Pontifícia Universidade Católica
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Rodrigo Ferreira Inocencio Silva

Assessment of a derivative management policy for risk-averse corporations: a stochastic dynamic programming approach

Dissertação de Mestrado

Dissertation presented to the Programa de Pós-Graduação em Engenharia de Produção da PUC-Rio in partial fulfillment of the requirements for the degree of Mestre em Engenharia de Produção.

Advisor : Prof. Davi Michel Valladão
Co-Advisor: Prof. Thuener Armando da Silva

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To my wife and son.

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Abstract

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Corporate finance comprises investment, financing and dividend policies aimed at maximizing shareholder value. In particular, the results of commodity producers and, consequently, the value to their shareholders are subject to high volatility, resulting from the variation of prices of these products in the global market. However, the risk of this variation can be mitigated by exploiting the broad derivatives market that is generally available for commodities. This work proposes to calculate the value increase that a commodity-producing company can provide to its shareholders through the use of an optimal derivatives management policy, by buying or selling forward contracts. To this end, it seeks to maximize shareholder returns via dividends in a risk-averse environment. The model assumes that the commodity price follows a discrete state Markov process. Since the model is applied in several stages, the problem becomes quite complex, and it is necessary to use a decomposition method to obtain the solution, so we used the method known as stochastic dynamic dual programming. The results show that by trading forward contracts, a company increases the value perceived by the shareholder, measured by the payment of dividends, to any level of risk aversion. The average value increase, considering different levels of risk aversion and an unbiased pricing assumption, is higher than 320% when compared to companies that do not have access to such instruments. In addition to measuring the value increase, we also analyzed which factors determine the optimal derivatives management policy. It was possible to identify that the derivatives management policy is very determined by the prices, which in turn are associated with the state of the Markov chain in force at each stage.

Keywords

Derivatives management policy   Forward contracts   Markov chain   Stochastic dynamic programming

Resumo

Silva, Rodrigo Ferreira Inocencio; Valladão, Davi Michel; Silva, Thuener Armando da. **Avaliação de uma política de gestão de derivativos em empresas avessas a risco: uma abordagem de programação dinâmica estocástica**. Rio de Janeiro, 2020. 70p. Dissertação de Mestrado – Departamento de Engenharia Industrial, Pontifícia Universidade Católica do Rio de Janeiro.

Finanças corporativas compreendem políticas de investimento, financiamento e dividendo cujo objetivo é maximizar o valor do acionista. Em particular, os resultados de empresas produtoras de commodities e, consequentemente, o valor para seus acionistas estão sujeitos a alta volatilidade, decorrentes da variação dos preços destes produtos no mercado global. Entretanto, o risco dessa variação pode ser mitigado ao se explorar o amplo mercado de derivativos que, em geral, está disponível para commodities. Este trabalho propõe calcular o acréscimo de valor que uma empresa produtora de commodities pode fornecer ao seu acionista pelo uso de uma política ótima de gestão de derivativos, por meio da compra ou venda de contratos a termo. Para tanto, busca maximizar o retorno aos acionistas via dividendos em um ambiente avesso a risco. O modelo assume que o preço da commodity segue um processo de Markov de estados discretos. Como o modelo é aplicado em vários estágios, o problema torna-se bastante complexo, sendo necessário usar um método de decomposição para obter a solução, sendo assim, utilizou-se o método conhecido como programação dual dinâmica estocástica. Os resultados demonstram que, ao comercializar contratos forward, uma empresa aumenta o valor percebido pelo acionista, medido pelo pagamento de dividendos, para qualquer nível de aversão a risco. A média de acréscimo de valor, considerando diferentes níveis de aversão a risco e uma premissa de precificação não viesada, é superior a 320% quando comparado a empresas que não possuem acesso a tais instrumentos. Além de medir o acréscimo de valor, analisou-se também quais os fatores determinantes para a política ótima de gestão de derivativos. Foi possível identificar que a política de gestão de derivativos é muito determinada pelos preços, que por sua vez estão associados ao estado da cadeia de Markov vigente em cada estágio.

Palavras-chave

Política de gestão de derivativos Contratos a termo Cadeias de Markov Programação dinâmica estocástica

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1
Introduction

Corporate finance is the study of all economic and financial aspects related to the operations of a company. It comprises the market analysis of the corporation's sector, the capital structure monitoring, the decisions that are made by managers, and the correct use of tools to maintain financial sustainability. In particular, the financial manager must choose whether the company acquires debt with third parties to finance investments, what is the appropriate level of investment (or divestment) under a given market condition, identify budgetary constraints, assess the impact of debt on the cost of capital considering the associated loan tax shield. All of these actions aim at providing the shareholder with the highest possible return.

In the case of commodity-producing companies, the decision-making process becomes more complicated due to the high volatility of commodity prices, which compromises the company's financial performance and, consequently, reduces the return to the shareholder. Given the investor risk aversion, market-price risk tends to reduce the shareholder value of a commodity company. However, the commodities business has a broad market of financial instruments (derivatives) that could mitigate the risks associated with price volatility. In the literature, several authors have tried to identify the impact of derivatives on the value of companies. Since the first work that models derivatives in the financial analysis of a company in 1984 (Stulz [32]), the models have become more realistic and complex, moving from static models with very restrictive premises to dynamic models that represent economic and financial decisions of a company in a very reliable way.

In practice, we observe some examples of companies that use derivative instruments in their financial planning, such as Petrobras – Brazilian state-controlled oil company and one of the largest worldwide – that, over the past two years, has announced partial hedging programs against oil price volatility. Similarly, Mexico's state-owned oil company (Pemex) also regularly discloses hedging for its activities, as do small US oil companies. However, the industry standard is to use simplified models in their medium and long term planning, usually spreadsheet-based and with predefined business and financing decisions. To correctly assess the value of using derivatives to the shareholders, an analytical tool capable of jointly simulate the optimal investment, financing, and dividend payment policies, together with the company's derivatives management policy, is required. To this end, this dynamic decision-making process under uncertainty should be modeled as accurately as possible.

In this work, we use Waga [36]'s capital structure model as a basis (see Chapter 3), which proposes a dynamic risk-averse model in which the most relevant variables for a company's financial analysis are represented, with emphasis on variable debt cost with leverage (based on the model proposed by Valladão *et al.* [34]), sale of assets and shares issuing. We expanded the Waga [36]'s model by inserting the possibility of contracting derivatives, in addition to changing the modeling of the problem uncertainty (see Chapter 4). Besides we applied the Bolton *et al.* [8]'s broader risk management concept: we consider a joint optimization of derivatives management together with financing and investment policies. The objective function is to maximize the risk-adjusted shareholder value under commodity-price uncertainty.

Our model assumes that prices follow a discrete-state Markov process characterized by a transition probability matrix and differing conditional probability distribution given each Markov state. The distributions and the Markov process parameters can be estimated from historical data using an Expectation and Maximization (EM) algorithm under a Hidden Markov Model (HMM) learning framework. Considering the number of variables and the long-term application, the model is quite complex and suffers from the well-known curse of dimensionality. We circumvent this issue using a sampling-based decomposition method, the Stochastic Dual Dynamic Programming (SDDP, Pereira & Pinto [26]).

In the case study, the optimization model is applied to an oil-producing company, which uses Brent as the base for oil pricing. We applied HMM over a historical series with data from the last 20 years of Brent prices. The results show that by trading forward contracts, a company increases the value perceived by the shareholder, measured by the payment of dividends, to any level of risk aversion. The average value increase, considering different levels of risk aversion and an unbiased pricing assumption, is higher than 320% when compared to companies that do not have access to such instruments. In addition to measuring the value increase, we also analyzed which factors determine the optimal derivatives management policy. We identify that the policy is highly determined by price levels, which in turn are associated with the state of the Markov chain in force at each stage.

The objective of this work is to assess the value of derivatives management policies (for hedging or speculative purposes) to risk-averse shareholders of a commodity-producing company. The main contributions are:

- A realistic and computationally tractable risk-averse dynamic model that jointly optimize investment, funding and dividend decisions, along with the company derivatives management strategy through forward contracts;
- A computational tool to simulate policies for a better understanding of the interplay of derivatives using decisions with investment, funding, and dividends in a commodity company;
- A case study for an oil company, with data from the last 20 years of Brent prices, allowing to verify that the derivatives management policy increases the value perceived by the shareholder (higher than 320%, on average under an unbiased pricing assumption). The policy is very determined by the commodity prices, which in turn are associated with the state of the Markov chain in force at each stage.

This document is divided into six chapters. After this introduction (Chapter 1), the theoretical background is presented in Chapter 2 with the most relevant topics employed in this dissertation. Chapter 3 briefly summarizes the Waga's model and Chapter 4 details the extensions and modifications made from it. The results are presented in Chapter 5 as a case study. Finally, the conclusions are in Chapter 6, with references right after. There is also an Appendix A, with a description of stochastic programming and SDDP algorithm.

2
Theoretical Background

In this chapter, we review some basic concepts on finance and optimization techniques used in this work, as well as briefly present the literature review of the topics studied, with emphasis on derivative instruments. Initially, a fundamental review of the capital structure literature and concepts of derivatives is presented, additionally, we made a description of a generic discrete Markovian process used later to model price dynamics.

2.1
Capital Structure

The foundations of capital structure studies, a corporate finance research area, were established in 1958, with the work of Modigliani & Miller [22]. In their article, by adopting as premises efficient markets, absence of taxes and costs (bankruptcy, transaction or agency), as well as equal conditions between companies and individuals in the capital market, the authors establish two propositions:

I The market value of any company is independent of its capital structure;

II The required return on equity is positively related to financial leverage due to increased risk faced by equity holders.

Proposition I is demonstrated through the concept of non-arbitration. For example, assume a leveraged company (company L) with identical assets to an unleveraged company (company U). Since an investor can replicate a portfolio of shares of company L, by buying shares of company U and borrowing (assuming the same interest rate for individuals and companies), both companies must have the same value. Otherwise, an investor could sell the company with the highest value, build a synthetic portfolio with the lowest value company, and obtain a risk-free return.

Proposition II implies the higher the leverage, the higher the return required for the company's equity. Thus, despite the cost of debt raised with third parties is lower than the cost of equity, the reduction in the cost of capital with the increased leverage is perfectly offset by the increasing return required by the shareholder as the company is riskier.

In later work, Modigliani & Miller [23] increase the model's reality level by eliminating the tax-free premise of the original one. In this context, a leveraged company has an advantage because financial debt provides a rebate on the income taxes, thereby increasing its value. This effect is commonly known as tax-shield. A direct consequence of the tax shield is that the higher a company's leverage, the greater its value. This unrealistic result is justified on the simplifying assumptions adopted by Modigliani & Miller [23]. Considering that there is no cost associated with rising leverage, the result is quite different from what we find in practice, where a heavily leveraged company has little or no value.

Baxter [5] changed another premise of Modigliani & Miller [22] 's work by now considering bankruptcy costs (also known as financial distress costs), i.e., the costs incurred by a company in the event of impending bankruptcy. As debts to be paid to the remaining creditors will be discounted from these costs, the interest rate charged increases with leverage. With this assumption, the cost of capital of the company increases if leverage is too high. So, there is a minimum value where the company valuation is maximized.

From the treatment of the assumptions adopted by Modigliani & Miller [22], mentioned above, comes a variety of articles seeking to identify an optimal capital structure based on similar concepts. Some of these studies are: Bradley *et al.* [9], Titman & Wessels [33], Fama & French [15], Hennessy & Whited [19] and Decamps & Villeneuve [12].

The theory that studies the balance between the benefit of reducing the amount to be paid in taxes due to the payment of interest on debts against the costs associated with increasing debt from third parties, was called by Myers [24] as trade-off theory. Other theories aim to explain the capital structure of a company. Myers [24] himself suggests one called pecking order, in which there is an order of preference for fundraising: operating generation, debt and, finally, issuance of shares.

2.2 Derivatives

According to the Basel Committee [3] liquidity risk can be defined as the risk that a company will not be able to efficiently settle its current and future, expected or not, cash flow obligations without affecting its daily operations or financial condition. Market risk is the possibility of an investor suffering losses due to factors that affect the performance of the markets in which it operates. To mitigate such risks, one of the main techniques applied is derivative contracting (an operation known as hedging).

Derivatives are instruments that always correspond (or derive on) to an underlying asset (the most common are stocks, bonds, commodities, currencies, interest rates, and market indexes). They are capable of providing, based on certain contractual premises, a financial return, also known as the payoff, to its buyer. Some derivative instruments have an initial cost for their acquisition, usually called a premium. In general, the value of this premium can be obtained by estimating the value of payoff in several future scenarios, calculating the expected value of these payoffs, and discounting it at an appropriate interest rate (r). Thus, let v_t be the premium value paid by any derivative in t and φ_{t+1} its payoff in $t+1$,

$$v_t = \frac{\mathbb{E}_t\left[\varphi_{t+1}\right]}{1+r}. \tag{2-1}$$

In general, derivative instrument types are divided into three major types: forwards, options, and swaps. In this work only forward contracts will be used, which is an agreement between two parties to buy or sell an asset at a specified price on a future date. The buyer of the contract must pay for the asset, at the time of maturity, exactly the value contracted at signing (we say that the buyer assumes the long position). The other party (short position) must sell the asset at the agreed price. Note that no matter how much the price of the asset varies between signing and maturity, the product will be traded at the agreed price at the time of signing.

However, in many contracts there is no physical delivery of the product, there is a financial compensation between the forward contract and the spot price at the time of maturity. In this way, if the price of an asset falls after the contract is signed, the seller is entitled to receive compensation. For instance, assume that a producer has signed (sold) a forward contract, which has the contractual price for a given asset equal to f. At contract maturity, the asset has a spot price equal to p. In this case, the seller will receive of the counterparty an amount equivalent to ($-p+\mathrm{f}$). Note that regardless of the amount the seller receives for the derivative instrument contract, he will always receives the value f for the asset, because he receives ($-p+\mathrm{f}$) for the contract and sells the asset at market price (p), totaling a revenue equal to f.

In the example above, the producer took the short position of the contract. In this case, the derivative was used for hedging purposes, as it ensured the sale price of its product at a future time, eliminating uncertainty. However, suppose that the seller has assumed the long position in this contract. At the contract maturity, he would still receive p for his product on the market, but would have to *pay* $(-p+\mathrm{f})$ for the derivative contract. In this case, the revenue is equal to $(2p-\mathrm{f})$. Thus, if the spot price at the time of maturity is higher than the contract price ($p \geq \mathrm{f}$), the producer increases his earnings, otherwise, he may have a loss. Given this context, when assuming the long position, the producer bets on the price rise, using the derivative instrument to leverage his gains, but incurring in risks of large losses, which configures a speculative operation.

Considering that there are risk-neutral agents that would eliminate any arbitrage and disregarding transaction costs, forward contracts have no cost to the buyer, that is, the premium is null ($v_t = 0$). As we assume no-arbitrage, the risk-neutral probability can be applied to equation (2-1) to calculate the contractual price (f). So we have

$$\frac{1}{1+r_f}\sum_{n=1}^{N_s}\pi_n\hat{\varphi}^{f}_{n,t+1} = 0, \tag{2-2}$$

where N_s is the number of future scenarios in which the payoff was calculated, $\hat{\varphi}^{f}_{n,t+1}$ represents the n samples of a forward contract payoff in $t+1$, π_n is the risk-neutral probability of each scenario n and r_f is the risk-free interest rate.

Note that in the previous equation (2-1) the discount rate was r, whereas in the equation (2-2) the discount rate is r_f, which represents the risk-free rate. This change is necessary because the expected value was calculated based on the risk-neutral probabilities, so no risk must be embedded in the discount rate.

Assuming that a forward contract has been signed at t, with the agreed acquisition price of the asset at $t+1$ equal to f_t, then the payoff ($\hat{\varphi}^{f}_{n,t+1}$) of this contract is equal to ($p_{t+1} - \mathrm{f}_t$), where p_{t+1} is the asset price at $t+1$. Replacing the payoff in equation (2-2), we have:

$$\begin{aligned}
&\frac{1}{1+r_f}\sum_{n=1}^{N_s}(\pi_n\hat{p}_{n,t+1} - \mathrm{f}_t) = 0\\
&\sum_{n=1}^{N_s}(\pi_n\hat{p}_{n,t+1}) - \sum_{n=1}^{N_s}(\pi_n\mathrm{f}_t) = 0\\
&\sum_{n=1}^{N_s}(\pi_n\hat{p}_{n,t+1}) - \mathrm{f}_t\sum_{n=1}^{N_s}(\pi_n) = 0,
\end{aligned} \tag{2-3}$$

where $\hat{p}_{n,t+1}$ represents each of the N_s realizations of p_{t+1}.

As the price of the underlying asset is subject to the same risk-neutral probabilities, which have a sum equal to 1, we have two additional restrictions:

$$p_t = \frac{1}{1+r_f}\sum_{n=1}^{N_s}\pi_n\hat{p}_{n,t+1} \tag{2-4}$$

$$\sum_{n=1}^{N_s} \pi_n = 1. \tag{2-5}$$

Replacing the equations (2-4) and (2-5) in the equation (2-3), we have:

$$\mathrm{f}_t = p_t \left(1 + r_f\right). \tag{2-6}$$

Now that we know the price set at the time of signing a forward contract (f_t), so it is possible to calculate the gain or loss produced by that instrument at maturity (f_t^h). Assuming a contract that is signed at $t-1$ (with price f_{t-1}) and matures at t, the flow received by the buyer at t will be:

$$f_t^h = h_{t-1} \left(p_t - \mathrm{f}_{t-1}\right). \tag{2-7}$$

Note that the greater the number of contracts (h_{t-1}) bought (for $h_{t-1} \geq 0$ or sold for $h_{t-1} \leq 0$) at $t-1$, the greater the magnitude of the monetary value to be paid or received.

In the literature, many works study the impact of derivatives on the value of a company. Among these, Stulz [32] focuses on managing exposure to foreign currencies using forward contracts. The author, through a continuous-time model, evaluates derivatives management policies in two scenarios: value maximization and risk-averse agents. As a result, it is possible to calculate the optimal level of hedge for the company under certain simplifying assumptions: that level remains fixed over time, and the investment and financing decisions have to be previously defined by the company's managers. Besides, it ignores the cost of financial distress, which impacts the optimal derivatives management policy. Similarly, Fehle & Tsyplakov [16] developed a model with limitations on its capital structure, assuming that the company holds no cash and maintains a constant debt level over time. They study a dynamic and infinite horizon model that seeks to reduce the uncertainty related to the selling price of a product. In this way, they could obtain a risk management policy by assessing issues such as the initiation and maturity of derivative contracts. Their main finding is that the policy is associated with financial distress costs.

Following previous continuous-time models, Rochet & Villeneuve [27] consider a constant size company, i.e., no new investments, null depreciation, and no tax shield considered. They show that the company must cover all its risk exposure if the cash value is below a certain level; otherwise, derivatives should not be acquired. From liquidity (cash) point of view, they state that the company must accumulate a certain amount of cash, from which all excess must be paid in the form of dividends. They assume investor risk neutrality and also the possibility of buying insurance for high impact risks and low probability of occurrence. Bolton *et al* [8] expanded the Rochet & Villeneuve's model [27] , adding the possibility of investment and refinancing, and proposed a dynamic corporate risk management flowchart combining cash, investment, external financing, dividend payments, and derivatives management policies. According to the authors, "risk management is not just a financial hedging; in fact, it is closely connected to liquidity management via daily operations". Their model ignores taxes and the influence of leverage on the company bankruptcy. Like Rochet & Villeneuve [27], they find that the company only distributes dividends if the cash level is above a certain threshold. They conclude that the optimal hedging policy is one that balances the marginal benefit of the hedge with its associated cost.

Considering the discrete-time models, one of the most important works focusing on hedging policies is the Froot *et al.* [17]. The authors present a single-period model whose uncertainty may arise from investment or funding sources. The goal of the model is to maximize the company's net income. The analysis takes place around the investment that can be funded from internal or external sources. As a result, the authors find that, if there is no correlation between investment opportunities and internal sources of funding (own funds), the company must fully protect its risk exposure. As the hedging is used as an instrument to ensure cash flow for the next stage, if the correlation is high, then there is no need to ensure own funds, because if the needy for investment is high, the amount of internal resources will also be. If the need for investment is low, there is no need to ensure a minimum amount of internal resources. Therefore, with the increase in the correlation between internal sources of financing and investment, the hedging is reduced.

Based on the research by Graham & Harvey [18], who found that 75% of CFO's claim to use the net present value (NPV) of free cash flows to measure value, Léautier *et al.* [20] builds a multi-stage discrete-time model that contemplates uncertainties associated with cash flow and investments. The authors used simplifying assumptions such as fixed cost of capital, no dividends distribution, and no share issuing. Amaya *et al.* [1] expand this model, especially by adding dividend distribution and the possibility of bankruptcy. However, the authors consider debt as the only option for obtaining external resources, excluding the possibility of selling assets or issuing shares. Besides, the authors do not consider cash holdings since all excess money is paid as dividends. By implementing an analytical solution, they claim that the company uses derivatives for all of its risk exposure until its leverage reaches a certain high level (above approximately 80%). By exceeding this limit, the company bets on its financial recovery and stop contracting hedge. Although the last two papers have an increasing financing cost with leverage, the cost of capital does not take into account investment risks.

2.3 Markov Chain

In a multi-stage optimization problem, assume that $\xi_1, \xi_2, \ldots, \xi_T$ is the uncertain information that is gradually revealed at each stage. The sequence ξ_t of the information is called a stochastic process, that is, a sequence of random variables with defined probability distribution. Considering that the notation $K_{[t]}$ characterizes the "history" of this process to the stage t $(\xi_1, \ldots, \xi_t)$, a process is stochastically independent between stages when ξ_t is independent of $K_{[t-1]}$. On the other hand, a Markov process occurs when the conditional probability distribution of ξ_t, given $K_{[t-1]}$ is the same as the conditional probability distribution of ξ_t, given ξ_{t-1}.

The concept of independent and Markovian processes can be illustrated through the diagrams of Figure 2.1. For better visualization, it was considered that the uncertain variable (ξ) has only two possible realizations (ξ_u and ξ_d). The probabilities of occurrence (P_u and P_d) of each achievement are described in the figure. Note that in Figure 2.1 (a) this probability is independent of the state.

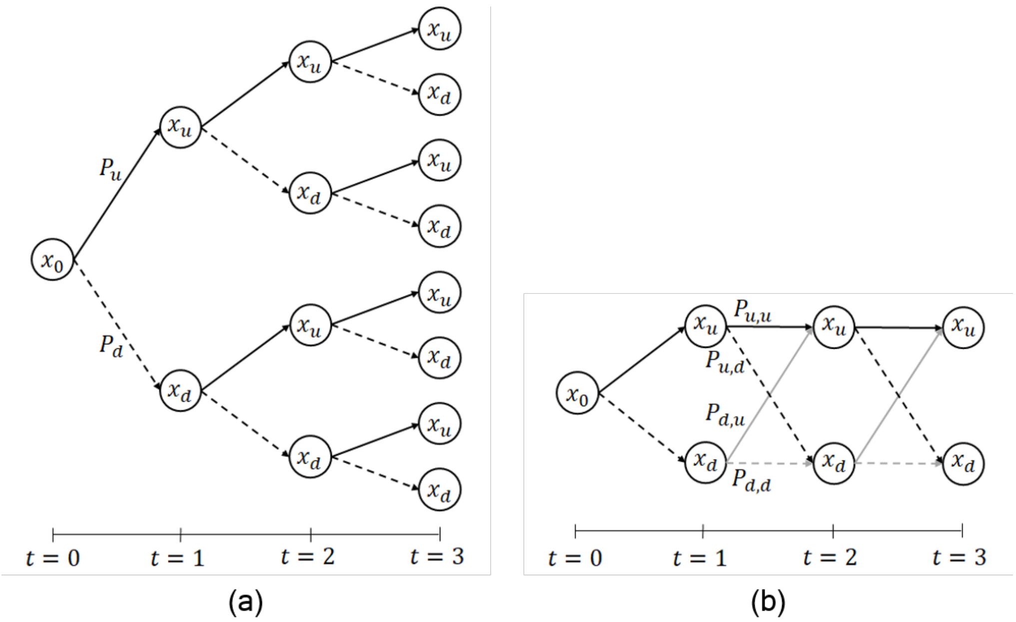

Figure 2.1: Comparison between independent stochastic process (a) and Markovian process (b).

The concept of Markov process was incorporated into the equations of dynamic programming assuming a Markov chain of discrete states of space $\mathcal{K}$ ($K_t \in \mathcal{K}$, where $K_t = u$ or $K_t = d$), in which ξ_{t+1} depends only of the state of

Markov K_t at t and does not depend on the past realizations $\xi_t, \ldots, \xi_1$. In the equation (3-1), the notation $P_{k|j}$, where $j, k \in \mathcal{K}$, represents the probability that the Markov state in the next stage K_{t+1} is equal to k, since the current state (K_t) is j.

2.4 Markov Chained Stochastic Dual Dynamic Programming

The stochastic dual dynamic programming (SDDP) – see Appendix A for a brief description of stochastic programming, as well as details of the SDDP technique algorithm – was proposed by Pereira & Pinto [26], in an application to energy planning (for more example of SDDP application, see Shapiro *et al.* [31]). In this methodology, the "curse of dimensionality" is overcome by assuming independence of the random variable between stages. Before SDDP, even with the use of some decomposition method, such as L-shaped, it is necessary a future cost function for each of the trajectories in the scenario tree, causing the problem to grow exponentially with the number of stages. Figure 2.2 illustrates the concept, considering a generic time dependency. Without time dependency of the uncetainty variable, it is possible to obtain a single convex future cost function for each stage, as shown in Figure 2.3.

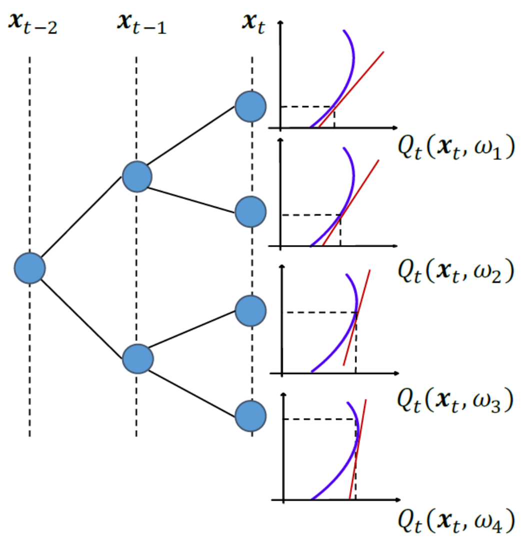

Figure 2.2: Illustrative value functions considering generic time dependence. Source: Valladão *et al.* [34].

Despite the reduction in computational cost, the stage-wise independence premise reduces the scope of real problems in which this method can be applied. That is a critical point for using the SDDP technique: it is difficult to argue that the return of assets from a portfolio, the flow of water in a reservoir or, in our case, a commodity price follows a stochastic process completely

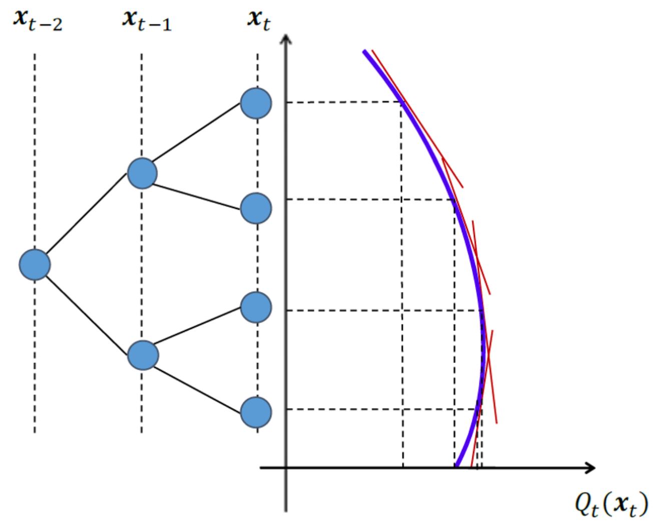

Figure 2.3: Illustrative SDDP value function considering stage-wise independence. Source: Valladão *et al.* [34].

independent of previous stages. This restriction is partially overcome by the works of Mo *et al.* [21] and Phillpot & De Matos [25], who demonstrate in their article that it is possible to work with dependence on SDDP as long as uncertainties follow Markovian process. In this case, we have a future cost function for each time stage and each Markov state, making the model more realistic than its stage-wise independence counterpart but still maintaining the model computationally tractable. Figure 2.4 presents the concept for a future cost function that is equal to the weighted average between the functions of two Markov states.

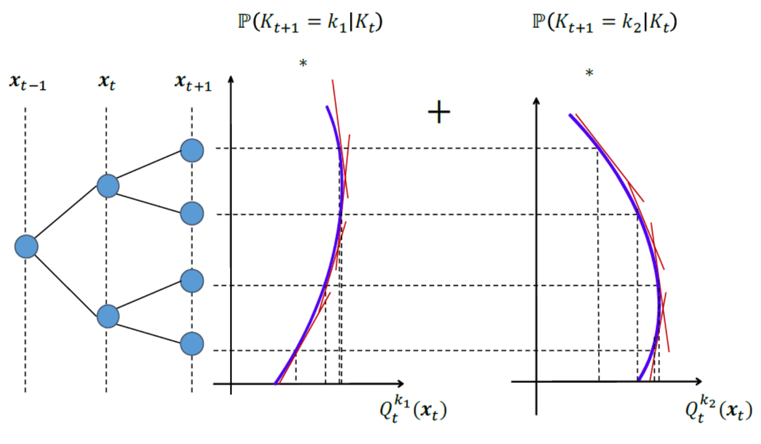

Figure 2.4: Illustrative weighted average of value functions for Markovian price model. Source: Valladão *et al.* [34].

3
Base Model

For investment, funding and dividend payment decisions, the capital structure model developed by Waga [36] is the a basis of our work. For clarity purposes, we present a compact representation of the model by defining the dynamic equations.

$$Q_t^j(\upsilon_{t-1}, p_t) = \max_{(e_t, \upsilon_t) \in \mathcal{X}(\upsilon_{t-1}, p_t)} \sum_{k \in \mathcal{K}} \left\{ e_t + \frac{\psi_t \left[Q_{t+1}^k (\upsilon_t, p_{t+1}) \,|K_{t+1} = k \right]}{1 + r_f} \right\} P_{k|j}. \tag{3-1}$$

where

$$Q_{T+1}^j(\upsilon_{T-1}, p_T) = 0. \tag{3-2}$$

The model objective is to maximize the risk-adjusted present-valued dividend payments (e_t). A time-consistent dynamic risk measure (ψ_t) is defined as the recursive formulation of the convex combination between expected value and CVaR (see Shapiro [28]), and the payments are discounted by the risk-free rate (r_f). The risk-adjusted shareholder value is a function of the previous state vector (υ_{t-1}) and the uncertainty realization (p_t). The state of system $\upsilon_t = (a_t, c_t, d_t, \iota_t)^\top$ includes the asset value (a_t), cash holdings (c_t) and debt raised (d_t), as well as other auxiliary variables (represented generically by ι_t). The variable $\mathcal{X}$ denotes the feasible set as a consequence of investment (i_t), debt (d_t) and dividend payment (e_t) decisions. The indexes j and k represent, respectively, the current and future states of a Markovian process. For the last stage, in addition to the end of recursion due to equation (3-2), the company is not able to raise debt or have operational generation.

In the following subsections, we detail some important features of the model proposed by Waga [36].

3.1
Coherent Risk Measures

A risk measure is a function $\phi : \mathbb{R}^n \to \mathbb{R}$ that associates a value to a realization of an uncertain scenario $\omega \in \Omega$. The risk measure used in this work is based on conditional value at risk (CVaR), which is extensively applied to optimization problems. Considering R as the random return of an asset, the

CVaR can be defined as:

$$\phi_\alpha(R) = -CVaR_\alpha(R) = \sup_z \left\{ z - \frac{\mathbb{E}\left[(z-R)^+\right]}{1-\alpha} \right\}, \tag{3-3}$$

where $(z-R)^+ = \max(z-R;0)$ and $\alpha \in (0,1)$ is the confidence level.

According to the above formulation, it can be understood that the CVaR is the average of $100 \times (1-\alpha)$ % worst scenarios.

As stated by Artzner *et al.* [2] a coherent risk measure respects the following properties:

- Monotonicity: for all random returns R_1 and R_2, if $R_1(\omega) \geq R_2(\omega), \forall \omega \in \Omega$, then $\phi_\alpha(R_1) \geq \phi_\alpha(R_2)$;
- Translation invariance: for all random return R and constant $m \in \mathbb{R}$, $\phi_\alpha(R+m) = \phi_\alpha(R) + m$;
- Positive homogeneity: for all random return R e constant $m \geq 0$, $\phi_\alpha(mR) = m\phi_\alpha(R)$;
- Subadditivity: for all random returns R_1 e R_2, $\phi_\alpha(R_1+R_2) \geq \phi_\alpha(R_1) + \phi_\alpha(R_2)$.

The coherent risk measures used in this work (ψ_t) is the convex combination expected value and CVaR, as proposed by Shapiro [28], defined as:

$$\psi_t[R] = (1-\lambda)\,\mathbb{E}_t[R] + \lambda\phi_{\alpha,t}[R], \tag{3-4}$$

where $0 \leq \lambda \leq 1$ is the risk averse parameter.

3.2 Cash Balance Constraint

The problem variables are related to each other through the cash equation, which is a result of the sum of the cash inflows and outflows. Cash inflows are the cash value of the previous period (c_{t-1}), plus risk-free income, operational generation (g_t), defined as EBITDA, and debt raised in the present period. The cash outflows are: amount paid from the debt raised in the previous period (plus interest) (q_t), investments (i_t), dividends paid to shareholder (e_t) and taxes (x_t). The variables m_t and b_t represent the cost of selling assets and issuing shares, respectively, which will be discussed in detail in section 3.3.

$$c_t = c_{t-1}(1+r_f) + g_t(a_{t-1}, p_t) + d_t - q_t(d_{t-1}) - (i_t + m_t) - (e_t + b_t) - x_t. \tag{3-5}$$

3.3
Asset and Equity

In addition to raising debt, there are two ways to obtain cash instantly: sale of assets (negative investment) or fundraising with partners (issuance of shares, i.e., negative dividend). However, both the sale of assets and the issue of shares are subject to a cost, respectively represented by m_t and b_t (for more details see Waga [36]), which consumes part of the amounts to be added to cash. These two variables only have value when the investment and dividend variables, respectively, are negative. Otherwise, as there is no cost on an operation that has not occurred, they must have null value. So, mathematically they are equal to:

$$m_t = \max\left(-\kappa_i i_t, 0\right). \tag{3-6}$$

$$b_t = \max\left(-\kappa_e e_t, 0\right). \tag{3-7}$$

To keep the linearity of the model, the equations can be rewritten by the set of equations:

$$m_t \geq -\kappa_i i_t. \tag{3-8}$$

$$b_t \geq -\kappa_e e_t. \tag{3-9}$$

$$m_t, b_t \geq 0. \tag{3-10}$$

Where κ_i and κ_e represent the percentage cost of the asset sale and issuance of shares, respectively.

Considering a depreciation rate δ, the assets (a_t) are updated according to the equation: $a_t = (1 - \delta)\, a_{t-1} + i_t$.

3.4
Borrowing Cost Function

In many models present in the literature, interest on debt acquisition varies linearly with debt, i.e. the interest rate paid is constant, regardless of the financial conditions in which the company is. This simplified approach makes it easy to apply mathematical models, but it is far from the reality of the financial market, where the interest rate for a loan can vary greatly when comparing companies in different financial situations.

According to Valladão *et al.* [34], simplified linear functions are not appropriate to represent the costs involved in raising new debt. To correctly consider different interest rates and credit limits, their paper proposes a convex piecewise-linear borrowing cost function. In their model, the interest rate of the loan ($\kappa_{d,n}$) increases proportionally to the amount of debt (d_t). The Figure 3.1 shows this concept.

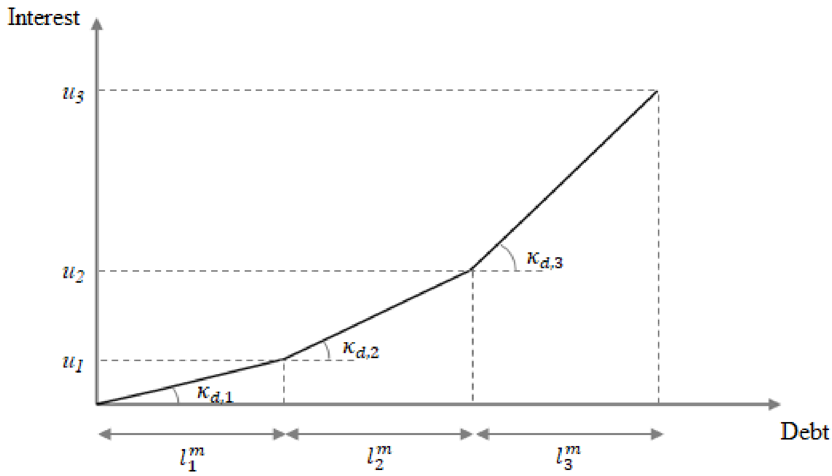

Figure 3.1: Borrowing cost function.

The total value to be paid at the present period (q_t), which is the sum of debt raised at the previous period and the interest, is obtained through an optimization problem. Given that $s_{t,n}$ represents the value of debt captured at each level of risk, limited to the maximum l_n^m (with the exception of the last level), on which there is an interest rate $\kappa_{d,n}$, and considering only debt with maturity equal to one stage (for example, 1 year), the optimization problem is:

$$q_t\left(d_{t-1}\right) = \min_{s_{t,n}} \quad \sum_{n=1}^{N_R} \left(1 + \kappa_{d,n}\right) s_{t,n} \tag{3-11}$$

$$\text{s.t.} \quad s_{t,n} \leq l_n^m, \quad \forall n = 1, \ldots, (N_R - 1) \tag{3-12}$$

$$\sum_{n=1}^{N_R} s_{t,n} = d_{t-1} \tag{3-13}$$

$$s_{t,n} \geq 0, \quad \forall n = 1, \ldots, N_R, \tag{3-14}$$

where N_R is the number of different risk classes. With such formulation, the increase in the total amount of debt implies an increase in the cost of marginal funding.

The direct consequence of this function is that, from a certain amount of debt (defined by the limits l_1^m, l_2^m and l_3^m in Figure 3.1), a company should pay more interest for each 1 \$ of debt raised. However, a company's debt amount is not the best measure of the company's risk. Leverage is a more appropriate measure because it takes into account the ability to generate revenue. Accordingly, Waga [36] proposed a change in constraint (3-12), including operating generational, defined as EBITDA ($g_t\left(a_{t-1}, p_t\right)$). As defined in equation (3-15).

$$s_{t,n} \leq g_t\left(a_{t-1}, p_t\right) l_n, \quad \forall n = 1, \ldots, \left(N_R - 1\right). \tag{3-15}$$

Note that with the change, the generation of the company will act as a multiplier of the original limits proposed by Valladão *et al.* [34]. Thus, we changed the symbol that represents the limits of each level of risk, as these are no longer monetary values and become multipliers of operational generation. The larger the firm's generation, the greater its capability to repay debt, so it is not appropriate to set the limits to absolute values for all firms in the market Thus, the company's risk measure becomes a limit of the debt to EBITDA ratio. This ratio was chosen because of its central role in the ratings of the international agency Standard & Poor's (S&P).

With this assumptions, the total amount of interest paid (u_t) can be calculated as follow:

$$u_t = \sum_{n=1}^{N_R} \kappa_{d,n} s_{t,n}. \tag{3-16}$$

3.5 Operational Generation and Taxes

According to Hennessy & Whited [19], operational generation (g_t) depends on the asset and a shock (z_t). The operating function can be described as strictly concave, twice continuously differentiable and strictly increasing. The profit shock $z_t \in [\underline{z}, \overline{z}]$ corresponds to the uncertainty on demand, prices or productivity. A function with such properties can be described by the following properties and equation:

$$\lim_{a_{t-1} \to \infty} g_t\left(a_{t-1}, z_t\right) = \infty; \quad z_t \in [\underline{z}, \overline{z}] \tag{3-17}$$

$$\lim_{a_{t-1} \to 0} g_t\left(a_{t-1}, z_t\right) = 0; \quad z_t \in [\underline{z}, \overline{z}] \tag{3-18}$$

$$g_t\left(a_{t-1}, z_t\right) = z_t (a_{t-1})^{\gamma}, \tag{3-19}$$

where $\gamma \in (0, 1)$ is the term that provides concavity to the function.

It is important to observe that the premise of positive operational generation is also valid for the model that we will propose in Chapter 4, since, in general, the cost of oil extraction is well below the selling price of this product.

Note that was used a_{t-1}, rather than a_t, since the quantity of the asset responsible for production at time t is defined at $t-1$. The asset a_t will generate the production over $t+1$.

The function in equation (3-19) is not linear. However, it is possible to

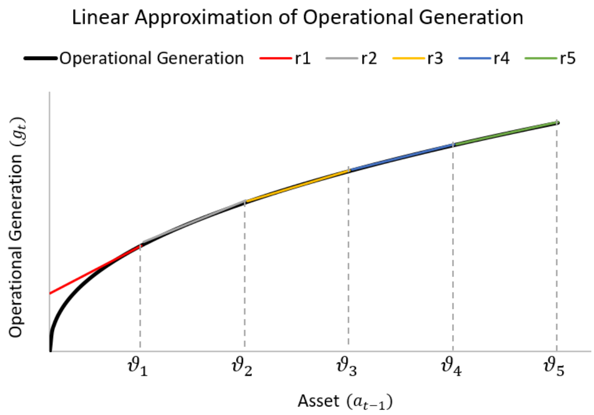

Figure 3.2: Linear approximation of operational genration by tangents lines.

approximate it by tangent lines from an optimization problem. The Figure 3.2 demonstrates the concept, the lines ($r1, r2, r3, r4$ and $r5$) are the tangent lines at points ϑ_n, for $n = 1, 2, 3, 4$ and 5. Notice that the greater the number of defined tangency points, the better is the approximation accuracy. To find the general equation of the tangent lines first is defined an auxiliary function as

$$f_g(a_{t-1}) = (a_{t-1})^{\gamma}. \tag{3-20}$$

The angular coefficients of the tangent lines are:

$$\beta_n = \frac{\partial f_g(a_{t-1})}{\partial a_{t-1}}|_{a_{t-1}=\vartheta_n} = \gamma(\vartheta_n)^{\gamma-1}. \tag{3-21}$$

Finally, the linear coefficients of the tangent lines are:

$$\alpha_n = f_g(\vartheta_n) - \beta_n\vartheta_n = (1-\gamma)(\vartheta_n)^{\gamma}. \tag{3-22}$$

The optimization problem will be a maximization, for which it will be necessary to define an auxiliary variable (y_t),

$$\begin{aligned} \max \quad & y_t \\ \text{s.t.} \quad & y_t \leq \alpha_n + \beta_n a_{t-1}, \quad \forall n = 1, \ldots, N_L \\ & g_t = z_t y_t, \end{aligned} \tag{3-23}$$

where N_L is the number of tangency points.

Operational generation is the basis for calculating the corporate taxable income (o_t), which is equal to operating profits added to the financial profit from the cash ($c_{t-1}r_f$), subtracted from the depreciation (δa_t), interest (u_t), which will be defined later, and the cost of asset sale (m_t).

$$o_t = g_t(a_{t-1}, p_t) + c_{t-1}r_f - \delta a_t(i_t) - u_t - m_t. \tag{3-24}$$

From the above equation, the taxes could be calculated by entering a constraint in the model $x_t \geq \tau o_t$, where $x_t \geq 0$ and τ is the corporate tax rate. That is, there is only taxes payment when the company has a profit.

4
Proposed Model

In this chapter, we propose a dynamic stochastic programming model that jointly optimizes derivatives management strategies along with investment, financing and dividend decisions for a commodity-producing company. From the concept of integrated analysis between liquidity and derivatives of Bolton *et al.* [8] and based on the capital structure model developed by Waga [36], we incorporate the possibility of trading forward contracts, allowing a broader analysis of the companies' financing policy.

In this work, the objective is to maximize the risk-adjusted shareholder value of a company producing a commodity whose price uncertainty can be hedged with forward contracts, although these instruments can also be used for speculative purposes, seeking greater gains for shareholders. For clarity purposes, we present the a compact representation of the proposed model by defining the dynamic equations

$$Q_t^j\left(\upsilon_{t-1}, p_t\right) = \max_{(e_t,\upsilon_t)\in\mathcal{X}(\upsilon_{t-1},p_t)} \sum_{k\in\mathcal{K}} \left\{ e_t + \frac{\psi_t\left[Q_{t+1}^k\left(\upsilon_t, p_{t+1}\right)|K_{t+1}=k\right]}{1+r_f} \right\} P_{k|j}. \quad (4\text{-}1)$$

where

$$Q_{T+1}^j\left(\upsilon_{T-1}, p_T\right) = 0. \quad (4\text{-}2)$$

The model objective is to maximize the risk-adjusted present-valued dividend payments (e_t). A time-consistent dynamic risk measure (ψ_t) is defined as the recursive formulation of the convex combination between expected value and CVaR (see Shapiro [28]), and the payments are discounted by the risk-free rate (r_f). The risk-adjusted shareholder value is a function of the previous state vector (υ_{t-1}) and the uncertainty realization (price p_t). The state of system $\upsilon_t = (a_t, c_t, d_t, h_t, \iota_t)^\top$ includes the derivatives contracted (h_t), asset value (a_t), cash holdings (c_t) and debt raised (d_t), as well as other auxiliary variables (represented generically by ι_t). The variable $\mathcal{X}$ denotes the feasible set as a consequence of investment (i_t), debt (d_t), derivatives (h_t) and dividend payment (e_t) decisions. The indexes j and k represent, respectively, the current and future states of a Markovian process. For the last stage, in addition to the end of recursion due to equation (4-2), the company is not able to raise debt or have operational generation or contract derivatives.

The complete optimization problem is presented throughout the equations (4-3) - (4-20) and the sections 4.1, 4.2, 4.3 and 4.4 detail changes to the Waga's model [36].

$$Q_t^j\left(a_{t-1}, c_{t-1}, d_{t-1}, h_{t-1}, p_t\right) =$$

$$\max_{\substack{a_t,c_t,d_t,h_t,u_t,i_t,g_t,y_t,\\ q_t,o_t,x,s_{n,t},e_t,m_t,b_t}} \sum_{k\in\mathcal{K}} \left\{ e_t + \frac{\psi_t\left[Q_{t+1}^k\left(a_t, c_t, d_t, h_t, p_{t+1}\right)|K_{t+1}=k\right]}{1+r_f} \right\} P_{k|j} \quad (4\text{-}3)$$

s.t.

$$c_t = c_{t-1}\left(1+r_f\right) + g_t\left(a_{t-1}, p_t\right) + d_t - q_t\left(d_{t-1}\right) - \left(i_t + m_t\right) - \left(e_t + b_t\right)$$

$$- x_t + h_{t-1}\left(p_t - \mathrm{f}_{t-1}\right) \quad (4\text{-}4)$$

$$m_t \geq -\kappa_i i_t \quad (4\text{-}5)$$

$$b_t \geq -\kappa_e e_t \quad (4\text{-}6)$$

$$a_t\left(i_t\right) = \left(1-\delta\right) a_{t-1} + i_t \quad (4\text{-}7)$$

$$o_t = g_t\left(a_{t-1}, p_t\right) + c_{t-1} r_f - \delta a_t\left(i_t\right) - u_t - m_t + h_{t-1}\left(p_t - \mathrm{f}_{t-1}\right) \quad (4\text{-}8)$$

$$x_t \geq \tau o_t \quad (4\text{-}9)$$

$$q_t\left(d_{t-1}\right) = \sum_{n=1}^{N_R} (1+\kappa_n) s_{t,n} \quad (4\text{-}10)$$

$$\sum_{n=1}^{N_R} s_{t,n} = d_{t-1} \quad (4\text{-}11)$$

$$s_{t,n} \leq \mathbb{E}\left[g_t\left(a_{t-1}, \tilde{p}_t\right)|K_t = j\right] l_n, \quad \forall n = 1, \ldots, (N_R - 1) \quad (4\text{-}12)$$

$$u_t = \sum_{n=1}^{N_R} \kappa_n s_{t,n} \quad (4\text{-}13)$$

$$y_t \leq \alpha_n + \beta_n a_{t-1}, \quad \forall n = 1, \ldots, N_L \quad (4\text{-}14)$$

$$g_t\left(a_{t-1}, p_t\right) = \frac{p_t^j}{\overline{p}} y_t \quad (4\text{-}15)$$

$$e_t \leq c_{t-1}\left(1+r_f\right) + g_t\left(a_{t-1}, p_t\right) \quad (4\text{-}16)$$

$$h_t \leq \eta_t \quad (4\text{-}17)$$

$$h_t \geq -\eta_t \quad (4\text{-}18)$$

$$\eta_t = \frac{y_t}{\overline{p}\theta} \quad (4\text{-}19)$$

$$a_t, c_t, d_t, x_t, s_{t,n}, u_t, y_t, m_t, b_t \geq 0. \quad (4\text{-}20)$$

Where $\psi_t\left[Q_{t+1}^k\right] = (1-\lambda)\,\mathbb{E}_t\left[Q_{t+1}^k\right] + \lambda\phi_{\alpha,t}\left[Q_{t+1}^k\right]$ and $0 \leq \lambda \leq 1$ is the risk averse parameter.

The constraints (4-15), (4-16), (4-17), (4-18) and (4-19) will be detailed in the following sections.

4.1 Cash balance and derivatives management policy

Although we used the Waga [36]'s model as a base, some modifications are necessary to incorporate derivatives management strategies via forward contracts. The Figure 4.1 presents the variables through their flow in time assumed in this work, for a better understanding of the moment in which the decisions are made. The variable f_t^h (from equation (2-7), $f_t^h = h_{t-1}\,(p_t - \mathrm{f}_{t-1})$) is the financial impact of the derivative (it is worth noting that the model assumes the use of forward contracts with maturity equal to 1 stage). It can be observed that the decisions made in t are made immediately before t, that is, the consequences of these decisions impact the period $t+1$.

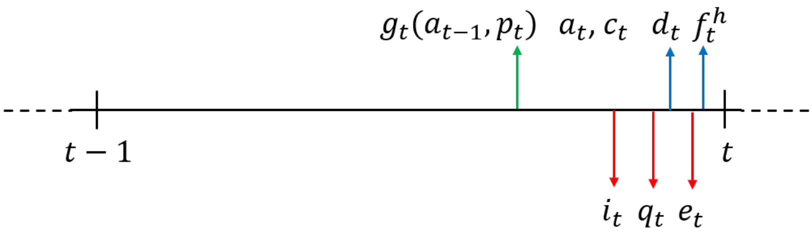

Figure 4.1: Dynamics of payments and receipts flows over period t.

As can be seen from equation (4-1), the variable h_t was inserted to represent the number of derivative contracts that were traded by the company at t. This cash flow (f_t^h), received or lost, contributes to increase or reduce the cash at the end of stage t and must be summed up on original cash balance equation (3-5), resulting in equation (4-4). The costs with asset sales and share issuing are represented in the equations (4-5) and (4-6), respectively. Equation (4-7) refers to the updated asset given a certain invested amount (see section (3.3)).

The cash flow from derivatives also impacts the corporate taxable income (o_t) calculation, so equation (3-24) must also be changed, resulting in equation (4-8). The taxes are presented in the equation (4-9).

4.2 Debt and Operational Generation Changes

In this work, the uncertain variable p_t denote the price of a commodity which is a Markovian stochastic process characterized by: (i) a transition probability matrix; (ii) given each Markov state, we estimate a different conditional probability distribution for the commodity price (see details in the section 4.4). Given this assumption, changes to some parameters of the base model are required. First, a company's operating inflow is directly related to the prices of products sold. As the price is a random variable, it is not possible to know for sure how much the company will generate in the next stage. This is an issue for calculating interest to be paid at the forward stage when raising a debt at t. Our modeling choice is to use the expected amount of operation inflow to calculate the interest payable on a debt. This solution is in line with market practices, as rating agencies, as well as financial institutions, project the company's financial condition so that it is possible to assess the rating or viability of a loan.

Thus, for better representation of the equations that define debt raising in the model, an additional change in constraint (3-15), which defines the value allocated at each risk level, is required. The new constraint is:

$$s_{t,n} \leq \mathbb{E}\left[g_t\left(a_{t-1}, \tilde{p}_t\right)|K_t\right] l_n, \quad \forall n = 1, \ldots, \left(N_R - 1\right). \tag{4-21}$$

Note again that in equation (4-21) the limits of each portion of the debt ($s_{t,n}$), which will be associated with increasing interest rates, are proportional to the operational generation. That means companies with greater productive capacity can raise larger amounts of debt with lower interest rates, which is also in line with market practices.

Choosing the uncertain variable as price will also impact the equations drawn from Hennessy & Whited [19]'s paper. Since they defined their uncertain variable (z_t, see section 3.5) as a profit shock, it is necessary to define what will be the shock to our problem, so:

$$z_t = \frac{p_t}{\overline{p}}, \tag{4-22}$$

where $\overline{p}$ is the unconditional expected value of prices.

As seen in section 3.4, the correct value to be paid for the debt (principal plus interest) can be obtained in the form of a linear optimization problem. This optimization problem is a minimization, so the amount to be paid of debt (q_t) is the minimum possible. The equation (4-4) shows the lower value for q_t implies a higher value for e_t, whose objective is to be maximized. So we can conclude the two optimization problems ((3-11) value to be paid for debt and (4-1) dividends to shareholders) have similar objectives. Therefore, the optimization problem of the variable q_t can be inserted into the optimization problem of dividends to shareholders, becoming constraints of the model, which are presented in equations (4-10), (4-11) and (4-12). To complete the constraints on debt acquisition, the definition of the amount paid in interest (u_t), presented in section 3.4, is in constraint (4-13).

With a similar analysis of the debt optimization problem, it is concluded that the optimization problem that defines the operational generation function (equation (3-23)) can also be added to the original problem. As it is a problem of maximization, the increase of the auxiliary variable y_t, increases the operational generation g_t, which in turn, through equation (4-4), implies in increasing the amount paid in dividends. That is, the two problems also have similar objectives. So we have added the two constraints (4-14) and (4-15).

4.3 Derivatives and dividend constraints

Besides the constraints already described above and detailed throughout the last sections, the final model has four additional constraints ((4-16) - (4-19)). These constraints were necessary for operational reasons of the optimization model simulation. Without them, the results obtained were extremely abnormal (for example, all variables were null up to stage $T-2$, from which very high debts are raised for the payment of dividends in $T-1$) or even obtaining unbounded models. Regardless of optimization issues, the payment of dividends could be made with resources from various sources that the model provides (debt, sale of assets or gains on derivatives), however, the market generally tends to criticize companies that, for example, raise debt for the sole purpose of maintaining their dividend payout level. Thus, we limit the payment of dividends to the company's ability to generate profits from their production, as represented in the equation (4-16).

As we are working with a commodity producer, the number of derivative contracts sold will be limited to to 100% of the quantity produced (η_t), that is, the company will be able to protect its entire exposure, depending on its hedging strategy. As for the long position in the contracts, this will also be limited to the company's total production. Strictly speaking, as we are allowing speculative strategies, it would not be necessary to impose the limits established for the number of contracts (note that even for a production company that sells forward contracts, if the number sold exceeds the total produced, this operation is also speculative). However, as explained in the previous paragraph for the constraint in the payment of dividends, not defining any limitation for derivatives trading implies abnormal results, with a very difficult interpretation. Therefore, the constraints (4-17) and (4-18) were added.

To calculate the quantity produced, we used a widely known ratio in the financial market, named as EBITDA margin (θ, defined as EBITDA divided by revenue). With our work assumptions, this ratio is equal to $g_t/p_t\eta_t$. Using the equation (4-15), the produced quantity it is represented by the equation (4-19). According to information provided by Damodaran [11], for oil companies, this ratio has an average value equal to 25%.

4.4
A Hidden Markov Model for commodity price process

In this section, we will present the basic concept of the technique known as Hidden Markov Model (HMM), which allows the parameters of a Markov chain to be calculated from a time series. Despite the use of HMM, in this case, the chain will not be exactly hidden due to the computational complexity of this approach. Thus, HMM serves as a tool for calculating chain parameters.

The HMM method is widely used when one has a time series. The basic structure of the HMM has two random variables (K and ξ). Let $K_1, K_2, \ldots, K_n \in \mathcal{K} = \{1, \ldots, N_s\}$, called hidden variables that follow a Markov process, and $\xi_1, \xi_2, \ldots, \xi_n \in \Xi$, where Ξ could be a set of discrete, continuous or n-dimension variables, which are the observable variables. These random variables are related through the graphical model presented in Figure 4.2, known as trellis diagram.

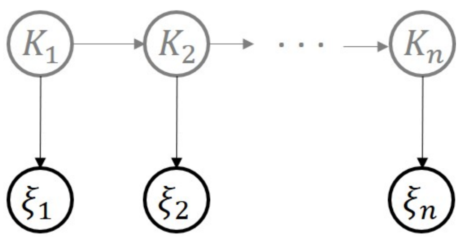

Figure 4.2: Graphic representation of Hidden Markov Model.

It is important to note that the observation of the diagram leads us to conclude that K_2 does not depend on ξ_1, only on K_1. The mathematical model that represents the diagram of Figure 4.2 is given by the joint distribution equation below:

$$\mathbb{P}\left(\xi_1, \ldots, \xi_n, K_1, \ldots, K_n\right) = \mathbb{P}\left(K_1\right) \mathbb{P}\left(\xi_1 | K_1\right) \prod_{t=2}^{n} \mathbb{P}\left(K_t | K_{t-1}\right) \mathbb{P}\left(\xi_t | K_t\right), \quad (4\text{-}23)$$

where $\mathbb{P}$ represents the probability density function of a random variable.

From equation (4-23) it is possible to observe the parameters that define the HMM: the transition probability $P_{j|k} = \mathbb{P}\left(K_t = k | K_{t-1} = j\right)$ and the density functions $\mathbb{P}\left(\xi_t | K_t\right)$ of the uncertainty variable. The transition probability $P_{j|k} = \mathbb{P}\left(K_t = k | K_{t-1} = j\right)$ is defined for all $j, k \in \mathcal{K}$ and can be represented by a matrix $N_s \times N_s$, called the transition matrix. One highlight of the transition matrix is that the sum of the elements of its rows must equal

1 and there can be no negative elements. The density functions $\mathbb{P}(\xi_t|K_t)$, known as emission probabilities, are the distribution of the uncertain variable conditioned to the state of the Markov chain. $\mathbb{P}(K_1)$ is the initial distribution.

Note that, from equation (4-23), it is possible to simulate new realizations for the random variables K and ξ that describe the process observed throughout ξ series. However, it is necessary to establish the correct values of the parameters described in the previous paragraph. In practice, such parameters are defined previously (for example, using a Gaussian distribution for the probabilities of emission, in case the observed data are real numbers) and then an estimation algorithm is applied to make the best fit to the observed data. Among these, one of the most known is the Baum-Welch's algorithm [4], which can be seen as an expectation-maximization algorithm applied to HMM.

4.4.1 Commodity Pricing

Our case study in Section 5, the uncertainty (ξ_t) is associated with the Brent oil price (p_t). In the literature, many papers are dedicated to the development of the best model to represent commodity prices, among which stand out the random walk and the mean reversion process. The purpose of this dissertation is not to discuss the price models. This, it is assumed that commodity price follows a mean reversion process, as in Schwartz [29] (considering constant convenience yield) and Schwartz & Smith [30] (short term).

For computational tractability, the price follows a Markov chained process of discrete states $\mathcal{K}$, in which the probability distribution of p_{t+1} given the state of Markov K_t at t does not depend on the past prices $p_t, \ldots, p_1$, only the state K_t. Thus, we are following the concept of Hidden Markov Models as explained in section 4.4. A similar approach was used by Valladão *et al.* [35], but applied to an optimal portfolio allocation problem.

Given a state, there is a probability distribution for the price, unlike what was assumed in item 2.3, where the uncertain variable has only a single value for each state. Besides, the probability distribution of p_t and p_{t+1}, conditioned to the same state j, are identical. That is, the distributions depend only on the Markov state and do not vary over time.

Figure 4.3 shows the transition probability between the different states of the Markov chain (K_r, K_s, $K_g \in \mathcal{K}$). It stands out that the probability values do not vary over time. In the present model, these states can be interpreted as scenarios of the financial market (recession, stability or growth, for instance). There is no temporal relationship for the probabilities, which means that these

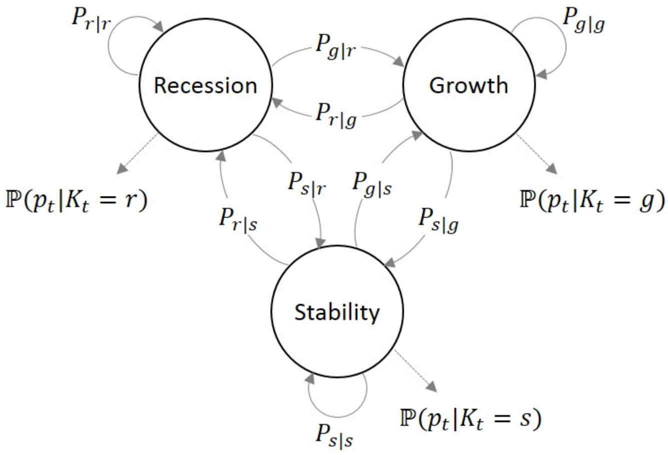

Figure 4.3: Transition probability between states of the Markov chain.

are constants for any stage t. In the figure, $\mathbb{P}(p_t|K_t = j)$ corresponds to the probability distribution of the price p_t conditioned to each state j. Besides that, consider that the notation $P_{k|j}$, where $j, k \in \mathcal{K}$, represents the probability that the Markov state in the next stage K_{t+1} is equal to k, since the current state (K_t) is j.

5
Case Study

Considering the model described in the previous section, a case study is presented, based on an application example of a company focused on oil exploration, that is, it does not refine its oil, being necessary to sell it in the market at international prices. The essential parameters for the model simulation will be presented throughout the chapter, highlighting the cost of raising debt and the selling price of crude oil, which was assumed to be the Brent oil.

The first discussion of the results addresses a view of the value added by using derivatives, comparing cases with and without available derivative instruments. In the following section, the optimal derivative management policy will be studied in detail to identify the variables that most contribute to the results of buy/sell forward instruments.

5.1
HMM estimation for Oil prices

As mentioned in section 4.4.1, the commodity price will be simulated using HMM. Even though a very long data history is available, the latest information better reflects the trading characteristics of this commodity, especially regarding the liquidity of the operations. In fact, one of the factors that contributed to the increase in liquidity was the consolidation of derivative instruments in the last 20 years, which allow almost all the agents inserted in the global financial market to trade oil. The oldest price information reflects transactions carried out only among oil companies. In addition, as the data are in nominal value, the monetary update of these values, which is not a trivial task, would have very relevant impacts on the oldest data and can be neglected for the most recent ones. Thus, the history of the last 20 years of Brent crude oil prices was considered (the chart on the left side of Figure 5.1).

With the HMM algorithm, it is possible to obtain the transition probabilities between the states and the probability distributions of the prices conditioned to each state. In this case study, we assume the presence of three states (*recession*, *stability*, and *growth*), as in the example of Figure 4.3. Since the series is a price sample, it is convenient to analyze the logarithm of prices to

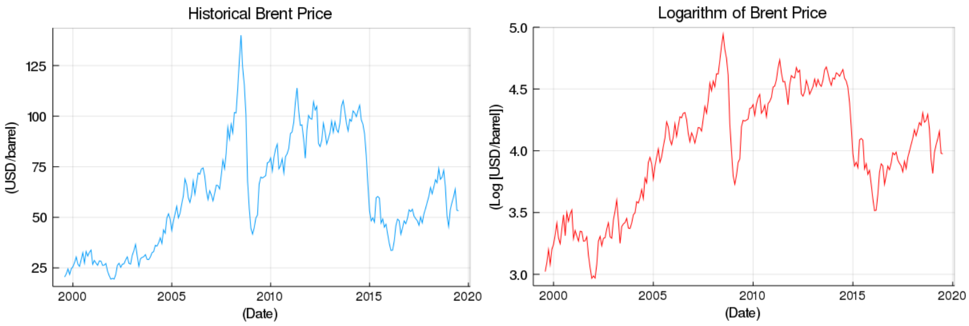

Figure 5.1: Historical monthly brent prices series.

avoid negative prices (the chart on the right side of Figure 5.1).

The series used presents monthly frequency and the present study is based on an annual analysis. With a 20-year history, the available series has only 20 annual price data, which is insufficient for the proper application of HMM in the estimation of Markov chain parameters. However, we have 240 monthly price data. Therefore, first, we carry out the monthly data forecasting with 12000 points, using the HMM. From this newly estimated series, we sampled every 12 months, obtaining 1000 points from a synthetic annual series. This synthetic series was used as a basis to obtain, using the HMM, the necessary Markov chain parameters. The result is presented by equations (5-1) and (5-2) which show the transition matrix between states and parameters of the conditioned distributions, respectively. A histogram of the prices distributions conditioned to each Markov state is shown in Figure 5.2. Two examples (simulation 01 and 02) of a Markov chain scenario for the period of 60 years is presented in Figure 5.3, in this graph, beyond average, there are the percentiles 10%, 25%, 75% and 90%, of the Brent series.

$$\begin{bmatrix} P_{r|r} & P_{r|s} & P_{r|g} \\ P_{s|r} & P_{s|s} & P_{s|g} \\ P_{g|r} & P_{g|s} & P_{g|g} \end{bmatrix} = \begin{bmatrix} 0.63 & 0.34 & 0.03 \\ 0.12 & 0.72 & 0.16 \\ 0.02 & 0.25 & 0.73 \end{bmatrix}, \tag{5-1}$$

$$\begin{aligned} \mathbb{P}\left(\log(p_t)|K_t = r\right) &\sim N\left(\mu = 3.34\ ,\ \sigma = 0.18\right) \\ \mathbb{P}\left(\log(p_t)|K_t = s\right) &\sim N\left(\mu = 4.02\ ,\ \sigma = 0.17\right) \\ \mathbb{P}\left(\log(p_t)|K_t = g\right) &\sim N\left(\mu = 4.55\ ,\ \sigma = 0.13\right). \end{aligned} \tag{5-2}$$

Instead of performing the procedure described above, we could simply have analyzed the annual historical series, but since we are considering only 20 years from the past, the series would have only 20 sample points, being insufficient for the analysis.

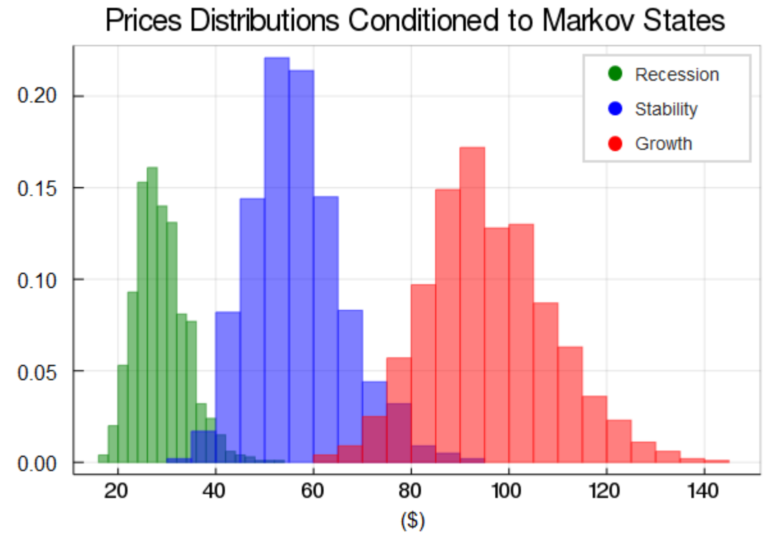

Figure 5.2: Histogram of the prices distributions conditioned to Markov states.

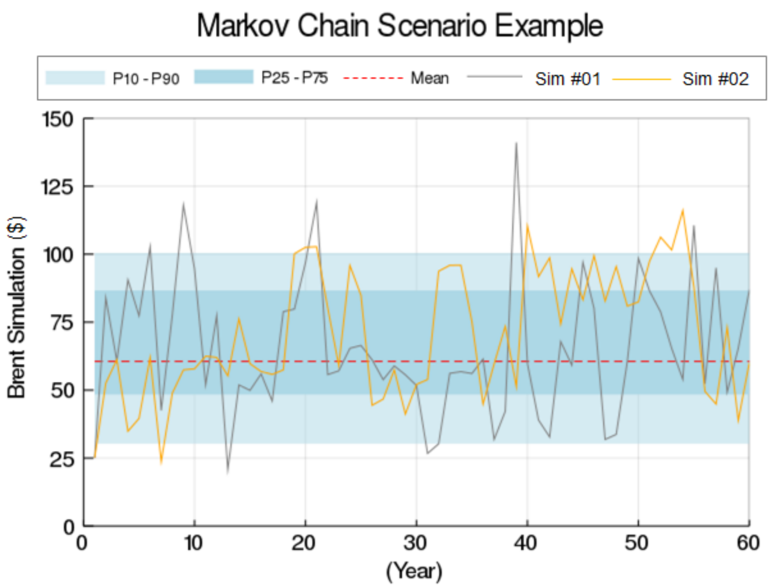

Figure 5.3: Examples of a Markov chain scenario.

5.2 Debt cost estimates

To complete the correspondence between risk and the effective debt cost, the data provided by Damodaran [11] will be applied. The data map S&P's rating of debt securities issued by companies and the average spread over the US government bond rates for the same maturity.

Note that the table presents six "risk classes" (rating column), which corresponds to the value of the variable N_R in the equations introduced in item 3.4. The table describes the spread over the risk-free rate.

At first stage, the company has no operational generation, assets, cash or equity. Therefore, to ensure that there is no debt too, there is a restriction that

Rating	Average (D/EBITDA)	spread
AAA/AA	< 1.5	0.7%
A	< 2.0	1.1%
BBB	< 3.0	1.6%
BB	< 4.0	3.0%
B	< 5.0	4.5%
Default	≥ 5.0	15.0%

Table 5.1: Relation among spread, leverage e rating.

its cost of funding in the first moment is quite high. The idea of "forbidding" debts at the first stage comes from the observation of companies known as start-ups, which first seek investors before issuing debts. The restriction could even be explicitly written in the model, but, a high cost of funding has the same result. Thus, for $t = 1$, $\kappa_n = 15\%, \quad \forall n = 1, 2, \ldots, N_R$.

5.2.1 Basic Parameters

In order to be possible the simulation of the model, it is necessary to define the parameters below:

Parameter	Symbol	Value
Risk free rate	r_f	2.5% p.a.
Tax rate	τ	30%
Depreciation rate	δ	14.5% p.a.
EBITDA margin	θ	25%
Initial cash	c_0	0.0
Initial asset	a_0	0.0
Initial debt	d_0	0.0
Total stages	T	60

Table 5.2: Basic parameter values.

The percentage costs, associated with the sale of assets and issuance of shares, are the same as in Waga [36] ($\kappa_i = 25\%$ and $\kappa_e = 5\%$) and, for the operational generation function (as in equation (3-19)), the same parameter of the Hennessy & Whited's work [19] is used, so $\gamma = 0.689$.

For the function approximation (as in equation (3-23)), seventeen tangency points ($N_L = 17$) were chosen, ϑ_n =[0.01, 1.0, 1.5, 2, 3, 5, 10, 15, 20, 30, 40, 50, 60, 100, 150, 200, 500].

The non-conditional average price $\overline{p}$ is calculated from the price samples made for all states, without taking into account the transition matrix (5-1). As a result, we find $\overline{p} = 60.6$ (constant for all stages).

As for the states of the Markov chain, the SDDP algorithm (Dowson & Kapelevich [14]) allows choosing the state of the first stage that will be kept constant in all scenarios of the Markov chain. For the case study, we define that the first stage is the recession state.

In the case study, each stage represents 1 year, so the forward contract has a maturity equal to 1 year, with no possibility of contracts with shorter terms.

5.3 Results

For the case study, several experiments were simulated which will be explained along the following sections. For each case, 5,000 iterations of the SDDP algorithm (Dowson & Kapelevich [14]) were run, with a maximum execution time limit of 24 hours. For cases without using a derivative instrument, the 5,000 iterations were reached after about 12 hours of simulation. For cases with derivative, after 24 hours, they reached about 4,000 iterations of the algorithm. The computer used was an Intel Core i7 1.8 GHz, 16 Gb RAM. From the optimal policy calculated by the algorithm, 1,000 Markov chain scenarios were simulated for observation and analysis of the results.

In all graphs displayed in this section, the last 10 stages are not being displayed, so as not to contaminate the analysis with possible end-effects.

5.3.1 Value of derivative management

Throughout the introduction, we have seen that many works in the literature are dedicated to studying if the use of derivative instruments can increase the value of a company. In this section, we will present the results focusing on the increase in value provided by the use of derivatives. It is worth remembering that in this study, the value of the company is measured by the sum of the dividends paid to shareholders, adjusted for risk. The optimal value of the objective function occurs when the lower and upper bounds are equal (given some margin of tolerance). However, the lower bound is statistical (see Appendix A), so for the purpose of displaying results, the upper bound will be used as the company's value.

To analyze the company's value increase, the model was simulated with and without the possibility of contracting derivatives, each one evaluated under three different levels of risk aversion ($\lambda = 0.00, 0.50$ and 0.99, see equation (3-4)). The results are presented in Table 5.3.

Risk Aversion (λ)	Value **with** derivatives ($)	Value **without** derivatives ($)	Value increasing (%)
0.00	716.33	155.15	361%
0.50	145.94	23.68	516%
0.99	8.94	4.48	99%

Table 5.3: Value increasing by the use of derivatives.

Simulations performed with different levels of risk aversion have distinct objective functions (see equations (3-4) and (4-1)). Therefore a more appropriate analysis of Table 5.3 is to compare the differences between the columns (with and without derivatives) along the same row. Thus, it can be observed that there is a significant difference in the simulations in which derivative instruments were available for use, with an increase in company value in all cases, reaching 516% for $\lambda = 0.50$.

A closer look at the table allows us to infer two possible interpretations for its reading. A manager with a certain risk aversion (observe a line in the table) concludes that using derivatives instruments increases shareholder value. On the other hand, when considering two managers with different risk aversions, i.e., same column and different lines in the table, they see that the company has different risk-adjusted values.

In addition to the value measured by the upper bound of the objective functions presented in Table 5.3, the value obtained by using derivatives can be observed by analyzing the main decision variables of the optimization model. Thus, to better analyze the decisions of the model and their impacts on the company's value, the following will be presented some area charts, in which the regions comprising the interval between the 10% and 90% percentiles of the simulations (lighter region), as well as the interval between the 25% and 75% percentiles will be displayed (darker region), in addition to the 50% percentile (continuous line) and expected value (dashed line).

The graphs in Figures 5.4 and 5.5 have the same risk aversion ($\lambda = 0.5$) but differ by the use of derivatives, in the first figure the use is allowed, in the second, the instrument is not available. The graphs show the decisions for the following variables: assets, debts raised, dividends paid and hedge ratio. The latter is equal to the quantity of acquired derivative contracts divided by the entire production of the company (see equation (4-19)). As explained in section 2.2, the company may assume the short or long position in derivative contracts.

In addition to the value comparison presented in Table 5.3, which can also be seen by the higher dividend amounts paid in the company that uses forward

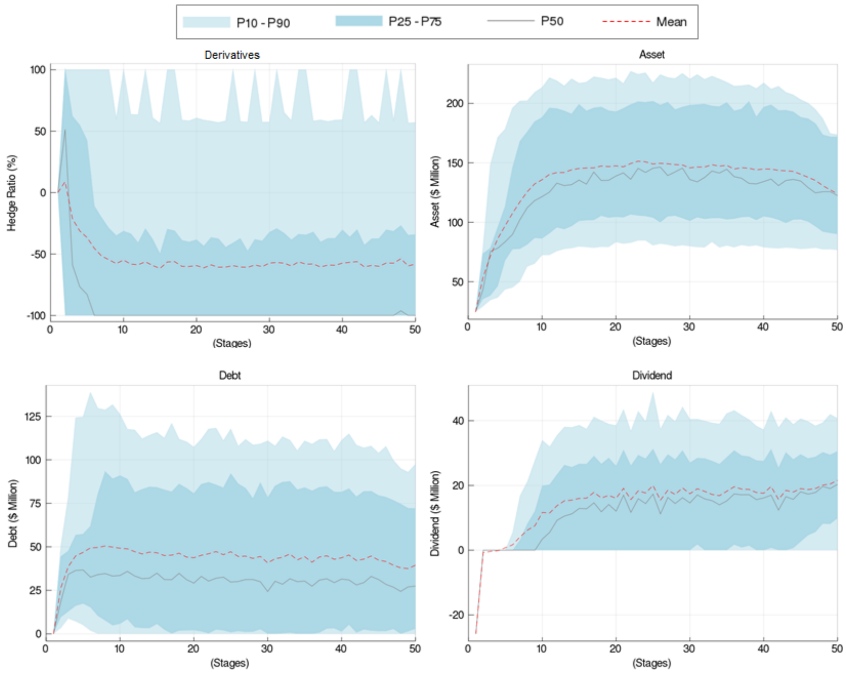

Figure 5.4: Percentiles graphs of asset, debt, dividend and derivatives contracts. Case with derivatives available and $\lambda = 0.5$.

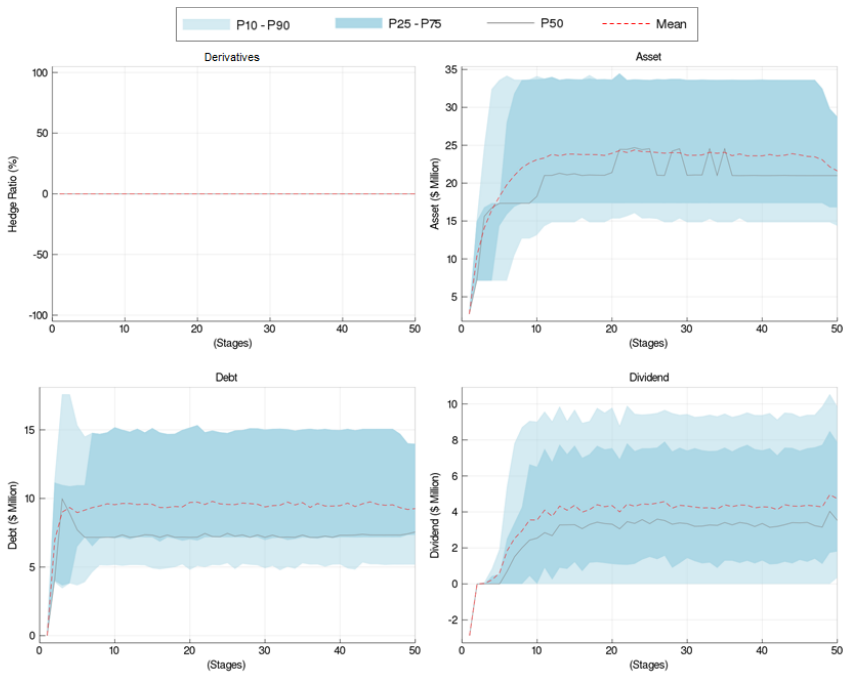

Figure 5.5: Percentiles graphs of asset, debt, dividend and derivatives contracts. Case with no derivative available and $\lambda = 0.5$.

contracts (Figure 5.4), one can observe the significant effect that derivative instruments have on other decision variables of model. The company that uses derivative contracts has a size, measured by the amount of assets, much larger than the company represented in Figure 5.5. Even the curve P10 of the company that uses forward contracts is larger than the P90 of the company that does not have access to this instrument. Another variable that stands out in the order of magnitude is debt. The company that uses derivatives has more assets, which leads to higher production and, consequently, higher operational generation (EBITDA), so it is able to acquire more debt without paying excessive interest, taking advantage of the tax shield described in item 2.1.

Observing the graphs in Figure 5.4, one can also conclude interesting points of the optimal policies for the main variables. In general, the percentile and average curves show stable behavior, suggesting that convergence was obtained in the calculation of the optimal policy. Assets grow steadily until stage 10, from which the stabilization movement begins, resulting in a company whose size ranges from approximately 75 to 220 ($ million). This growth and stabilization of the asset implies similar debt behavior, as the size of the asset directly influences the operational generation (item 3.5), which in turn impacts the interest to be paid for the debt raised (section 3.4). Another important point to note from the debt chart is that, in all policy simulations, first-stage debt is zero due to the high initial interest rate (as detailed in item 5.2).

A direct consequence of the debt analysis made in the previous paragraph is represented in the dividend chart. In all simulations, the dividend in the first stage is negative (issuance of shares) as it is the cheapest way to obtain funds at this time. Also, there is usually no dividend payment until stage 5, indicating that the firm first grows to a point beyond which it is sufficient to begin distributing the earnings. This behavior is in line with a general corporate in the market. Once the payment of dividends to the shareholder begins, it presents a steady small growth and there are no negative values after the initial investor contribution.

Regarding the derivatives management policy, it is observed that, in most simulations, the decision was to sell the forward contract (negative values in the chart, short position). As seen in section 2.2, the seller of this type of contract benefits from the price fall in the stage ahead. In the next section, it will be possible to identify that the average is not good for derivatives management policy analysis. This analysis is more appropriate when looking at each simulated scenario instead of all the scenarios together.

5.3.2
Derivatives management policy

Given the model developed in chapter 4 and the parameters defined at the beginning of chapter 5, the purpose of this section is to study in detail the optimal policy to “translate” the policy into some simple guidelines, i.e., identify which variables or relationships among them most influence the policy, making it easy to understand and apply.

The method used to carefully study how the variables influence or not the optimal policy was to evaluate the number of derivatives traded for different realizations of the studied variable, keeping the other variables constant, thus isolating the effect of each variable of the problem in the optimal decision for derivatives.

The analysis is focused on the model in which forward contracts are available for purchase or sale and the risk aversion level is equal to 0.5. It stands out that, for the variables that will remain constant during the analyzes, the chosen values are always equal to their respective values in the 50% percentiles (P50) of the simulations. Also, it was decided to study the policy at stage 30, as it was considered that, at this intermediate stage, the policy is stabilized, without suffering the differential effects of the early and late stages.

The first variable used to analyze the optimal policy was the exogenous variable that represents the price of the product (p_t). With this initial assessment, it is possible to identify the fundamental characteristics of the optimal policy, the graph in Figure 5.6 presents the results. To construct this graph, keeping all other variables constant, prices at stage 30 (x-axis) were varied, observing the results that the optimal policy provided for the number of derivatives traded (y-axis), which was measured as the number of contracts bought (positive values) or sold (negative values).

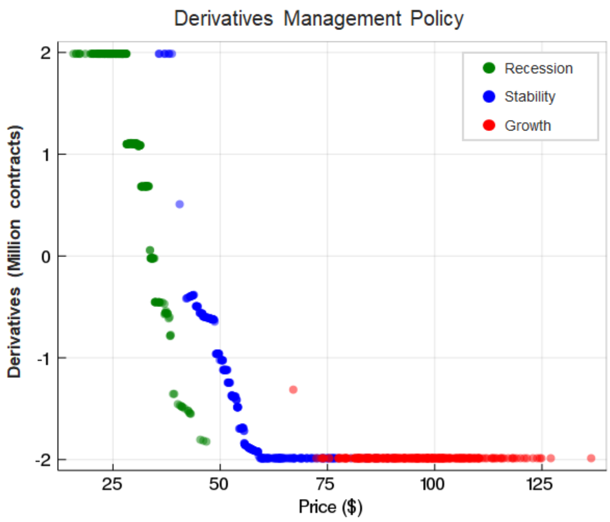

Figure 5.6: Influence of price changes in number of contracts traded.

Observing Figure 5.6, it can be seen that the *recession* state has mostly (approximately 82%) a positive number of derivatives (long position), which means that the company would benefit from a price increase at the forward stage. This behavior is very consistent with the price dynamics that were introduced in the model (section 5.1), because the *recession* state is the one with the lowest price realizations, so it is expected that a higher probability of price increase to the next stage. In particular, when price values are very low (below about 28 $), the derivatives management policy is very well established, buying as many contracts as possible (around 2 million). In terms of hedge ratio, this means that the amount purchased from the derivative instrument equals all the amount that will be produced at the current stage, i.e., the hedge ratio is equal to 100%. As there is a price increase (greater than 28 $), the company will buy fewer contracts until it starts selling contracts as it begins to protect itself for events where the price may fall.

For the *stability* state, the interpretation is similar to the *recession* state, but contract buying occurs only for very low price realizations for this state. Remember that for each state there is a different probability distribution of prices, so a high price in one state may not be a high price for a different state. From a slight price increase, still in the *stability* state, the company is already beginning to position itself for a fall in prices, increasingly protecting a larger portion of production (increasingly negative hedge ratio), until it protects all the amount produced. For the *growth* state, which has the highest price realizations, the number of derivatives traded is practically all negative, protecting 100% of the company's production for a possible price drop in the next stage.

In the previous paragraphs, the analysis was divided for each state to make the explanation easier. However, note that while price has a major influence on derivatives management policy, the state is also important. For example, assume the price is 40 $. In this case, looking at Figure 5.6, it can be concluded that the number of contracts can be very negative if the present state is the *recession* or slightly negative if the state is the *stability*. This is mainly due to the transition matrix (5-1) and state-conditioned price distributions (5-2). If the *recession* state is in effect, the transition matrix reports that is likely to remain in that state in the next stage, but for the *recession* state, a price in the 40 $ range is already a high price, i.e., the state is likely to continue and the next price realization will be lower than the current one, so the policy indicates the sale of the derivative, so that the company benefits from the likely drop in price. Considering that the *stability* state is in effect, despite the likelihood that it will remain is high, in this state price realizations are similarly distributed between values greater or less than 40 $, guiding to, approximately, no use of derivatives because the price may rise or fall with similar probability.

As seen, the price, as expected, has a very high influence on the number of forward contracts traded by the company. However, one should investigate whether other variables interfere the optimal decision. The following discussion goes further in assessing the impact that asset has on the derivatives management. For other decision variables present in the model, no direct influence on the optimal derivatives management policy was identified. Given such impact of state and price, the analysis will be segmented into states and observed at different price levels for a more complete interpretation of the policy.

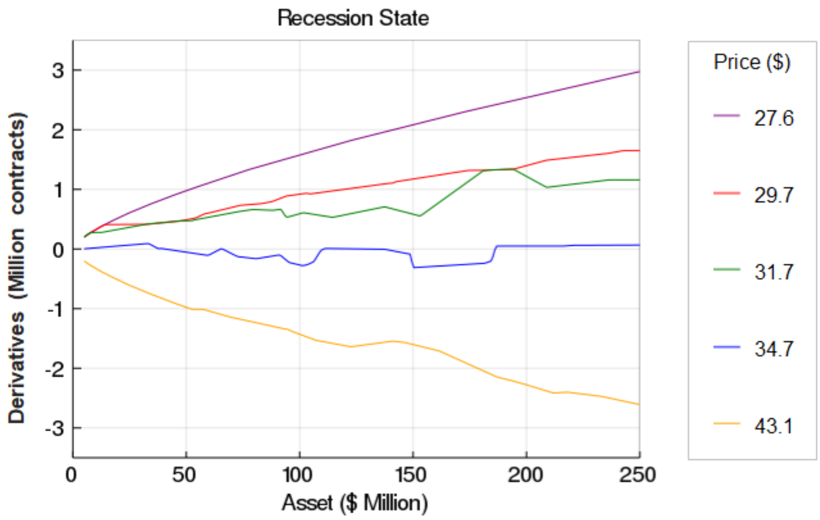

Figure 5.7: Derivatives management policy analysis for asset variations. *Recession* state.

Observing firstly the *recession* state, and remembering that the other variables in the model were kept constant in each analysis, Figure 5.7 shows the assessment of the asset variable. Each curve of the graph represents a test for each different price realization within the *recession* state, whose values presented in the legend.

For the price of 27.6 $, the policy is very clear, increasing the amount of assets, the number of contracts to be bought increases in the same proportion. However, the line is not straight (remember that the quantity produced is not a linear function of the asset, see equations (3-19) and (4-19)). Besides the number of derivative contracts traded is limited by production, see constraints (4-18) and (4-17)), for all points of this curve, the hedge ratio value is exactly equal to 100%, that is, derivatives are bought in the same company's production quantity. It was also found that for all *recession* state prices lower than 27.6 $ exhibit precisely the same behavior, corroborating the conclusion reached by analyzing Figure 5.6, in which, for prices less than approximately 28 $, the hedge ratio was constant (in fact in Figure 5.6 the number of derivatives contracts traded was constant, but, as the asset was kept constant, the hedge ratio is constant too).

An additional point of attention is to compare the contracted amount by observing Figures 5.6 and 5.7. As already informed, Figure 5.6 was built keeping the variables equal to their values in P50, for the asset, representing something close to 140 $ million. In Figure 5.7, for asset equal to 140 $ million, the quantity of contracts purchased is equal to 2 million, exactly the same value found in Figure 5.6. The curves shown in Figure 5.7 are just a few samples of the *recession* state prices. Generally, the behavior displayed by these samples holds for all realizations of this state, i.e. from lowest to highest price, the derivatives management policy starts from positive values (long position) to negative values (short position), with curve shapes following the same pattern seen in the chart.

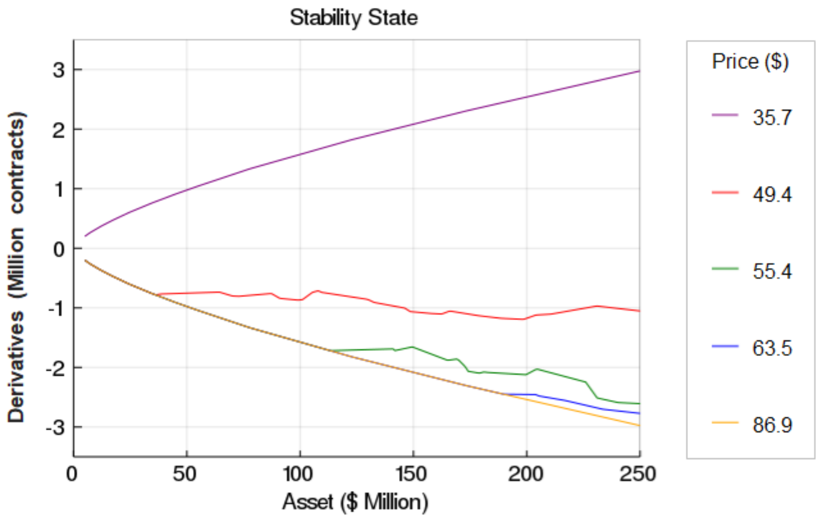

Figure 5.8: Derivatives management policy analysis for asset variations. *Stability* state.

Analyzing the derivatives management policy for the *stability* state, presented in Figure 5.8, again it is possible to confirm the behavior observed in Figure 5.6, in the lowest price range of *stability* state, the company buys the derivatives, selling them as prices rise in the samples chosen. Nevertheless, there is an interesting behavior of the relationship between derivatives management policy and quantity of assets for the stability state. See the curve representing the price of 55.4 $ (green line), as the assets increase, the number of contracts sold also increases as the company is seeking protection for 100% of its production. However, from a certain point (asset greater than approximately 110 $ million), the green line no longer follows the curve representing protecting 100% of production (yellow line), resulting in a declining modulus hedge ratio. Despite increasing the number of contracts, the proportion to the quantity produced is reduced. One possible interpretation is that the company has reached sufficient size (asset) being no longer necessary to protect 100% of its production and, as the company increases, the proportion of production to be protected becomes smaller and smaller. Also, this trigger, from which the hedge ratio becomes higher than -100%, is different for each price level.

From Figure 5.8 it is possible to draw one more important conclusion about the derivatives management policy for the *stability* state. For the lowest price realizations in this state, the policy guides the purchase of all available derivatives (see the curve representing price 35.7 $ in the graph). However, as the price increases, contrary to what we saw in the *recession* state, the company does not gradually reduce the number of contracts purchased, there is a discontinuity. For a price realization greater than about 42 $, the company stops buying all available derivatives and starts trading nothing. After this trigger, as the price increases, the company begins to sell more and more contracts, with a roughly continuous relationship between price, assets, and quantity of contracts sold. In short, for the *stability* state lowest price realizations, the policy recommends buying the full amount of derivatives available, betting on price increases. From a certain price level, the policy guides to follow the complex relationships between price and asset described in the previous paragraph. This behavior can be confirmed in Figure 5.6. There is a sharp change between buying about 2 million contracts, for prices below 40 $, and selling about 0.5 million contracts for prices slightly higher in the *stability* state (blue dots).

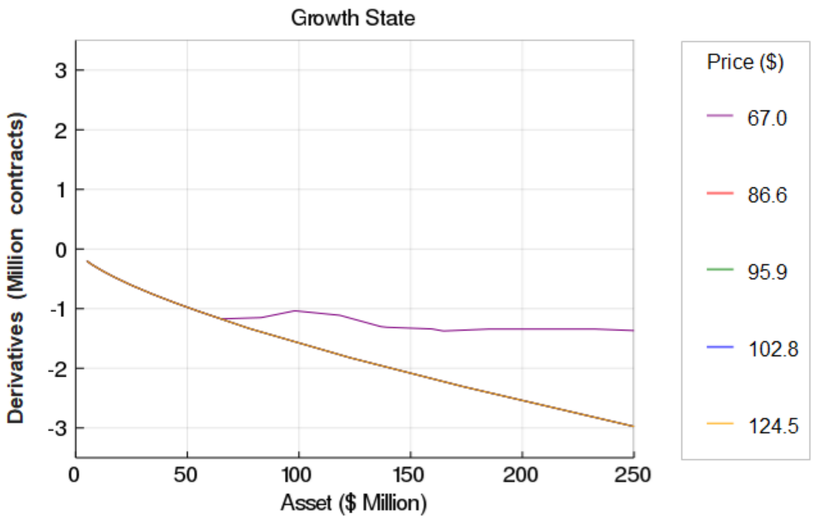

Figure 5.9: Derivatives management policy analysis for asset variations. *Growth* state.

Finally, for the *growth* state, shown in Figure 5.9, the conclusion obtained from Figure 5.6 is maintained. Only for a single state price realization, the policy offers a hedge ratio different from 100%. For the price equals to 67 $, full protection is no longer required for assets greater than approximately 70 $

million, which is similar to the behavior described in the previous paragraphs when analyzing the *stability* state. For all other price realizations in *growth* state, the derivatives management policy is always the same, forward contracts are sold to obtain a hedge ratio of exactly -100%. As already mentioned, this result is expected since the *growth* state has high price realizations, the company seeks to ensure this high price level in the next stage, protecting all its production.

As seen in section 5.3.1, the graph with percentiles of optimal policy simulations is useful for getting some idea of model decisions, but it does not clarify sufficiently the details of optimal policy. Continuing the analysis performed with each decision variable separately, in order to identify possible correlations between the derivatives management policy and the problem decision variables, Figure 5.10 shows the results of the simulation of a specific price scenario, with forward instruments available and risk-averse equal to 0.5.

Comparing the graphs of Figure 5.10 with the graphs in Figures 5.4 and 5.5, it is clear that some of the observations made are still valid here, such as no debt and share emission in the first stage. However, the chart presenting the decisions for derivatives shows that the derivatives management policy should be evaluated in a single scenario instead of using statistics from a set of scenarios.

By analyzing the chart containing the traded quantities of forward contracts (Figure 5.10), the behavior discussed above can be confirmed. Until stage 15, the hedge ratio assumes only two values +100% or -100%, indicating that the company buys or sells enough contracts for the entire quantity produced. The decision to buy or sell is influenced by the Markov states, buying in the *recession* state and selling in the *growth* state. For the *stability* state, in cases where the price is slightly higher the contract is sold, otherwise is bought.

Notice the period between states 15 and 25 approximately. The hedge ratio fluctuates between low negative values and close to 50%. In this period, the state is mostly the *recession*, which is reflected in the prices realized. As pointed out in the comments on Figures 5.6 and 5.7, in the *recession* state the company tends to buy contracts (positive values) and, from a certain price, the hedge ratio decreases, including an inversion, in which the company sells contracts. It is exactly this behavior that can be observed in the period between stages 15 and 25. With this price range, the hedge ratio is between 50% and values close to -10%.

For the interval between stages 25-32 and 43-50, approximately, the predominant state is the *growth* and, as already noted, the hedge ratio is exactly equal to -100%, as already concluded by Figures 5.6 and 5.8. However,

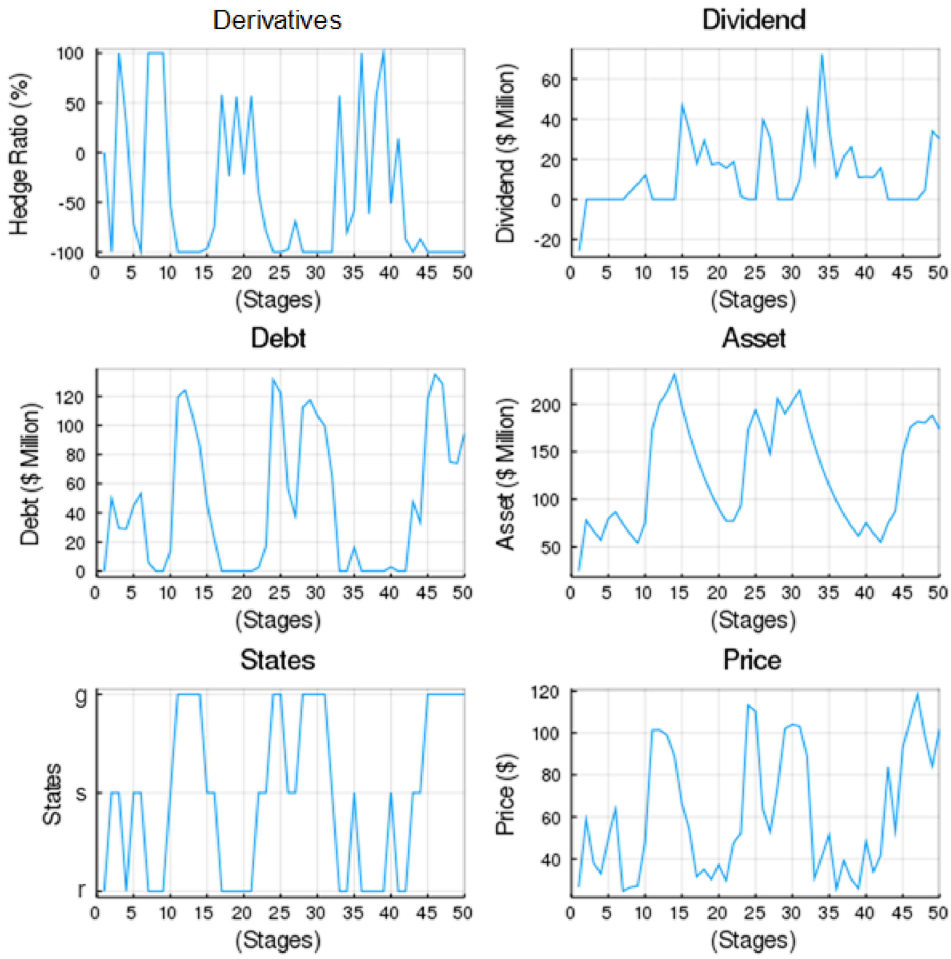

Figure 5.10: Analysis of model variables for a single price scenario. Case with derivatives.

note that in these two highlighted intervals, there are times when the state is the *stability* and the hedge ratio is not equal to -100%. This stems from the effect found in Figure 5.8, where for a quantity of assets above a certain value, the hedge ratio is no longer equal to -100% and begins to gradually decrease with increasing assets. Just to remember, this effect can best be seen by the red and green curves in Figure 5.8.

In the period between stages 33 and 42, the states observed are *recession* and *stability*. In turn, the hedge ratio fluctuates a lot, because of price fluctuation higher than that seen between stages 16 and 24. As already pointed out, these two states allow both long and short positions in forward contracts.

In addition to the analysis of the derivatives management policy, some points of the model's decision variables can be highlighted. After the acquisition of debt in the early stages, intended to finance the initial investment in the company, the debt grows significantly around stage 10. This growth is coincident to third state achieved. In this state the prices are very high, the company takes advantage to ensure a great selling price for its product in the following stage (short position in forward contracts), as well as invests heavily in its asset. As soon as the price state does not hold, around stage 15, the company identifies the change in the price context and stops investing, letting its asset depreciate (notice how the asset fall follows approximately an exponential, due to the annual rate depreciation of 14.5%). Profits in this period are then distributed to their shareholders through dividends. This dynamic among the variables, except for some slight variations, is repeated over the entire period presented.

6
Conclusion

The purpose of this dissertation is to asses the value of derivatives management policies for risk-averse shareholders of a commodity-producing company. To this end, we use Waga [36]'s capital structure model as a basis and include the possibility of using forward contracts, in addition to other modifications, as the uncertainty variable modeling. Some constraints were added to the original model to obtain a viable solution with interpretable results, keeping the model computationally tractable. With this, we obtain a computational tool to simulate policies, applying Stochastic Dual Dynamic Programming (SDDP), for a better understanding of the interplay of derivatives management with investment, funding, and dividends in a commodity company.

We simulated the model in a case study for an oil company, for which we adopted Brent as a price reference. From historical data with the last 20 years of Brent prices, we made a price projection using the Hidden Markov Model (HMM) method, assuming three Markov states. The results show that derivative contracts, traded following an optimal policy, can add value to the shareholder, by increasing the dividends to be distributed to the company's partners. In all risk aversion cases studied, it can be identified that the use of derivatives instruments increases the value of the company substantially (up to 500% increase). Moreover, the difference in scale when analyzing other variables of the problem, such as assets and debt, also allows us to identify that companies that can contract forward instruments have a larger size.

In addition to assessing the company's value, it was possible to identify some guidelines for the optimal derivatives management, i.e., which would be simple drivers that companies could follow to achieve the same results, translating the policy algorithm into simple statements. This analysis showed that prices have a major influence on derivatives management policy, which in turn is greatly influenced by the Markov states. For the *growth* state, the hedge ratio was found to be equal to -100% for all price realizations except for the lowest in this state. For the *recession* state, the hedge ratio is 100% up to a certain price, from which the price increase produces a reduction in the hedge ratio and may even be negative. For the *stability* state, the hedge ratio ranges from 100% to -100%, but for a range of price realizations, for asset values greater than a certain value, the amount of asset is already high enough to no longer hedging the entire amount produced, causing the hedge ratio, starting at -100%, to rise progressively as the asset increases.

In this work, we note the importance of derivatives management being made in conjunction with the financing, investment, and dividend payment decisions. Some oil companies have already identified the value that a derivatives management policy can add to their stakeholders, such as Petrobras, Pemex and some smaller companies in the United States. However, this assessment is usually done with models that are still very simplified (spreadsheet-based), so we highlight the importance of developing more sophisticated models for better assessment of the value of derivatives management in companies.

As an expansion of this work, we propose to insert other derivative contracts, such as options, to provide more flexibility for the optimal derivatives management policy. Besides, real market contracts may be used for pricing forward derivatives, even conditioning future oil prices to the defined Markov states. It is also possible to increase the number of constraints on the model so that the results are more in line with reality, for example by reducing volatility in dividend payments that are not well-liked by shareholders, as well as inserting long-term investment commitments.

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A
Stochastic Programming

The vast majority of real-world problems have uncertainty about their parameters, meaning that their objective function or constraints are not deterministic, with one or more random variables. Stochastic programming is a dynamic programming approach to solving problems that present such behavior.

Dynamic programming is an optimization technique applied to problems that have a recursive structure and require sequential decisions. Its purpose is to subdivide complex problems into smaller problems, called subproblems. Solutions to subproblems that have already been solved help in calculating the remaining subproblems. For the application of this technique, the problems, in general, present the following elements:

- Stage: Defines the number of subproblems to solve;
- State variables: are the variables that carry information from one stage to the next;
- Decision variables: represent the decision made at each stage;
- Exogenous information: data external to the problem that influences decisions;
- Transition Function: Function that describes how state and exogenous variables influence the forward stage;
- Objective function: a function that describes the problem's objective.

The mathematical equation of a two-stage problem, which aims to minimize cost, with the factors detailed above and recursive structure, is:

$$\begin{aligned} \min_{x} \quad & c^\top x + \mathbb{E}\left[Q\left(x, \xi\right)\right] \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned} \tag{A-1}$$

The first stage problem is represented by equation (A-1). The second stage problem, which is being evaluated by the expected value, is represented by the Q function, defined below.

$$Q(x,\xi) = \min_{y} \quad q^{\top} y \qquad \text{(A-2)}$$
$$\text{s.t.} \quad T(\xi)\, x + W(\xi)\, y = h(\xi)$$
$$y \geq 0.$$

Explaining in detail each variable of the problem:

– x: decision variable from the first stage that carries information to the second stage, is the state variable;

– $c^{\top}$: represents the immediate cost associated with the "x" decision;

– Q: is the second stage problem and represents the future cost, which depends on the first stage decision;

– A e b: define first stage constraints;

– $q^{\top}$: immediate cost in the second stage;

– T: represents the transition function, i.e. how the decision of the first stage will influence the second stage problem. It can be influenced by exogenous information;

– W e h: complete the constraints of the second stage problem. It can be influenced by exogenous information;

– ξ: variable containing exogenous information. It is the random variable of the problem.

The first stage decision (variable x) occurs before the definition of exogenous information (ξ), while the second stage decisions are made after the disclosure of the information contained in ξ.

From the two-stage problem, the extension to multiple stages occurs naturally. Adopting the extended form the objective function of the problem would be:

$$\min_{x_1 \in \chi_1} Q_1(x_1) + \mathbb{E}_1\left[\min_{x_2 \in \chi_{2(x_1,\xi_2)}} Q_2(x_2,\xi_2) + \mathbb{E}_2\left[\ldots + \mathbb{E}_{T-1}\left[\min_{x_T \in \chi_{T(x_{T-1},\xi_T)}} Q_T(x_T,\xi_T) \right]\right]\right]. \qquad \text{(A-3)}$$

Where:
x_t are the problem's variable in each stage t;
Q_t are the future cost functions in each stage t;
ξ_t represents the uncertainties of each stage t;
χ_t is the viable set of decisions.

Additionally, the equation (A-3) could be rewritten in a recursive way, like the follow Bellman [6] equation:

$$Q_t\left(x_{t-1},\xi_t\right) = \min_{x_t \in \chi_t(x_{t-1},\xi_t)} f_t\left(x_t,\xi_t\right) + \mathbb{E}_t\left[Q_{t+1}\left(x_t,\xi_{t+1}\right)\right], \tag{A-4}$$

where $f_t\left(x_t,\xi_t\right)$ is a generic function of the variables at stage t.

The basic concept behind dynamic programming is recursive optimization, in which the problem is solved from its last stage to its first stage. This concept is defined by Bellman [6] in his optimality principle "for a given state of the system, the optimal policy for the remaining states is independent of the decision policy adopted in previous states." In this principle, "policy" means the set of decisions taken at each stage.

The expected value measure has been used to assess the cost of the forward stages, however, there is also the possibility of applying some risk metric as proposed by Shapiro [28]. Considering the risk measure defined in (3-4), the equation of the multi-stage optimization problem can be changed:

$$Q_t\left(x_{t-1},\xi_t\right) = \min_{x_t \in \chi_t(x_{t-1},\xi_t)} f_t\left(x_t,\xi_t\right) + \psi_t\left[Q_{t+1}\left(x_t,\xi_{t+1}\right)\right]. \tag{A-5}$$

Considering the stochastic nature of the problems faced in real life, in addition to a large number of variables and stages, even with the use of some decomposition method, such as L-shaped, causing the problem to grow exponentially with the number of stages. The representation of the stochastic optimization model through linear programming may generate a very large problem (called the curse of dimensionality), which makes it impossible to obtain a solution in practical applications, since there is a clear conflict between the number of discrete samples (realizations) sufficient to satisfactorily describe the entire probability space versus the algorithm execution time. Therefore, to solve these problems efficiently, more appropriate methods such as decomposition methods are needed.

Among the decomposition methods, there is the so-called Progressive Hedging, which is decomposed in scenarios and known as Stochastic Dual Dynamic Programming (SDDP), in which the decomposition is based on stages. This dissertation used the SDDP method, because it is faster and has been successfully applied to energy and finance problems. The main points of the SDDP technique will be presented in the next section.

A.1 Stochastic Dual Dynamic Programming

The stochastic dual dynamic programming (SDDP) was proposed by Pereira & Pinto [26], in an application to energy planning. According to the authors, their algorithm seeks to solve the optimization problem by sampling various sequences of uncertain variable realizations, that is, several scenarios are sampled. Another example of SDDP application is available in Shapiro *et al.* [31]. Other methods based on sampling can also be studied in Donohue & Birge [13] and Chen & Powell [10].

In this methodology, the "curse of dimensionality" is overcome by assuming independence of the random variable between stages, i.e. the realization of the variable at any stage "t" is independent of other stages' realizations. Thus, it is possible to separate the future cost function by stage.

To better explain the basic process on which the SDDP technique is based, below is an example, applied to finance, of the algorithm executed to obtain the optimal policy for a multi-stage optimization problem.

For this example, a very simplified model of a company's cash flow will be used. Cash at the end of the stage (c_t) will be equal to cash from the previous stage (c_{t-1}), plus free cash flow at the end of the stage (f_t) and subtracted from the amount distributed to shareholders (e_t). In this two-stage problem, in which it is desired to maximize the total shareholder return via dividends over two periods, the company's free cash flow is the uncertain variable, the amount paid in dividends is the decision variable and the cash it is a state variable since the decision made in the t stage depends on the cash value in the $t-1$ stage, that is, it is the variable responsible for making the "connection" between the stages.

In the first stage, the free cash flow is already defined (f_1). For the second stage, two equiprobable realizations (*up*; *down*) can occur, in which the cash flow can be f_2^u or f_2^d. Figure A.1 schematically presents the two realizations.

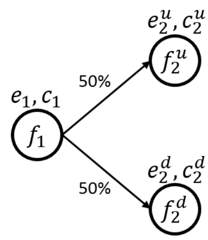

Figure A.1: Representation of a two-stage problem with two realizations.

Considering that the objective is to maximize the expected value of dividends distributed to shareholders, the optimization problem can be written as shown below. It is worth remembering that this is still the linear problem in its extended form, as yet Bellman's formulation is not being used.

$$\begin{aligned}
\max_{e_t} \quad & \left[e_1 + 0,5\left(\frac{e_2^u}{1+r_f}\right) + 0,5\left(\frac{e_2^d}{1+r_f}\right)\right] \\
\text{s.t.} \quad & e_1 + c_1 = c_0 + f_1 \\
& e_2^u + c_2^u = c_1 + f_2^u \\
& e_2^d + c_2^d = c_1 + f_2^d \\
& c_1, c_2^u, c_2^d \geq 0.
\end{aligned} \tag{A-6}$$

Where:
r_f is the risk-free rate;
c_0 is the cash value in stage $t = 0$, defined as a known constant;
e_1 is the optimal decision to be made in the first stage;
c_1 is the cash value at the end of the first stage;
f_2^u, f_2^d are the possible occurrence realizations for free cash flow in the second stage;
e_2^u, e_2^d are the optimal decisions to be made in the occurrence of each possible realization for free cash flow in the second stage;
c_2^u, c_2^d are the resulting cash values at the end of the second stage, considering each realization that occurred;

Realize that, given the first stage solution, the second stage problem, assessed in the first stage, can be treated as:

$$\begin{aligned}
\max_{e_t} \quad & \left[0,5\left(\frac{e_2^u}{1+r_f}\right) + 0,5\left(\frac{e_2^d}{1+r_f}\right)\right] \\
\text{s.t.} \quad & e_2^u + c_2^u = \tilde{c_1} + f_2^u \\
& e_2^d + c_2^d = \tilde{c_1} + f_2^d \\
& c_2^u, c_2^d \geq 0.
\end{aligned} \tag{A-7}$$

Where $(\tilde{c_1})$ is the resulting cash value after deciding on the dividend to be distributed in the first stage.

Note that the above problem can be subdivided into two independent problems.

$$\begin{array}{ll}
\textit{Up}\text{ realization} & \textit{Down}\text{ realization} \\
\begin{aligned}
\zeta_1 = \max_{e_t} \quad & \left[0,5\left(\frac{e_2^u}{1+r_f}\right)\right] \\
\text{s.t.} \quad & e_2^u + c_2^u = \tilde{c_1} + f_2^u \\
& c_2^u \geq 0
\end{aligned}
&
\begin{aligned}
\zeta_2 = \max_{e_t} \quad & \left[0,5\left(\frac{e_2^d}{1+r_f}\right)\right] \\
\text{s.t.} \quad & e_2^d + c_2^d = \tilde{c_1} + f_2^d \\
& c_2^d \geq 0
\end{aligned}
\end{array} \tag{A-8}$$

For the second stage problems (ζ_1 and ζ_2), the value of $\tilde{c_1}$ is defined and does not multiply any decision or state variables, so it was placed on the right side of the equation. It is easy to notice that, depending on the value of $\tilde{c_1}$, the problems will present different answers, so ζ_1 and ζ_2 can be interpreted as functions whose dependent variable is the first stage decision. In this way, the two-stage problem can be rewritten as:

$$\begin{aligned} \max_{e_t} \quad & \left[e_1 + \overline{\zeta}\left(c_1\right)\right] \\ \text{s.t.} \quad & e_1 + c_1 = c_0 + f_1 \\ & c_1 \geq 0 \end{aligned} \tag{A-9}$$

Where $\overline{\zeta}\left(c_1\right) = 0,5\zeta_1\left(c_1\right) + 0,5\zeta_2\left(c_1\right)$.

The $\overline{\zeta}$ function is known as the “future cost function”. For this example, which is about maximizing profits and not reducing costs, it can be considered as a “future profit function”, that is, it represents the expected profit value, obtained by the shareholder, in the stages ahead.

The determination of $\overline{\zeta}$ could, for example, be done by calculating the numerical derivative of ζ_1 and ζ_2. However, Pereira & Pinto [26] use the Benders [7] decomposition, adapted to stochastic cases, to obtain the approximation of the $\overline{\zeta}$ function.

The algorithm proposed by Pereira & Pinto [26] is presented below.

1. Set an initial value for $\overline{\zeta}$. As this is a maximization problem, it can be a very high value, that is, an upper limit for the profit of the next stages;

2. Solve the first stage problem by replacing $\overline{\zeta}$ with its approximate value $\widehat{\zeta}$ (estimated in item 1).

$$\begin{aligned} \max_{e_1} \quad & \left[e_1 + \widehat{\zeta}\left(c_1\right)\right] \\ \text{s.t.} \quad & e_1 + c_1 = c_0 + f_1 \\ & c_1 \geq 0; \end{aligned} \tag{A-10}$$

3. According to Pereira & Pinto [26], it is possible to demonstrate that a lower bound for the multi-stage optimization problem is given by $\widetilde{e_1} + \widehat{\zeta}\left(\widetilde{c_1}\right)$. Where $\widetilde{e_1}$ and $\widetilde{c_1}$ are the solutions of the first stage problem obtained in step 2;

4. Solve the next stage problem using the solution obtained in the previous stage (similar to step 2). This step is called “forward simulation”. If the analyzed problem had T stages, this procedure should be repeated for $t = 2, 3, \ldots, T$.

   This step is not performed for all possible realizations of the forward stages, because, depending on the number of stages and uncertain variables, the number of scenarios would grow exponentially. Since this technique is based on sampling, one realization will be selected for each stage, thus generating a scenario from which the decision variables of the problem will be calculated. As this example has only two stages, it will be performed only for one of the realizations of the second stage, whose equations are given below, considering that the realization called *up* was selected.

$$\begin{aligned} \zeta_1 = \max_{e_t} \quad & \left[0,5\left(\frac{e_2^u}{1+r_f}\right)\right] \\ \text{s.t.} \quad & e_2^u + c_2^u = \widetilde{c_1} + f_2^u \\ & c_2^u \geq 0; \end{aligned} \quad \text{(A-11)}$$

5. Let $\widetilde{e_2^u}$ be the solution to the second stage problem obtained in step 4. Therefore a feasible solution was obtained, but not necessarily the optimal one. This way you can calculate an upper bound for the multi-stage problem.

   In this example, there are only two realizations and two stages, however, for bigger problems, the lower limit is reached by performing step 4 several times (sorting different realizations and, consequently, different scenarios). With these various calculated solutions, it is possible to obtain a distribution of the results for the problem, thus constituting a stochastic upper bound;

6. Since the upper bound is stochastic, one must choose a metric to compare it to the lower bound. The so-called conservative upper limit is generally used, which is equal to the upper limit average minus one standard deviation. Thus, if the difference between the conservative upper limit and the lower limit is below a defined tolerance for the problem, then convergence is reached and the algorithm is terminated. Otherwise, proceed to the next step;

7. If convergence has not been achieved, a more accurate future cost function approximation ($\overline{\zeta}$) is necessary. This method is called backward recursion. If the problem analyzed has T stages, this step should be repeated for $t = T, T-1, \ldots, 2$. Unlike step 4, this procedure should be performed for all possible realizations of the stage immediately ahead. At this point, the SDDP technique has its differential to others in the literature. The approximation of the future cost function is performed by adding Benders cuts to the previous stage problem, using the $(T, T-1, \ldots, 2)$ dual problem as a basis to obtain the necessary parameters to the cut. In practice, problems of all stages will be solved with one additional constraint (Benders cut). With each iteration of the

algorithm, as long as the desired convergence is not reached, a further constraint will be added. A more detailed explanation of the calculation of Bender cuts is available in the next section.

A.2 Benders cuts in SDDP

This section speeds up the calculation of the Benders cut through the dual problem that is necessary to step 7 of the procedure described in the previous section. Continuing with the same example that was treated above, which has only two stages, this step will be performed only for the second stage.

To improve the approximation of the future cost function, you must improve the approximation of the functions that represent the second stage problem in each realization (ζ_1 and ζ_2). The procedure below is the same for all realizations, therefore, will be presented only for the function ζ_1 (see equation (A-11)).

First, it is necessary to obtain the dual of the problem represented by ζ_1. The equations below describe the dual.

$$\begin{aligned} \min_{\nu_2^u} \quad & \left[\left(f_2^u + c_1 \right) \nu_2^u \right] \\ \text{s.t.} \quad & \nu_2^u = \frac{0,5}{1 + r_f} \\ & c_2^u \geq 0, \end{aligned} \tag{A-12}$$

where ν_2^u is the dual variable.

Note that the restrictions now no longer depend on the first stage solution (c_1). Therefore, the viable region for this problem does not depend on c_1, only on the restrictions of the dual problem. As is known, the solution to a linear optimization problem is found at the vertices of the viable region. For this very simplified example, the restrictions define only a single point as a viable region. However, for bigger problems, there would be a polyhedron defined by a set of points that determine the vertices of the viable region. Just as a way of presenting the complete procedure of the SDDP technique, suppose that the viable region (Γ) was represented by the following set of n vertices: $\Gamma = \left\{ \nu_2^{u,1}; \nu_2^{u,2}; \ldots; \nu_2^{u,n} \right\}$. Once these points are identified, the dual problem can be solved by finding the point that corresponds to the minimum value of the objective function. Thus, the dual problem can be presented as follows:

$$\begin{aligned}\zeta_1 &= \min_{\nu_2^u} \left[\left(f_2^u + c_1 \right) \nu_2^u \right] \\ &= \min \left\{ \left(f_2^u + c_1 \right) \nu_2^{u,1} ; \left(f_2^u + c_1 \right) \nu_2^{u,1} ; \ldots ; \left(f_2^u + c_1 \right) \nu_2^{u,n} \right\}\end{aligned} \tag{A-13}$$

Note that the same variable ζ_1 was used as the previous problem, because, by the principle of strong duality, the optimal solution of the dual and primal problems are the same.

The problem of finding the minimum value among a list of values is widely known in the literature and can be solved using the formulation below:

$$\begin{aligned}\max_{\epsilon} \quad & \epsilon \\ \text{s.t.} \quad & \epsilon \leq \left(f_2^u + c_1 \right) \nu_2^{u,1} \\ & \epsilon \leq \left(f_2^u + c_1 \right) \nu_2^{u,2} \\ & \vdots \\ & \epsilon \leq \left(f_2^u + c_1 \right) \nu_2^{u,n}.\end{aligned} \tag{A-14}$$

Where ϵ is just an auxiliary variable.

The optimum point of the above problem, for each value of c_1, is exactly equal to the value of $\zeta_1(c_1)$, that is, the value of the future cost function measured at the solution point of the previous stage.

The set of restrictions above can still be described differently. Assume w_1 is the value of the optimal solution of the second stage problem, considering $\tilde{c_1}$ as the first stage solution, so the value of w_1 is:

$$w_1 = \left(f_2^u + \tilde{c_1} \right) \nu_2^u. \tag{A-15}$$

Organizing the equation, you can write:

$$f_2^u \nu_2^u = w_1 - \tilde{c_1} \nu_2^u \tag{A-16}$$

Replacing the equation (A-16) in the restrictions in (A-14), these can be written in general way as:

$$\epsilon \leq w_1 + \nu_2^u \left(c_1 - \tilde{c_1} \right) \tag{A-17}$$

Below is Taylor's serial expansion, up to first order, of the real function ζ_1, around the point $c_1 = \tilde{c_1}$.

$$\zeta_1(c_1) \approx \zeta_1(\tilde{c_1}) + \left(\frac{\partial \zeta_1}{\partial c_1} | \left(c_1 = \tilde{c_1} \right) \right) \left(c_1 - \tilde{c_1} \right) \tag{A-18}$$

Remembering that $\epsilon = \zeta_1(c_1)$ and that $w_1 = \zeta_1(\tilde{c_1})$, by construction, when comparing the two equations (A-17) and (A-18), it can be verified that the variable of the dual problem (ν_2^u) corresponds to the first derivative of the function ζ_1, at the point $c_1 = \tilde{c}_1$.

Thus, it can be concluded that the restrictions of the above problem are, in fact, an approximation of the future cost function through its tangent lines. As the number of chosen tangency points increases, the accuracy of the approach increases. Figure A.2 illustrates this concept in a geometric form.

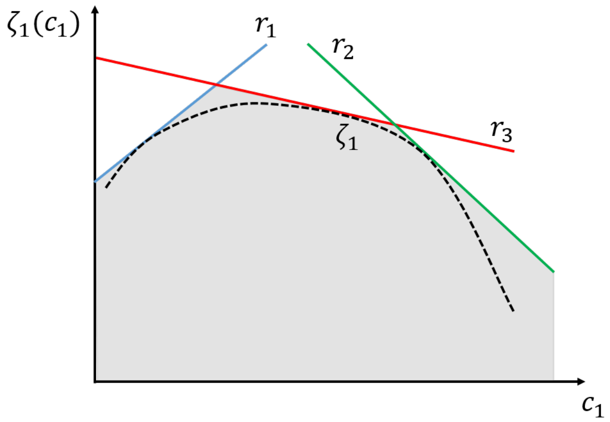

Figure A.2: Approximation of the future cost function by tangent lines.

Thus it is concluded that the set of some of the above restrictions (A-14) (which correspond, for example, to the lines r_1, r_2 and r_3) form an approximation to the function ζ_1. Such a function would be completely known if all restrictions were used, however, it is not necessary to describe the function completely, only its approximation. Besides, in a problem that has a large number of restrictions, some of them are likely inactive. Thus, one way to increase the precision of the approximation of the ζ_1 function is to add one constraint at a time to the original problem. These restrictions are also known as "Bender cuts".

Just remember, the approximation calculation shown for ζ_1 should also be done for the function ζ_2, which represents the future cost of the second realization (*down*). If there were other realizations, they should all have their functions approximated by the same method. To conclude the procedure, just return to the problem of the previous stage (first in this case) and solve it again, but with the addition of a constraint (Benders cut), to improve the approximation of $\overline{\zeta}$, through its estimator $\widehat{\zeta}$. The original problem would then be:

$$\begin{aligned} \max_{e_1} \quad & \left[e_1 + \widehat{\zeta}\right] \\ \text{s.t.} \quad & e_1 + c_1 = c_0 + f_1 \\ & \widehat{\zeta} \leq \overline{w} + \overline{\nu_2}\left(c_1 + \tilde{c}_1\right) \\ & c_1 \geq 0. \end{aligned} \tag{A-19}$$

Where $\overline{\nu_2}$ and $\overline{w}$ are the average coefficients obtained from the cuts made for all the stages in the future. In this example, there are only two realizations, so:

$$\overline{\nu_2} = 0,5\left(\nu_2^u + \nu_2^d\right) \tag{A-20}$$

$$\overline{w} = 0,5\left(w_1 + w_2\right), \tag{A-21}$$

Where ν_2^u and ν_2^d are the dual variables of the problems of the realizations *up* and *down*, referring to the problems ζ_1 and ζ_2, whose solutions are w_1 and w_2, respectively.

For each iteration, add a new constraint (associated with new $\overline{\nu_2}$ and $\overline{w}$) to the problem of the previous stage, to reduce, for the multi-stage problem, the gap between the lower conservative and upper limits to an acceptable value.