Pontifícia Universidade Católica
do Rio de Janeiro

Rafael Antonio Sanabria Villalobos

Inelastic Boltzmann equation driven by a particle thermal bath

TESE DE DOUTORADO

Thesis presented to the Programa de Pós–graduação em Matemática of PUC–Rio in partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Ricardo José Alonso Plata

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Abstract

Sanabria Villalobos, Rafael Antonio; Alonso Plata, Ricardo José (Advisor). **Inelastic Boltzmann equation driven by a particle thermal bath**. Rio de Janeiro, 2020. 87p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

We consider the spatially inhomogeneous Boltzmann equation for inelastic hard-spheres, with constant restitution coefficient $\alpha \in (0,1)$, under the thermalization induced by a host medium with a fixed Maxwellian distribution and any fixed $e \in (0,1]$. When the restitution coefficient α is close to 1 we prove existence of global solutions considering the close-to-equilibrium regime. We also study the long-time behaviour of these solutions and prove a convergence to equilibrium with an exponential rate.

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Keywords

Kinetic theory; Boltzmann equation; Non homogeneus equation; Inelastic equation; Forcing term; Thermal bath.

Resumo

Sanabria Villalobos, Rafael Antonio; Alonso Plata, Ricardo José. **Equação inelastica de Boltzmann com banho térmico**. Rio de Janeiro, 2020. 87p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Consideramos a equação de Boltzmann espacialmente não homogênea para esferas duras inelásticas, com coeficiente de restituição constante $\alpha \in (0,1)$, sob a termalização induzida por um meio hospedeiro com uma distribuição Maxwelliana fixa e fixando $e \in (0,1)$ qualquer. Quando o coeficiente de restituição α é próximo de 1, comprovamos a existência de soluções globais considerando o regime próximo ao equilíbrio. Também estudamos o comportamento de longo prazo dessas soluções e comprovamos uma convergência para o equilíbrio com uma taxa exponencial.

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Palavras–chave

Teoría kinética; Equação de Boltzmann; Equação não homogênea; Equação inelástica; Termo de força; Banho térmico;

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1
Introduction

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The subject of granular matter is a relatively new and expanding field. Examples of granular matters are very abundant: salt, pepper, sand, cold, pebbles, planetary rings composed of ice "grains", and more. Understanding the flow and packing of such objects is of great practical importance to several industries, such as construction, chemical, pharmaceutical and agricultural for instance, and more recently to NASA's planned Moon and Mars programs. They need to store, handle and transport granular materials, any problem that arises in the processing of granular materials can translate into enormous monetary losses and tragedies. A collection of granular objects is also an intriguing physical system. The qualitative and quantitative description of this type of materials pose physical and mathematical challenges which turn them very appealing to scientists (see [Goldhirsch, 2005] for a more detail description).

Granular materials exhibit different phases. The grains in a sand pile are practically stationary. Weak forces may be resisted by a "near elastic" response. Stronger forces may cost deformation of the material. Meanwhile, stronger forces will fluidized the entire granular system, resulting in a slow flow. With even stronger forces it may become a "granular gas", where the grains are subject of binary collisions.

The goal of kinetic gas theory is to model any system consisting of a large amount of particles. There exist different ways to describe such a system.The microscopic description, for example, studies trajectories of each particle. Although, this has drawbacks due to the large amount of particles considered and, for the fact that it does not give access to observables like mass, mean velocity and temperature. On the other hand, the macroscopic description focuses on study macroscopic quantities of this system, where the amounts mentioned above can be effectively measure. Kinetic theory is precisely on one level intermediary between these two descriptions, microscopic and macroscopic that is called the *mesoscopic description*. This is a statistical description, whose purpose is to describe the "typical" behavior of a particle, thus simplifying detailed study of the trajectories of each particle while preserving the information of the system.

Classical kinetic gas theory considers elastic particle collisions, which implies conservation of mechanical energy. Therefore, it takes a molecular gas just a few collisions per particle to relax to its equilibrium state, characterized by a Maxwellian velocity distribution function.

It is tempting to consider the grains of a granular system as molecules in a solid or fluid. Although, there are fundamental differences between them. Whereas molecules interact by reversible forces, grains experience friction and inelastic collisions. This means that in grain collisions energy is lost to internal degrees of freedom and their description is irreversible. In the pioneering articles by Goldhirsch, Zanetti [Goldhirsch et al., 1993] and McNamara, Young [McNamara and Young, 1993] in 1993 they established that a gas of particles which collide inelastically behaves qualitatively different compared to a molecular gas. This type of gases are called *granular gases* and the inelasticity is based on the fact that these particles are themselves macroscopic bodies with many degrees of freedom.

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1.1 Kinetic model

Dilute granular flows are commonly modelled by Boltzmann equation for inelastic hard-spheres interacting through binary collisions [Brilliantov and Pöschel, 2010]. Due to dissipative collisions, energy continuously decreases in time which implies that, in absence of energy supply, the corresponding dissipative Boltzmann equation admits only trivial equilibria. This is no longer the case if the spheres are forced to interact with a forcing term, in which case the energy supply may lead to a non-trivial steady state. For such a driven system we consider hard spheres particles described by their distribution density $f = f(t, x, v) \geq 0$, $x \in \mathbb{T}^3$, $t > 0$ satisfying

$$\partial_t f + v \cdot \nabla_x f = \mathcal{Q}_\alpha(f, f) + \mathcal{L}(f). \tag{1.1}$$

where $\mathcal{Q}_\alpha(f, f)$ is the inelastic quadratic Boltzmann collision operator, and $\mathcal{L}(f)$ models the forcing term. The parameter $\alpha \in (0, 1)$ is the so called "restitution coefficient" that characterized the inelasticity of the binary collisions. The purely elastic case is recover when $\alpha = 1$.

We assume the granular particles to be perfectly smooth hard spheres of mass $m = 1$ performing inelastic collisions. In the model at stake, the inelasticity is characterized by the so-called normal restitution coefficient $\alpha \in (0, 1)$. The Figure 1.1 below provides a schematic picture of what goes on. The incoming velocities are v and v_*, while v' and v'_* represent the outgoing ones and $n \in \mathbb{S}^2$ stands for the unit vector that points from the v-particle center to the v_*-particle center at the moment of impact, that is the impact direction. In the elastic case the outgoing velocities would be given by the dashed arrows. However, because of inelasticity effect, there is some loss of momentum in the impact direction, resulting in the boldface arrows indicating the outgoing velocities v' and v'_*.

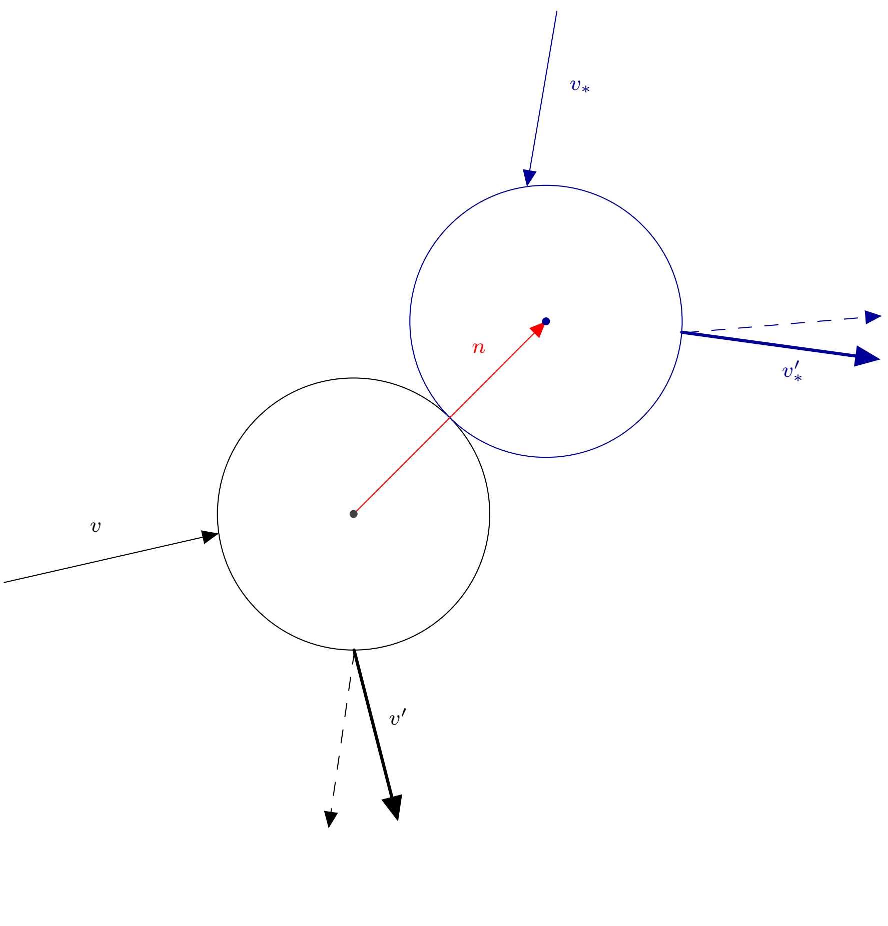

Figure 1.1: Binary Collision.

The restitution coefficient quantifies the loss of relative normal velocity of a pair of colliding particles after the collision with respect to the impact

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velocity (see [Brilliantov and Pöschel, 2010, Chapter 2]). More precisely, we have the following equalities

$$\begin{cases} v + v_* = v' + v'_*, \\ u' \cdot n = -\alpha(u \cdot n), \end{cases} \tag{1.2}$$

where $u = v - v_*$ and, $u' = v' - v'_*$. The velocities after collision are then given by

$$v' = v - \frac{1+\alpha}{2}(u \cdot n)n, \quad v'_* = v_* + \frac{1+\alpha}{2}(u \cdot n)n.$$

In particular, the rate of kinetic energy dissipation is

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 \leq -\frac{1-\alpha^2}{4}(u \cdot n)^2 \leq 0. \tag{1.3}$$

It is worth to mention that there exist two limit regimes $e = 1$ and $e = 0$ respectively correspond to elastic collisions, where there is no loss of energy, and sticky collisions, where after collision particles travel together.

The velocities v' have to lie on a certain sphere S with center c' and radius $r \leq |v - v_*|/2$. It is often convenient to parameterize collisions by the direction σ of the vector $v' - c'$. This is called the σ parametrization. See Figure 1.2 where the dashed line represent the elastic rules of collision. Moreover, σ varies in the unit sphere $\mathbb{S}^2$ and, if we set $\hat{u} = u/|u|$, performing in (1.2) the change of unknown $\sigma = \hat{u} - 2(\hat{u} \cdot n)n \in \mathbb{S}^2$, the post-collisional velocities v' and v'_* are given by

$$v' = v - \frac{1+\alpha}{2} \cdot \frac{u - |u|\sigma}{2}, \quad v'_* = v_* + \frac{1+\alpha}{2} \cdot \frac{u - |u|\sigma}{2}.$$

Given a constant restitution coefficient $\alpha \in (0, 1)$, one defines the weak form of the bilinear Boltzman operator $\mathcal{Q}_\alpha$ for inelastic interactions and hard spheres by its action on test functions $\phi(v)$ (see for example [Cañizo and Lods, 2016, Section 2]),

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, g)\phi(v)dv &= \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} f(v)g(v_*)[\phi(v') - \phi(v)]|v - v_*|d\sigma dv_* dv, \\ &= \frac{1}{2}\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} f(v)g(v_*)[\phi(v'_*) + \phi(v') - \phi(v_*) - \phi(v)]|v - v_*|d\sigma dv_* dv. \end{aligned} \tag{1.4}$$

Here the factor $|v - v_*|$ is characteristic of the "hard sphere model": the mean number of collisions between particles of given velocities is proportional to the difference of velocities. Furthermore, the parameters x and t do not explicitly appear in (1.4), this is because collisions are supposed to be localized in space

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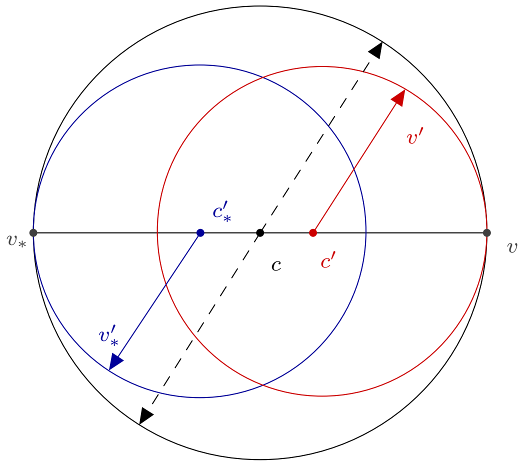

Figure 1.2: σ parametrization.

and time.

On the other hand, we can consider $('v, 'v_*)$ the pre-collisional velocities which result in (v, v_*) after collision, that is, the velocities given by

$$'v = v - \frac{1+\alpha}{2\alpha}(u \cdot n)n, \quad 'v_* = v_* + \frac{1+\alpha}{2\alpha}(u \cdot n)n.$$

Note carefully that $'v$ and $'v_*$ do not coincide with v' and v'_*, this means that the collisions are not reversible.

Using this pre-collisional velocities we can write the Boltzmann operator in its strong form (see for example [Arlotti and Lods, 2007, Villani, 2006])

$$\mathcal{Q}_\alpha(f,g) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} \left(\frac{1}{\alpha^2} f('v) g('v_*) - f(v) g(v_*) \right) |u \cdot n| dv_* dn.$$

1.2 Forcing term

In the literature there exist several possible physically meaningful choices for the forcing term $\mathcal{L}$ in order to avoid the cooling of the granular gas. The first one is the pure diffusion thermal bath, studied in [Gamba et al., 2004,

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Mischler and Mouhot, 2009b, Tristani, 2016], for which

$$\mathcal{L}_1(f) = \mu\Delta_v f,$$

where $\mu > 0$ is a given parameter and Δ_v the Laplacian in the velocity variable. Other fundamental examples of forcing terms are the thermal bath with linear friction

$$\mathcal{L}_2(f) = \mu\Delta_v f + \lambda\,\mathrm{div}(vf),$$

where μ and λ are positive constants and div is the divergence operator with respect to the velocity variable. Also, we have to mention the fundamental example of anti-drift forcing term which is related to the existence of self-similar solution to the inelastic Boltzmann equation:

$$\mathcal{L}_3(f) = -\lambda\,\mathrm{div}(vf), \quad \lambda > 0.$$

This problem has been treated in [Mischler and Mouhot, 2009a, Mischler and Mouhot, 2006] for hard spheres.

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We consider a situation in which the system of inelastic hard spheres is immersed into a thermal bath of particles, so that the forcing term $\mathcal{L}$ is given by a linear scattering operator describing inelastic collisions with the background medium. More precisely, the forcing term $\mathcal{L}$ is given by a linear Boltzmann collision operator of the form

$$\mathcal{L}(f) := \mathcal{Q}_e(f, \mathcal{M}_0), \tag{1.5}$$

where $\mathcal{Q}_e(\cdot,\cdot)$ is the Boltzmann collision operator associated to the fixed restitution coefficient $e \in (0,1)$, and $\mathcal{M}_0$ stands for the distribution function of the host fluid which we assume to be a given Maxwellian with unit mass, bulk velocity u_0 and temperature $\theta > 0$:

$$\mathcal{M}_0(v) = \left(\frac{1}{2\pi\theta_0}\right)^{\frac{3}{2}} \exp\left(-\frac{(v-u_0)^2}{2\theta_0}\right), \quad v \in \mathbb{R}^3\,. \tag{1.6}$$

An important feature of the collision operators $\mathcal{Q}_\alpha(f,f)$ and $\mathcal{L}(f)$ is that they both preserve mass. That is

$$\int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f,f)dv = \int_{\mathbb{R}^3} \mathcal{L}(f)dv = 0.$$

However, only the operator $\mathcal{Q}_\alpha$ preserves momentum. Neither the momentum nor the energy are conserved by the $\mathcal{L}$ operator.

1.3 Description of the problem and strategy of the proof

Our main result is the proof of existence of solutions for the non-linear problem (1.1) as well as stability and relaxation to equilibrium for these solutions.

The existence of smooth stationary solutions F_α for the inelastic Boltzmann equation under the thermalization given by the forcing term $\mathcal{L}$ has been proved in [Bisi et al., 2008]. Moreover, uniqueness of the steady state is proven in [Bisi et al., 2011] for a smaller range of parameters α (see Section 2.4). We are able to prove an existence theorem in a close to-equilibrium regime (a precise statement is given in Chapter 5):

Theorem 1.3.1. *Consider the functional spaces* $\mathcal{E} = W_x^{s,1} L_v^1(e^{b\langle v\rangle^\beta})$ *and* $\mathcal{E}_1 = W_x^{s,1} L_v^1(\langle v\rangle\, e^{b\langle v\rangle^\beta})$ *where* $b > 0$*,* $\beta \in (0,1)$ *and* $s > 6$*. For* α *close to 1, for any* $e \in (0,1]$*, and for an initial datum* $f \in \mathcal{E}_1$ *close enough to the equilibrium* F_α*, there exists a unique global solution* $f \in L_t^\infty(\mathcal{E}) \cap L_t^1(\mathcal{E}_1)$ *to* (1.1) *which furthermore satisfies for all* $t \geq 0$*,*

$$\|f_t - F_\alpha\|_{\mathcal{E}_0} \leq C e^{-at} \|f_{in} - F_\alpha\|_{\mathcal{E}_0},$$

for some constructive constants C *and* a*.*

The relevance of this work relies on the fact that it is one of the few existence result obtain in the spatially inhomogeneous case in an inelastic collision regime with thermal bath. Most of the previous results have been established in an homogeneous framework. The works of Bisi, Carrillos, Lods [Bisi et al., 2008]; Bisi, Cañizo, Lods [Bisi et al., 2011, Bisi et al., 2015] and Cañizo Lods [Cañizo and Lods, 2016] are the most relevant ones. On the other hand, for the inhomogeneous inelastic Boltzmann equation the literature is scarce. It is worth mentioning the work of Alonso [Alonso, 2009] that treats the Cauchy problem in the case of near-vacuum data. Recently, Tristani [Tristani, 2016] studied the existence, stability and relaxation to equilibrium for the solutions of the spatially inhomogeneous diffusively driven inelastic Boltzmann equation. That is, the equation (1.1) taking $\mathcal{L}_3$ as the forcing term.

In the inhomogeneous elastic case the Cauchy problem is usually handled by the theory of perturbative solutions. This is based on the study of the linearized associated operator. However, this strategy was not available in the inelastic case, due to the absence of precise spectral study of the linearized

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problem. Another well-known theory in the elastic case is the one of DiPerna-Lions renormalized solutions [DiPerna and Lions, 1989] which is no longer available in the inelastic case due to the lack of entropy estimates for the inelastic Boltzmann equation.

The recent work of Gualdani, Mischler, Mouhot [Gualdani et al., 2017] presented a new technique to the spectral study of the elastic inhomogeneous regime. They presented an abstract method for deriving decay estimates on the resolvents and semigroups of non-symmetric operators in Banach spaces in terms of estimates in another smaller reference Banach space. As a consequence, they obtained the first constructive proof of exponential decay, with sharp rate, towards global equilibrium for the full nonlinear Boltzmann equation for hard spheres, conditionally to some smoothness and (polynomial) moment estimates. Furthermore, they strategy inspired several works in the kinetic theory of granular gases like [Alonso et al., 2019, Alonso et al., 2017, Bisi et al., 2011, Bisi et al., 2015, Cañizo and Lods, 2016]. Using Gualdani et. al. approach, Tristani [Tristani, 2016] was able to develop a perturbative argument around the elastic case in the same line as the one developed by Mischler, Mouhot [Mischler and Mouhot, 2009a, Mischler and Mouhot, 2009b].

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The strategy in this paper consists in combine the main ideas adopted in [Tristani, 2016] with the arguments given by [Bisi et al., 2011] and [Cañizo and Lods, 2016]. To develop a Cauchy theory for the equation (1.1), we first study the linearized problem around the equilibrium. Thus, we linearize our equation with the ansatz $f = F_\alpha + h$. Let us denote by $\mathscr{L}_\alpha$ the linearized operator obtained by this ansatz. That is $\mathscr{L}_\alpha(h) = \mathcal{Q}_\alpha(F_\alpha, h) + \mathcal{Q}_\alpha(h, F_\alpha) + \mathcal{L}(h) - v \cdot \nabla_x h$ (see Section 2.5).

The study of the elastic case consists in deducing the spectral properties in L^1 from the well-known spectral analysis in L^2. This can be done thanks to a suitable splitting of the linearized operator as $\mathscr{L}_1 = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is bounded and $\mathcal{B}$ is “dissipative” operator, which are defined through an appropriate mollification-truncation process. This process is done in the same line as [Gualdani et al., 2017] but incorporating the ideas of [Bisi et al., 2011] for the splitting of the forcing term $\mathcal{L}$. We conclude that the spectrum of the linearized elastic operator is well localized. That means, it has an spectral gap in a large class of Sobolev spaces.

A crucial point in our approach is that it strongly relies on the understanding of the elastic problem corresponding to $\alpha = 1$. Due to the properties of the equilibrium F_α presented in [Bisi et al., 2011], we are able to prove that

$$\mathscr{L}_\alpha - \mathscr{L}_1 = O(1-\alpha), \tag{1.7}$$

for a suitable norm operator. Thus, one deduces the spectral properties of $\mathscr{L}_\alpha$ from those of the elastic operator by a perturbation argument valid for α close enough to 1. Notice that we only restrict the range of α, e is independent of α and can be taking in $(0, 1]$.

Moreover, as in the case of the linearized operator, we obtain an splitting $\mathscr{L}_\alpha = \mathcal{A}_\alpha + \mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ enjoys some dissipative properties and $\mathcal{A}_\alpha$ some regularity properties. Combining this with the well localization of the spectrum of $\mathscr{L}_1$ and (1.7) allow us to deduce some properties of the spectrum of $\mathscr{L}_\alpha$ valid for α close to 1. Moreover, we are able to obtain an estimate on the semigroup thanks to a spectral mapping theorem.

Respect to the nonlinear problem, one can build a solution by the use of an iterative scheme whose convergence is ensure due to a priori estimates coming from estimates of the semigroup of the linearized operator. A key element is an estimate for the bilinear collision operator established by Tristani [Tristani, 2016]. For a sufficiently close to the equilibrium initial datum, the nonlinear part of the equation is small with respect to the linear part which dictates the dynamic. Therefore, we can recover an exponential decay to equilibrium for the nonlinear problem.

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1.4 Organization of the thesis

The organization of this paper is as follows. In Chapter 2, we proceed to define the function spaces as well as some spectral notations and definitions. The main known results are presented in Section 2.4. In Chapter 3 we introduce the splitting of our forcing term $\mathcal{L}$ as the sum of a regularizing part and a dissipative part. Moreover, we prove existence of a spectral gap for the elastic linearized operator as well as decay rate for the linearized semigroup.

In Chapter 4 we see the inelastic linearized operator is a small perturbation of the elastic one. We also make a fine study of spectrum close to 0, which allows us to prove existence of a spectral gap. Furthermore, we obtain a property of semigroup decay in $W^{s,1}_x W^{2,1}_v(\langle v\rangle\, e^{b\langle v\rangle^\beta})$ with $b > 0$ and $\beta \in (0, 1)$.

Finally, we go back to the nonlinear Boltzmann equation in Chapter 5 and prove our main result.

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2
Preliminaries

We begin by fixing our notation and presenting the Sobolev spaces we are considering through this work. We also present here the main known results regarding the existence and uniqueness of the equilibrium state. Finally we present the linearization of the equation (1.1) around the equilibrium.

2.1
Function spaces

Let us introduce the Sobolev spaces we shall use in the sequel. Throughout the paper we shall use the notation $\langle v \rangle = \sqrt{1+|v|^2}$. For any $p,q,r \geq 1$ and any weight $m > 0$ on $\mathbb{R}^3$ we define the weighted Lebesgue space

$$L^p_x L^r_v(\langle v \rangle^q m) := \{ f : \mathbb{T}^3 \times \mathbb{R}^3 \to \mathbb{R} \text{ measurable} : \|f\|_{L^p_x L^r_v(\langle v \rangle^q m)} < +\infty \},$$

where the norm $\|f\|_{L^p_x L^r_v(\langle v \rangle^q m)}$ is defined by

$$\|f\|_{L^p_x L^r_v(\langle v \rangle^q m)} := \| \|f(\cdot, v)\|_{L^p_x} \langle v \rangle m(v) \|_{L^q_v}.$$

The weighted Sobolev space $W^{s,p}_x W^{\sigma,r}_v(\langle v \rangle^q m(v))$ for any $p,q,r \geq 1$ and $\sigma, s \in \mathbb{N}$ is defined by the norm

$$\|f\|_{W^{s,p}_x W^{\sigma,r}_v(\langle v \rangle^q m)} := \sum_{0 \leq s' \leq s, 0 \leq \sigma' \leq \sigma, s'+\sigma' \leq \max(s,\sigma)} \| \|\nabla^{s'}_x \nabla^{\sigma'}_v f(\cdot, v)\|_{L^p_x} \langle v \rangle^q m(v) \|_{L^r_v}.$$

Now we want to define the fractional weighted Sobolev spaces $W^{s,p}_x W^{\sigma,r}_v(\langle v \rangle^q m)$ with $s > 1$ any real number. We start by fixing the fractional exponent $\gamma \in (0,1)$. Consider a function $g : \mathbb{T}^3 \to \mathbb{R}$, and let us define the function

$$G_\gamma(g)(x,y) := \frac{|g(x) - g(y)|}{|x-y|^{4/p+\gamma}},$$

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Hence, we define $W_x^{\gamma,p}$ as follows

$$W_x^{\gamma,p} := \left\{ g \in L_x^p : G_\gamma(g) \in L^p_{(x,y)}(\mathbb{T}^3 \times \mathbb{T}^3) \right\}.$$

That is, an intermediary Banach space between L_x^p and $W_x^{1,p}$, endowed with the natural norm

$$\|g\|_{W_x^{\gamma,p}} := \left(\int_{\mathbb{R}^3} |f|^p dx + \int_{\mathbb{R}^3 \times \mathbb{R}^3} |G_\gamma(g)|^p dxdy \right)^{1/p},$$

where the term

$$[g]_{W_x^{\gamma,p}} := \int_{\mathbb{R}^3 \times \mathbb{R}^3} |G_\gamma(g)|^p dxdy = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|g(x) - g(y)|^p}{|x-y|^{4+\gamma p}} dxdy,$$

is the so-called Gagliardo (semi) norm of g.

Moreover, if $s > 1$ and is not an integer then we write $s = t + \gamma$, where $t \in \mathbb{Z}$ and $\gamma \in (0,1)$. In this case, the space $W_x^{s,p}$ consist of those equivalence classes of functions $g \in W_x^{t,p}$ whose distributional derivatives $\nabla_x^\beta g$, with $|\beta| = t$ belong to $W_x^{\gamma,p}$. This is a Banach space with respect to the norm

$$\|g\|_{W_x^{s,p}} := \left(\|f\|_{W_x^{t,p}} + \int_{\mathbb{R}^3 \times \mathbb{R}^3} |G_\gamma(\nabla_x^\beta g)|^p dxdy \right)^{1/p}.$$

The weighted Sobolev space $W_x^{s,p} W_v^{\sigma,r}(\langle v \rangle^q \, m(v))$ for any $p, q, r \geq 1$ and $\sigma \in \mathbb{N}$ and $s = t + \gamma$ with $t \in \mathbb{N}$ and $\gamma \in [0,1)$, is defined by the norm

$$\|f\|_{W_x^{s,p} W_v^{\sigma,r}(\langle v \rangle^q m)} := \sum_{0 \leq s' \leq s, 0 \leq \sigma' \leq \sigma, s' + \sigma' \leq \max(s,\sigma)} \left\| \|\nabla_x^{s'} \nabla_v^{\sigma'} f(\cdot, v)\|_{W_x^{\gamma,p}} \langle v \rangle^q \, m(v) \right\|_{L_v^r},$$

where $W_x^{0,p} := L_x^p$. We refer the reader to [Di Nezza et al., 2012] for a further discussion about the fractional Sobolev spaces.

2.2 Notations and definitions

We now proceed to give the basic notations and definitions of spectral theory that will be needed along this work.

Given a real number $a \in \mathbb{R}$ let us defined

$$\Delta_a := \{ z \in \mathbb{C} : \mathfrak{Re}\, z > a \}.$$

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For some given Banach spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, we denote the space of bounded linear operators from E to F by $\mathscr{B}(E,F)$ and we denote by $\|\cdot\|_{\mathscr{B}(E,E)}$ or $\|\cdot\| : E \to E$ the associated operator norm. We write $\mathscr{B}(E) = \mathscr{B}(E,E)$ when $F = E$. Moreover, we denote by $C(E,F)$ the space of closed unbounded linear operators from E to F with dense domain, and $C(E) = C(E,E)$ in the case $E = F$.

For a Banach space X and $\Lambda \in C(X)$ its associated semigroup is denoted by $S_\Lambda(t)$, for $t \geq 0$, when it exists. Also denote by $D(\Lambda)$ its domain, by $N(\Lambda)$ its null space and by $R(\Lambda)$ its range. Let us introduce the D(Λ)-norm defined as

$$\|f\|_{D(\Lambda)} = \|f\|_X + \|\Lambda f\|_X \text{ for } f \in D(\Lambda).$$

More generally, for every $k \in \mathbb{N}$, we define

$$\|f\|_{D(\Lambda^k)} = \sum_{j=0}^{k} \|\Lambda^j f\|_X, \quad f \in D(\Lambda^k).$$

Its spectrum is denoted by $\Sigma(\Lambda)$, and the resolvent set $\rho(\Lambda) := \mathbb{C} \setminus (\Sigma(\Lambda))$. So for any $z \in \rho(\Lambda)$ the operator $\Lambda - z$ is invertible and the resolvent operator

$$R_\Lambda(z) := (\Lambda - z)^{-1}$$

is well-defined, belongs to $\mathscr{B}(X)$ and has range equal to $D(\Lambda)$. A number $\xi \in \Sigma(\Lambda)$ is said to be an eigenvalue if $N(\Lambda - \zeta) = \{0\}$. Moreover, an eigenvalue $\xi \in \Sigma(\Lambda)$ is said to be isolated if there exists $r > 0$ such that

$$\Sigma(\Lambda) \cap \{|z \in \mathbb{C} : |z - \xi| < r\} = \{\xi\}.$$

If ξ is an isolated eigenvalue we define the associated spectral projector

$$\Pi_{\Lambda,\xi} := -\frac{1}{2\pi i} \int_{|z-\xi|=r'} R_\Lambda(z) dz \in \mathscr{B}(X),$$

which is independent of $0 < r' < r$ since $z \mapsto R_\Lambda(z)$ is holomorphic. It is well-known that $\Pi_{\Lambda,\xi}^2 = \Pi_{\Lambda,\xi}$ so it is a projector and the “associated projected semigroup” is

$$S_{\Lambda,\xi}(t) := -\frac{1}{2\pi i} \int_{|z-\xi|=r'} e^{tz} R_\Lambda(z) dz, \quad t > 0;$$

which satisfies that for all $t > 0$

$$S_{\Lambda,\xi}(t) = \Pi_{\Lambda,\xi} S_{\Lambda,\xi}(t) = S_{\Lambda,\xi} \Pi_{\Lambda,\xi}(t).$$

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When the “algebraic eigenspace” $R(\Pi_{\Lambda,\xi})$ is finite dimensional we say that ξ is a discrete eigenvalue, written as $\xi \in \Sigma_d(\Lambda)$. For more about these results we refer the reader to [Kato, 2013, Chapter III-6].

Finally for any $a \in \mathbb{R}$ such that $\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, ..., \xi_k\}$ where $\xi_1, ..., \xi_k$ are distinct discrete eigenvalues, we define without ambiguity

$$\Pi_{\Lambda,a} = \Pi_{\Lambda,\xi_1} + \cdots + \Pi_{\Lambda,\xi_k}.$$

If one considers some Banach spaces X_1, X_2, X_3, for two given functions $S_1 \in L^1(\mathbb{R}_+, \mathscr{B}(X_1, X_2))$ and $S_2 \in L^1(\mathbb{R}_+, \mathscr{B}(X_2, X_3))$, the convolution

$$S_1 * S_2 \in L^1(\mathbb{R}_+, \mathscr{B}(X_1, X_3)),$$

is defined for all $t \geq 0$ as

$$(S_1 * S_2)(t) := \int_0^t S_2(s) S_1(t-s) ds.$$

When $S_1 = S_2$ and $X_1 = X_2 = X_3$, $S^{(*l)}$ is defined recursively by $S^{(*1)} = S$ and for any $l \geq 2$, $S^{(*l)} = S * S^{(*(l-1))}$.

2.3 Hypodissipative operators

Let us introduce the notion of *hypodissipative* operators. Consider a Banach space $(X, \|\cdot\|_X)$ and some operator $\Lambda \in C(X)$, $(\Lambda - a)$ is said to be hypodissipative on X if there exists some norm $|||\cdot|||_X$ on X equivalent to the initial norm $\|\cdot\|_X$ such that for every $f \in D(\Lambda)$ there exist $\phi \in F(f)$ such that

$$\mathfrak{Re}\,\langle \phi, (\Lambda - a) f \rangle \leq 0,$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket for the duality in X and X^* and, $F(f) \subset X^*$ is the dual set of f defined by

$$F(f) := \{\phi \in X^* : \langle \phi, f \rangle = |||\phi|||_{X^*} = |||f|||_X\}. \tag{2.1}$$

The following theorem is a non standard formulation of the classical Hille-Yosida theorem on m-dissipative operators and semigroups. It summarizes the link between PDE’s, the semigroup theory and spectral analysis.

Theorem 2.3.1. *Consider X a Banach space and Λ the generator of a C^0-semigroup S_Λ. We denote by R_Λ its resolvent. For given constants $a \in \mathbb{R}$, $M > 0$ the following assertions are equivalent:*

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1. $\Lambda - a$ *is hypodissipative;*

2. *the semigroup satisfies the growth estimate for every* $t \geq 0$

$$\|S_\Lambda(t)\|_{\mathscr{B}(X)} \leq Me^{at};$$

3. $\Sigma(\Lambda) \cap \Delta_a = \emptyset$ *and for all* $z \in \Delta_a$

$$\|R_\Lambda(z)^n\| \leq \frac{M}{(\mathfrak{Re}\, z - a)^n};$$

4. $\Sigma(\Lambda) \cap (a, \infty) = \emptyset$ *and there exist some norm* $|||\cdot|||$ *on* X *equivalent to the norm* $\|\cdot\|$ *such that for all* $f \in X$

$$\|f\| \leq |||f||| \leq M\|f\|,$$

and such that for every $\lambda > a$ *and every* $f \in D(\Lambda)$

$$|||(\Lambda - \lambda)f||| \geq (\lambda - a)|||f|||.$$

For the proof of this result we refer the reader to [Pazy, 2012, Chapter 1], and for more about hypodissipative operators see [Gualdani et al., 2017, Section 2.3] and [Mischler and Scher, 2016, Section 2.1].

2.4 Main known results

The existence of smooth stationary solutions for the inelastic Boltzmann equation under the thermalization given by (1.5) has already been proved by Bisi, Carrillo, Lods in [Bisi et al., 2008, Theorem 5.1] for any choice of restitution coefficient α. Moreover, the uniqueness of the solution was obtained by Bisi, Cañizo, Lods in [Bisi et al., 2011, Theorem 1.1]. These results can be summarized as follows:

Theorem 2.4.1. *For any* $\rho > 0$ *and* $\alpha \in (0, 1]$*, there exists a steady solution* $F_\alpha \in L^1_v(\langle v \rangle^2)$*,* $F_\alpha(v) \geq 0$ *to the problem*

$$\mathcal{Q}_\alpha(F_\alpha, F_\alpha) + \mathcal{L}(F_\alpha) = 0, \tag{2.2}$$

with $\int_{\mathbb{R}^3} F_\alpha(v)dv = \rho$*.*

Moreover, there exists $\alpha_0 \in (0, 1]$ *such that such a solution is unique for* $\alpha \in (\alpha_0, 1]$*. This (unique) steady state is radially symmetric and belongs to* $C^\infty(\mathbb{R}^3)$*.*

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Let us denote by $\mathscr{G}_\alpha$ the set of functions F_α solutions of (2.2) with mass 1, that satisfies $\int_{\mathbb{R}^3} F_\alpha(v)dv = 1$. We recall a quantitative estimate on the distance between F_α and the Maxwellian $\mathcal{M}$:

$$\mathcal{M}(v) = \left(\frac{1}{2\pi\theta^\sharp}\right)^{3/2} \exp\left\{-\frac{(v-u_0)}{2\theta^\sharp}\right\}, \tag{2.3}$$

with $\theta^\sharp = \frac{1+e}{3-e}\theta_0$ where θ_0 is defined in (1.6). This Maxwellian is the unique solution of (2.2) in the elastic case, i.e. when $\alpha = 1$, (see [Bisi et al., 2011, Theorems 2.3 and 5.5]).

Theorem 2.4.2. *There exist an explicit function $\eta_1(\alpha)$ such that $\lim_{\alpha\to 1}\eta_1(\alpha) = 0$ and such that for any $\alpha_0 \in (0,1]$*

$$\sup_{F_\alpha \in \mathscr{G}_\alpha} \|F_\alpha - \mathcal{M}\|_{\mathcal{Y}} \leq \eta_1(\alpha), \quad \forall \alpha \in (\alpha_0, 1].$$

Here, $\mathcal{Y} = L^1_v(\langle v\rangle\, e^{a|v|^s})$ with $a > 0$ and $s \in (0,1)$.

The weak form of the collision operator $\mathcal{Q}_e$ suggests the natural splitting between gain and loss parts $\mathcal{L} = \mathcal{L}^+ - \mathcal{L}^-$. For the loss part notice that

$$\left\langle \mathcal{L}^-(f,g), \psi \right\rangle = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} g(v_*)f(v)\psi(v)|v - v_*|d\sigma dv_* dv = \langle fL(g), \psi\rangle\,,$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in L^2 and L is the convolution operator

$$L(g)(v) = 4\pi(|\cdot| * g)(v). \tag{2.4}$$

Remark 2.4.3. *Notice that L and $\mathcal{L}^- = \mathcal{Q}_e^-$ are independent of the restitution coefficient.*

Let us introduce the collision frequency $\nu_e := L(\mathcal{M}_0)$ and consider

$$\nu_e^0 = \inf_{v\in\mathbb{T}^3} \nu_e(v) > 0. \tag{2.5}$$

It is easy to see that $\nu(v) \approx \langle v\rangle$. In other words, there exist some constants $\nu_{e,0}, \nu_{e,1} > 0$ such that for every $v \in \mathbb{R}^3$

$$0 < \nu_{e,0} \leq \nu_{e,0}\langle v\rangle \leq \nu_e(v) \leq \nu_{e,1}\langle v\rangle\,. \tag{2.6}$$

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Indeed, notice that

$$\begin{aligned}
\frac{\nu(v)}{\langle v\rangle} &= \frac{4\pi}{|v|}\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)|v-v_*|dv_*,\\
&\geq \frac{4\pi}{\langle v\rangle}\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)(|v|-|v_*|)dv_*,\\
&= \left(4\pi\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)dv_*\right)\frac{|v|}{\langle v\rangle} - \frac{2\pi}{\langle v\rangle}\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)|v_*|dv_*,\\
&= K_0\frac{|v|}{\langle v\rangle} - K_1\frac{1}{\langle v\rangle},
\end{aligned}$$

where $K_0 = 4\pi\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)dv_*$ and $K_1 = 4\pi\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)|v_*|dv_*$. Therefore, if $K' = \min\{K_0, K_1\}$, taking R_0 big enough for every $R > R_0$ and if $|v| > R$ we have

$$\frac{\nu(v)}{\langle v\rangle} \geq K_2 := K'\left(\frac{R^2}{R^2+1} - \frac{1}{R}\right) > 0.$$

If $|v| \leq R$ then taking ν_e^0 defined in (2.5) we have

$$\frac{\nu(v)}{\langle v\rangle} \geq \frac{\nu_e^0}{\sqrt{1+R^2}} =: K_3 > 0.$$

Let $\nu_{e,0} := \min\{K_2, K_3\} > 0$ hence $\nu_e(v) \geq \nu_{e,0}\langle v\rangle$. On the other hand, since $|v-v_*| \leq 2\langle v\rangle\langle v_*\rangle$

$$\nu_e(v) \leq 8\pi\langle v\rangle\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)\langle v_*\rangle dv_* \leq \nu_{e,1}\langle v\rangle,$$

where $\nu_{e,1} = 8\pi\int_{\mathbb{R}^3}\mathcal{M}_0(v_*)\langle v_*\rangle dv_*$.

Consider the space $\mathscr{H} = L^2_v(\mathcal{M}^{-1/2})$. Arlotti and Lods in [Arlotti and Lods, 2007, Theorem 3.7] performed the spectral analysis of $\mathcal{L}$ in $\mathscr{H}$. Moreover, Mouhot, Lods and Toscani in [Lods et al., 2008] give some quantitative estimates of the spectral gap. These results can be summarized in the following:

Theorem 2.4.4. *Consider $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ the inner product in the Hilbert space $\mathscr{H}$. Then we have the following description of the spectrum of $\mathcal{L}$:*

1. *$\mathcal{L}^+$ is compact in $\mathscr{H}$.*

2. *The spectrum of $\mathcal{L}$ as an operator in $\mathscr{H}$ consist of the spectrum of $-\mathcal{L}^-$ and of, at most, eigenvalues of finite multiplicities. Precisely, we have*

$$\Sigma(\mathcal{L}) = \{\lambda\in\mathbb{R} : \lambda \leq -\nu_e^0\} \cup \{\lambda_n : n \in I\},$$

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where $I \subset \mathbb{N}$ and $(\lambda_n)_n$ is a decreasing sequence of real eigenvalues of $\mathcal{L}$ with finite algebraic multiplicities: $\lambda_0 = 0 > \lambda_1 > \lambda_2 \cdots > \lambda_n > \cdots$, which unique possible cluster point is $-\nu_e^0$.

3. *$\mathcal{L}$ is a nonnegative self-adjoint operator and there exists $\mu_e > 0$ (the spectral gap) such that*

$$-\langle h, \mathcal{L}(h)\rangle_{\mathscr{H}} \geq \mu_e \|h - \rho_h \mathcal{M}\|_{\mathscr{H}},$$

where $\rho_h = \int h dv$.

4. *0 is a simple eigenvalue of $\mathcal{L}$ with $N(\mathcal{L}) = Span\{\mathcal{M}\}$*

2.5 Preliminaries on steady states

As mentioned in Section 2.4, there exist $\alpha_0 \in (0,1]$ such that for every $\alpha \in (\alpha_0, 1]$, F_α is the unique solution of (2.2) in $L^1_v(\langle v\rangle^2)$ with mass 1. We linearize our equation around the equilibrium F_α with the perturbation $f = F_\alpha + h$. That is, by substituting f in (1.1) we obtain

$$\partial_t h = \mathcal{Q}_\alpha(h,h) + \mathscr{L}_\alpha(h). \tag{2.7}$$

where $\mathscr{L}_\alpha(h) = \mathcal{Q}_\alpha(F_\alpha, h) + \mathcal{Q}_\alpha(h, F_\alpha) + \mathcal{L}(h) - v \cdot \nabla_x h$.

If we consider only the linear part we obtain the first order linearized equation around the equilibrium F_α

$$\partial_t h = \mathscr{L}_\alpha(h). \tag{2.8}$$

Throughout the paper, we shall consider

$$m(v) := \exp\left(b \langle v\rangle^\beta\right),$$

with $b > 0$ and $\beta \in (0,1)$. Our weights will be of the form $\langle v\rangle^q m(v)$ for some $q \geq 0$. The polynomial weights are important in kinetic theory for the study of stadistical moments of the density function. Meanwhile, optimal tails for the inelastic case are exponential. Thats why we consider this kind of weights.

Let us state several lemmas on steady states F_α that will we needed in the future.

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First of all, we prove an estimate for the Sobolev norm

Lemma 2.5.1. *Let $k, q \in \mathbb{N}$. Then there exist $\beta \in (0,1)$ and $C > 0$ such that*

$$\|F_\alpha\|_{W_v^{k,p}(\langle v\rangle^q m)} \leq C.$$

Proof. First we recall that by [Bisi et al., 2011, Theorem 3.3] there exist some constants $A > 0$ and $M > 0$ such that, for any $\alpha \in (0,1]$ and any solution F_α to (2.2) one has

$$\int_{\mathbb{R}^3} F_\alpha(v) e^{A|v|^2} dv < M.$$

Using the inequalities of Cauchy and Bernoulli we get that, if $6\beta < A$ we obtain

$$\int_{\mathbb{R}^3} F_\alpha(v) m(v) dv \leq e^{(b^2+1)/2} M =: C_1,$$

and

$$\int_{\mathbb{R}^3} F_\alpha(v) m^{12}(v) dv \leq e^{6(b^2+1)} M =: C_2.$$

Moreover, from [Bisi et al., 2011, Corollary 3.6] we know that for any $k \in \mathbb{N}$, there exist $C_k > 0$ such that $\|F_\alpha\|_{H_v^k} \leq C_k$. Thus, if we denote $k' = 8k + 7(1 + 3/2)$, using Lemma A.2.1 we get

$$\|F_\alpha\|_{W_v^{k,1}(\langle v\rangle^q m)} \leq C C_{k'}^{1/8} C_1^{1/8} C_2^{3/4} =: C',$$

that concludes our proof. □

Here and subsequently we will consider $\beta \in (0, \min\{1, A\})$. We now proceed to estimate the difference between F_α and the elastic equilibrium $\mathcal{M}$, which is the Maxwellian given in (2.3).

Lemma 2.5.2. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$. Then there exists a function $\eta(\alpha)$ such that for any $\alpha \in (\alpha_0, 1]$*

$$\|F_\alpha - \mathcal{M}\|_{W_v^{k,1}(\langle v\rangle^q m)} \leq \eta(\alpha),$$

with $\eta(\alpha) \to 0$ when $\alpha \to 1$.

Proof. By Theorem 2.4.2 there exists an explicit function $\eta_1(\alpha)$ such that $\lim_{\alpha\to 1} \eta_1(\alpha) = 0$ and such that for any $\alpha_0 \in (0,1]$

$$\|F_\alpha - \mathcal{M}\|_{L_v^1(\langle v\rangle m)} \leq \eta_1(\alpha).$$

Since, $1 \leq \langle v\rangle$ for every $v \in \mathbb{R}^3$

$$\|F_\alpha - \mathcal{M}\|_{L_v^1(m)} \leq \eta_1(\alpha).$$

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We denote $k' = 8k + 7(1 + 3/2)$. Then, using Lemma A.2.1 and the proof of Lemma 2.5.1 we get

$$\|F_\alpha - \mathcal{M}\|_{W_v^{k,1}(\langle v\rangle^q m)} \leq C(2C_{k'})^{1/8}(2C_2)^{1/8}\eta_1^{3/4}(\alpha) =: \eta(\alpha),$$

which concludes our proof. □

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3
Spectral properties of the linearized elastic operator

The purpose of this chapter is to prove the localization of the spectrum of the elastic linearized operator $\mathscr{L}_1$. The key ingredient in our proof will be the Enlargement Theorem C.0.1. So we aim to find a splitting of the linearized operator $\mathscr{L}_1 = \mathcal{A}_1 + \mathcal{B}_1$ where $\mathcal{A}_1$ and $\mathcal{B}_1$ satisfy the assumptions of the Enlargement Theorem. Namely, hipodissipativity of $\mathcal{B}_1$ and boundedness $\mathcal{A}_1$ for suitable Sobolev spaces.

Consider the following Banach spaces for $s \geq 2$

$$
\begin{aligned}
&E_1 := W_x^{s+2,1} W_v^{4,1}(\langle v \rangle^2 m), \\
&E_0 = E := W_x^{s,1} W_v^{2,1}(\langle v \rangle m), \\
&\mathcal{E} := W_x^{s,1} L_v^1(m).
\end{aligned}
$$

As mention in the previous chapter, it is well-known that 0 is a simple eigenvalue associated to the eigenfunction $\mathcal{M}$ for the operator $\mathcal{L}$ and it admit a positive spectral gap in $\mathscr{H} = L_v^2(\mathcal{M}^{-1/2})$, where $\mathcal{M}$ is defined in (2.3) . The aim of this chapter is to show that the same is true for $\mathscr{L}_1$ in the larger spaces E_0, E_1.

3.1
The forcing term and its splitting.

Before given the decomposition of the linearized operator, we focus on the study of the forcing term $\mathcal{L}$. First, we are going to define a decomposition for $\mathcal{L}$ ans then, use this decomposition to defined the one of $\mathscr{L}_1$.

Recall the definition of the collision frequency $\nu_e := L(\mathcal{M}_0)$ given in Section 2.4 where L is the convolution operator defined by (2.4).

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Let us decompose the operator $\mathcal{L}$ in the following way $\mathcal{L} = \mathcal{A}_{e,\delta} + \mathcal{B}_{e,\delta}$. We want to choose this operator so they satisfy the assumption (B.2) and (B.1) in the Enlargement Theorem C.0.1. In order to do this, for any $\delta \in (0,1)$ consider the bounded (by one) operator $\Theta_\delta = \Theta_\delta(v, v_*, \sigma) \in C^\infty$, which equals one

$$\left\{|v| \leq \delta^{-1}, 2\delta \leq |v - v_*| < \delta^{-1} \quad \text{and} \quad |\cos\theta| \leq 1 - 2\delta\right\},$$

and whose support is included in

$$\left\{|v| \leq 2\delta^{-1}, \delta \leq |v - v_*| < 2\delta^{-1} \quad \text{and} \quad |\cos\theta| \leq 1 - \delta\right\}.$$

The introduction of the parameter δ in this truncation function will allow as to prove the properties we are looking for. Namely, regularity of $\mathcal{A}_{e,\delta}$ and hypodissivativity of $\mathcal{B}_{e,\delta}$.

We can now give the following decomposition of the forcing term cL:

$$\mathcal{L}(h)(v) = \mathcal{L}_S^+(h)(v) + \mathcal{L}_R^+(h)(v) - \nu_e(v)h(v),$$

where $\mathcal{L}_S^+$ is the truncated operator given by Θ_δ and $\mathcal{L}_R^+$ the corresponding reminder. By this we mean that for any test function ψ:

$$\left\langle \mathcal{L}_S^+(h), \psi \right\rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \Theta_\delta h(v) \mathcal{M}_0(v_*) \psi(v) |v - v_*| d\sigma dv_* dv,$$

and,

$$\left\langle \mathcal{L}_R^+(h), \psi \right\rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (1 - \Theta_\delta) h(v) \mathcal{M}_0(v_*) \psi(v) |v - v_*| d\sigma dv_* dv.$$

Hence we can give a decomposition for the linearized operator $\mathcal{L}(h) = \mathcal{A}_{e,\delta}(h) + \mathcal{B}_{e,\delta}(h)$ where $\mathcal{A}_{e,\delta}(h) := \mathcal{L}_S^+(h)$ and $\mathcal{B}_{e,\delta}(h)$ is the remainder. That is $\mathcal{B}_{e,\delta}(h) = \mathcal{L}_R^+(h) - \nu_e h$.

By the Carleman representation for the inelastic case (see [Arlotti and Lods, 2007, Theorem 1.4] and [Mischler and Mouhot, 2006, Proposition 1.5])

$$\mathcal{L}^+(h)(v) = \int_{\mathbb{R}^3} k_e(v, v_*) h(v_*) dv_*,$$

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where k_e is given by

$$k_e(v, v_*) \tag{3.1}$$
$$= \frac{C_e}{|v - v_*|} \exp\left\{ \frac{-1}{8\theta_0} \left((1+\mu)|v - v_*| + \frac{|v - u_0|^2 - |v_* - u_0|^2}{|v - v_*|} \right)^2 \right\},$$

for some constants $C_e, \mu > 0$ depending only on e. Furthermore, we can write the truncated operator $\mathcal{L}_S^+$ as

$$\mathcal{A}_{e,\delta}(h)(v) = \int_{\mathbb{R}^3} k_{e,\delta}(v, v_*) h(v_*) dv_*, \tag{3.2}$$

for some smooth kernel $k_{e,\delta} \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. The smoothness of the kernel allow us to prove a the following regularity estimate on the truncated operator $\mathcal{A}_{e,\delta}$.

Lemma 3.1.1. *For any $s \geq 0$ and any $e \in (0, 1]$, the operator $\mathcal{A}_{e,\delta}$ maps L^1_v into H^{s+1}_v functions with compact support, with explicit bounds (depending on δ) on the $L^1_v \mapsto H^{s+1}_v$ norm and on the size of the support. More precisely, there are two constants $C_{s,\delta}$ and R_δ such that for any $h \in L^1_v$*

$$K := \operatorname{supp} \mathcal{A}_{e,\delta} h \subset B(0, R_\delta), \quad \text{and} \quad \|\mathcal{A}_{e,\delta}\|_{H^{s+1}_v(K)} \leq C_{s,\delta} \|h\|_{L^1_v}.$$

In particular, we deduce that $\mathcal{A}_{e,\delta}$ is in $\mathscr{B}(E_j)$ for $j = 0, 1$ and $\mathcal{A}_{e,\delta}$ is in $B(\mathcal{E}, E)$.

Proof. It is clear that the range of the operator $\mathcal{A}_{e,\delta}$ is included into a compactly supported functions thanks to the truncation. Moreover, the bound on the size of the support is related to δ.

On the other hand, the proof of the smoothing estimate follows from [Alonso and Lods, 2013b, Proposition 2.4] since $\mathcal{L}_S^+$ is the gain part of the collision operator associated to the mollified collision kernel $B = \Theta_\delta |v - v_*|$. Even though their original statement is for functions in $L^1_v(\langle v \rangle^{2\eta+s+4})$ for any $\eta \geq 0$, as they mention in their proof, in the case of compact support of the collision kernel it can be proved for functions in $L^1_v(\langle v \rangle^\eta)$. Therefore, taking $\eta = 0$ we obtain our result.

□

Furthermore, notice that from [Tristani, 2016, Lemma 2.6] we have that the operator $\mathcal{L}$ is bounded from $W^{s,1}_x W^{k,1}_v(\langle v \rangle^{q+1} m)$ to $W^{s,1}_x W^{k,1}_v(\langle v \rangle^q m)$ for any $q > 0$. Hence, is bounded from E_1 to E_0.

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3.2
Hypodissipativity of $\mathcal{B}_{e,\delta}$

Lemma 3.2.1. *Let us consider $k \geq 0$, $s \geq k$ and $q \geq 0$. Then, there exist $\delta \geq 0$ and $a_0 > 0$ such that for any $e \in (0,1]$, the operator $\mathcal{B}_{e,\delta} + a_0$ is hypodissipative in $W^{s,1}_x W^{k,1}_v(\langle v\rangle^q m)$.*

Proof. We consider the case $W^{1,1}_{x,v}(\langle v\rangle^q m)$. The higher-order cases are treated in a similar way.

Consider a solution h to the linear equation $\partial_t h = \mathcal{B}_{e,\delta}(h)$ given an initial datum h_0. The main idea of the proof is to construct a positive constant a_0 and a norm $\|\cdot\|_*$ equivalent to the norm on $W^{1,1}_{x,v}(\langle v\rangle^q m)$ such that there exist ψ in the dual space $\left(W^{1,1}_{x,v}(\langle v\rangle^q m)\right)^*$ such that $\psi \in F(h)$, where $F(h)$ is defined in (2.1), and

$$\mathfrak{Re}\,\langle \psi, \mathcal{B}_{e,\delta} h\rangle \leq -a_0\|h\|_*. \tag{3.3}$$

We have divided the proof into four steps. The first one deals with the hypodissipativity of $\mathcal{B}_{e,\delta}$ in $L^1_x L^1_v(\langle v\rangle^{q+1} m)$, while the second and third deal with the x and v-derivatives respectively. In the last step we construct the $\|\cdot\|_*$ norm and prove that it satisfies (3.3).

Step 1: Notice that, for $k = 0$, the hypo–dissipativity of $\mathcal{B}_{e,\delta}$ simply reads

$$\int_{\mathbb{R}^3\times\mathbb{T}^3} \mathcal{B}_{e,\delta}(h)\text{sign}(h)\,\langle v\rangle^q\, m(v)dxdv \leq -a_0'\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)},$$

for some positive constant a_0'. Which means that, for $k = 0$, $\mathcal{B}_{e,\delta}$ is actually dissipative.

Let us recall that $\mathcal{B}_{e,\delta}(h) = \mathcal{L}^+_R(h) - \nu_e h$. Consider Ω_δ the set where Θ_δ equals 1 that is

$$\Omega_\delta := \left\{|v| \leq \delta^{-1}, 2\delta \leq |v - v_*| < \delta^{-1} \quad \text{and} \quad |\cos\theta| \leq 1 - 2\delta\right\}.$$

Our proof starts with the observation that $\Omega^c_\delta = \Omega^1_\delta \cup \Omega^2_\delta \cup \Omega^3_\delta \cup \Omega^4_\delta$ where

$$\begin{aligned}
\Omega^1_\delta &= \left\{|v| > \delta^{-1}\right\},\\
\Omega^2_\delta &= \left\{2\delta > |v - v_*|\right\},\\
\Omega^3_\delta &= \left\{|v - v_*| \geq \delta^{-1}\right\},\\
\Omega^4_\delta &= \left\{|\cos\theta| > 1 - 2\delta\right\}.
\end{aligned}$$

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Hence, we have that $1 - \Theta_\delta \leq \mathbb{1}_{\Omega_\delta^c}$. Thus, using the weak form of the collision operator for $\psi(v) = \operatorname{sign}(h) \langle v \rangle^q m(v)$

$$\begin{aligned}\int_{\mathbb{R}^3} \mathcal{L}_R^+(h)\psi(v)dv &= \int_{\mathbb{R}^3\times\mathbb{R}^3} (1-\Theta_\delta)h(v)\mathcal{M}_0(v_*)|v-v_*|\psi(v')d\sigma dv dv_* \\ &\leq \sum_{j=1}^4 \int_{\Omega_\delta^j} |h(v)|\mathcal{M}_0(v_*)|v-v_*| \langle v' \rangle^q m(v') d\sigma dv dv_* \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Let us first prove that there exist a constant $C_q > 0$ such that

$$\langle v' \rangle^q m(v') \leq C_q \langle v \rangle^q m(v) \langle v_* \rangle^q m(v_*). \tag{3.4}$$

Indeed, by the kinetic rate of energy dissipation (1.3) we have that

$$|v'|^2 \leq |v|^2 + |v_*|^2$$

and, since $\beta \in (0,1)$,

$$\langle v' \rangle^\beta \leq \langle v \rangle^\beta + \langle v_* \rangle^\beta. \tag{3.5}$$

Moreover, for every $q \geq 0$ there exist a constant $C_q' > 0$ such that

$$\langle v' \rangle^q \leq C_q'(\langle v \rangle^q + \langle v_* \rangle^q) \leq 2C_q' \langle v \rangle^q \langle v_* \rangle^q \tag{3.6}$$

Thus, using (3.5) and (3.6) we get (3.4) with $C_q = 2C_q'$.

We first deal with the integral I_2. Notice that in this case, using the inequality presented in (3.4), one has

$$\begin{aligned}I_2 &= \int_{\Omega_\delta^2} |h(v)|\mathcal{M}_0(v_*)|v-v_*| \langle v' \rangle^q m(v') d\sigma dv dv_*, \\ &\leq 2\delta \int_{\Omega_\delta^2} |h(v)|\mathcal{M}_0(v_*) \langle v' \rangle^q m(v') d\sigma dv dv_*, \\ &\leq 2\delta C_q \int_{\Omega_\delta^2} |h(v)|\mathcal{M}_0(v_*) \langle v \rangle^q m(v) \langle v_* \rangle^q m(v_*) d\sigma dv dv_*, \\ &\leq 8\pi C_q \delta \int_{\mathbb{R}^3} \mathcal{M}_0(v_*) \langle v_* \rangle^q m(v_*) dv_* \int |h(v)| \langle v \rangle^q m(v) dv, \\ &\leq C_2 \delta \|h\|_{L^1_v(\langle v \rangle^q m)},\end{aligned} \tag{3.7}$$

where $C_2 = 8\pi C_q \|\mathcal{M}_0\|_{L^1_v(\langle v \rangle^q m)}$.

We now turn to I_4. Notice that $|u| \leq 2\langle v \rangle \langle v_* \rangle$ combining this with (3.4) there exist some positive constant C_4' such that

$$|v-v_*| \langle v' \rangle^q m(v') \leq C_4' \langle v \rangle^{q+1} m(v) \langle v_* \rangle^{q+1} m(v_*). \tag{3.8}$$

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Then using (3.8) we have

$$
\begin{aligned}
I_4 &= \int_{\Omega_\delta^4} |h(v)| \mathcal{M}_0(v_*) |v - v_*| \langle v' \rangle^q m(v') d\sigma dv dv_*, \qquad (3.9)\\
&\le C_4' \int_{\Omega_\delta^4} |h(v)| \mathcal{M}_0(v_*) \langle v \rangle^{q+1} m(v) \langle v_* \rangle^{q+1} m(v_*) d\sigma dv dv_*,\\
&\le C_4' \Lambda(\delta) \cdot \int_{\mathbb{R}^3} \mathcal{M}_0(v_*) \langle v_* \rangle^q m(v_*) dv_* \cdot \int |h(v)| \langle v \rangle^q m(v) dv,\\
&\le C_4 \Lambda(\delta) \|h\|_{L^1_v(\langle v \rangle^{q+1} m)},
\end{aligned}
$$

where $C_4 = C_4' \|\mathcal{M}_0\|_{L^1_v(\langle v \rangle^q m)}$ and $\Lambda(\delta)$ is the measure of Ω_δ^4 on the sphere $\mathbb{S}^2$ which goes to zero as δ goes to zero.

We can now proceed to analyze the integral I_3. The general case follows in a similar manner. According to the kinetic rate of energy dissipation (1.3) and Lemma A.1.1

$$
\begin{aligned}
|v'|^\beta &\le \left(|v|^2 + |v_*|^2 - \frac{|v_*'|^2}{2} \right)^{\beta/2},\\
&\le \left(|v|^2 + |v_*|^2 \right)^{\beta/2} - C_\beta \left(\frac{|v_*'|^2}{2} \right)^{\beta/2},\\
&\le \left(|v|^\beta + |v_*|^\beta \right) - C_\beta' |v_*'|^\beta,
\end{aligned}
$$

where in the last inequality we use the fact that $\beta \in (0,1)$. Therefore, we have that for some positive constant C_3'

$$
\mathcal{M}_0(v_*) m(v') \le C_3' m(v) \exp\left\{ b|v_*|^\beta - \beta_0 |v_*|^2 - bC_\beta' |v_*'|^\beta \right\},
$$

here we are using the fact that $\mathcal{M}_0(v) \le C_e e^{-\beta_0 |v|^2}$ for some positive constants β_0 and C_e. Thus, on account of Lemma A.1.2, for any $\gamma > 0$, we get

$$
\mathcal{M}_0(v_*) m(v') \le C_3' e^{C_\gamma} m(v) \exp\left\{ -\gamma |v_*|^\beta - bC_\beta' |v_*'|^\beta \right\}.
$$

Take $\gamma = 2bC_\beta'$, and since $((1+e)|u \cdot n|)/2 \le |v_*| + |v_*'|$, where $u = v - v_*$, we can conclude that

$$
\begin{aligned}
\exp\left\{ -\gamma |v_*|^\beta - C_\beta' |v_*'|^\beta \right\} &\le \exp\left\{ -bC_\beta' |v_*|^\beta - bC_\beta' (|v_*| + |v_*'|)^\beta \right\},\\
&\le \exp\left\{ -bC_\beta' |v_*|^\beta - \frac{bC_\beta'}{2^\beta} |u \cdot n|^\beta \right\}.
\end{aligned}
$$

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Hence, using (3.6), we have that for $C_{3,\beta} = 4C_3' \exp(2bC_\beta')$

$$
\begin{aligned}
I_3 &= \int_{\Omega_\delta^3} |h(v)|\mathcal{M}_0(v_*)|v-v_*| \langle v' \rangle^q m(v') d\sigma dv dv_*, \\
&\le C_{3,\beta} \int \mathbb{1}_{\{|u|>\delta^{-1}\}} |h(v)| \langle v \rangle^q \langle v_* \rangle^q m(v) \exp\left\{ -bC_\beta' |v_*|^\beta - \frac{bC_\beta'}{2^\beta} |u\cdot n|^\beta \right\} |u| dn dv_* dv, \\
&= C_{3,\beta} \int \mathbb{1}_{\{|u|>\delta^{-1}\}} |h(v)| \langle v \rangle^q \langle v_* \rangle^q m(v) \exp\left\{ -bC_\beta' |v_*|^\beta \right\} |u| J dv_* dv, \qquad (3.10)
\end{aligned}
$$

where J is the integral $J = \int_{\mathbb{S}} \exp\left\{ -(bC_\beta'/2^\beta)|u\cdot n|^\beta \right\} dn$. Hence, recalling that $\hat{u} = u/|u|$ and $(\hat{u}\cdot n) = \cos\theta$ we get that

$$
J = 2\int_0^{\pi/2} \exp\left\{ -(C_\beta'/2^\beta)|u|^\beta |\cos\theta|^\beta \right\} \sin\theta dn.
$$

Take $C_J = bC_\beta'/2^\beta$ Now consider the change of variables $z = \cos\theta$ and then $w = |u|z$ which transform the integral J into

$$
J = 2\int_0^1 e^{-C_J|u|^\beta z^\beta} dz = 2|u|^{-1} \int_0^{|u|} e^{-C_J w^\beta} dw \le C_J' |u|^{-1}.
$$

It is easy to see that $\mathbb{1}_{\{|u|>\delta^{-1}\}} \le \delta|u|$. Substituting the above inequality into the integral in (3.10) and using (3.8) we obtain

$$
\begin{aligned}
I_3 &\le C_{3,\beta} C_J' \int \mathbb{1}_{\{|u|>\delta^{-1}\}} |h(v)| \langle v \rangle^q \langle v_* \rangle^q m(v) \exp\left\{ -C_\beta |v_*|^\beta \right\} dv_* dv, \\
&\le C_3' \delta \int \exp\left\{ -C_\beta |v_*|^\beta \right\} \langle v_* \rangle^{q+1} dv_* \int |h(v)| \langle v \rangle^{q+1} m(v) dv, \\
&\le C_3 \delta \|h\|_{L^1(\langle v \rangle^{q+1} m)}. \qquad (3.11)
\end{aligned}
$$

Concerning the term I_1 we recall the idea from the proof of [Bisi et al., 2011, Theorem 5.3] that for all $h \in L_v^1(\langle v \rangle^q m)$ there holds

$$
\begin{aligned}
I_1 &\le \int_{\Omega_\delta^1} |h(v)|\mathcal{M}_0(v_*)|v-v_*| \langle v' \rangle^q m(v') d\sigma dv dv_*, \\
&= \int_{\mathbb{R}^3} \mathcal{L}^+\left(\mathbb{1}_{\{|v|>\delta^{-1}\}} |h| \right)(v) \langle v \rangle^q m(v) dv, \\
&= \int_{\mathbb{R}^3} \langle v \rangle^q m(v) \int_{\{|v_*|>\delta^{-1}\}} k_e(v,v_*)|h(v_*)| dv_* dv, \\
&\le \int_{\{|v_*|>\delta^{-1}\}} |h(v_*)| H(v_*) dv_*,
\end{aligned}
$$

where $H(v_*) = \int_{\mathbb{R}^3} k_e(v,v_*) \langle v \rangle^q m(v) dv$ for every $v_* \in \mathbb{R}^3$. Hence, using Proposition A.2.4, there exists some positive constant K such that

$$
I_1 \le K \int_{\{|v_*|>\delta^{-1}\}} |h(v_*)|(1+|v_*|^{1-\beta}) \langle v_* \rangle^q m(v_*) dv_*. \qquad (3.12)
$$

Let us recall the by (2.6) we have $0 < \nu_{e,0}|v| \le \nu_{e,0} \langle v \rangle \le \nu_e(v)$. Therefore, using (3.12), we have that

$$\begin{aligned}
I_1 - \int_{\mathbb{R}^3} \nu_e(v)|h(v)| \langle v\rangle^q m(v)dv, \\
\leq & K \int_{\{|v_*|>\delta^{-1}\}} |h(v_*)|(1+|v_*|^{1-\beta}) \langle v_*\rangle^q m(v_*)dv_* \\
& - \nu_{e,0} \int_{\mathbb{R}^3} \langle v\rangle |h(v)| \langle v\rangle^q m(v)dv, \\
\leq & - \nu_{e,0} \int_{\{|v|\leq\delta^{-1}\}} |h(v)| \langle v\rangle^{q+1} m(v)dv \\
& + \int_{\{|v|>\delta^{-1}\}} |h(v)| \Big(K(1+|v|^{1-\beta}) - \nu_{e,0}|v|\Big) \langle v\rangle^q m(v)dv.
\end{aligned}$$

We claim that, since $\beta > 0$ there exists δ_0 sufficiently small such that

$$K(1+|v|^{1-\beta}) - \nu_{e,0}|v| \leq -\frac{\nu_{e,0}}{2} \langle v\rangle ,$$

for every $|v| > \delta^{-1}$ with $0 < \delta < \delta_0$. Indeed, if we take δ_0 small enough so

$$\frac{\delta_0 + \delta_0^\beta}{1-\delta_0} \leq \frac{\nu_{e,0}}{2K},$$

we have that for every $|v| > \delta^{-1}$ with $0 < \delta < \delta_0$ since $|v| - \langle v\rangle \geq -1$

$$K(1+|v|^{1-\beta}) \leq \nu_{e,0}(|v|-1) \leq \frac{\nu_{e,0}}{2}(2|v| - \langle v\rangle),$$

we can conclude our claim. Therefore,

$$\int_{\mathbb{T}^3} I_1 dx - \int_{\mathbb{T}^3\times\mathbb{R}^3} \nu_e(v)|h(v)| \langle v\rangle^q m(v)dxdv \leq -\frac{\nu_{e,0}}{2}\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}. \tag{3.13}$$

Gathering (3.7), (3.9) (3.11) and (3.13) we obtain that for any $0 < \delta < \delta_0$

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{T}^3} \mathcal{B}_{e,\delta}(h)\mathrm{sign}(h) \langle v\rangle^q m(v)dxdv \\
&\qquad \leq \left((C_2+C_3)\delta + C_4\Lambda(\delta) - \frac{\nu_{e,0}}{2}\right) \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}.
\end{aligned}$$

Hence, we choose $0 < \delta_1 \leq \delta_0$ close enough to 0 in order to have

$$a_0' := -\left((C_2+C_3)\delta_1 + C_4\Lambda(\delta_1) - \frac{\nu_{e,0}}{2}\right) > 0. \tag{3.14}$$

Therefore, for any $0 < \delta < \delta_1$, we have

$$\int_{\mathbb{R}^3\times\mathbb{T}^3} \mathcal{B}_{e,\delta}(h)\mathrm{sign}(h)dx \langle v\rangle^q m(v)dv \leq -a_0'\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)},$$

where we deduce that $\mathcal{B}_{e,\delta} + a_0'$ is dissipative in $L^1_x L^1_v(\langle v\rangle^q m)$.

Step 2: Since the x-derivatives commute with $\mathcal{B}_{e,\delta}$, using the proof of

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Step 1 we have

$$\int_{\mathbb{R}^3\times\mathbb{T}^3} \partial_x\left(\mathcal{B}_{e,\delta}(h)\right)\operatorname{sign}(\partial_x h)\,\langle v\rangle^q\, m(v)dxdv \le -a_0'\|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}.$$

Step 3: In order to deal with the v-derivatives, let us recall the following property

$$\partial_v\mathcal{Q}_e^{\pm}(f,g) = \mathcal{Q}_e^{\pm}(\partial_v f, g) + \mathcal{Q}_e^{\pm}(f,\partial_v g). \tag{3.15}$$

Using this and the fact that $\mathcal{L}_R^+ = \mathcal{L}^+(h) - \mathcal{L}_S^+(h) = \mathcal{L}^+(h) - \mathcal{A}_{e,\delta}(h)$ we compute

$$\begin{aligned}\partial_v\mathcal{B}_{e,\delta}(h) &= \partial_v\left(\mathcal{L}_R^+(h) - \nu_e h\right),\\ &= \partial_v\left(\mathcal{Q}_e^+(h,\mathcal{M}_0) - \mathcal{A}_{e,\delta}(h) - \nu_e h\right),\\ &= \mathcal{Q}_e^+(\partial_v h,\mathcal{M}_0) + \mathcal{Q}_e^+(h\partial_v,\mathcal{M}_0) - \partial_v\mathcal{A}_{e,\delta}(h) - \nu_e\cdot\partial_v h.\end{aligned}$$

Therefore, adding and subtracting the term $\mathcal{A}_{e,\delta}(\partial_v h)$ we obtain

$$\partial_v\mathcal{B}_{e,\delta}(h) = \left(\mathcal{L}^+(\partial_v h) - \mathcal{A}_{e,\delta}(\partial_v h) - \nu_e\cdot\partial_v h\right) + \mathcal{R}(h),$$

where

$$\mathcal{R}(h) = \mathcal{Q}_e^+(h,\partial_v\mathcal{M}_0) - \partial_v\mathcal{A}_{e,\delta}(h) + \mathcal{A}_{e,\delta}(\partial_v h). \tag{3.16}$$

Performing one integration by parts and using Lemma 3.1.1, we have

$$\|\left(\partial_v\mathcal{A}_{e,\delta}\right)(h)\|_{L^1_xL^1_v(\langle v\rangle^q m)} + \|\left(\mathcal{A}_{e,\delta}\right)(\partial_v h)\|_{L^1_xL^1_v(\langle v\rangle^q m)} \le C_\delta'\|h\|_{L^1_xL^1_v(\langle v\rangle^q m)}.$$

Using this and the estimates of Proposition A.2.2 on the operator $\mathcal{Q}_e^+$ we obtain, for some constant $C_\delta > 0$

$$\|\mathcal{R}(h)\|_{L^1_xL^1_v(\langle v\rangle^q m)} \le C_\delta\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}. \tag{3.17}$$

Hence, using the proof presented in **Step 1** and (3.17) we have

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{T}^3}\partial_v\left(\mathcal{B}_{e,\delta}(h)\right)\operatorname{sign}(\partial_v h)\,\langle v\rangle^q\, m(v)dv\\ &\qquad= \int_{\mathbb{R}^3\times\mathbb{T}^3}\left(\mathcal{B}_{e,\delta}(\partial_v h) + \mathcal{R}(h)\right)\operatorname{sign}(\partial_v h)\,\langle v\rangle^q\, m(v)dv,\\ &\qquad\le -a_0'\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)} + C_\delta\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}.\end{aligned}$$

where a_0' is defined in (3.14). Notice that here δ is fixed small enough to guarantee that $a_0' > 0$.

Step 4: For some $\varepsilon > 0$ to be fixed later, we define the norm

$$\|h\|_* = \|h\|_{L^1_xL^1_v(\langle v\rangle^q m)} + \|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^q m)} + \varepsilon\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^q m)}.$$

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Notice that this norm is equivalent to the classical $W^{1,1}_{x,v}(\langle v\rangle^q m)$-norm. We deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{T}^3}\mathcal{B}_{e,\delta}(h)\mathrm{sign}(h)\langle v\rangle^q m(v)dxdv\\
&\qquad+\int_{\mathbb{R}^3\times\mathbb{T}^3}\partial_x\left(\mathcal{B}_{e,\delta}(h)\right)\mathrm{sign}(\partial_x h)\langle v\rangle^q m(v)dxdv\\
&\qquad+\varepsilon\int_{\mathbb{R}^3\times\mathbb{T}^3}\partial_v\left(\mathcal{B}_{e,\delta}(h)\right)\mathrm{sign}(\partial_v h)\langle v\rangle^q m(v)dxdv\\
&\qquad\le -a_0'\left(\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}+\|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}+\varepsilon\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}\right)\\
&\qquad+\varepsilon\left(C_\delta\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}\right),\\
&\qquad\le(-a_0'+o(\varepsilon))\left(\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}+\|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}\varepsilon\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}\right),\\
&\qquad\le(-a_0'+o(\varepsilon))\|h\|_*,
\end{aligned}$$

where $o(\varepsilon)=\varepsilon\cdot C_\delta$ and goes to 0 as ε goes to 0 and we choose ε close enough to 0 so that $a_0=a_0'-o(\varepsilon)>0$. Hence, we obtain that $\mathcal{B}_{e,\delta}+a_0$ is dissipative in $W^{1,1}_{x,v}(\langle v\rangle^q m)$ for the norm $\|\cdot\|_*$ and thus hypodissipative in $W^{1,1}_{x,v}(\langle v\rangle^q m)$.

□

The following result comprises what we proved above:

Lemma 3.2.2. *For any $e\in(0,1]$, there exist $a_0>0$ and $\delta>0$ such that the operator $\mathcal{B}_{e,\delta}+a_0$ is hypodissipative in E_j, $j=0,1$ and $\mathcal{E}$.*

3.3 Splitting of the linearized elastic operator $\mathscr{L}_1$

We now focus into the study of the linear equation $\partial_t h=\mathscr{L}_1(h)$ introduced in (2.8) for $h=h(t,x,v)$ with $x\in\mathbb{T}^3$, $v\in\mathbb{R}^3$ and $\alpha=1$. First of all let us recall the definition of $\mathscr{L}_1$ given by (2.7):

$$\mathscr{L}_1(h):=\mathcal{Q}_1(\mathcal{M},h)+\mathcal{Q}_1(h,\mathcal{M})+\mathcal{L}(h)-v\cdot\nabla_x h, \tag{3.18}$$

where $\mathcal{M}$ is defined in (2.3).

In the elastic case, i.e. $\alpha=1$, we can define the collision operator in strong form

$$\mathcal{Q}_1(f,g)=\int_{\mathbb{R}^3\times\mathbb{S}^2}[f(v')g(v'_*)-f(v)g(v_*)]|v-v_*|dv_*d\sigma.$$

Therefore, using the function Θ_δ defined in Section 3.1 we can give the following decomposition of the linearized collision operator $\mathcal{T}_1(h):=\mathcal{Q}_1(\mathcal{M},h)+$

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$\mathcal{Q}_1(h, \mathcal{M})$:

$$\mathcal{T}_1(h) = \mathcal{T}_{1,R}(h) + \mathcal{T}_{1,S}(h) - \nu h,$$

where ν is a collision frequency defined in a similar way as ν_e. More precisely, $\nu := L(\mathcal{M})$, with L defined as in (2.4) and $\mathcal{M}$ is given by (2.3). $\mathcal{T}_{1,S}$ is the truncated operator given by Θ_δ while $\mathcal{T}_{1,R}$ is the corresponding reminder. By this we mean

$$\mathcal{T}_{1,S}(h) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} \Theta_\delta \left[\mathcal{M}(v'_*)h(v') + \mathcal{M}(v')h(v'_*) - \mathcal{M}(v)h(v_*)\right] |v - v_*| dv_* d\sigma$$

and,

$$\mathcal{T}_{1,R}(h) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} (1 - \Theta_\delta) \left[\mathcal{M}(v'_*)h(v') + \mathcal{M}(v')h(v'_*) - \mathcal{M}(v)h(v_*)\right] |v - v_*| dv_* d\sigma. \tag{3.19}$$

Hence we can give a decomposition for the linearized operator $\mathscr{L}_1$:

$$\begin{aligned}
\mathscr{L}_1(h) &= \mathcal{T}_1(h) + \mathcal{L}(h) - v \cdot \nabla_x h, \\
&= \mathcal{T}_{1,R}(h) + \mathcal{T}_{1,S}(h) - (\nu + \nu_e)h + \mathcal{L}_R^+(h) + \mathcal{L}_S^+(h) - v \cdot \nabla_x h, \\
&= \left(\mathcal{T}_{1,S}(h) + \mathcal{A}_{e,\delta}(h)\right) + \left(\mathcal{T}_{1,R}(h) - \nu h + \mathcal{B}_{e,\delta}(h) - v \cdot \nabla_x h\right), \\
&= \mathcal{A}_{1,\delta}(h) + \mathcal{B}_{1,\delta}(h).
\end{aligned}$$

where $\mathcal{A}_{1,\delta}(h) := \mathcal{T}_{1,S}(h) + \mathcal{A}_{e,\delta}(h)$ and $\mathcal{B}_{1,\delta}(h)$ is the remainder.

As in the case of the operator $\mathcal{L}$, by the Carleman representation for the elastic case (see [Bisi et al., 2011, Theorem 5.4], [Villani, 2002, Chapter 1] or [Gamba et al., 2009, Appendix C])

$$\mathcal{T}_1^+(h)(v) = \int_{\mathbb{R}^3} k_1(v, v_*)h(v_*)dv_*,$$

where k_1 is given by

$$k_1(v, v_*) = \frac{C_1}{|v - v_*|} \exp\left\{\frac{-1}{8\theta^\sharp}\left(|v - v_*| + \frac{|v - u_0|^2 - |v_* - u_0|^2}{|v - v_*|}\right)^2\right\}, \tag{3.20}$$

for some constant $C_1 > 0$. Moreover, following the same construction from [Gamba et al., 2009, Appendix C], we can write the truncated operator

$$\mathcal{T}_{1,S}(h)(v) = \int_{\mathbb{R}^3} k_{1,\delta}(v, v_*)h(v_*)dv_*,$$

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where $k_{1,\delta} = k^1_{1,\delta} + k^2_{1,\delta} - k^3_{1,\delta} \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3)$ with

$$k^i_{1,\delta}(v, v_*) = \frac{C^i}{|v - v_*|} \int_{V_2 \cdot (v' - v) = 0} \Theta_\delta \cdot \mathcal{M}\,(V_2 + \Omega)\, dV_2 dv_*, \qquad \text{for } i = 1, 2;$$
$$k^3_{1,\delta}(v, v_*) = \int_{\mathbb{R}^3} \Theta_\delta \cdot \mathcal{M}(v_*)|v - v_*| dv_*;$$

where $\Omega = v + ((\alpha - 1)/(\alpha + 1))\,(v_* - v)$. Therefore,

$$\mathcal{A}_{1,\delta}(h)(v) = \int_{\mathbb{R}^3} k_\delta(v, v_*) h(v_*) dv_*, \tag{3.21}$$

where $k_\delta = k_{1,\delta} + k_{e,\delta}$ with $k_{e,\delta}$ defined by (3.2).

Gualdani, Mischler and Mouhot [Gualdani et al., 2017, Lemma 4.16] proved a regularity estimate on the truncated operator $\mathcal{T}_{1,S}$. Gathering this result with the one presented in Lemma 3.1.1 we get the following regularity result for $\mathcal{A}_{1,\delta}$.

Lemma 3.3.1. *For any $s \in \mathbb{N}$ and any $e \in (0, 1]$, the operator $\mathcal{A}_{1,\delta}$ maps $L^1_v(\langle v\rangle)$ into H^s_v functions with compact support, with explicit bounds (depending on δ) on the $L^1_v(\langle v\rangle) \mapsto H^s_v$ norm and on the size of the support. More precisely, there are two constants $C_{s,\delta}$ and R_δ such that for any $h \in L^1_v(\langle v\rangle)$*

$$K := \operatorname{supp} \mathcal{A}_{1,\delta} h \subset B(0, R_\delta), \quad \text{and} \quad \|\mathcal{A}_{1,\delta}\|_{H^s_v(K)} \le C_{s,\delta} \|h\|_{L^1_v(\langle v\rangle)}.$$

In particular, we deduce that $\mathcal{A}_{1,\delta}$ is in $\mathscr{B}(E_j)$ for $j = 0, 1$ and $\mathcal{A}_{1,\delta}$ is in $B(\mathcal{E}, E)$.

Furthermore, Gualdani, Mischler and Mouhot [Gualdani et al., 2017, Lemma 4.14] study the hypositivativity of the operator $\mathcal{B}^1_{1,\delta} := \mathcal{T}_{1,R}(h) - \nu h - v \cdot \nabla_x h$. More precisely, they proved that there exist a constant $\lambda_0 > 0$ such that $\mathcal{B}^1_{1,\delta} + \lambda_0$ is hypodissivative in E_j with $j = 0, 1$ and in $\mathcal{E}$. Thus, following their proof jointly with the one of Lemma 3.2.2, we are led with the following:

Lemma 3.3.2. *For any $e \in (0, 1]$, there exist $a_1 > 0$ and $\delta > 0$ such that the operator $\mathcal{B}_{1,\delta} + a_1$ is hypodissipative in E_j, $j = 0, 1$ and $\mathcal{E}$.*

Proof. It is enough to see that, by the definition of $\mathcal{B}_{e,\delta}$

$$\mathcal{B}_{1,\delta}(h) = \Big(\mathcal{T}_{1,R}(h) + \mathcal{L}^+_R(h)\Big) - (\nu + \nu_e)h - v \cdot \nabla_x h.$$

Notice the divergence structure of the last term in the x-coordinate. Hence, when integrating over $\mathbb{T}^3$ the last term vanish. Therefore, proceeding as in the proof of Lemma 3.2.1 we conclude our proof. □

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3.4 Regularization properties of T_n

In this subsection we prove the key regularity results for our factorization and enlargement theory. Consider the operators

$$T_n(t) := (\mathcal{A}_{1,\delta} S_{\mathcal{B}_{1,\delta}})^{(*n)}(t),$$

for $n \geq 1$, where $S_{\mathcal{B}_{1,\delta}}$ is the semigroup generated by the operator $\mathcal{B}_{1,\delta}$ and $*$ denotes the convolution. We remind the reader that the $T_n(t)$ operators are merely time-indexed family of operators which do not have the semigroup property in general.

Remark 3.4.1. *As we will see below, thanks to Theorem 2.4.4 and Theorem 3.5.1, the operator $\mathscr{L}_1$ generates a C_0-semigroup in $H^s_x H^\sigma_v(\mathcal{M}^{-1})$. Therefore, jointly with the hypodissipativity of $\mathcal{B}_{1,\delta}$ this guarantees that the operator $\mathcal{B}_{1,\delta}$ also generates a C_0-semigroup in E_j for $j = 0, 1$. We refer the reader to the proof of [Gualdani et al., 2017, Theorem 2.13] for a further discussion about this matter. Moreover, a direct proof can be perform as the one presented in Appendix B for a similar operator.*

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Thanks to Lemma 3.3.1 we know that the operator $\mathcal{A}_{1,\delta}$ provides as much regularity as we want in the v-coordinate. We now want to see how much regularity we gain in the x-coordinate with the T_n operator. More precisely:

Lemma 3.4.2. *Let us consider a_1 as in Lemma 3.3.2. The time indexed family T^n of operators satisfies the following: for any $a' \in (0, a_1)$ and any $e \in (0, 1]$, there are some constructive constants $C_\delta > 0$ and R_δ such that for any $t \geq 0$*

$$\operatorname{supp} T_n(t)h \subset K := B(0, R_\delta),$$

and

– *If $s \geq k \geq 1$ then*

$$\|T_1(t)h\|_{W^{s+1,1}_x W^{k,1}_v(K)} \leq C \frac{e^{-a't}}{t} \|h\|_{W^{s,1}_x W^{1,1}_v(\langle v \rangle m)}. \tag{3.22}$$

– *If $s \geq k \geq 0$ then*

$$\|T_2(t)h\|_{W^{s+1/2,1}_x W^{k,1}_v(K)} \leq C e^{-a't} \|h\|_{W^{s,1}_x L^1_v(\langle v \rangle m)}. \tag{3.23}$$

Proof. Let us consider $h_0 \in W^{s,1}_x L^1_v(\langle v\rangle\, m)$, $s \in \mathbb{N}$. Since the x-derivatives commute with $\mathcal{A}_{1,\delta}$ and $\mathcal{B}_{1,\delta}$, using Lemma 3.3.1 we have

$$\begin{aligned}\|T_1(t)h_0\|_{W^{s,1}_x W^{k,1}_v(K)} &= \|\mathcal{A}_{1,\delta} S_{\mathcal{B}_{1,\delta}}(t)h_0\|_{W^{s,1}_x W^{k,1}_v(K)},\\ &\leq C\|S_{\mathcal{B}_{1,\delta}}(t)h_0\|_{W^{s,1}_x L^1_v(K)},\\ &\leq C\|S_{\mathcal{B}_{1,\delta}}(t)h_0\|_{W^{s,1}_x L^1_v(\langle v\rangle m)}.\end{aligned}$$

Since $\mathcal{B}_{1,\delta} + a_1$ is hypodissipative in $W^{s,1}_x L^1_v(\langle v\rangle\, m)$ from Theorem 2.3.1 we have

$$\|T_1(t)h_0\|_{W^{s,1}_x W^{k+1,1}_v(K)} \leq Ce^{-a_1 t}\|h_0\|_{W^{s,1}_x W^{k,1}_v(\langle v\rangle m)}. \tag{3.24}$$

Now let us assume $h_0 \in W^{s,1}_x W^{1,1}_v(\langle v\rangle\, m)$ and consider the function $g_t = S_{\mathcal{B}_{1,\delta}}(t)(\partial^\beta_x h_0)$, for any $|\beta| \leq s$. Notice that, since $\partial_t(S_{\mathcal{B}_{1,\delta}}) = \mathcal{B}_{1,\delta} S_{\mathcal{B}_{1,\delta}}$, this operator satisfies

$$\partial_t g_t = \mathscr{L}_1 g_t - \mathcal{A}_{1,\delta} g_t = \mathcal{T}_1(g_t) + \mathcal{L}(g_t) - v\cdot\nabla_x g_t - \mathcal{A}_{1,\delta} g_t.$$

Introducing the differential operator $D_t := t\nabla_x + \nabla_v$, we observe that D_t commutes with the free transport operator $\partial_t + v\cdot\nabla_x$, so that using the property (3.15) we have

$$\begin{aligned}\partial_t(D_t g_t) + v\cdot\nabla_x(D_t g_t) &= D_t\left(\partial_t g_t + v\cdot\nabla_x g_t\right),\\ &= D_t\left(\mathcal{T}_1(g_t) + \mathcal{L}(g_t) - \mathcal{A}_{1,\delta}(g_t)\right),\\ &= \mathcal{T}_1(D_t g_t) + \mathcal{L}(D_t g_t) + \mathcal{Q}_1(g_t, \nabla_v\mathcal{M}) + \mathcal{Q}_1(\nabla_v\mathcal{M}, g_t)\\ &\quad + \mathcal{Q}_e(g_t, \nabla_v\mathcal{M}_0) - D_t(\mathcal{A}_{1,\delta} g_t).\end{aligned}$$

Using the notation in (3.21) we obtain

$$\begin{aligned}D_t(\mathcal{A}_{1,\delta} g_t) &= \int_{\mathbb{R}^3} k_\delta(v,v_*)t\nabla_x g_t(v_*)d_{v_*} + \int_{\mathbb{R}^3}\left(\nabla_v k_\delta(v,v_*)\right)g_t(v_*)d_{v_*},\\ &= \int_{\mathbb{R}^3} k_\delta(v,v_*)(D_t g_t)(v_*)d_{v_*} + \int_{\mathbb{R}^3}\left(\nabla_v k_\delta(v,v_*)\right)g_t(v_*)d_{v_*}\\ &\quad - \int_{\mathbb{R}^3} k_\delta(v,v_*)\nabla_{v_*} g_t(v_*)d_{v_*},\\ &= \int_{\mathbb{R}^3} k_\delta(v,v_*)(D_t g_t)(v_*)d_{v_*} + \int_{\mathbb{R}^3}\left(\nabla_v k_\delta(v,v_*)\right)g_t(v_*)d_{v_*}\\ &\quad + \int_{\mathbb{R}^3}\left(\nabla_{v_*} k_\delta(v,v_*)\right)g_t(v_*)d_{v_*},\\ &= \mathcal{A}_{1,\delta}(D_t g_t) + \mathcal{A}^1_{1,\delta}(g_t) + \mathcal{A}^2_{1,\delta}(g_t)\end{aligned}$$

where in the third equality we have performed one integration by parts. All

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together, we may write $\partial_t(D_t g_t) = \mathcal{B}_{1,\delta}(D_t g_t) + \mathcal{I}_\delta(g_t)$, where

$$\begin{aligned}\mathcal{I}_\delta(g_t) =&\mathcal{Q}_1(g_t, \nabla_v \mathcal{M}) + \mathcal{Q}_1(\nabla_v \mathcal{M}, g_t) + \mathcal{Q}_e(g_t, \nabla_v \mathcal{M}_0)\\ &- \mathcal{A}^1_{1,\delta}(g_t) - \mathcal{A}^2_{1,\delta}(g_t).\end{aligned}$$

Hence, since $\mathcal{A}^1_{1,\delta}$ stands for the integral operator associated with the kernel $\nabla_v k_\delta$ and $\mathcal{A}^2_{1,\delta}$ stands for the integral operator associated with the kernel $\nabla_{v^*} k_\delta$, using Lemma 3.3.1 and Proposition A.2.2 $\mathcal{I}_\delta$ satisfies

$$\|\mathcal{I}_\delta(g_t)\|_{L^1_v(\langle v\rangle m)} \leq C\|g_t\|_{L^1_v(\langle v\rangle^2 m)}.$$

Arguing as in Lemma 3.2.1, and by the hypodissipativity of $\mathcal{B}_{1,\delta}$, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{T}^3} |D_t g_t| \langle v\rangle m(v) dx dv \\ &\quad= \int_{\mathbb{R}^3\times\mathbb{T}^3} \partial_t(D_t g_t)\, \mathrm{sign}(D_t g_t) \langle v\rangle m(v) dx dv,\\ &\quad\leq \int_{\mathbb{R}^3\times\mathbb{T}^3} \mathcal{B}_{1,\delta}(D_t g_t)\, \mathrm{sign}(D_t g_t) \langle v\rangle m(v) dx dv + \|\mathcal{I}_\delta(g_t)\|_{L^1_x L^1_v(\langle v\rangle m)},\\ &\quad\leq -a_1 \int_{\mathbb{R}^3\times\mathbb{T}^3} |D_t g_t| \langle v\rangle^2 m(v) dx dv + C_\delta \|g_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)},\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{T}^3} |g_t| \langle v\rangle m(v) dx dv &= \int_{\mathbb{R}^3\times\mathbb{T}^3} \partial_t(g_t)\, \mathrm{sign}(g_t) \langle v\rangle m(v) dx dv,\\ &\leq \int_{\mathbb{R}^3\times\mathbb{T}^3} \mathcal{B}_{1,\delta}(g_t)\, \mathrm{sign}(g_t) \langle v\rangle m(v) dx dv,\\ &\leq -a_1 \int_{\mathbb{R}^3\times\mathbb{T}^3} |g_t| \langle v\rangle^2 m(v) dx dv.\end{aligned} \tag{3.26}$$

Combining the differential inequalities (3.25) and (3.26) we obtain, for any $a' \in (0, a_1]$ and for ε small enough

$$\frac{d}{dt}\left(e^{a't}\int (\varepsilon|D_t g_t| + |g_t|) \langle v\rangle m(v) dx dv\right) \leq 0,$$

which implies

$$\|D_t g_t\|_{L^1_x L^1_v(\langle v\rangle m)} + \|g_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \varepsilon^{-1} e^{-a't} \|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \tag{3.27}$$

Now, notice that since the x-derivatives commute with $\mathcal{A}_{1,\delta}$ and $\mathcal{B}_{1,\delta}$ we

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have

$$\begin{aligned} t\nabla_x T_1(t)\left(\partial_x^\beta h_0\right) &= t\nabla_x \mathcal{A}_{1,\delta} S_{\mathcal{B}_{1,\delta}}(t)\left(\partial_x^\beta h_0\right), \\ &= \mathcal{A}_{1,\delta}\left(t\nabla_x g_t\right), \\ &= \mathcal{A}_{1,\delta}\left(D_t g_t - \nabla_v g_t\right), \\ &= \int_{\mathbb{R}^3} k_\delta(v,v_*)\left(D_t g_t\right)(v_*)dv_* + \int_{\mathbb{R}^3}\left(\nabla_{v_*} k_\delta(v,v_*)\right) g_t(v_*)dv_*, \\ &= \mathcal{A}_{1,\delta}(D_t g_t) + \mathcal{A}^2_{1,\delta}(g_t). \end{aligned}$$

Thus, using (3.27) we get

$$\begin{aligned} t\|\nabla_x T_1(t)\left(\partial_x^\beta h_0\right)\|_{L^1_x L^1_v(K)} &\leq C_\delta\left(\|D_t g_t\|_{L^1_x L^1_v(\langle v\rangle m)} + \|g_t\|_{L^1_x L^1_v(\langle v\rangle m)}\right), \\ &\leq C_\delta \varepsilon^{-1} e^{-a't}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \end{aligned}$$

Since $0 < a' \leq a_1$, using the last inequality together with (3.24) and Lemma 3.3.1, for $s \geq 0$ we have

$$\|T_1(t)\left(\partial_x^\beta h_0\right)\|_{W^{1,1}_x W^{k,1}_v(K)} \leq \frac{Ce^{-a't}}{t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}.$$

This concludes the proof of (3.22).

In order to prove (3.23) we interpolate between the last inequality for a given s i.e.

$$\|T_1(t)h_0\|_{W^{s+1,1}_x W^{k,1}_v(K)} \leq \frac{Ce^{-a't}}{t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)},$$

and

$$\|T_1(t)h_0\|_{W^{s,1}_x W^{k,1}_v(K)} \leq Ce^{-a_1 t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)},$$

obtained in (3.24) written for the same s, it gives us

$$\begin{aligned} &\|T_1(t)h_0\|_{W^{s+1/2,1}_x W^{k,1}_v(K)} \\ &\quad \leq C\left(\frac{e^{-a't}}{t}\right)^{1/2}\left(e^{-a_1 t}\right)^{1/2}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}, \\ &\quad \leq C\frac{e^{-a't}}{\sqrt{t}}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \end{aligned} \tag{3.28}$$

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Putting together (3.24) and (3.28) for $s \geq 0$ we get

$$
\begin{aligned}
&\|T_2(t)h_0\|_{W_x^{s+1/2,1}W_v^{k,1}(K)} \\
&\quad = \int_0^t \|T_1(t-\tau)T_1(\tau)h_0\|_{W_x^{s+1/2,1}W_v^{k,1}(K)} d\tau, \\
&\quad \leq C \int_0^t \frac{e^{-a'(t-\tau)}}{\sqrt{t-\tau}} \|T_1(\tau)h_0\|_{W_x^{s,1}W_v^{1,1}(\langle v\rangle m)} d\tau, \\
&\quad \leq C \left(\int_0^t \frac{e^{-a'(t-\tau)}}{\sqrt{t-\tau}} e^{-a_1\tau} d\tau \right) \|h_0\|_{W_x^{s,1}L_v^1(\langle v\rangle m)}, \\
&\quad \leq C e^{-a't} \left(\int_0^t \frac{e^{-(a_1-a')\tau}}{\sqrt{t-\tau}} d\tau \right) \|h_0\|_{W_x^{s,1}L_v^1(\langle v\rangle m)}, \\
&\quad \leq C' e^{-a't} \|h_0\|_{W_x^{s,1}L_v^1(\langle v\rangle m)}.
\end{aligned}
$$

This concludes our proof.

□

Summarizing we just proved that every time we apply T_1 we gain one derivative in the x-coordinate, while when we apply T_2 we gain 1/2. Hence, combining Lemma 3.3.2 and Lemma 3.4.2 we get the assumptions of Lemma C.0.2. We have thus proved the following result:

Lemma 3.4.3. *Let us consider a_1 as in Lemma 3.3.2. For any $a' \in (0, a_1)$, there exist some constructive constants $n \in \mathbb{N}$ and $C_{a'} \geq 1$ such that for all $t \geq 0$ and*

$$\|T_n(t)\|_{\mathscr{B}(E_0,E_1)} \leq C_{a'} e^{-a't}.$$

3.5 Semigroup spectral analysis of the linearized elastic operator

This section will be devoted to prove some hypodissipative results for the semigroup associated to the linearized elastic Boltzmann equation. Namely, we are going to prove the localization of the spectrum of the operator $\mathscr{L}_1$, as well as the decay estimate for the semigroup. This result will be used in the next chapter to prove similar properties for the spectrum of $\mathscr{L}_\alpha$.

Let us first recall an important result due to Gualdani, Mischler, Mouhot [Gualdani et al., 2017].

Theorem 3.5.1. *[Gualdani et al., 2017, Theorem 4.2] Consider the operator*

$$\hat{\mathscr{L}}_1(h) := \mathcal{T}_1(h) - v \cdot \nabla_x.$$

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Let $\mathcal{E}' = H^s_x H^\sigma_v(\mathcal{M}^{-1})$ where $s, \sigma \in \mathbb{N}$ with $\sigma \leq s$. Then there exist constructive constants $C \geq 1$, $\lambda > 0$, such that the operator $\hat{\mathscr{L}}_1$ satisfies in $\mathcal{E}'$:

$$\Sigma(\hat{\mathscr{L}}_1) \subset \{z \in \mathbb{C} : \mathfrak{Re}(z) \leq -\lambda\} \cup \{0\},$$
$$N(\hat{\mathscr{L}}_1) = Spam\{\mathcal{M}, v_1\mathcal{M}, v_2\mathcal{M}, v_3\mathcal{M}, |v|^2\mathcal{M}\}.$$

It is also the generator of a strongly continuous semigroup $h_t = S_{\hat{\mathscr{L}}_1}(t)h_{in}$ in $\mathcal{E}'$, solution to the initial value problem $\partial_t h = \hat{\mathscr{L}}_1(h)$, which satisfies for every $t \geq 0$:

$$\|h_t - \Pi h_{in}\|_{\mathcal{E}'} \leq Ce^{-\lambda t}\|h_{in} - \Pi h_{in}\|_{\mathcal{E}'},$$

where Π stands for the projection over $N(\hat{\mathscr{L}}_1)$. Moreover λ can be taken equal to the spectral gap of $\hat{\mathscr{L}}_1$ in $H^s(\mathcal{M}^{-1})$ with $s \in \mathbb{N}$ as large as wanted.

We can now formulate the main result of this section.

Theorem 3.5.2. *For any $e \in (0, 1]$, there exist constructive constants $C \geq 1$, $a_2 > 0$ such that the operator $\mathscr{L}_1$ satisfies in E_0 and E_1:*

$$\Sigma(\mathscr{L}_1) \cap \Delta_{-a_2} = \{0\} \quad \textit{and} \quad N(\mathscr{L}_1) = Span\{\mathcal{M}\}.$$

Moreover, $\mathscr{L}_1$ is the generator of a strongly continuous semigroup $h(t) = S_{\mathscr{L}_1}h_{in}$ in E_0 and E_1, solution to the initial value problem (2.8) with $\alpha = 1$, which satisfies that for all $t \geq 0$ and $j = 0, 1$:

$$\|S_{\mathscr{L}_1}(t)(\mathrm{id} - \Pi_{\mathscr{L}_1,0})\|_{\mathscr{B}(E_j)} \leq Ce^{-a_2 t}$$

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Proof. The idea of the proof consist in deducing the spectral properties in E_j from the much easier spectral analysis in $H^{s'}_{x,v}(\mathcal{M}^{-1/2})$. More precisely, we will see that the assumptions of a more abstract theorem regarding the enlargement of the functional space semigroup decay are satisfied. We enunciate this result in Theorem C.0.1.

Consider $\mathcal{E}' = E_j$ and $E' = H^{s'}_{x,v}(\mathcal{M}^{-1})$ with s' large enough so $E' \subset \mathcal{E}'$. The assumptions in (B.) in Theorem C.0.1 are a direct consequence of the Lemmas 3.3.2, 3.3.1 and 3.4.3. Indeed, from Lemma 3.4.2 and Lemma C.0.2 we have for instance

$$\|T_n(t)h\|_{H^{s'}_{x,v}} \leq Ce^{-a't}\|h\|_{L^1_{x,v}(\langle v\rangle m)},$$

and so

$$\|T_{n+1}\|_{E'} \leq Ce^{-a't}\|h\|_{\mathcal{E}'}.$$

So we are left with the task of verifying that (A.) is satisfied. However, this is a direct consequence of Theorem 2.4.4 and Theorem 3.5.1 taking $a_2 \in (0, a_1)$, where a_1 is given by Lemma 3.3.2. Therefore, the result follows by the Enlargement Theorem C.0.1.

□

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4
Properties of the linearized operator

We now focus into the study of the linear equation

$$\partial_t h = \mathscr{L}_\alpha(h) = \mathcal{Q}_\alpha(F_\alpha, h) + \mathcal{Q}_\alpha(h, F_\alpha) + \mathcal{L}(h) - v \cdot \nabla_x h,$$

introduced in (2.8) for $h = h(t, x, v)$ with $x \in \mathbb{T}^3$ and $v \in \mathbb{R}^3$.

First, we want to find a splitting of the linearized operator $\mathscr{L}_\alpha = \mathcal{A}_\alpha + \mathcal{B}_\alpha$ where $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ satisfy the assumptions of the Spectral Mapping Theorem C.0.3 for the Sobolev spaces E_1, E_0 defined in the previous chapter. Namely, hipodissipativity of $\mathcal{B}_\alpha$ and boundedness $\mathcal{A}_\alpha$ in E_j for $j = 0, 1$. We are able to do this by a perturbative argument around the elastic case. Once we obtain the localization in the spectrum and exponential decay of $\mathscr{L}_\alpha$ in $E = E_0$ we, once again, apply the Enlargement Theorem C.0.1 to obtain this properties in the larger space $\mathcal{E}$.

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4.1
The linearized operator and its splitting.

In this section we give a decomposition of the linear operator $\mathscr{L}_\alpha$. In order to do this, for any $\delta \in (0, 1)$ consider the bounded (by one) operator Θ_δ defined in Section 3.1. We also need to consider the collision frequency $\nu_\alpha = L(F_\alpha)$, where F_α is given by Theorem 2.4.1.

Let us define the operator $\mathcal{T}_\alpha$ by $\mathcal{T}_\alpha(h) = \mathcal{Q}_\alpha(F_\alpha, h) + \mathcal{Q}_\alpha(h, F_\alpha)$. Therefore, using the weak formulation we have for any test function ψ

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathcal{T}_\alpha(h)\psi dv \\ &= \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} F_\alpha(v)h(v_*)|v - v_*| \left[\psi(v'_*) + \psi(v') - \psi(v_*) - \psi(v)\right] d\sigma dv_* dv.\end{aligned}$$

Now we can give the following decomposition of the linearized collision operator $\mathcal{T}_\alpha(h)$:

$$\mathcal{T}_\alpha(h) = \mathcal{T}_{\alpha,R}(h) + \mathcal{T}_{\alpha,S}(h) - \nu_\alpha h,$$

where $\mathcal{T}_{\alpha,S}$ is the truncated operator given by Θ_δ and $\mathcal{T}_{\alpha,R}$ the corresponding reminder. By this we mean:

$$\mathcal{T}_{\alpha,S}(h) = \mathcal{Q}^+_{\alpha,S}(h, F_\alpha) + \mathcal{Q}^+_{\alpha,S}(F_\alpha, h) - \mathcal{Q}^-_{\alpha,S}(F_\alpha, h),$$

where $\mathcal{Q}^+_{\alpha,S}$ (resp. $\mathcal{Q}^-_{\alpha,S}$) is the gain (resp. loss) part of the collision operator associated to the mollified collision kernel $\Theta_\delta B$. More precisely, for any test function ψ

$$\left\langle \mathcal{Q}^+_{\alpha,S}(h, F_\alpha), \psi \right\rangle = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} h(v)F_\alpha(v_*)\left(\Theta_\delta \cdot |v - v_*|\right)\psi(v)d\sigma dv_* dv.$$

In a similar way

$$\mathcal{T}_{\alpha,R}(h) = \mathcal{Q}^+_{\alpha,R}(h, F_\alpha) + \mathcal{Q}^+_{\alpha,R}(F_\alpha, h) - \mathcal{Q}^-_{\alpha,R}(F_\alpha, h),$$

where $\mathcal{Q}^+_{\alpha,R}$ (resp. $\mathcal{Q}^-_{\alpha,R}$) is the gain (resp. loss) part of the collision operator associated to the mollified collision kernel $(1 - \Theta_\delta)B$.

Hence we can give a decomposition for the linearized operator $\mathscr{L}_\alpha$ in the following way:

$$\begin{aligned}
\mathscr{L}_\alpha(h) &= \mathcal{T}_\alpha(h) + \mathcal{L}(h) - v\cdot\nabla_x h,\\
&= \mathcal{T}_{\alpha,R}(h) + \mathcal{T}_{\alpha,S}(h) - (\nu_\alpha + \nu_e)h + \mathcal{L}^+_R(h) + \mathcal{L}^+_S(h) - v\cdot\nabla_x h,\\
&= \left(\mathcal{T}_{\alpha,S}(h) + \mathcal{A}_{e,\delta}(h)\right) + \left(\mathcal{T}_{\alpha,R}(h) - \nu_\alpha h + \mathcal{B}_{e,\delta} - v\cdot\nabla_x h\right),\\
&= \mathcal{A}_{\alpha,\delta}(h) + \mathcal{B}_{\alpha,\delta}(h).
\end{aligned}$$

where $\mathcal{A}_{\alpha,\delta}(h) := \mathcal{T}_{\alpha,S}(h) + \mathcal{A}_{e,\delta}(h)$ and $\mathcal{B}_{\alpha,\delta}(h)$ is the remainder.

Moreover, by the Carleman representation for the inelastic case given by Arlotti Lods in [Arlotti and Lods, 2007, Theorem 2.1] we can write the truncated operator as

$$\mathcal{A}_{\alpha,\delta}(h)(v) = \int_{\mathbb{R}^3} k_\delta(v, v_*)h(v_*)dv_*, \tag{4.1}$$

where $k_\delta = k_{\alpha,\delta} + k_{e,\delta}$ with $k_{e,\delta}$ is defined in (3.2), and $k_{\alpha,\delta}$ is the kernel associated to $\mathcal{T}_{\alpha,S}$. Furthermore, from the proof of [Arlotti and Lods, 2007,

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Theorem 2.1] we have that $k_{\alpha,\delta} = k^1_{\alpha,\delta} + k^2_{\alpha,\delta} - k^3_{\alpha,\delta} \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^3)$ with

$$k^i_{\alpha,\delta}(v, v_*) = \frac{C_e}{|v - v_*|} \int_{V_2\cdot(v'-v)=0} \Theta_\delta \cdot F_\alpha\left(V_2 + \Omega\right) dV_2 dv_*, , \qquad \text{for } i = 1, 2;$$
$$k^3_{\alpha,\delta}(v, v_*) = \int_{\mathbb{R}^3} \Theta_\delta \cdot F_\alpha(v_*)|v - v_*| dv_*;$$

where $\Omega = v + ((\alpha - 1)/(\alpha + 1))\,(v_* - v)$.

Notice that by Lemma 3.1.1 we obtain a regularity estimate on the truncated operator $\mathcal{A}_{\alpha,\delta}$.

Lemma 4.1.1. *For any $s \in \mathbb{N}$, any $\alpha \in (0, 1]$ and any $e \in (0, 1]$, the operator $\mathcal{A}_{\alpha,\delta}$ maps $L^1_v(\langle v\rangle)$ into H^{s+1}_v functions with compact support, with explicit bounds (depending on δ) on the $L^1_v(\langle v\rangle) \mapsto H^{s+1}_v$ norm and on the size of the support. More precisely, there are two constants $C_{s,\delta}$ and R_δ such that for any $h \in L^1_v(\langle v\rangle)$*

$$K := \operatorname{supp} \mathcal{A}_{\alpha,\delta} h \subset B(0, R_\delta), \quad \textit{and} \quad \|\mathcal{A}_{\alpha,\delta}\|_{H^{s+1}_v(K)} \leq C_{s,\delta} \|h\|_{L^1_v(\langle v\rangle)}.$$

In particular, we deduce that $\mathcal{A}_{\alpha,\delta}$ is in $\mathscr{B}(E_j)$ for $j = 0, 1$ and $\mathcal{A}_{\alpha,\delta}$ is in $B(\mathcal{E}, E)$, where the spaces E_j and $\mathcal{E}$ were defined in Chapter 3.

Proof. It is clear that the range of the operator $\mathcal{A}_{\alpha,\delta}$ is included into a compactly supported functions thanks to the truncation. Moreover, the bound on the size of the support is related to δ.

Notice that,the proof of the smoothing estimate for the gain terms $\mathcal{Q}^+_{\alpha,S}$ in the definition of $\mathcal{A}_{\alpha,\delta}$ follows as in [Alonso and Lods, 2013b, Proposition 2.4]. On the other hand, the regularity estimate is trivial for the loss term since we can decompose the truncation as $\Theta_\delta = \Theta^1_\delta(v)\Theta^2_\delta(v - v_*)\Theta^3_\delta(\cos\theta)$, and we can write

$$\left\langle \mathcal{Q}^-_{\alpha,S}(F_\alpha, h), \psi \right\rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} F_\alpha(v)h(v_*)\psi(v)\left(\Theta_\delta|v - v_*|\right) d\sigma dv_* dv,$$
$$= \int_{\mathbb{R}^3} \Theta^1_\delta(v) F_\alpha(v)\left(h * \zeta_\delta\right)\psi(v) dv,$$

where $\zeta_\delta := \Theta^2_\delta(v - v_*)\Theta^3_\delta(\cos\theta)|v - v_*|$ which clearly has compact support.

Moreover, since the regularity of $\mathcal{A}_{e,\delta}$ is given by Lemma 3.1.1 we conclude our proof. □

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Notice that from [Tristani, 2016, Lemma 2.6] we have that the operators $\mathcal{T}_\alpha$ and $\mathcal{L}$ are bounded from $W^{s,1}_x W^{k,1}_v(\langle v\rangle^{q+1} m)$ to $W^{s,1}_x W^{k,1}_v(\langle v\rangle^{q} m)$. Hence, using the fact that the operator $v\cdot\nabla_x$ is bounded from E_1 to E_0, we can conclude that the operator $\mathscr{L}_\alpha$ is bounded from E_1 to E_0.

4.2 Hypodissipativity of $\mathcal{B}_{\alpha,\delta}$

The aim of this section is to prove the hypodissipativity of $\mathcal{B}_{\alpha,\delta}$.

Lemma 4.2.1. *Let us consider $k\ge 0$, $s\ge k$ and $q\ge 0$. Let $\delta>0$ be given by Lemma 3.3.2. Then, there exist $\alpha_1\in(\alpha_0,1]$ and $a_3>0$ such that for any $\alpha\in[\alpha_1,1]$ and any $e\in(0,1]$, the operator $\mathcal{B}_{\alpha,\delta}+a_3$ is hypodissipative in $W^{s,1}_x W^{k,1}_v(\langle v\rangle^{q} m)$.*

Proof. We consider the case $W^{1,1}_{x,v}(\langle v\rangle^{q} m)$. The higher-order cases are treated in a similar way.

Consider a solution h to the linear equation $\partial_t h = \mathcal{B}_{\alpha,\delta}(h)$ given initial datum h_0. The main idea of the proof is to construct a positive constant a_3 and a norm $\|\cdot\|_*$ equivalent to the norm on $W^{1,1}_{x,v}(\langle v\rangle^{q} m)$ such that there exists ψ in the dual space $\left(W^{1,1}_{x,v}(\langle v\rangle^{q} m)\right)^*$ such that $\psi\in F(h)$, where $F(h)$ is defined in (2.1), and

$$\mathfrak{Re}\,\langle\psi,\mathcal{B}_{e,\delta}h\rangle \le -a_3\|h\|_*. \tag{4.2}$$

We have divided the proof into four steps. The first one deals with the hypodissipativity of $\mathcal{B}_{\alpha,\delta}$ in $L^1_x L^1_v(\langle v\rangle^{q+1} m)$, while the second and third deal with the x and v-derivatives respectively. In the last step we construct the $\|\cdot\|_*$ norm and prove that it satisfies (4.2).

Step 1: The main idea of the proof is to compare $\mathcal{B}_{\alpha,\delta}$ with $\mathcal{B}_{1,\delta}$ defined in Section 3.3. In order to do this notice that

$$\mathcal{B}_{\alpha,\delta}-\mathcal{B}_{1,\delta} = (\mathcal{T}_{\alpha,R}-\mathcal{T}_{1,R}) - (\nu_\alpha-\nu)\,h.$$

Moreover, it is easy to see that the definition of $\mathcal{T}_{1,R}$ given in (3.19) coincides with the definition of $\mathcal{T}_{\alpha,R}$ with $\alpha=1$. Thus, we can write

$$\begin{aligned}\mathcal{T}_{\alpha,R}(h)-\mathcal{T}_{1,R}(h) &= \Big(\mathcal{Q}^+_{\alpha,R}(h,F_\alpha)-\mathcal{Q}^+_{1,R}(h,F_\alpha)\Big)+\Big(\mathcal{Q}^+_{\alpha,R}(F_\alpha,h)-\mathcal{Q}^+_{1,R}(F_\alpha,h)\Big)\\ &\quad -\Big(\mathcal{Q}^-_{\alpha,R}(F_\alpha,h)-\mathcal{Q}^-_{1,R}(\mathcal{M},h)\Big)+\mathcal{Q}^+_{1,R}(h,F_\alpha-\mathcal{M})+\mathcal{Q}^+_{1,R}(F_\alpha-\mathcal{M},h).\end{aligned}$$

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Therefore, if we take $\psi(v) = \text{sign}\, h(v)\, \langle v\rangle^q\, m(v)$ and denote by v'_α the post collision velocity for the restitution coefficient α, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\mathcal{Q}^+_{\alpha,R}(h,F_\alpha) - \mathcal{Q}^+_{1,R}(h,F_\alpha)\right)\psi(v)dv \\
&\quad= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} (1-\Theta_\delta)h(v)F_\alpha(v_*)|v-v_*|\left(\psi(v'_\alpha)-\psi(v'_1)\right)d\sigma dv_* dv, \\
&\quad\leq \int_{\mathbb{R}^3} \left|\mathcal{Q}^+_\alpha(h,F_\alpha) - \mathcal{Q}^+_1(h,F_\alpha)\right| \langle v\rangle^q\, m(v)dv, \\
&\quad= \|\mathcal{Q}^+_\alpha(h,F_\alpha) - \mathcal{Q}^+_1(h,F_\alpha)\|_{L^1_v(\langle v\rangle^q m)}, \\
&\quad\leq p(\alpha-1)\|F_\alpha\|_{W^{k,1}_v(\langle v\rangle^{q+1}m)}\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}, \\
&\quad\leq \eta_1(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}.
\end{aligned} \tag{4.3}$$

Here we used Proposition A.2.3, where $p(r)$ is a polynomial going to 0 as r goes to zero. Hence, $\eta_1(\alpha)\to 0$ as $\alpha\to 1$.

In the same manner we can see that there exist functions $\eta_2(\alpha)$ going to 0 as α goes to 1, such that

$$\int_{\mathbb{R}^3} \left(\mathcal{Q}^+_{\alpha,R}(F_\alpha,h) - \mathcal{Q}^+_{1,R}(F_\alpha,h)\right)\psi(v)dv \leq \eta_2(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}. \tag{4.4}$$

Furthermore, using Proposition A.2.2 we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left(\mathcal{Q}^-_{\alpha,R}(F_\alpha,h) - \mathcal{Q}^-_{1,R}(\mathcal{M},h)\right)\psi(v)dv \\
&\quad= \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} (1-\Theta_\delta)h(v_*)\left(F_\alpha(v)-\mathcal{M}\right)|v-v_*|\psi(v)d\sigma dv_* dv, \\
&\quad\leq \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} |h(v_*)|\,|F_\alpha(v)-\mathcal{M}\|\,v-v_*|\psi(v)d\sigma dv_* dv, \\
&\quad\leq \int_{\mathbb{R}^3} \mathcal{Q}^-_1(|F_\alpha-\mathcal{M}|,h)\,\langle v\rangle^q\, m(v)dv, \\
&\quad= \|\mathcal{Q}^-_1(|F_\alpha-\mathcal{M}|,h)\|_{L^1(\langle v\rangle^q m)}, \\
&\quad\leq C_{k,m}\|F_\alpha-\mathcal{M}\|_{L^1_v(\langle v\rangle^{q+1}m)}\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}, \\
&\quad= \eta_3(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}.
\end{aligned} \tag{4.5}$$

By Lemma 2.5.2 we know that $\eta_3(\alpha)$ converges to 0 when $\alpha\to 1$.

Analogously, applying Lemma 2.5.2 we obtain a function $\eta_4(\alpha)$ converging to 0 when $\alpha\to 1$ such that

$$\int_{\mathbb{R}^3} \left(\mathcal{Q}^+_{1,R}(h,F_\alpha-\mathcal{M}) + \mathcal{Q}^+_{1,R}(F_\alpha-\mathcal{M},h)\right)\psi(v)dv \leq \eta_4(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}. \tag{4.6}$$

Hence, taking $\eta_5 := \eta_1+\cdots+\eta_4$, gathering (4.3), (4.4), (4.5) and (4.6) we have that

$$\int_{\mathbb{R}^3} \left(\mathcal{T}_{\alpha,R}(h) - \mathcal{T}_{1,R}(h)\right)\psi(v)dv \leq \eta_5(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}. \tag{4.7}$$

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We have proved that there exist $\eta_5(\alpha)$ converging to 0 when $\alpha \to 1$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{B}_{\alpha,\delta}(h)\text{sign}(h) \langle v\rangle^q m(v)dv \\
&\quad \le \int_{\mathbb{R}^3} \mathcal{B}_{1,\delta}(h)\text{sign}(h) \langle v\rangle^q m(v)dv + \eta_5(\alpha)\|h\|_{W_v^{k,1}(\langle v\rangle^{q+1}m)} \\
&\quad + \int_{\mathbb{R}^3} |\nu_\alpha(v) - \nu(v)| \, \|h\| \langle v\rangle^{q+1} m(v)dv.
\end{aligned}$$

Finally, since $|v - v_*| \le \langle v\rangle \langle v_*\rangle$ we have

$$|\nu_\alpha(v) - \nu(v)| \le \int_{\mathbb{R}^3} |v - v_*\|F_\alpha(v_*) - \mathcal{M}(v_*)|dv_* \le \langle v\rangle \|F_\alpha - \mathcal{M}\|_{L^1(\langle v\rangle^q m)}.$$

Then, we deduce from Lemma 2.5.2 that

$$\int_{\mathbb{R}^3} |\nu_\alpha(v) - \nu(v)\| \, h(v)| \langle v\rangle^q m(v)dv \le \eta_6(\alpha)\|h(v)\|_{L^1(\langle v\rangle^{q+1}m)},$$

with $\lim_{\alpha\to 1} \eta_6(\alpha) = 0$. To summarize, there exists a function $\eta = \eta_5 + \eta_6$ converging to 0 when $\alpha \to 1$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathcal{B}_{\alpha,\delta}(h)\text{sign}(h) \langle v\rangle^q m(v)dv \\
&\quad \le \int_{\mathbb{R}^3} \mathcal{B}_{1,\delta}(h)\text{sign}(h) \langle v\rangle^q m(v)dv + \eta(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1}m)}.
\end{aligned}$$

Thus, fixing $\delta > 0$ as in Lemma 3.3.2, this inequality becomes

$$\int_{\mathbb{R}^3 \times \mathbb{T}^3} \mathcal{B}_{\alpha,\delta}(h)\text{sign}(h) \langle v\rangle^q m(v)dxdv \le (\eta(\alpha) - a_1)\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1}m)}.$$

Taking α_1 big enough we can suppose that for any $\alpha \in [\alpha_1, 1]$ we have $\eta(\alpha) < a_1$ and therefore

$$a_3' := a_1 - \eta(\alpha) > 0. \tag{4.8}$$

With this we can conclude

$$\int_{\mathbb{R}^3 \times \mathbb{T}^3} \mathcal{B}_{\alpha,\delta}(h)\text{sign}(h) \langle v\rangle^q m(v)dxdv \le -a_3'\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1}m)},$$

where we deduce that for any $\alpha \in [\alpha_1, 1]$, $\mathcal{B}_{\alpha,\delta} + a_3'$ is dissipative in $L^1_x L^1_v(\langle v\rangle^{q+1} m)$.

Step 2: Since the x-derivatives commute with $\mathcal{B}_{\alpha,\delta}$, using the proof of

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Step 1 we have

$$\int_{\mathbb{R}^3\times\mathbb{T}^3} \partial_x\left(\mathcal{B}_{\alpha,\delta}(h)\right)\operatorname{sign}(\partial_x h)\langle v\rangle^q m(v)dxdv \le -a_3'\|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}.$$

Step 3: In order to deal with the v-derivatives, we proceed analogously to the proof of Lemma 3.2.1 to see that

$$\partial_v\mathcal{B}_{\alpha,\delta}(h) = \partial_v\left(\mathcal{T}_{\alpha,R} - \nu_\alpha h + \mathcal{B}_{e,\delta} - v\cdot\nabla_x h\right) = \mathcal{B}_{\alpha,\delta}(\partial_v h) - \partial_x h + \mathcal{R}'(h),$$

where, recalling that $\mathcal{R}(h)$ is given by (3.16),

$$\begin{aligned}\mathcal{R}'(h) &= \mathcal{Q}_\alpha^+(h,\partial_v F_\alpha) + \mathcal{Q}_\alpha^+(\partial_v F_\alpha, h) - \mathcal{Q}_\alpha^-(\partial_v F_\alpha, h)\\ &\quad - \left(\partial_v\mathcal{T}_{\alpha,S}\right)(h) + \mathcal{T}_{\alpha,S}(\partial_v h) + \mathcal{R}(h).\end{aligned} \tag{4.9}$$

Proceeding as in the proof of Lemma 4.1.1 and performing one integration by parts, we have

$$\|\left(\partial_v\mathcal{T}_{\alpha,S}\right)(h)\|_{L^1_xL^1_v(\langle v\rangle^q m)} + \|\mathcal{T}_{\alpha,S}(\partial_v h)\|_{L^1_xL^1_v(\langle v\rangle^q m)} \le C_\delta\|h\|_{L^1_xL^1_v(\langle v\rangle^q m)}.$$

Using this, the estimates of Proposition A.2.2 on the inelastic operators $\mathcal{Q}_\alpha^\pm$ and the bound of $\mathcal{R}(h)$ given in (3.17), we obtain for some constant $C_{\alpha,\delta}>0$

$$\|\mathcal{R}'(h)\|_{L^1_xL^1_v(\langle v\rangle^q m)} \le C_{\alpha,\delta}\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}. \tag{4.10}$$

Thus, using the bounds found in (4.10) and the proof presented in **Step 1** we have

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{T}^3} \partial_v\left(\mathcal{B}_{\alpha,\delta}(h)\right)\operatorname{sign}(\partial_v h)\langle v\rangle^q m(v)dxdv\\ &\qquad\le -a_3'\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)} + C_{\alpha,\delta}\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)} + \|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}.\end{aligned}$$

where a_3' is defined in (4.8).

Step 4: Now, for some $\varepsilon>0$ to be fixed later, we define the norm

$$\|h\|_* = \|h\|_{L^1_xL^1_v(\langle v\rangle^q m)} + \|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^q m)} + \varepsilon\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^q m)}.$$

Notice that this norm is equivalent to the classical $W^{1,1}_{x,v}(\langle v\rangle^q m)$-norm. We

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deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{T}^3} \mathcal{B}_{\alpha,\delta}(h)\mathrm{sign}(h)\,\langle v\rangle^q\, m(v)dxdv \\
&\qquad + \int_{\mathbb{R}^3\times\mathbb{T}^3} \partial_x\left(\mathcal{B}_{\alpha,\delta}(h)\right)\mathrm{sign}(\partial_x h)\,\langle v\rangle^q\, m(v)dxdv \\
&\qquad + \varepsilon\int_{\mathbb{R}^3\times\mathbb{T}^3} \partial_v\left(\mathcal{B}_{\alpha,\delta}(h)\right)\mathrm{sign}(\partial_v h)\,\langle v\rangle^q\, m(v)dxdv \\
&\qquad \leq -a_3'\left(\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)} + \|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)} + \varepsilon\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}\right) \\
&\qquad + \varepsilon\left(C_{\alpha,\delta}\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)} + \|\nabla_x h\|\right), \\
&\qquad \leq (-a_3' + o(\varepsilon))\left(\|h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)} + \|\nabla_x h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}\right. \\
&\qquad \left. + \varepsilon\|\nabla_v h\|_{L^1_xL^1_v(\langle v\rangle^{q+1}m)}\right),
\end{aligned}$$

where $o(\varepsilon) \to 0$ as ε goes to 0. We choose ε close enough to 0 so that $a_3 = a_3' - o(\varepsilon) > 0$. Hence, we obtain that $\mathcal{B}_\delta + a_3$ is dissipative in $W^{1,1}_{x,v}(\langle v\rangle^q\, m)$ for the norm $\|\cdot\|_*$ and thus hypodissipative in $W^{1,1}_{x,v}(\langle v\rangle^q\, m)$. □

The following result comprises what we proved above:

Lemma 4.2.2. *There exist $\alpha_1 \in (\alpha_0, 1]$, $\delta \geq 0$ and $a_3 > 0$ such that for any $\alpha \in [\alpha_1, 1]$ and any $e \in (0,1]$, the operator $\mathcal{B}_{\alpha,\delta} + a_3$ is hypodissipative in E_j, $j = 0, 1$ and $\mathcal{E}$.*

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4.3 Regularization properties of T_n and estimates on $\mathscr{L}_\alpha - \mathscr{L}_1$

Let us recall the notation

$$T_n(t) := (\mathcal{A}_{\alpha,\delta} S_{\mathcal{B}_{\alpha,\delta}})^{(*n)}(t),$$

for $n \geq 1$, where $S_{\mathcal{B}_{\alpha,\delta}}$ is the semigroup generated by the operator $\mathcal{B}_{\alpha,\delta}$ and $*$ denotes the convolution. The proof of the fact that $\mathcal{B}_{\alpha,\delta}$ generates a C_0-semigroup can be found in B.

Notice that the proof of Lemma 3.4.2 remains valid in this case. Therefore, combining Lemma 4.2.2 and Lemma 3.4.2 we get the assumptions of Lemma C.0.2, so applying it we get the following result:

Lemma 4.3.1. *Let us consider α_1 and a_3 as in Lemma 4.2.2 and let α be in $[\alpha_1, 1)$. For any $a' \in (0, a_3)$ and for any $e \in (0,1]$, there exist some constructive constants $n \in \mathbb{N}$ and $C_{a'} \geq 1$ such that for all $t \geq 0$*

$$\|T_n(t)\|_{\mathscr{B}(E_0,E_1)} \leq C_{a'} e^{-a't}.$$

Moreover, using estimates from the proof of Lemma 4.2.1, we can prove the following result:

Lemma 4.3.2. *There exists a function $\eta(\alpha)$ such that it tends to 0 as α tends to 1 and the difference $\mathscr{L}_\alpha - \mathscr{L}_1$ satisfies for any $e \in (0,1]$*

$$\|\mathscr{L}_\alpha - \mathscr{L}_1\|_{\mathscr{B}(E_1,E_0)} \leq \eta(\alpha).$$

Proof. First of all notice that

$$\begin{aligned}
\mathscr{L}_\alpha - \mathscr{L}_1 &= \mathcal{T}_\alpha - \mathcal{T}_1, \\
&= \Big(\mathcal{Q}_\alpha^+(h,F_\alpha) - \mathcal{Q}_1^+(h,F_\alpha)\Big) + \Big(\mathcal{Q}_\alpha^+(F_\alpha,h) - \mathcal{Q}_1^+(F_\alpha,h)\Big) \\
&\quad + \mathcal{Q}_1(h, F_\alpha - \mathcal{M}) + \mathcal{Q}_1(F_\alpha - \mathcal{M}, h), \\
&= \Big(\mathcal{T}_\alpha^+(h) - \mathcal{T}_1^+(h)\Big) + \mathcal{Q}_1(h, F_\alpha - \mathcal{M}) + \mathcal{Q}_1(F_\alpha - \mathcal{M}, h).
\end{aligned}$$

Therefore, by Proposition A.2.3 we have $\eta_1(\alpha)$

$$\|\mathcal{T}_\alpha^+(h) - \mathcal{T}_1^+(h)\|_{L^1_v(\langle v\rangle^q m)} \leq \eta_1(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1} m)}, \tag{4.11}$$

with $\eta_1(\alpha) \to 0$ when $\alpha \to 1$. Moreover, by Proposition A.2.2 and Lemma 2.5.2 there exist $\eta_2(\alpha)$ such that

$$\|\mathcal{Q}_1(h, F_\alpha - \mathcal{M})\|_{L^1_v(\langle v\rangle^q m)} + \|\mathcal{Q}_1(F_\alpha - \mathcal{M}, h)\|_{L^1_v(\langle v\rangle^q m)} \leq \eta_2(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1} m)},$$

with $\eta_2(\alpha) \to 0$ when $\alpha \to 1$. Thus, taking $\eta' = \eta_1 + \eta_2$ we have

$$\|\mathcal{T}_\alpha - \mathcal{T}_1\|_{L^1_v(\langle v\rangle^q m)} \leq \eta'(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1} m)}.$$

Furthermore, using (3.15) we have that

$$\begin{aligned}
\partial_v\left(\mathscr{L}_\alpha^+(h) - \mathscr{L}_1^+(h)\right) &= \partial_v\left(\mathcal{T}_\alpha^+(h) - \mathcal{T}_1^+(h)\right), \\
&= \mathcal{T}_\alpha^+(\partial_v h) - \mathcal{T}_1^+(\partial_v h) + \Big(\mathcal{Q}_\alpha^+(h,\partial_v F_\alpha) - \mathcal{Q}_1^+(h,\partial_v F_\alpha)\Big) \\
&\quad + \Big(\mathcal{Q}_\alpha^+(\partial_v F_\alpha, h) - \mathcal{Q}_1^+(\partial_v F_\alpha, h)\Big) + \mathcal{Q}_1(\partial_v h, F_\alpha - \mathcal{M}) \\
&\quad + \mathcal{Q}_1(F_\alpha - \mathcal{M}, \partial_v h).
\end{aligned}$$

Thus, by (4.11) we have that

$$\|\mathcal{T}_\alpha^+(\partial_v h) - \mathcal{T}_1^+(\partial_v h)\|_{L^1_v(\langle v\rangle^q m)} \leq \eta_1(\alpha)\|\partial_v h\|_{L^1_v(\langle v\rangle^{q+1} m)}.$$

Proceeding as before by Proposition A.2.3 we have that there exist η_3

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converging to 0 as α goes to 1 such that

$$\|\mathcal{Q}^+_\alpha(h,\partial_v F_\alpha) - \mathcal{Q}^+_1(h,\partial_v F_\alpha)\|_{L^1_v(\langle v\rangle^q m)} + \|\mathcal{Q}^+_\alpha(\partial_v F_\alpha, h) - \mathcal{Q}^+_1(\partial_v F_\alpha, h)\|_{L^1_v(\langle v\rangle^q m)} \\ \leq \eta_3(\alpha)\|h\|_{L^1_v(\langle v\rangle^{q+1} m)}.$$

And using again Proposition A.2.2 and Lemma 2.5.2 there exist $\eta_4(\alpha)$ such that

$$\|\mathcal{Q}_1(h,\partial_v(F_\alpha - \mathcal{M})) + \mathcal{Q}_1(\partial_v h, F_\alpha - \mathcal{M})\|_{L^1_v(\langle v\rangle^q m)} \\ \leq \eta_4(\alpha)\left(\|h\|_{L^1_v(\langle v\rangle^{q+1} m)} + \|\partial_v h\|_{L^1_v(\langle v\rangle^{q+1} m)}\right).$$

For higher-order derivatives we proceed in the same way and we can conclude that there exist η such that it tends to 0 as α tends to 1, and satisfies

$$\|\mathscr{L}_\alpha - \mathscr{L}_1\|_{E_0} \leq \eta(\alpha)\|h\|_{E_1}.$$

In a similar way we obtain

$$\|\mathscr{L}_\alpha - \mathscr{L}_1\|_{E_{-1}} \leq \eta(\alpha)\|h\|_{E_0}.$$

□

For now on we fix δ as in Lemma 4.2.2 and we write $\mathcal{A} = \mathcal{A}_{\alpha,\delta}$ and $\mathcal{B} = \mathcal{B}_{\alpha,\delta}$.

4.4 Semigroup spectral analysis of the linearized operator

This section is dedicated to present some results regarding on the geometry of the spectrum of the linearized inelastic collision operator for a parameter close to 1.

Proposition 4.4.1. *There exists $\alpha_2 \in [0,1)$ such that for any $\alpha \in [\alpha_2, 1)$ and any $e \in (0,1]$, $\mathscr{L}_\alpha$ satisfies the following properties in $E = W^{s,1}_x W^{2,1}_v(\langle v\rangle\, m)$, $s \in \mathbb{N}^*$:*

(a) $\Sigma(\mathscr{L}_\alpha) \cap \Delta_{-a_2} = \{0\}$ where a_2 is given by Theorem 3.5.2. Moreover, 0 simple eigenvalue of $\mathscr{L}_\alpha$ and $N(\mathscr{L}_\alpha) = Span\{F_\alpha\}$.

(b) For any $a \in (0, \min(a_2, a_3))$, where a_3 is given by Lemma 4.2.2, the semigroup generated by $\mathscr{L}_\alpha$ has the following decay property

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$$\|S_{\mathscr{L}_\alpha}(t)(I - \Pi_{\mathscr{L}_\alpha,0})\|_{\mathscr{B}(E)} \leq Ce^{-at}, \tag{4.12}$$

for all $t \geq 0$ and for some $C > 0$.

The proof of the proposition stated above is a straightforward adaptation of one presented in [Tristani, 2016, Section 2.7]. We shall only mention the main steps of the proof and we emphasize the few points which differs here (due to the replacement of the diffusive term by a linear scattering operator).

Proof. ***Step 1: Localization of the spectrum of $\mathscr{L}_\alpha$ and dimension of eigenspaces.***

Notice that, by the result [Tristani, 2016, Lemma 2.16] (which we get due to Lemmas 4.2.2, 4.3.2, 4.3.1 and Theorem 3.5.2) we know that there exist $\alpha' > \alpha_1$ such that $\mathscr{L}_\alpha - z$ is invertible for any $z \in \Omega_\alpha = \Delta_{-a_2} \backslash \{0\}$ and any $\alpha \geq \alpha'$. Moreover, we have that

$$\Sigma(\mathscr{L}_\alpha) \cap \Delta_{-a_2} \subset B(0, \eta'(\alpha)),$$

where η' goes to 0 as α goes to 1. Furthermore, by [Tristani, 2016, Lemma 2.17] which remains true in our context as a result of Lemmas 4.2.2 and 4.3.2, there exist a function $\eta''(\alpha)$ such that

$$\|\Pi_{\mathscr{L}_\alpha,-a_1} - \Pi_{\mathscr{L}_1,-a_1}\|_{\mathfrak{B}(E_0)} \leq \eta''(\alpha), \tag{4.13}$$

with $\eta''(\alpha) \to 0$ as $\alpha \to 0$. Hence, by Theorem 3.5.2, it implies that for α close to 1, we have

$$\dim R(\Pi_{\mathscr{L}_\alpha,-a_1}) = \dim R(\Pi_{\mathscr{L}_1,-a_1}) = 1.$$

Therefore, there exist $\alpha_2 > \alpha'$ such that $\eta''(\alpha) < 1$ for every $\alpha \in (\alpha_2, 1]$. Also there exist $\xi_\alpha \in \mathbb{C}$ such that

$$\Sigma(\mathscr{L}_\alpha) \cap \Delta_{-a_2} = \{\xi_\alpha\}.$$

Let us prove that $\xi_\alpha = 0$. We argue by contradiction and assume that for α close to 1 we have $\xi_\alpha \neq 0$. Let φ_α be some normalized eigenfunction of $\mathscr{L}_\alpha$ associated to ξ_α, i.e. satisfies $\mathscr{L}_\alpha \varphi_\alpha = \xi_\alpha \varphi_\alpha$. Integrating over $\mathbb{R}^3$ we get that

$$\int_{\mathbb{R}^3} \varphi_\alpha(v) dv = 0.$$

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For any $h \in E_0$ there exist $\rho = \rho(\alpha, h)$ and $\rho' = \rho'(h)$ such that $\Pi_{\mathscr{L}_\alpha,\xi_\alpha} h = \rho\varphi_\alpha$ while $\Pi_{\mathscr{L}_1,0} h = \rho'\mathcal{M}$. Hence, we have

$$\int_{\mathbb{R}^3} \Pi_{\mathscr{L}_\alpha,\xi_\alpha} h dv = 0 \quad \text{and} \quad \int_{\mathbb{R}^3} \Pi_{\mathscr{L}_1,0} h dv = \rho',$$

which contradicts (4.13). Therefore, $\xi_\alpha = 0$. Furthermore, 0 is a simple eigenvalue of $\mathscr{L}_\alpha$ since F_α is the unique steady state of $\mathscr{L}_\alpha$ satisfying $\int_{\mathbb{R}^3} F_\alpha dv = 1$.

Step 2: Semigroup decay.

In order to prove the estimate on the semigroup decay (4.12) we apply the Spectral Mapping Theorem C.0.3 with $a = \max\{-a_2, -a_3\} < 0$.

First of all notice that $E_1 \subset D(\mathscr{L}_\alpha^2) \subset E_0$. Moreover, by the results presented in Lemmas 4.2.2, 4.1.1 and 4.3.1 the assumptions (i), (ii), and (iii) of Theorem C.0.3 are satisfied.

Furthermore, the condition (A) in Theorem C.0.3 is also satisfied by **Step 1**. Thus we have the decay result (4.12) for any $a' \in (0, \min\{a_2, a_3\})$. This concludes the proof of Proposition 4.4.1.

□

Combining the results of Lemmas 4.2.2, 4.1.1 and Proposition 4.4.1 we fulfilled the assumptions of Theorem C.0.1. Therefore, we have a localized spectrum and exponential decay of the semigroup of $\mathscr{L}_\alpha$ in a larger space:

Theorem 4.4.2. *There exist $\alpha_2 \in (0, 1]$ such that for any $\alpha \in [\alpha_2, 1)$ and any $e \in (0, 1]$, $\mathscr{L}_\alpha$ satisfies the following properties in $\mathcal{E} = W_x^{s,1} L_v^1(m)$, $s \geq 2$:*

1. *The spectrum $\Sigma(\mathscr{L}_\alpha)$ satisfies the separation property: $\Sigma(\mathscr{L}_\alpha) \cap \Delta_{-a_2} = \{0\}$ where a_2 is given by Theorem 3.5.2 and $N(\mathscr{L}_\alpha) = \{F_\alpha\}$.*
2. *For any $a \in (0, \min\{a_2, a_3\})$, where a_3 is provided by Lemma 4.2.2, the semigroup generated by $\mathscr{L}_\alpha$ has the following decay property for every $t \geq 0$*
$$\|S_{\mathscr{L}_\alpha}(t)(\mathrm{id} - \Pi_{\mathscr{L}_\alpha,0})\|_{\mathscr{B}(\mathcal{E})} \leq Ce^{-at}, \tag{4.14}$$
for some $C > 0$.

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5
The nonlinear Boltzmann equation

The aim of this chapter is to prove our main result. Namely, the existence of solutions of (1.1) in the close-to-equilibrium regime:

Theorem 5.0.1. *Consider constant restitution coefficients* $\alpha \in [\alpha_0, 1]$*, where* α_0 *is given by Theorem 4.4.2 and any* $e \in (0, 1]$*. There exist constructive constant* $\varepsilon > 0$ *such that for any initial datum* $f_{in} \in \mathcal{E}$ *satisfying*

$$\|f_{in} - F_\alpha\|_{\mathcal{E}} \leq \varepsilon,$$

and f_{in} *has the same global mass as the equilibrium* F_α*, there exist a unique global solution* $f \in L^\infty_t(\mathcal{E}) \cap L^1_t(\mathcal{E}_1)$ *to* (1.1)*.*

Moreover, consider $a \in (0, \min\{a_2, a_3\})$*, where* a_2 *is given by Theorem 3.5.2 and* a_3 *by Lemma 4.2.2. This solution satisfies that for some constructive constant* $C \geq 1$ *and for every* $t \geq 0$

$$\|f - F_\alpha\|_{\mathcal{E}} \leq Ce^{-at}\|f_{in} - F_\alpha\|_{\mathcal{E}}.$$

We are going to build a solution by the use of an iterative scheme whose convergence is ensure due to a priori estimates coming from estimates of the semigroup of the linearized operator. First, we need to see that for a sufficiently close to the equilibrium initial datum, the nonlinear part of the equation is small with respect to the linear part which dictates the dynamic. Hence, we begin by given some bilinear estimates and a dissipative norm for the full linearized operator. Then, we can define the iterative scheme, see its convergence and, thus, prove our result.

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Due to the results presented in Chapter 4 the proof of this theorem follows as [Tristani, 2016, Theorem 3.2]. For the convenience of the reader we present the relevant material here, with some adaptations for our case, thus making our exposition self-contained.

5.1 A dissipative norm and bilinear estimates.

Let us fix the integer $s > 6$. Consider the Banach spaces

$$\mathcal{E}_1 := W^{s,1}_x L^1_v(\langle v\rangle\, m),$$
$$\mathcal{E} := W^{s,1}_x L^1_v(m).$$

Consider the following norm in $\mathcal{E}$

$$|||h|||_{\mathcal{E}} := \eta\|h\|_{\mathcal{E}} + \int_0^{+\infty} \|S_{\mathscr{L}_\alpha}(\tau)(I - \Pi_{\mathscr{L}_\alpha,0})h\|_{\mathcal{E}} d\tau, \tag{5.1}$$

for $\eta > 0$. This norm is well-defined thanks to estimate (4.14) for α close to 1. Furthermore, we define $|||\cdot|||_{\mathcal{E}_1}$ as in (5.1) for the space $\mathcal{E}_1$.

For this Banach norm the semigroup is not only dissipative, it also has a stronger dissipativity property: the damping term in the energy estimate controls the norm of the graph of the collision operator. More precisely:

Proposition 5.1.1. *Consider $\alpha \in [\alpha_2, 1)$. There exist $\eta > 0$ and $K > 0$ such that for any initial datum $h_{in} \in \mathcal{E}$ satisfying $\Pi_{\mathscr{L}_\alpha,0}h_{in} = 0$, the solution $h_t := S_{\mathscr{L}_\alpha}(t)h_{in}$ to the initial value problem* (2.8) *satisfies for every $t \geq 0$*

$$\frac{d}{dt}|||h_t|||_{\mathcal{E}} \leq -K|||h_t|||_{\mathcal{E}_1}.$$

Proof. The proof of this result follows as in [Tristani, 2016, Proposition 2.23] and [Gualdani et al., 2017, Proposition 5.15]. Let us present their arguments here.

First of all, notice that from the decay property of $\mathscr{L}_\alpha$ provided in (4.14) we deduce that the norms $\|\cdot\|_{\mathcal{E}}$ and $|||\cdot|||_{\mathcal{E}}$ are equivalent for any $\eta > 0$. Indeed,

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from (4.14) we have

$$\begin{aligned}
|||h|||_{\mathcal{E}} &= \eta\|h\|_{\mathcal{E}} + \int_0^{+\infty} \|S_{\mathscr{L}_\alpha}(\tau)(I - \Pi_{\mathscr{L}_\alpha,0})h\|_{\mathcal{E}} d\tau, \\
&\leq \eta\|h\|_{\mathcal{E}} + \int_0^{+\infty} Ce^{-a\tau}\|h\|_{\mathcal{E}} d\tau, \\
&\leq C(\eta)\|h\|_{\mathcal{E}}.
\end{aligned}$$

Let us now compute the time derivative of the norm $|||\cdot|||_{\mathcal{E}}$ along h_t which solves the linear evolution problem 2.8. A fundamental observation in our case is that

$$\Pi_{\mathscr{L}_\alpha,0} h_t = \Pi_{\mathscr{L}_\alpha,0} S_{\mathscr{L}_\alpha} h_{in} = S_{\mathscr{L}_\alpha} \Pi_{\mathscr{L}_\alpha,0} h_{in} = 0.$$

Since the x-derivatives commute with the linearized operator, without loss of generality we can set $s = 0$. We thus first treat the case $L^1_x L^1_v(m)$. We have that

$$\begin{aligned}
\frac{d}{dt}|||h_t|||_{L^1_x L^1_v(m)} &= \eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} \mathscr{L}_\alpha(h_t) \ \mathrm{sign}(h_t) dx \right) m dv + \int_0^\infty \frac{\partial}{\partial_t} \|h_{t+\tau}\|_{L^1_x L^1_v(m)} d\tau, \\
&= I_1 + I_2.
\end{aligned}$$

Concerning the first term I_1, due to the dissipativity of $\mathcal{B}$ proved in Lemma 4.2.1 and the bounds on $\mathcal{A}$ in Lemma 4.1.1 we get

$$\begin{aligned}
I_1 &= \eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} (\mathcal{A}(h_t) + \mathcal{B}(h_t)) \ \mathrm{sign}(h_t) dx \right) m dv, \\
&= \eta(C\|h_t\|_{L^1_x L^1_v(m)} - K\|h_t\|_{L^1_x L^1_v(\langle v \rangle m)}),
\end{aligned}$$

for some constants $C, K > 0$.

The second term is computed exactly:

$$\begin{aligned}
I_2 &= \int_0^\infty \frac{\partial}{\partial_t} \|h_{t+\tau}\|_{L^1_x L^1_v(m)} d\tau, \\
&= \int_0^\infty \frac{\partial}{\partial_\tau} \|h_{t+\tau}\|_{L^1_x L^1_v(m)} d\tau, \\
&= -\|h_t\|_{L^1_x L^1_v(m)}.
\end{aligned}$$

By choosing η small enough, the combination of the last two equations yields to

$$\frac{d}{dt}|||h_t|||_{\mathcal{E}} \leq -K|||h_t|||_{\mathcal{E}_1}.$$

Finally the case of a higher-order v-derivative is treated by an argument close to the one in Lemma 4.2.1. For instance, the case $s = \sigma = 1$ is proved by

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introducing the norms

$$|||h|||_{\mathcal{E},\varepsilon} = |||h|||_{\mathcal{E}} + |||\nabla_x h|||_{\mathcal{E}} + \varepsilon |||\nabla_v h|||_{\mathcal{E}},$$
$$|||h|||_{\mathcal{E}_1,\varepsilon} = |||h|||_{\mathcal{E}_1} + |||\nabla_x h|||_{\mathcal{E}_1} + \varepsilon |||\nabla_v h|||_{\mathcal{E}_1},$$

for some $\varepsilon > 0$.

Therefore, arguing as before one has

$$\frac{d}{dt}\left(|||h_t|||_{L^1_{x,v}(m)} + |||\nabla_x h_t|||_{L^1_{x,v}(m)}\right), \\ \leq -K_1\left(|||h_t|||_{L^1_{x,v}(\langle v\rangle m)} + |||\nabla_x h_t|||_{L^1_{x,v}(\langle v\rangle m)}\right).$$

And for the v derivatives we have

$$\frac{d}{dt}|||\nabla_v h_t|||_{L^1_{x,v}(m)}, \\ \leq -K_2\left(|||\nabla_v h_t|||_{L^1_{x,v}(\langle v\rangle m)} + |||\nabla_x h_t|||_{L^1_{x,v}(m)} + |||\mathcal{R}(h_t)|||_{L^1_{x,v}(m)}\right),$$

where $\mathcal{R}(h)$ is given by (4.9). Proceeding as in the proof of Lemma 4.2.1 and using the equivalence of the norms $\|\cdot\|_{\mathcal{E}}$ and $|||\cdot|||_{\mathcal{E}}$ we have for some constant $C > 0$

$$|||\mathcal{R}(h_t)|||_{L^1_{x,v}(m)} \leq C|||h_t|||_{L^1_{x,v}(\langle v\rangle m)}.$$

Therefore, we deduce that for $\varepsilon > 0$ small enough

$$\frac{d}{dt}|||h_t|||_{\mathcal{E},\varepsilon} \leq -K_3|||h_t|||_{\mathcal{E}_1,\varepsilon}.$$

The higher-order v-derivatives are treated in a similar way. Thus, we conclude our proof. □

In order to guarantee the convergence of the iterative scheme defined in the proof of Theorem 5.0.1, we need to recall the bilinear estimates on the nonlinear term $\mathcal{Q}_\alpha(h,h)$ of the equation (2.7). This estimates have been established in [Tristani, 2016, Lemma 3.1]:

Lemma 5.1.2. *Denoting $X^q := W_v^{\sigma,1}W_x^{s,1}(\langle v\rangle^q m)$ with $s,\sigma \in \mathbb{N}$, $s > 6$ and $q \in \mathbb{N}$, and X^{q+1} in a similar way. The collision operator $\mathcal{Q}_\alpha$ satisfies*

$$\|\mathcal{Q}_\alpha(f,g)\|_{X^q} \leq C\left(\|g\|_{X^{q+1}}\|f\|_{X^q} + \|g\|_{X^q}\|f\|_{X^{q+1}}\right),$$

for some constant $C > 0$.

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Remark 5.1.3. *The assumption $s > 6$ is a technique condition that guarantees the continuous embedding of $W^{s/2,1}_x \subset L^\infty_x(\mathbb{T}^3)$ in the proof of the lemma.*

5.2 Proof of Theorem 5.0.1

Let us now begin with the proof of our main result. We begin the proof by giving a key priori estimate.

Lemma 5.2.1. *[Tristani, 2016, Lemma 3.5] Under the assumptions of Theorem 5.0.1 a solution f_t to the Boltzmann equation* (1.1) *formally writes $f_t = F_\alpha + h_t \in \mathcal{E}$ with $\Pi_{\mathscr{L}_\alpha,0} h_t = 0$. Then, h_t satisfies the estimate*

$$\frac{d}{dt}|||h_t|||_{\mathcal{E}} \leq \left(C|||h_t|||_{\mathcal{E}} - K\right)|||h_t|||_{\mathcal{E}_1},$$

for some constants $C, K > 0$.

Proof. We are going to consider only the case $L^1_x L^1_v(m)$ since the other cases follow in a similar way.

Notice that, as in the proof of Proposition 5.1.1 we have

$$\frac{d}{dt}|||h_t|||_{L^1_x L^1_v(m)} = I_1 + I_2,$$

with

$$\begin{aligned} I_1 :=& \eta \int_{\mathbb{R}^3}\left(\int_{\mathbb{T}^3} \mathscr{L}_\alpha(h_t)\ \mathrm{sign}(h_t) dx\right) m dv \\ &+ \int_0^\infty \int_{\mathbb{R}^3}\left(\int_{\mathbb{T}^3} S_{\mathscr{L}_\alpha}(\tau)\mathscr{L}_\alpha(h_t)\ \mathrm{sign}(S_{\mathscr{L}_\alpha}(\tau)h_t)dx\right) m dv d\tau, \end{aligned}$$

and,

$$\begin{aligned} I_2 :=& \eta \int_{\mathbb{R}^3}\left(\int_{\mathbb{T}^3} \mathcal{Q}_\alpha(h_t,h_t)\ \mathrm{sign}(h_t) dx\right) m dv \\ &+ \int_0^\infty \int_{\mathbb{R}^3}\left(\int_{\mathbb{T}^3} S_{\mathscr{L}_\alpha}(\tau)\mathcal{Q}_\alpha(h_t,h_t)\ \mathrm{sign}(S_{\mathscr{L}_\alpha}(\tau)h_t)dx\right) m dv d\tau. \end{aligned}$$

From Proposition 5.1.1 we know that for η small enough

$$I_1 \leq -K|||h_t|||_{L^1_x L^1_v(\langle v\rangle m)},$$

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for some $K > 0$.

For the second term, since $\Pi_{\mathscr{L}_\alpha,0}\mathcal{Q}_\alpha(h_t, h_t) = 0$, we have that

$$\begin{aligned} I_2 &\leq \eta \int_{\mathbb{R}^3} \|\mathcal{Q}_\alpha(h_t, h_t)\|_{L^1_x} m dv + \int_0^\infty \int_{\mathbb{R}^3} \|S_{\mathscr{L}_\alpha}(\tau)\mathcal{Q}_\alpha(h_t, h_t)\|_{L^1_x} m dv d\tau, \\ &\leq \eta\|\mathcal{Q}_\alpha(h_t, h_t)\|_{L^1_x L^1_v(m)} + \int_0^\infty \|S_{\mathscr{L}_\alpha}(\tau)\mathcal{Q}_\alpha(h_t, h_t)\|_{L^1_x L^1_v(m)}\, d\tau, \\ &= |||\mathcal{Q}_\alpha(h_t, h_t)|||_{L^1_x L^1_v(m)}. \end{aligned}$$

Therefore, we deduce that

$$\frac{d}{dt}|||h_t|||_{L^1_x L^1_v(m)} \leq -K|||h_t|||_{L^1_x L^1_v(\langle v\rangle m)} + |||\mathcal{Q}_\alpha(h_t, h_t)|||_{L^1_x L^1_v(m)}.$$

Using the bilinear estimate presented in Lemma 5.1.2 and the semigroup decay from (4.14) we have

$$\begin{aligned} |||\mathcal{Q}_\alpha(h_t, h_t)|||_{L^1_x L^1_v(m)} &\leq \eta C_1\|h_t\|_{L^1_x L^1_v(m)}\|h_t\|_{L^1_x L^1_v(\langle v\rangle m)} \\ &\quad + \int_0^\infty C_2 e^{-a\tau}\|\mathcal{Q}_\alpha(h_t, h_t)\|_{L^1_x L^1_v(m)} d\tau, \\ &\leq \left(C_1\eta + C_3\int_0^\infty e^{-a\tau}d\tau\right)\|h_t\|_{L^1_x L^1_v(m)}\|h_t\|_{L^1_x L^1_v(\langle v\rangle m)}, \\ &\leq C\|h_t\|_{L^1_x L^1_v(m)}\|h_t\|_{L^1_x L^1_v(\langle v\rangle m)}. \end{aligned}$$

Therefore, we conclude that

$$\frac{d}{dt}|||h_t|||_{L^1_x L^1_v(m)} \leq -K|||h_t|||_{L^1_x L^1_v(\langle v\rangle m)} + C\|h_t\|_{L^1_x L^1_v(m)}\|h_t\|_{L^1_x L^1_v(\langle v\rangle m)},$$

which concludes our proof. □

We now proceed to prove Theorem 5.0.1. We will construct solutions through the following iterative scheme

$$\partial_t h^{n+1} = \mathscr{L}_\alpha h^{n+1} + \mathcal{Q}_\alpha(h^n, h^n), \quad n \geq 1,$$

with the initialization $\partial_t h^0 = \mathscr{L}_\alpha h^0$, $h^0_{in} = h_{in}$. We also assume $|||h_{in}|||_{\mathcal{E}} \leq \varepsilon/2$.

Due to the study of the semigroup in Theorem 4.4.2, the functions h^n are well-defined in $\mathcal{E}$. We are now in a position to show the stability estimates. In order to do this we we divide our proof into three steps. The first two steps establish the stability and convergence of the iterative scheme. The third step consists of a bootstrap argument in order to recover the optimal decay rate of

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the linearized semigroup.

Step 1: Stability of the scheme.

Let us prove by induction the following control: if $\varepsilon \leq K/(2C)$ then, for every $n \geq 0$

$$\sup_{t\geq 0}\left(|||h_t^n|||_{\mathcal{E}} + K\int_0^t |||h_\tau^n|||_{\mathcal{E}_1} d\tau \right) \leq \varepsilon. \tag{5.2}$$

The case $n = 0$ follows from Lemma 5.2.1 and the assumption that $|||h_{in}|||_{\mathcal{E}} \leq \varepsilon/2$. More precisely, for every $t \geq 0$ we have

$$\left|\left|\left| h_t^0 \right|\right|\right|_{\mathcal{E}} + K\int_0^t \left|\left|\left| h_\tau^0 \right|\right|\right|_{\mathcal{E}_1} d\tau \leq |||h_{in}|||_{\mathcal{E}} \leq \varepsilon.$$

Thus we can conclude,

$$\sup_{t\geq 0}\left(\left|\left|\left| h_t^0 \right|\right|\right|_{\mathcal{E}} + K\int_0^t \left|\left|\left| h_\tau^0 \right|\right|\right|_{\mathcal{E}_1} d\tau \right) \leq \varepsilon.$$

We now proceed by induction. Let us assume (5.2) is satisfied for any $0 \leq n \leq N$ for some $N \in \mathbb{N}^*$. The task is now to prove it for $n = N+1$. A similar computation as in Lemma 5.2.1 yields

$$\frac{d}{dt}\left|\left|\left| h_t^{N+1} \right|\right|\right|_{\mathcal{E}} + K\left|\left|\left| h_t^{N+1} \right|\right|\right|_{\mathcal{E}_1} \leq C\left|\left|\left| \mathcal{Q}_\alpha(h_t^N, h_t^N) \right|\right|\right|_{\mathcal{E}},$$

for some constants $C, K > 0$. Thus, since $\varepsilon \leq K/(2C)$ and by the induction hypothesis, we can deduce

$$\begin{aligned}
\left|\left|\left| h_t^{N+1} \right|\right|\right|_{\mathcal{E}} + K\int_0^t \left|\left|\left| h_\tau^0 \right|\right|\right|_{\mathcal{E}_1} d\tau &\leq |||h_{in}|||_{\mathcal{E}} + \int_0^t \left|\left|\left| \mathcal{Q}_\alpha(h_\tau^N, h_\tau^N \right|\right|\right|_{\mathcal{E}} d\tau, \\
&\leq |||h_{in}|||_{\mathcal{E}} + C\left(\sup_{\tau\geq 0} \left|\left|\left| h_\tau^N \right|\right|\right|_{\mathcal{E}} \right) \int_0^\infty \left|\left|\left| h_\tau^N \right|\right|\right|_{\mathcal{E}_1} d\tau, \\
&\leq \frac{\varepsilon}{2} + \frac{C}{K}\varepsilon^2, \\
&\leq \frac{\varepsilon}{2}.
\end{aligned}$$

Hence, the induction is proven.

Step 2: Convergence of the scheme.

Let us now denote $d^n := h^{n+1} - h^n$ and $s^n := h^{n+1} + h^n$ for $n \geq 0$. They satisfied

$$\partial_t d^{n+1} = \mathscr{L}_\alpha d^{n+1} + \mathcal{Q}_\alpha(d^n, s^n) + \mathcal{Q}_\alpha(s^n, d^n),$$

for every $n \geq 0$ and, $\partial_t d^0 = \mathscr{L}_\alpha d^0 + \mathcal{Q}_\alpha(h^0, h^0)$.

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Let us denote

$$A^n(t) := \sup_{0\le r\le t}\left(|||d_r^n|||_{\mathcal{E}} + K\int_0^r |||d_\tau^n|||_{\mathcal{E}_1} d\tau \right).$$

By induction, we can prove that for every $t, n \geq 0$,

$$A^n(t) \leq (C_0\varepsilon)^{n+2},$$

for some constant $C_0 > 0$.

Hence for ε small enough, the series $\sum_{n\geq 0} A^n(t)$ is summable for any $t \geq 0$, and the sequence h^n has the Cauchy property in $L_t^\infty(\mathcal{E})$, which proves the convergence of the iterative scheme. Let us call the limit of h^n as n goes to infinity by h. This limit satisfies the equation in the strong sense in $\mathcal{E}$.

Step 3: Rate of decay.

From ***Step 1***, by letting n goes to infinity in the stability estimate, one can deduce that

$$\sup_{t\geq 0}\left(|||h_t|||_{\mathcal{E}} + K\int_0^t |||h_\tau|||_{\mathcal{E}_1} d\tau \right) \leq \varepsilon.$$

Now we can apply the a priori estimate from Lemma 5.2.1 to our solution h with $\varepsilon < K/(2C)$ we get that

$$\begin{aligned}\frac{d}{dt}|||h_t|||_{\mathcal{E}} &\leq \left(C|||h_t|||_{\mathcal{E}} - K\right)|||h_t|||_{\mathcal{E}_1},\\ &\leq (C\varepsilon - K)|||h_t|||_{\mathcal{E}_1},\\ &\leq -\frac{K}{2}|||h_t|||_{\mathcal{E}_1}.\end{aligned} \tag{5.3}$$

Hence, this implies that

$$|||h_t|||_{\mathcal{E}} \leq \exp\left(-\frac{K}{2}t\right)|||h_{in}|||_{\mathcal{E}}.$$

Since $|||h_t|||_{\mathcal{E}}$ converges to zero as t goes to infinity, integrating (5.3) above from t to ∞ we have

$$\frac{K}{2}\int_t^\infty |||h_\tau|||_{\mathcal{E}_1} d\tau \leq |||h_t|||_{\mathcal{E}} \leq \exp\left(-\frac{K}{2}t\right)|||h_{in}|||_{\mathcal{E}}.$$

Which implies that

$$\int_t^\infty \|h_\tau\|_{\mathcal{E}_1} d\tau \leq \frac{2}{K\eta}\|h_t\|_{\mathcal{E}} \leq \exp\left(-\frac{K}{2}t\right)\|h_{in}\|_{\mathcal{E}}. \tag{5.4}$$

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For the decay rate of the solution h_t we will perform a bootstrap argument in order to ensure that it enjoys to same optimal decay rate as the linearized semigroup in Theorem 4.4.2. That is, $O(e^{-at})$ with $a \in (0, \min\{a_2, a_3\})$, where a_3 is provided by Lemma 4.2.2 and a_2 is given by Theorem 3.5.2. Assuming the solution is known to decay

$$\|h_t\|_{\mathcal{E}} \leq C_1 e^{-a't}, \tag{5.5}$$

for some constant $C_1 > 0$. Let us prove, using Theorem 4.4.2 and Lemma 5.1.2, that indeed it decays

$$\|h_t\|_{\mathcal{E}} \leq C e^{-a''t},$$

with $a'' = \min\{a' + K/4, a\}$.

Assume (5.5) and write a Duhamel formulation:

$$h_t = S_{\mathscr{L}_\alpha}(t)h_{in} + \int_0^t S_{\mathscr{L}_\alpha}(t-\tau)\mathcal{Q}_\alpha(h_\tau, h_\tau)d\tau.$$

From Theorem 4.4.2 and Lemma 5.1.2, for any $a \in (0, \min\{a_2, a_3\})$ we have

$$\|h_t\|_{\mathcal{E}} \leq Ce^{-at}\|h_{in}\|_{\mathcal{E}} + C\int_0^t e^{-a(t-\tau)}\|h_\tau\|_{\mathcal{E}}\|h_\tau\|_{\mathcal{E}_1}d\tau.$$

Assume $a' < a$ and $a'' = \min\{a' + K/4, a\}$. Hence we estimate

$$\begin{aligned}
&\int_0^t e^{-a(t-\tau)}\|h_\tau\|_{\mathcal{E}}\|h_\tau\|_{\mathcal{E}_1}d\tau \\
&\qquad \leq \int_0^t e^{-a''(t-\tau)}\|h_\tau\|_{\mathcal{E}}\|h_\tau\|_{\mathcal{E}_1}d\tau, \\
&\qquad \leq Ce^{-a''t}\left(\int_0^t e^{(a''-a')\tau}\|h_\tau\|_{\mathcal{E}_1}d\tau\right)\|h_{in}\|_{\mathcal{E}}.
\end{aligned}$$

Integrating by parts and using (5.4) for some positive constant C' we obtain

$$\begin{aligned}
&\int_0^t e^{(a''-a')\tau}\|h_\tau\|_{\mathcal{E}_1}d\tau \\
&\leq \int_0^t |h_\tau\|_{\mathcal{E}_1}d\tau + (a''-a')\int_0^t e^{(a''-a')\tau}\left(\int_\tau^t \|h_{\tau'}\|_{\mathcal{E}_1}d\tau'\right)d\tau, \\
&\leq C\|h_{in}\|_{\mathcal{E}} + (a''-a')\left(\int_0^t (t-\tau)e^{(a''-a'-K/2)\tau}d\tau\right)\|h_{in}\|_{\mathcal{E}}, \\
&\leq C'\|h_{in}\|_{\mathcal{E}}.
\end{aligned}$$

Therefore, we deduce

$$\|h_t\|_{\mathcal{E}} \leq Ce^{-a''t}\|h_{in}\|_{\mathcal{E}}.$$

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This proves the claim and concludes the proof of the estimate

$$\|h_t\|_{\mathcal{E}} \leq Ce^{-at}\|h_{in}\|_{\mathcal{E}},$$

where $a > 0$ is the sharp rate of the linearized semigroup in Theorem 4.4.2.

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A
Inequalities

In this chapter we are going to present several inequalities and norm bounds that we need along this work.

A.1
Useful inequalities

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Lemma A.1.1. *Let* $\mathbb{C}_\beta = 2^{\beta/2} - 1$*, then if* $x \leq a/2$ *we have*

$$(a-x)^{\beta/2} \leq a^{\beta/2} - C_\beta x^{\beta/2}.$$

Proof. Since $x \leq a/2$ then $2 \leq y = a/x$ and

$$(a-x)^{\beta/2} = x^{\beta/2}(y-1)^{\beta/2}.$$

Recall that if $0 < p < 1$ then for every $A, B > 0$ we have that $|A^p - B^p| \leq |A-B|^p$. Thus

$$(A-B)^p \leq -|A^p - B^p| \leq -B^p + A^p.$$

Hence, taking $A = y \geq 2 = B$ we get

$$|y-1|^p - 1 \leq |(y-1)^p - 1^p| \leq |y-2|^p \leq y^p - 2^p.$$

Therefore, if $p = \beta/2$

$$(a-x)^{\beta/2} \leq x^{\beta/2}(y^{\beta/2} - 2^p + 1),$$

taking $C_\beta = 2^{\beta/2} - 1$ we conclude our result.

□

Lemma A.1.2. *Given any $\gamma > 0$ there exist a positive constant $C_\gamma > 0$ such that*

$$b|v_*|^\beta - \beta_0|v_*|^2 \leq C_\gamma - \gamma|v_*|^\beta.$$

Proof. Given $\gamma > 0$ for $x \geq 0$ consider the function

$$f(x) = (b+\gamma)x^\beta - \beta_0 x^2.$$

It is easy to see that this function attains its maximum when $x^{2-\beta} = \beta(b+\gamma)/(2\beta_0)$. Therefore, it is enough to take C_γ as this maximum. □

A.2 Interpolation Inequalities and norm bounds

Let's recall an interpolation inequality given by [Mischler and Mouhot, 2009a, Lemma B.1] which can be easily extended to other weights of type $\langle v\rangle^q m$.

Lemma A.2.1. *For any $k, q \in \mathbb{N}$, there exists $C > 0$ such that for any $h \in H^{k'} \cap L^1(m^{12})$ with $k' = 8k + 7(1 + 3/2)$*

$$\|h\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq C\|h\|^{1/8}_{H^{k'}_v}\|h\|^{1/8}_{L^1_v(m^{12})}\|h\|^{3/4}_{L^1_v(m)}.$$

Now we present an useful result given by Mischler Mouhot in [Mischler and Mouhot, 2009a, Proposition 3.1]. Although, they proved it for a different type of weights it can we extended to our case.

Proposition A.2.2. *For any $k, q \in \mathbb{N}$ there exist $C > 0$ such that for any smooth functions f, g (say $f, g \in \mathcal{S}(\mathbb{R}^N)$) and any $\alpha \in [0,1]$ there holds*

$$\|\mathcal{Q}^\pm_\alpha(f,g)\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq C_{k,m}\|f\|_{W^{k,1}_v(\langle v\rangle^{q+1} m)}\|g\|_{W^{k,1}_v(\langle v\rangle^{q+1} m)}.$$

Proof. Let us begin by considering $\mathcal{Q}^-_\alpha$. Recall that

$$\left\langle \mathcal{Q}^-_\alpha(f,g), \psi \right\rangle = \int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} g(v_*)f(v)\psi(v)|v-v_*|d\sigma dv_* dv = \langle fL(g), \psi\rangle,$$

where $L(g)(v) = 4\pi(|\cdot| * g)(v)$. Hence, since

$$|L(g)| \leq 4\pi\langle v\rangle \|g\|_{L^1_v(\langle v\rangle)},$$

we have that

$$\|\mathcal{Q}^-_\alpha(f,g)\|_{L^1_v(\langle v\rangle^q m)} \leq 4\pi\|g\|_{L^1_v(\langle v\rangle^{q+1} m)}\|f\|_{L^1_v(\langle v\rangle^{q+1} m)}.$$

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Moreover, using (3.15) we obtain a constant $C > 0$ such that

$$\|\partial_v \mathcal{Q}_\alpha^-(f,g)'\|_{L^1_v(\langle v\rangle^q m)} \le C\|g\|_{W^{1,1}_v(\langle v\rangle^{q+1} m)}\|f\|_{W^{1,1}_v(\langle v\rangle^{q+1} m)}.$$

We continue in this fashion obtaining our result.

We proceed to estimate the gain term. By duality we have

$$\begin{aligned}\|\mathcal{Q}_\alpha^+(f,g)\|_{L^1_v(\langle v\rangle^q m)} &= \sup_{\|\psi\|_{L^{p'}}=1} \int_{\mathbb{R}^3} \mathcal{Q}_\alpha^+(f,g)\psi(v)\langle v\rangle^q m(v)dv,\\ &= \sup_{\|\phi\|_{L^{p'}}=1} \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v_*)|v-v_*| \int_{\mathbb{S}^2} \psi(v')\langle v'\rangle^q m(v')d\sigma dv_* dv,\\ &= \sup_{\|\phi\|_{L^{p'}}=1} I(\psi).\end{aligned}$$

Hence, using (3.8) we can assert that for some positive constant C_1

$$I(\psi) \le C_1 \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v_*)\langle v\rangle^{q+1} m(v)\langle v_*\rangle^q m(v_*) \int_{\mathbb{S}^2} \psi(v')d\sigma dv_* dv.$$

Let us call $S(\psi) = \int_{\mathbb{S}^2}\psi(v')d\sigma$. In fact, we split $S(\psi)$ into two parts $S_+(\psi)$ and $S_-(\psi)$ where

$$S_\pm(\psi) = \int_{\pm u\cdot\sigma>0} \psi(v')d\sigma.$$

By [Gamba et al., 2004, Proposition 4.2] we have that the operators

$$\begin{aligned}S_+ &: L^r(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3_v, L^r(\mathbb{R}^3_{v_*})),\\ S_- &: L^r(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3_{v_*}, L^r(\mathbb{R}^3_v))\end{aligned}$$

are bounded for every $1 \le r \le \infty$. Therefore, we conclude

$$\begin{aligned}&\|\mathcal{Q}_\alpha^+(f,g)\|_{L^1_v(\langle v\rangle^q m)}\\ &\le \sup_{\|\psi\|_{L^{p'}}=1} C_1 \int_{\mathbb{R}^3\times\mathbb{R}^3} f(v)g(v_*)\langle v\rangle^{q+1} m(v)\langle v_*\rangle^q m(v_*)(S_+(\psi)+S_-(\psi))dv_* dv,\\ &\le C_2\|f\|_{L^1_v(\langle v\rangle^{q+1} m)}\|g\|_{L^1_v(\langle v\rangle^{q+1} m)}.\end{aligned}$$

For the derivatives we proceed analogously to the proof for the loss term. □

The following result may be proved in much the same way as [Mischler and Mouhot, 2009a, Proposition 3.2].

Proposition A.2.3. *For any $\alpha, \alpha' \in (0,1]$, and any $g \in L^1_v(\langle v\rangle^{q+1} m)$,*

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$f \in W^{1,1}_v(\langle v\rangle^{q+1} m)$ there holds

$$\|\mathcal{Q}^+_\alpha(f,g) - \mathcal{Q}^+_{\alpha'}(f,g)\|_{L^1_v(\langle v\rangle^q m)} \le p(\alpha-\alpha')\|f\|_{W^{1,1}_v(\langle v\rangle^{q+1} m)}\|g\|_{L^1_v(\langle v\rangle^{q+1} m)},$$

and

$$\|\mathcal{Q}^+_\alpha(g,f) - \mathcal{Q}^+_{\alpha'}(g,f)\|_{L^1_v(\langle v\rangle^q m)} \le p(\alpha-\alpha')\|f\|_{W^{1,1}_v(\langle v\rangle^{q+1} m)}\|g\|_{L^1_v(\langle v\rangle^{q+1} m)}.$$

Where $p(r)$ is an explicit polynomial converging to 0 if r goes to 0.

Proof. The proof of this proposition follows exactly as in [Mischler and Mouhot, 2009a, Proposition 3.2], due to (3.8). Let us just remark that in our case [Mischler and Mouhot, 2009a, Lemma 3.3] reads as follows. For any $\delta > 0$ and $\alpha \in (0,1)$ there holds that if $\sigma \in \mathbb{S}^2$ and $\sin^2\xi \ge \delta$, where $\cos\xi = |\sigma\cdot(w/|w|)|$ and $w = v + v_* \neq 0$, then

$$m(v') \le e^b m^k(v) m^k(v_*), \tag{A.1}$$

where $k = (1-\delta/160)^{\beta/2}$.

Moreover, from the proof of [Mischler and Mouhot, 2009a, Lemma 3.3] we have that

$$|v|^2 \le (1-\delta/160)(|v|^2 + |v_*|^2).$$

This, together with $\langle v\rangle^\beta \le 1 + |v|^\beta$ concludes the proof of (A.1). □

The next result is a reformulation of [Bisi et al., 2011, Proposition A.2] for our weight function.

Proposition A.2.4. *Set $m(v) = \exp(b\langle v\rangle^\beta)$ with $b > 0$ and $\beta \in (0,1)$ and let*

$$H(v_*) = \int k_e(v,v_*)\langle v\rangle^q m(v)dv \quad \forall w \in \mathbb{T}^3.$$

Then, there exist a constant $K = K(e,b,\beta) > 0$ such that

$$H(v_*) \le K(1 + |v_*|^{1-\beta})\langle v_*\rangle^q m(v_*),$$

for every $v_* \in \mathbb{T}^3$.

Proof. Consider $\hat{m}(v) = \exp(b|v|^\beta)$. It is easy to see that

$$\hat{m}(v) \le m(v) \le e^b \hat{m}(b),. \tag{A.2}$$

Furthermore, there exist a constant $C'_{q,b} > 0$ such that $\langle v\rangle^q \le C'_{q,b}(1+|v|^q)$, we have that

$$H(v_*) \le C_{q,b}\int k_e(v,v_*)(1+|v|^q)\hat{m}(v)dv,$$

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where $C_{q,b} = e^b C'_{q,b}$.

Let us recall that k_e is given by (3.1) with $c_0 = 1/(8\theta_0)$. Here we will assume $u_0 = 0$ to simplify the computations, the general case follows in a similar manner. Taking into account that $\frac{|v|^2-|v_*|^2}{|v-v_*|} - |v-v_*| = 2\frac{v-v_*}{|v-v_*|}\cdot v_*$, we can rewrite k_e as

$$k_e(v_*) = \frac{C_e}{|v-v_*|}\exp\left\{-c_0\left((2+\mu)|v-v_*| + 2\frac{v-v_*}{|v-v_*|}\cdot v_*\right)^2\right\}.$$

Moreover, performing the change of variables $u = v - v_*$ and using spherical coordinates with $\rho = |u|$ and $\rho|v_*|y = u\cdot v_*$, one gets

$$H(v_*) = C_0\int_A F(\rho,y)d\rho dy$$

with $A = [0,+\infty)\times[-1,1]$ and

$$F(\rho,y) = \left[1 + (\rho^2 + |v_*|^2 + 2\rho|v_*|y)^{q/2}\right]\rho\times$$
$$\exp\left\{-c_0\left((2+\mu)\rho + 2|v_*|y\right)^2 + b\left(\rho^2 + |v_*|^2 + 2\rho|v_*|y\right)^{q/2}\right\}$$

Let us split A into two regions $A = A_1\cup A_2$ where

$$A_1 := \{(\rho,y)\in A : 3|v_*|y \geq -2\rho\} \quad \text{and} \quad A_2 := A\backslash A_1.$$

We first compute the integral over A_1. Notice that since $y\leq 1$ and $\beta\in(0,1)$ we get that for every $(\rho,y)\in A$

$$\begin{aligned}\exp\left\{b(\rho^2 + |v_*|^2 + 2\rho|v_*|y)^{q/2}\right\} &\leq \exp\left(b(\rho+|v_*|)^\beta\right)\\ &\leq \exp\left(b\rho^\beta\right)\exp\left(b|v_*|^\beta\right).\end{aligned}$$

Moreover, since $(2+\mu)\rho + 2|v_*|y \geq (\mu + 2/3)\rho$ for any $(\rho,y)\in A_1$ we have

$$\begin{aligned}\int_{A_1} F(\rho,y)d\rho dy &\leq C_{1,q}|v_*|^q e^{b|v_*|^\beta}\int_0^{+\infty}\rho\exp\left(-c_0(2/3+\mu)^2\rho^2 + b\rho^\beta\right)dyd\rho\\ &+ C_{2,q}e^{b|v_*|^\beta}\int_0^{+\infty}\rho^{q+1}\exp\left(-c_0(2/3+\mu)^2\rho^2 + b\rho^\beta\right)dyd\rho,\\ &\leq C_1\langle v_*\rangle^q\,\hat{m}(v_*),\\ &\leq C_1\langle v_*\rangle^q\,m(v_*),\end{aligned} \tag{A.3}$$

for some constant $C_1 > 0$, since the integrals are convergent.

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On the other hand, for every $(\rho, y) \in A_2$ we have

$$\rho^2 + |v_*|^2 + 2\rho|v_*|y < |v_*|^2 - \frac{1}{3}\rho^2 \quad \text{and} \quad \rho < \frac{3}{2}|v_*|y.$$

In that case, using the change of variables $z = (2+\mu)\rho + 2|v_*|y$ we get

$$\begin{aligned}
&\int_{A_2} F(\rho, y) d\rho dy \\
&\leq C_{3,q} \langle v_* \rangle^q \int_0^{(3/2)|v_*|} \rho \exp\left(b(|v_*|^2 - \frac{1}{3}\rho^2)^{\beta/2}\right) d\rho \cdot \int_{-1}^{1} \exp\left(-c_0((2+\mu)\rho + 2|v_*|y)^2\right) dy, \\
&\leq C_{3,q} \cdot \frac{1}{2|v_*|} \cdot \langle v_* \rangle^q \int_0^{(3/2)|v_*|} \rho \exp\left(b(|v_*|^2 - \frac{1}{3}\rho^2)^{\beta/2}\right) d\rho \cdot \int_{-\infty}^{+\infty} \exp\left(-c_0 z^2\right) dz, \\
&\leq C_{4,q} \cdot \frac{1}{2|v_*|} \cdot \langle v_* \rangle^q \int_0^{(3/2)|v_*|} \rho \exp\left(b(|v_*|^2 - \frac{1}{3}\rho^2)^{\beta/2}\right) d\rho,
\end{aligned}$$

since the integral over z is finite. Finally, setting $w = |v_*|^2 - \rho^2/3$ we obtain,

$$\begin{aligned}
&\int_{A_2} F(\rho, y) d\rho dy \\
&\leq C_{4,q} \cdot \frac{3}{2|v_*|} \cdot \langle v_* \rangle^q \int_{|v_*|^2/4}^{|v_*|^2} \exp\left(b w^{\beta/2}\right) dw, \\
&\leq C_{4,q} \cdot \frac{3}{2|v_*|} \cdot \langle v_* \rangle^q \int_0^{|v_*|^2} \exp\left(b w^{\beta/2}\right) dw, \\
&\leq C_{4,q} \cdot \frac{3}{2|v_*|} \cdot \langle v_* \rangle^q \cdot \frac{2}{b\beta} |v_*|^{2-\beta} \exp(b|v_*|^\beta), \\
&\leq C_2 \langle v_* \rangle^q \hat{m}(v_*) |v_*|^{1-\beta}, \\
&\leq C_2 \langle v_* \rangle^q m(v_*) |v_*|^{1-\beta},
\end{aligned} \tag{A.4}$$

for some positive constant C_2. Thus, putting together (A.3) and (A.4) we get the result.

□

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B
Semigroup Generators

The aim of this section is to prove that the operator $\mathcal{B}_{\alpha,\delta}$ generates a C_0-semigroup in $L^1_{x,v}(\langle v\rangle^q m)$ for any $q \geq 0$. Moreover, the same proof remains true in the spaces E_j with $j = -1, 0, 1$ and $\mathcal{E}$.

The proof presented here follows the one in [Alonso et al., 2019, Appendix C], with some adaptations due to the definition and splitting of the operator $\mathcal{B}_{\alpha,\delta}$ presented here.

Let us recall the definition of the operator $\mathcal{B}_{\alpha,\delta}$:

$$\mathcal{B}_{\alpha,\delta}(h) := \mathcal{T}_{\alpha,R}(h) + \mathcal{L}^+_R(h) - \Sigma h - v \cdot \nabla_x h,$$

where $\Sigma = \nu_\alpha + \nu_e$, with domain $D(\mathcal{B}_{\alpha,\delta}) = W^{1,1}_{x,v}(\langle v\rangle^{q+1} m)$. Moreover, recall the definition of $\mathcal{T}_{\alpha,R}$ and $\mathcal{L}^+_R$

$$\begin{aligned}
\mathcal{T}_{\alpha,R}(h) &= \mathcal{Q}^+_{\alpha,R}(h, F_\alpha) + \mathcal{Q}^+_{\alpha,R}(F_\alpha, h) - \mathcal{Q}^-_{\alpha,R}(F_\alpha, h),\\
\mathcal{L}^+_R(h) &= \mathcal{Q}^+_{e,R}(h, \mathcal{M}_0),
\end{aligned}$$

where $\mathcal{Q}^+_{\alpha,R}$, $\mathcal{Q}^+_{e,R}$ (resp. $\mathcal{Q}^-_{\alpha,R}$) is the gain (resp. loss) part of the collision operator associated to the mollified collision kernel $(1 - \Theta_\delta)B$.

Consider the operator

$$A_0(h) := -\Sigma h - v \cdot \nabla_x h.$$

Notice that, by a similar argument as in (2.6), we have that there exist $\sigma_0, \sigma_1 > 0$ such that

$$0 < \sigma_0 \leq \sigma_0 \langle v\rangle \leq \Sigma(v) \leq \sigma_1 \langle v\rangle\,,$$

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for every $v \in \mathbb{R}^3$. Therefore, the domain of A_0 coincides with the domain of $v \cdot \nabla_x h$ wich is $W^{1,1}_{x,v}(\langle v \rangle^{q+1} m)$. It is easy to see that A_0 generates a C_0-semigroup $\{U(t) : t \geq 0\}$ given by

$$U(t)h(x,v) := e^{-\Sigma(v)t} h(x - tv, v),$$

which satisfies

$$\|U(t)h\|_{L^1_{x,v}(\langle v \rangle^q m)} \leq e^{-\sigma_0 t} \|h\|_{L^1_{x,v}(\langle v \rangle^q m)}.$$

In particular $\{U(t) : t \geq 0\}$ is a nonnegative contractive semigroup in $L^1_{x,v}(\langle v \rangle^q m)$.

Lemma B.0.1. *For any $\alpha > 0$, $q \geq 0$ and $\lambda > 0$*

$$\|R_{A_0}(\lambda)\|_{\mathscr{B}(L^1_{x,v}(\langle v \rangle^q m), L^1_{x,v}(\langle v \rangle^{q+1} m))} \leq \frac{1}{\sigma_0}, \tag{B.1}$$

and,

$$\|R_{A_0}(\lambda)\|_{\mathscr{B}(L^1_{x,v}(\langle v \rangle^q m))} \leq \frac{1}{\lambda + \sigma_0}. \tag{B.2}$$

Proof. Since, $\{U(t) : t \geq 0\}$ is a nonnegative semigroup $R_{A_0}(\lambda)$ is also nonnegative. Moreover, since the positive cone of $L^1_{x,v}(\langle v \rangle^q m)$ is generating (i.e. every element in $L^1_{x,v}(\langle v \rangle^q m)$ is the difference of two elements in the positive cone), it is enough to consider h nonnegative.

Let $g = R_{A_0}(\lambda)h$. Thus, we have

$$\begin{aligned} h &= (\lambda I - A_0) R_{A_0}(\lambda) h = (\lambda I - A_0) g, \\ &= (\lambda + \Sigma) g + v \cdot \nabla_x g. \end{aligned}$$

Multipling by the weight and integrating over $\mathbb{R}^3 \times \mathbb{T}^3$ we get

$$\int_{\mathbb{R}^3 \times \mathbb{T}^3} (\lambda + \Sigma(v)) g(x,v) \langle v \rangle^q m(v) dv dx = \|h\|_{L^1_{x,v}(\langle v \rangle^q m)}.$$

Hence, since $\sigma_0 \langle v \rangle \leq \Sigma(v)$ we have

$$\lambda \|g\|_{L^1_{x,v}(\langle v \rangle^q m)} + \sigma_0 \|g\|_{L^1_{x,v}(\langle v \rangle^{q+1} m)} \leq \|h\|_{L^1_{x,v}(\langle v \rangle^q m)},$$

which concludes our proof. □

The next step is to split $\mathcal{T}_R := \mathcal{T}_{\alpha,R} + \mathcal{L}^+_R$ into positive and negative parts

$$\begin{aligned} \mathcal{T}^+_R(h) &:= \mathcal{Q}^+_{\alpha,R}(h, F_\alpha) + \mathcal{Q}^+_{\alpha,R}(F_\alpha, h) + \mathcal{L}^+_R(h), \\ \mathcal{T}^-_R(h) &:= \mathcal{Q}^-_{\alpha,R}(F_\alpha, h). \end{aligned}$$

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By a similar argument as the one presented in [Cañizo and Lods, 2016, Proposition B.2] one has the following lemma:

Lemma B.0.2. *For any* $q \geq 0$ *there exist* $\kappa(\delta)$ *going to 0 as* $\delta \to 0$*, and such that for ever* $h \in L^1_{x,v}(\langle v \rangle^q m)$

$$\|\mathcal{T}_R^+(h)\|_{L^1_{x,v}(\langle v\rangle^q m)} \leq \kappa(\delta)\|h\|_{L^1_{x,v}(\langle v\rangle^{q+1} m)},$$

and that $\mathcal{T}_R^-$ *is bounded in* $L^1_{x,v}(\langle v \rangle^q m)$.

Let us introduce the operator $A_1 := A_0 + \mathcal{T}_R^+$. We want to prove that $\lambda I - A_1$ is invertible. In order to do this, notice that

$$\lambda I - A_1 = (\lambda I - A_0)(I - R_{A_0}\mathcal{T}_R^+),$$

so it is enought to see that $I - R_{A_0}\mathcal{T}_R^+$ is invertible. By the lemmas B.1 and B.0.2 we have that for every $\lambda > 0$

$$\begin{aligned}\|\mathcal{T}_R^+ R_{A_0}(\lambda) h\|_{L^1_{x,v}(\langle v\rangle^q m)} &\leq \kappa(\delta)\|R_{A_0}(\lambda) h\|_{L^1_{x,v}(\langle v\rangle^{q+1} m)}, \\ &\leq \frac{\kappa(\delta)}{\sigma_0}\|h\|_{L^1_{x,v}(\langle v\rangle^q m)}.\end{aligned} \tag{B.3}$$

Thus, taking δ small enough we have $\kappa(\delta) < \sigma_0$, so one deduces that $\mathbb{R}^+ \subset \rho(A_1)$ and, for every $\lambda > 0$, $I - R_{A_0}\mathcal{T}_R^+$ is invertible. Moreover, by the Neumann series (see [Banasiak and Arlotti, 2006, Remark 2.34]) one has

$$R_{A_1}(\lambda) = R_{A_0}\sum_{j=0}^{\infty}\left(\mathcal{T}_R^+ R_{A_0}(\lambda)\right)^j.$$

Therefore, according to (B.2)

$$\lim_{\lambda\to\infty}\|R_{A_1}(\lambda)\|_{\mathscr{B}(L^1_{x,v}(\langle v\rangle^q m))} \leq \lim_{\lambda\to\infty}\frac{1}{\lambda+\sigma_0} = 0.$$

Finally, notice that $\mathcal{B}_{\alpha,\delta} = A_1 - \mathcal{T}_R^-$. Since $\mathcal{T}_R^-$ is bounded, one can take λ large enough such that

$$\|\mathcal{T}_R^-\|_{\mathscr{B}(L^1_{x,v}(\langle v\rangle^q m))}\|R_{A_1}(\lambda)\|_{\mathscr{B}(L^1_{x,v}(\langle v\rangle^q m))} < 1.$$

Hence, one deduces that $\lambda I - \mathcal{B}_{\alpha,\delta}$ is invertible for λ large enough. This, together with the hypo-dissipativity ensures that $\mathcal{B}_{\alpha,\delta}$ generates a C_0-semigroup due to Lumer-Phillips theorem (see [Banasiak and Arlotti, 2006, Theorem 3.19] or [Pazy, 2012, Theorem 4.3]).

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C
Spectral theorems

In this section we present a more abstract theorem regarding enlargement of the functional space semigroup decay. More specifically:

Theorem C.0.1. *[Gualdani et al., 2017, Theorem 2.13] Let* $E', \mathcal{E}'$ *be two Banach spaces with* $E' \subset \mathcal{E}'$ *dense with continuous embedding, and consider* $L \in \mathcal{C}(E')$, $\mathcal{L} \in \mathcal{C}(\mathcal{E}')$ *with* $\mathcal{L}|_{E'} = L$ *and* $a \in \mathbb{R}$. *Assume*

A. L *generates a semigroup* e^{tL} *in* E', $L - a$ *is hypodissipative on* *Range*$(\mathrm{id} - \Pi_{L,a})$ *and*

$$\Sigma(L) \cap \Delta_a := \{\xi_1, ..., \xi_k\} \subset \Sigma_d(L).$$

B. There exist $\mathcal{A}, \mathcal{B} \in \mathcal{C}(\mathcal{E}')$ *such that* $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{A}|_{E'} = A$, $\mathcal{B}|_{E'} = B$, *some* $n \geq 1$ *and* $C_a > 0$ *such that*

(B.1) $\mathcal{B} - a$ *is hypodissipative in* $\mathcal{E}'$,

(B.2) $\mathcal{A} \in \mathscr{B}(\mathcal{E}')$ *and* $A \in \mathscr{B}(E')$,

(B.3) $T_n := (\mathcal{A}S_{\mathcal{B}})^{(*n)}$ *satisfies* $\|T_n(t)\|_{\mathscr{B}(\mathcal{E}', E')} \leq C_a e^{at}$.

Then $\mathcal{L}$ *is hypodissipative in* $\mathcal{E}'$ *with*

$$\|S_{\mathcal{L}}(t) - \sum_{j=1}^{k} S_L(t)\Pi_{\mathcal{L},\xi_j}\|_{\mathscr{B}(\mathcal{E}')} \leq C'_a t^n e^{at},$$

for all $t \geq 0$ *and for some* $C'_a > 0$.

Actually, the assumption (B.3) follows from [Gualdani et al., 2017, Lemma 2.17] which yields an estimate on the norms $\|T_n\|_{\mathscr{B}(E_j, E_{j+1})}$ for $j = -1, 0$:

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Lemma C.0.2. *Let X, Y be two Banach space with $X \subset Y$ dense with continuous embedding, and consider $L \in \mathscr{B}(X)$, $\mathcal{L} \in \mathscr{B}(Y)$ such that $\mathcal{L}|_X = L$, $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $a \in \mathbb{R}$. We assume that there exist some intermediate spaces*

$$X = \mathcal{E}_J \subset \mathcal{E}_{J-1} \subset \cdots \mathcal{E}_2 \subset \mathcal{E}_1 = Y,$$

with $J \geq 2$, such that if we denote $\mathcal{A}_j = \mathcal{A}|_{\mathcal{E}_j}$ and $\mathcal{B}_j = \mathcal{B}|_{\mathcal{E}_j}$

1. *$(B_j - a)$ is hypodissipative and $\mathcal{A}_j$ is bounded on $\mathcal{E}_j$ for $j = 1, .., J$.*
2. *There are some constants $l \in \mathbb{N}^*$, $C \geq 1$, $K \in \mathbb{R}$, $\gamma \in [0, 1)$ such that for all $t \geq 0$ for $j = 1, .., J - 1$*

$$\|T_l(t)\|_{\mathscr{B}(\mathcal{E}_j, \mathcal{E}_{j+1})} \leq C \frac{e^{Kt}}{t^\gamma}.$$

Then for any $a' > a$, there exist some constructive constants $n \in \mathbb{N}$, $C_{a'} \geq 1$ such that for all $t \geq 0$

$$\|T_n(t)\|_{\mathscr{B}(\mathcal{E}_j, \mathcal{E}_{j+1})} \leq C_{a'} e^{a't}.$$

Furthermore, we state a quantitative spectral mapping theorem. A proof for this result can be found in [Tristani, 2016, Proposition 2.20]. A more general version of this theorem can be found in [Mischler and Scher, 2016].

Theorem C.0.3. *Consider a Banach space X and an operator $\Lambda \in \mathscr{C}(X)$ so that $\Lambda = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \in \mathscr{B}$ and $\mathcal{B} - a$ is hypodissipative on X for some $a \in \mathbb{R}$. We assume furthermore that there exist a family X_j, $1 \leq j \leq m$, $m \geq 2$ of intermediate spaces such that*

$$X_m \subset D(\Lambda^2) \subset X_{m-1} \subset \cdots X_2 \subset X_1 = X,$$

and a family of operators Λ_j, $\mathcal{A}_j$, $\mathcal{B}_j \in \mathscr{C}(X_j)$ such that

$$\Lambda_j = \mathcal{A}_j + \mathcal{B}_j, \quad \Lambda_j = \Lambda|_{X_j}, \quad \mathcal{A}_j = \mathcal{A}|_{X_j}, \quad \mathcal{B}_j = B|_{X_j},$$

and that there holds

(i) *$(B_j - a)$ is hypodissipative on X_j;*

(ii) *$\mathcal{A}_j \in \mathscr{B}(X_j)$;*

(iii) *there exist $n \in \mathbb{N}$ such that $T_n(t) := (\mathcal{A}S_{\mathcal{B}}(t))^{*n}$ satisfies*

$$\|T_n(t)\|_{\mathscr{B}(X, X_m)} \leq C e^{at}.$$

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Hence, the following localization of the principal part of the spectrum

(A) There are some distinct complex numbers $\xi_1, ..., \xi_k \in \Delta_a$, $k \in \mathbb{N}$ *such that*

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, ..., \xi_k\} \subset \Sigma_d(\Lambda);$$

implies the following quantitative growth estimate on the semigroup:

(B) for any $a' \in (a, \infty) \backslash \{\mathfrak{Re}\, \xi_j, j = 1, ..., k\}$*, there exist some constructive constant* $C_{a'} > 0$ *such that for every* $t \geq 0$

$$\left\| S_\Lambda(t) - \sum_{j=1}^{k} S_\Lambda(t) \Pi_{\Lambda, \xi_j} \right\|_{\mathscr{B}(X)} \leq C_{a'} e^{a' t}.$$

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