\psi_2(x_2) = v(x_2) \ge \psi_1(x_2)$. Now, by taking $M_0 > 1/(8d)$ we obtain that $\{\psi_2 - \psi_1 \ge 0\}$ is a convex set. Hence $\psi_2 - \psi_1 < 0$ in $\mathbb{R}^d \backslash B_{4\sqrt{d}}(x_1)$. It implies that

$$\psi_2 \le \psi_1 \le v \quad \text{in} \quad \Omega \backslash B_{4\sqrt{d}}(x_1).$$

From (3.6) we get $\psi_2 \le v$ in Ω. Thus

$$8d\psi_2 - 8da_1 \le u \quad \text{in} \quad \Omega,$$

with equalities at x_2. Therefore $x_2 \in \underline{G}_{8dM_0}(u, \Omega) \cap Q_1$. ■

At this point, we connect the former information on the measure of $\underline{G}_M \cap Q_1$ with the corollary of Calderón-Zygmund decomposition presented in Lemma 3.

Lemma 6 *Let $u \in \mathcal{C}(B_{6\sqrt{d}})$ be a viscosity solution to* (3.1) *in $B_{6\sqrt{d}}$. Suppose A1 and A3 are in force and Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Define*

$$A = \underline{A}_{M^{k+1}}(u, \Omega) \cap Q_1$$

and

$$B = (\underline{A}_{M^k}(u, \Omega) \cap Q_1) \cup \{x \in Q_1 : m(f^d)(x) \ge (c_1 M^{k(q+1)})^d\}.$$

Then

$$|A| \le \sigma |B|,$$

where $0 < \sigma < 1$, $c_1 > 0$ and $M > 1$ are positive constants.

Proof. We start by noticing that $(\underline{G}_M(\Omega) \cap Q_1) \subset (\underline{G}_N(\Omega) \cap Q_1)$, whenever $M \le N$. Hence,

$$|\underline{G}_{M^{k+1}}(\Omega) \cap Q_1| \ge |\underline{G}_{M^k}(\Omega) \cap Q_1| \ge 1 - \sigma.$$

It follows that $|A| \leq \sigma$. For $i \geq 1$, let $Q = Q_{1/2^i}(x_0)$ be a dyadic cube satisfying

$$|\underline{A}_{M^{k+1}}(\Omega) \cap Q| = |A \cap Q| > \sigma|Q|. \tag{3.7}$$

To complete the proof, we resort to Lemma 3. That is, we verify that $\widetilde{Q} \subset B$, where $\widetilde{Q}$ is a predecessor of Q.

We argue by contradiction and suppose that $\widetilde{Q} \not\subset B$. Then, there exists x_1 such that

$$x_1 \in \widetilde{Q} \cap \underline{G}_{M^k}(\Omega) \tag{3.8}$$

and

$$\sup_{r>0} \frac{1}{|Q_r(x_1)|} \int_{Q_r(x_1)} |f|^d dx \leq \left(c_1 M^{k(q+1)}\right)^d.$$

Now, consider the transformation

$$x = x_0 + \frac{1}{2^i} y, \quad y \in Q_1, \quad x \in Q \tag{3.9}$$

and define the function

$$v(x) = \frac{2^{(1+1/(1+q))i}}{M^k} u\left(x_0 + \frac{1}{2^i} y\right).$$

We will check that v satisfies the hypothesis of Lemma 5 with Ω replaced by $\widetilde{\Omega}$, where

$$\widetilde{\Omega} := x_0 + \frac{1}{2^i}\Omega.$$

Observe that $x \in Q$ (respectively, $Q_{3/2^i}(x_0)$, $B_{6\sqrt{d}/2^i}(x_0)$, and Ω) if and only if $y \in Q_1$ (respectively, Q_3, $B_{6\sqrt{d}}$, $\widetilde{\Omega}$). Because $B_{6\sqrt{d}/2^i}(x_0) \subset B_{6\sqrt{d}}$ and $\widetilde{Q} \subset Q_{3/2^i}(x_0)$, transformation (3.9) leads to

$$B_{6\sqrt{d}} \subset \widetilde{\Omega} \qquad \text{and} \qquad |x_1 - x_0| \leq \frac{3}{2^{i+1}}.$$

In addition, v solves

$$|Dv(y)|^q \widetilde{F}(D^2 v) - \widetilde{f}(y) = 0,$$

where

$$\widetilde{F}(P) := \frac{2^{-qi/(q+1)}}{M^k} F\left(\frac{M^k}{2^{-qi/(q+1)}} P\right),$$

and

$$\widetilde{f}(y) = \frac{1}{M^{k(q+1)}} f\left(x_0 + \frac{1}{2^i} y\right).$$

Since $B_{6\sqrt{d}/2^i}(x_0) \subset Q_{15\sqrt{d}/2^i}(x_1)$, we obtain

$$\begin{aligned}
\|\widetilde{f}\|^d_{L^d(B_{6\sqrt{d}})} &= \frac{2^{id}}{M^{kd(q+1)}} \int_{B_{6\sqrt{d}/2^i}(x_0)} |f(x)|^d dx \\
&\leq \frac{c(d)}{M^{kd(q+1)}} \frac{1}{|Q_{15\sqrt{d}/2^i}|} \int_{Q_{15\sqrt{d}/2^i}(x_1)} |f(x)|^d dx \\
&\leq c(d) c_1^d \\
&\leq \delta_0^d
\end{aligned}$$

if we choose c_1 sufficiently small.

Recall that $x_1 \in \underline{G}_{M^k}(u, \Omega) \cap \widetilde{Q}$. Then there exits a concave $C^{1,\frac{1}{1+q}}$-cone of opening M^k and vertex x_1

$$\psi(x) = a - \frac{M^k}{2}|x - x_1|^{1+\frac{1}{1+q}},$$

that touches u from bellow at x_1. Define

$$\widetilde{\psi}(y) = \frac{2^{i(1+\frac{1}{1+q})}}{M^k} \psi\left(x_0 + \frac{1}{2^i} y\right).$$

It is easy to see that $\widetilde{\psi}$ touches v from bellow at y_1, where y_1 is such that $x_1 = x_0 + \frac{1}{2^i} y_1$. In addition, by definition of $\widetilde{\psi}$, we have that

$$\tilde{\psi}(y) = \tilde{L}(x) - \frac{1}{2}|y - y_1|^{1+\frac{1}{1+q}},$$

where $\tilde{L}(x)$ is an affine function.

Hence, $y_1 \in \underline{G}_1(v, \widetilde{\Omega})$ which implies that $\underline{G}_1(v, \widetilde{\Omega}) \cap Q_3 \neq \emptyset$. Therefore, then by Lemma 5,

$$|\underline{G}_M(v, \widetilde{\Omega}) \cap Q_1| \geq 1 - \sigma = (1-\sigma)|Q_1|.$$

Hence

$$|\underline{G}_{M^{k+1}}(u, \Omega) \cap Q_1| \geq (1-\sigma)|Q_1|,$$

which is a contradiction with (3.7). ■

The next lemma concerns a decay rate of the measure of the sets $A_t(u, \Omega) \cap Q_1$.

Lemma 7 *Let $u \in \mathcal{C}(B_{6\sqrt{d}})$ be a viscosity solution to* (3.1) *in $B_{6\sqrt{d}}$. Suppose A1 and A3 are in force and Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Extend f by zero outside $B_{6\sqrt{d}}$. Then*

$$|A_t(u,\Omega)\cap Q_1| \le c_2 t^{-\mu}, \quad \forall\, t>0, \tag{3.10}$$

where c_2 and μ are positive universal constants.

Proof. We start by defining the quantities α_k and β_k as follows:

$$\alpha_k = |\underline{A}_{M^k}(u,\Omega)\cap Q_1|$$

and

$$\beta_k = \Big|x\in Q_1 : m(f^d)(x) \ge c_1^d M^{kd(q+1)}\Big|.$$

Because of Lemma 6, we have $\alpha_{k+1}\le \sigma(\alpha_k+\beta_k)$. Hence

$$\alpha_k \le \sigma^k + \sum_{i=0}^{k-1}\sigma^{k-i}\beta_i.$$

Since $\|f\|_{L^d(B_{6\sqrt{d}})}\le \delta_0$, we have that $\|f^d\|_{L^1(B_{6\sqrt{d}})}\le \delta_0^d$. Thus, the definition of maximal operator (2.1) yields

$$\beta_k \le c(d)\delta_0^d\left(c_1 M^{k(q+1)}\right)^{-d} = CM^{-dk(q+1)}.$$

Hence

$$\sum_{i=0}^{k-1}\sigma^{k-i}\beta^i \le Ckm_0^k,$$

where $m_0=\max\{\sigma, M^{-d(q+1)}\}<1$. Therefore

$$\alpha_k \le \sigma^k + Ckm_0^k \le (1+Ck)m_0^k$$

and, since $m_0<1$, we conclude

$$|\underline{A}_t(u,\Omega)\cap Q_1| \le c_2 t^{-\mu}, \quad \forall\, t>0. \tag{3.11}$$

Notice that $v=-u$ solves

$$-|Dv|^q F(-D^2 v) = -f(x).$$

Since $G(x,p,M) = -|Dv|^q F(-M)+f(x)$ satisfies (2.2) and (2.3), Lemma 4 is available for v. Hence,

$$|\overline{A}_t(u,\Omega)\cap Q_1| \le c_2 t^{-\mu}, \quad \forall\, t>0. \tag{3.12}$$

By gathering (3.11) and (3.12) we conclude

$$|A_t(u,\Omega)\cap Q_1| \le c_2 t^{-\mu}, \quad \forall\, t>0,$$

and complete the proof. ■

At this point, we have all the necessary ingredients to establish Proposition 5.

Proof of Proposition 5. Without loss of generality, we may assume that u solves

$$|Du|^q F(D^2u) = f(x) \qquad \text{in } B_{6\sqrt{d}},$$

with $\|u\|_{L^\infty(B_{6\sqrt{d}})} \leq 1$ and

$$\|f\|_{L^d(B_{6\sqrt{d}})} \leq \delta_0.$$

We will prove that $\|\theta\|_{L^\delta(B_{1/2})} \leq C$, for some $0 < \delta \ll 1$. We resort to (3.10), with $\Omega = B_{6\sqrt{d}}$, and obtain

$$|A_{M^k}(u, B_{6\sqrt{d}}) \cap Q_1| \leq c_2 M^{-k\mu}.$$

Hence,

$$\sum_{k \geq 1} M^{\frac{\mu}{2}k} |A_{M^k}(u, B_{6\sqrt{d}}) \cap Q_1| \leq C,$$

for a constant $C > 0$. Since $B_{1/2} \subset Q_1 \subset B_{6\sqrt{d}}$, we get

$$A_{M^k}(u, B_{1/2}) \subset A_{M^k}(u, B_{6\sqrt{d}}) \cap Q_1.$$

Hence

$$\sum_{k \geq 1} M^{\frac{\mu}{2}k} |A_{M^k}(u, B_{1/2})| \leq C.$$

Recall that $\mu_\theta(t) \leq |A_t(u, B_{1/2})|$. Therefore, Lemma 2 implies

$$\|\theta\|_{L^{\mu/2}(B_{1/2})} \leq C;$$

by taking $\delta = \mu/2$, the proof is complete. ■

In the next section, we produce an approximation lemma relating the solutions to (1.1) with viscosity solutions to $F = 0$. Improved regularity available for the latter refines the decay rate for $|A_{M^k}(u, B_{6\sqrt{d}}) \cap Q_1|$, ultimately yielding improved integrability for the aperture function θ.

3.3
Improved integrability for the aperture function

In what follows, we refine the decay rate of the measure of certain sets, leading to improved integrability of the aperture function. In the sequel we state the main result in this section.

Proposition 6 (**$p(q+1)$-integrability of θ**) *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to* (3.1). *Suppose A1-A3 are in force. Suppose further that $\|f\|_{L^d(B_1)} < \varepsilon$, for some $\varepsilon > 0$ to be determined. Then $\theta \in L^{p(q+1)}(B_1)$ and there exists a universal constant $C > 0$ such that*

$$\|\theta\|_{L^{p(q+1)}(B_{1/2})} \leq C.$$

The key ingredient in establishing Proposition 6 is an approximation lemma importing information from the solutions to $F = 0$, under A2.

Lemma 8 (Approximation Lemma) *Let $u \in \mathcal{C}(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (3.1) *in $B_{8\sqrt{d}}$. Suppose that A1-A3 are in force. Given $\delta > 0$, there exists $0 < \varepsilon < \delta^{q+1}$ such that, if $\|f\|_{L^d(B_{8\sqrt{d}})} \leq \varepsilon$, one can find a function $h \in C^{1,1}(B_{6\sqrt{d}})$ satisfying*

$$\|u - h\|_{L^\infty\left(\overline{B}_{6\sqrt{d}}\right)} \leq \delta.$$

Proof. We argue by contradiction. Suppose that the statement of the proposition is false. Then, there exists $\widetilde{\delta}_0$ and sequences $(u_n)_n, (f_n)_n$, such that

$$\begin{aligned} |Du_n|^q F(D^2 u_n) = f_n \quad \text{in} \quad B_{8\sqrt{d}}, \\ \|f_n\|_{L^p(B_{8\sqrt{d}})} \leq \frac{1}{n} \end{aligned} \tag{3.13}$$

and

$$\|u_n - h\|_{L^\infty(B_{6\sqrt{d}})} > \widetilde{\delta}_0, \tag{3.14}$$

for all $h \in C^{1,1}(\overline{B}_{6\sqrt{d}})$.

From the regularity available for (3.13), see [37], we have that there exists a function $u_\infty \in C^{1,\beta}_{\text{loc}}(B_{7\sqrt{d}})$, for some $0 < \beta < 1$, such that $u_n \longrightarrow u_\infty$ in $C^{1,\beta}_{\text{loc}}(B_{7\sqrt{d}})$, through a subsequence if necessary. Note that u_∞ solves

$$|Du_\infty|^q F(D^2 u_\infty) = 0 \quad \text{in} \quad B_{7\sqrt{d}}$$

in the viscosity sense. It follows that u_∞ solves

$$F(D^2 u_\infty) = 0 \quad \text{in} \quad B_{6\sqrt{d}}.$$

Since F has $C^{1,1}_{loc}$-estimates, $u_\infty \in C^{1,1}(\overline{B}_{6\sqrt{d}})$. By taking $h \equiv u_\infty$ we get a contradiction with (3.14) and the proof is complete. ■

Lemma 9 *Let $u \in \mathcal{C}(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (3.1) *in $B_{8\sqrt{d}}$. Suppose that A1-A3 are in force and*

$$-|x|^{1+1/(q+1)} \leq u(x) \leq |x|^{1+1/(q+1)} \qquad in \qquad \Omega\backslash B_{6\sqrt{d}}.$$

Then

$$|G_M(u,\Omega) \cap Q_1| \geq 1-\rho, \tag{3.15}$$

for some $\rho \in (0,1)$, where $M > 1$ depends only on d and q, and δ in Lemma 8 will be determined by ρ.

Proof. Fix $0 < \delta < 0$, yet to be determined. Then let h be the δ-approximating function whose existence is ensured by Lemma 8, restricted to $\overline{B}_{6\sqrt{d}}$. We know from $A2$ that $h \in C^{1,1}(\overline{B}_{6\sqrt{d}})$ and

$$\|h\|_{C^{1,1}(\overline{B}_{6\sqrt{d}})} \leq C.$$

Extend h outside $\overline{B}_{6\sqrt{d}}$ continuously, as to have $h = u$ in $\Omega\backslash B_{7\sqrt{d}}$ and $\|u-h\|_{L^\infty(\Omega)} = \|u-h\|_{L^\infty(B_{6\sqrt{d}})}$. Recall that $\|h\|_{L^\infty(B_{6\sqrt{d}})} = \|u\|_{L^\infty(B_{6\sqrt{d}})}$. It is clear that

$$\|u-h\|_{L^\infty(\Omega)} \leq 2.$$

It follows that

$$-2-|x|^{1+\frac{1}{q+1}} \leq h(x) \leq 2+|x|^{1+\frac{1}{q+1}} \qquad \text{in} \qquad \Omega\backslash B_{6\sqrt{d}}.$$

Therefore, there exists $1 < N = N(d,q,C)$ such that

$$Q_1 \subset G_N(h,\Omega). \tag{3.16}$$

Define

$$w(x) = \frac{\min(1,\delta_0)^{1/(q+1)}}{2\delta}(u-h)(x),$$

where δ_0 is the constant in Lemma 4. Notice that w solves

$$\left|Du + \frac{\min(1,\delta_0)^{1/(q+1)}}{2\delta}Dh\right|^q \tilde{F}(D^2u) - \tilde{f}(x) = 0,$$

where

$$\tilde{F}(M) := \frac{\min(1,\delta_0)^{1/(q+1)}}{2\delta}F\left(\frac{2\delta}{\min(1,\delta_0)^{1/(q+1)}}M + D^2h\right),$$

and

$$\tilde{f}(x) := \frac{\min(1,\delta_0)}{(2\delta)^{q+1}}f(x).$$

As a consequence of the former inequality, we have

$$\|\tilde{f}\|_{L^d} \leq \frac{\min(1,\delta_0)}{(2\delta)^{q+1}}\|f\|_{L^d} \leq \delta_0.$$

Hence, w is entitled to the conclusions of Lemma 4 in Ω. Because of Lemma 7, we obtain

$$|A_t(w,\Omega)\cap Q_1| \leq t^{-\mu}, \quad \text{for all} \quad t>0.$$

It follows that

$$|A_s(u-h,\Omega)\cap Q_1| \leq cs^{-\mu}\delta^{\mu} \quad \text{for all} \quad s>0.$$

By choosing δ small enough we get

$$|G_N(u-h,\Omega)\cap Q_1| \geq 1-\delta^{\mu} \geq 1-\rho.$$

■

The proof of Lemma 9 sets the proximity-regime encoded by $\delta > 0$. As a by-product it sets the smallness condition on the L^d-norm of the source term f, encoded by $\varepsilon > 0$ in the statement of Lemma 8. In the remainder of this chapter, these constants are fixed.

Lemma 10 *Let $u \in \mathcal{C}(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (3.1) *in $B_{8\sqrt{d}}$. Suppose that A1-A3 are in force. If*

$$G_1(u,\Omega)\cap Q_3 \neq \emptyset, \tag{3.17}$$

then

$$|G_M(u,\Omega)\cap Q_1| \geq 1-p,$$

with M and ρ as in Lemma 9.

Proof. Let $x_1 \in G_1(u,\Omega)\cap Q_3$. Hence, there exists an affine function $L(x)$, such that

$$-\frac{1}{2}|x-x_1|^{1+1/(q+1)} \leq u(x)-L(x) \leq \frac{1}{2}|x-x_1|^{1+1/(q+1)} \quad \text{in} \quad \Omega.$$

Define

$$v(x) = \frac{u(x)-L(x)}{c(d)},$$

where $c(d)$ is a constant depending only on d, large enough as to guarantee $|v(x)| \leq 1$ and

$$|v(x)| \leq |x|^{1+1/(q+1)} \quad \text{in} \quad \Omega\backslash B_{6\sqrt{d}}.$$

In addition, v solves
$$|Dv|^q \tilde{F}(D^2u) - \tilde{f}(x) = 0,$$
where
$$\tilde{F}(M) := \frac{1}{c(d)} F(c(d)M),$$
and
$$\tilde{f}(x) := \frac{1}{c(d)^{q+1}} f(x).$$
Lemma 9 yields
$$|G_M(v,\Omega) \cap Q_1| \geq 1 - \rho,$$
and, therefore
$$|G_{c(d)M}(u,\Omega) \cap Q_1| \geq 1 - \rho.$$

■

The next result resorts once again to the Calderón-Zygmund decomposition.

Lemma 11 *Let $u \in \mathcal{C}(B_{8\sqrt{d}})$ be a normalized viscosity solution to* (3.1) *in $B_{8\sqrt{d}}$. Suppose that A1-A3 are in force. Extend f by zero outside $B_{8\sqrt{d}}$ and set*
$$A := A_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q_1,$$
$$B := \left\{ A_{M^k}(u, B_{8\sqrt{d}}) \cap Q_1 \right\} \cup \left\{ x \in Q_1 : m(f^d)(x) \geq c_3^d M^{kd(q+1)} \right\},$$
for $k \in \mathbb{N}$. Then
$$|A| \leq \rho |B|,$$
where $M > 1$ depends on d and q, and $c_3 > 0$ depends only on d, λ, Λ and ρ.

Proof. We start by noticing that $|u| \leq 1 \leq |x|^{1+1/(q+1)}$ in $B_{8\sqrt{d}} \backslash B_{6\sqrt{d}}$. Hence Lemma 9 applied with $\Omega = B_{8\sqrt{d}}$, implies
$$|G_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q_1| \geq |G_{M^k}(u, B_{8\sqrt{d}}) \cap Q_1| \geq 1 - \rho.$$

It leads to $|A| \leq \rho$.

The remainder of the proof relies on the Calderón-Zygmund decomposition, as stated in Lemma 3. Hence, we need to show that if $Q = Q_{1/2^i}(x_0)$ is a dyadic cube Q_1 such that
$$|A_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q| = |A \cap Q| > \rho |Q|, \tag{3.18}$$
we have $\tilde{Q} \subset B$. We suppose otherwise and produce a contradiction. Suppose that $\tilde{Q} \not\subset B$ and let x_1 be such that

$$x_1 \in \tilde{Q} \cap G_{M^k}(u, B_{8\sqrt{d}}) \tag{3.19}$$

and

$$m(f^d)(x_1) \leq \left(c_3 M^{k(q+1)}\right)^d. \tag{3.20}$$

Now, we proceed as in Lemma 6. Consider as before the transformation

$$x = x_0 + \frac{1}{2^i} y, \quad x \in B_{8\sqrt{d}}, \tag{3.21}$$

and define

$$v(y) = \frac{2^{(1+\frac{1}{q+1})i}}{M^k} u\left(x_0 + \frac{1}{2^i} y\right).$$

Finally, let $\tilde{\Omega}$ be the image of $B_{8\sqrt{d}}$ under the transformation (3.21).

We need to verify that v satisfies the hypothesis of Lemma 10. Note that v solves

$$|Dv(y)|^q \tilde{F}(D^2u) - \tilde{f}(x) = 0 \;\; in \;\; \tilde{\Omega},$$

where

$$\tilde{F}(N) := \frac{1}{2^{iq/(q+1)} M^k} F\left(2^{iq/(q+1)} M^k N\right)$$

and

$$\tilde{f}(x) := \frac{1}{M^{k(q+1)}} f\left(x_0 + \frac{1}{2^i} x\right).$$

Since $B_{8\sqrt{d}} \subset \tilde{\Omega}$, the function v satisfies the equation in $B_{8\sqrt{d}}$, in the viscosity sense. Furthermore $|x_1 - x_0|_\infty \leq 3/2^{i+1}$ implies that $B_{8\sqrt{d}/2^i}(x_0) \subset Q_{19\sqrt{d}/2^i}(x_1)$. Hence

$$\begin{aligned}
\|\tilde{f}\|^d_{B_{8\sqrt{d}}} &= \frac{2^{id}}{M^{kd(q+1)}} \int_{B_{8\sqrt{d}/2^i}(x_0)} |f(x)|^d dx \\
&\leq \frac{c(d)}{M^{kd(q+1)}} \frac{1}{|Q_{19\sqrt{d}/2^i}|} \int_{Q_{19\sqrt{d}/2^i}(x_1)} |f(x)|^d dx \\
&\leq c(d) c_3^d \\
&\leq \varepsilon,
\end{aligned}$$

for c_3 small enough.

Now, by (3.19) there exist a convex and a concave $C^{1,\alpha}$-cones of opening M^k, ψ_1 and ψ_2 respectively, such that ψ_1 touches u from above at x_1 and ψ_2 touches u from bellow at x_1. Define

$$\tilde{\psi_1}(y) := \psi_1\left(x_0 + \frac{1}{2^i} y\right)$$

and

$$\tilde{\psi_2}(y) := \psi_2\left(x_0 + \frac{1}{2^i} y\right).$$

It is easy to see that $\tilde{\psi_1}$ (resp. $\tilde{\psi_2}$) touches v from above (respectively from bellow) in a point y_1 such that $x_1 = x_0 + \frac{1}{2^i} y_1$. Therefore $G_1(v, \tilde{\Omega}) \neq \emptyset$. By Lemma 10 we obtain

$$|G_M(v, \tilde{\Omega}) \cap Q_1| \geq 1 - \rho = (1-\rho)|Q_1|.$$

Hence

$$|G_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q| \geq (1-\rho)|Q|,$$

which implies

$$|A_{M^{k+1}}(u, B_{8\sqrt{d}}) \cap Q| \leq \rho |Q|.$$

This is a contradiction with (3.18).

■

Proof of Proposition 6. Let M be as in Lemma 11 and take ρ such that

$$\rho M^{p(q+1)} = \frac{1}{2}.$$

For $k \geq 0$, define

$$\alpha_k := |A_{M^k}(u, B_{8\sqrt{d}}) \cap Q_1|$$

and

$$\beta_k := \left| \left\{ x \in Q_1 : m(f^d)(x) \geq \left(c_3 M^{k(q+1)} \right)^d \right\} \right|.$$

By Lemma 11 we obtain $\alpha_{k+1} \leq \rho(\alpha_k + \beta_k)$. Hence

$$\alpha_k \leq \rho^k + \sum_{i=0}^{k-1} \rho^{k-i} \beta_i. \tag{3.22}$$

Since $f^d \in L^{p/d}(B_{8\sqrt{d}})$, we have that $m(f^d) \in L^{p/d}(B_{8\sqrt{d}})$ and

$$\|m(f^d)\|_{L^{p/d}(B_{8\sqrt{d}})} \leq c \|f\|^d_{L^p(B_{8\sqrt{d}})} \leq C.$$

Therefore, by Lemma 2 we obtain

$$\sum_{k \geq 0} \left(M^{d(q+1)} \right)^{\frac{pk}{d}} |x \in Q_1 : m(f^d)(x) \geq c_3^d M^{dk(q+1)}| \leq C.$$

The former inequality implies

$$\sum_{k \geq 0} M^{p(q+1)k} \beta_k \leq C. \tag{3.23}$$

Since $B_{1/2} \subset Q_1$, the distribution function of θ is bounded from above as follows:

$$\mu_\theta(t) \leq |A_t(u, B_{1/2})| \leq |A_t(u, B_{8\sqrt{d}}) \cap Q_1|.$$

Hence

$$\begin{aligned}
\sum_{k\geq 1} M^{p(q+1)k}\alpha_k &\leq \sum_{k\geq 1}\left(\rho M^{p(q+1)}\right)^k + \sum_{k\geq 1}\sum_{i=0}^{k-1}\rho^{k-i}M^{p(q+1)k}\beta_i \\
&= \sum_{k\geq 1} 2^{-k} + \sum_{k\geq 1}\sum_{i=0}^{k-1}\rho^{k-i}M^{p(q+1)(k-i)}M^{p(q+1)i}\beta_i \\
&= \sum_{k\geq 1} 2^{-k} + \sum_{k\geq 1}\sum_{i=0}^{k-1} 2^{-(k-i)}M^{p(q+1)i}\beta_i \\
&= \sum_{k\geq 1} 2^{-k} + \left(\sum_{i\geq 0} M^{p(q+1)i}\beta_i\right)\left(\sum_{j\geq 1} 2^{-j}\right) \\
&\leq C.
\end{aligned}$$

Applying Lemma 2 once again we conclude that $\|\theta\|_{L^{p(q+1)}(B_{1/2})} \leq C$ and complete the proof. ■

At this point we relate the (improved) integrability of the aperture function θ with the regularity of solutions in fractional Sobolev spaces. For completeness, we restate Theorem 1.

Theorem 6 (Restatement of Theorem 1) *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to* (3.1)*. Suppose that A1-A3, to be determinate later, hold true. Then $u \in W^{\sigma,p(q+1)}_{loc}(B_1)$, for every*

$$\sigma < 1 + \frac{1}{q+1}$$

In addition, there exists a positive constant $C > 0$ such that

$$\|u\|_{W^{\sigma,p(q+1)}(\bar{B}_{1/2})} \leq C.$$

Proof. Let ψ is a $C^{1,\alpha}$-cone of opening $\pm M$ and vertex x_0, we have:

$$\begin{aligned}
\Delta_h^{1+\alpha}\psi(x_0) &:= \frac{\psi(x_0+h)+\psi(x_0-h)-2\psi(x_0)}{|h|^{1+\alpha}} \\
&= \pm M.
\end{aligned}$$

Also, notice that touching u strictly in $B_{\frac{1}{10}}(x_0)$ from above at x_0 by a convex $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 gives for all $0 < h < \frac{1}{10}$

$$\begin{aligned}\Delta_h^{1+\alpha} u(x_0) &:= \frac{u(x_0+h)+u(x_0-h)-2u(x_0)}{|h|^{1+\alpha}}\\ &< \frac{\psi(x_0+h)+\psi(x_0-h)-2\psi(x_0)}{|h|^{1+\alpha}}\\ &\leq \theta(u, B_{1/2})(x_0).\end{aligned}$$

Similarly, touching u strictly in $B_{\frac{1}{10}}(x_0)$ from below at x_0 by a concave $C^{1,\alpha}$-cone ψ of opening M and vertex x_0 gives, for all $0 < h < \frac{1}{10}$:

$$-\theta(u, B_{1/2})(x_0) < \Delta_h^{1+\alpha} u(x_0).$$

Hence, by hypothesis,

$$\|\Delta_h^{1+\alpha} u\|_{L^p(B_{1/2})} \leq C,$$

uniformly for all $0 < h < \frac{1}{10}$, for $\alpha = \frac{1}{1+q}$. At this point, we set $\varphi := u\chi_{B_{1/2}}$ in $\mathbb{R}^d$. Then we have that $\varphi \in L^\infty(\mathbb{R}^d)$. Next, for

$$\sigma < 1 + \frac{1}{q+1},$$

we define the singular integral operator:

$$I_{\sigma/2}(v)(x_0) := \int_{\mathbb{R}^d} \frac{v(x_0+y)+v(x_0-y)-2v(y)}{|y|^{d+\sigma}}.$$

Notice that $I_{\sigma/2}(v) = \Delta^{\sigma/2}(v)$. For $x_0 \in B_{1/2}$ we estimate

$$\begin{aligned}I_{\sigma/2}(\varphi)(x_0) &= \int_{\mathbb{R}^d} \frac{\varphi(x_0+y)+\varphi(x_0-y)-2\varphi(y)}{|y|^{d+\sigma}}\\ &= \int_{B_{1/10}} \frac{u(x_0+y)+u(x_0-y)-2u(y)}{|y|^{d+\sigma}}\\ &\quad + \int_{B_1\setminus B_{1/10}} \frac{u(x_0+y)+u(x_0-y)-2u(y)}{|y|^{d+\sigma}}\\ &\leq \theta(u, B_{1/2})(x_0) \int_{B_{1/10}} \frac{1}{|y|^{d-\varepsilon}} + C\|u\|_{L^\infty(B_1)}\\ &= \frac{C}{\varepsilon 10^d} \cdot \theta(u, B_{1/2})\end{aligned}$$

where

$$\varepsilon = 1 + \frac{1}{q+1} - \sigma,$$

and C is a universal constant. Hence, we have proven that

$$I_{\sigma/2}(\varphi) \in L^{p(q+1)}(B_{1/2}).$$

Therefore,

$$(-\Delta)^{\sigma/2}\varphi \in L^{p(q+1)}(B_{1/2}).$$

By setting $g := (-\Delta)^{\sigma/2}\varphi$ in $B_{1/2}$ we conclude that φ satisfies

$$\begin{cases} (-\Delta)^{\sigma/2}\varphi &= g \text{ in } B_{1/2} \\ \varphi &= 0 \text{ in } \mathbb{R}^d \setminus B_{1/2} \end{cases}$$

Now, we want to show that $\varphi \in W^{\sigma/2,2}(\mathbb{R}^d)$. Extend g by zero outside $B_{1/2}$. It is clear that $g \in L^2(\mathbb{R}^d)$. Hence $\mathcal{F}(g) \in L^2(\mathbb{R}^d)$. In addition, since $\varphi \in L^2(\mathbb{R}^d)$ (φ is bounded in $\mathbb{R}^d$), $\mathcal{F}(\varphi) \in L^2(\mathbb{R}^d)$ as well. By Proposition 2 (applied to functions in $L^2(\mathbb{R}^d)$)

$$\mathcal{F}(g)(x) = (|x|)^{\sigma}\mathcal{F}(\varphi).$$

Furthermore, we have that

$$(1 + |x|^{\sigma})\mathcal{F}(\varphi) = \mathcal{F}(\varphi) + \mathcal{F}(g).$$

It follows that $(1 + |x|^{\sigma})\mathcal{F}(\varphi) \in L^2(\mathbb{R}^d)$. In particular,

$$\int_{\mathbb{R}^d} (1 + |x|^{\sigma})|\mathcal{F}(\varphi)(x)|^2 dx < \infty.$$

Hence, by Proposition 3 we conclude that $\varphi \in W^{\sigma/2,2}(\mathbb{R}^d)$. Finally, by Theorem 2.4 we obtain that $\varphi \in W^{\sigma,p(q+1)}_{loc}(B_{1/2})$. Therefore

$$u \in W^{\sigma,p(q+1)}_{loc}(B_{1/2}),$$

which ends the proof. ■

By combining Theorem 6 with standard embedding results for fractional Sobolev spaces, we obtain Theorem 2, restated in what follows. See [31, Section 4.6] for an account of embedding results for fractional Sobolev spaces $W^{\sigma,p}$, for $\sigma \geq 1$.

Theorem 7 (Restatement of Theorem 2) *Let $u \in \mathcal{C}(B_1)$ be a normalized viscosity solution of* (3.1). *Suppose A1-A3, to be detailed further, are in force. Then $u \in \mathcal{C}^{1,\beta}_{loc}(B_1)$ for all*

$$\beta < \frac{1}{q+1}\left(q + 2 - \frac{d}{p}\right) - 1.$$

The *almost optimality* of the exponent $\beta \in (0,1)$ can be derived from the analysis of an explicit example, combined with intrinsic, scaling-related, constraints of the equation.

In fact, consider $v(x) := |x|^{1+\beta}$. A straightforward computation yields

$$|Dv|^q \Delta v \sim |x|^{(1+\beta)(q+1)-q-2};$$

to ensure the right hand side in the former expression belongs to $L^p(B_1)$, we must secure

$$\beta \geq \frac{1}{q+1}\left(q+2-\frac{d}{p}\right)-1. \tag{3.24}$$

Conversely, we examine the $\mathcal{C}^{1,\beta}$-scaling of (3.1). Let

$$v(x) := \frac{u(rx)}{r^{1+\beta}},$$

for some fixed $0 < r \ll 1$. We find that v solves

$$|Dv|^q \overline{F}(D^2 v) = \overline{f} \qquad \text{in } B_1,$$

where

$$\overline{F}(M) := r^{2-(1+\beta)} F(r^{(1+\beta)-2} M)$$

and

$$\overline{f}(x) := r^{2-(1+\beta)-((1+\beta)-1)q} f(rx).$$

To ensure that $\overline{f} \in L^p(B_1)$, we require

$$\beta \leq \frac{1}{q+1}\left(q+2-\frac{d}{p}\right)-1. \tag{3.25}$$

By gathering (3.24) and (3.25), one finds the constraint on the Hölder exponent prescribed in Theorem 2 to be almost optimal.

4
A fully nonlinear free transmission problem

In this chapter, we study the fully nonlinear free transmission problem

$$F_1(D^2u)\chi_{\{u>0\}} + F_2(D^2u)\chi_{\{u<0\}} = 1 \qquad \text{in} \qquad \Omega^+(u) \cup \Omega^-(u), \qquad (4.1)$$

where F_1, $F_2 : \mathcal{S}(d) \to \mathbb{R}$ are (λ, Λ)-elliptic operators, $\Omega^+(u) := \{u > 0\}$ and $\Omega^-(u) := \{u < 0\}$.

We start by noticing that a $W^{2,d}$-strong solution to (4.1) solves a uniformly elliptic PDE in B_1. Moreover, the source term for such equation is bounded. Then we establish quadratic growth for the solutions *away from the free boundary.*

Proposition 7 *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (4.1)*. Suppose A4-A5 hold true. There exists a (λ, Λ)-elliptic operator $G : \mathcal{S}(d) \to \mathbb{R}$ and a function $g \in L^\infty(B_1)$ such that u is a strong solution to*

$$G(D^2u(x)) = g(x) \qquad \textit{in} \qquad B_1. \qquad (4.2)$$

Proof. Because $u \in W^{2,d}(B_1)$, we have $D^2u(x) = 0$ *a.e.* $- x \in \{u = 0\}$. Without loss of generality, consider $G := F_1$. For *a.e.* $- x \in \{u > 0\}$ we have $G(D^2u(x)) = 1$. In addition, the last condition in A4 yields $G(D^2u(x)) = 0$ for almost every $x \in \{u = 0\}$. Finally, we consider $x \in \{u < 0\}$; because of (4.1), we know that $F_2(D^2u(x)) = 1$ for *a.e.* $- x \in \{u < 0\}$. It follows from A5 that

$$|F_1(D^2u(x))| \leq C + 1 \qquad a.e. - x \in \{u < 0\}.$$

By defining $g : B_1 \to \mathbb{R}$ as

$$g(x) := \begin{cases} 1 & \text{if } x \in \{u > 0\} \\ 0 & \text{if } x \in \{u = 0\} \\ F_1(D^2u(x)) & \text{if } x \in \{u < 0\}, \end{cases}$$

we have $g \in L^\infty(B_1)$ and $G(D^2u(x)) = g(x)$ almost everywhere in B_1. ■

The next result states that u is an L^d-viscosity solution to $G = g$ in B_1; it follows from [20, Lemma 2.5]. For the sake of completeness, we include it here as a Proposition.

Proposition 8 *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (4.1)*. Suppose A4-A5 hold true. Then, u is an L^d-viscosity solution to* (4.2)*.*

As mentioned in [20], Proposition 8 follows from the ellipticity of G and the maximum principle for $W^{2,d}$-functions, as stated in [46]; see also [15]. At this point, by requiring F_1 to be convex, our analysis produces a first regularity result concerning the solutions to (4.1). In fact, the convexity of F_1 turns u into a viscosity solution to a convex equation with bounded right-hand side. From [23] we infer that $D^2u \in \mathrm{BMO}_{loc}(B_1)$, with the appropriate estimates. As yet a further consequence, we also have $u \in \mathcal{C}^{1,\text{Log-Lip}}_{loc}(B_1)$; for a direct proof of this fact, see [57]. An alternative argument would be to relate functions with derivatives in BMO-spaces with the Zygmund class [47] and then notice the Zygmund class is a subset of the space of Log-Lipschitz functions [66].

4.1
Some context on transmission problems

Transmission problems comprise a class of models aimed at examining a variety of phenomena in heterogeneous media. The problems under the scope of this formulation include thermal and electromagnetic conductivity, composite materials and, more generally, diffusion processes driven by discontinuous laws.

Given a domain $\Omega \subset \mathbb{R}^d$, there exist distinct subregions $\Omega_1, \Omega_2, \ldots, \Omega_k$ satisfying $\Omega_i \subset \Omega$ for every $i = 1, \ldots, k$, for some $k \in \mathbb{N}$ so that the mechanism governing the problem is smooth within Ω_i, though possibly discontinuous across $\partial\Omega_i$. A paramount, subtle, aspect of the theory concerns the nature of those subregions.

In fact, $(\Omega_i)_{i=1}^k$ and the geometry of $\partial\Omega_i$ can be prescribed a priori. The alternative is $(\Omega_i)_{i=1}^k$ to be determined endogenously. The latter setting frames the theory in the context of free boundary problems. Both cases differ substantially; as a consequence, their analysis also requires distinct techniques. The vast majority of former studies on transmission problems presupposes *a priori knowledge* of the subregions Ω_i and their geometric properties. A work-horse of the theory is the divergence-form equation

$$\operatorname{div}(a(x)Du) = 0 \quad \text{in} \quad \Omega, \tag{4.3}$$

where the matrix-valued function $a(\cdot)$ is defined as

$$a(x) := a_i \ \text{ for } \ x \in \Omega_i,$$

for constant matrices a_i and $i = 1, \ldots, k$. Though smooth within every Ω_i, the coefficients of (4.3) can be discontinuous across $\partial\Omega_i$. This feature introduces genuine difficulties in the analysis.

The first formulation of a transmission problem appeared in [51] and addressed a topic in the realm of material sciences. More precisely, in elasticity theory. In that paper, the author proves the uniqueness of solutions for a model consisting of two subregions, which are known a priori. The existence of solutions is discussed in [51], although not examined in detail. See also [50].

The formulation in [51] motivated a number of subsequent studies [17, 26, 27, 28, 45, 35, 48, 55, 65, 58]. Those papers present a wide range of developments, including the existence of solutions for the transmission problem in [51] and the analysis of several variants. We refer the reader to [16] for an account of those results and methods.

Estimates and regularity results for the solutions to transmissions problems have also been treated in the literature. In [44] the authors consider a bounded subdomain $\Omega \subset \mathbb{R}^d$, which is split into a finite number of subregions $\Omega_1, \Omega_2, \ldots, \Omega_k$, known a priori. The motivation is in the study of composite materials with closely spaces inclusions. A two-dimensional example is the cross-section of a fiber-reinforced material; see Figure 4.1.

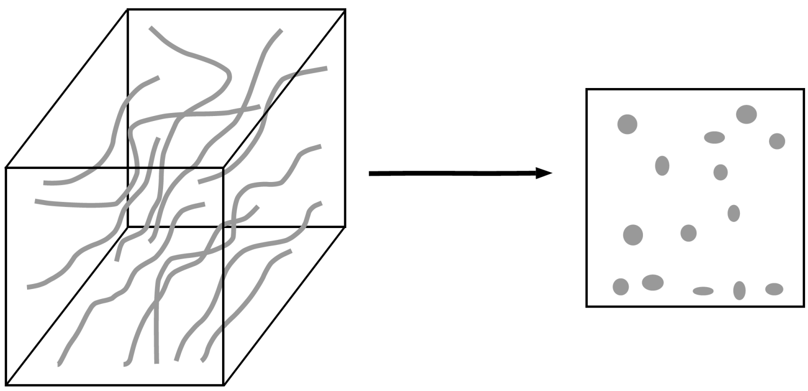

Figure 4.1: The cross-section of a fiber-reinforced material provides an example in $\mathbb{R}^2$ of a bounded domain with a finite number of inclusions. The grey subregions in the cross-section represent the fibers, whereas the remainder of the material is the matrix.

The mathematical analysis amounts to the study of

$$\frac{\partial}{\partial x_i}\left(a(x)\frac{\partial}{\partial x_j}u\right) = f \quad \text{in} \quad \Omega, \tag{4.4}$$

where

$$a(x) := \begin{cases} a_i(x) & \text{for} \quad x \in \Omega_i, \;\; i = 1, \ldots, k \\ a_{k+1}(x) & \text{for} \quad x \in \Omega \setminus \cup_{i=1}^k \Omega_i. \end{cases}$$

Under natural assumptions on the data, the authors establish local Hölder continuity for the gradient of the solutions. From the applied perspective, the gradient encodes information on the stresses of the material. Their findings imply bounds on the gradient *independent of the location of the fibers*. C.f. [14].

The vectorial setting is the subject of [43]. In that paper the authors extend the developments reported in [44] for systems. Moreover, they produce bounds for higher derivatives of the solutions.

In [5] the authors consider a domain with two subregions, which are supposed to be ε-apart, for some $\varepsilon > 0$. Within each subregion, the divergence-form equation is governed by a constant coefficient k, whereas outside the coefficient is equal to 1. By setting $k = +\infty$, the authors frame the problem in the context of perfect conductivity.

In this setting, it is known that bounds on the gradient deteriorate as the two subregions approach each other. The analysis in [5] yields blow up rates for the gradient bounds as $\varepsilon \to 0$. The case of multiple inclusions, covering perfect conductivity and insulation ($k = 0$), is discussed in [6]. See also [18].

Recently, new developments have been obtained under minimal regularity requirements for the transmission interfaces. In [25] the authors consider a smooth and bounded domain Ω and fix $\Omega_1 \Subset \Omega$, defining $\Omega_2 := \Omega \setminus \overline{\Omega}_1$. They suppose the boundary of the transmission interface $\partial\Omega_1$ to be of class $\mathcal{C}^{1,\alpha}$ and prove existence, uniqueness and $\mathcal{C}^{1,\alpha}(\overline{\Omega}_i)$-regularity of the solutions to the problem, for $i = 1, 2$. Their argument imports regularity from flat problems, through a new stability result; see [25, Theorem 4.2].

Another class of transmission problems concerns models where the subregions of interest are determined endogenously. For example, given $\Omega \subset \mathbb{R}^d$, one would consider

$$\Omega_1 : \{x \in \Omega \mid u(x) < 0\} \qquad \text{and} \qquad \Omega_2 : \{x \in \Omega \mid u(x) > 0\},$$

where $u : \Omega \to \mathbb{R}$ solves a prescribed equation. Roughly speaking, knowledge of the solution is required to determine the subregions of the domain where distinct diffusion phenomena take place. In this context, a further structure arises, namely, the free interface, or free boundary. Here, in addition to the analysis of the solutions, properties of the free boundary are also of central interest.

In [1] the authors examine a transmission problem with free interface. They consider the functional

$$I(v) := \int_\Omega \frac{1}{2}\langle A(x,v)Dv, Dv\rangle + \Lambda(v) + fv\, dx, \tag{4.5}$$

where

$$
\begin{aligned}
A(x,u) &:= A_+(x)\chi_{\{u>0\}} + A_-(x)\chi_{\{u\leq 0\}},\\
\Lambda(u) &:= \lambda_+(x)\chi_{\{u>0\}} + \lambda_-(x)\chi_{\{u\leq 0\}},\\
f &:= f_+(x)\chi_{\{u>0\}} + f_-(x)\chi_{\{u\leq 0\}},
\end{aligned}
$$

with $A_\pm$ matrix-valued mappings and $\lambda_\pm$ and $f_\pm$ given functions. Local minimizers for (4.5) satisfy

$$
\begin{aligned}
\operatorname{div}(A_+(x)Du) &= f_+ \quad \text{in} \quad \Omega_+ := \{u>0\},\\
\operatorname{div}(A_-(x)Du) &= f_- \quad \text{in} \quad \Omega_- := \{u>0\}^\circ,
\end{aligned}
$$

while Hadamard's-type of arguments yield a flux condition across the free interface $F(u) := \partial\Omega_+ \cap \Omega$, depending on λ_+ and λ_-. The authors prove the existence of minimizers, with L^∞-bounds. In fact the proof of existence bypasses the lack of convexity of the functional and yield estimates in L^∞ as a by-product. Those local minima are proved to have a local modulus of continuity. Under the assumption that A_+ and A_- are close, in a sense made precise in that paper, the authors prove that solutions are indeed asymptotically Lipschitz. We emphasize that improved regularity under such small-jumps condition follows through a set of methods known as *geometric tangential analysis*. We refer the reader to the following survey papers on this class of techniques [59, 52, 60].

The problem examined in [1] profits from the existence of an associated functional and the properties derived for its minima. We remark those structures are not available in the context of (4.1).

4.2 Quadratic growth away from the free boundary

Let $x^* \in B_1$ be fixed. Consider the maximal subset of $\mathbb{N}$ whose elements j are such that

$$
\sup_{x\in B_{2^{-j-1}}(x^*)} |u(x)| \geq \frac{1}{16} \sup_{x\in B_{2^{-j}}(x^*)} |u(x)|; \tag{4.6}
$$

we denote such set by $\mathcal{M}(x^*,u)$.

Proposition 9 *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (4.1)*. Suppose A4-A7 hold true. Let $x^* \in \partial\Omega$. There exists $C_0 > 0$ such that, if*

$$
V_{2^{-j}}(x^*,u) < C_0, \tag{4.7}
$$

for every $j \in \mathcal{M}(x^*, u)$, then

$$\sup_{x \in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{1}{C_0} 2^{-2j}, \qquad \forall\, j \in \mathcal{M}(x^*, u).$$

Proof. For ease of notation, we set $x^* = 0$ and $\mathcal{M}(u) := \mathcal{M}(0, u)$. We resort to a contradiction argument; suppose the statement of the proposition is false. Then, there exist sequences $(u_n)_{n \in \mathbb{N}}$ and $(j_n)_{n \in \mathbb{N}}$ such that u_n is a normalized strong solution to (4.1),

$$V_{\frac{1}{2^n}}(u_n) < \frac{1}{n}, \tag{4.8}$$

with

$$\sup_{x \in B_{2^{-j_n}}} |u_n(x)| > \frac{n}{2^{2j_n}}, \tag{4.9}$$

for every $j_n \in \mathcal{M}(u_n)$, and $n \in \mathbb{N}$. Because $\|u_n\|_{L^\infty(B_1)}$ is uniformly bounded, it follows from (4.9) that $j_n \longrightarrow \infty$. In particular, we may re-write (4.8) as

$$V_{\frac{1}{2^{j_n}}}(u_n) < \frac{1}{j_n}. \tag{4.10}$$

Now, we introduce an auxiliary function $v_n : B_1 \to \mathbb{R}$, given by

$$v_n(x) := \frac{u_n(2^{-j_n} x)}{\|u_n\|_{L^\infty\left(B_{2^{-(j_n+1)}}\right)}}.$$

Clearly, $v_n(0) = 0$. In addition, $V_1(v_n) \longrightarrow 0$. Moreover, it follows from the definition of v_n that

$$\sup_{B_{1/2}} |v_n(x)| = 1 \tag{4.11}$$

and

$$\sup_{B_1} |v_n(x)| \leq 16.$$

We notice that A6 yields

$$G(D^2 v_n) = \frac{2^{-2j_n}}{\|u_n\|_{L^\infty\left(B_{2^{-(j_n+1)}}\right)}} G(D^2 u_n(2^{-j_n} x)).$$

Therefore,

$$\left| G(D^2 v_n) \right| \leq \frac{1}{n} \frac{C \|u_n\|_{L^\infty\left(B_{2^{-j_n}}\right)}}{\|u_n\|_{L^\infty\left(B_{2^{-(j_n+1)}}\right)}} \leq \frac{C}{n} \longrightarrow 0, \tag{4.12}$$

as $n \to \infty$.

It follows from the Krylov-Safonov theory that $(v_n)_{n \in \mathbb{N}}$ is equibounded in $\mathcal{C}^{1,\alpha}_{loc}(B_1)$, for some $\alpha \in (0,1)$. Therefore, there exists v_∞ such that $v_n \longrightarrow v_\infty$ in $\mathcal{C}^{1,\beta}_{loc}(B_1)$, for every $0 < \beta < \alpha$. Since $v_n(0) = 0$ for every $n \in \mathbb{N}$ we infer that $v_\infty(0) = 0$, whereas (4.11) leads to $\|v_\infty\|_{L^\infty(B_{1/2})} = 1$. Because $V_1(v_n) \longrightarrow 0$, we conclude that $v_\infty \geq 0$ in B_1.

By the same token, standard stability results for viscosity solutions build upon (4.12) to ensure

$$G(D^2 v_\infty) = 0 \qquad \text{in} \qquad B_1.$$

We conclude that v_∞ is a viscosity solution to a homogeneous equation which attains an interior local minimum at the origin. As a consequence of the strong maximum principle, we obtain a contradiction and complete the proof. ■

In Proposition 9 the constant $C_0 > 0$ is determined. This quantity remains unchanged throughout the thesis.

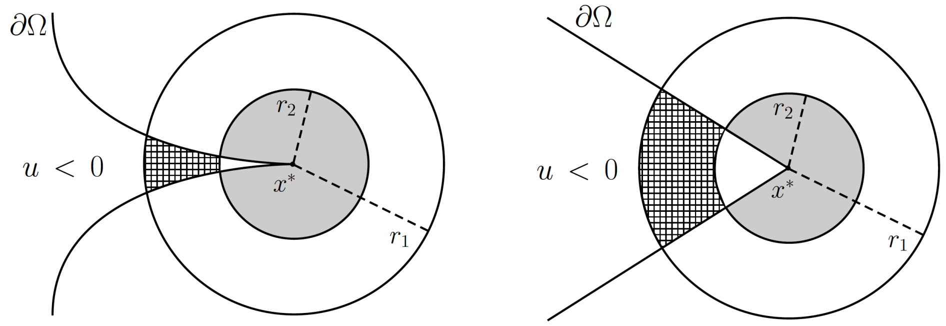

Figure 4.2: The geometry depicted on the left is within the scope of (4.7). In fact, as the radii of the balls centered at x^* decrease from r_1 to r_2, $V(x^*, r)$ decreases even faster. The case on the right behaves differently. Here, the normalized volume is constant, independent of the radii of the ball; hence, it might fail to satisfy a prescribed smallness regime as in (4.7).

The next result extrapolates the former analysis from $\mathcal{M}(x^*, u)$ to the entire set of natural numbers.

Proposition 10 *Let $u \in W^{2,d}(B_1)$ be a strong solution to (4.1). Suppose A4-A7 hold true. Let $x^* \in \partial\Omega$. Suppose further that for every $j \in \mathcal{M}(x^*, u)$ we have*

$$V_{2^{-j}}(x^*, u) < C_0,$$

for $C_0 > 0$ fixed in (4.7). Then

$$\sup_{x \in B_{2^{-j}}(x^*)} |u(x)| \leq \frac{4}{\varepsilon} 2^{-2j}, \qquad \forall j \in \mathbb{N}.$$

Proof. As before we set $x^* = 0$ and argue through a contradiction argument. Suppose the proposition is false. Let $m \in \mathbb{N}$ be the smallest natural number such that

$$\sup_{B_{2^{-m}}} |u(x)| > \frac{4}{\varepsilon} 2^{-2m}. \tag{4.13}$$

We claim that $m - 1 \in \mathcal{M}(u)$. Indeed,

$$\sup_{B_{2^{1-m}}} |u(x)| \le \frac{4}{\varepsilon} 2^{-2(m-1)} = \frac{16}{\varepsilon} 2^{-2m} < 4 \sup_{B_{2^{-m}}} |u(x)|.$$

We conclude

$$\sup_{B_{2^{-m}}} |u(x)| \le \sup_{B_{2^{1-m}}} |u(x)| \le \frac{1}{\varepsilon} 2^{-2(m-1)} = \frac{4}{\varepsilon} 2^{-2m},$$

which contradicts (4.13) and completes the proof. ■

Consequential to Proposition 10 is the quadratic growth of u away from the free boundary. This is the content of the next corollary.

Corollary 1 (Quadratic growth) *Let $u \in W^{2,d}(B_1)$ be a strong solution to (4.1). Suppose A4-A7 hold true. Let $x^* \in \partial\Omega \cap B_{1/2}$. Suppose further that, for every $j \in \mathcal{M}(x^*, u)$, we have*

$$V_{2^{-j}}(x^*, u) < C_0,$$

for $C_0 > 0$ as in Proposition 9. Then, for $0 < r < 1/2$ there exists $C > 0$ such that

$$\sup_{x \in B_r(x^*)} |u(x)| \le Cr^2,$$

where $C = C\left(d, \lambda, \Lambda, \|u\|_{L^\infty(B_1)}\right)$.

Proof. Find $j \in \mathbb{N}$ satisfying $2^{-(j+1)} \le r < 2^{-j}$. It is straightforward to notice that

$$\sup_{B_r} |u(x)| \le \sup_{B_{2^{-j}}} |u(x)| \le C \left[\left(\frac{1}{2}\right)^{j+1-1} \right]^2 \le Cr^2,$$

which ends the proof. ■

We close this section with the proof of Theorem 3; We restate it in what follows for completeness.

Theorem 8 (Restatement of Theorem 3) *Let $u \in W^{2,d}(B_1)$ be a strong solution to (1.2). Suppose A4-A7 hold true. Suppose further that $V_r(x, u) < C_0$ for every $x \in \partial(\Omega^+(u) \cup \Omega^-(u)) \cap B_{1/2}$, for some $C_0 > 0$. Then, $u \in \mathcal{C}^{1,1}_{loc}(B_1)$ and there exists a universal constant $C > 0$ such that*

$$\left\| D^2 u \right\|_{L^\infty(B_{1/2})} \le C.$$

Proof. Suppose $0 \in \partial\Omega$. Corollary 1 leads to

$$|u(x)| \leq C\,[\mathrm{dist}(x, \partial\Omega)]^2$$

for every $x \in B_{1/2}$. Consider the auxiliary function $v : B_1 \to \mathbb{R}$ given by

$$v(y) := \frac{u(x + y\,\mathrm{dist}(x, \partial\Omega))}{[\mathrm{dist}(x, \partial\Omega)]^2};$$

clearly, $|D^2u(x)| = |D^2v(0)|$. Notice that

$$\{z \in B_1 \mid y \in B_1 \quad \text{and} \quad z := x + y\,\mathrm{dist}(x, \partial\Omega)\}$$

is contained in the same connected component to which x belongs. Therefore, $F_i(D^2v) = 1$ or $F_1(D^2v) = 0$ in the unit ball. Hence, standard results in elliptic regularity theory produce

$$\left|D^2v(0)\right| \leq C,$$

for some universal constant $C > 0$, not depending on x, and the proof is complete. ■

In the next section, we turn our attention to the analysis of the free interface. We start working under the assumption $\{Du \neq 0\} \subset \Omega$ and produce a characterization of global solutions.

4.3
Classification of global solutions

In this section we examine the non-degeneracy of the free boundary. In addition we study properties of the global solution. We start with the non-degeneracy property. This is the content of the next proposition.

Proposition 11 (Non-degeneracy of the free boundary) *Let* $u \in W^{2,d}(B_1)$ *be a strong solution to* (1.2). *Suppose that A4-A5 and A8 are in force. Let* $x^* \in \partial\Omega \cap B_{1/2}$. *There exists* $C > 0$ *such that*

$$\sup_{x \in \partial B_r(x^*)} u(x) \geq Cr^2$$

for every $0 < r < 1/2$.

Proof. For ease of presentation, we split the proof in three steps.

Step 1. Without loss of generality, we take $x^* \in \Omega$. Furthermore, the set $\{Du \neq 0\}$ is dense in Ω. It follows from the fact that $D^2u(x) = 0$ for almost every $x \in \{Du = 0\}$, the last condition in A4 and (1.2). Therefore, we assume $x \in \{Du \neq 0\} \cap \Omega$.

Step 2. We introduce an auxiliary function $w \in W^{2,d}(B_1)$, given by

$$w(x) := u(x) - \frac{|x - x^*|^2}{2d\lambda}.$$

We claim that

$$\max_{x \in \partial B_r(x^*)} w(x) = \sup_{x \in B_r(x^*)} w(x), \tag{4.14}$$

for every $0 < r < 1/2$. It follows from (4.14) that

$$\max_{x \in B_r(x^*)} u(x) \geq u(x^*) + Cr^2,$$

where $C := 4d\Lambda$. By approximation, the former inequality yields the results. It remains to establish (4.14).

Step 3. Suppose (4.14) is false. There exists a maximum point $y \in B_r(x^*)$ for w. Hence,

$$Dw(y) = Du(y) - \frac{|y - x^*|^2}{4d\Lambda} = 0.$$

Were $y = x^*$, it would be $Du(x^*) = 0$; we conclude that $y \neq x^*$ and, in addition,

$$Du(y) \neq 0.$$

By assumption, we have $y \in \Omega$.

On the other hand, w is a subsolution for $G = 0$ in Ω. In fact, for $x \in \Omega$,

$$G(D^2 w(x)) \geq 1 - \mathcal{M}^+\left(\frac{I_d}{2d\lambda}\right) = \frac{1}{2}.$$

Since $y \in \Omega$, there exists a neighborhood of y where w is constant. We conclude w is constant in $B_r(x^*)$ and (4.14) follows. ■

Remark 1 The proof of Proposition 11 follows closely the ideas put forward in [34, Lemma 3.1].

The former result has a number of standard consequences. Of particular interest is the negligibility of the free boundary in the sense of Lebesgue.

Corollary 2 (Lebesgue negligibility of the free boudnary) *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (1.2)*. Suppose that A4-A5 and A8 are in force. Then $\partial\Omega$ has Lebesgue measure zero.*

Next, we establish the classification of global solutions. For completeness, we recall the definition of thickness $\delta_r(u, x_0)$: let $u \in W^{2,d}(B_1)$ be a strong solution to (1.2) and suppose $x_0 \in \partial\Omega(u)$. Then,

$$\delta_r(u, x_0) \;:=\; \frac{\mathrm{MD}\,(\Sigma(u) \cap B_r(x_0))}{r}.$$

We proceed with a proposition on the geometry of u.

Proposition 12 *Let $u \in W^{2,d}(B_1)$ be a strong solution to (1.2) in $\mathbb{R}^d$. Suppose A4-A8 are in force. Suppose further that there exists $\varepsilon_0 > 0$ such that*

$$\delta_r(u, x_0) \;\geq\; \varepsilon_0 \tag{4.15}$$

for all $r > 0$ and every $x_0 \in \partial\Omega$. Then u is a convex function.

Proof. We argue by contradiction. Suppose that u is not convex and define

$$-m \;:=\; \inf_{z\in\Omega,\, e\in\mathbb{S}^{d-1}} \partial_{ee}u(z) < 0. \tag{4.16}$$

We claim $m > 0$ is finite. In fact, because of Theorem 8, we have $u \in C^{1,1}(\mathbb{R}^d)$. Now, consider a minimizing sequence $(y_n, e_n) \subset \Omega \times \mathbb{S}^{d-1}$ for (4.16); that is,

$$\partial e_n e_n u(y_n) \;\longrightarrow\; -m$$

as $n \to \infty$. Define the rescaled function

$$u_n(x) \;:=\; \frac{u(d_n x + y_n) \;-\; u(y_n) \;-\; d_n Du(y_n)\cdot x}{d_n^2},$$

where $d_n := \mathrm{dist}(y_n, \partial\Omega)$. Define further

$$\Omega_n \;:=\; \frac{\Omega \;-\; y_n}{d_n} \qquad \text{and} \qquad \ell_n \;:=\; -\frac{Du(y_n)}{d_n}.$$

Since $Du = 0$ on $\partial\Omega$, we conclude $Du_n = \ell_n$ on $\partial\Omega_n$.

Now, observe that given $y_n \in \Omega$, we either have $y_n \in \Omega^+$ or $y_n \in \Omega^-$. It follows that $z = d_n x + y_n$ belongs either to Ω^+ or Ω^-. Therefore, we have

$$F_1(D^2 u_n) = 1 \qquad \text{or} \qquad F_2(D^2 u_n) \;=\; 1$$

in Ω_n. In any case u_n is $C^{2,\alpha}$-regular; hence, $\ell_n < C$ for every $n \in \mathbb{N}$ and some $C > 0$, universal. As a consequence, we have $\ell_n \to \ell_\infty$, through some subsequence if necessary.

Without loss of generality, suppose $e_n \to e_1$, as $n \to \infty$, through a subsequence, if required. Since $(u_j)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}^{2,\alpha}(B_{1/2})$ there exists $u_\infty \in \mathcal{C}^{2,\beta}(B_{1/2})$ such that $u_n \to u_\infty$ in the $\mathcal{C}^{2,\beta}$-topology, for $0 < \beta < \alpha$. Moreover, $\partial_{11} u_\infty(0) = -m$.

Because G is convex, $\partial_{11}u_\infty$ is a supersolution of the equation driven by the linear operator $G_{ij}(D^2u_\infty)\partial_{ij}$. Let Ω_∞ be the connected component containing B_1. Since $\partial_{11}u_\infty(z) \geq -m$ in B_1, the strong maximum principle yields $\partial_{11}u_\infty \equiv -m$ in Ω_∞.

Without loss of generality we can assume $Du_\infty(x) = 0$ on $\partial\Omega_\infty$; indeed, it follows from an affine transformation of u_∞. For any $e \in \mathbb{S}^{d-1}$ we have $\partial_{ee}u_\infty(z) \geq -m$ in B_1; in addition, the directional Hessian along e_1 attains $-m$. We conclude e_1 is an eigenvector for D^2u at every point, associated with the smallest eigenvalue. It follows that $\partial_{1j}u_\infty = 0$ along $\partial\Omega_\infty$, for any $j = 2, \ldots, d$. Integrating u_∞ in the direction of e_1 we deduce deduce

$$u_\infty(x) = P(x) := -m\frac{x_1^2}{2} + ax_1 + b(x') \quad \text{in} \quad \Omega_\infty,$$

where $x' = (x_2, \ldots, x_d)$, $a \in \mathbb{R}$ is a fixed constant and $b : \mathbb{R}^{d-1} \to \mathbb{R}$.

Observe that

$$\frac{\partial}{\partial x_1}P(x) = -mx_1 + a;$$

hence, $\partial_1 P$ vanishes along the set $\{x_1 = a/m\}$. On the other hand, the fact that $Du_\infty = 0$ on $\partial\Omega_\infty$ yields $\partial_1 u_\infty = \partial_1 P = 0$ on $\partial\Omega_\infty$. As a consequence, we infer $\partial\Omega_\infty \subset \{x_1 = a/m\}$. At this point we distinguish two cases related to the former inclusion.

Case 1, $\partial\Omega_\infty \neq \{x_1 = a/m\}$ **-** It follows that $\mathbb{R}^d \backslash \{x_1 = a/m\} \subset \Omega_\infty$, and a further alternative is available, i.e.:

$$F_1(D^2u_\infty) = 1 \qquad \text{or} \qquad F_2(D^2u_\infty) = 1$$

almost everywhere in $\mathbb{R}^d$. The Evans-Krylov Theorem applies to $u_r(y) := u_\infty(ry)/r^2$ inside B_1 to produce

$$\sup_{x,z\in B_r} \frac{|D^2u_\infty(x) - D^2u_\infty(z)|}{|x-z|^\alpha} \leq \frac{C}{r^\alpha}.$$

Letting $r \to \infty$ we deduce that D^2u_∞ is constant. Hence $u_\infty(x)$ is a second order polynomial.

Case 2, $\partial\Omega_\infty = \{x_1 = a/m\}$ **-** In this case, we have $D_{x'}P = 0$ on $\{x_1 = a/m\}$ (recall that $Du_\infty = 0$ on $\partial\Omega_\infty$). Thus, b is constant and we obtain

$$u_\infty(x) = -m\frac{x_1^2}{2} + ax_1 + b \quad \text{in} \quad \{x_1 > a/m\},$$

which implies $D^2u_\infty \equiv -m\,\mathrm{Id}$. Being negative-definite, D^2u_∞ cannot satisfy either $F_1(D^2u_\infty) = 1$ or $F_2(D^2u_\infty) = 1$, which leads to a contradiction.

Therefore, if u is not convex, it has to be a second order polynomial. By combining (4.15) and Proposition 4 we conclude that u_∞ can not be a second degree polynomial, which lead us to a contradiction. ■

Corollary 3 *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (1.2) *in $\mathbb{R}^d$. Suppose A4-A8 are in force. Suppose further* (4.15) *is in force. Then $\Omega = \{Du \neq 0\}$.*

Proof. Because u is convex, its set of critical points coincide with its set of minima; in addition, the set of minima of a convex function is trivially convex. Hence, $\{Du = 0\}$ is convex. Since $F_1(D^2u) = 1$ in Ω^+ and $F_2(D^2u) = 1$ in Ω^- we have that $|\Omega\backslash\{Du \neq 0\}| = 0$. As a consequence, the convex set $\{Du = 0\}$ has measure zero in Ω; if the former in nonempty, it must have co-dimension 1 and, therefore, violates the thickness condition (4.15). Hence, $\Omega = \{Du \neq 0\}$. ■

Next, we prove Theorem 4.

Theorem 9 (Restatement of Theorem 4) *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (1.2) *in $\mathbb{R}^d$. Suppose A4-A8 are in force. Suppose further there exists $\varepsilon_0 > 0$ such that*

$$\frac{MD\left((B_1 \setminus \Omega) \cap B_r(x)\right)}{r} > \varepsilon_0,$$

where $0 < r \ll 1$ and $x \in \partial\Omega$. Then u is a half-space solution. That is, up to a rotation,

$$u(x) = \frac{\gamma[(x_1)_+]^2}{2} + C,$$

where $C \in \mathbb{R}$ and $\gamma \in (1/\Lambda, 1/\lambda)$ is such that either $F_1(\gamma e_1 \otimes e_1) = 1$ or $F_2(\gamma e_1 \otimes e_1) = 1$.

Proof of Theorem 4. For simplicity we suppose that $0 \in \partial\Omega$. For $r > 0$ define

$$u_r(x) := \frac{u(rx)}{r^2}$$

and let u_∞ be the limit, up to a sequence if necessary, of u_r as $r \to \infty$. Notice that

$$\Sigma(u_\infty) = \{\Sigma(u) : tx \in \Sigma(u) \quad \forall t > 0\}.$$

Next, we prove that $\Sigma(u_\infty)$ is a half-space. As before, we resort to a contradiction argument. Suppose $\Sigma(u_\infty)$ is a not a half-space; then in some system of coordinates, we have

$$\Sigma(u_\infty) \subset \mathcal{C}_{\theta_0} := \{x \in \mathbb{R}^d; x = (\rho\cos\theta, \rho\sin\theta, x_3, \ldots, x_d), \theta_0 \leq |\theta| \leq \pi\},$$

for some $\theta_0 > \pi/2$.

Choose $\theta_1 \in (\pi/2, \theta_0)$ and set $\alpha := \pi/\theta_1$. Then for $\beta > 0$ sufficiently large, the function

$$v := r^\alpha(e^{-\beta\sin(\alpha\theta)} - e^{-\beta})$$

is a positive subsolution for the linear operator $G_{ij}(D^2u)\partial_{ij}$ inside $\mathbb{R}^d\backslash\mathcal{C}_1$, which vanishes on $\partial\mathcal{C}_{\theta_1}$. From Proposition 12 we have that u_∞ is convex; thus we deduce that $\partial_1 u_\infty > 0$ in $\mathbb{R}^d\backslash\mathcal{C}_{\theta_0}$. In addition $\theta_0 > \theta_1$. Hence, by the comparison principle we obtain that

$$v \leq \partial_1 u_\infty.$$

Therefore, $\Sigma(u_\infty)$ is a half-space that happens to be convex. As a consequence, it follows that that $\Sigma(u)$ is also a half space.

Finally, we apply global $C^{2,\alpha}$-estimates to u inside the half-ball $B_1\backslash\Sigma(u)$; see, for instance, [56]. We obtain

$$\sup_{x,z\in B_r\backslash\Sigma(u)} \frac{|D^2u(x) - D^2u(z)|}{|x-z|^\alpha} \leq \frac{C}{r^\alpha}.$$

Thus, letting $r \to \infty$ we conclude that D^2u is constant and hence u is a second order polynomial inside the half-space $\mathbb{R}^d\backslash\Sigma(u)$. Recall that $Du = 0$ on the hyperplane $\partial\Sigma(u)$, because we have supposed $\{Du \neq 0\} \subset \Omega$. Hence, we conclude u is a half-space solution and complete the proof. ■

In what follows we produce information on the regularity of the free boundary *dropping the condition* $\{Du \neq 0\} \subset \Omega$. Our analysis focuses on the non-degenerate points $x \in \mathcal{N}(u)$.

4.4 Regularity of the free boundary

In this section we focus on non-degenerate points. For a point $x^* \in \mathcal{N}(u)$ we define the normal vector at x^* as the direction $\nu : \mathcal{N}(u) \to \mathbb{S}^{n-1}$ given by

$$\nu_{x^*} := \frac{Du(x^*)}{|Du(x^*)|}.$$

In the sequel our arguments build upon the $\mathcal{C}^{1,1}$-regularity of the solutions to (4.1). Hence, throughout this section, we suppose (4.7) is in force with $\varepsilon > 0$ as determined in Proposition 9.

Proposition 13 *Let $u \in W^{2,d}(B_1)$ be a solution to* (4.1)*. Suppose A4-A7 hold true. Suppose further that $x^* \in \mathcal{N}(u)$ and define $\delta > 0$ as*

$$\limsup_{x \to x^*} \frac{|u(x)|}{|x - x^*|} = \delta.$$

There exists a universal constant $C > 0$, such that

$$B_r(x^*) \cap \Sigma(u) \subset \left\{ x \in B_1 \mid |(x - x^*) \cdot \nu_{x^*}| \leq \frac{C}{\delta}|x - x^*|^2 \right\}, \tag{4.17}$$

for every $0 < r \ll 1$.

Proof. Notice that for $x^* \in \Sigma(u)$ we have

$$\limsup_{x \to x^*} \frac{|u(x) - u(x^*)|}{|x - x^*|} = \delta > 0.$$

It implies that $|Du(x^*)| \geq \delta$. From the $\mathcal{C}^{1,1}$-regularity of u we infer that, for $x \in \Sigma(u)$, is holds

$$|Du(z) \cdot (x - x^*)| \leq C|x - x^*|^2.$$

Therefore

$$|\nu_{x^*} \cdot (x - x^*)| \leq \frac{C}{\delta}|x - x^*|^2.$$

$\blacksquare$

Proposition 14 *Let $u \in W^{2,d}(B_1)$ be a solution to* (4.1)*. Suppose A4-A7 hold true. Suppose further that $x^* \in \mathcal{N}(u)$ and*

$$\limsup_{x \to x^*} \frac{|u(x)|}{|x - x^*|} = \delta.$$

Then, there are universal constants $C > 0$ and $r > 0$ such that $\Sigma(u) \cap B_r(x^*) \subset \mathcal{N}(u)$ and that for any $\xi \in \Sigma(u) \cap B_r(x^*)$,

$$\limsup_{x \to \xi} \frac{|u(x)|}{|x - \xi|} \geq C\delta.$$

Proof. Without loss of generality we can assume $x^* = 0$. Take $\varepsilon > 0$, to be fixed later. The definition of limit superior yields the existence of $r_0 > 0$ such that, for $0 < r \leq r_0$, one can find $x_r \in B_r \setminus \{0\}$ satisfying $|u(x_r)| > (\delta - \varepsilon)|x_r|$. By taking $\varepsilon := \delta/2$ we obtain

$$\sup_{B_r} |u(x)| > \frac{\delta}{2} r,$$

for $0 < r \leq r_0$. Fix r and choose $\xi \in (B_r(0)\backslash\{0\}) \cap \Gamma(u)$. Set $\rho = |\xi|$; then

$$\sup_{B_{2\rho}(\xi)} |u(x)| \geq \delta\rho.$$

Hence, there exists $x_0 \in B_{2\rho}(\xi)$ such that

$$|u(x_0)| \geq \delta\rho. \tag{4.18}$$

The regularity of the solutions and Proposition 13 yield

$$|u(x_0) - Du(\xi) \cdot (x_0 - \xi)| \leq C|x_0 - \xi|^2.$$

In addition,

$$|Du(\xi) \cdot (x_0 - \xi)| \leq 2\rho|Du(\xi)|.$$

The inequality in (4.18) and a straightforward use of the triangle inequality produce

$$|Du(\xi)| \geq \frac{1}{2}(\delta - 4C\rho) \geq \frac{\delta}{4},$$

provided

$$\rho \leq \frac{\delta}{8C}.$$

Set $x_r := \xi + r\nu_\xi$ for $r > 0$ and notice that

$$|u(x_r)| \geq |Du(\xi)|r - Cr^2 \geq \left(\frac{\delta}{4} - Cr\right) r \geq \frac{\delta}{8}r,$$

provided

$$r \leq \frac{\delta}{8C}.$$

Hence

$$\sup_{B_r(\xi)} |u(x)| \geq |u(x_r)| \geq \frac{\delta}{8}r,$$

which, in turn, implies

$$\sup_{B_r(z)} \frac{|u(x)|}{|x - \xi|} \geq \frac{\delta}{8} > 0.$$

Therefore $\xi \in \mathcal{N}(u)$ and the proof is complete. ■

Now we detail the proof of Theorem 5.

Theorem 10 (Restatement of Theorem 5) *Let $u \in W^{2,d}(B_1)$ be a strong solution to* (1.2)*. Suppose A4-A7 hold true. Then $\mathcal{N}(u)$ is, locally, a graph of class $\mathcal{C}^{1,1}$. In addition, there exists a universal constant $C > 0$ such that for* *all* $z \in \mathcal{N}(u)$*, we have*

$$|\nu_x - \nu_y| \leq C|x - y|$$

for every $x,\ y \in B_r(z) \cap \Sigma(u)$ *and every* $0 < r \ll 1$*.*

Proof. Without loss of generality we may assume that

$$\limsup_{x \to 0} \frac{|u(x)|}{|x|} = 1.$$

Let $z \in B_r \cap \Sigma(u)$. For some $e \in \mathbb{S}^{d-1}$ and $\rho = |z|$, set $x := z + \rho e$. Observe that $x \in \partial B_\rho(z) \cap \Sigma(u)$. The regularity of u yields

$$|\nu_0 \cdot x| \leq C|x|^2,$$

for all $x \in B_r \cap \Sigma(u)$. Notice that

$$\rho|e \cdot \nu_0| \leq |(\rho e + z) \cdot \nu_0| + |z \cdot \nu_0| \leq C|x|^2 + C\rho^2 \leq C\rho^2.$$

Now, select a $(d-1)$-uple $(e^1, e^2, \ldots, e^{d-1})$ such that $z + \rho e^i \in \Sigma(u)$ and that $(e^1, \ldots, e^{d-1}, \nu_z)$ spans $\mathbb{R}^d$. Since

$$\nu_0 = \sum_{i=1}^{d-1} (\nu_0 \cdot e^i) e^i + (\nu_0 \cdot \nu_z)\nu_z,$$

we obtain

$$1 = \sum_{i=1}^{d-1} (\nu_0 \cdot e^i)^2 + (\nu_0 \cdot \nu_z)^2$$

Hence,

$$|\nu(z) - \nu(0)|^2 \leq 2\left[1 - \left(1 - \sum_{i=1}^{d-1} (\nu_0 \cdot e^i)^2\right)^{1/2}\right] = 2\sum_{i=1}^{d-1} (\nu_0 \cdot e^i)^2 \leq C\rho^2,$$

which leads to

$$|\nu_z - \nu_0| \leq C\rho$$

and completes the proof. ∎

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