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Tolerance Intervals for Sample Variances Applied to the Study of the Phase II Performance and Design of S^2 Charts with Estimated Parameters

Tese de Doutorado

Thesis presented to the Programa de Pós-graduação em Engenharia de Produção of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Engenharia de Produção.

Advisor: Prof. Eugenio Kahn Epprecht

Co-advisor: Prof. Subhabrata Chakraborti

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Abstract

Sarmiento, Martin Guillermo Cornejo; Epprecht, Eugenio Khan (Advisor); Chakraborti, Subhabrata (Co-advisor). **Tolerance Intervals for Sample Variances Applied to the Study of the Phase II Performance and Design of S^2 Charts with Estimated Parameters**. Rio de Janeiro, 2019. 155p. Tese de Doutorado - Departamento de Engenharia Industrial, Pontifícia Universidade Católica do Rio de Janeiro.

The S^2 control charts are fundamental tools widely used to monitor the process dispersion in applications of Statistical Process Monitoring and Control. Phase II performance of different types of control charts, including the S^2 chart, with unknown process parameters may be significantly different from the nominal performance due to the effect of parameter estimation. In the last few years, this effect has been addressed predominantly under the *conditional* perspective, which considers the variability of parameter estimates obtained from different Phase I reference samples instead of the traditional unconditional performance measures based on the marginal (unconditional) run length (RL) distribution, such as the unconditional average run length. In light of this new *conditional* perspective, the analysis of the Phase II performance and design of control charts is frequently undertaken using the *Exceedance Probability Criterion* for the conditional (given the parameter estimates) in-control average run length ($CARL_0$), that is, the criterion that ensures a high probability that the $CARL_0$ is at least a specified minimum tolerated value. Tolerance intervals for sample variances are useful when the main concern is the precision of the values of the quality characteristic and then they can be used in decision-making on lot acceptance sampling. Motivated by the fact that these tolerance intervals, specifically in the case of the two-sided intervals, have not been addressed in the literature so far, exact and approximate two-sided tolerance limits for the population of sample variances are derived and presented in this work. The mathematical-statistical relationship between tolerance interval for the sample variance and the exceedance probability (survival probability) of the $CARL_0$ for the S^2 control chart with estimated parameter is recognized, highlighted and used in this work in such a way that the study of the Phase II performance and design of this chart can be based on tolerance interval for the sample variance, and vice versa. Works on performance and design of S^2 chart with estimated parameter generally

focused on only one perspective (either unconditional or conditional) and considered only one type of chart (either upper one-sided chart or two-sided chart). The existence of both perspectives and two types of charts may be confusing for practitioners. For that reason, the performance and design of S^2 control chart according to these two perspectives are compared, considering each type of chart. Similarly, these two types of charts are also compared for each perspective. Some important results related to the S^2 chart design, which are not yet available in the literature, were required and obtained in this work to provide a comprehensive comparative study that enables practitioners to be aware of the significant differences between these two perspectives and the two types of charts so that proper informed decisions about the chart design to choose can be made. Furthermore, because the conditional RL distribution is usually highly right-skewed, the median and some extreme quantiles of the conditional RL distribution are proposed as complementary performance measures to the customary mean ($CARL_0$). Finally, some practical recommendations are offered.

Keywords

Phase II Performance of S^2 Control Charts; Design of S^2 Control Charts; Tolerance Limits for Sample Variances; Conditional Performance; Run Length Distribution and Quantiles.

Resumo

Sarmiento, Martin Guillermo Cornejo; Epprecht, Eugenio Khan (Orientador); Chakraborti, Subhabrata (Co-orientador). **Intervalos de Tolerância para Variâncias Amostrais Aplicados ao Estudo do Desempenho na Fase II e Projeto de Gráficos de S^2 com Parâmetros Estimados**. Rio de Janeiro, 2019. 155p. Tese de Doutorado - Departamento de Engenharia Industrial, Pontifícia Universidade Católica do Rio de Janeiro.

Os gráficos de controle de S^2 são ferramentas fundamentais amplamente utilizados para monitoramento da dispersão do processo em aplicações de CEP. O desempenho na Fase II de diferentes tipos de gráficos de controle, incluindo o gráfico de S^2, com parâmetros desconhecidos pode ser significativamente diferente do desempenho nominal por causa do efeito da estimação de parâmetros. Nos anos mais recentes, este efeito tem sido abordado predominantemente sob a perspectiva condicional, que considera a variabilidade das estimativas de parâmetros obtidas a partir de diferentes amostras de referência da Fase I em vez das típicas medidas de desempenho baseadas na distribuição marginal (incondicional) do número de amostras até o sinal (*Run Length-RL*), como sua média. À luz dessa nova perspectiva condicional, a análise do desempenho da Fase II e do projeto de gráficos de controle é frequentemente realizada usando o *Exceedance Probability Criterion* para a média da distribuição condicional do RL ($CARL_0$), isto é, o critério que garante uma alta probabilidade de que $CARL_0$ seja pelo menos um valor mínimo tolerado e especificado. Intervalos de tolerância para variâncias amostrais são úteis quando o maior interesse está focado na precisão dos valores de uma característica de qualidade, e podem ser usados na tomada de decisões sobre a aceitação de lotes por amostragem. Motivado pelo fato de que estes intervalos, especificamente no caso dos intervalos bilaterais, não foram abordados na literatura, limites bilaterais de tolerância exatos e aproximados de variâncias amostrais são derivados e apresentados neste trabalho. A relação matemática-estatística entre o intervalo de tolerância para a variância amostral e o *Exceedance Probability* (função de sobrevivência) da $CARL_0$ do gráfico de S^2 com parâmetro estimado é reconhecida, destacada e usada neste trabalho de tal forma que o estudo do desempenho na Fase II e o projeto desse gráfico pode ser baseado no intervalo de tolerância para a variância amostral, e vice-versa. Os trabalhos sobre o desempenho e projeto do

gráfico de S^2 com parâmetro estimado focaram-se em apenas uma perspectiva (incondicional ou condicional) e consideraram somente um tipo de gráfico (unilateral superior ou bilateral). A existência de duas perspectivas e dois tipos de gráficos poderia ser confusa para os usuários. Por esse motivo, o desempenho e o projeto do gráfico de S^2 de acordo com essas duas perspectivas são comparados, considerando cada tipo de gráfico. Da mesma forma, esses dois tipos de gráficos também são comparados para cada perspectiva. Alguns resultados importantes relacionados ao projeto do gráfico de S^2, que ainda não estão disponíveis na literatura, foram necessários e obtidos neste trabalho para fornecer um estudo comparativo completo que permita aos usuários estarem cientes das diferenças significativas entre as duas perspectivas e os dois tipos de gráficos para tomar decisões informadas sobre a escolha do projeto do gráfico de S^2. Além disso, dado que a distribuição condicional do RL é em geral fortemente enviesada à direita, a mediana e alguns quantis extremos desta distribuição são propostos como medidas de desempenho complementares à sua tradicional média ($CARL_0$). Finalmente, algumas recomendações práticas são oferecidas.

Palavras – chave

Desempenho na Fase II de Gráficos de Controle de S^2; Projeto de Gráficos de Controle de S^2; Intervalos de Tolerância para Variâncias Amostrais; Desempenho Condicional; Distribuição e Quantis do Número de Amostras Até Um Alarme.

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GLOSSARY

α	Nominal false alarm rate
α^*	Adjusted nominal false alarm rate
ARL	Unconditional average run length ($E(RL) = ARL = E(CARL)$)
ARL_0	Unconditional in-control average run length
ARL_0^*	Desired and specified unconditional in-control average run length in the case of unconditional adjustment
$1-\beta$	Nominal proportion or content of the tolerance interval for the sample variance
β^*	Adjusted value of β, which is the complement of the nominal proportion of the tolerance interval for the sample variance
β_{CE}^*	Approximate of β^* based on the CE method
β_{KMM}^*	Approximate of β^* based on the KMM method
cdf	Cumulative distribution function
$CARL$	Conditional average run length
CE	Conditional expectation
$CFAR$	Conditional false alarm rate
$CMRL$	Conditional median run length
CPS	Conditional probability of a signal
CRL	Conditional run length
CRL_q	Conditional run length q-quantile
df	Degrees of freedom
EP	Exceedance Probability
EPC	Exceedance Probability Criterion
EX	Exact method
$E(X)$	Expected value of the random variable X
ε	Tolerance factor to determine a proper conditional performance threshold using the EPC

f_X	Probability distribution function (or probability mass function) of the continuous (or discrete) random variable X
F_X	Cumulative distribution function of the random variable X
F_X^{-1}	Inverse of the cumulative distribution function (quantile function) of the random variable X
IC	In control
k	Tolerance factor for a normal distribution
KMM	Krishnamoorthy, Mathew and Mukherjee
$G(Y)$	Actual coverage of the two-sided tolerance interval for the sample variance expressed as a function of Y
γ	Confidence level of the tolerance interval for the sample variance
L^*	Exact two-sided lower tolerance factor for the sample variance
L_{CE}	Approximate two-sided lower tolerance factor for the sample variance based on the CE method
L_{KMM}	Approximate two-sided lower tolerance factor for the sample variance based on the KMM method
L_{two}	Lower control limit factor of two-sided S^2 chart when σ_0^2 is known
L_{two}^*	Adjusted lower control limit factor of two-sided S^2 chart when σ_0^2 is unknown and estimated
LCL_{two}	Lower two-sided S^2 control limit when σ_0^2 is known
$\widehat{LCL}_{two}$	Lower two-sided S^2 control limit when σ_0^2 is unknown and estimated
$\widehat{LCL}_{two}^*$	Adjusted lower two-sided S^2 control limit when σ_0^2 is unknown and estimated
m	Number of Phase I samples (in the context of both tolerance intervals for sample variances and S^2 control charts)
$max(T)$	Maximum value of T
μ	Mean of the normal population, from which samples are collected to generate sample variances that make up the inferred population using tolerance interval

μ_0	Phase I (in-control) process mean (in the context of S^2 chart)
n	Phase I sample size and sample size required to obtain the sample variances that make up the inferred population (in the context of tolerance intervals for sample variances), and size of each Phase I and Phase II sample (in the context of S^2 control charts)
one	Used as a subscript to indicate that upper one-sided S^2 chart is considered
OOC	Out of control
p	The risk (probability) that the $CARL_0$ is smaller than a specified tolerated value
PD	Percentage difference between the values related to the approximate tolerance interval for the sample variance (based on either the CE method or the KMM method) and the exact ones ($1-\beta^*$, L^*, U^*, $\hat{S}^2_{L^*}$, $\hat{S}^2_{U^*}$, WI)
pdf	Probability distribution function
pmf	Probability mass function
RL	Run length
ρ	Ratio between the process standard deviations in Phases II and I
σ^2 (or σ)	Variance (or standard deviation) of the normal population, from which samples are collected to generate sample variances (or standard deviations) that make up the inferred population using tolerance interval
σ_0^2	Phase I (in-control) process variance
σ_1^2	Phase II process variance
$\hat{\sigma}^2$	Estimator for σ^2
$\hat{\sigma}_0^2$	Estimator for σ_0^2
$SDARL$	Standard deviation of the $CARL$
SPC	Statistical Process Monitoring and Control
S^2 (or S)	Sample variance (or standard deviation)

S_i^2	i-*th* Phase I sample variance ($i = 1,2,\ldots,m$) obtained from a sample of size n (in the context of both tolerance intervals for sample variances and S^2 control charts)
S_l^2	l-*th* Phase II sample variance ($l = m+1, m+2,\ldots$) obtained from a sample of size n (in the context of S^2 control charts)
S_p^2	Pooled sample variance
$\hat{S}_{L^*}^2$	Exact two-sided lower tolerance limit for the sample variance
$\hat{S}_{L_{CE}}^2$	Approximate two-sided lower tolerance limit for the sample variance based on the CE method
$\hat{S}_{L_{KMM}}^2$	Approximate two-sided lower tolerance limit for the sample variance based on the KMM method
$\hat{S}_{U^*}^2$	Exact two-sided upper tolerance limit for the sample variance
$\hat{S}_{U_{CE}}^2$	Approximate two-sided upper tolerance limit for the sample variance based on the CE method
$\hat{S}_{U_{KMM}}^2$	Approximate two-sided upper tolerance limit for the sample variance based on the KMM method
two	Used as a subscript to indicate that two-sided S^2 chart is considered
$\lceil t \rceil$	The smallest integer greater or equal to t
$\lfloor t \rfloor$	The largest integer less than or equal to t
U	Uniform random variable between 0 and 1
u	A observed value or realization of a uniform random variable U. It also denotes the order of the quantiles of the Y distribution
U^*	Exact two-sided upper tolerance factor for the sample variance
U_{CE}	Approximate two-sided upper tolerance factor for the sample variance based on the CE method
U_{KMM}	Approximate two-sided upper tolerance factor for the sample variance based on the KMM method

U_{one} (U_{two})	Upper control limit factor of one-sided (two-sided) S^2 chart when σ_0^2 is known
U_{one}^* (U_{two}^*)	Adjusted upper control limit factor of one-sided (two-sided) S^2 chart when σ_0^2 is unknown and estimated
UCL_{one} (UCL_{two})	Upper one-sided (or two-sided) S^2 control limit when σ_0^2 is known
$\widehat{UCL}_{one}$ ($\widehat{UCL}_{two}$)	Upper one-sided (or two-sided) S^2 control limit when σ_0^2 is unknown and estimated
$\widehat{UCL}_{one}^*$ ($\widehat{UCL}_{two}^*$)	Adjusted upper one-sided (or two-sided) S^2 control limit when σ_0^2 is unknown and estimated
W	Chi-square random variable with $(n-1)$ degrees of freedom
WH	Wilson-Hilferty
WI	Width of the exact tolerance interval for the sample variance
WI_{CE}	Width of the approximate tolerance interval for the sample variance based on the CE method
WI_{KMM}	Width of the approximate tolerance interval for the sample variance based on the KMM method
X_{ij}	j-*th* observation of the i-*th* Phase I sample ($i = 1,2,\dots,m$ and $j = 1,2,\dots,n$). X_{ij} are iid $N(\mu,\sigma^2)$ in the context of tolerance intervals for sample variances and $N(\mu_0,\sigma_0^2)$ in the context of S^2 control charts
X_j'	j-*th* (test or future) observation in the context of tolerance intervals for sample variances. X_j' are iid $N(\mu,\sigma^2)$, $j = 1,2,\dots,n$
$\bar{X}$	Sample mean
$\bar{X}_i$	i-*th* Phase I sample mean
χ_{df}^2	Chi-square random variable with df degrees of freedom
$\chi_{df,q}^2$	q-quantile of the distribution of a chi-square random variable with df degrees of freedom. This is also denoted as $F_{\chi_{df}^2}^{-1}(q)$
$\chi_{df,q}^2(nc)$	q-quantile of the distribution of a non-central chi-square random variable with df degrees of freedom and non-centrality parameter nc

Y	Chi-square random variable with $m(n-1)$ degrees of freedom
y	A observed value or realization of a chi-square random variable Y
0	Used as a subscript to indicate that in-control process is considered

1
Introduction

1.1
Motivation and Contributions

Control charts are one of the main techniques of Statistical Process Monitoring and Control (SPC) for quality control and improvement. These charts, which were proposed in the 1920s and 1930s by Walter A. Shewhart (see Shewhart, 1931), are powerful tools to monitor traditional manufacturing processes as well as service and recent communication processes, for instance, nowadays, social networks and public health are being monitored using control charts. Statistical tolerance intervals, which were studied for the first time by Samuel S. Wilks in the early 1940s (see Wilks, 1941), are used in statistical inference-based decision making. These intervals are useful tools extensively utilized in manufacturing (quality control), engineering (reliability) and different scientific fields. In the case of quality control and improvement applications, tolerance intervals are usually constructed to be examined in conformity assessment and acceptance of products (or processes). It is worth to note that these intervals are generally employed in the release of production lots (incoming raw materials or components and finished products), while control charts are used in on-line (in real time) process monitoring.

Three types of statistical intervals, which are computed from a random sample and quantify the uncertainty related to sampling variability, are the most important in statistical inference. The most well-known interval is the confidence interval that is analyzed to infer about unknown parameters of the corresponding distribution of the studied population. Prediction intervals are used when the interest is in predicting one or more future observations from this population. However, sometimes the user desires to get information on a relatively large number of such future observations or (given that this number could be unknown or theoretically infinite) a large proportion of the entire population. In such a case, the construction of tolerance intervals can be useful, that is, setting limits that cover at least a specified large proportion of a population of interest with a preassigned high confidence level can provide the required information.

In several applications of quality assessment, the major concern is the precision of the measurements of the quality characteristic of interest, and thus the process variability must be small. For instance, in mass production, processes based on the manufacture of interchangeable parts require to achieve highly accurate and precise measurements of certain quality characteristics of manufactured parts. In addition, in many real applications, the related decision-making must be based on a specific sample. In this situation, the construction of statistical tolerance intervals for the population of sample variances can be of great utility because it can bring significant gains in knowledge of the product/process variability. For that reason, Tietjen & Johnson (1979) proposed the construction of one-sided upper tolerance limits for the population of sample variances. They justified their proposal by pointing out that the control of increases in the process variability is the principal interest in applications. Accordingly, the process deterioration can be monitored. However, although one-sided upper tolerance limits are useful in examining how large the process variability is, they disregard the information on how small the process variability is, which is a fundamental objective pursued in quality control and improvement. Motivated by this fact, the study of two-sided tolerance intervals for sample variances, which allows us to examine the process improvement as well as the process deterioration and are yet not available in the literature, represents a research gap.

In the context of SPC, the S^2 process control chart is one of the most well-known and used tool to monitor the variability of the quality characteristic of interest. In real S^2 control chart applications, the in-control process variance (σ_0^2) is generally unknown and needs to be estimated from a reference dataset, which is constituted of m independent samples (or subgroups) of size n and is collected from the Phase I process. With this estimate of σ_0^2, the control limits of the S^2 chart are constructed to be used in prospective Phase II process monitoring, where Phase II sample variances (plotting statistics), obtained from samples (also of size n) collected at regular intervals, are compared with the estimated control limit(s). The Phase II chart performance is usually measured by some property of the distribution of the run length (RL), which is the number of sample variances until an alarm (i.e., the number of plotting statistics plotted until getting the first one outside the control limits), such as its expected value, the so-called average run length (ARL). When σ_0^2 is known or specified (unrealistic case), the RL follows a geometric distribution with parameter equal to the probability of an alarm (signal). If the process is in-control (IC), the signal probability is called the false alarm rate (the probability of type I error, denoted as α). Any associated property of the in-control run length (RL_0) distribution is a constant value that depends on a specified α, for instance, the ARL is the reciprocal value of α.

Several works have revealed that the Phase II performance of control charts with process parameters estimated in Phase I (including the S^2 chart) may be extremely different from the theoretical (or nominal) performance that it would have in the (unrealistic) parameters-known case, in which no Phase I is needed, or if the Phase I estimate were "absolutely accurate". It is worth to note that most of these works about the effect of parameter estimation on the performance of control charts have concentrated on the marginal (or unconditional) distribution of the RL (called the *unconditional* perspective), particularly on the expected value of the marginal IC RL (the unconditional in-control average run length, denoted as ARL_0).

With regard to the S^2 chart, the unconditional RL distribution can be found by averaging the conditional RL probabilities (conditioned on the estimator of σ_0^2, denoted as $\hat{\sigma}_0^2$) over the distribution of $\hat{\sigma}_0^2$. So, the unconditional RL distribution (which, differently from the σ_0^2-known case, isn't geometrically distributed), and any associated (derived) unconditional performance measure as well, represent an "average" performance of the chart, rather than the actual (or attained) performance of any particular control chart in a given application. However, this *unconditional* perspective can be considered as theoretical and not practical because, in practice, there generally exists only a unique reference dataset to estimate σ_0^2 and different instances of a same control chart constructed from different reference datasets would have different performances (the so-called "practitioner-to-practitioner variability"). A "new" *conditional* perspective that take into account that this variability may be significantly large has emerged in the last few years (since mid-2015). So, the focus moved from, the prevailing approach until then, the unconditional RL distribution to the conditional RL (CRL) distribution. The CRL when the process is IC (denoted as CRL_0) follows a geometric distribution, parameter of which is a particular realization of the conditional false-alarm rate ($CFAR$), that is, the false alarm rate conditioned on (given) a particular $\hat{\sigma}_0^2$ obtained from a certain Phase I reference dataset. The estimation of σ_0^2 turns the associated properties of the CRL_0 distribution (constant values in the σ_0^2-known case) into random variables expressed as functions of $\hat{\sigma}_0^2$ for a given value of α (the specified or nominal false alarm rate in the σ_0^2-known case). For instance, the conditional in-control average run length ($CARL_0$), which is the reciprocal of $CFAR$, and some of its properties, such as its standard deviation ($SDARL_0$) and exceedance probability (EP), are used rather than the unconditional performance measures. Since small realizations of $CARL_0$ are undesired, the EP of the $CARL_0$, which is the probability that the $CARL_0$ is at least a minimum tolerated value, is a useful conditional performance measure.

Moreover, since the CRL distribution is generally highly right-skewed, especially when the specified α is small (e.g., traditional value of 0.0027), the sole use of the mean of this CRL distribution (i.e., the $CARL$) as a performance measure may convey limited information about this distribution and, therefore, the examination and knowledge of the entire RL distribution, by means of its quantiles (the conditional run length q-quantile, denoted as CRL_q) are convenient. These quantiles, which are random variables rather than constant values (the parameters known case), better depict the location of a skewed distribution, and particular quantiles such as the conditional median run length ($CMRL$) may be considered more robust measures of centrality than the $CARL$ since the mean is more impacted by extreme CRL values than the median. When the process is IC, the analysis of lower quantiles of the CRL_0 (that is, the random variable called q-quantile of the conditional in-control run length, denoted as $CRL_{0,q}$), such as $CRL_{0,0.05}$, could be of practical interest and more useful for practitioners. In this way, large realizations of $CRL_{0,q}$ are preferred, because they correspond to smaller values of $CFAR$ or larger values of $CARL_0$.

Studies on the Phase II performance of control charts with estimated parameter(s) have also pointed out that the required amount of Phase I reference data to ensure a desired (unconditional or conditional) IC performance of the Phase II chart could be fairly larger than the recommended amounts in many books and manuals (traditionally, $m = 20$ to 30 and $n = 4$ or 5, see, for instance, Montgomery, 2009). This desired IC performance can be the unconditional one close to the nominal performance that the chart would have in the parameter known case, such as the nominal ARL_0 (*unconditional* perspective) or a specified minimum tolerated IC performance based on the *Exceedance Probability Criterion* (EPC) for the $CARL_0$ (*conditional* perspective). This EPC takes into account the "practitioner-to-practitioner variability" and can be defined as the required high probability that the $CARL_0$ is at least a specified minimum tolerated value (a high value of the EP of the $CARL_0$) in a given application. Note that the cumulative distribution function (cdf) of the $CARL_0$ is needed to apply this EPC. Since the required large amount of Phase I reference data is most often infeasible in practice, some authors have proposed to adjust the control limit(s), given a practical and available amount of data, in order to achieve a desired (either unconditional or conditional) IC performance as outlined above.

The close mathematical-statistical relationship between the two-sided tolerance intervals for sample variances and the cdf of the $CFAR$ or, equivalently, the EP (survival probability or the complement of the cdf) of the $CARL_0$ of the two-sided S^2 chart is exploited in this work. For instance, we highlight the use of exact tolerance limits as adjusted control limits to ensure a specified minimum tolerated value of the $CARL_0$ with a high probability. Hence, characteristics associated with the $CARL_0$ (its cdf and EP) and the values related to the design of the S^2 control chart are obtained on the basis of exact tolerance intervals for the population of sample variances. In this work, the "design" of S^2 control chart entails the decisions about the amount of Phase I samples and/or the use of unadjusted or adjusted control limits and, if adjusting them, the values of the adjusted control limits factors required to achieve a specified Phase II performance.

Similarly to the construction and implementation of tolerance intervals, the decision to use either upper one-sided or two-sided S^2 (or S) control chart must be based on the context and the objective of the particular application, and so both configurations have their place in practice. When the detection of increases in the process variance (or standard deviation) is the principal concern of the monitoring process variability (information about process deterioration), S^2 (or S) chart can be constructed without the lower control limit. It is well known that, given a specified significance level, the power of the one-tailed hypothesis test outperforms the power of the two-sided one, likewise, the upper one-sided S^2 (or S) chart is greater power to detect increases in the process variability. However, when the user is interested in detecting increases along with decreases in the process variability, S^2 (or S) chart designed with two-sided control limits is appropriate (information about process deterioration as well as process improvement). Furthermore, it is important to detect whether the process dispersion decreases, through a two-sided S^2 (or S) chart, because the limits of $\bar{X}$ control chart depend on the estimated process standard deviation. For example, if this standard deviation has shifted to a smaller value (possibly due to a process improvement during the Phase II monitoring), the IC and out-of-control (OOC) performances of the $\bar{X}$ chart deteriorate and improve, respectively. However, there would be mistaken measurements of these both performances unless such decrease is detected and the control limits are recalculated using the new estimate of the process standard deviation.

The existing studies on the performance and design of S^2 (and S) control charts with estimated parameters have been conducted considering either upper one-sided chart or two-sided chart, alternately, but never on both, and have focused on either *unconditional* perspective or *conditional* one. In addition, all works on the performance and design of the S^2 (and S) control charts under the *unconditional* perspective have considered only two-sided charts. On the other hand, works under the *conditional* perspective considered only one-sided charts.

Failing to distinguish between the particularities of the two existing perspectives and between the characteristics of the two types of S^2 charts can give rise to a likelihood of confusion among practitioners, and thus lead to making wrong conclusions about Phase II performance and design of S^2 chart. Accordingly, a comparison study between both perspectives enables practitioners to gain information for constructing S^2 control charts that guarantee a specified chart performance based on each perspective so that the differences of the design of S^2 chart can be clearly identified. A comparison study between both types of charts can also be useful for practitioners to understand the performance differences between these charts, in order to be aware of them in making decision about chart design of a particular application. Additionally, these comparisons are relevant because there are some involved cost, including cost of sampling, cost of unnecessary production stops due to false alarms and costs of producing defective units as well as low-quality units.

Under some assumptions (namely, normally distributed data, charts with probability limits, the use of the pooled sample variance (S_p^2) and its square root as estimators of σ_0^2 and σ_0 for the S^2 and S control charts, respectively), the control limit(s) of the S chart equals the square root of the corresponding limit(s) of the S^2 chart (for the same specified false alarm rate (α), and same number and size of Phase I samples (m and n)), considering one-sided chart as well as equal-tailed two-sided chart. Consequently, these charts are completely equivalent and could be used interchangeably: they would signal exactly at the same monitoring time (the same samples), and will have the same performance; for this reason, the focus is restricted, for simplicity, to the S^2 chart; the reader should keep in mind that all results presented in this work apply to the S charts with limits that are the square root of the S^2 chart limits. Based on similar assumptions, but in the context of our proposed tolerance intervals, if the construction of exact two-sided tolerance intervals for the populations of sample standard deviations is required, the pooled sample standard deviation (S_p) must be used as the estimator of the standard deviation of the analyzed normal population σ (which has been suggested in the literature, see, for example, Mahmoud et al., 2010) so that the required tolerance factors can be obtained by simply taking the square root of the corresponding tolerance factors of tolerance intervals for sample variances.

With this background outlined above, this work seeks to achieve the following objectives:

1. To derive the formulas for the tolerance factors required to construct exact two-sided statistical tolerance intervals for the population of sample variances for data that come from a normal distribution. These obtained tolerance factors are tabulated for various cases (settings) to be implemented in practice.

2. To derive approximate tolerance factors to obtain approximations for the aforementioned tolerance intervals (item 1) since the computation of the exact tolerance limits for sample variances is a non-trivial problem. Indeed,

the corresponding exact tolerance factors cannot be provided in closed-form equations and a numerical method is necessary to solve a system of nonlinear equations. With this in mind, two approximate methods are proposed to facilitate the construction of tolerance intervals in practice. To the best of our knowledge, there is no study in the literature about exact and approximate two-sided tolerance intervals for sample variances;

3. To derive the formula of the exact cdf of the conditional average run length $CARL$ (and the exact cdf of its reciprocal, the conditional probability of a signal (CPS)) of the two-sided S^2 control chart. Since we identified and highlighted the close mathematical-statistical relationship between the two-sided tolerance intervals for sample variances and the exceedance probability (EP) of the $CARL$ (or, equivalently, the cdf of the CPS), we exploit this relationship so that these exact cdf's are derived on the basis of the exact tolerance intervals for sample variances, which are addressed in item 1. These cdf's had been identified as research gaps in the literature review we undertook. However, when the present work was in progress and these cdf's were already derived, the paper by Guo & Wang (2017) appeared providing these cdf's.

4. To compare the *conditional* and *unconditional* perspectives for designing the Phase II S^2 control charts, considering two-sided chart as well as one-sided chart. Furthermore, we compare the one-sided and two-sided S^2 control charts, in terms of their Phase II performance and design, considering unadjusted limits and adjusted limits (aiming to guarantee a desired IC performance of the chart) according to each perspective (*conditional* or *unconditional*). The cdf of the $CARL_0$, which is addressed in item 3, is required in these comparisons, namely, when the EP and the *Exceedance Probability Criterion* (EPC) for the $CARL_0$ are used. Some results unavailable in the previous literature and necessary for the detailed comparisons are presented, such as

- for the two-sided S^2 chart: the minimum numbers of Phase I samples (m) required to guarantee a specified conditional IC performance based on the EPC for the $CARL_0$;
- for the two-sided S^2 chart and adjustments under the *conditional* perspective: for a given amount of Phase I data, the adjusted control limit factors that guarantee a specified conditional IC performance based on the EPC for the $CARL_0$. As noted above, when the present work was in progress and the adjusted control limit factors were already obtained, the paper by Guo & Wang (2017) appeared providing these factors;
- for the one-sided S^2 chart and adjustments under the *unconditional* perspective: the adjusted control limit factors that guarantee the desired unconditional average run length (ARL_0) for a given amount of Phase I data;
- for the one-sided and two-sided S^2 charts and adjustments under the *unconditional* perspective: assessment of the resulting IC performance of the S^2 chart using the EP of the $CARL_0$ (*conditional* perspective) when the control limit(s) is(are) adjusted according to the *unconditional* perspective (to attain a specified unconditional ARL_0); and
- the analysis of the characteristics and differences between the $CARL_0$ distributions of the one-sided and two-sided S^2 charts, which is fundamental for this comparative study.

5. To derive the formulas of the exact cdf's of the conditional run length q-quantile (CRL_q) of the one- and two-sided S^2 chart. The derivation of this distribution for the one-sided chart is based on exact analytical derivations. The CRL_q distribution for the two-sided chart is based on the exact two-sided tolerance intervals for sample variances, which are addressed in item 1. These two distributions are presented for the first time in the literature. The $CRL_{0,0.05}$ and $CRL_{0,0.5}$ (the conditional in-control median run length) may serve as complementary performance measures to the traditional mean of the conditional in-control run length distribution ($CARL_0$) since the distribution of the conditional run length is generally highly right-skewed.

1.2
Methodology of this work

First of all, a thorough literature review on the effect of parameter estimation on the Phase II performance of control charts as well as on tolerance intervals were made. Therefore, some research gaps on this topic regarding S^2 (and S) control charts along with the tolerance intervals for sample variances were identified and the practical relevance of them were verified. In this way, formulas, results and comparisons related to the aforementioned five objectives have not been conducted yet in the previous literature and, therefore, they are addressed throughout this work.

To achieve the objectives of the present work, some techniques of mathematical statistics were used to obtain the required formulas and the results of the five objectives:

- Differential and integral calculus. For instance, the derivation of the formula of the exact tolerance factors of the two-sided tolerance limits for sample variances (and thus the exact cdf of the $CARL$) requires the analysis of first and second derivatives of the actual coverage of the tolerance interval. Moreover, the computation of the expected value and the standard deviation of the $CARL_0$ requires the use of integral calculus.
- Probability theory. Analytical derivations based on properties of continuous and discrete distributions are used. For instance, the derivation of the exact cdf of $CARL$ (and hence the corresponding EPC for $CARL_0$) and the exact cdf of CRL_q are obtained using some properties of chi-squared and geometric distributions, respectively.
- Numerical analysis. A root-finding algorithm called the secant method is used because numerical solutions of systems of nonlinear equations are required. For instance, exact tolerance factors of two-sided tolerance limits for sample variances, the exact cdf 's of the $CARL$, minimum number of Phase I samples to ensure a conditional IC performance and adjusted control limits factors of the two-sided S^2 control chart using the EPC for the $CARL_0$ (*conditional* perspective), which aren't expressed in closed-form equations, required this secant method.

- Normal approximation (or transformation) methods. Two approximate methods for computing two-sided tolerance limits for the population of sample variances are used. These methods are based on the Wilson-Hilferty normal approximation (see Wilson and Hilferty, 1931) and the Krishnamoorthy-Mathew-Mukherjee approximation (see Krishnamoorthy et al., 2008).

R-codes were written to apply these techniques of mathematical statistics indicated above and make all the computations and plots presented in this work. Some of the main R-codes are provided in Appendix F.

1.3
Organization of the Thesis

The remainder of this work is organized as follows:

- Chapter 2 presents a comprehensive literature review on the previous works about exact and approximate tolerance intervals (especially, for a gamma distribution), the effect of parameter estimation on the Phase II performance of S^2 and S control charts, the design of these charts, and the recently published works about these topics under the *unconditional* and *conditional* perspectives. The definition of two-sided tolerance intervals for sample variances, the formulas of control limits and the Phase II performance of the S^2 charts are also presented.
- Chapter 3 provides the derivations of the exact two-sided tolerance limits for the population of sample variances and also two approximate methods for computing these tolerance limits. These methods are based on the Wilson-Hilferty (WH) normal approximation for a chi-square random variable. The first method is called the conditional expectation (CE) method. The second method, which is based on an adaptation of approximate tolerance intervals for a gamma distribution, is called the Krishnamoorthy-Mathew-Mukherjee (KMM) method. A comparison of the tolerance factors based on the proposed (CE and KMM) approximate methods and the exact tolerance factors is made. To illustrate these methods,

real data from an application is used to compare the exact and approximate tolerance limits for sample variances.

- Chapter 4 presents the relationship between the two-sided tolerance intervals for sample variances and the EP of the $CARL$ (or the cdf of the CPS) of the S^2 control chart so that the EPC for the $CARL_0$, required for designing S^2 control chart under the *conditional* perspective, is applied on the basis of the derivation of the exact two-sided tolerance intervals for sample variances. Furthermore, formulas of the conditional run length q-quantile (CRL_q) of the upper one-sided S^2 chart and its distribution are provided.
- Chapter 5 provides the comparison between the *conditional* and *unconditional* perspectives for designing the Phase II S^2 control charts, considering two-sided chart as well as upper one-sided chart. Moreover, the one-sided and two-sided S^2 control charts are compared, in terms of their Phase II performance and design, considering unadjusted limits and adjusted limits (in order to guarantee a desired IC performance of the chart) according to each perspective (*conditional* and *unconditional*).
- Chapter 6 presents some conclusions of this work and some practical recommendations.

To complement the understanding of this work, appendixes provide tables of exact tolerance factors of two-sided tolerance intervals for sample variances considering different settings (cases), and some extra proofs and derivations to obtain some of the proposed formulas.

2
Literature review and basic concepts

2.1
Previous works

Most of the published literature on statistical inference considers three principal types of statistical intervals, namely, the confidence interval, the prediction interval and the tolerance interval (see, for example, Hahn, 1970a, b; Vardeman, 1992; Ryan, 2007; Meeker et al., 2017). The proper election of an interval depends on the underlying problem. Nevertheless, there is sometimes a mistaken or confused use of statistical intervals in practice. Due to this issue, Meeker et al. (2017) provide a useful guideline to choice a suitable statistical interval for a specific application.

The first studies on the statistical tolerance interval (addressed in this work) were carried out by Wilks (1941, 1942), Wald (1943) and Wald and Wolfowitz (1946) in the traditional case of a normal population. From then on, several researches have been conducted to construct tolerance intervals parametrically, considering different distributions, either continuous (e.g., Shirke et al., 2005; Chen & Ye, 2017) or discrete (e.g., Cai & Wang, 2009; Wang & Tsung, 2009) as well as nonparametrically that apply to any continuous distribution (e.g., Ahmadi & Arghami, 2003; Young & Mathew, 2014). Bayesian tolerance intervals have been studied, for example, by Aitchison (1964) and Hamada et al. (2004). There are also some works on tolerance intervals for regression and multivariate normal settings (see, for instance, Lee & Mathew, 2004; Krishnamoorthy & Mondal, 2006). Earlier literature reviews were carried out by Jílek (1981), Patel (1986) and Jílek & Ackermann (1989). In more recent literature, the books by Krishnamoorthy & Mathew (2009) and Meeker et al. (2017) provide complete developments, applications and theory of tolerance intervals. Tolerance intervals have wide applicability in manufacturing (quality control), engineering (reliability) and contemporary applications of several fields of science, such as Economy, Medicine, Pharmacy, Biochemistry, Hydrology, Meteorology, Environmental Science, and

Occupational and Industrial Hygiene (see, for instance, Millard & Neerchal, 2000; Gibbons et al., 2001, 2009; Fernandez, 2010; Lee & Liao, 2012).

Tolerance intervals are routinely used in quality control of manufacturing applications, namely, in conformity assessment and acceptance of products or processes (product acceptance sampling). The quality assessment can be carried out by checking whether the tolerance interval obtained for a quality characteristic of a product (or a process) is contained within its defined specification limits (see, for example, Lai et al., 2012; Dong et al., 2015b), that is, tolerance intervals are often useful to analyze the process capability. Montgomery (2009, p. 389) states in this regard, "…*unless the product specifications exactly coincide with or exceed the tolerance limits of the process, an extremely high percentage of the production will be outside specifications, resulting in a high loss or rework rate*…" Moreover, tolerance intervals are sometimes used in setting or revising product specification limits since them are not well-defined and there is a lack of previous knowledge of the process (see, for instance, Dong et al., 2015a).

In the construction of tolerance intervals, similarly to the case of the hypothesis test and confidence intervals, the user should decide to use either a directional (one-sided) or nondirectional (two-sided) interval. This choice is made according to the context of application, the questions of interest, and the nature of the quality characteristic. Although both types of tolerance intervals have received attention in practice, some authors, including Meeker at al. (2017) and Fraser (2011), have suggested the construction and report of two-sided intervals since the examination of the two tails of the corresponding distribution can provide more complete information. Indeed, Meeker et al. (2017, p. 33) state, "…*there are many applications for which one is primarily interested in either a lower bound or an upper bound. Even in such situations it is often still convenient to report a two-sided interval*…" As noted before, tolerance intervals for sample variances can be useful when the main interest is in the precision of the values of the quality characteristic of interest. Tietjen & Johnson (1979) considered one-sided upper tolerance limits, while the two-sided ones, which have not been yet studied, are proposed in this work.

Setting tolerance limits, especially in the case of two-sided limits that usually cannot be expressed in a closed-form, is mathematically more challenging than setting confidence and prediction limits because the computation of exact tolerance limits is generally based on numerical methods. Special statistical software packages, which are available to find tolerance factors (or limits) for different probability distributions (e.g., the R package “tolerance” by Young, 2010, 2014b), may be complicated for a non-expert user. Furthermore, these exact factors are sometimes provided in tables to expedite the construction of tolerance intervals, for instance, the one-sided and two-sided tolerance factors for the normal distribution, the one-sided tolerance factors for the Weibull distribution and the one-sided tolerance factors for the two-parameter exponential distribution are tabulated by Krishnamoorthy & Mathew (2009). Nevertheless, tables of factors may offer incomplete information because a limited number of settings (combinations of sample size, nominal proportion, and confidence level) is frequently made available. Due to this difficulty, several authors have proposed approximate methods for computing tolerance limits for some probability distributions. For example, Howe (1969) and Jensen (2009) worked on approximations of tolerance limits for the normal population, and Bain & Engelhardt (1981) and Rinne (2008) for the Weibull distribution. In the case of discrete distributions, author such as Krishnamoorthy et al. (2011), Young (2014a) provided approximate tolerance intervals for binomial, Poisson and negative binomial distributions.

The computation of exact one-sided upper tolerance limit for the population of sample variances is not a demanding task (see Tietjen & Johnson, 1979) because this is equivalent to the computation of a one-sided upper confidence limit on a certain quantile of the distribution of the sample variance. On the other hand, since the construction of exact two-sided tolerance limits (the focus of this work) is not reduced to the computation of a two-sided confidence interval for a certain quantile, and then the corresponding computation is not straightforward, approximate tolerance limits for sample variances are proposed so that they can easily be implemented in practice. For that purpose, first of all, notice that the population of sample variances follows a multiple of a chi-square distribution or, equivalently, a gamma distribution. Consequently, previous studies of approximate tolerance limits for a gamma distribution can be useful. First works in this line assumed that at least one of the two distribution parameters is known or accurately estimated from a specific reference sample (see Bain et al., 1984; Ashkar & Ouarda, 1998). Because both parameters are almost always unknown in practice, other approximate intervals that consider the effect of parameter estimation have arisen. Namely, Aryal et al. (2008) proposed one-sided tolerance limits for a gamma distribution using a log-normal approximation, while Krishnamoorthy et al. (2008) proposed one- and two-sided tolerance limits using the Wilson-Hilferty (WH) normal approximation (see Wilson and Hilferty, 1931). WH approximation can be described as follows: the cube-root of the ratio between a chi-square random variable and its degrees of freedom approximately follows a normal distribution, whose mean and variance depending on these degrees of freedom. In recent years, Chen & Ye (2017) presented the construction of approximate one-sided tolerance limits, which are not expressed in closed-form, using the generalized fiducial method (see, for more details of this method, Hannig, 2009). With this framework, two methods to compute approximate two-sided tolerance limits for sample variances are presented in this work. One of these proposed methods is based on the WH transformation, and the other one is adapted from the article by Krishnamoorthy et al. (2008) to obtain a closed-form formula.

With regard to SPC, the majority of works about control charts with estimated parameters obtained from a Phase I reference dataset (for more information about Phase I analysis, see Chakraborti et al., 2009; Jones-Farmer et al., 2014) and the effect of parameters estimation on the Phase II chart performance have focused on the *unconditional* perspective, i.e., on the marginal (unconditional) distribution of the RL distribution (detailed literature reviews on this effect is conducted by Jensen et al., 2006; Psarakis et al., 2014). Jensen et al. (2006) identified three main issues on this topic and asked the following related questions (i) "…*Just how poorly (or well) might a chart perform if designed with estimates in place of known parameters?*", (ii) "*What sample size is needed in Phase I to ensure adequate performance in Phase II?*", and (iii) "*How should the Phase II limits be adjusted to compensate for the size of the Phase I sample?*"

Regarding the answers to the first and second questions for two-sided S^2 and S charts designed with probability and equal-tailed limits, namely, to achieve "proper" mean and standard deviation of the unconditional IC RL distribution (denoted ARL_0 and $SDRL_0$, respectively) close to the ones of the chart with known parameter, Chen (1998) suggested that the user should have at least 75 samples of size 5. In a similar way, but using simulated values of the unconditional ARL_0 and $SDRL_0$, Maravelakis et al. (2002) recommended at least $m = 200$ with $5 \leq n \leq 20$ or at least $m = 100$ with $n > 20$. Castagliola et al. (2009) assessed the absolute percentage difference between the cdf of the $IC\ RL$ in the variance-known case and its unconditional cdf in the variance-unknown case, and suggested at least $m = 200$ to achieve absolute differences of no more than 2%.

The three main issues, indicated by Jensen et al. (2006) above, have also been addressed according to the prominent *conditional* perspective since mid-2015. Epprecht et al. (2015) presented the first work of the conditional performance of the upper one-sided S^2 and S charts, while Saleh et al. (2015), who introduced the term "practitioner-to-practitioner variability", worked on the $\bar{X}$ and X charts according to the *conditional* perspective for the first time. With respect to the first two issues, Epprecht et al. (2015) provided the cdf of the $CFAR$ (or, equivalently, the cdf of the $CARL_0$) and the required amounts of Phase I reference data to ensure a specified minimum tolerated IC performance in terms of the *Exceedance Probability Criterion* EPC (proposed by Albers & Kallenberg, 2005; Albers et al., 2005) for the $CARL_0$ of the upper one-sided S^2 (and S) charts. Their findings revealed that these amounts should be even larger (several hundreds or even some thousands) than the amounts found on the basis of the *unconditional* perspective mentioned earlier. Table 1 summarizes our literature review related to the Jensen's second issue according to the perspective adopted, that is, *unconditional* (an unconditional ARL_0 close to a nominal value) or *conditional* (using the EPC for the $CARL_0$), and type of chart considered (one-sided or equal-tailed two-sided charts with probability limits). From Table 1, it is important to note that all authors that addressed the *unconditional* perspective considered only two-sided charts, while the unique authors that focused on the *conditional* perspective considered only upper one-sided charts. Therefore, works on S^2 and S charts, which are shown in Table 1, partially address the second major question (mentioned by Jensen et al., 2006) that will be completed and seen with more detail in Subchapter 5.2.

Table 1 - Overview of the literature on the amount of Phase I reference data required for guaranteeing a desired Phase II performance under the *unconditional* and *conditional* perspectives for the one-sided and two-sided S^2 and S charts

Unconditional Perspective			**Conditional Perspective**		
Limits	**Reference**	**Chart/ Method**	**Limits**	**Reference**	**Chart/ Method**
Two-sided	Chen (1998)	S and S^2/ Numerical integration (exact)	**One-sided**	Epprecht et al. (2015)	S and S^2/ Search algorithm (exact)
	Castagliola et al. (2009)	S^2 / Numerical integration (exact)			
	Maravelakis et al. (2012)	S / Simulation (approximation)			

Since the amount of Phase I reference data is usually large and often unfeasible, some authors proposed adjustments of control limits to overcome this problem (third question quoted by Jensen et al., 2006). Under the *unconditional* perspective, Castagliola et al. (2009) and Diko et al. (2017) proposed adjusted control limits based on a desired unconditional ARL_0 for the two-sided S^2 chart (using a numerical method) and for the two-sided S chart (using a first-order Taylor approximation and a numerical method), respectively. Our literature review on S^2 and S charts with control limits adjustments considers a desired unconditional ARL_0 as the aim in the *unconditional* perspective. There are other authors who considered other aims, for instance, Yang and Hillier (1970) and Schoonhoven et al. (2011) found adjusted factors of the two-sided S^2 and S control limits to achieve a specified unconditional false-alarm rate, however, as argued by Quesenberry (1993), initially for the case of $\bar{X}$ charts, the adjustments proposed were not reliable, since the authors disconsidered that the signaling events are dependent. Along the same lines as in Quesenberry (1993), Maravelakis et al. (2002) demonstrated this dependence in the case of the S control chart.

On the other hand, regarding the *conditional perspective*, Faraz et al. (2015, 2018) and Goedhart et al. (2017a) proposed adjustments to the upper one-sided S and S^2 control limits for guaranteeing a specified minimum tolerated IC performance (also using the EPC for the $CARL_0$). Faraz et al. (2015) used the bootstrap approach proposed by Gandy & Kvaløy (2013); while Goedhart et al. (2017a) and Faraz et al. (2018) used analytical derivations. To complement the overview, Table 2 organizes the literature review (related to the third question quoted by Jensen et al., 2006) by perspective adopted, namely, *unconditional* (for achieving a desired unconditional ARL_0) or *conditional* (using the EPC for the

$CARL_0$), and type of chart considered (one-sided or equal-tailed two-sided charts with probability limits). Note that all works proposing adjustments to the control limits under the *unconditional* perspective have considered only two-sided charts. On the other hand, works under the *conditional* perspective considered only one-sided charts. However, as noted earlier, when this work was in progress and the adjusted control limit factors were already obtained, Guo & Wang (2017), which were not shown in Table 2, appeared providing the adjustments to control limits (Jensen's third issue) of the two-sided S^2 chart. Therefore, works, which are shown in Table 2, partially deal with the third main question (mentioned by Jensen et al., 2006) that will be completed and examined with more detail in Subchapter 5.3. Motivated by this background that is summarized in Tables 1 and 2, the present work seeks to fill the indicated gaps. This entails that this work pursues to examine the three issues indicated by Jensen et al. (2006) for the two-sided S^2 chart with estimated parameter, but according to the *conditional* perspective. Note that these three issues can be addressed on the basis of the tolerance limits for sample variances, that is, taking advantage of the close mathematical-statistical relationship between the tolerance intervals for sample variances and the EP of the $CARL_0$ (or, equivalently, the cdf of the $CFAR$) of the S^2 control chart.

Table 2 - Overview of the literature on adjustments of probability control limit(s) of the one-sided and two-sided S^2 and S charts under the *unconditional* and *conditional* perspectives

Unconditional Perspective			**Conditional Perspective**		
Limits	**Reference**	**Chart/ Method**	**Limits**	**Reference**	**Chart/ Method**
Two-sided	Castagliola et al. (2009)	S^2 / Numerical integration (exact)	**One-sided**	Faraz et al. (2015)	S^2 / Bootstrap simulation (approximation)
	Diko et al. (2017)	S / Numerical integration (exact) and Taylor approximation		Goedhart et al. (2017a), Faraz et al. (2018)	S and S^2 / Analytical derivation (exact)

For the two-sided charts (under the *unconditional* and *conditional* perspectives), the presented literature review focuses on the equal-tailed design (see Table 2). However, other alternative chart designs have been proposed in the literature. For instance, for the S^2 chart design, Zhang et al. (2005) proposed the "smallest area criterion" (SAC) and the "two points criterion" (TPC), and Guo & Wang (2015) presented the ARL-unbiased criterion (applied initially by Champ & Lowry, 1994; Pignatiello et al., 1995 for S charts when σ_0 is known). Moreover, for the S^2 chart and using EPC, a comparative study of the equal-tailed design and the

ARL-unbiased design was made by Guo & Wang (2017). The chosen equal-tailed design is the most common one and presents shorter length of the interval than those of other mentioned types of designs (see, for instance, comparison studies by Zhang et al., 2005; Guo & Wang, 2017 under the *unconditional* and *conditional* perspectives, respectively).

Additionally, several authors, including Chakraborti (2007), Mei (2008), Zhou et al. (2012), Graham et al. (2014), Woodall & Montgomery (2014) and Teoh et al. (2016), have recommended the analysis of the entire RL distribution and its quantiles (such as some extreme quantiles and the median), and not only the ARL, because this distribution is usually right-skewed. The first works about the quantiles of the RL distribution as performance measures of control charts are due to Barnard (1959) and Bissell (1969). Radson & Boyd (2005) provided a short review of the literature and proposed a modified boxplot, which is very useful for displaying the complete RL distribution and some of its main quantiles. Khoo (2004) and Chakraborti (2007) studied the RL distribution and its quantiles for the Shewhart $\bar{X}$ chart when the process parameters are known and unknown (considering the unconditional RL quantiles), respectively. The latter author also proposed the adjustments of control limits based on a desired in-control median run-length (MRL_0). During recent years, in the same line, some authors have recommended this MRL_0 criterion for different types of control charts, for instance, Lee & Khoo (2006), Khoo et al. (2011) and Teoh et al. (2014, 2016). These last mentioned studies have addressed only the unconditional RL distribution. Since, as far as we know, there has been no study in the literature to date focusing on the conditional run length quantiles (CRL_q), these are derived and examined for the one- and two-sided S^2 charts in this work.

2.2 Definition of two-sided tolerance intervals for sample variances

Sarmiento et al. (2018), which is one of the research articles that arises from the present work, provide a study of the construction of exact two-sided statistical tolerance intervals for the population of sample variances based on the assumption of normally distributed data. In this subchapter, a briefly definition of these tolerance intervals is presented.

Let S^2 be the (test or future) variance of a random sample of size n, observations of which come from the normal distribution (with unknown mean μ and unknown variance σ^2) of the random variable X that represents a product (or a process) quality characteristic of interest. This S^2 is obtained from a (test or future) sample $\{X'_1, X'_2, \dots, X'_n\}$ and is given by $S^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X'_j - \bar{X}'\right)^2$, where the (test or future) observations X'_j are iid ($X'_j \sim N(\mu, \sigma^2), j = 1, 2, \dots, n$) and $\bar{X}' = \frac{1}{n}\sum_{j=1}^{n} X'_j$.

In the same way that other types of tolerance intervals (see, for instance, Krishnamoorthy & Mathew, 2009; Meeker et al., 2017), the construction of tolerance intervals for sample variances depends on the estimation of the unknown parameters of the underlying distribution, namely, on the estimation of σ^2. This makes sense since, in most practical applications, the parameters of the population of interest are unknown and must be estimated on the basis of a reference dataset collected from a process that must be in a state of statistical control (similarly to the context of SPC, in the so-called Phase I analysis).

The $(1-\beta,\ \gamma)$ two-sided tolerance interval for the population of (test or future) sample variance S^2 $\left(\hat{S}^2_{L^*}, \hat{S}^2_{U^*}\right)$ is defined as the interval that contains at least a specified $(1-\beta)100\%$ of the population with a specified probability γ, and is given as follows

$$P_{\hat{\sigma}^2}\left(P_{S^2}\left(\hat{S}^2_{L^*} \le S^2 \le \hat{S}^2_{U^*} \mid \hat{\sigma}^2\right) \ge 1-\beta\right) = \gamma, \qquad (1)$$

where $\hat{S}^2_{L^*}$ and $\hat{S}^2_{U^*}$ are the lower and upper tolerance limits, respectively, $1-\beta$ is the nominal proportion or content ($0 < 1-\beta < 1$), γ is the confidence level ($0 < \gamma < 1$) of the tolerance interval, and $P_{S^2}\left(\hat{S}^2_{L^*} \le S^2 \le \hat{S}^2_{U^*} \mid \hat{\sigma}^2\right)$ is the actual coverage. In practical applications, $1-\beta$ and γ are large, and usually greater than or equal to 0.90.

The distribution of S^2 is a multiple $\left(\frac{\sigma^2}{n-1}\right)$ of the chi-square distribution with $(n-1)$ degrees of freedom (df). Since the S^2 distribution is most often right-skewed, we consider "equal-tailed" (lower and upper) probability tolerance limits rather than the traditional "k-sigma" limits. In the unrealistic (theoretical) case where the parameter σ^2 is known, the lower and upper tolerance limits are constant values: $\frac{\sigma^2}{n-1}\chi^2_{n-1,\frac{\beta}{2}}$ and $\frac{\sigma^2}{n-1}\chi^2_{n-1,1-\frac{\beta}{2}}$, respectively, so that the actual coverage attains its specified minimum value (nominal proportion) of $1-\beta$. Note that the proposed "equal-tailed" lower and upper tolerance limits consider the $\left(\frac{\beta}{2}\right)$ and the $\left(1-\frac{\beta}{2}\right)$ quantiles of a chi-square distribution with $(n-1)$ df. However, the concern focuses on the σ^2-unknown case, where σ^2 is estimated from the Phase I reference data, namely, from m independent random samples of equal size n. Therefore, the tolerance interval is a random interval, and the lower and upper tolerance limits are random variables that are defined, respectively, by

$$\hat{S}^2_{L^*} = \frac{\hat{\sigma}^2}{n-1}\chi^2_{n-1,\frac{\beta^*}{2}} \text{ and } \hat{S}^2_{U^*} = \frac{\hat{\sigma}^2}{n-1}\chi^2_{n-1,1-\frac{\beta^*}{2}},$$

where the quantity β^* is the "adjusted" value of β. This β^* takes into account the error of the variance estimation and, hence, β^* depends on m and n (the amount of Phase I reference data) in order to guarantee an interval that covers at least the nominal proportion $1-\beta$ of the population of sample variances with a specified confidence level γ (so, β^* also depends on β and γ).

The lack of knowledge of σ^2 is taken into consideration by the specified γ associated with the tolerance interval. Note that the actual coverage of the S^2 distribution contained within the tolerance interval is a random variable because this coverage depends on the estimator $\hat{\sigma}^2$. We consider that σ^2 is estimated by the unbiased pooled sample variance (S_p^2) as follows

$$S_p^2 = \frac{1}{m}\sum_{i=1}^{m} S_i^2 = \frac{1}{m(n-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij}-\bar{X}_i\right)^2, \qquad (2)$$

where S_i^2 is the i-th Phase I sample variance, $\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ is the i-th sample mean ($i = 1,2,\dots,m$ and $j = 1,2,\dots,n$) and X_{ij} is the j-th observation of the i-th sample in Phase I (the observations X_{ij} are iid $N(\mu,\sigma^2)$). Then, using $\hat{\sigma}^2 = S_p^2$, the two-sided (lower and upper) tolerance limits for sample variances ($\hat{S}^2_{L^*}$, $\hat{S}^2_{U^*}$) are given by

$$\hat{S}^2_{L^*} = \frac{\chi^2_{n-1,\frac{\beta^*}{2}}}{n-1} S_p^2 \text{ and } \hat{S}^2_{U^*} = \frac{\chi^2_{n-1,1-\frac{\beta^*}{2}}}{n-1} S_p^2.$$

Note that the tolerance limits are expressed as multiples of the variance estimator S_p^2. These multiples are denoted by L^* and U^* and are called the lower and upper tolerance factors, respectively, of the $(1-\beta,\ \gamma)$ two-sided tolerance interval for the sample variance:

$$L^* = \frac{\chi^2_{n-1,\frac{\beta^*}{2}}}{n-1} \text{ and } U^* = \frac{\chi^2_{n-1,1-\frac{\beta^*}{2}}}{n-1}. \tag{3}$$

Next, the corresponding lower and upper tolerance limits $\left(\hat{S}^2_{L^*}, \hat{S}^2_{U^*}\right)$ can be computed as follows:

$$\hat{S}^2_{L^*} = L^* S^2_p \text{ and } \hat{S}^2_{U^*} = U^* S^2_p. \tag{4}$$

Accordingly, the value of β^* should be found to compute the tolerance factors so that the tolerance limits can be established.

2.3
The Control Limits of the S^2 Charts

The control limits of one-sided and two-sided S^2 charts are shown in this subchapter. As in the case of tolerance intervals for sample variances described above, recall that normality and independence of the underlying data are assumed, probability limits are used rather than "three-sigma" limits (as have been suggested and justified by Epprecht et al., 2015; Woodall, 2017; Diko et al., 2017) and, for the two-sided chart, the equal-tailed design is considered. When the process is operating with natural (or inherent) variability, the process is in a state of statistical control. This is called the in-control (IC) process, whose sample observations (X_{ij}) follow a normal distribution with in-control process mean μ_0 and variance σ_0^2, that is, X_{ij} are iid $N(\mu_0, \sigma_0^2)$ when the process is IC. On the other hand, when the process operates with the presence of assignable causes of variation, the process is out-of-control (OOC). In this situation, since the focus is the monitoring and control of the process variability, given a certain shift in the process variance at a given time, the process variance becomes σ_1^2 and the process mean is assumed to remain unchanged at the IC value μ_0, that is, X_{ij} are iid $N(\mu_0, \sigma_1^2)$ when the process is OOC.

In the ideal (but not typical) case that σ_0^2 is specified or known exactly (the so-called variance-known case), the upper control limit of the one-sided S^2 control chart (UCL_{one}) is given by

$$UCL_{one} = \frac{\chi^2_{n-1,1-\alpha}}{(n-1)} \sigma_0^2 = U_{one} \sigma_0^2, \tag{5}$$

and the lower and upper control limits of the equal-tailed two-sided S^2 control chart (LCL_{two} and UCL_{two}, respectively) are given by

$$LCL_{two} = \frac{\chi^2_{n-1,\frac{\alpha}{2}}}{(n-1)}\sigma_0^2 = L_{two}\sigma_0^2 \text{ and } UCL_{two} = \frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{(n-1)}\sigma_0^2 = U_{two}\sigma_0^2. \quad (6)$$

In Equations (5) and (6), n is the size of samples (subgroups) in prospective process monitoring and α denotes the nominal false alarm rate or the preassigned probability of type I error of the S^2 control chart (the traditional α is 0.0027). The upper one-sided and two-sided limits are multiples of σ_0^2, which are the upper one-sided control limit factor (U_{one}) and the lower and upper two-sided control limit factors (L_{two}, U_{two}), respectively. LCL_{two} and UCL_{two} are called equal-tailed limits because are multiples of the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles of a chi-square distribution with $(n-1)$ df (denote as $\chi^2_{n-1,\frac{\alpha}{2}}$ and $\chi^2_{n-1,1-\frac{\alpha}{2}}$, respectively). In the remainder of this work, subscripts one and two will be used when necessary to indicate that we are considering the one-sided or the two-sided S^2 control chart, when it is the case, and will be omitted when a symbol or equation applies to both.

As highlighted earlier, in most real applications, the IC process parameters are unknown and need to be estimated from the Phase I reference data, which are collected when the process is IC and are made up of m independent samples (subgroups) each of size n. Similarly to the defined tolerance interval, we consider that the Phase I estimator of σ_0^2 ($\hat{\sigma}_0^2$) is the efficient (unbiased) multi-sample estimator, the so-called pooled sample variance (S_p^2) given by Equation (2). Recall that, differently from the parameters of the normal population in the context of tolerance interval, each j-*th* observation of the i-*th* sample (X_{ij}) in Phase I ($i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$) is $N(\mu_0,\sigma_0^2)$. Then, considering $\hat{\sigma}_0^2 = S_p^2$, the traditional (unadjusted) upper control limit of the one-sided S^2 control chart with estimated σ_0^2 ($\widehat{UCL}_{one}$) is given by

$$\widehat{UCL}_{one} = \frac{\chi^2_{n-1,1-\alpha}}{(n-1)}S_p^2 = U_{one}S_p^2, \quad (7)$$

and the traditional (unadjusted) lower and upper control limits of the equal-tailed two-sided S^2 control chart with estimated σ_0^2 ($\widehat{LCL}_{two}$ and $\widehat{UCL}_{two}$, respectively) are given by

$$\widehat{LCL}_{two} = \frac{\chi^2_{n-1,\frac{\alpha}{2}}}{(n-1)} S_p^2 = L_{two} S_p^2 \text{ and } \widehat{UCL}_{two} = \frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{(n-1)} S_p^2 = U_{two} S_p^2. \qquad (8)$$

where n is the size of Phase I samples as well as Phase II samples. Note that U_{one} and (L_{two}, U_{two}), defined by n and α, are the (unadjusted) control limit factors of charts with limits that are not adjusted or corrected. These control limits are random variables expressed as functions of the IC process variance estimator (S_p^2).

As explained before, adjustments to the control limits are needed to ensure a specified IC chart performance, according to a certain performance criterion, with a practical amount of Phase I reference data at hand. These adjustments seek to compensate for the effects of the variance estimation on IC performance of Phase II S^2 chart, avoiding the deterioration of it. For that purpose, adjustments to control limits for one-sided and two-sided charts consist in substituting the unadjusted control limit factors, i.e., U_{one} and (L_{two}, U_{two}) obtained given a specified α, by the corresponding adjusted upper one-sided control limit factor (U^*_{one}) and the adjusted lower and upper two-sided control limit factors (L^*_{two}, U^*_{two}), respectively, so that a specified (either unconditional or conditional) IC performance can be achieved. These adjusted factors, which are required for setting the control limits of both charts, depend on the value of the adjusted false alarm rate (the adjusted α) that is denoted as α^*. Similarly to β^* in the context of tolerance intervals for sample variances, the quantity α^* recognize the effect of the σ_0^2 estimation. This means that α^*varies depending on the number (m) of Phase I samples (subgroups), the size (n) of each subgroup, and the estimator ($\hat{\sigma}_0^2$) chosen to estimate σ_0^2 ($\hat{\sigma}_0^2 = S_p^2$ is used here). The found value of α^* must result in a specified IC chart performance, which is in turn defined based on a specific criterion (*unconditional* or *conditional* perspectives). U^*_{one} is defined as a multiple $1-\alpha^*$ quantile of a chi-square distribution with ($n-1$) df, while L^*_{two} and U^*_{two} are defined as multiples of the $\frac{\alpha^*}{2}$ and $1-\frac{\alpha^*}{2}$ quantiles, respectively, so that the adjusted two-sided control limits are equal-tailed (probability) control limits, as were originally defined in the case of charts with unadjusted control limits.

Next, the Phase II performance measures of the S^2 chart with estimated parameter are examined in the following subchapter. These performance measures, which are based on the unconditional and conditional point of view, take into account the effect of the process variance estimation in such a way that the design of Phase II S^2 chart can be analyzed and implemented considering this effect.

2.4
Phase II performance of S^2 control chart based on the conditional and unconditional average run lengths

The run length (RL) is a random variable defined as the number of samples or charting statistics until a signal (alarm). Control chart performance is typically measured in terms of the RL distribution and some of its attributes, such as the expected value (the mean), that is, the so-called average run length (ARL). When the process is in control, the ARL (in this case denoted by ARL_0) is the expected number of samples until a false alarm.

In the ideal (typically unrealistic) case that the IC process parameters were known exactly, in which no Phase I is needed, or if the Phase I estimates were "absolutely accurate", the RL follows a geometric distribution with parameter equal to the specified probability of a signal so that any associated property of the RL (such as the mean, the standard deviation and the quantiles) would be a constant value, which depends on the probability of a signal. In particular, the nominal false alarm rate (α) is the distribution parameter of the run length when the process is in-control (denoted RL_0) and the corresponding mean ARL_0 equals $1/\alpha$.

In most real applications, the IC process parameters are unknown and they need to be estimated using a Phase I reference data so that the marginal (or unconditional) RL no longer follows a geometric distribution and the probability of a signal becomes a random variable. Under the conditional point of view, that is, taking account of the "practitioner-to-practitioner variability", the conditional false alarm rate $CFAR$ (i.e., conditional probability of a signal CPS when the process is in-control) and, its reciprocal, the conditional in-control average run length $CARL_0$ are random variables expressed as functions of the estimators of the IC process parameters because the chart's limits are set as functions of those estimators. Thus, for any Shewhart-type chart in a particular application, the conditional (given the parameters estimates) in-control run length (denoted as CRL_0) is also geometrically distributed, so its distribution parameter is a certain realization (given the parameters estimates) of the $CFAR$ and its mean — the $CARL_0$ — is the reciprocal of the $CFAR$.

First, the probability of a signal, which is an important measure of the Phase II performance of S^2 control chart, is examined. The CPS is the probability that each l-th charting statistic (Phase II sample variance of size n, denoted as S_l^2) falls outside the estimated (given the particular estimate $\hat{\sigma}_0^2$) control limits range during the Phase II monitoring (where $l = m+1, m+2, \ldots$). Therefore, the conditional run length (CRL) distribution (i.e., given $\hat{\sigma}_0^2$, the distribution of the number of S_l^2 until a signal) is geometric with probability of success equal to CPS.

To express the CPS in mathematical terms, let's define the process variance of the Phase II monitoring as σ_1^2. Thus, according to equations of the S^2 control limits (Equations (7) and (8)) and considering the fact that $S_l^2(n-1)/\sigma_1^2$ follows a chi-square distribution with $n-1$ df, the CPS's of the one-sided and two-sided S^2 control charts, respectively, are given by Equations (9) and (10):

$$CPS_{one}(\rho^2) = P\left(S_l^2 > \widehat{UCL}_{one}\right) = 1 - F_{\chi^2_{n-1}}\left(\frac{S_p^2}{\sigma_0^2}\frac{\chi^2_{n-1,1-\alpha}}{\rho^2}\right), \tag{9}$$

$$CPS_{two}(\rho^2) = P\left(S_l^2 < \widehat{LCL}_{two} \cup S_l^2 > \widehat{UCL}_{two}\right)$$

$$= 1 - \left(F_{\chi^2_{n-1}}\left(\frac{S_p^2}{\sigma_0^2}\frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{\rho^2}\right) - F_{\chi^2_{n-1}}\left(\frac{S_p^2}{\sigma_0^2}\frac{\chi^2_{n-1,\frac{\alpha}{2}}}{\rho^2}\right)\right), \tag{10}$$

where $F_{\chi^2_{n-1}}$ denotes the cdf of a chi-square random variable with $n-1$ df, and ρ^2 represents a shift in the process variance, i.e., the ratio between the process variances in Phases II and I (or, equivalently, ρ is a shift in the process standard deviation), and is defined by

$$\rho^2 = \sigma_1^2/\sigma_0^2. \tag{11}$$

When $\sigma_1^2 = \sigma_0^2$ ($\rho^2 = 1$), the process is IC, otherwise, the process is OOC. From Equations (9) and (10), let Y be defined as follows: $Y = m(n-1)\,S_p^2/\sigma_0^2$. Note that Y follows a chi-square distribution with $m(n-1)$ df and the error of the process variance estimation is quantified by a particular realization of the multiple of the random variable Y, namely, $Y/m(n-1)$. Next, given a value of the variance ratio ρ^2, from Equations (9) and (10), the CPS's are functions of the random variable Y. Thus, the cdf's of the geometric CRL's of the one-sided and two-sided S^2 charts are given by Equations (12) and (13), respectively, as follows:

$$F_{CRL_{one}(\rho^2)}(t) = P(CRL_{one}(\rho^2) \leq t) = P(RL_{one}(\rho^2) \leq t \mid Y)$$
$$= 1 - \left(1 - CPS_{one}(Y;\rho^2)\right)^t = 1 - \left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,1-\alpha}\right)\right)^t, \quad (12)$$

$$F_{CRL_{two}(\rho^2)}(t) = P(CRL_{two}(\rho^2) \leq t) = P(RL_{two} \leq t \mid Y)$$
$$= 1 - \left(1 - CPS_{two}(Y;\rho^2)\right)^t$$
$$= 1 - \left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,1-\frac{\alpha}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\frac{\alpha}{2}}\right)\right)^t, \quad (13)$$

where $t \geq 1$ is a real number. Because the CRL of the S^2 chart is geometrically distributed, the conditional average run length ($CARL(Y;\rho^2)$) is the reciprocal of $CPS(Y;\rho^2)$; therefore, from Equations (14) and (15), the $CARL$'s are given by

$$CARL_{one}(Y;\rho^2) = [CPS_{one}(Y;\rho^2)]^{-1}$$
$$= \left[1 - F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,1-\alpha}\right)\right]^{-1}, \quad (14)$$

$$CARL_{two}(Y;\rho^2) = [CPS_{two}(Y;\rho^2)]^{-1}$$
$$= \left[1 - \left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,1-\frac{\alpha}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\frac{\alpha}{2}}\right)\right)\right]^{-1}. \quad (15)$$

Since the CRL, by definition, is a discrete (geometric) random variable, the q-quantile of the conditional run length (CRL_q), where $0 < q < 1$, is defined as the smallest positive integer CRL_q so that the cdf of CRL at CRL_q is larger than or equal to q (i.e., $F_{CRL}(CRL_q) \geq q$). The CRL_q can be found, from Equations (12) and (13), as being the smallest integer that satisfies: $1 - \left(1 - CPS(Y;\rho^2)\right)^{CRL_q} \geq q$. Thus, from Equations (16) and (17), the CRL_q's are given by

$$CRL_{q,one}(Y;\rho^2) = \left\lceil ln(1-q)/ln\left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,1-\alpha}\right)\right)\right\rceil, \quad (16)$$

$$CRL_{q,two}(Y;\rho^2)$$
$$= \left\lceil ln(1-q)/ln\left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,1-\frac{\alpha}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\frac{\alpha}{2}}\right)\right)\right\rceil, \quad (17)$$

where $\lceil b \rceil$ denotes the smallest integer greater or equal to b (ceiling function) and then $CRL_q = \{1, 2, 3, \ldots\}$.

The r^{th} moment of the $CARL(\rho^2)$ can be obtained as the r^{th} moment of a function of the chi-square random variable Y with $m(n-1)$ df (as shown in Equations (14) and (15)). So, the first and second moments of the $CARL(Y;\rho^2)$ distribution are given by

$$E_Y\big(CARL(Y;\rho^2)\big) = ARL(\rho^2) = \int_0^\infty CARL(y;\rho^2)\, f_Y(y)dy\,, \quad (18)$$

$$E_Y(CARL(Y;\rho^2)^2) = \int_0^\infty CARL(y;\rho^2)^2\, f_Y(y)dy, \quad (19)$$

where $f_Y(y)$ denotes the probability density function (pdf) of Y. From Equation (18), an interesting point to note is that the expected value of the $CARL$ (the expected value of the marginal (unconditional) RL distribution) equals the unconditional ARL. This arises from the Law of Total Expectation, that is, $E_Y\big(CARL(Y)\big) = E_Y(E(RL|Y)) = E(RL) = ARL$. From Equations (18) and (19), the standard deviation (SD) can be found using the first and second moments of the $CARL(Y;\rho^2)$, denoted as $SDARL(\rho^2)$, as follows

$$SD\big(CARL(Y;\rho^2)\big) = SDARL(\rho^2)$$
$$= \sqrt{E_Y(CARL(Y;\rho^2)^2) - \Big(E_Y\big(CARL(Y;\rho^2)\big)\Big)^2}, \quad (20)$$

when the process is IC ($\rho^2 = 1$), the CPS represents the conditional false alarm rate (denoted $CFAR$), i.e., $CFAR(Y) = CPS(Y;\rho^2 = 1)$, the CRL_q represents the conditional in-control run length q-quantile (denoted $CRL_{0,q}$) and the $CARL$ represents the conditional in-control average run length (denoted $CARL_0$). Formally, $CARL_0(Y) = CARL(Y;\rho^2 = 1) = \big(CFAR(Y)\big)^{-1}$. Its expected value

$E_Y\big(CARL_0(Y)\big)$ and standard deviation $SDARL_0 = SD\big(CARL_0(Y)\big)$ can be obtained substituting Equations (14) and (15) into Equations (18)-(20) with $\rho = 1$. Note that the subscripts one and two do not appear in Equations (18)-(20), because this equation applies for both the one-sided and two-sided charts. We will keep this pattern throughout this work.

Figure 1 shows the curves of $CARL_0$ of one-sided and two-sided S^2 charts (obtained from Equations 14 and 15, respectively) with traditional control limits that are not adjusted ($\alpha = 0.0027$, i.e., nominal $ARL_0 = 370.4$). These curves of $CARL_0$ are parameterized by different values of m and $n = 5$. The curves of $CARL_{0,one}$ were already examined by Epprecht et al. (2015). However, as far as we know, this is the first time that the curves of $CARL_{0,two}$ are examined, published and compared with the $CARL_{0,one}$. The $CARL_0{}'$s are shown as functions of a uniform random variable (U), which is the order quantile of Y distribution. This relation is established through the probability integral transformation, namely, $Y = F^{-1}_{\chi^2_{m(n-1)}}(U)$, where $F^{-1}_{\chi^2_{m(n-1)}}$ denotes the inverse of the cdf (or the quantile function) of the chi-squared distribution with $m(n-1)$ df. Since U only takes values between 0 and 1, realizations of $CARL_0$ can be visualized clearly: $CARL_{0,one}(U)$ and $CARL_{0,two}(U)$ in the left and right panels, respectively. Two additional horizontal lines are added. These correspond to values of $CARL_0$ equal to the nominal $ARL_0 = 370.4$ and the maximum value of the $CARL_{0,two}$, denoted $max\big(CARL_{0,two}\big)$, which depends only on the values of n and α, regardless of the value of m. This last property is shown later in the cdf of the $CARL_{two}$. In this analyzed case ($n = 5$ and $\alpha = 0.0027$), the $max\big(CARL_{0,two}\big)$ equals 459.1. The fact that $CARL_{0,two}$ cannot be larger than $max\big(CARL_{0,two}\big)$ is a particular and remarkable property of the two-sided chart.

The plots of Figure 1 describe some interesting characteristics of the $CARL_0$. $CARL_{0,one}$ is a monotonic increasing function of U so the $CARL_{0,one}$ quantiles can be directly obtained from it, while the $CARL_{0,two}$ is non-monotonic. The same difference holds between $CFAR_{one}$ and $CFAR_{two}$ since $CFAR = 1/CARL_0$. The shape of the curves of $CARL_0{}'$s can be verified mathematically by their first and second derivatives, similarly to the proof given in Appendix A in the context of tolerance intervals, which is required in the next chapter.

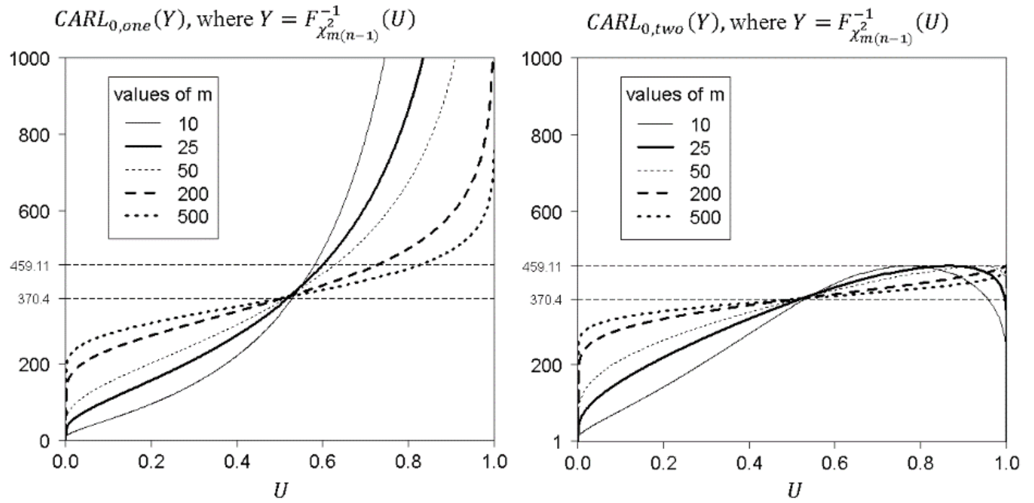

Figure 1 - $CARL_{0,one}$ and $CARL_{0,two}$ of S^2 charts with unadjusted limits ($\alpha = 0.0027$) as functions of U for different values of m and $n = 5$. Nominal $ARL_0 = 370.4$ and maximum value of $CARL_{0,two}$ ($max(CARL_{0,two}) = 459.1$)

The effect of the amount of Phase I samples (m) on the Phase II IC performance of the one- and two-sided S^2 control charts are shown in the plots of Figure 1: the curves of $CARL_0$ are closer to the horizontal line of nominal $ARL_0 = 370.4$ as m increases (for example, note the difference between the curves for $m = 10$ and for $m = 500$). In other words, the discrepancy between the nominal $ARL_0 = 370.4$ and the actual $CARL_0$ is considerably more likely to be larger when m is small. It is also interesting to note the opposite effects on the two sides of the 0.5-quantile of Y ($u = 0.5$), especially for small values of m. When σ_0^2 is underestimated (left tails of the $CARL_0$'s plots, e.g., $u < 0.4$), the actual $CARL_0$ may be smaller than the nominal ARL_0. On the other hand, when σ_0^2 is overestimated (right tails of the $CARL_0$'s plots, e.g., $u > 0.6$), the actual $CARL_{0,one}$ may be quite larger than the nominal ARL_0, differently from the actual $CARL_{0,two}$ that may be either somewhat larger or smaller than the nominal ARL_0, as can be seen in the plots of Figure 1. This finding is related to the answer to the first major question indicated by Jensen et al. (2006).

3
Derivation of the two-sided tolerance limits for sample variances

In this chapter, the derivation and formulas of the proposed exact and the proposed approximate two-sided tolerance factors for constructing two-sided tolerance intervals for sample variances are presented. Two approximate methods of tolerance factors are presented. The first method is called the CE method and is based on the Wilson–Hilferty (WH) transformation (see Wilson and Hilferty, 1931) and the use of conditional expectation. The second one is called the KMM method and is adapted from the approximate two-sided tolerance interval for a gamma distribution (see Krishnamoorthy et al., 2008). The performances of the two proposed approximate tolerance intervals are examined by means of the assessment of the accuracy of the corresponding approximate tolerance factors and an illustration of a dataset from a real application. For more details on the exact method and the second approximate method, see Sarmiento et al. (2018) and Yao et al. (2019), respectively, which are two resulting articles of the present thesis.

3.1
Derivation of the exact two-sided tolerance limits for sample variances

In this part, we show how to obtain the equations to find the exact value of β^*, which is required to compute the exact tolerance factors from Equation (3). Once the exact factors are obtained, the corresponding exact tolerance limits can be computed from Equation (4) and, therefore, the exact tolerance interval can be established. In order to gain a better understanding of our obtained formulation, some mathematical details of our derivations are provided in Appendix A. Tolerance factors, for several cases (settings) extensively used in practice, are tabulated and presented in Appendix B (Tables B.1-B.3) so that the two-sided tolerance intervals for sample variances can be constructed in practice.

First, substituting Equations (2)-(4) into Equation (1), the actual coverage $P_{S^2}\left(\hat{S}_L^2 \leq S^2 \leq \hat{S}_U^2 \mid S_p^2\right)$ is denoted as the function G that can be expressed in terms of the chi-squared random variable $Y = m(n-1)\,S_p^2/\sigma^2$ with $m(n-1)$ df. So, the actual coverage $G(Y)$ is defined by

$$G(Y;\beta^*,m,n) = P_W\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}} \leq W \leq \frac{Y}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}} \mid Y\right), \qquad (21)$$

where $W = (n-1)S^2/\sigma^2$ follows also a chi-square distribution with $(n-1)$ df. Y and W are independent random variables since the (test or future) sample and the Phase I reference samples are independent. Thus, the $(1-\beta,\gamma)$ two-sided tolerance interval for sample variances, given by Equation (1), can also be defined as follows

$$\begin{aligned} &P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) \\ &= P_Y\left(P_W\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}} \leq W \leq \frac{Y}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}} \mid Y\right) \geq 1-\beta\right) = \gamma. \end{aligned} \qquad (22)$$

Analyzing Equation (22), note that, when Y equals its expected value $m(n-1)$ or, equivalently, when there is no error in the variance estimation (that is, $\hat{\sigma}^2 = S_p^2 = \sigma^2$), the actual coverage $G(Y)$ equals $1-\beta^*$. Moreover, given a specified confidence level that is usually high (say, $\gamma = 0.95$), we can point out that $1-\beta^* > 1-\beta$ because, first, the tolerance interval must cover at least the specified proportion $1-\beta$ and, second, because of the effect of variance estimation (measured through the random variable Y or, equivalently, through the ratio S_p^2/σ^2). As the amount of reference data (mn) to estimate σ^2 increases, the value of $1-\beta^*$ decreases and converges to $1-\beta$, otherwise a large discrepancy between them is found. These indicated relations can be verified in Tables B.1-B.3 (Appendix B).

Since the real coverage $G(Y)$ is a concave function, increasing over $(0, y_0)$ and decreasing over (y_0, ∞) with a maximum at y_0 (see Note 1 in Appendix A), the coverage probability $P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$, from Equation (22), can be rewritten as (see Note 2 in Appendix A):

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) = F_{\chi^2_{m(n-1)}}(y_2) - F_{\chi^2_{m(n-1)}}(y_1), \qquad (23)$$

where $F_{\chi^2_\tau}$ denotes the cdf of a chi-square distribution with τ df, and y_1 and y_2 $(y_1 < y_2)$ are the solutions of $G(Y;\beta^*,m,n) = 1-\beta$ (see Equation 21). Hence, given the values of m, n, $1-\beta$ and γ, the required β^* is found by solving a system of three nonlinear equations for β^*, y_1 and y_2:

$$\begin{cases} F_{\chi^2_{n-1}}\left(\frac{y_1}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{y_1}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}}\right) = 1-\beta \\ F_{\chi^2_{n-1}}\left(\frac{y_2}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{y_2}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}}\right) = 1-\beta \\ F_{\chi^2_{m(n-1)}}(y_2) - F_{\chi^2_{m(n-1)}}(y_1) = \gamma, \end{cases} \tag{24}$$

using a numerical (search) method, where $0 < y_1 < y_0 < y_2 < \infty$, $1-\beta < Max(G(Y))$, $y_0 = m(n-1)\ln(U^*/L^*)/(U^*-L^*)$ and

$$Max\big(G(Y)\big) = G(y_0) = F_{\chi^2_{n-1}}\left(\frac{\ln(U^*/L^*)}{(U^*-L^*)}\chi^2_{n-1,1-\frac{\beta^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{\ln(U^*/L^*)}{(U^*-L^*)}\chi^2_{n-1,\frac{\beta^*}{2}}\right).$$

Recall that L^* and U^* are the exact two-sided lower and upper tolerance factors, respectively, defined in Equation (3). From Equation (24), as noted earlier, the value of β^* depends on the amount of data that is used to estimate σ^2 (m and n), the specified proportion ($1-\beta$) and the specified confidence level (γ).

Tables B.1-B.3 (provided in Appendix B) show the exact $(1-\beta,\gamma)$ lower and upper tolerance factors for different combinations of the specified proportion $1-\beta = \{0.90, 0.95, 0.99\}$ (for Tables B.1, B.2 and B.3, respectively) and the specified confidence level $\gamma = \{0.90, 0.95, 0.99\}$, considering different amounts of reference data $m = \{5, 10, 15, 20, 25, 30, 50, 75, 100, 200, 250, \infty\}$ and $n = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25\}$, which are used to estimate the process variance σ^2 using $\hat{\sigma}^2 = S_p^2$. In addition, the values of $1-\beta^*$ related to each setting are also provided so that these values can be compared with the nominal value $1-\beta$. Note that these tables of factors (Appendix B) are an extension of the ones provided by Sarmiento et al. (2018), that is, Tables B.1-B.3 present a larger amount of settings (cases) than tables provided in the cited paper.

For instance, let suppose that the construction of the exact ($1-\beta = 0.90$, $\gamma = 0.95$) two-sided tolerance interval for sample variances, which arise from 5 observations ($n = 5$), is required. Suppose also that 30 samples ($m = 30$) each of size 5 to estimate σ^2 is available. From Table B.1, we find $1-\beta^* = 0.9348$ and thus the required exact (lower and upper) tolerance factors are $L^* = 0.1401$ and $U^* = 2.6282$, respectively. Accordingly, the exact lower and upper tolerance limits, given by $\hat{S}^2_{L^*} = 0.1401 * S_p^2$ and $\hat{S}^2_{U^*} = 2.6282 * S_p^2$, respectively, make up the required two-sided tolerance interval.

Note that, from Tables B.1- B.3, given a reference dataset used to estimate σ^2 (m and n) and a specified proportion $(1-\beta)$, when the values of the specified confidence level (γ) is larger, the width of the tolerance interval ($\hat{S}^2_{U^*} - \hat{S}^2_{L^*}$) increases. For example, for $m = 10$ and $n = 5$, the exact $(1-\beta = 0.90, \gamma = 0.90)$ two-sided (lower and upper) tolerance factors are 0.1193 and 2.8018, respectively. On the other hand, for the same (m, n) and $1-\beta$, the exact $(1-\beta = 0.90, \gamma = 0.99)$ two-sided tolerance factors are 0.0610 and 3.5349 such that the resulting width is approximately 30% larger than the one of the $(1-\beta = 0.90, \gamma = 0.90)$ tolerance interval.

It can also be seen from Tables B.1-B.3 that, given the values of n, $1-\beta$ and γ, the values of $1-\beta^*$ decrease and converge to $1-\beta$ as the number of subgroups (m) increases. Accordingly, the exact lower and upper tolerance factors (increases and decreases, respectively) converge to the ones for the variance known case ($\lim_{m\to\infty} L^* = \frac{\chi^2_{n-1,\frac{\beta}{2}}}{n-1}$ and $\lim_{m\to\infty} U^* = \frac{\chi^2_{n-1,1-\frac{\beta}{2}}}{n-1}$), which are provided in the last row of Tables B.1-B.3 ($m \to \infty$), so that the width of the exact t olerance interval becomes narrower and converges to the one for the variance known case. For example, the width of a two-sided $(1-\beta = 0.95, \gamma = 0.90)$ tolerance interval with $n = 5$ and $m = 5$ is $3.7978 * S_p^2$, however, when m increases to 250, this width decreases to $2.7305 * S_p^2$ which is 28.1% smaller.

Sarmiento et al. (2018) also made a comparison between the proposed two-sided tolerance limits and the one-sided upper tolerance limits for sample variances, which was studied in Tietjen & Johnson (1979). For the same setting (same values of $m \geq 5$, $n \geq 5$, $1-\beta \geq 0.90$ and $\gamma \geq 0.90$), Sarmiento et al. (2018) revealed that the width of the two-sided tolerance interval is narrower than the one of the corresponding one-sided upper tolerance interval.

The construction of these proposed exact two-sided tolerance intervals is not straightforward and can be a complicated task for beginner users because of the challenging computation of the tolerance factors, which cannot be provided in closed-form formulas, and the fact that the available tables of tolerance factors may not include a required setting (case). With that in mind, from a practical point of view, we propose two approximation methods for computing tolerance limits for sample variances more easily in practice, and then the approximate factors are compared with the exact ones in the next subchapter.

3.2 Derivation of the approximate two-sided tolerance limits for sample variances

3.2.1 Derivation of the approximate two-sided tolerance limits for sample variances based on the CE method

Since the value of β^* cannot be found directly by solving Equation (22), an approximation of the coverage probability, that is $P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$, could help to find the value of β^* in an easier way. To this end, this approximation should be a reduced and tractable expression. Note that, in this situation, the found value of β^* would be an approximate value. Therefore, we try to find the approximation of the coverage probability.

First, on the basis of the Wilson–Hilferty (WH) transformation (see Wilson & Hilferty, 1931), from Equation (22), the chi-square random variable W with $(n-1)$ df is transformed into $WT = \sqrt[3]{\frac{W}{n-1}}$ that follows a normal distribution with mean $1-\frac{2}{9(n-1)}$ and variance $\frac{2}{9(n-1)}$. Let $d = \frac{2}{9(n-1)}$ so that $WT \sim N(1-d, d)$. Next, the approximate coverage probability can be obtained by means of some analytical derivations (for more details, see Appendix C) such that

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) \cong 1 - F_{\chi^2_{m(n-1)}}\big(R(Y;\beta^*,m,n,1-\beta)\big), \quad (25)$$

where R is a function of the chi-square random variable Y with $m(n-1)$ df (see this R function in Appendix C). Thus, from Equations (22) and (25), we have:

$$1 - F_{\chi^2_{m(n-1)}}(R(Y;\beta^*,m,n,1-\beta)) \cong \gamma.$$

The left side of the latter equation can be obtained by conditioning it on Y using the conditional expectation. For that reason, this first proposed approximation method is called the Conditional Expectation (CE) method. Then, we get

$$E_Y\left(1 - F_{\chi^2_{m(n-1)}}\big(R(Y;\beta^*,m,n,1-\beta)\big) \mid Y = y\right) =$$

$$1-\int_0^\infty F_{\chi^2_{m(n-1)}}\big(R(y;\beta^*,m,n,1-\beta)\big)f_Y(y)dy \cong \gamma.$$

Finally, based on the probability integral transformation (namely, $U = F_{\chi^2_{m(n-1)}}(Y)$, where U is uniformly distributed between $u = 0$ and $u = 1$), the approximate value of β^* (denoted β^*_{CE}) can be found from the following equation

$$1-\int_0^1 F_{\chi^2_{m(n-1)}}\big(R(u;\beta^*_{CE},m,n,1-\beta)\big)f_U(u)du = \gamma,$$

thus, given the values of m, n, $1-\beta$ and γ, β^*_{CE} can be found from Equation (26) by a numerical (search) method

$$1-\int_0^1 F_{\chi^2_{m(n-1)}}\left(\frac{16\left(\sqrt{2m^2(n-1)}\right)\left(\sqrt{\chi^2_{1,1-\beta}\left(\left(\frac{A(u)+B(u)}{2}\right)^2\right)}\right)^3}{27\left(\sqrt[3]{\chi^2_{n-1,1-\frac{\beta^*_{CE}}{2}}}-\sqrt[3]{\chi^2_{n-1,\frac{\beta^*_{CE}}{2}}}\right)^3}\right)du = \gamma, \qquad (26)$$

where $\chi^2_{1,1-\beta}\left(\left(\frac{A(u)+B(u)}{2}\right)^2\right)$ denotes the $(1-\beta)$-quantile of a non-central chi-square distribution with 1 df and non-centrality parameter $\left(\frac{A(u)+B(u)}{2}\right)^2$. A and B, which are functions of a realization of the uniform random variable U, are given by

$$A(u) = \frac{\sqrt[3]{\frac{\chi^2_{m(n-1),u}}{m(n-1)^2}\chi^2_{n-1,1-\frac{\beta^*_{CE}}{2}}}-(1-d)}{\sqrt{d}},\; B(u) = \frac{\sqrt[3]{\frac{\chi^2_{m(n-1),u}}{m(n-1)^2}\chi^2_{n-1,\frac{\beta^*_{CE}}{2}}}-(1-d)}{\sqrt{d}}.$$

Recall that $d = \frac{2}{9(n-1)}$ and $\chi^2_{\tau,\upsilon}$ denotes the υ-quantile of a central chi-square distribution with τ df. The found value of β^*_{CE} is used to compute the approximate lower and upper tolerance factors (L_{CE} and U_{CE}, respectively) and limits ($\hat{S}^2_{L_{CE}}$ and $\hat{S}^2_{U_{CE}}$, respectively) based on the CE method from Equations (27) and (28), and hence the approximate $(1-\beta, \gamma)$ equal-tailed two-sided tolerance interval for the sample variance based on the CE method can be constructed.

$$L_{CE} = \frac{\chi^2_{n-1,\frac{\beta^*_{CE}}{2}}}{n-1} \text{ and } U_{CE} = \frac{\chi^2_{n-1,1-\frac{\beta^*_{CE}}{2}}}{n-1}, \qquad (27)$$

$$\hat{S}^2_{L_{CE}} = L_{CE}S^2_p \text{ and } \hat{S}^2_{U_{CE}} = U_{CE}S^2_p. \qquad (28)$$

3.2.2 Derivation of the approximate two-sided tolerance limits for sample variances based on the KMM method

The determination of the exact tolerance interval is complex because solutions of a system of three nonlinear equations are required. Although the CE method is less complicated than the exact one because requires the solution of a unique nonlinear equation, from a practical point of view, it is useful to find another fairly simpler approximation with a closed-form formula. To this end, in this second approximation method, we adapt the Normal-Based Method proposed by Krishnamoorthy, Mathew and Mukherjee (see, for more details, Krishnamoorthy et al., 2008) and we called it the KMM method. The Normal-Based Method, which considers the WH transformation, is used in the construction of approximate (one- and two-sided) tolerance intervals for a gamma distribution and can be applied in this work because the population of sample variances (S^2) is distributed as a multiple $\left(\frac{\sigma^2}{n-1}\right)$ of the chi-square distribution with $(n-1)$ df which, in turn, is equivalent to a gamma distribution with a shape parameter $\left(\frac{n-1}{2}\right)$ and scale parameter $\left(\frac{2\sigma^2}{n-1}\right)$, that is,

$$S^2 \sim \left(\frac{\sigma^2}{n-1}\right) \chi^2_{n-1} \sim Gamma\left(\frac{n-1}{2}, \frac{2\sigma^2}{n-1}\right).$$

The approximate tolerance limits are obtained in three steps. In the first step, we apply the WH transformation on each of the m Phase I sample variances $\{S_1^2, S_2^2, \dots, S_m^2\}$. Since the transformed variables are approximately normally distributed, in the second step, we use the tolerance intervals for the normal distribution as described in Krishnamoorthy et al. (2008) for the transformed data. Finally, in the third step, the tolerance limits for the original data are obtained by back transforming, that is, by cubing the tolerance limits found in step two (see, for more details on these three steps, Appendix D).

Thus, denoting $T_i = (S_i^2)^{\frac{1}{3}}$ and letting $\bar{T}$ and S_T denote, respectively, the mean and the standard deviation of $(T_1, T_2, \dots, T_m)$, the approximate $(1-\beta,\ \gamma)$ lower and upper tolerance limits for S^2 based on the KMM method ($\hat{S}^2_{L_{KMM}}$, $\hat{S}^2_{U_{KMM}}$) are simply given by

$$\hat{S}^2_{L_{KMM}} = [\bar{T} - k\,S_T]^3 \text{ and } \hat{S}^2_{U_{KMM}} = [\bar{T} + k\,S_T]^3, \tag{29}$$

where k is the well-known tolerance factor for the two-sided tolerance interval for a normal distribution that depends on m, $1-\beta$ and γ (see, for instance, exact values of k from Table B2 in Krishnamoorthy & Mathew, 2009, p. 365). Note that these factors can also be obtained from the R package 'tolerance' provided by Young (2010, 2014b). Furthermore, an approximation of k can be found in the literature, namely, $k \cong \left(\frac{(m-1)\chi^2_{1,1-\beta}(1/m)}{\chi^2_{m-1,1-\gamma}}\right)^{\frac{1}{2}}$ (see, for instance, Krishnamoorthy & Mathew, 2009). More details of the proof of the resulting Equation (29) are outlined in Appendix D.

Note that the tolerance limits based on the KMM method (Equation 29) are obtained directly from the Phase I data and there are not tolerance factors in our formulation. However, for purposes of comparison with the exact tolerance factors, we can define the approximate tolerance limits as $\hat{S}^2_{L_{KMM}} = L_{KMM}S^2_p = \frac{\chi^2_{n-1,\beta^*_L}}{n-1}S^2_p$ and $\hat{S}^2_{U_{KMM}} = U_{KMM}S^2_p = \frac{\chi^2_{n-1,1-\beta^*_U}}{n-1}S^2_p$ so that we can find (e.g., via simulation) the corresponding approximate (lower and upper) tolerance factors and β^* based on the KMM method (L_{KMM}, U_{KMM} and β^*_{KMM}, respectively) as

$$L_{KMM} = \frac{\hat{S}^2_{L_{KMM}}}{S^2_p} = \frac{\chi^2_{n-1,\beta^*_L}}{n-1} = \frac{[\bar{V}-k\,S_V]^3}{\bar{H}},$$

$$U_{KMM} = \frac{\hat{S}^2_{U_{KMM}}}{S^2_p} = \frac{\chi^2_{n-1,1-\beta^*_U}}{n-1} = \frac{[\bar{V}+k\,S_V]^3}{\bar{H}} \text{ and } \beta^*_{KMM} = \beta^*_L + \beta^*_U, \tag{30}$$

where each H_i $(H_i = (n-1)S^2_i/\sigma^2, i = 1,2,\dots m)$ follows a chi-square distribution with $(n-1)$ df, $\bar{H}$ is the mean of $(H_1, H_2, \dots, H_m)$, $V_i = (H_i)^{\frac{1}{3}}$ with mean $\bar{V}$ and standard deviation S_V, respectively.

Yao et al. (2019) examine the second KMM method and complement this present thesis by providing a simulation study of the accuracy (assessment through the width of the tolerance interval) and the robustness (over different values of parameters and types of distributions of sample variances) of the approximate two-sided tolerance limits based on the KMM method, which are compared with the ones of the exact tolerance limits.

3.3
Accuracy of the approximate tolerance intervals for sample variances based on CE and KMM methods

3.3.1
Comparison of the exact (EX) and approximate (based on CE and KMM methods) tolerance factors for sample variances

For $n = 5$, different number of samples to estimate σ^2 using S_p^2 ($m = 10, 25, 50, 100, 250$) and $1 - \beta, \gamma = \{0.90, 0.95, 0.99\}$, Table 3 gives the approximate values of $1 - \beta^*$ and the corresponding approximate $(1 - \beta, \gamma)$ lower and upper tolerance factors based on the CE method ($1 - \beta_{CE}^*$, L_{CE} and U_{CE} from Equations (26) and (27)) and the KMM method ($1 - \beta_{KMM}^*$, L_{KMM} and U_{KMM} obtained by simulation from Equation (30), specifically, by generation of 100,000 random samples of size m (H_1, H_2, …, H_m)). These approximate values are compared to the exact (EX) ones ($1 - \beta^*$, L^* and U^*, respectively). We can note that the approximations of the tolerance factors based on the CE method are quite satisfactory most of the time and outperform the KMM approximation, which is often acceptable or moderately satisfactory in settings examined in Table 3. For instance, in the case of the CE method, the absolute differences between L_{CE} and L^* vary between 0.0003 and 0.0199 for all considered settings. For the upper tolerance factors (U_{CE} and U^*), these absolute differences are smaller than 0.1 for the settings ($1 - \beta \leq 0.95, \gamma \leq 0.99$), except for the case of the settings ($1 - \beta \leq 0.95, \gamma = 0.99$) with $m \leq 25$, where absolute differences achieve values of up to 0.3186. In the case of the KMM method, for the settings ($1 - \beta \leq 0.95,\ \gamma \leq 0.95$) with $m \geq 25$, the absolute differences between the lower tolerance factors (L_{KMM} and L^*) and between the upper tolerance factors (U_{KMM} and U^*) are smaller than 0.05 and 0.64, respectively.

The accuracies of the proposed methods are also measured in terms of the relative percentage difference (PD) between the approximate (lower and upper) tolerance factors based on each method and the EX (lower and upper) tolerance factors, which are provided in Table 3, denoted as ($PD(L_{CE})$, $PD(U_{CE})$) and ($PD(L_{KMM})$, $PD(U_{KMM})$), respectively. These values of PD are presented in Table 4. In general, the PD of an approximate value related to tolerance interval for sample variances is computed as follows:

$$PD(Approximate\ value) = 100\% \frac{(Approximate\ value - Exact\ value)}{Exact\ value}. \quad (31)$$

For instance, the PD of U_{CE} is given by $PD(U_{CE}) = 100\%\,(U_{CE} - U^*)/U^*$. From Table 4, for the settings ($1-\beta \leq 0.95$, $\gamma \leq 0.95$) using m of at least 25 samples, the absolute values of $PD(L_{CE})$ and $PD(U_{CE})$ are no larger than 4.1% and 1.6%, respectively, while the absolute values of $PD(L_{KMM})$ and $PD(U_{KMM})$ are smaller than 45.3% and 20.1%, respectively. If the nominal confidence and the nominal proportion are reduced up to ($1-\beta \leq 0.90$, $\gamma \leq 0.90$) and the minimum value of m is increased up to 50 samples, the absolute values of $PD(L_{KMM})$ and $PD(U_{KMM})$ won't exceed 19.1% and 8.5%, respectively.

Moreover, in Table 4, we provide the PD between the approximate and the EX values of $1-\beta^*$ ($PD(1-\beta^*_{CE})$ and $PD(1-\beta^*_{KMM})$), and the PD between the width of the approximate tolerance interval based on each proposed method (WI_{CE} and WI_{KMM}) and the width of the EX tolerance interval (WI). For instance, from Equation (31), the $PD(WI_{CE})$ is given by $PD(WI_{CE}) = 100\%\,(WI_{CE} - WI)/WI$, where $WI = (U^* - L^*)S_p^2$ and $WI_{CE} = (U_{CE} - L_{CE})S_p^2$. Since the values of $PD(1-\beta^*_{CE})$ vary between negative and positive values (namely, -0.99% and 0.33%), the approximate tolerance intervals based on the CE method are sometimes contained within the EX tolerance intervals ($WI_{CE} < WI$) and sometimes not ($WI_{CE} > WI$). In the case of the proposed KMM method, the value of the $PD(1-\beta^*_{KMM})$ is always positive and we get $L_{KMM} < L^*$ and $U_{KMM} > U^*$ for all settings considered here. Therefore, the width of the approximate tolerance interval based on the KMM method is always wider than the one of the EX tolerance interval ($WI_{KMM} > WI$). It means that the KMM approximation of the tolerance interval is conservative. For example, for the settings ($1-\beta \leq 0.95, \gamma \leq 0.95$) using m of at least 25 samples, the values of $PD(WI_{KMM})$ vary between 3.76% and 21.87%.

Since our two proposed approximate methods are based on the WH approximation, as the sample size n increases, the accuracy of the approximate tolerance limits gets better. Zar (1978) presented an assessment of the accuracy of the approximate quantiles of the chi-square distribution obtained according to the WH transformation. For instance, in the case of the CE method, when $n = 3$ and 4, the approximation is acceptable, and quite satisfactory when $n = 10$. Although the KMM approximation method does not work very well as the CE method, the accuracy of the KMM approximation is usually satisfactory for large values of n, e.g., for the settings $(1-\beta \le 0.95, \gamma \le 0.95)$ with $n = 10$ and $25 \le m \le 250$, the $PD(L_{KMM})$, $PD(U_{KMM})$ and $PD(WI_{KMM})$ vary between $(-27.3\%, -5.0\%)$, $(2.8\%, 17.3\%)$ and $(4.7\%, 23.0\%)$, respectively.

Table 3 - Comparison between the exact and approximate $(1-\beta, \gamma)$ two-sided tolerance factors for S^2 (which come from $n = 5$ observations) using m subgroups each of size $n = 5$ to estimate σ^2

EX method					$1-\beta^*$	L^*	U^*			
CE method					$1-\beta^*_{CE}$	L_{CE}	U_{CE}			
KMM method					$1-\beta^*_{KMM}$	L_{KMM}	U_{KMM}			
$1-\beta$	m	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
		0.9513	0.1193	2.8018	0.9660	0.0984	3.0115	0.9863	0.0610	3.5349
	10	0.9511	0.1196	2.7990	0.9605	0.1066	2.9249	0.9765	0.0809	3.2258
		0.9848	0.0733	3.6606	0.9934	0.0491	4.2071	0.9993	0.0177	5.7725
		0.9290	0.1467	2.5780	0.9389	0.1351	2.6673	0.9573	0.1111	2.8793
	25	0.9303	0.1453	2.5886	0.9366	0.1378	2.6454	0.9486	0.1229	2.7698
		0.9605	0.1080	2.9385	0.9720	0.0896	3.1358	0.9877	0.0574	3.5960
		0.9188	0.1581	2.4973	0.9253	0.1510	2.5472	0.9383	0.1359	2.6609
0.90	50	0.9205	0.1562	2.5102	0.9249	0.1514	2.5441	0.9332	0.1419	2.6139
		0.9438	0.1281	2.7089	0.9539	0.1145	2.8214	0.9701	0.0899	3.0621
		0.9123	0.1650	2.4512	0.9165	0.1606	2.4805	0.9250	0.1513	2.5447
	100	0.9142	0.1630	2.4646	0.9173	0.1597	2.4863	0.9230	0.1535	2.5289
		0.9312	0.1422	2.5811	0.9391	0.1326	2.6506	0.9526	0.1149	2.7921
		0.9072	0.1704	2.4171	0.9095	0.1679	2.4325	0.9143	0.1629	2.4649
	250	0.9092	0.1683	2.4304	0.9111	0.1663	2.4431	0.9147	0.1625	2.4675
		0.9196	0.1546	2.4859	0.9249	0.1486	2.5252	0.9345	0.1373	2.6027
		0.9812	0.0719	3.3551	0.9883	0.0560	3.6282	0.9964	0.0305	4.2929
	10	0.9822	0.0698	3.3872	0.9868	0.0597	3.5592	0.9937	0.0408	3.9744
		0.9959	0.0393	4.5282	0.9986	0.0239	5.2834	0.9999	0.0070	7.4668
		0.9687	0.0942	3.0592	0.9744	0.0845	3.1778	0.9843	0.0655	3.4566
	25	0.9710	0.0904	3.1041	0.9747	0.0841	3.1827	0.9812	0.0718	3.3562
		0.9867	0.0599	3.5519	0.9918	0.0463	3.8160	0.9974	0.0248	4.4407
		0.9623	0.1040	2.9517	0.9664	0.0978	3.0182	0.9741	0.0852	3.1693
0.95	50	0.9651	0.0997	2.9974	0.9678	0.0955	3.0445	0.9728	0.0873	3.1423
		0.9784	0.0749	3.2414	0.9836	0.0641	3.3913	0.9912	0.0452	3.7174
		0.9581	0.1100	2.8904	0.9608	0.1061	2.9294	0.9662	0.0981	3.0149
	100	0.9612	0.1056	2.9351	0.9632	0.1027	2.9651	0.9667	0.0972	3.0247
		0.9714	0.0862	3.0703	0.9759	0.0780	3.1640	0.9831	0.0636	3.3539
		0.9548	0.1147	2.8452	0.9563	0.1125	2.8657	0.9594	0.1082	2.9087
	250	0.9580	0.1102	2.8883	0.9592	0.1085	2.9058	0.9615	0.1051	2.9397
		0.9645	0.0965	2.9435	0.9677	0.0914	2.9956	0.9734	0.0818	3.0992
		0.9980	0.0228	4.6137	0.9990	0.0156	5.0268	0.9998	0.0064	5.9927
	10	0.9988	0.0175	4.8989	0.9994	0.0126	5.2558	0.9999	0.0047	6.3221
		0.9997	0.0112	6.5862	0.9999	0.0057	7.8571	$\cong 1$	0.0013	11.6261
		0.9954	0.0348	4.1501	0.9967	0.0293	4.3378	0.9985	0.0197	4.7687
	25	0.9970	0.0279	4.3931	0.9977	0.0241	4.5512	0.9989	0.0170	4.9348
		0.9991	0.0143	4.9757	0.9996	0.0091	5.4167	0.9999	0.0031	6.4576
		0.9937	0.0407	3.9782	0.9948	0.0369	4.0847	0.9966	0.0297	4.3246
0.99	50	0.9958	0.0332	4.2013	0.9964	0.0305	4.2952	0.9976	0.0251	4.5051
		0.9980	0.0189	4.4777	0.9988	0.0140	4.7209	0.9996	0.0069	5.2578
		0.9925	0.0445	3.8801	0.9933	0.0421	3.9424	0.9948	0.0371	4.0794
	100	0.9948	0.0368	4.0885	0.9954	0.0349	4.1474	0.9963	0.0311	4.2714
		0.9968	0.0237	4.1990	0.9976	0.0195	4.3489	0.9988	0.0128	4.6600
		0.9915	0.0475	3.8082	0.9920	0.0461	3.8406	0.9929	0.0433	3.9093
	250	0.9940	0.0398	4.0030	0.9943	0.0386	4.0367	0.9950	0.0363	4.1040
		0.9953	0.0290	3.9934	0.9961	0.0260	4.0772	0.9972	0.0209	4.2445

Table 4 - Percentage difference (PD) between the approximate and exact values related to the $(1-\beta,\gamma)$ two-sided tolerance intervals for S^2 provided in Table 3

CE method		$PD(1-\beta^*_{CE})$	$PD(L_{CE})$	$PD(U_{CE})$	$PD(WI_{CE})$								
KMM method		$PD(1-\beta^*_{KMM})$	$PD(L_{KMM})$	$PD(U_{KMM})$	$PD(WI_{KMM})$								
$1-\beta$	m	$\gamma=0.90$				$\gamma=0.95$				$\gamma=0.99$			
0.90	10	-0.02%	0.26%	-0.10%	-0.12%	-0.56%	8.27%	-2.88%	-3.25%	-0.99%	32.63%	-8.75%	-9.47%
		3.51%	-38.60%	30.65%	33.73%	2.84%	-50.12%	39.70%	42.74%	1.32%	-70.99%	63.30%	65.66%
	25	0.13%	-0.97%	0.41%	0.49%	-0.24%	2.04%	-0.82%	-0.97%	-0.91%	10.60%	-3.80%	-4.38%
		3.39%	-26.38%	13.98%	16.42%	3.52%	-33.69%	17.56%	20.30%	3.17%	-48.37%	24.89%	27.83%
	50	0.19%	-1.19%	0.52%	0.63%	-0.04%	0.28%	-0.12%	-0.15%	-0.54%	4.44%	-1.77%	-2.10%
		2.73%	-19.01%	8.47%	10.33%	3.10%	-24.16%	10.76%	12.96%	3.39%	-33.87%	15.08%	17.71%
	100	0.21%	-1.24%	0.55%	0.68%	0.09%	-0.54%	0.23%	0.29%	-0.22%	1.48%	-0.62%	-0.76%
		2.07%	-13.85%	5.30%	6.69%	2.47%	-17.46%	6.86%	8.54%	2.99%	-24.08%	9.72%	11.86%
	250	0.22%	-1.23%	0.55%	0.69%	0.17%	-0.98%	0.44%	0.54%	0.04%	-0.24%	0.11%	0.13%
		1.37%	-9.24%	2.85%	3.76%	1.69%	-11.53%	3.81%	4.95%	2.21%	-15.73%	5.59%	7.10%
0.95	10	0.10%	-2.89%	0.96%	1.04%	-0.15%	6.50%	-1.90%	-2.03%	-0.27%	33.78%	-7.42%	-7.72%
		1.50%	-45.38%	34.96%	36.72%	1.04%	-57.35%	45.62%	47.24%	0.35%	-77.13%	73.93%	75.01%
	25	0.24%	-4.03%	1.47%	1.64%	0.02%	-0.45%	0.16%	0.17%	-0.31%	9.60%	-2.90%	-3.14%
		1.86%	-36.38%	16.11%	17.77%	1.78%	-45.22%	20.09%	21.87%	1.34%	-62.18%	28.47%	30.22%
	50	0.29%	-4.10%	1.55%	1.75%	0.15%	-2.38%	0.87%	0.98%	-0.13%	2.50%	-0.85%	-0.94%
		1.67%	-28.00%	9.81%	11.19%	1.79%	-34.49%	12.36%	13.93%	1.76%	-46.97%	17.29%	19.07%
	100	0.32%	-4.02%	1.55%	1.77%	0.24%	-3.22%	1.22%	1.39%	0.06%	-0.89%	0.32%	0.37%
		1.38%	-21.66%	6.22%	7.33%	1.57%	-26.49%	8.01%	9.30%	1.76%	-35.16%	11.24%	12.80%
	250	0.33%	-3.88%	1.51%	1.74%	0.30%	-3.62%	1.40%	1.60%	0.22%	-2.80%	1.07%	1.21%
		1.01%	-15.81%	3.45%	4.26%	1.19%	-18.80%	4.54%	5.49%	1.45%	-24.37%	6.55%	7.74%
0.99	10	0.08%	-23.01%	6.18%	6.33%	0.03%	-19.00%	4.56%	4.63%	0.01%	-26.34%	5.50%	5.53%
		0.17%	-50.89%	42.75%	43.22%	0.09%	-63.14%	56.30%	56.68%	0.02%	-79.63%	94.00%	94.19%
	25	0.16%	-19.92%	5.86%	6.07%	0.11%	-17.74%	4.92%	5.07%	0.04%	-14.14%	3.48%	3.56%
		0.37%	-58.88%	19.89%	20.56%	0.29%	-68.96%	24.87%	25.51%	0.15%	-84.41%	35.41%	35.91%
	50	0.21%	-18.43%	5.61%	5.86%	0.16%	-17.50%	5.15%	5.36%	0.09%	-15.22%	4.17%	4.31%
		0.43%	-53.63%	12.56%	13.24%	0.40%	-62.22%	15.58%	16.29%	0.30%	-76.75%	21.58%	22.26%
	100	0.23%	-17.33%	5.37%	5.63%	0.21%	-17.07%	5.20%	5.44%	0.15%	-16.09%	4.71%	4.90%
		0.43%	-46.77%	8.22%	8.86%	0.44%	-53.52%	10.31%	11.00%	0.40%	-65.50%	14.23%	14.97%
	250	0.25%	-16.30%	5.12%	5.39%	0.24%	-16.39%	5.11%	5.37%	0.21%	-16.29%	4.98%	5.22%
		0.38%	-39.01%	4.86%	5.42%	0.41%	-43.56%	6.16%	6.76%	0.43%	-51.83%	8.58%	9.25%

3.3.2 Illustration

A dataset presented by Tietjen & Johnson (1979) is used here to illustrate the construction of the exact two-sided tolerance interval for sample variances as well as the approximate intervals based on the two proposed methods. They illustrated the one-sided tolerance interval for sample variances using a dataset from the field of explosives engineering, namely, a dataset of detonation (ignition) times related to a detonation process. This type of process requires a number of nearly simultaneous explosions that are initiated by detonators in such a way that the variance of the detonation (or initiation or ignition) times (e.g., a set of n detonators that are fired/pulsed simultaneously) should be small enough according to certain specified limits. This requires a thorough evaluation and monitoring of the corresponding process variability because it must be quite small. Note that the assessment of the process variability can be based on the available samples (dataset), namely, we can construct tolerance interval for sample variances of the detonation times to carry out the conformity assessment and therefore we can make a decision about the acceptance or rejection of this process (or production lot) in terms of quality. So we use the available dataset, reproduced in Table 5, to compare the approximate and EX tolerance intervals. There are 20 shots for each of 14 detonators, that is, the data were collected as 20 subgroups or samples ($m = 20$) each of size $n = 14$ to estimate σ^2 using S_p^2. Next, the values of each of the twenty sample variances (S_i^2) and S_p^2 are provided.

Tietjen & Johnson (1979) focused on the construction of EX one-sided upper tolerance limits for the sample variances of the detonation times that is useful when the detection of an increase in the variance is of interest to prevent the worsening of the quality of the detonation process. However, note that we can get more comprehensive information about process variability using the two-sided tolerance intervals instead of the one-sided ones. This is because, in addition to detecting the process deterioration, we can identify the quality improvement of the detonation process due to the inclusion of the corresponding lower tolerance limits, that is, we can gain information about how small the variance of detonation times is as well as how large this variance is (see Sarmiento et al., 2018).

Even though the construction of the EX two-sided tolerance intervals for the sample variances can provide meaningful information, this is technically challenging because the computation of tolerance factors depends on a numerical method to solve a system of nonlinear equations. Furthermore, there is not available software packages that provide directly the required tolerance factors and the available tables of tolerance factors do not include all the possible settings. In light of this difficulty, we construct the approximate two-sided tolerance intervals for sample variances (which come from 14 observations) of the detonation times based on our two proposed (CE and KMM) methods using the dataset given in Table 5 to estimate σ^2. The approximate values of $1-\beta^*$ and tolerance (lower and upper) factors and limits are provided and compared with the corresponding EX values of tolerance intervals in Table 6 (using Equations (3), (4), (24) and (26)-(30)) for all possible pairs of settings $(1-\beta, \gamma)$ from $\{0.90, 0.95, 0.99\}$.

Similarly to Table 4, for purposes of evaluation of the accuracy of the two proposed approximation methods, using the values provided in Table 6 and Equation (31), Table 7 presents the relative percentage differences (PD) between the approximate (lower and upper) tolerance limits (based on the CE and KMM methods) and the EX tolerance limits, denoted as $\left(PD\left(\hat{S}^2_{L_{CE}}\right), PD\left(\hat{S}^2_{U_{CE}}\right)\right)$ and $\left(PD\left(\hat{S}^2_{L_{KMM}}\right), PD\left(\hat{S}^2_{U_{KMM}}\right)\right)$, respectively. Additionally, the PD between the width of the approximate tolerance intervals and the width of the EX tolerance intervals (denoted as $PD(WI_{CE})$ and $PD(WI_{KMM})$ for the CE and KMM methods, respectively) are also given in Table 7. As noted earlier in the comparison of approximate and EX tolerance factors from Tables 3 and 4 of Subchapter 3.3.1 ($n = 5$), the approximate tolerance limits based on the CE method works very well in the considered settings. Even though the KMM approximation method is less accurate than the CE approximation method, the accuracy of the KMM approximation is satisfactory for most of the cases. This is because, in this illustration, the sample size is not small ($n = 14$).

Regarding the CE method, the approximate lower and upper tolerance limits are rather close to the EX lower and upper tolerance limits, respectively. We get $\hat{S}^2_{L_{CE}} > \hat{S}^2_{L^*}$, $\hat{S}^2_{U_{CE}} < \hat{S}^2_{U^*}$, and then the width of the approximate tolerance interval based on the CE method is a little bit narrower than the width of the EX tolerance interval ($WI_{CE} < WI$) except for the setting $(1-\beta = 0.99, \gamma = 0.90)$. For a fixed value of $1-\beta$, as the value of γ decreases, the absolute values of $PD(\hat{S}^2_{L_{CE}})$, $PD(\hat{S}^2_{U_{CE}})$ and $PD(WI_{CE})$ become smaller, that is, the accuracy of the CE approximation gets better. For example, for the settings $(1-\beta \leq 0.99, \gamma = 0.99)$, the absolute values of $PD(\hat{S}^2_{L_{CE}})$ and $PD(\hat{S}^2_{U_{CE}})$ do not exceed 6% and the absolute values of $PD(WI_{CE})$ are no larger than 3.7%. If the confidence level decreases to $\gamma = 0.95$, the absolute values of $PD(\hat{S}^2_{L_{CE}})$, $PD(\hat{S}^2_{U_{CE}})$ and $PD(WI_{CE})$ are smaller than 1.5%, and if the reduction is up to $\gamma = 0.90$, the absolute values of $PD(\hat{S}^2_{L_{CE}})$, $PD(\hat{S}^2_{U_{CE}})$ are no longer than 0.6% and the absolute values of $PD(WI_{CE})$ do not exceed 0.4%.

The results in Tables 6 and 7 indicate that the approximate tolerance interval based on the proposed KMM method, as opposed to the CE approximation, is conservative, that is, $\hat{S}^2_{L_{KMM}} < \hat{S}^2_{L^*}$, $\hat{S}^2_{U_{KMM}} > \hat{S}^2_{U^*}$ and thus $WI_{KMM} > WI$. For a given value of $1-\beta$ (or γ), as the value of γ (or $1-\beta$) decreases, the absolute values of $PD(\hat{S}^2_{L_{KMM}})$, $PD(\hat{S}^2_{U_{KMM}})$ and $PD(WI_{KMM})$ become smaller, that is, the accuracy of the KMM approximation gets better and the tolerance interval is less conservative. For instance, the absolute values of $PD(\hat{S}^2_{L_{KMM}})$ and $PD(\hat{S}^2_{U_{KMM}})$ are no larger than 8.90% and $PD(WI_{KMM})$ equals 7.22% for the setting $(1-\beta = 0.99, \gamma = 0.90)$. For a reduction of the nominal proportion to $1-\beta = 0.90$ in the last setting, the absolute values of $PD(\hat{S}^2_{L_{KMM}})$ and $PD(\hat{S}^2_{U_{KMM}})$ are smaller than 4.16% and the value of $PD(WI_{CE})$ is 6.41%.

Table 5 - Detonation times (μsec) of 20 shots with 14 detonators per shot (Tietjen & Johnson, 1979)

Shot number (i)	Detonator number (j)														S_i^2
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1	2.689	2.677	2.675	2.691	2.698	2.694	2.702	2.698	2.706	2.692	2.691	2.681	2.700	2.698	0.000081
2	2.687	2.683	2.683	2.693	2.693	2.702	2.687	2.683	2.702	2.687	2.683	2.677	2.687	2.704	0.000065
3	2.701	2.690	2.701	2.719	2.711	2.707	2.711	2.711	2.709	2.705	2.723	2.703	2.727	2.709	0.000081
4	2.681	2.679	2.677	2.687	2.691	2.689	2.679	2.693	2.689	2.685	2.695	2.683	2.695	2.702	0.000048
5	2.690	2.688	2.674	2.694	2.680	2.686	2.676	2.690	2.684	2.692	2.692	2.676	2.686	2.694	0.000042
6	2.689	2.685	2.689	2.698	2.704	2.689	2.690	2.708	2.704	2.692	2.704	2.690	2.710	2.702	0.000067
7	2.695	2.684	2.689	2.695	2.689	2.691	2.691	2.678	2.695	2.676	2.689	2.676	2.686	2.688	0.000043
8	2.715	2.690	2.721	2.698	2.700	2.698	2.701	2.698	2.705	2.682	2.698	2.696	2.696	2.698	0.000085
9	2.678	2.678	2.692	2.694	2.690	2.701	2.690	2.698	2.690	2.674	2.707	2.694	2.703	2.707	0.000101
10	2.690	2.676	2.694	2.705	2.697	2.688	2.690	2.713	2.699	2.705	2.697	2.703	2.711	2.699	0.000091
11	2.715	2.698	2.706	2.727	2.715	2.715	2.706	2.719	2.715	2.702	2.713	2.706	2.715	2.723	0.000063
12	2.710	2.700	2.727	2.710	2.712	2.723	2.708	2.718	2.700	2.714	2.718	2.700	2.718	2.725	0.000077
13	2.700	2.708	2.698	2.714	2.718	2.704	2.698	2.698	2.700	2.714	2.712	2.710	2.714	2.730	0.000077
14	2.699	2.685	2.693	2.697	2.699	2.691	2.707	2.703	2.685	2.697	2.697	2.701	2.712	2.703	0.000052
15	2.722	2.701	2.710	2.720	2.709	2.707	2.703	2.701	2.709	2.709	2.720	2.709	2.714	2.718	0.000048
16	2.708	2.691	2.710	2.726	2.710	2.707	2.695	2.697	2.697	2.705	2.710	2.726	2.707	2.705	0.000098
17	2.690	2.694	2.690	2.692	2.707	2.694	2.709	2.699	2.705	2.699	2.728	2.715	2.721	2.723	0.000157
18	2.697	2.709	2.713	2.713	2.716	2.701	2.713	2.703	2.711	2.697	2.722	2.693	2.720	2.724	0.000090
19	2.701	2.688	2.703	2.711	2.711	2.695	2.699	2.689	2.707	2.699	2.709	2.693	2.699	2.699	0.000050
20	2.704	2.692	2.706	2.706	2.704	2.676	2.684	2.706	2.692	2.700	2.702	2.704	2.715	2.696	0.000093
														S_p^2	0.000075

Table 6 - Comparison between the exact and approximate $(1-\beta, \gamma)$ two-sided tolerance factors and limits for S^2 of detonation times, using the reference dataset from Table 5 to estimate σ^2

Method															
EX						$1-\beta^*$	L^*	U^*	$\hat{S}^2_{L^*} * 10^4$	$\hat{S}^2_{U^*} * 10^4$					
CE						$1-\beta^*_{CE}$	L_{CE}	U_{CE}	$\hat{S}^2_{L_{CE}} * 10^4$	$\hat{S}^2_{U_{CE}} * 10^4$					
KMM						$1-\beta^*_{KMM}$	L_{KMM}	U_{KMM}	$\hat{S}^2_{L_{KMM}} * 10^4$	$\hat{S}^2_{U_{KMM}} * 10^4$					
$1-\beta$	$\gamma = 0.90$					$\gamma = 0.95$					$\gamma = 0.99$				
	0.9253	0.4226	1.7983	0.3189	1.3568	0.9348	0.4094	1.8342	0.3089	1.3839	0.9534	0.3793	1.9205	0.2862	1.4490
0.90	0.9246	0.4236	1.7958	0.3196	1.3549	0.9311	0.4146	1.8198	0.3128	1.3730	0.9451	0.3936	1.8787	0.2969	1.4175
	0.9404	0.4174	1.9108	0.3087	1.4132	0.9572	0.3873	1.9971	0.2864	1.4771	0.9809	0.3259	2.1971	0.2411	1.6250
	0.9662	0.3533	2.0014	0.2666	1.5100	0.9718	0.3397	2.0464	0.2563	1.5440	0.9818	0.3098	2.1524	0.2337	1.6240
0.95	0.9660	0.3536	2.0003	0.2668	1.5093	0.9699	0.3446	2.0298	0.2600	1.5315	0.9774	0.3241	2.1005	0.2445	1.5848
	0.9755	0.3433	2.1368	0.2539	1.5804	0.9844	0.3123	2.2468	0.2310	1.6617	0.9949	0.2500	2.5057	0.1849	1.8532
	0.9947	0.2424	2.4377	0.1829	1.8393	0.9960	0.2294	2.5023	0.1731	1.8880	0.9979	0.2027	2.6478	0.1530	1.9978
0.99	0.9948	0.2410	2.4448	0.1818	1.8446	0.9957	0.2327	2.4857	0.1756	1.8755	0.9972	0.2147	2.5801	0.1620	1.9467
	0.9970	0.2253	2.6266	0.1666	1.9426	0.9986	0.1951	2.7926	0.1443	2.0654	0.9998	0.1379	3.1867	0.1020	2.3569

Table 7 - Percentage difference (PD) between the approximate and exact tolerance limits and between the width of the approximate tolerance interval and the width of the exact tolerance interval for S^2 of detonation times obtained from Table 6

Method									
CE				$PD(\hat{S}^2_{L_{CE}})$	$PD(\hat{S}^2_{U_{CE}})$	$PD(WI_{CE})$			
KMM				$PD(\hat{S}^2_{L_{KMM}})$	$PD(\hat{S}^2_{U_{KMM}})$	$PD(WI_{KMM})$			
$1-\beta$	$\gamma = 0.90$			$\gamma = 0.95$			$\gamma = 0.99$		
0.90	0.23%	-0.14%	-0.26%	1.29%	-0.78%	-1.38%	3.76%	-2.18%	-3.64%
	-3.20%	4.16%	6.41%	-7.27%	6.73%	10.76%	-15.77%	12.14%	19.02%
0.95	0.09%	-0.05%	-0.08%	1.45%	-0.81%	-1.26%	4.61%	-2.41%	-3.59%
	-4.76%	4.66%	6.68%	-9.88%	7.63%	11.11%	-20.89%	14.11%	20.00%
0.99	-0.60%	0.29%	0.39%	1.42%	-0.66%	-0.87%	5.92%	-2.56%	-3.26%
	-8.90%	5.62%	7.22%	-16.64%	9.40%	12.02%	-33.32%	17.97%	22.23%

4
Derivation of the cumulative distribution functions of the $CARL$, CPS and CRL_q of S^2 chart

In this chapter, we present the cumulative distribution functions (cdf's) of the main conditional performance measures of the two-sided S^2 chart, specifically, the cdf's of the conditional average run length ($CARL$), the conditional probability of a signal (CPS) and the conditional run length q-quantile (CRL_q). In addition, the CRL_q distribution of the upper one-sided S^2 chart is provided.

The distributions of the performance measures for the two-sided S^2 chart are derived on the basis of their mathematical-statistical relationship with the two-sided tolerance interval for the sample variance. First, this relationship is revealed showing the equation of the *Exceedance Probability* (EP) of the $CARL$ (or, equivalently, the cdf of the CPS) and the equation of two-sided tolerance interval for the sample variance, and then the equivalence between the "components" of these two equations are identified and highlighted. We focus on this relationship when the process is IC since a specified IC performance measure (using $CARL_0$ or $CFAR$) is required to design the Phase II S^2 chart. By "components" we mean tolerance factors, proportion and confidence level (equation of the tolerance interval), control limit factors, tolerated lower bound of the $CARL_0$ and the value of F_{CARL_0} (equation of the EP or complement of the cdf of the $CARL_0$), and the number and size of Phase I samples (m and n) for both equations. Accordingly, the exact cdf of $CARL_0$ (and cdf of the $CFAR$) and thus the Exceedance Probability Criterion (EPC), which is used to obtain the adjusted control limits and minimum number of Phase I samples m (design of Phase II two-sided S^2 chart), are obtained exploiting the derivations of the exact two-sided tolerance intervals for sample variances (provided in Subchapter 3.1). The exact cdf of the CRL_q for the two-sided S^2 chart is also obtained using this studied tolerance interval.

In the case of the upper one-sided S^2 chart, Epprecht et al. (2015) derived the exact cdf's of the $CARL$ and CPS. The exact cdf of the CRL_q, which also turns into a random variable due to estimation of σ_0^2, is derived on the basis of exact analytical derivations and provided in the present thesis for the first time in the literature.

4.1 Relation between two-sided tolerance interval for the sample variance and the *Exceedance Probability* of the $CARL$ of the two-sided S^2 control chart

The cdf of the $CARL_{two}$ is defined as the probability that the random variable $CARL_{two}$ is less than or equal to a real positive value t ($t \geq 1$). The cdf of the $CARL_{two}$ (denoted $F_{CARL_{two}}(t)$) can be expressed in terms of the cdf of its reciprocal CPS_{two} (denoted $F_{CPS_{two}}$) as follows

$$F_{CARL_{two}}(t) = P(CARL_{two} \leq t) = P\left(CPS_{two} \geq \frac{1}{t}\right) = 1 - F_{CPS_{two}}\left(\frac{1}{t}\right).$$

The *Exceedance Probability* (EP) of the $CARL_{two}$ is the complement of its cdf ($1 - F_{CARL_{two}}(t)$). Thus, using the CPS_{two} defined in Equation (10), the EP of the $CARL_{two}$ is given by

$$\begin{aligned} 1 - F_{CARL_{two}}(t) &= F_{CPS_{two}}\left(\frac{1}{t}\right) \\ &= P\left(P\left(S_i^2 < \widehat{LCL}_{two} \cup S_i^2 > \widehat{UCL}_{two}\right) \leq \frac{1}{t}\right) \qquad (32) \\ &= P\left(P\left(\widehat{LCL}_{two} \leq S_i^2 \leq \widehat{UCL}_{two}\right) \geq 1 - \frac{1}{t}\right). \end{aligned}$$

Note that, from Equation (32), the expression of the EP of the $CARL_{two}$ (or the cdf of the CPS_{two}) is similar to the expression of the two-sided tolerance interval for the sample variance, which is given by Equation (1).

The Phase II sample variances (S_i^2), which arise from a sample of size n, can be deemed as the sample variances that make up the inferred population in the context of tolerance intervals (S^2). Therefore, each “component” of the expression of the EP can be associated with the corresponding one of the expression of tolerance interval for the sample variance.

In the case when the process is IC ($\rho = 1$), substituting Equations (10) and (15) into Equation (32), we have the EP of the $CARL_{0,two}$ of the two-sided S^2 chart, denoted as $1 - F_{CARL_{0,two}}(t)$ (Equation 33), and from Equation (22), the two-sided tolerance interval for the sample variance can be rewritten in Equation (34) as follows

$$1 - F_{CARL_{0,two}}(t) = F_{CPS_{two}}\left(\frac{1}{t}\right)$$
$$= P_Y\left(F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,1-\frac{\alpha}{2}} | Y\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\frac{\alpha}{2}} | Y\right) \geq 1 - \frac{1}{t}\right). \quad (33)$$

$$\gamma = P_Y\left(F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}} | Y\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}} | Y\right) \geq 1 - \beta\right), \quad (34)$$

Since Phase II sample variances (S_i^2) and sample variances (S^2) of the inferred population (through tolerance intervals), from Equations (32) and (1), respectively, come from samples of the same size n (observations that come from normal distributions with variance σ_0^2 and σ^2, respectively), four "components" of the EP of the $CARL_{0,two}$ are associated with the corresponding ones of the tolerance interval (see Equations 33 and 34):

a) the amount of Phase I reference samples (m): the number of Phase I reference data (mn) to estimate σ_0^2 (S^2 chart) can be associated with the required one to estimate σ^2 (tolerance interval). Because the size of Phase I samples in the context of S^2 chart (n) is equivalent to the one in the context of tolerance interval for the sample variance, the corresponding numbers of Phase I samples (m's) are equivalent. We use the same notation of m and n in both contexts;
b) $F_{CARL_{0,two}}(t) = 1 - \gamma$: the cdf of the $CARL_{0,two}$ at t ($t > 1$) is equivalent to the complement of the specified confidence level of the tolerance interval ($0 < \gamma < 1$);
c) $t = \frac{1}{\beta}$: the point $1 - \frac{1}{t}$ (where $t = F^{-1}_{CARL_{0,two}}(1 - \gamma)$) is equivalent to the specified proportion of the tolerance interval ($1 - \beta$); and
d) $\alpha = \beta^*$: since the corresponding control limits and tolerance limits were defined as equal-tailed limits in previous chapters, the nominal false alarm rate α is equivalent to the "adjusted" value of β (denoted β^*). Hence, the

unadjusted (lower and upper) control limits factors (L_{two} and U_{two} from Equation 8) are equivalent to the (lower and upper) tolerance factors (L^* and U^* from Equation 3), respectively:

$$L_{two} = L^* = \frac{\chi^2_{n-1,\frac{\alpha}{2}}}{n-1} = \frac{\chi^2_{n-1,\frac{\beta^*}{2}}}{n-1} \text{ and } U_{two} = U^* = \frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{n-1} = \frac{\chi^2_{n-1,1-\frac{\beta^*}{2}}}{n-1}.$$

Note that this equivalence (d) can also be considered in the case of adjusted control limits, that is, considering α^*, L^*_{two} and U^*_{two} rather than α, L_{two} and U_{two}, respectively. These described four equivalences can be exploited to obtain some properties of the examined performance measures (such as the cdf and the EPC for the $CARL_{0,two}$) and the values related to the design of the S^2 chart based on the two-sided tolerance interval for the population of sample variances (Subchapter 3.1), and vice versa.

In the present thesis, given the sample size n (in both contexts, S^2 chart and tolerance intervals for sample variances), if we know three of the four "components" of the S^2 chart indicated above (a-d, see also Equations 33 and 34), the unknown "component" in the context of S^2 can be found using the derivations obtained in the context of exact two-sided tolerance interval for the sample variance (Equation 24 in Subchapter 3.1). Therefore, we are interested in three main issues or computations that will be addressed later, namely, the cdf of the $CARL_{0,two}$, the number of Phase I samples (m) and adjusted control limits that guarantees a conditional IC performance in Subchapters 4.2, 5.2.2 and 5.3.2, respectively.

4.2 Distributions of the conditional average run length ($CARL$) and conditional probability of a signal (CPS)

In the previous subchapter, we focus on the performance measures when the process is IC ($\rho = 1$), however, in this part, we present the cdf's of the $CARL$ and CPS when ρ can be different than 1 (OOC process). The exact cdf of the CPS (or, equivalently, the complement of the cdf of the $CARL_0$) for the upper one-side S^2 chart, i.e., the $F_{CARL_{0,one}}(t)$ (or $F_{CFAR_{one}}(t^{-1})$), was derived and studied by Epprecht et al. (2015). For the two-sided S^2 chart, as noted before, the cdf of the $CARL_{two}$, i.e., $F_{CARL_{two}}(t)$ (or $F_{CPS_{two}}(t^{-1})$) was presented by Guo & Wang (2017) when this was already derived in the present thesis. Differently from the one-sided chart, the cdf of $CARL_{two}$ is not presented in a closed form since a numerical method is required.

From Equations (33) and (34) and considering the four "components" (a)-(d) in Subchapter 4.1 (recall that the sample size n is the same in both contexts, i.e., for S^2 chart and tolerance intervals for sample variances), we can derive the cdf of the $CARL_{two}$ as follows:

- **In the context of S^2 control chart:** given the values of m, t and α (unadjusted limits), we want to obtain $F_{CARL_{0,two}}(t)$ (Equation 36).
- **In the context of tolerance interval for the sample variance:** the three known "components" are m, $\beta = \frac{1}{t}$ and $\beta^* = \alpha$, while the unknown "component" is γ. Thus, γ is found using two-sided tolerance interval (Equation (24)) so that $F_{CARL_{0,two}}(t) = 1 - \gamma$

Thus, these expressions of the cdf's are given by

$$F_{CARL_{one}}(t) = 1 - F_{CPS_{one}}(t^{-1}) = P(CARL_{one} \leq t)$$

$$= \begin{cases} 0, \; t \leq 1 \\ F_{\chi^2_{m(n-1)}}\left(\frac{\rho^2 m(n-1)\chi^2_{n-1,\,1-t^{-1}}}{\chi^2_{n-1,\,1-\alpha}}\right), \; t > 1 \end{cases}, \tag{35}$$

and

$$F_{CARL_{two}}(t) = 1 - F_{CPS_{two}}(t^{-1}) = P(CARL_{two} \leq t)$$

$$= \begin{cases} 0, & t \leq 1 \\ 1 - (u_2 - u_1), & 1 < t < max(CARL_{two}), \\ 1, & t \geq max(CARL_{two}) \end{cases} \tag{36}$$

where u_1 and u_2 are the two solutions of $CARL_{two}(U; \rho^2) = t$, that is

$$\left[1 - \left(F_{\chi^2_{n-1}}\left(\frac{F^{-1}_{\chi^2_{m(n-1)}}(U)}{\rho^2 m(n-1)}\chi^2_{n-1,1-\alpha/2}\right) - F_{\chi^2_{n-1}}\left(\frac{F^{-1}_{\chi^2_{m(n-1)}}(U)}{\rho^2 m(n-1)}\chi^2_{n-1,\,\alpha/2}\right)\right)\right]^{-1} = t,$$

for U, being $u_1 < u_0 < u_2$, $u_0 = F_{\chi^2_{m(n-1)}}\left(\frac{\rho^2 m(n-1)^2 \ln(\chi^2_{n-1,1-\alpha/2}/\chi^2_{n-1,\,\alpha/2})}{\chi^2_{n-1,1-\alpha/2} - \chi^2_{n-1,\,\alpha/2}}\right)$ and

$max(CARL_{two}) = CARL_{two}(u_0)$

$$= \left[1 - \left(F_{\chi^2_{n-1}} \left((n-1) \frac{\ln\left(\frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)}{\chi^2_{n-1,1-\frac{\alpha}{2}} - \chi^2_{n-1,\frac{\alpha}{2}}} \chi^2_{n-1,1-\frac{\alpha}{2}} \right) - F_{\chi^2_{n-1}} \left((n-1) \frac{\ln\left(\frac{\chi^2_{n-1,1-\frac{\alpha}{2}}}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)}{\chi^2_{n-1,1-\frac{\alpha}{2}} - \chi^2_{n-1,\frac{\alpha}{2}}} \chi^2_{n-1,\frac{\alpha}{2}} \right) \right) \right]^{-1}.$$

Similarly to the two-sided tolerance interval for sample variances (Equation 24), Equation (36) can be explained as follows: there are only two solutions for U of $CARL_{two}(U;\rho^2) = t$ (from Equation 15, where $Y = F^{-1}_{\chi^2_{m(n-1)}}(U)$) because $CARL_{two}$ is a concave function of U, increasing on $[0, u_0]$ and decreasing on $[u_0, 1]$ in such a way that $CARL_{0,two}$ varies in the interval $[1, max(CARL_{two})]$, where $max(CARL_{two}) = CARL_{two}(u_0)$. See this behavior when the process is IC ($\rho = 1$) and the control limits are unadjusted ($\alpha = 0.0027$) in Figure 1 (Chapter 2, page 52). In addition, from Equation (36), note that the value of $max(CARL_{two})$ depends only on the values of n and α, as highlighted and shown in Figure 1 in the case of IC process. The $CARL_{two}(U;\rho^2)$ converges to one, in probability, when U converges to zero or one (see, e.g., Figure 1). Hence, the probability that $CARL_{two} > t$, which is the complement of the cdf of $CARL_{two}$, is the probability that U belongs to the interval between u_1 and u_2, the two solutions of $CARL_{two}(U) = t$.

Note that the cdf's of $CARL$ of the one- and two-sided S^2 charts from Equations (35) and (36), respectively, depend on m, n, ρ^2 and α values. From these expressions, we can obtain (numerically) the corresponding probability density functions (pdf's) of both $CARL$. Plots of Figure 2 show the cdf's and pdf's of the $CARL_{0,one}$ and $CARL_{0,two}$ of the one- and two-sided S^2 charts with unadjusted limits, for $m = \{10, 25, 50, 200, 500\}$, $n = 5$ and $\alpha = 0.0027$. Similarly to Figure 1, there are two additional vertical lines: $nominal\, ARL_0 = 370.4$ and $max\left(CARL_{0,two}\right) = CARL_{0,two}(u_0) = 459.11$.

From Figure 2, note that the difference between the shapes of the distributions of $CARL_{0,one}$ and $CARL_{0,two}$ is evident. It is remarkable that the pdf´s of $CARL_{0,one}$ are most often highly skewed to the right (especially, when m is small), differently from the pdf´s of $CARL_{0,two}$ which are left-skewed and have a finite maximum value. This in turn leads to the variability of the $CARL_{0,two}$ is much smaller than that of the $CARL_{0,one}$. Note that when m gets larger (such as $m = 500$), the cdf´s curves are much "closer" to the vertical line of the nominal value 370.4. From the cdf´s of $CARL_0$ in Figure 2, it is seem that the median of the $CARL_0$ distribution and the nominal ARL_0 (=370.4) are approximately equal (i.e., $P(CARL_0 \leq 370.4)$ is around 50%), especially, when m is large. This is because, from Equations (14) and (15), as the value of m increases (i.e., the median of Y tends to its mean), $Y/m(n-1)$ converges to 1 in probability (error factor of the variance estimate converges to 1 or, equivalently, S_p^2 converges to σ_0^2), and thus the value of the true $CARL_0$ converges to the nominal ARL_0 $(1/\alpha)$ in probability.

The examination of the differences in the $CARL_0$ distribution between both charts, that is, the $CARL_{0,one}$ and $CARL_{0,two}$ distributions, will enable a better understanding of the differences in performance between the one-sided and two-sided S^2 charts, either with adjusted or unadjusted limits.

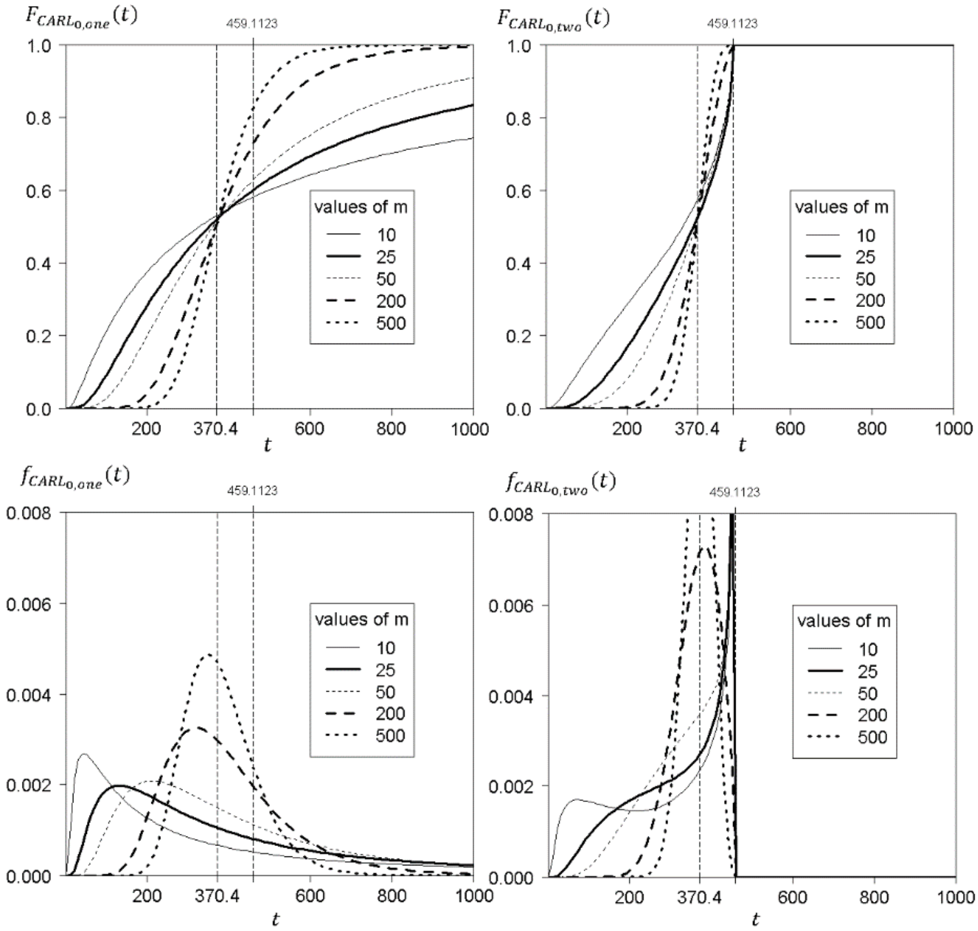

Figure 2. Cdf's and pdf's of the $CARL_0$ (F_{CARL_0} and f_{CARL_0}, respectively) of the one- and two-sided charts with unadjusted control limits ($\alpha = 0.0027$) for different values of m and $n = 5$. Nominal $ARL_0 = 370.4$ and maximum value of $CARL_{0,two}$ ($max(CARL_{0,two}) = 459.11$).

4.3 Exceedance Probability Criterion (EPC) for the $CARL_0$

Since the $CARL_0$ is a random variable and quite small values are undesired, the Exceedance Probability (EP) of the $CARL_0$ can be useful, that is, the assessment of the probability that the $CARL_0$ exceeds a specified minimum value can provide valuable information. Indeed, the Exceedance Probability Criterion (EPC), which was proposed by Albers and Kallenberg (2005) and Albers et al. (2005), has become the performance measure most used by the authors who adopted the conditional perspective. The EPC for the $CARL_0$ is defined as requiring a high probability of $1-p$ that the chart has a $CARL_0$ greater or equal to a specified minimum tolerated value. It means, given a minimum tolerated of the $CARL_0$, the EP of the $CARL_0$ must be $1-p$. Note that this criterion depends on the cdf's of the $CARL_0$, which, in turn, is expressed as a function of the random variable Y and depends on m, n and α. In formal notation, this EPC can be stated as: given the amount of Phase I data (m and n), the unadjusted or adjusted control limits (defined by α or α^*, respectively) and the smallest (minimum) tolerated value of the $CARL_0$ (defined by α and ε), the EP of the $CARL_0$ must results in $(1-p)$, that is

$$P\left(CARL_0 \geq \frac{1}{(1+\varepsilon)}\left(\frac{1}{\alpha}\right)\right) = P(CFAR \leq (1+\varepsilon)\alpha) = 1-p, \qquad (37)$$

where: α is the nominal false alarm rate; $\varepsilon \geq 0$ is a factor that is defined to determine a proper performance threshold, meaning that the smallest tolerated value for the $CARL_0$ is $100\left(\frac{\varepsilon}{1+\varepsilon}\right)\%$ smaller than the nominal $ARL_0 = 1/\alpha$ (this is equivalent to saying that the largest tolerated value for the $CFAR$ is $(100\varepsilon)\%$ larger than the nominal α, as used by Epprecht et al., 2015); and p is the risk (probability) accepted by the practitioner that the true $CARL_0$ be smaller than the specified tolerated value (or the true $CFAR$ be larger than the specified tolerated value).

4.4 Distribution of the conditional run length quantile (CRL_q)

The cumulative distribution function (cdf) of the conditional run length quantiles for the one- and two-sided S^2 charts ($CRL_{q,one}(\rho^2)$ and $CRL_{q,two}(\rho^2)$), that is, $F_{CRL_{q,one}}(t)$ and $F_{CRL_{q,two}}(t)$ (Equations (38) and (39), respectively) are provided. Substituting $q = 0.05$ and $q = 0.50$ (conditional median run length $CMRL$) into both equations, the cdf's of both important order quantiles can be obtained. The proof of the obtained cdf of the $CRL_{q,one}$ ($F_{CRL_{q,one}}(t)$) is based on exact analytical derivations and is presented step by step in Appendix E. Similarly to the one-sided chart, the cdf of the $CRL_{q,two}$ ($F_{CRL_{q,two}}(t)$) is obtained using the same analytical derivations of the one-sided case as well as the derivations provided in the context of two-sided tolerance interval for the sample variance (Equation 23 in Subchapter 3.1). Hence, the cdf's of the $CRL_{q,one}(\rho^2)$ and $CRL_{q,two}(\rho^2)$ are given by Equations (38) and (39), respectively:

$$F_{CRL_{q,one}}(t) = P\big(CRL_{q,one}(\rho^2) \leq t\big)$$
$$= \begin{cases} 0,\ t < 1 \\ F_{\chi^2_{m(n-1)}}\left(\dfrac{\rho^2 m(n-1)\chi^2_{n-1,\,(1-q)^{1/\lfloor t \rfloor}}}{\chi^2_{n-1,\,1-\alpha}}\right),\ t \geq 1 \end{cases}, \tag{38}$$

where t is a real value and $\lfloor t \rfloor$ denotes the largest integer less than or equal to t. Note that, from Equation (38), the cdf of $CRL_{q,one}(\rho^2)$, which is a non-decreasing step function, depends on m, n and α values.

$$F_{CRL_{q,two}}(t) = P\big(CRL_{q,two}(\rho^2) \leq t\big) = 1 - (u_2 - u_1),\ t \geq 1, \tag{39}$$

where u_1 and u_2 are the two solutions of $CARL_{two}(U;\rho^2) = \frac{1}{1-(1-q)^{1/\lfloor t \rfloor}}$, that is

$$F_{\chi^2_{n-1}}\left(\frac{F^{-1}_{\chi^2_{m(n-1)}}(U)}{\rho^2 m(n-1)}\chi^2_{n-1,1-\frac{\alpha}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{F^{-1}_{\chi^2_{m(n-1)}}(U)}{\rho^2 m(n-1)}\chi^2_{n-1,\frac{\alpha}{2}}\right) = (1-q)^{1/\lfloor t \rfloor},$$

for U, being $u_1 < u_2$. As opposed to the distribution of $CRL_{q,one}$, this is not possible to obtain a closed-form equation for the cdf of the $CRL_{q,two}$, so numerical (search) method is required, similarly to the cdf of the $CARL_{two}$ (Equation 36).

To the best of our knowledge, this is the first time that the distributions of the CRL_q of the one- and two-sided S^2 charts are examined. As noted before (Introduction), previous works have studied the unconditional RL quantiles (including the unconditional median run-length MRL_0) only for the $\bar{X}$ chart.

5
Performance and design of the phase II S^2 control chart with estimated parameter

The study of control chart performance is a very important concern for researchers and practitioners involved in SPC not just because they can analyze the chart's ability to detect quickly shifts in the process parameter(s), but also because the examination of chart performance enables them to be able to make decisions about control chart design. In this work, the term "design" must be understood as the decision-making related to the choice of the amount of Phase I reference data and/or the decision of adjusting or not the control limits for ensuring a specified IC performance of the Phase II chart and, if adjusting them, the value of the adjusted control limit factor, which are used to construct the control charts.

As noted in previous chapters, authors studying the design and performance of S^2 and S control charts with estimated parameters have followed principally either the *unconditional* perspective or the *conditional* perspective separately, i.e., according to the marginal (unconditional) and conditional distributions of the RL, respectively, and considered only one of two configurations, namely, either the one-sided chart (without a lower control limit) or the two-sided chart, alternately, but never on both.

Under the *unconditional* perspective, the design of control charts is based on a specified unconditional ARL_0. However, since differences in the actual performance of control charts of a same application but constructed based on different Phase I reference samples (the "practitioner-to-practitioner variability") may be significantly large, the focus has moved from the unconditional RL distribution, which had been the prevailing approach until then, to the conditional RL distribution and associated performance measures (we called the *conditional* perspective). In this chapter, we focus on the random variable $CARL_0$ as the main conditional performance measure, and then its distribution and properties are examined.

Each type of S^2 chart (one-sided or two-sided charts) has a particular purpose and is widely used in monitoring the process dispersion, specifically, the upper one-sided chart is used to know only how large the process dispersion is (interest in process deterioration) and the two-sided chart is used to know how large as well as how small the process dispersion is (interest in process deterioration and improvement).

The indicated differences, related to type of chart configuration and perspectives for designing chart, could become confusing for the practitioner. Jardim et al. (2019) presented a comparative study on the two mentioned perspectives for the $\bar{X}$ chart. However, considering S^2 and S control charts, which are the principal charts for monitoring the process variability, there is no similar study guiding the practitioner for decision making of the design of S^2 and S charts. For that reason, a complete comparative study of the *unconditional* and *conditional* perspectives for designing one- and two-sided Phase II S^2 charts is undertaken, and some important gaps (in the studies of previous authors) are filled. This proposed study is useful to assess if Phase II charts with adjusted control limits that guarantee a desired IC performance based on one perspective are able to satisfy the IC performance under the other perspective so that a better and complete understanding of chart performance can be gained and a proper chart design can be chosen. The chart design and the resulting Phase II performance of charts with adjusted limits are also compared to the ones with unadjusted limits and the ones for the known-parameter case (nominal chart performance). Such a comparative study is relevant due to the costs involved in designing control charts (e.g., cost of sampling, cost of unnecessary production stops due to false alarms, and costs of producing defective units).

First of all, the Phase II performance of S^2 control chart with traditional control limits that are not adjusted is examined. This Phase II performance depends on a specified nominal false alarm rate (α), the amount of Phase I reference data (m and n) to estimate σ_0^2 and the chosen estimator of σ_0^2. The discrepancy between the Phase II performance of charts with unadjusted control limits, measured according to the *unconditional* and *conditional* perspectives, and the one in the known-σ_0^2 case (nominal chart performance) is presented and discussed. Next, we find the amount of Phase I reference data (several combinations of values of m and

n) that leads to a non-significant level of this discrepancy. Specifically, under the *unconditional* perspective, we find the number of Phase I samples (m) required to achieve an unconditional ARL_0 close to the one of the chart with known σ_0^2; and under the *conditional* perspective, the minimum m required to guarantee a tolerated lower bound for $CARL_0$ with a high probability (i.e., the EPC for $CARL_0$). It is important to note that all works on number of Phase I samples that have been focused on the *unconditional* perspective considered only two-sided charts, while the sole work that has been conducted under the *conditional* perspective is about upper one-sided charts (see Table 1). To complement this assessment, the required minimum values of m for one-sided versus two-sided charts (under each perspective) are compared, as well as the required minimum m for the unconditional versus conditional perspectives (regarding each type of chart).

Practitioners deal with a practical problem because the required number of Phase I samples (m) to ensure a desired (conditional or unconditional) IC performance is large and usually infeasible. Due to this difficulty, adjusting the control limits represents an effective solution to guarantee a desired IC performance with a practical amount of m at hand. To this end, formulas for the adjusted limits factors of the one- and two-sided S^2 charts obtained under the *unconditional* and *conditional* perspectives are provided. Studies on adjustments under the *unconditional* perspective have been addressed only on the two-sided limits and no one has worked on the upper one-sided chart, while, under the *conditional* perspective, works have been focused on the upper one-sided chart (see Table 2), with the exception of Guo & Wang (2017) that presented their paper when the present work was in progress and the formulation of the adjusted control limits were already obtained (as indicated before in Chapter 1). In addition, researchers neither compared the *unconditional* and *conditional* perspectives for adjusting control limits of the S^2 (or S) charts, regarding each chart (one-sided and two-sided charts), nor the adjustments of the one-sided and the two-sided S^2 (or S) charts under each perspective. Thus, in the next subchapters, we pursue to fill the gaps mentioned (related to the three major questions from Jensen et al. (2006), which are indicated in Subchapter 2.1) and do such comparative analysis of the adjustments between these two perspectives and types of charts.

Finally, it is worth to note that, for designing two-sided S^2 chart under the *conditional* perspective, we exploit the relationship between the exact two-sided tolerance interval for the sample variance and the EP of the $CARL_0$ (or, equivalently, the cdf of the $CFAR$) of the two-sided S^2 chart and, therefore, the minimum Phase I samples (m) and the adjusted control limits to ensure a tolerated lower bound of the $CARL_0$ (provided in Subchapters 5.2.2 and 5.3.2, respectively) are obtained on the basis of the derivations of the exact two-sided tolerance intervals for samples variances, which were given in Subchapter 3.1.

5.1 Phase II performance of one-sided and two-sided S^2 charts with unadjusted limits

Table 8 shows values of certain measures associated with the $CARL_0$, namely, expected value ARL_0, standard deviation $SDARL_0$, and the Exceedance Probability EP (from Equation 37 for $\varepsilon = 0$ and $\varepsilon = 0.20$) for the one- and two-sided S^2 charts with unadjusted limits ($\alpha = 0.0027$, i.e., with a nominal ARL_0 of 370.4) considering several values of m and n to estimate σ_0^2. The percentage ratio between the unconditional ARL_0 and the nominal ARL_0 $\left(\frac{ARL_0}{nominal\ ARL_0}100\%\right)$ is also provided in Table 8. The exact values of ARL_0 and $SDARL_0$ are calculated numerically, substituting Equations (14) and (15) into Equations (18)-(20) with $\rho = 1$, and the plots of them as functions of m, $n = 5$ and $\alpha = 0.0027$ are shown in the panels (a) and (b) of Figure 3, respectively. In addition, Figure 4 provides modified boxplots that show the distributions and some percentiles ($1th$, $5th$, $25th$, $50th$, $75th$, $95th$ and $99th$) of the $CARL_0$ of one- and two-sided charts, for $m = 25$ and $m = 250$ (panels a and b of Figure 4, respectively), $n = 5$ and $\alpha = 0.0027$.

From Table 8, the Phase II performance difference between the two charts is noticeable: the values of $ARL_{0,one}$ are always larger than the nominal value 370.4, while the $ARL_{0,two}$ values are always somewhat smaller. As the amount of Phase I samples (m) increases, the $ARL_{0,one}$ decreases and converges to 370.4, while the $ARL_{0,two}$ increases and converges to 370.4 (see the plots of panel a of Figure 3). For instance, when $n = 3$ and $m = 25$, the $ARL_{0,one} = 852.9$ is more than twice larger than the nominal value, while $ARL_{0,two} = 336.4$ is only 9.2% smaller than this nominal one.

Table 8 - Associated measures of the $\boldsymbol{CARL_0}$ of the one- and two-sided $\boldsymbol{S^2}$ charts with unadjusted limits (nominal $\boldsymbol{ARL_0}$ of $\mathbf{370.4}$) as functions of of m and n

		$ARL_0 = E(CARL_0)$		$\frac{ARL_0}{nominal\ ARL_0}100\%$		$SDARL_0 = SD(CARL_0)$		$P(CARL_0 \geq Tol)$ $Tol = 370.4$		$P(CARL_0 \geq Tol)$ $Tol = 308.6$	
m	n	One-sided	Two-sided	One-sided	Two-sided	One-sided	Two-sided	One-sided	Two-sided	One-sided	Two-sided
25	3	852.9	336.4	230.3%	90.8%	2889.9	141.8	47.3%	47.3%	53.5%	58.8%
	5	674.2	331.9	182.0%	89.6%	1292.9	113.4	48.1%	47.7%	55.3%	62.4%
	9	587.4	327.1	158.6%	88.3%	823.5	90.6	48.7%	45.0%	56.7%	65.6%
50	3	541.6	351.1	146.2%	94.8%	658.9	116.0	48.1%	48.1%	56.9%	64.2%
	5	490.8	348.3	132.5%	94.0%	458.1	91.2	48.7%	48.7%	58.7%	68.5%
	9	461.7	345.4	124.7%	93.2%	357.1	70.9	49.1%	48.4%	60.4%	73.1%
75	3	473.8	356.9	127.9%	96.3%	406.5	100.8	48.5%	48.5%	59.1%	67.9%
	5	445.2	354.8	120.2%	95.8%	308.6	78.7	48.9%	48.9%	61.2%	72.7%
	9	428.1	352.7	115.6%	95.2%	252.5	60.4	49.2%	49.1%	63.0%	77.8%
100	3	444.4	360.0	120.0%	97.2%	309.7	90.5	48.7%	48.7%	61.0%	70.8%
	5	424.6	358.4	114.6%	96.8%	244.1	70.3	49.1%	49.1%	63.2%	75.9%
	9	412.6	356.7	111.4%	96.3%	204.1	53.5	49.3%	49.3%	65.2%	81.3%
150	3	417.5	363.2	112.7%	98.1%	224.0	76.9	48.9%	48.9%	63.8%	75.3%
	5	405.3	362.1	109.4%	97.8%	182.6	59.4	49.2%	49.2%	66.3%	80.8%
	9	397.8	361.0	107.4%	97.5%	155.8	44.8	49.5%	49.5%	68.6%	86.3%
200	3	404.9	365.0	109.3%	98.5%	183.0	68.1	49.1%	49.1%	66.2%	78.7%
	5	396.2	364.1	107.0%	98.3%	151.6	52.4	49.3%	49.3%	68.9%	84.4%
	9	390.7	363.2	105.5%	98.1%	130.6	39.4	49.5%	49.5%	71.3%	89.8%
250	3	397.7	366.0	107.4%	98.8%	158.1	61.7	49.2%	49.2%	68.1%	81.5%
	5	390.8	365.3	105.5%	98.6%	132.2	47.4	49.4%	49.4%	71.0%	87.2%
	9	386.5	364.6	104.3%	98.4%	114.5	35.5	49.6%	49.6%	73.6%	92.2%
∞	3	370.4	370.4	100%	100%	0.0	0.0	100%	100%	100%	100%
	5	370.4	370.4	100%	100%	0.0	0.0	100%	100%	100%	100%
	9	370.4	370.4	100%	100%	0.0	0.0	100%	100%	100%	100%

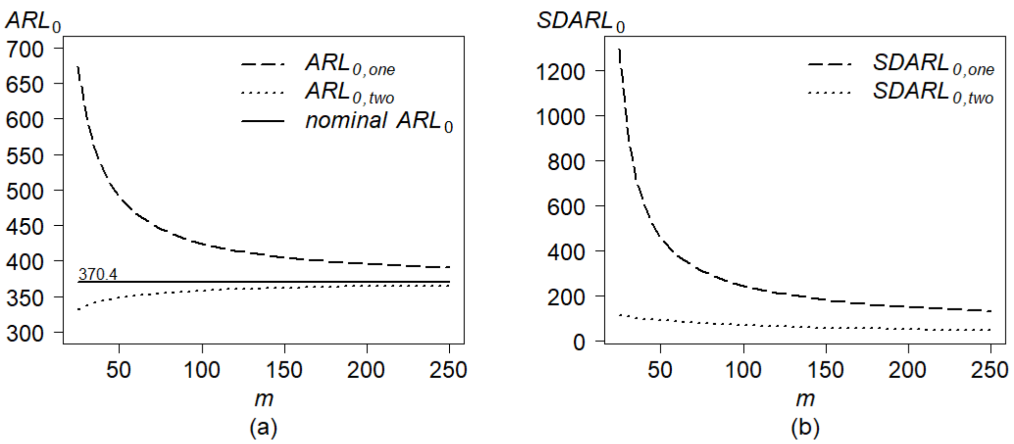

Figure 3 - a) $\boldsymbol{ARL_{0,one}}$ and $\boldsymbol{ARL_{0,two}}$, and (b) $\boldsymbol{SDARL_{0,one}}$ and $\boldsymbol{SDARL_{0,two}}$ of the $\boldsymbol{S^2}$ control charts with unadjusted limits (nominal $\boldsymbol{ARL_0 = 370.4}$) as a function of $\boldsymbol{m}$ and $\boldsymbol{n = 5}$

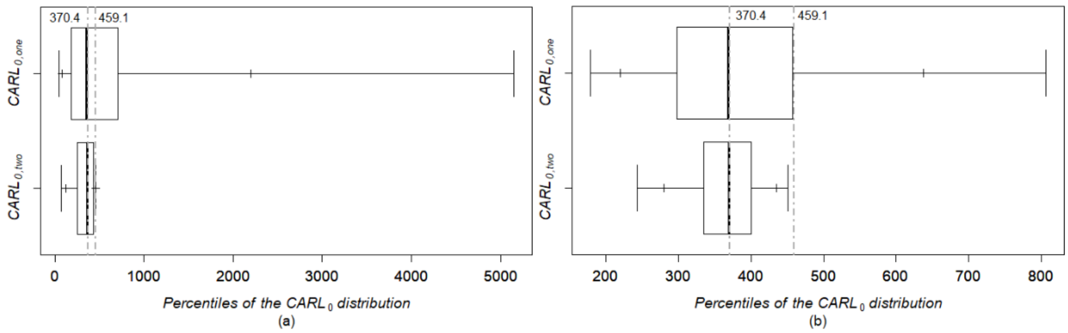

Figure 4 - Distributions and some percentiles of $\boldsymbol{CARL_{0,one}}$ and $\boldsymbol{CARL_{0,two}}$ of the $\boldsymbol{S^2}$ charts with unadjusted limits (nominal $\boldsymbol{ARL_0 = 370.4}$) for $\boldsymbol{n = 5}$ and (a) $\boldsymbol{m = 25}$, $\boldsymbol{max(CARL_{0,two}) = 459.1}$ and (b) $\boldsymbol{m = 250}$, $\boldsymbol{max(CARL_{0,two}) = 459.1}$

This significant difference in the values of $ARL_{0,one}$ and $ARL_{0,two}$ (expected values of the $CARL_{0,one}$ and $CARL_{0,two}$, respectively), which is shown in Table 8 and plots of Figure 3(a), can be explained by the shapes and variabilities of their corresponding distributions. Regarding the shapes (see the plots of Figure 4), the distribution of the $CARL_{0,one}$ is highly right-skewed, while the distribution of the $CARL_{0,two}$ is left-skewed with a finite upper bound (from Figure 4, note that $max(CARL_{0,two}) = 459.1$ depends only on $n = 5$ and $\alpha = 0.0027$, regardless the value of m), as was described in Subchapter 4.2. With respect to the variability of the $CARL_0$ distribution, the dispersion of the $CARL_{0,one}$ distribution, measured in terms of the $SDARL_0$, is much larger than the one of the $CARL_{0,two}$ distribution (see Table 8 and also the plots of Figure 3b): the values of $SDARL_{0,one}$ vary approximately between twice and eleven-times the $SDARL_{0,two}$ values. For instance, for $m = 25$ and $n = 5$, $SDARL_{0,one} = 1292.9$ and $SDARL_{0,two} = 113.4$. The variability of the $CARL_0$ gets smaller when m and n increase, for example, for $m = 250$ and $n = 9$, $SDARL_{0,one} = 114.5$ and $SDARL_{0,two} = 35.5$.

For the sake of obtaining more information on Phase II performance of charts with unadjusted limits under the conditional point of view, from Table 8, the $EP's$ of the $CARL_0$ of charts with unadjusted control limits are computed. When $\varepsilon = 0$, $\alpha = 0.0027$ and all values of m and n, note that $P(CARL_0 \geq 370.4)$ is a little bit less than 50% for the one- and two-sided charts, as can be seen in Figure 4 (and also in Figure 2). These results represent unwanted performance measures of charts with unadjusted limits since excessive false alarms may be expected compared with the nominal level. Even though the value of ε is increased up to 20%, these $EP's$ (for both charts) are not large enough, such as 90% or 95%, to avoid too many false alarms in most of the cases. For example, $P(CARL_0 \geq 308.6)$ is 58.7% and 68.5% for the one-sided and two-sided charts, respectively, for $m = 50$ and $n = 5$. Put another way, values of the $CARL_0$ smaller than the tolerated minimum (308.6) are expected in more than 41% and 31% of the cases (for one- and two-sided charts, respectively) albeit the nominal ARL_0 is 370.4. These high risks (probabilities) are undesirable outcomes of IC performance that could lead to a serious problem in designing charts. Only when m and n are large, $P(CARL_0 \geq 308.6)$ increases substantially in some cases, specially, for the two-sided chart. For instance, given $n = 5$, when m is increased up to 250, $P(CARL_0 \geq 308.6)$ attains 87.2% for the two-sided chart, however, the probabilities for the one-sided chart do not exceed 71%. From Table 8, given certain values of n, α and ε and a specific chart (one- or two-sided chart), the values of the $EP's$ increases when the variability of the $CARL_0$ (measured by the $SDARL_0$) decreases, that is, when m takes on larger values. For the same values of m, n, α and ε in both types of charts, the values of the $EP's$ of the two-sided chart are larger than the ones of the one-sided chart since the variability of the $CARL_{0,one}$ distribution is much larger than the one for the $CARL_{0,two}$ (see, for instance, Figures 3b and 4). Accordingly, since the risk of poor IC performance of one-sided and two-sided charts with unadjusted limits, based on the complement of the $EP's$ of the $CARL_0$, is generally high, the practitioners face a serious problem that can be solved by adjusting the control limits, as will be examined later.

5.2
Number of Phase I samples (m) that guarantees an IC performance of Phase II S^2 chart

One of the more interesting practical question related to the effect of parameter estimation on the chart performance, as was highlighted by Jensen et al. (2006), is the determination of the amount of the Phase I reference data that enables to achieve a desired IC performance of Phase II control charts, in terms of either *conditional* or *unconditional* performance measure, using the traditional uncorrected control limits (also called unadjusted control limits). As we have noted previously in Table 1, note that all works on this question under the *unconditional* point of view have considered only two-sided charts, while the unique work under the *conditional* point of view has considered only upper one-sided charts. In light of this, the distribution of the $CARL_0$ and its properties, such as its expected value (the unconditional ARL_0) and its EP, are first used to find the number of Phase I samples (m) that leads an unconditional ARL_0 of the one-sided S^2 chart close to the advertised nominal ARL_0, which is typical in the known-σ_0^2 case, and second, to compute the minimum m that guarantees a conditional IC performance of the two-sided S^2 chart using the EPC for the $CARL_0$.

5.2.1
Number of Phase I samples (m) that guarantees an unconditional IC performance

From Table 8, the findings revealed by Chen (1998), Maravelakis et al. (2012) and Castagliola et al. (2009) for the two-sided charts can be verified, that is, the minimum number of Phase I samples (m) must be around 200 (with a common n of 5) for obtaining an unconditional ARL_0 (expected value of the $CARL_0$) of the two-sided charts close to the nominal $ARL_0 = 370.4$.

Given specified values of n and α (for instance, see plots of Figure 3(a) when $n = 5$ and $\alpha = 0.0027$), as the value of m increases, the $ARL_{0,two}$ increases and converges to the nominal ARL_0 faster than the $ARL_{0,one}$ which, differently from the two-sided chart, decreases. Accordingly, $ARL_{0,two}$ reaches a value close to the nominal 370.4 with much less number of Phase I samples (m) than $ARL_{0,one}$. For instance, when $n = 5$ and $m = 25$, the $ARL_{0,one} = 674.2$ is 82% larger than the nominal value, while $ARL_{0,two} = 331.9$ is just 10.4% smaller than the nominal 370.4. If m increases to 50, $ARL_{0,two} = 348.3$ is now 6% smaller, while $ARL_{0,one} = 490.8$ is still not close to the nominal one, indeed, this is 32.5% larger.

The values of $\left(\frac{ARL_0}{nominal\ ARL_0} 100\%\right)$ in Table 8 bring to light an interesting finding, given $n = 5$ and a value of m of at least 200 (the minimum recommended), the $ARL_{0,two}$ is no more than 1.7% lower than the nominal $ARL_0 = 370.4$. In the case of the one-sided S^2 chart, the $ARL_{0,one}$ is no more than 7% higher than the nominal ARL_0. Hence, given the same n and α (nominal ARL_0) for the two types of charts, the minimum m to achieve a value close to a desired unconditional ARL_0

for the one-sided chart (computed for the first time in the literature) is moderately larger than the minimum m for the two-sided chart.

5.2.2 Number of Phase I samples (m) that guarantees a conditional IC performance

A new perspective for designing control charts with estimated parameters, the so-called *conditional* perspective, has arisen in the last few years. This is the predominant point of view in the most recent researches that take into account the "practitioner-to-practitioner" variability inherent to any control chart with estimated parameters. This perspective is based on the use of the EPC (Equation 37), where the randomness of the $CARL_0$ is considered through its distribution. Under this perspective, the question of interest is: What is the minimum number of Phase I reference samples (m) that ensures a minimum tolerated value of the $CARL_0$ with a high probability of at least $1-p$ (e.g., 0.9)? Such minimum tolerated value is generally equal to or slightly smaller than the nominal ARL_0, e.g., 95% of 370.4.

For the case of the upper one-sided S and S^2 charts, with the same notation considered in the present thesis, Epprecht et al. (2015) presented the inequality $m(n-1)\chi^2_{n-1,\,[1-(1+\varepsilon)\alpha]} \leq \chi^2_{m(n-1),\,p}\left(\chi^2_{m(n-1),\,1-\alpha}\right)$ that is used to obtain the smallest value of m by a search algorithm. It is worth to remark that the expression of this inequality takes the relational operator "less than or equal to" ($\leq$) instead of "greater than or equal to" ($\geq$) that appears in Equation (17) in Epprecht et al. (2015). This was a typographical error in the cited paper.

From what we know, there is no similar research in the literature about the two-sided chart. Motivated by this, using the cdf of the $CARL_{0,two}$ (Equation 36) and the EPC for the $CARL_{0,two}$ (Equation 37), given the chart with unadjusted limits ($\alpha = 0.0027$), the minimum tolerated value of the $CARL_0$ (defined by α and ε), the sample size n and a specified high probability $1-p$, the minimum m is obtained by a search algorithm so that $F_{CARL_{0,two}}\left(t = \frac{1}{(1+\varepsilon)}\left(\frac{1}{\alpha}\right); m\right) \leq p$. Note that the last "=" operator in Equation (37) must be replaced by the "$\geq$" operator because m is an integer and a perfect match of the probability $(1-p)$ is generally not possible.

As highlighted above, equations to find the minimum m for the two-sided chart can be derived on the basis of the exact two-sided tolerance interval for the sample variance. From Equations (33) and (34) and considering the four "components" (a)-(d) in Subchapter 4.1 (recall that the sample size n is the same for the S^2 chart and tolerance intervals for sample variances), we can find the minimum number of Phase I reference samples (m) according to the EPC for the $CARL_{0,two}$ (Equation 37) as follows:

- **In the context of S^2 control chart:** given the values of $F_{CARL_{two}}(t) = p$, $t = \frac{1}{(1+\varepsilon)\alpha}$ and α (unadjusted limits), we want to obtain m (Equation 37).
- **In the context of tolerance interval for the sample variance:** the three known "components" are $\gamma = 1 - p$, $\beta = (1+\varepsilon)\alpha$ and $\beta^* = \alpha$, while the unknown "component" is m. Thus, m is found using two-sided tolerance interval (Equation 24) .

Table 9 provides the minimum m needed to guarantee a specified conditional IC performance (in terms of the EPC for the $CARL_0$) of the one-sided and two-sided S and S^2 charts. The value of m is computed for several values of n, $\alpha = 0.005$, $\varepsilon = \{10\%, 20\%\}$ and $p = \{0.05, 0.10\}$. In addition, the percentage ratio of the minimum values of m for both charts, which is given by $\%Rm = \frac{m \text{ for two-sided chart}}{m \text{ for one-sided chart}} 100\%$, is presented.

As was shown by Epprecht et al. (2015), the required amount of Phase I reference samples (m) is rather large (and almost always impractical). Furthermore, an interesting outcome is revealed in Table 9, for the same size of samples n (in Phase I and II), the same specified minimum tolerated of the $CARL_0$ to be guaranteed with the same specified high probability (the same specified values of ε, α and p), the minimum number of m of the two-sided chart is much smaller than the corresponding one of the one-sided chart, even with large sample sizes such as $n = 30$. More specifically, from the values of $\%Rm$, the minimum number of m needed by the two-sided charts are between 10% and 33% of the ones by one-sided charts. Despite these reductions, often, the amounts of m for two-sided charts are still infeasible to collect in practice. For example, from Table 9, given $n = 5$, the minimum m for ensuring $P(CARL_0 \geq 181.8) \geq 95\%$ ($\varepsilon = 10\%$ and $p = 0.05$) is 6337 for the one-sided chart, while for the two-sided chart, the minimum m is 1325 samples: a reduction of 79%. However, note that, when n is increased and/or the desired (specified) IC performance is less rigorous (namely, when the values of ε and/or p are increased), the minimum m could decrease significantly. For instance, given $n = 20$, the minimum m for ensuring $P(CARL_0 \geq 166.7) \geq 90\%$ ($\varepsilon = 20\%$ and $p = 0.10$) is 668 for the one-sided chart, while for the two-sided chart, the minimum m is 106 samples, in this case, a reduction of 84%. Hence, the results in Table 9 reveal that there are some cases in the two-sided chart (*conditional* perspective) where the minimum m is no more than 120, for instance, when $n \geq 16$, $\varepsilon \geq 20\%$ and $p \geq 0.10$.

Table 9. Minimum number of Phase I samples ($\boldsymbol{m}$) required to guarantee a conditional IC performance of the one- and two-sided $\boldsymbol{S^2}$ charts with unadjusted limits ($\boldsymbol{\alpha = 0.005}$)

	$\varepsilon = 0.10$						$\varepsilon = 0.20$					
	p						p					
	0.05			0.10			0.05			0.10		
n	one-sided	two-sided	$\%Rm$	one-sided	two-sided	$\%Rm$	one-sided	two-sided	$\%Rm$	one-sided	two-sided	$\%Rm$
2	11224	3366	30%	6838	2056	30%	3046	1002	33%	1862	616	33%
3	8298	2144	26%	5052	1309	26%	2252	652	29%	1375	400	29%
4	7053	1623	23%	4293	991	23%	1914	503	26%	1168	309	26%
5	6337	1325	21%	3856	809	21%	1719	419	24%	1049	257	24%
6	5861	1131	19%	3566	690	19%	1590	364	23%	970	223	23%
7	5518	994	18%	3357	607	18%	1497	325	22%	913	199	22%
8	5257	893	17%	3197	545	17%	1426	296	21%	869	181	21%
9	5049	814	16%	3071	497	16%	1370	273	20%	835	167	20%
10	4880	751	15%	2968	458	15%	1324	255	19%	806	156	19%
11	4738	700	15%	2881	427	15%	1285	240	19%	783	147	19%
12	4618	658	14%	2808	401	14%	1252	228	18%	763	140	18%
13	4514	622	14%	2745	379	14%	1224	218	18%	746	133	18%
14	4423	591	13%	2689	360	13%	1199	209	17%	730	128	18%
15	4342	564	13%	2640	344	13%	1178	201	17%	717	123	17%
16	4271	541	13%	2597	330	13%	1158	194	17%	705	119	17%
17	4206	520	12%	2557	317	12%	1141	188	16%	695	115	17%
18	4148	502	12%	2522	306	12%	1125	182	16%	685	112	16%
19	4095	485	12%	2489	296	12%	1110	178	16%	676	109	16%
20	4046	470	12%	2460	287	12%	1097	173	16%	668	106	16%
25	3854	413	11%	2342	252	11%	1045	156	15%	636	96	15%
30	3716	374	10%	2259	228	10%	1007	144	14%	613	89	15%

From Tables 8 and 9, given the same setting (namely, the same n and α), we can realize that the *conditional* perspective generates a larger number of Phase I samples (m) compared to the *unconditional* perspective, for the one-sided chart as well as for the two-sided one.

As mentioned earlier, since large amounts of Phase I reference data required to attain a desired (unconditional or conditional) IC performance of one-sided and two-sided S^2 charts, some authors have proposed adjustments to the control limit(s) in order to guarantee this desired IC performance with a practical amount of Phase I reference data. Furthermore, since the risk of poor IC performance (in terms of the $EP's$ of the $CARL_0$) of charts with unadjusted limits is most often high, the practitioners face a considerable issue that can be tackled by adjusting the control limits, as will be seen below.

5.3 Adjusting the control limits of S^2 control charts

Adjusted or corrected control limits, denoted as $\widehat{UCL}^*_{one}$ and ($\widehat{LCL}^*_{two}$, $\widehat{UCL}^*_{two}$) for the one- and two-sided charts, respectively, are determined by replacing the traditional unadjusted control limit factors of one-sided and two-sided charts (U_{one} and (L_{two}, U_{two}), respectively, from Equations 7 and 8) by the corresponding "new" adjusted or corrected control limit factors (U^*_{one} and (L^*_{two}, U^*_{two}), respectively, from Equations 40 and 41, which are shown below) in order to assure a specified (either unconditional or conditional) IC performance. In other words, the adjusted control limits are determined by replacing the nominal false alarm rate (α) of one-sided and two-sided charts by the "new" adjusted false alarm rate α^*.

The values of α^* should be found and are required to compute the adjusted upper one-sided control limit factor (U^*_{one}) and the adjusted (lower and upper) control limit factors (L^*_{two}, U^*_{two}) which, in turn, are used to obtain the corresponding adjusted upper one-sided S^2 control limit ($\widehat{UCL}^*_{one}$) and the adjusted (lower and upper) two-sided S^2 control limits ($\widehat{LCL}^*_{two}$, $\widehat{UCL}^*_{two}$), respectively, from Equations (40) and (41):

$$\widehat{UCL}^*_{one} = \frac{\chi^2_{n-1,1-\alpha^*}}{(n-1)} S_p^2 = U^*_{one} S_p^2, \quad (40)$$

$$\widehat{LCL}^*_{two} = \frac{\chi^2_{n-1,\alpha^*/2}}{(n-1)} S_p^2 = L^*_{two} S_p^2 \text{ and } \widehat{UCL}^*_{two} = \frac{\chi^2_{n-1,1-\alpha^*/2}}{(n-1)} S_p^2 = U^*_{two} S_p^2. \quad (41)$$

This subchapter is divided in two parts. We provide formulas of the adjusted false alarm rate α^* of the (one- and two-sided) S^2 charts that are obtained under both the *unconditional* and *conditional* perspectives, respectively, in Subchapters 5.3.1 and 5.3.2.

5.3.1 Adjusting the control limits under the *unconditional* perspective

Under the *unconditional* perspective (see Table 2), the adjusted limits (or, specifically, the adjusted factors), from Equations (40) and (41), are obtained given a desired unconditional in-control average run length, denoted as ARL_0^*. The traditional ARL_0^* is 370.4, i.e., the value of the ARL_0 in the σ_0^2-known case, given a nominal false alarm rate (α) of 0.0027. The adjusted control limits are calculated by first obtaining the value of α^* that yields this desired ARL_0^*. The value of α^* is found by solving the resulting equation of making Equation (18), for $\rho = 1$, equal to ARL_0^*.

The adjusted factor of one-sided S^2 control chart

Equation (14) with $\rho = 1$ and $\alpha = \alpha^*$ must be substituted into Equation (18). Thus, α^* is found by a search method from

$$ARL_{0,one}^* = \int_0^\infty \left(1 - F_{\chi_{n-1}^2}\left(\frac{Y}{m(n-1)}\chi_{n-1,1-\alpha^*}^2\right)\right)^{-1} f_Y(y)dy, \quad (42)$$

with the α^* value thus found, the adjusted upper one-sided factor (U_{one}^*) and limit ($\widehat{UCL}_{one}^*$) can be obtained using Equation (40). To the best of our knowledge, the adjusted factor U_{one}^* of the upper one-sided S^2 control chart has not been examined yet.

The adjusted factor of two-sided S^2 control chart

Equation (15) with $\rho = 1$ and $\alpha = \alpha^*$ must be substituted into Equation (18). Thus, α^* is found by a search method from

$$ARL_{0,two}^* = \int_0^\infty \left(1 - \left(F_{\chi_{n-1}^2}\left(\frac{Y}{m(n-1)}\chi_{n-1,1-\frac{\alpha^*}{2}}^2\right) - F_{\chi_{n-1}^2}\left(\frac{Y}{m(n-1)}\chi_{n-1,\frac{\alpha^*}{2}}^2\right)\right)\right)^{-1} f_Y(y)dy, \quad (43)$$

with the α^* value thus found, the adjusted (lower and upper) factors (L_{two}^*, U_{two}^*) and limits ($\widehat{LCL}_{two}^*$, $\widehat{UCL}_{two}^*$) can be obtained using Equation (41). Castagliola et al. (2009) and Diko et al. (2017) calculated these exact adjusted control limit factors of two-sided S^2 and S control charts, respectively.

5.3.2 Adjusting the control limits under the *conditional* perspective

Under the *conditional* perspective (see related works on Table 2), the purpose of the adjustment is to guarantee that, with a high probability (say 95%), the $CARL_0$ is at least a minimum tolerated value (or the $CFAR$ not larger than a maximum). This aim is achieved by the EPC, which was defined before (Equation 37). The EPC for the $CARL_0$ is used to find the value of α^*, and thus the adjusted control limits can be computed for given (m, n) combinations.

The adjusted factor of upper one-sided S^2 control chart

The adjusted false alarm rate (α^*) of the upper one-sided S^2 chart is obtained using the cdf of $CARL_{0,one}$ with $\rho = 1$ and $\alpha = \alpha^*$ from Equation (35) and the EPC from Equation (37), based on exact analytical derivations, by solving the following equation

$$F_{CARL_{0,one}}\left(t = \left(\frac{1}{1+\varepsilon}\right)\frac{1}{\alpha};\ \alpha^*\right) = F_{\chi^2_{m(n-1)}}\left(\frac{m(n-1)\chi^2_{n-1,\,1-(1+\varepsilon)\alpha}}{\chi^2_{n-1,\,1-\alpha^*}}\right) = p$$

for α^*, given the values of m, n, α, ε and p. Rearranging the terms above, the value of α^* can be obtained by

$$\alpha^* = 1 - F_{\chi^2_{n-1}}\left(\frac{m(n-1)\chi^2_{n-1,\,1-\alpha(1+\varepsilon)}}{\chi^2_{m(n-1),\,p}}\right), \qquad (44)$$

with the resulting value of α^*, the adjusted factor (U^*_{one}) and limit ($\widehat{UCL}^*_{one}$) of the upper one-sided S^2 chart can be found using Equation (40). For various settings, Goedhart (2017) and Faraz et al. (2018) provided exact values of U^*_{one}. Faraz et al. (2015) had computed U^*_{one} using the Bootstrap method proposed by Gandy and Kvaløy (2013). However, given the assumption of normality of the data, U^*_{one} can be found analytically, and thus the use of the Bootstrap method is no longer necessary and may be questionable. So, we focus on exact methods and do not explore this approximation approach here.

The adjusted factors of two-sided S^2 control chart

The value of α^* of the two-sided S^2 chart is obtained using the cdf of $CARL_{0,two}$ with $\rho = 1$ and $\alpha = \alpha^*$ from Equation (36) and the EPC from Equation (37). Or, more specifically, α^* is found by solving the following system of equations for α^*, u_1 and u_2 (where $u_1 < u_2$) using a numerical method (such as a search algorithm):

$$\begin{cases} u_2 - u_1 = 1 - p \\ F_{\chi^2_{n-1}}\left(\dfrac{F^{-1}_{\chi^2_{m(n-1)}}(u_1)}{m(n-1)}\chi^2_{n-1,1-\frac{\alpha^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\dfrac{F^{-1}_{\chi^2_{m(n-1)}}(u_1)}{m(n-1)}\chi^2_{n-1,\frac{\alpha^*}{2}}\right) = 1 - (1+\varepsilon)\alpha \\ F_{\chi^2_{n-1}}\left(\dfrac{F^{-1}_{\chi^2_{m(n-1)}}(u_2)}{m(n-1)}\chi^2_{n-1,1-\frac{\alpha^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\dfrac{F^{-1}_{\chi^2_{m(n-1)}}(u_2)}{m(n-1)}\chi^2_{n-1,\frac{\alpha^*}{2}}\right) = 1 - (1+\varepsilon)\alpha, \end{cases} \quad (45)$$

with the resulting value of α^*, the adjusted factors (U^*_{two} and L^*_{two}) and limits ($\widehat{LCL}^*_{two}$, $\widehat{UCL}^*_{two}$) of the two-sided S^2 chart can be found using Equation (41).

As indicated before, the relationship between the EP of the $CARL_{0,two}$ of the two-sided S^2 chart and the two-sided tolerance interval for the sample variance is used to derive the equations of the adjusted control limit (or, specifically the equations to find α^*). In other words, (lower and upper) tolerance factors can be used as the adjusted (lower and upper) control limit factors, respectively. From Equations (33) and (34) and considering the four "components" (a)-(d) in Subchapter 4.1 (recall that the sample size n is the same for the S^2 chart and tolerance intervals for sample variances), we can find the adjusted control limits according to the EPC for the $CARL_{0,two}$ (Equation 37) as follows:

- **In the context of S^2 control chart:** given the values of m, $F_{CARL_{two}}(t) = p$ and $t = \frac{1}{(1+\varepsilon)\alpha}$, we want to obtain α^* (Equation 45) and the corresponding adjusted (lower and upper) control limit factors: $L^*_{two} = \frac{\chi^2_{n-1,\frac{\alpha^*}{2}}}{n-1}$ and $U^*_{two} = \frac{\chi^2_{n-1,1-\frac{\alpha^*}{2}}}{n-1}$ (Equation 41).
- **In the context of tolerance interval for the sample variance:** the three known "components" are m, $\gamma = 1 - p$ and $\beta = (1+\varepsilon)\alpha$, while the

unknown "component" is β^*. Thus, β^* is found using two-sided tolerance interval (Equation 24) so that $\alpha^* = \beta^*$, $L^*_{two} = \frac{\chi^2_{n-1,\frac{\beta^*}{2}}}{n-1}$ and $U^*_{two} = \frac{\chi^2_{n-1,1-\frac{\beta^*}{2}}}{n-1}$.

5.4 Comparison of S^2 chart designs between the unconditional and conditional perspectives

Tables 10 and 11 provide the values of α^* and the corresponding adjusted control limit factors for the upper one-sided S^2 chart (U^*_{one}) and the two-sided S^2 chart (L^*_{two}, U^*_{two}), respectively, for $\alpha = 0.0027$ and various values of m and n. These values are obtained under the *unconditional* perspective (denoted UNC), which guarantees $ARL^*_0 = 370.4$, i.e., $E(CARL_0) = 370.4$, and *conditional* perspective that guarantees $P(CARL_0 \geq 370.4) = 95\%$ (EPC for ($\varepsilon = 0$, $p = 0.05$) denoted COND 1) and $P(CARL_0 \geq 308.6) = 80\%$ (EPC for ($\varepsilon = 0.20, p = 0.20$) denoted COND 2) from Equations (42)-(45). The last row of Tables 10 and 11 (for $m \to \infty$) corresponds to the case in which σ_0^2 is known (so that $\alpha = 0.0027$ and then the ARL_0 equals the nominal 370.4 value), and hence the control limit factors are the ones of the one- and two-sided charts with unadjusted control limits: U_{one} and (L_{two}, U_{two}), respectively (see Equations 7 and 8).

To gain a better comparative picture between the *unconditional* and *conditional* perspectives, plots depicted in panels (a) and (b) of Figure 5 show the control limit factors of the one-sided and two-sided charts, respectively, expressed as functions of m and $n = 5$, when the corresponding control limits are adjusted under each perspective as well as when they are not adjusted.

Tables 10 and 11 also provide the resulting IC performance measures according to one perspective of one-sided and two-sided charts, respectively, with adjusted control limits under the other perspective.

For charts with adjusted limits under the *unconditional* perspective, the EP's of the $CARL_0$ (probability that $CARL_0$ is at least the minimum tolerated value from Equation 37), for $\varepsilon = 0\%$ and $\varepsilon = 20\%$ (corresponding then to the probabilities $P(CARL_0 \geq 370.4)$ and $P(CARL_0 \geq 308.6)$, respectively), are obtained. The plots of Figure 6 show the comparison between the $EP's$ of the $CARL_0$ of charts with adjusted limits under the *unconditional* perspective (presented in Tables 10 and 11) and the ones with unadjusted limits (presented in Table 8) when $n = 5$. The panels

(a) and (b) of Figure 6 correspond to the $EP's$ of the one- and two-sided charts, respectively. These plots enable to know if the corresponding $EP's$ values of charts with unadjusted and unconditional adjusted limits achieve (or not achieve) the specified high probabilities in the *conditional* perspective ($1 - p = 95\%$ and 80%). The reason to evaluate the EP of the $CARL_0$ is that the adjustments considering only the desired unconditional ARL_0^* do not consider the risk of poor performance of a particular instance of the chart, and hence the consequences of this approach in terms of this risk can be examined.

For charts with adjusted limits under the *conditional* perspective, the unconditional ARL_0 (expected value of the $CARL_0$) is computed. The plots of Figure 7 show the unconditional $ARL_0's$ of the one- and two-sided charts (panels a and b, respectively) with unadjusted limits (Table 8) and with adjusted limits under each perspective (Tables 10 and 11) when $n = 5$.

In addition, Tables 10 and 11, for the one- and two-sided charts, respectively, provide the variability of the $CARL_0$, in terms of the standard deviation of the $CARL_0$ (denoted $SDARL_0$), and the relative percentage difference (PD) between the $SDARL_0$'s when the control limits are adjusted and unadjusted, which is given by

$$PD(SDARL_0) = 100\% \frac{SDARL_0(adjusted\ limits) - SDARL_0(unadjusted\ limits)}{SDARL_0(unadjusted\ limits)}. \quad (46)$$

The plots of Figure 8 show the $SDARL_0's$ of the one- and two-sided charts (panels a and b, respectively) with unadjusted limits (Table 8) and with adjusted limits under each perspective (Tables 10 and 11) when $n = 5$.

Table 10 - Adjusted control limit factors of upper one-sided $\boldsymbol{S^2}$ chart ($\boldsymbol{U^*_{one}}$) under the *unconditional* (UNC) perspective (with $\boldsymbol{ARL^*_0 = 370.4}$) and *conditional* perspective (COND 1: $\boldsymbol{P(CARL_0 \geq 370.4) = 95\%}$ and COND 2: $\boldsymbol{P(CARL_0 \geq 308.6) = 80\%}$ with $\boldsymbol{\alpha = 0.0027}$ and different values of $\boldsymbol{m}$ and $\boldsymbol{n}$), and some resulting associated properties of the $\boldsymbol{CARL_0}$

		α^*			U^*_{one}			$P(CARL_0 \geq Tol)$ Tol 370.4 ($\varepsilon = 0$)	$P(CARL_0 \geq Tol)$ Tol 308.6 ($\varepsilon = 0.20$)	$ARL_0 = E(CARL_0)$		$SDARL_0 = SD(CARL_0)$			$PD(SDARL_0)$		
m	*n*	UNC	COND 1	COND 2	UNC	COND 1	COND 2	UNC	UNC	COND 1	COND 2	UNC	COND 1	COND 2	UNC	COND 1	COND 2
	3	0.00516	0.00020	0.00099	5.2670	8.5066	6.9147	25.6%	31.0%	32789.7	3276.1	856.3	1391074.0	23394.0	-70.4%	48036.5%	709.5%
25	5	0.00448	0.00034	0.00123	3.7776	5.2134	4.5031	28.5%	35.1%	8600.4	1743.0	593.7	38432.4	4491.6	-54.1%	2872.6%	247.4%
	9	0.00406	0.00047	0.00141	2.8129	3.5023	3.1555	30.8%	38.4%	4343.8	1225.7	467.3	9593.7	2029.5	-43.3%	1065.0%	146.5%
	3	0.00378	0.00051	0.00148	5.5782	7.5896	6.5179	32.1%	40.5%	3757.2	1078.9	408.1	7599.2	1569.1	-38.1%	1053.4%	138.1%
50	5	0.00350	0.00068	0.00168	3.9170	4.8287	4.3281	34.4%	44.3%	2220.9	823.1	326.1	2797.7	853.3	-28.8%	510.7%	86.3%
	9	0.00332	0.00083	0.00184	2.8790	3.3237	3.0714	36.2%	47.4%	1618.6	695.3	275.6	1541.0	576.8	-22.8%	331.5%	61.5%
	3	0.00339	0.00072	0.00174	5.6874	7.2309	6.3567	35.2%	45.8%	2005.5	766.9	300.9	2339.0	730.7	-26.0%	475.4%	79.7%
75	5	0.00321	0.00090	0.00192	3.9649	4.6721	4.2551	37.2%	49.5%	1420.4	640.2	248.1	1207.4	474.1	-19.6%	291.2%	53.6%
	9	0.00310	0.00105	0.00205	2.9014	3.2491	3.0357	38.6%	52.6%	1141.8	569.6	213.4	782.5	351.6	-15.5%	209.9%	39.2%
	3	0.00320	0.00089	0.00190	5.7431	7.0294	6.2646	37.1%	49.4%	1463.7	645.1	248.6	1287.8	484.7	-19.7%	315.8%	56.5%
100	5	0.00308	0.00106	0.00207	3.9891	4.5824	4.2128	38.8%	53.1%	1123.6	561.3	207.8	759.9	338.7	-14.9%	211.3%	38.7%
	9	0.00300	0.00121	0.00219	2.9127	3.2058	3.0149	40.1%	56.3%	946.3	512.1	180.2	529.3	261.9	-11.7%	159.3%	28.3%
	3	0.00303	0.00111	0.00211	5.7995	6.8015	6.1589	39.4%	54.6%	1054.1	538.6	194.2	669.4	303.2	-13.3%	198.8%	35.4%
150	5	0.00295	0.00128	0.00225	4.0135	4.4795	4.1638	40.9%	58.4%	873.5	487.8	164.4	444.9	226.6	-10.0%	143.6%	24.1%
	9	0.00290	0.00141	0.00236	2.9240	3.1557	2.9905	41.9%	61.6%	770.5	456.3	143.6	331.9	182.5	-7.9%	113.0%	17.1%
	3	0.00294	0.00127	0.00225	5.8280	6.6710	6.0977	40.8%	58.3%	884.3	489.1	164.6	458.7	228.8	-10.0%	150.7%	25.0%
200	5	0.00288	0.00143	0.00237	4.0257	4.4199	4.1352	42.1%	62.2%	760.9	452.0	140.1	322.5	176.7	-7.5%	112.7%	16.6%
	9	0.00285	0.00155	0.00247	2.9297	3.1264	2.9762	43.0%	65.5%	687.2	428.3	122.8	248.8	145.1	-5.9%	90.5%	11.1%
	3	0.00289	0.00138	0.00234	5.8452	6.5842	6.0567	41.8%	61.3%	790.3	460.0	145.4	354.1	187.8	-8.1%	123.9%	18.8%
250	5	0.00285	0.00153	0.00246	4.0331	4.3799	4.1159	42.9%	65.2%	695.7	430.3	124.2	257.4	147.8	-6.0%	94.7%	11.8%
	9	0.00282	0.00165	0.00254	2.9331	3.1066	2.9665	43.7%	68.6%	637.6	411.0	109.1	202.8	122.9	-4.8%	77.0%	7.3%
	3	0.00270	0.00270	0.00270	5.9145	5.9145	5.9145	100.0%	100.0%	370.4	370.4	0	0.0	0.0	-	-	-
∞	5	0.00270	0.00270	0.00270	4.0628	4.0628	4.0628	100.0%	100.0%	370.4	370.4	0	0.0	0.0	-	-	-
	9	0.00270	0.00270	0.00270	2.9468	2.9468	2.9468	100.0%	100.0%	370.4	370.4	0	0.0	0.0	-	-	-

Table 11 - Adjusted control limit factors of two-sided $\boldsymbol{S^2}$ chart ($\boldsymbol{L^*_{two}}$, $\boldsymbol{U^*_{two}}$) under the *unconditional* (UNC) perspective (with $\boldsymbol{ARL^*_0 = 370.4}$) and *conditional* perspective (COND 1: $\boldsymbol{P(CARL_0 \geq 370.4) = 95\%}$ and COND 2: $\boldsymbol{P(CARL_0 \geq 308.6) = 80\%}$ with $\boldsymbol{\alpha = 0.0027}$ and different values of $\boldsymbol{m}$ and $\boldsymbol{n}$), and some resulting associated properties of the $\boldsymbol{CARL_0}$

		α^*			U^*_{two}			L^*_{two}			$P(CARL_0 \geq Tol)$ Tol 370.4 ($\varepsilon = 0$)	308.6 ($\varepsilon = 0.20$)	$ARL_0 = E(CARL_0)$		$SDARL_0 = SD(CARL_0)$			$PD(SDARL_0)$		
m	n	UNC	COND 1	COND 2	UNC	COND 1	COND 2	UNC	COND 1	COND 2	UNC	UNC	COND 1	COND 2	UNC	COND 1	COND 2	UNC	COND 1	COND 2
25	3	0.00245	0.00038	0.00153	6.7050	8.5780	7.1771	0.0012	0.0002	0.0008	53.7%	63.8%	2365.8	590.8	158.1	1216.6	266.3	11.4%	757.8%	87.7%
	5	0.00242	0.00062	0.00184	4.5119	5.2653	4.6624	0.0250	0.0125	0.0218	57.1%	68.8%	1429.9	484.2	128.4	578.9	173.8	13.3%	410.5%	53.3%
	9	0.00238	0.00085	0.00210	3.2104	3.5353	3.2506	0.1124	0.0849	0.1085	60.4%	74.0%	1025.1	419.6	104.3	327.5	120.2	15.2%	261.7%	32.8%
50	3	0.00256	0.00085	0.00204	6.6616	7.7584	6.8869	0.0013	0.0004	0.0010	53.3%	68.0%	1100.8	463.3	123.3	419.5	159.0	6.3%	261.7%	37.1%
	5	0.00254	0.00112	0.00228	4.4846	4.9353	4.5433	0.0256	0.0169	0.0243	56.2%	73.3%	831.3	411.4	97.9	245.3	110.4	7.3%	168.8%	21.0%
	9	0.00252	0.00136	0.00248	3.1926	3.3878	3.1975	0.1141	0.0964	0.1136	59.9%	78.9%	682.5	376.0	76.9	154.7	78.2	8.4%	118.1%	10.3%
75	3	0.00260	0.00115	0.00228	6.6451	7.4642	6.7789	0.0013	0.0006	0.0011	52.9%	71.0%	836.6	423.1	105.3	265.6	122.6	4.4%	163.4%	21.6%
	5	0.00259	0.00140	0.00248	4.4741	4.8134	4.4982	0.0259	0.0190	0.0253	55.5%	76.6%	681.0	386.7	82.7	166.2	86.9	5.1%	111.1%	10.5%
	9	0.00257	0.00162	0.00264	3.1858	3.3328	3.1774	0.1148	0.1011	0.1156	58.9%	82.3%	586.8	360.8	63.9	108.7	62.0	5.8%	79.9%	2.7%
100	3	0.00262	0.00134	0.00241	6.6363	7.3079	6.7200	0.0013	0.0007	0.0012	52.6%	73.4%	722.9	402.6	93.6	200.9	103.0	3.4%	122.0%	13.8%
	5	0.00261	0.00158	0.00259	4.4685	4.7479	4.4735	0.0260	0.0201	0.0259	55.0%	79.2%	611.7	373.7	73.1	130.2	73.8	3.9%	85.2%	5.0%
	9	0.00260	0.00178	0.00273	3.1822	3.3031	3.1664	0.1151	0.1037	0.1167	58.2%	85.0%	540.9	352.6	55.9	86.8	52.8	4.5%	62.0%	-1.3%
150	3	0.00265	0.00158	0.00257	6.6272	7.1404	6.6551	0.0013	0.0008	0.0013	52.3%	77.3%	617.8	380.8	78.7	141.8	81.3	2.3%	84.4%	5.6%
	5	0.00264	0.00179	0.00272	4.4627	4.6772	4.4461	0.0261	0.0215	0.0265	54.3%	83.2%	544.5	359.5	61.0	95.4	58.9	2.7%	60.5%	-0.8%
	9	0.00263	0.00197	0.00284	3.1784	3.2710	3.1543	0.1155	0.1066	0.1179	57.0%	88.8%	495.2	343.6	46.2	64.8	42.3	3.1%	44.5%	-5.6%
200	3	0.00266	0.00174	0.00267	6.6225	7.0494	6.6189	0.0013	0.0009	0.0013	52.0%	80.4%	567.1	369.1	69.3	113.4	69.0	1.7%	66.5%	1.3%
	5	0.00265	0.00192	0.00280	4.4597	4.6386	4.4308	0.0262	0.0223	0.0269	53.8%	86.3%	510.8	351.7	53.5	77.7	50.3	2.0%	48.3%	-4.0%
	9	0.00265	0.00208	0.00290	3.1764	3.2536	3.1475	0.1157	0.1083	0.1186	56.3%	91.6%	471.7	338.4	40.3	53.4	36.2	2.3%	35.7%	-8.0%
250	3	0.00267	0.00184	0.00273	6.6196	6.9910	6.5952	0.0013	0.0009	0.0014	51.8%	82.9%	536.6	361.5	62.6	96.2	60.9	1.4%	55.9%	-1.4%
	5	0.00266	0.00201	0.00285	4.4578	4.6137	4.4208	0.0263	0.0228	0.0272	53.4%	88.7%	490.2	346.6	48.2	66.8	44.6	1.6%	40.9%	-6.0%
	9	0.00266	0.00215	0.00294	3.1752	3.2424	3.1431	0.1158	0.1093	0.1191	55.7%	93.6%	457.2	335.1	36.2	46.3	32.2	1.9%	30.4%	-9.5%
∞	3	0.00270	0.00270	0.00270	6.6077	6.6077	6.6077	0.0014	0.0014	0.0014	100.0%	100.0%	370.4	370.4	0	0.0	0.0	-	-	-
	5	0.00270	0.00270	0.00270	4.4501	4.4501	4.4501	0.0264	0.0264	0.0264	100.0%	100.0%	370.4	370.4	0	0.0	0.0	-	-	-
	9	0.00270	0.00270	0.00270	3.1701	3.1701	3.1701	0.1163	0.1163	0.1163	100.0%	100.0%	370.4	370.4	0	0.0	0.0	-	-	-

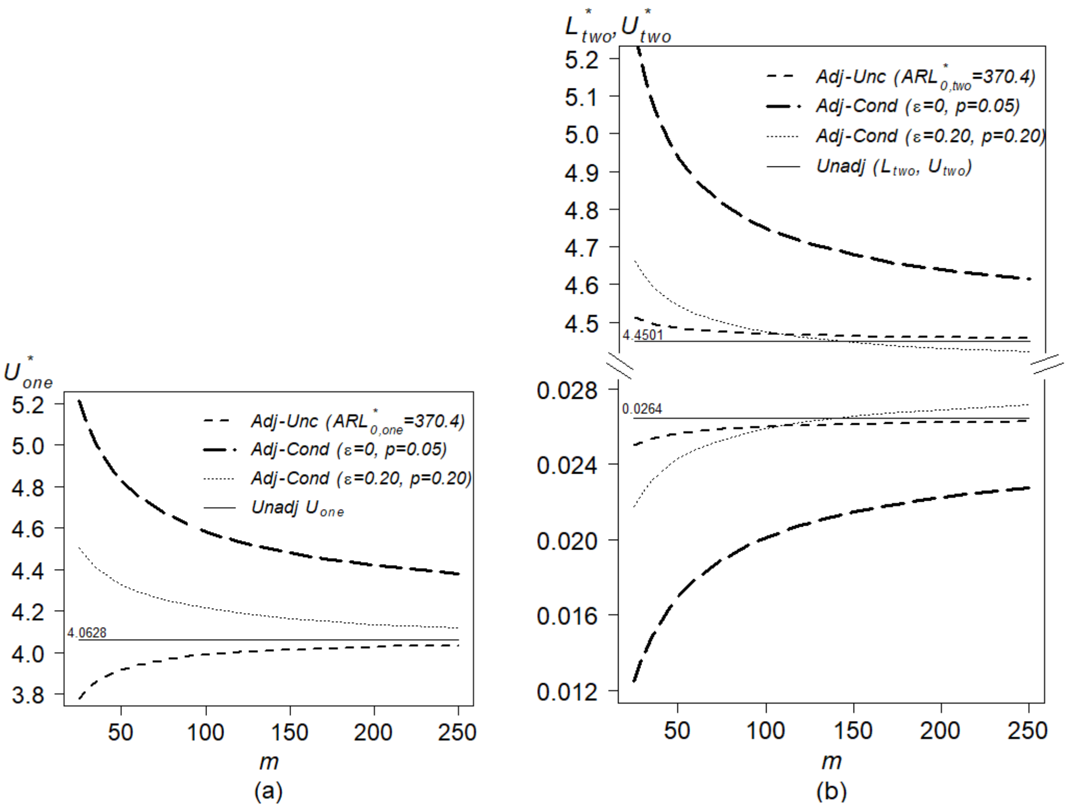

Figure 5 - Plots of control limits factors of the (a) one-sided ($\boldsymbol{U^*_{one}}$) and (b) two-sided ($\boldsymbol{L^*_{two}}, \boldsymbol{U^*_{two}}$) $\boldsymbol{S^2}$ charts with unadjusted (*Unadj*) limits and adjusted (*Adj*) limits under the *unconditional* (*Unc*) and *conditional* (*Cond*) perspective using $\boldsymbol{EPC}$ with $\boldsymbol{\alpha = 0.0027}, \boldsymbol{n = 5}$ and different values of $\boldsymbol{m}$.

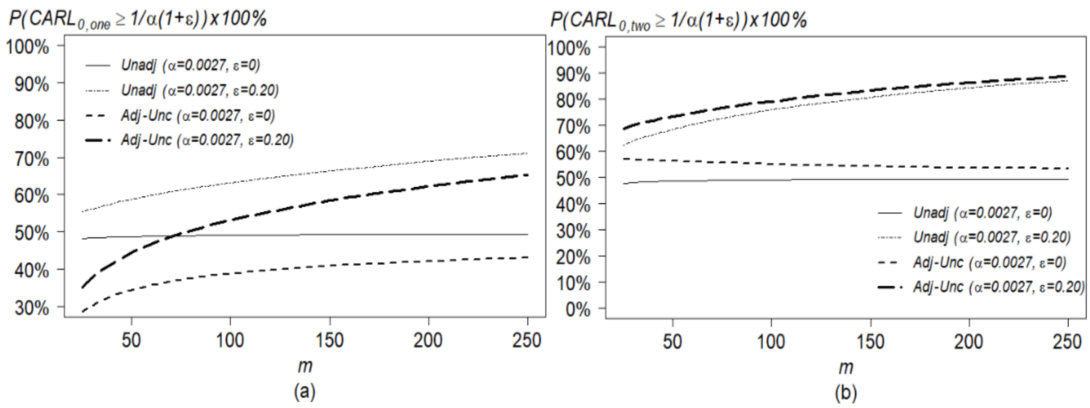

Figure 6 - Plots of the *Exceedance Probability* of the $\boldsymbol{CARL_0}$ for $\boldsymbol{\alpha = 0.0027}$ and $\boldsymbol{\varepsilon = 0}$ and $\boldsymbol{0.20}$ with $\boldsymbol{n = 5}$ and different values of $\boldsymbol{m}$ of the (a) one-sided and (b) two-sided $\boldsymbol{S^2}$ charts with unadjusted (*Unadj*) and adjusted (*Adj*) limits under the *unconditional* (*Unc*) perspective (with $\boldsymbol{ARL^*_0 = 370.4}$)

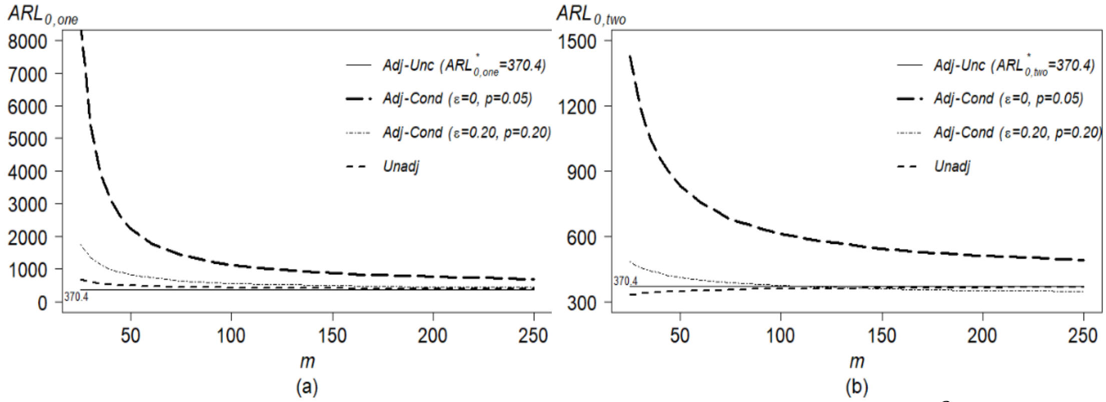

Figure 7 - Plots of the unconditional ARL_0 of the (a) one-sided and (b) two-sided S^2 charts with unadjusted (*Unadj*) and adjusted (*Adj*) limits under the *unconditional* (*Unc*) and *conditional* (*Cond*) perspectives using EPC with $\alpha = 0.0027$, $n = 5$ and different values of m

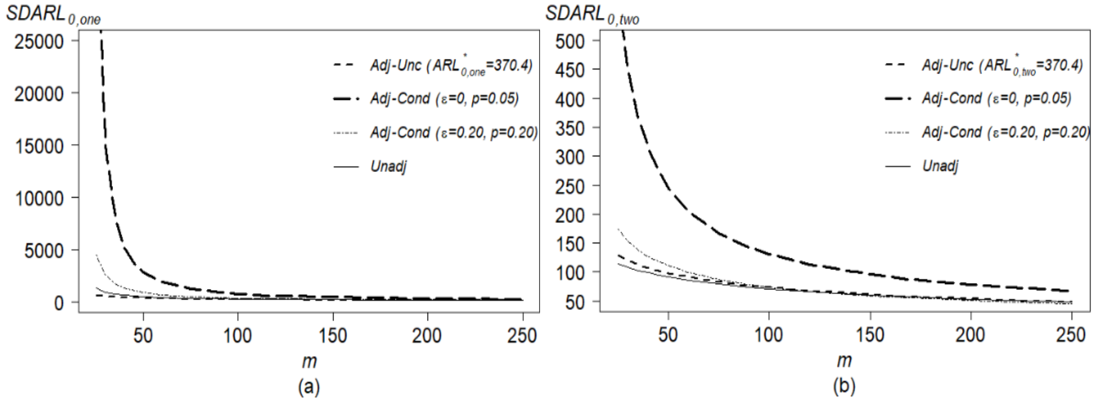

Figure 8 - Plots of the $\boldsymbol{SDARL_0}$ of the (a) one-sided and (b) two-sided $\boldsymbol{S^2}$ charts with unadjusted (*Unadj*) and adjusted (*Adj*) limits under the *unconditional* (*Unc*) and *conditional* (*Cond*) perspectives using $\boldsymbol{EPC}$ with $\boldsymbol{\alpha = 0.0027}$, $\boldsymbol{n = 5}$ and different values of $\boldsymbol{m}$

From Table 10, for the one-sided chart with adjusted limits under the *unconditional* perspective (UNC), the values of α^* are always larger than 0.0027 (the nominal false alarm rate), meaning that $U^*_{one} < U_{one}$. As the value of m increases, U^*_{one} increases and converges to U_{one}. On the other hand, under the conditional perspective (COND 1 and COND 2), the values of α^* are always smaller than the nominal 0.0027 and hence, $U^*_{one} > U_{one}$ (see the plots of panel a of Figure 5). Put another way, under the *unconditional* perspective, the adjustment corresponds to decreasing the height of the upper control limit of the one-sided chart, and, conversely, to increasing this height in the case of the *conditional* perspective.

The results of the adjustments to the one-sided control limits can be explained as follows: for the *unconditional* perspective, because the unconditional $ARL_{0,one}$

of the charts with unadjusted control limits is always larger than the specified $ARL^{*}_{0,one} = 370.4$ (see Table 8 and the plots of panel a of Figure 7), which is the aim of the adjustment, the distribution of $CARL_{0,one}$ must be pushed in the direction of the left tail until the unconditional $ARL_{0,one}$ value becomes the desired ARL^{*}_{0}. This only can be achieved when U_{one} is reduced. The variability of the $CARL_{0,one}$ distribution decreases (see the negatives values of $PD(SDARL_{0,one})$ in Table 10 and the plots of panel a of Figure 8) as a result of the adjustment.

By contrast, for the *conditional* perspective in the one-sided chart, since the adjustment pursues to attain the specified high probability $1 - p$ that the minimum tolerated $CARL_0$ value is exceeded, if this minimum tolerated $CARL_0$ value is larger than the p-quantile of the $CARL_0$ distribution of (one- or two-sided) charts with unadjusted limits, the $CARL_0$ distribution must be pushed to the right until the minimum tolerated $CARL_0$ value becomes its p-quantile. For instance, for a minimum tolerated value of 370.4 ($\alpha = 0.0027, \varepsilon = 0$), $n = 5$ and different values of m (COND 1), since 370.4 is close to (a little bit larger than) the 0.50-quantile of the $CARL_{0,one}$ distribution of one-sided charts with unadjusted limits (see Table 8), the center of the $CARL_{0,one}$ distribution must be pushed to the right until 370.4 becomes the p-quantile of this distribution (namely, $p = 0.05$). This outcome only can be reached increasing U_{one}. As a consequence of this conditional adjustment, where $P\left(CARL_{0,one} \geq 370.4\right) = 95\%$ (COND 1) is met, the variability of the $CARL_{0,one}$ distribution increases a lot and its respective mean (the unconditional $ARL_{0,one}$) is always rather larger than the nominal 370.4 (especially, when m is small). For example, from Table 10, when $m = 50$ and $n = 5$, $SDARL_{0,one} = 2797.7$ is 510.7% larger than the one of its counterpart with unadjusted control limits, and $ARL_{0,one} = 2220.9$ is 6 times more larger than then nominal 370.4 value (see also the plots of panels a of Figures 7 and 8). Considering that the $ARL_{0,one}$, which results from conditional adjusted limits, can be extremely larger than the desired $ARL^{*}_{0} = 370.4$ (based on the *unconditional* perspective), the values of $ARL_{0,one}$ and $SDARL_{0,one}$ can be reduced through modifications in the parameters of the EPC with the same amount of reference data, namely, relaxing (by being less rigorous in) the requirement of the conditional IC performance (reducing the specified tolerated lower bound of the $CARL_{0,one}$ and/or reducing the specified target of the EP). For instance, for conditional adjusted control limits that satisfy $P\left(CARL_{0,one} \geq 308.6\right) = 80\%$ (COND 2), $ARL_{0,one} = 823.1$ and $SDARL_{0,one} = 853.3$.

Following similar arguments as in the one-sided chart, the outcomes for the two-sided chart because of the adjustments under both perspectives can be explained. From Table 11, differently from the one-sided chart, the values of α^* for the two-sided chart with adjusted limits under the *unconditional* perspective (UNC) are always smaller than 0.0027 (the nominal false alarm rate), which means that $U_{two}^* > U_{two}$ and $L_{two}^* < L_{two}$. Note that, as m increases, U_{two}^* decreases and L_{two}^* increases, converging to the corresponding unadjusted values U_{two} and L_{two}, respectively (see the plots of panel b of Figure 5). For the unconditional adjustment, a small increase in the unconditional $ARL_{0,two}$ is required, and this yields an increase in the variability of the $CARL_{0,two}$, particularly, when m is small (see the panels b of Figures 7 and 8).

For adjustments under the *conditional* perspective in the two-sided charts (COND 1 and COND 2 from Table 11), excepting the case in which $\varepsilon = 0.20$, $p = 0.20$ (COND 2) and $mn \geq 750$ (then $\alpha^* > 0.0027$), α^* is always smaller than the nominal 0.0027 and thus the width of the adjusted control limit factors of the two-sided chart $U_{two}^* - L_{two}^*$ is larger than the one of its counterpart with unadjusted control limits (see the plots of panel b of Figure 5). The mean and the dispersion of the $CARL_{0,two}$ distribution increase (particularly, when m is small) due to the conditional adjustment, excepting the case indicated above (see also the plots of panels b of Figures 7 and 8). Increases in the dispersion of the $CARL_{0,two}$ distribution (in terms of the $SDARL_{0,two}$) as a result of conditional adjustments are rather larger than the one due to unconditional adjustments (see the values of $PD(SDARL_{0,two})$ in Table 11).

The panel (a) of Figure 6, for the upper one-sided chart with $n = 5$ and $m \leq 250$, note that the EP of the $CARL_{0,one}$ decreases due to adjustments under the *unconditional* perspective, namely, given ($\alpha = 0.0027, \varepsilon = 0$), $P\left(CARL_{0,one} \geq 370.4\right)$ decreases from values of approximately 50% (see Table 8) to values between 28% and 43% (Table 10). These last probabilities less than 50% were to be expected, since the $CARL_{0,one}$ distribution is usually highly right-skewed and the minimum tolerated 370.4 corresponds to the mean of this distribution, which is the purpose of the unconditional point of view. Similarly, given ($\alpha = 0.0027, \varepsilon = 0.20$), $P(CARL_{0,one} \geq 308.6)$ also decreases from values between 55% and 71% (Table 8) to values between 35% and 65% (Table 10) because, as noted above, the distribution of the $CARL_{0,one}$ is pushed to the left to achieve $ARL^{*}_{0,one} = 370.4$. It is worth to remark that, for the same setting ($m \leq 250$, n, $\alpha = 0.0027$ and $\varepsilon = 0$ or 0.20), these $EP's$ of the $CARL_0$ of the upper one-sided S^2 charts with both unadjusted limits and adjusted limits according to the unconditional perspective, that is, $P(CARL_{0,one} \geq 370.4)$ and $P(CARL_{0,one} \geq 308.6)$ are smaller than the ones high specified targets with adjusted limits under the *conditional* perspective ($1 - p = 95\%$ and 80%, respectively). It means that charts with unadjusted and unconditional adjusted limits are not satisfactory under the *conditional* perspective (see Table 10 and panel a of Figure 6).

The panel (b) of Figure 6, for the two-sided chart with $n = 5$ and $m \leq 250$, note that the EP of the $CARL_{0,two}$, differently from the case of the one-sided chart, increases as a result of the adjustments under the *unconditional* perspective, namely, given ($\alpha = 0.0027, \varepsilon = 0$), $P(CARL_{0,two} \geq 370.4)$ increases from values around 50% (Table 8) to values between 57% and 53% (Table 11). These last values greater than 50% were to be expected because the $CARL_{0,two}$ distribution is left-skewed and the minimum tolerated 370.4 corresponds to the mean of this distribution. Likewise, given ($\alpha = 0.0027, \varepsilon = 0.20$), $P(CARL_{0,two} \geq 308.6)$ increases from values between 62% and 87% (Table 8) to values between 68% and 89% (Table 11) because, as noted above, the distribution of the $CARL_{0,two}$ is pushed to the right to achieve $ARL^{*}_{0,two} = 370.4$. It is interesting to note that, for the same setting ($m \leq 250$, $n = 5$, $\alpha = 0.0027$ and $\varepsilon = 0$ or 0.20), these $EP's$ of the $CARL_0$ of the two-sided S^2 charts with unadjusted limits as well as with adjusted limits according to the *unconditional* perspective, that is, $P(CARL_{0,one} \geq 370.4)$ and $P(CARL_{0,one} \geq 308.6)$ are smaller than the ones high specified with adjusted limits according to the *conditional* perspective ($1 - p = 95\%$ and 80%, respectively), excepting the case in which the EP is $P(CARL_{0,one} \geq 308.6)$ with $m \geq 150$ (see Table 11 and panel b of Figure 6).

5.5 Comparison of designs between the one-sided and two-sided S^2 charts

In this section, we compare the performance and the design of the one- and two-sided charts, without and with control limit adjustments aiming to guarantee a desired IC performance of the chart under each perspectives (conditional and unconditional).

Tables 12-14 show, for $\alpha = 0.0027$ and several values of n and m, the values of α^* and the resulting adjusted factors for the upper one-sided (U^*_{one}) and two-sided (U^*_{two} and L^*_{two}) S^2 charts, obtained under the *unconditional* perspective (Table 12) and *conditional* perspective (Tables 13 and 14), respectively. The unconditional adjustments guarantee $ARL^*_0 = 370.4$, i.e., $E(CAR_0) = 370.4$ in Table 12, while the conditional adjustments guarantee $P(CARL_0 \geq 370.4) = 95\%$ and $P(CARL_0 \geq 308.6) = 80\%$ in Tables 13 and 14, respectively, using the EPC for ($\varepsilon = 0$, $p = 0.05$) and ($\varepsilon = 20\%$, $p = 0.20$). Similarly to Tables 10 and 11, Tables 12-14 also give the resulting IC performance measures of both charts, as defined before. Table 12 presents measures of spread of $CARL_0$, namely, the standard deviation $SDARL_0$ and the percentage difference between the $SDARL_0$ when the control limits are adjusted and unadjusted $PD(SDARL_0)$, as well as the probability that $CARL_0$ exceeds the minimum tolerated value according to the *conditional* perspective, for $\varepsilon = 0\%$ and $\varepsilon = 20\%$ (EP of the $CARL_0$ from Equation 37), corresponding thus to the probabilities $P(CARL_0 \geq 370.4)$ and $P(CARL_0 \geq 308.6)$ respectively. Tables 13 and 14 present the expected values (ARL_0), the standard deviations ($SDARL_0$), $PD(SDARL_0)$ and $PD(ARL_0)$ for comparing the performances of the two types of chart. The $PD(ARL_0)$ is the percentage difference between the ARL_0's when the control limits are adjusted and unadjusted, which are computed similarly to $PD(SDARL_0)$ in Equation (46).

From Table 12 (*unconditional* perspective), the values of α^* for the one-sided S^2 chart are always larger than the nominal 0.0027 false alarm rate, symmetrically, for the two-sided S^2 chart, the α^* values are always smaller than 0.0027: $\alpha^*(two - sided) < 0.0027 < \alpha^*(one - sided)$. Put another way, the adjustment under the *unconditional* perspective yield: $U^*_{one} < U_{one}$, $U^*_{two} > U_{two}$ and $L^*_{two} < L_{two}$.

From Tables 13 and 14 (*conditional* perspective), with the exception of the cases where $\varepsilon = 0.20$, $p = 0.20$ and $mn \geq 750$ (in which we get $\alpha^*(one - sided) < 0.0027 < \alpha^*(two - sided)$), the values of α^* for both charts are always smaller than the nominal 0.0027, specifically, the α^* values for the one-sided chart are always smaller than the α^* values for the two-sided chart: $\alpha^*(one - sided) < \alpha^*(two - sided) < 0.0027$). In other words, the height of the adjusted upper control limit factor of the one-sided chart U^*_{one} and the width of the adjusted control limit factors of the two-sided chart $U^*_{two} - L^*_{two}$ are larger than the ones of their counterparts with unadjusted control limits.

Even though the expected values of the $CARL_0$'s of both charts (unconditional ARL_0's) equal the nominal 370.4 due to unconditional adjustments, the variability of the $CARL_0$ is much larger for the one-sided chart than for the two-sided one. This difference is evidenced by the values of $SDARL_0$ in Table 12 and Figures 9(a) and 10. The variability of the $CARL_0$ gets smaller when m and n increase, and the values of $SDARL_{0,one}$ vary approximately between three and five-times the $SDARL_{0,two}$ values (see the plots of Figure 9a). For instance, for $m = 25$ and $n = 5$, $SDARL_{0,one} = 593.7$ and $SDARL_{0,two} = 128.4$. Figure 10, which provides modified boxplots that show the distributions and some percentiles ($1th, 5th, 25th, 50th, 75th, 95th$ and $99th$) of the $CARL_0$ for $m = 25$ and $m = 250$ (Figure 10a and b, respectively) and $n = 5$.

The comparison of the conditional IC performance between the one-sided and two-sided charts with adjusted control limits under the *unconditional* perspective is made. Given the same value of (m, n) and using the EP of the $CARL_0$ with $\varepsilon = 0$, from Table 12, note the following relation $25\% < P\big(CARL_{0,one} \geq 370.4\big) < 50\% < P\big(CARL_{0,two} \geq 370.4\big) < 61\%$, which can be seen in the plots of Figure 11(a) for the specific case of $n = 5$. This was to be expected, since the minimum tolerated 370.4 corresponds to the mean of the $CARL_0$ distribution that is right- and left-skewed for the one- and two-sided charts, respectively (see Figure 10), as has been highlighted before in the case of control charts with unadjusted limits (see Figure 2). When $\varepsilon = 20\%$, the EP of the $CARL_0$, i.e., $P(CARL_0 \geq 308.6)$, increases, especially for the two-sided chart, but it is not large as well unless the amount of Phase I data is large. For instance, from Table 12, with $m = 25$ and $n = 5$, $P(CARL_0 \geq 308.6)$ is 35.1% and 68.8% for the one- and two-sided charts, respectively. To achieve $P(CARL_0 \geq 308.6)$ greater than 80%, it is necessary that $m > 100$ for the two-sided chart; the probabilities for the one-sided chart remain, though, between 53.1% and 65.2% when $100 \leq m \leq 250$. Accordingly, the described difference in the variability of the $CARL_0$ between the charts explains why the values of $P(CARL_0 \geq 370.4)$ and $P(CARL_0 \geq 308.6)$ for the one-sided S^2 chart are smaller than those for the two-sided S^2 chart (see Figure 11a).

Moreover, we can examine and compare how the EP of the $CARL_0$ changes in the one- and two-sided charts, as a result of the control limit adjustments according to the *unconditional* perspective. As noted before, since the variability of the $CARL_{0,one}$ is reduced (see the negatives values of $PD(SDARL_{0,one})$ in Table 12) due to the adjustment, the values of $P\left(CARL_{0,one} \geq 370.4\right)$ and $P\left(CARL_{0,one} \geq 308.6\right)$ will decrease (see the panel a of Figure 6). Contrarily, these EP's values of the two-sided chart will increase (see the panel b of Figure 6) because of the increment of the variability of the $CARL_{0,two}$ distribution (see the positive values of $PD(SDARL_{0,two})$ in Table 12). Hence, due to the adjustments under the *unconditional* perspective, the resulting conditional IC performance (measured in terms of the EP of the $CARL_0$) of the two-sided chart improves, while the corresponding one of the one-sided chart deteriorates.

Table 12 - Adjusted control limit factors of one- and two-sided $\boldsymbol{S^2}$ charts obtained under the *unconditional* perspective (with $\boldsymbol{ARL_0^* = 370.4}$), and some resulting associated properties of the $\boldsymbol{CARL_0}$

		α^*		U_{one}^*	U_{two}^*	L_{two}^*	$SDARL_0 = SD(CARL_0)$		$PD(SDARL_0)$		$\varepsilon = 0$ $P(CFAR \leq 0.0027) = (1-p) = P(CARL_0 \geq 370.4)$		$\varepsilon = 0.20$ $P(CFAR \leq 0.0032) = (1-p) = P(CARL_0 \geq 308.6)$	
m	n	one-sided	two-sided	one-sided	two-sided	two-sided	one-sided	two-sided	one-sided	two-sided	one-sided	two-sided	one-sided	two-sided
25	3	0.00516	0.00245	5.2670	6.7050	0.0012	856.3	158.1	-70.4%	11.4%	25.6%	53.7%	31.0%	63.8%
	5	0.00448	0.00242	3.7776	4.5119	0.0250	593.7	128.4	-54.1%	13.3%	28.5%	57.1%	35.1%	68.8%
	9	0.00406	0.00238	2.8129	3.2104	0.1124	467.3	104.3	-43.3%	15.2%	30.8%	60.4%	38.4%	74.0%
50	3	0.00378	0.00256	5.5782	6.6616	0.0013	408.1	123.3	-38.1%	6.3%	32.1%	53.3%	40.5%	68.0%
	5	0.00350	0.00254	3.9170	4.4846	0.0256	326.1	97.9	-28.8%	7.3%	34.4%	56.2%	44.3%	73.3%
	9	0.00332	0.00252	2.8790	3.1926	0.1141	275.6	76.9	-22.8%	8.4%	36.2%	59.9%	47.4%	78.9%
75	3	0.00339	0.00260	5.6874	6.6451	0.0013	300.9	105.3	-26.0%	4.4%	35.2%	52.9%	45.8%	71.0%
	5	0.00321	0.00259	3.9649	4.4741	0.0259	248.1	82.7	-19.6%	5.1%	37.2%	55.5%	49.5%	76.6%
	9	0.00310	0.00257	2.9014	3.1858	0.1148	213.4	63.9	-15.5%	5.8%	38.6%	58.9%	52.6%	82.3%
100	3	0.00320	0.00262	5.7431	6.6363	0.0013	248.6	93.6	-19.7%	3.4%	37.1%	52.6%	49.4%	73.4%
	5	0.00308	0.00261	3.9891	4.4685	0.0260	207.8	73.1	-14.9%	3.9%	38.8%	55.0%	53.1%	79.2%
	9	0.00300	0.00260	2.9127	3.1822	0.1151	180.2	55.9	-11.7%	4.5%	40.1%	58.2%	56.3%	85.0%
150	3	0.00303	0.00265	5.7995	6.6272	0.0013	194.2	78.7	-13.3%	2.3%	39.4%	52.3%	54.6%	77.3%
	5	0.00295	0.00264	4.0135	4.4627	0.0261	164.4	61.0	-10.0%	2.7%	40.9%	54.3%	58.4%	83.2%
	9	0.00290	0.00263	2.9240	3.1784	0.1155	143.6	46.2	-7.9%	3.1%	41.9%	57.0%	61.6%	88.8%
200	3	0.00294	0.00266	5.8280	6.6225	0.0013	164.6	69.3	-10.0%	1.7%	40.8%	52.0%	58.3%	80.4%
	5	0.00288	0.00265	4.0257	4.4597	0.0262	140.1	53.5	-7.5%	2.0%	42.1%	53.8%	62.2%	86.3%
	9	0.00285	0.00265	2.9297	3.1764	0.1157	122.8	40.3	-5.9%	2.3%	43.0%	56.3%	65.5%	91.6%
250	3	0.00289	0.00267	5.8452	6.6196	0.0013	145.4	62.6	-8.1%	1.4%	41.8%	51.8%	61.3%	82.9%
	5	0.00285	0.00266	4.0331	4.4578	0.0263	124.2	48.2	-6.0%	1.6%	42.9%	53.4%	65.2%	88.7%
	9	0.00282	0.00266	2.9331	3.1752	0.1158	109.1	36.2	-4.8%	1.9%	43.7%	55.7%	68.6%	93.6%
∞	3	0.00270	0.00270	5.9145	6.6077	0.0014	0	0	-	-	100.0%	100.0%	100.0%	100.0%
	5	0.00270	0.00270	4.0628	4.4501	0.0264	0	0	-	-	100.0%	100.0%	100.0%	100.0%
	9	0.00270	0.00270	2.9468	3.1701	0.1163	0	0	-	-	100.0%	100.0%	100.0%	100.0%

Table 13 - Adjusted control limit factors of one- and two-sided $\boldsymbol{S^2}$ charts obtained under the *conditional* perspective (using $\boldsymbol{EPC}$ for $\boldsymbol{\varepsilon = 0, p = 0.05}$ and $\boldsymbol{\alpha = 0.0027}$) and some resulting associated properties of the $\boldsymbol{CARL_0}$

		Adjusted Limits to $P(CARL_0 \geq 370.4) = 95\%$ (EPC for $\varepsilon = 0,\ p = 0.05, \alpha = 0.0027$)												
		α^*		U^*_{one}	U^*_{two}	L^*_{two}	$ARL_0 = E(CARL_0)$		$PD(ARL_0)$		$SDARL_0 = SD(CARL_0)$		$PD(SDARL_0)$	
m	n	one-sided	two-sided	one-sided	two-sided	two-sided	one-sided	two-sided	one-sided	two-sided	one-sided	two-sided	one-sided	two-sided
25	3	0.00020	0.00038	8.5066	8.5780	0.0002	32789.7	2365.8	3744.4%	603.3%	1391074.0	1216.6	48036.5%	757.8%
	5	0.00034	0.00062	5.2134	5.2653	0.0125	8600.4	1429.9	1175.7%	330.9%	38432.4	578.9	2872.6%	410.5%
	9	0.00047	0.00085	3.5023	3.5353	0.0849	4343.8	1025.1	639.5%	213.4%	9593.7	327.5	1065.0%	261.7%
50	3	0.00051	0.00085	7.5896	7.7584	0.0004	3757.2	1100.8	593.7%	213.5%	7599.2	419.5	1053.4%	261.7%
	5	0.00068	0.00112	4.8287	4.9353	0.0169	2220.9	831.3	352.5%	138.7%	2797.7	245.3	510.7%	168.8%
	9	0.00083	0.00136	3.3237	3.3878	0.0964	1618.6	682.5	250.6%	97.6%	1541.0	154.7	331.5%	118.1%
75	3	0.00072	0.00115	7.2309	7.4642	0.0006	2005.5	836.6	323.3%	134.4%	2339.0	265.6	475.4%	163.4%
	5	0.00090	0.00140	4.6721	4.8134	0.0190	1420.4	681.0	219.0%	91.9%	1207.4	166.2	291.2%	111.1%
	9	0.00105	0.00162	3.2491	3.3328	0.1011	1141.8	586.8	166.7%	66.4%	782.5	108.7	209.9%	79.9%
100	3	0.00089	0.00134	7.0294	7.3079	0.0007	1463.7	722.9	229.4%	100.8%	1287.8	200.9	315.8%	122.0%
	5	0.00106	0.00158	4.5824	4.7479	0.0201	1123.6	611.7	164.6%	70.7%	759.9	130.2	211.3%	85.2%
	9	0.00121	0.00178	3.2058	3.3031	0.1037	946.3	540.9	129.4%	51.6%	529.3	86.8	159.3%	62.0%
150	3	0.00111	0.00158	6.8015	7.1404	0.0008	1054.1	617.8	152.5%	70.1%	669.4	141.8	198.8%	84.4%
	5	0.00128	0.00179	4.4795	4.6772	0.0215	873.5	544.5	115.5%	50.4%	444.9	95.4	143.6%	60.5%
	9	0.00141	0.00197	3.1557	3.2710	0.1066	770.5	495.2	93.7%	37.2%	331.9	64.8	113.0%	44.5%
200	3	0.00127	0.00174	6.6710	7.0494	0.0009	884.3	567.1	118.4%	55.4%	458.7	113.4	150.7%	66.5%
	5	0.00143	0.00192	4.4199	4.6386	0.0223	760.9	510.8	92.0%	40.3%	322.5	77.7	112.7%	48.3%
	9	0.00155	0.00208	3.1264	3.2536	0.1083	687.2	471.7	75.9%	29.9%	248.8	53.4	90.5%	35.7%
250	3	0.00138	0.00184	6.5842	6.9910	0.0009	790.3	536.6	98.7%	46.6%	354.1	96.2	123.9%	55.9%
	5	0.00153	0.00201	4.3799	4.6137	0.0228	695.7	490.2	78.0%	34.2%	257.4	66.8	94.7%	40.9%
	9	0.00165	0.00215	3.1066	3.2424	0.1093	637.6	457.2	65.0%	25.4%	202.8	46.3	77.0%	30.4%
∞	3	0.00270	0.00270	5.9145	6.6077	0.0014	370.4	370.4	-	-	0.0	0.0	-	-
	5	0.00270	0.00270	4.0628	4.4501	0.0264	370.4	370.4	-	-	0.0	0.0	-	-
	9	0.00270	0.00270	2.9468	3.1701	0.1163	370.4	370.4	-	-	0.0	0.0	-	-

Table 14 - Adjusted control limit factors of one- and two-sided $\boldsymbol{S^2}$ charts obtained under the *conditional* perspective (using $\boldsymbol{EPC}$ for $\boldsymbol{\varepsilon = 0.20, p = 0.20}$ and $\boldsymbol{\alpha = 0.0027}$) and some resulting associated properties of the $\boldsymbol{CARL_0}$

		Adjusted Limits to $P(CARL_0 \geq 308.6) = 0.80$ (EPC for $\varepsilon = 0.20,\ p = 0.20, \alpha = 0.0027$)												
		α^*		U^*_{one}	U^*_{two}	L^*_{two}	$ARL_0 = E(CARL_0)$		$PD(ARL_0)$		$SDARL_0 = SD(CARL_0)$		$PD(SDARL_0)$	
m	n	one-sided	two-sided	one-sided	two-sided	two-sided	one-sided	two-sided	one-sided	two-sided	one-sided	two-sided	one-sided	two-sided
25	3	0.00099	0.00153	6.9147	7.1771	0.0008	3276.1	590.8	284.1%	75.6%	23394.0	266.3	709.5%	87.7%
	5	0.00123	0.00184	4.5031	4.6624	0.0218	1743.0	484.2	158.5%	45.9%	4491.6	173.8	247.4%	53.3%
	9	0.00141	0.00210	3.1555	3.2506	0.1085	1225.7	419.6	108.7%	28.3%	2029.5	120.2	146.5%	32.8%
50	3	0.00148	0.00204	6.5179	6.8869	0.0010	1078.9	463.3	99.2%	32.0%	1569.1	159.0	138.1%	37.1%
	5	0.00168	0.00228	4.3281	4.5433	0.0243	823.1	411.4	67.7%	18.1%	853.3	110.4	86.3%	21.0%
	9	0.00184	0.00248	3.0714	3.1975	0.1136	695.3	376.0	50.6%	8.9%	576.8	78.2	61.5%	10.3%
75	3	0.00174	0.00228	6.3567	6.7789	0.0011	766.9	423.1	61.9%	18.6%	730.7	122.6	79.7%	21.6%
	5	0.00192	0.00248	4.2551	4.4982	0.0253	640.2	386.7	43.8%	9.0%	474.1	86.9	53.6%	10.5%
	9	0.00205	0.00264	3.0357	3.1774	0.1156	569.6	360.8	33.1%	2.3%	351.6	62.0	39.2%	2.7%
100	3	0.00190	0.00241	6.2646	6.7200	0.0012	645.1	402.6	45.2%	11.8%	484.7	103.0	56.5%	13.8%
	5	0.00207	0.00259	4.2128	4.4735	0.0259	561.3	373.7	32.2%	4.3%	338.7	73.8	38.7%	5.0%
	9	0.00219	0.00273	3.0149	3.1664	0.1167	512.1	352.6	24.1%	-1.2%	261.9	52.8	28.3%	-1.3%
150	3	0.00211	0.00257	6.1589	6.6551	0.0013	538.6	380.8	29.0%	4.8%	303.2	81.3	35.4%	5.6%
	5	0.00225	0.00272	4.1638	4.4461	0.0265	487.8	359.5	20.3%	-0.7%	226.6	58.9	24.1%	-0.8%
	9	0.00236	0.00284	2.9905	3.1543	0.1179	456.3	343.6	14.7%	-4.8%	182.5	42.3	17.1%	-5.6%
200	3	0.00225	0.00267	6.0977	6.6189	0.0013	489.1	369.1	20.8%	1.1%	228.8	69.0	25.0%	1.3%
	5	0.00237	0.00280	4.1352	4.4308	0.0269	452.0	351.7	14.1%	-3.4%	176.7	50.3	16.6%	-4.0%
	9	0.00247	0.00290	2.9762	3.1475	0.1186	428.3	338.4	9.6%	-6.8%	145.1	36.2	11.1%	-8.0%
250	3	0.00234	0.00273	6.0567	6.5952	0.0014	460.0	361.5	15.7%	-1.2%	187.8	60.9	18.8%	-1.4%
	5	0.00246	0.00285	4.1159	4.4208	0.0272	430.3	346.6	10.1%	-5.1%	147.8	44.6	11.8%	-6.0%
	9	0.00254	0.00294	2.9665	3.1431	0.1191	411.0	335.1	6.3%	-8.1%	122.9	32.2	7.3%	-9.5%
∞	3	0.00270	0.00270	5.9145	6.6077	0.0014	370.4	370.4	-	-	0.0	0.0	-	-
	5	0.00270	0.00270	4.0628	4.4501	0.0264	370.4	370.4	-	-	0.0	0.0	-	-
	9	0.00270	0.00270	2.9468	3.1701	0.1163	370.4	370.4	-	-	0.0	0.0	-	-

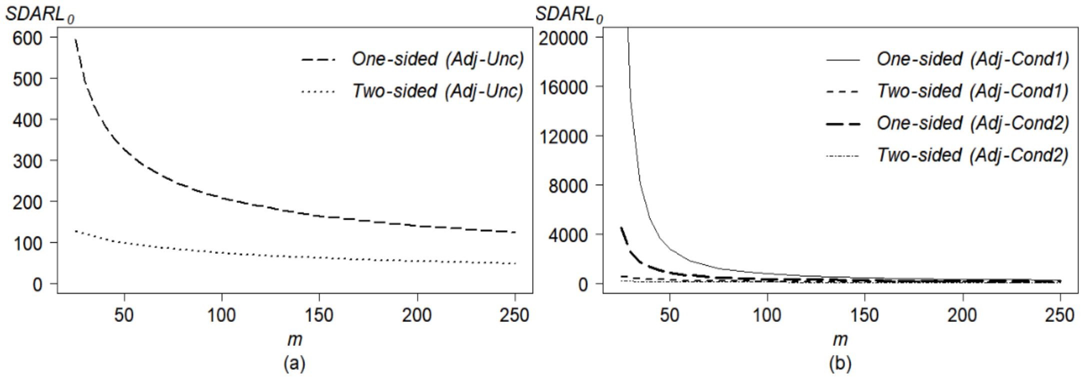

Figure 9 - Plots of the $\boldsymbol{SDARL_0}$ of the one- and two-sided $\boldsymbol{S^2}$ charts with adjusted limits under (a) the *unconditional* perspective (*Adj-Unc*) with $\boldsymbol{ARL_0^* = 370.4}$ and (b) the *conditional* perspective: *Adj-Cond1* (using $\boldsymbol{EPC}$ with $\boldsymbol{\varepsilon = 0}$ and $\boldsymbol{p = 0.05}$) and *Cond2* (using $\boldsymbol{EPC}$ with $\boldsymbol{\varepsilon = 0.20}$ and $\boldsymbol{p = 0.20}$) given $\boldsymbol{\alpha = 0.0027}$, $\boldsymbol{n = 5}$ and different values of $\boldsymbol{m}$

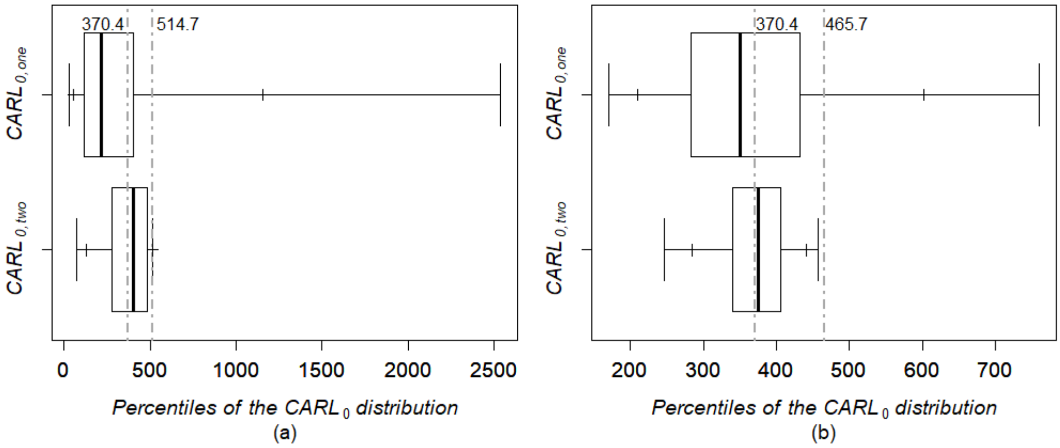

Figure 10 - Distributions and some percentiles of the $CARL_{0,one}$ and $CARL_{0,two}$ of S^2 charts with adjusted limits under the *unconditional* perspective (with $ARL_0^* = 370.4$) for $n = 5$ and (a) $m = 25$, $max(CARL_{0,two}) = 514.7$ and (b) $m = 250$, $max(CARL_{0,two}) = 465.7$

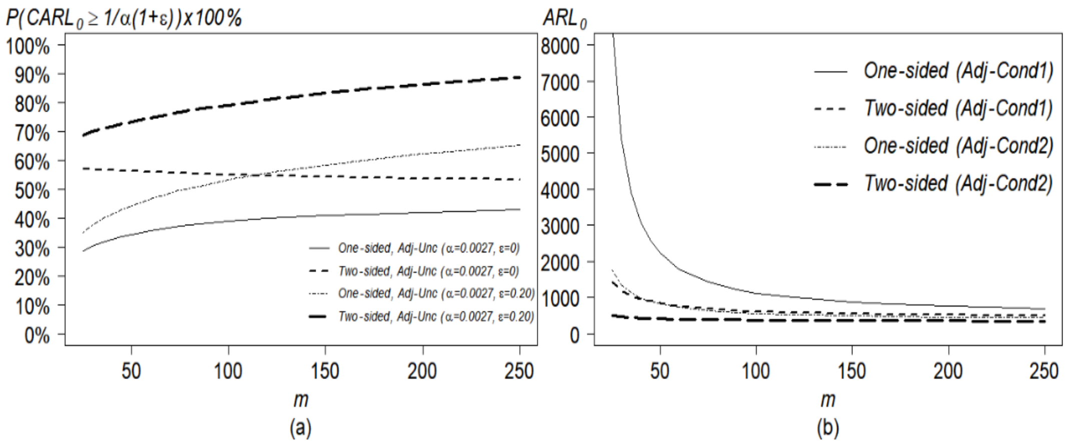

Figure 11. (a) Plots of EP's of the $CARL_0$ of S^2 charts with adjusted limits under the *unconditional* (*Adj-Unc*) perspective and (b) Plots of the unconditional ARL_0 of the S^2 charts with adjusted limits under the *conditional* perspective: *Adj-Cond1* and *Adj-Cond2* using EPC with ($\varepsilon = 0$ $p = 0.05$) and ($\varepsilon = 0.20$ $p = 0.20$), respectively, given $\alpha = 0.0027$, $n = 5$ and different values of m

Note that similarly to the cases of charts with unadjusted limits and charts with limits adjusted under the *unconditional* perspective, also when the limits are adjusted according to the *conditional* perspective, the resulting dispersion of the $CARL_0$ distribution for the one-sided chart is much larger than that for the two-sided chart. This is revealed by the values of $SDARL_0$ in Tables 13 and 14 and Figure 9(b). As a result of this conditional adjustments, the ARL_0 is very much larger for the one-sided than for the two-sided chart ($ARL_{0,one} > ARL_{0,two}$, see Figure 11b), specifically, in the case of the one-sided chart, the unconditional ARL_0 is always larger than 370.4 and, often very much larger. This is similar to what happens with the two-sided chart, except in the cases when $\varepsilon = 0.20$, $p = 0.20$ and $mn \geq 750$. To sum up, $ARL_{0,one}$ and $SDARL_{0,one}$ are very much larger than $ARL_{0,two}$ and $SDARL_{0,two}$, respectively, because the $CARL_{0,one}$ distribution is often highly right-skewed, while the $CARL_{0,two}$ distribution is left-skewed and its domain reaches a maximum at a finite value. These differences in the resulting IC performance are naturally greater when the amount of Phase I data is smaller or the minimum tolerated value of the $CARL_0$ is larger (i.e., smaller values of ε). This behavior can be verified in Figures 9(b) and 11(b).

Finally, we can examine, for the one- and two-sided charts separately, how the ARL_0 and $SDARL_0$ change when the limits are adjusted according to the *conditional* perspective, and next compare the impacts of this adjustment on the two types of chart. The ARL_0 and $SDARL_0$ will increase in both chart, with the exception of some cases in the two-sided chart (specifically, when $\varepsilon = 0.20$, $p = 0.20$ and $mn \geq 750$), in which these values will decrease (see the negative values of $PD(ARL_0)$ and $PD(SDARL_0)$ in Table 14). Note that the increases in the ARL_0 and $SDARL_0$ due to the adjustment are much larger in the case of the one-sided chart than in the case of the two-sided chart, as can be seen from the positive values of $PD(ARL_0)$ and $PD(SDARL_0)$ in Tables 13 and 14. Once again, this difference is due to the different shapes of the distribution of the $CARL_0$ of the two types of chart.

This work focused only on the charts' IC performance. Nevertheless, given the results presented here (namely, given the unadjusted and adjusted control limit factors), some interesting findings regarding the out-of-control (OOC) performance are provided beforehand (see plots of Figure 5). They are:

- If upper control limits of one-sided S^2 chart are adjusted according to the unconditional perspective, the adjustments lead to an improvement of the one-sided chart's OOC performance because the adjusted upper control limit is smaller than the unadjusted one. On the other hand, if the conditional perspective is considered, the upper limit increases, and thus the corresponding OOC performance will be worse than the one of the charts designed with unadjusted limits.
- If control limits of two-sided S^2 chart are adjusted according to the unconditional perspective, the adjustment yields a deterioration of the OOC performance of the two-sided chart since the adjusted control limits interval is wider than the unadjusted control limits interval. Regarding the conditional perspective, the OOC performance may either deteriorate or improve, depending on the case (see also Table 6): if the specified minimum IC performance is relaxed or less strict (namely, $\varepsilon = 0.20$ and $p = 0.20$) and the amount of Phase I data is large (namely, $mn \geq 750$, such as $m = 150$ and $n = 5$), the two-sided chart's OOC performance will improve (because the adjusted control limits interval is narrower than the unadjusted control limits interval); otherwise, the OOC performance will deteriorate, in that case, the conditional adjustment has a greater negative impact on the OOC performance than the unconditional adjustment.

When the control limits are adjusted under the conditional perspectives and the amount of Phase I data (m, n) is small, the deterioration of the OOC performance of (one-sided and two-sided) charts is large. On the other hand, when the values of m and n are large, the OOC performances of charts with unadjusted and adjusted limits are similar. For instance, in the case of upper one-sided chart and for ($\varepsilon \geq 0.10$, $p \geq 0.05$ and $\rho \geq 1.5$), Goedhart et al. (2017a) showed that, for the traditional $m = 25$ and $n = 5$, the $CARL_{one}(Y = m(n-1);\ \rho)$ could be up to 2.6 times more larger than the one for the unadjusted limits, however, for ($m \geq 200$ and $n \geq 5$) or ($m \geq 100$ and $n \geq 10$), increases in the $CARL_{one}(Y = m(n-1);\ \rho)$ do not exceed the 20% of the one for the unadjusted limits.

6
Conclusions and recommendations

In several real industrial and different field of science applications, when the precision of a desired value of a quality characteristic of a process or a product is the main concern and the quality assessment (conformity assessment and acceptance of products) is based on sampling acceptance, the construction of tolerance intervals for samples variances can be very useful. The two-sided (lower and upper) tolerance factors are computed and tabulated for some settings (cases) used widely in practice. Unlike one-sided upper tolerance limits, these two-sided tolerance intervals provide valuable information not only about process degradation, but also about process improvement due to the addition of the lower tolerance limits. However, these two-sided tolerance intervals are not obtained directly since the computation of the corresponding tolerance factors are found by solving a system of three nonlinear equations using numerical methods. With that in mind, under normality assumption, two approximation methods that facilitate the computation of two-sided tolerance limits for sample variances are proposed here.

Our findings have shown that the proposed CE approximation method for two-sided tolerance interval for sample variances is satisfactory for the number of samples (m) and sample sizes (n) at least 10 and 5, respectively, and is quite satisfactory for even larger values of m and n. The computation of the CE approximate tolerance limits is easier than that of the exact tolerance limits, and the required computer run time of the CE method is shorter than that of the exact method. Despite the fact that the proposed KMM approximation method is less accurate than the proposed CE approximation method, this method turns out reasonable for moderate to large sample sizes (n) and appealing because the KMM approximate two-sided tolerance limits can be obtained easily using a calculator without the use of advanced computational tools. In addition, the KMM approximation of the tolerance limits is more robust than that based on the CE and exact methods, and the performance of the KMM method is more satisfactory than the CE method in terms of the simulated coverage probabilities (as was revealed in

Yao et al., 2019). For that reason, based on our study, we would just recommend the use of this KMM method when the number of samples (m) is large (say $m \geq 25$) and sample size (n) is moderate to large (say $n \geq 10$). Otherwise, the use of the CE approximation method may be more proper.

The relationship between two-sided tolerance intervals for sample variances and the *Exceedance Probability* (EP) of the $CARL_0$ (cdf of the $CFAR$) of the S^2 control charts is highlighted and exploited in this work. Accordingly, the analysis of the IC performance and decision making about the design of the S^2 control charts (minimum m required to achieve a desired IC performance and the values of adjusted control limits) are based on tolerance limits for the population of sample variances, for instance, the (lower and upper) tolerance factors can be used as the adjusted control limit factors, which are required for guaranteeing a specified conditional IC performance.

Since the construction of the S^2 and S charts with estimated parameters is based on a specified in-control performance, which in turn is measured according to either *unconditional* or *conditional* perspectives, and upper one-sided or two-sided charts can be considered, it is fundamental that the practitioner can be aware of chart design differences between these two perspectives, and the two types of charts, in order to avoid, e.g., excessive false alarms, high cost of sampling and defective products. With this motivation, we studied side by side the designs and the resulting Phase II IC performances under both the unconditional and the conditional perspectives, regarding the upper one-sided and the two-sided S^2 and S charts. This comparison study can provide relevant information to support the choice of the most appropriate chart design and, to the best of our knowledge, there is no such study for the S^2 and S charts available in the literature, as made by Jardim et al. (2019) in the context of the $\bar{X}$ chart. Previous authors focused on the design and the Phase II performance according to one or other of these two perspectives but never on both, corresponding to S^2 and S charts with either upper one-sided limits or two-sided limits. We also obtained some results, not yet available in the literature, for completing this proposed study properly. The examination of the exact distribution of $CARL_0$ play a key role in this work.

As in the case of $\bar{X}$ chart (see, e.g., Saleh et al., 2015), for the S^2 chart with estimated parameter, the amount of Phase I reference data required for guaranteeing a specified Phase II IC performance according to the conditional perspective is larger than the one according to the unconditional perspective, for the one-sided chart as well as for the two-sided chart. Epprecht et al. (2015) warned this difference of the required amount of dataset between these two perspectives, but they considered different types of charts, namely, two-sided chart (unconditional) and one-sided chart (conditional). In our work, providing tabulated values of m and n, a complete comparison is made for each chart. Since the required Phase I dataset is often large and infeasible in practical terms, for the two perspectives and regarding one- and two-sided charts, control chart limits are adjusted to ensure a specified and desired Phase II IC performance under each perspective with a reasonable amount of Phase I data.

Due to the unconditional adjustments, the OOC performance of the one-sided chart improves, while it deteriorates as a result of the conditional adjustments. For the two-sided chart, the OOC performance of charts with adjusted limits under the unconditional perspective is less deteriorated than the one under the conditional perspective, with the exception of some cases, that is, $\varepsilon = 0.20$, $p = 0.20$, $\alpha = 0.0027$ when ($n = 3$, $m \geq 200$), ($n = 5$, $m \geq 150$) or ($n = 9$, $m \geq 75$). Owing to these results involved in chart design (required minimum m and control limits adjustments), one could perhaps think that the unconditional perspective may be preferable, with respect to one-sided charts as well as two-sided one. However, this is a misleading conclusion because of the particularity of each criterion followed in these perspectives. Namely, under the *unconditional* perspective, the goal is to achieve a specified level of the unconditional ARL_0. This means that the concern is the expected value of the $CARL_0$ distribution rather than the true $CARL_0$ (a certain realization of the $CARL_0$, i.e., given a parameter estimate, the average number of subgroups until a false alarm). Under this unconditional adjustment, this true $CARL_0$ may be much smaller than a nominal value (say, 370.4) with a high probability, which is undesirable in practical applications. This is because the variability of the $CARL_0$ distribution, which is overlooked in the *unconditional* perspective, is usually large (especially, in the case of the one-sided chart and/or when m is small).

It is worth to note that, for the same Phase I reference data (m and n), nominal false alarm rate (α) and minimum tolerated of the $CARL_{0,one}$, the risk of poor IC performance of a particular instance of the upper one-sided chart with adjusted control limit under the *conditional* perspective is smaller than the one under the *unconditional* perspective, that is, the EP of the $CARL_{0,one}$ of the chart with adjusted limit under the *conditional* perspective is larger than the one under its counterpart unconditional. In a similar way, this risk for the two-sided chart with adjusted control limits under the *conditional* perspective (in terms of the EP of the $CARL_{0,two}$) is smaller than the one under the *unconditional* perspective, with the exception of some particular cases indicated above (previous paragraph). On the other hand, even though adjustments under the *conditional* perspective guarantee a specified small risk (probability) that the $CARL_0$ is smaller than a minimum tolerated value, it generates rather larger mean and standard deviation of the $CARL_0$ distribution (unconditional ARL_0 and $SDARL_0$, respectively) compared to the nominal ARL_0 (say, 370.4), which demonstrates that the *conditional* perspective does not guarantee desired levels of the mean and standard deviation of the $CARL_0$ distribution. In other words, the "conservative" adjustments under the *conditional* perspective take into account the variation of the parameter estimates from different reference samples of a same application (practitioner-to-practitioner variability), however, these adjustments may lead to an unacceptable unconditional ARL_0 (compared to the nominal ARL_0, such as 370.4), especially, in the case of the one-sided chart and/or when m is small.

Our results help the practitioner, for instance, first, gain a better understanding of the meaningful differences between the S^2 (or S) chart design under to the two perspectives, and second, help them be aware that adjustments to control limits for guaranteeing a specified and desired chart performance under one perspective do not necessarily result in a "satisfactory" performance under the other perspective. Accordingly, the results and discussion of the design of the S^2 and S charts with estimated parameter based on the unconditional and conditional points of view reveal that each perspective is useful and has advantages and disadvantages in practical implementation, and thus both perspectives have their place in practice.

On the basis of the results presented in Tables 10 and 11, some practical recommendations are offered when the control limits are adjusted under the *conditional* perspective. The values of the desired IC performance (settings used in the EPC: ε and p) in conjunction with the required amount of Phase I reference data

(m and n) are recommended in order to choose a range of adequate adjusted control limit factors so that this desired conditional IC performance, in terms of the EPC for the $CARL_0$, can be guaranteed. In the same line with Jardim et al. (2019), the aim is to balance the EPC for the $CARL_0$ and the resulting values of the ARL_0 and $SDARL_0$. The choice of $\varepsilon \leq 0.20$ and $p \leq 0.20$ leads to a minimum tolerated value of the $CARL_0$ that won't be smaller than 83.33% of the nominal ARL_0 $(= 1/\alpha)$ with a probability of at least 80%. This yields a low risk of constructing charts with an unacceptably small value of the $CARL_0$ (compared to the specified nominal value). In addition, a choice of a proper combination of (m, n) enables to avoid large values of ARL_0 and $SDARL_0$ via the following rules: the values of m and n are found so that the ARL_0 must not be larger than 1.5 times the nominal ARL_0 $(1/\alpha)$, that is, $ARL_0 \leq 1.5\ (1/\alpha)$ and the $SDARL_0$ must not be larger than 50% of the nominal ARL_0, that is, $SDARL_0 \leq 0.5(1/\alpha)$. Accordingly, from Tables 10 and 11, for ($\varepsilon = 0, p = 0.05$) denoted as COND 1, we recommended ($n \geq 9, m \gg 250$) for the one-sided chart, and ($n \geq 9$, $m \geq 100$) or ($n \geq 5$, $m \geq 150$) or ($n \geq 3$, $m \geq 250$) for the two-sided chart. For ($\varepsilon = 0.20$, $p = 0.20$) denoted as COND 2, we recommended ($n \geq 9$, $m \geq 150$) or ($n \geq 5$, $m \geq 200$) or ($n \geq 3$, $m > 250$) for the one-sided chart, and ($n \geq 9$, $m \geq 25$) or ($n \geq 5$, $m \geq 25$) or ($n \geq 3$, $m \geq 50$) for the two-sided chart. It is worth to note that, for the same setting (α, ε and p), the recommended amount of Phase I data (mn) for the $\bar{X}$ chart is smaller than the one for the upper one-sided S^2 chart, but larger than the one for the two-sided S^2 chart.

The results of this study reveal that the differences in IC performance between the one-sided and the two-sided charts are large and we believe that the users of these charts should be aware of this.

Considering the S^2 chart with probability limits (either adjusted or unadjusted, and under any of the two perspectives, the unconditional or the conditional one), our results revealed that the IC performance of the one-sided chart is more strongly affected by parameter estimation than that of the two-sided chart. With unadjusted control limits, the number of Phase I samples (m) that guarantees an unconditional ARL_0 close to a given nominal value (e.g., 370.4) or that guarantees a minimum tolerated $CARL_0$ value (in terms of the EPC) is much smaller in the case of the two-sided chart than that in the case of the one-sided chart. Also, the standard deviation and the coefficient of variation of the $CARL_0$ of the one-sided chart are much larger than those of the two-sided chart. Such differences in variability (which are considerably more pronounced under the conditional perspective and when m is small) are explained by the substantial differences between the distributions of the $CARL_0$ of the two types of chart (namely, $CARL_{0,two}$ has a finite maximum value and a left-skewed distribution, while $CARL_{0,one}$ has generally a highly right-skewed distribution with no finite maximum). Accordingly, when the charts have their limits adjusted under the *unconditional* perspective (e.g., to achieve a desired unconditional $ARL_0 = 370.4$), the resulting EP's for the $CARL_0$ (the probabilities that the $CARL_0$ is not smaller than a minimum tolerated value specified, e.g. 308.6) are larger for the two-sided chart than for the one-sided chart. Correspondingly, having found the adjusted limits under the *conditional* perspective using the EPC, the resulting values of the unconditional ARL_0 are much larger for the one-sided than for the two-sided chart. Considering the effect of the adjustments on the performance of the charts:

- If we adjust the control limits under the *unconditional* perspective and our goal is just to achieve a certain unconditional ARL_0, both charts attain the same IC performance. This adjustment leads to an improvement of the OOC performance of the one-sided chart and to a deterioration of the OOC performance of the two-sided chart, especially for smaller number of Phase I samples (m). Although the performance of the one-sided chart with adjusted control limits may seem better than the two-sided one, we should keep in mind that in the adjustments under this perspective, the sole concern is to attain the expected value of the $CARL_0$, overlooking the practitioner-to-practitioner variability (i.e., the variability of the $CARL_0$ distribution); our results show that the two-sided chart has smaller practitioner-to-practitioner variability. Namely, with the adjustment, the values of $P(CARL_0 \geq 370.4)$ and $P(CARL_0 \geq 308.6)$ decrease for the one-sided chart and increase for the two-sided chart.
- When the adjustments are made under the *conditional* perspective, even though both charts achieve the same EP $(1 - p)$ of the $CARL_0$ (metric used in the conditional perspective), the increase in the variability and the mean of the $CARL_{0,one}$ distribution due to the adjustment is much larger than those of the $CARL_{0,two}$ distribution. As to the OOC performance, it may be deteriorated or

improved in the case of the two-sided chart (depending on the tolerances and on the amount of Phase I data, specifically, it improves when $\varepsilon = 0.20$, $p = 0.20$ and $mn \geq 750$, otherwise it deteriorates); in the case of the one-sided chart, the OOC performance always deteriorates, and more pronouncedly than the two-sided chart's performance. It is well known that the one-sided chart with unadjusted limits detects increments in the process variance faster than the two-sided chart. Even though adjustments to control limits under the conditional perspective could deteriorate and improve the one- and two-sided charts, respectively, these corrections don't lead to obtaining a worse OOC performance (e.g., the $CARL(Y, \rho^2)$) of the one-sided chart than that of the two-sided chart. Indeed, since the control limits factors of the two-sided chart with adjusted limits (under any perspective) are always larger than those of the one-sided chart ($U^*_{two} > U^*_{one}$), increases in the process variance ($\rho^2 > 1$) in a specific chart application are detected by the one-sided chart faster than the two-sided chart (e.g., $CARL_{one}(y, \rho^2) < CARL_{two}(y, \rho^2)$). On the other hand, as expected, decreases in the process variance ($\rho^2 < 1$) can be satisfactorily detected by the two-sided chart.

Finally, future directions of research are proposed. First, the mathematical-statistical relationship between the tolerance intervals and the EPC for the $CARL_0$ highlighted in the present work can also be seen in other types of control charts, for instance, X (individual control) chart and the two-sided tolerance interval for a normal population (see, Goedhart et al., 2017b and Krishnamoorthy & Mathew, 2009) as well as the upper one-sided S^2 chart and one-sided tolerance interval for sample variance (see, Goedhart et al., 2017a and Tietjen & Johnson, 1979). Hence, this relationship could continue to be exploited in more types of control charts. Since, in this work, the CRL_q is a recommended performance measure and the cdf of the conditional run length q-quantile (CRL_q) of S^2 chart is derived for the first time in the literature, this cdf could be exploited for designing S^2 charts based on the EPC for the $CRL_{0,q}$ (in a similar way that the EPC for the $CARL_0$, which is the focus of the present work). Moreover, the impact of control limit adjustments on the OOC performance of the S^2 chart can be thoroughly examined.

7
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Appendix A - Proof of Equation (24): exact two-sided tolerance limits for sample variances

In this Appendix A, the proof of Equation (24), which is required to find the value of β^*, is provided so that the exact two-sided tolerance factors for sample variances can be obtained. First, the shape of the actual coverage $G(Y)$ is examined on the basis of its first and the second derivatives. Using Equations (3) and (21), $G(Y;\beta^*,m,n)$ can be re-written as follows

$$G(Y;\beta^*,m,n) = F_{\chi^2_{n-1}}\left(\frac{Y}{m}U^* \mid Y\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{m}L^* \mid Y\right).$$

Then, the first derivative of $G(Y)$ is given by

$$G'(Y) = f_{\chi^2_{n-1}}\left(\frac{Y}{m}U^*\right).\frac{U^*}{m} - f_{\chi^2_{n-1}}\left(\frac{Y}{m}L^*\right).\frac{L^*}{m},$$

where $f_{\chi^2_{n-1}}(x)$ is the probability distribution function (pdf) of the chi-square random variable with $n-1$ df, that is, $f_{\chi^2_{n-1}}(x) = \frac{x^{\frac{n-1}{2}-1}e^{\frac{-x}{2}}}{\Gamma((\mathrm{n}-1)/2).2^{(\mathrm{n}-1)/2}}$. Let $a = \Gamma((\mathrm{n}-1)/2).2^{(\mathrm{n}-1)/2}$ so that $G'(Y)$ can be rewritten as

$$G'(Y) = \frac{Y^{\frac{n-1}{2}-1}}{a}\left(\frac{1}{m}\right)^{\frac{n-1}{2}}\left(U^{*\frac{n-1}{2}}.e^{\frac{-Y}{2m}U^*} - L^{*\frac{n-1}{2}}.e^{\frac{-Y}{2m}L^*}\right).$$

Only one stationary point $(y_0, G(y_0))$ of G is found by solving the equation $G'(Y) = 0$, where

$$y_0 = m(n-1)\ln(U^*/L^*)/(U^* - L^*).$$

Next, let $b = \frac{n-1}{2} - 1$ so that the second derivative of $G(Y)$ is given by

$$G''(Y) = \frac{d\big(G'(Y)\big)}{dY}$$

$$= \frac{Y^b}{2a}\left[\left(\left(\frac{L^*}{m}\right)^{b+2}.e^{\frac{-L^*Y}{2m}}\right) - \left(\left(\frac{U^*}{m}\right)^{b+2}.e^{\frac{-U^*Y}{2m}}\right)\right]$$

$$+\frac{bY^{b-1}}{a}\left[\left(\left(\frac{U^*}{m}\right)^{b+1}.e^{\frac{-U^*Y}{2m}}\right) - \left(\left(\frac{L^*}{m}\right)^{b+1}.e^{\frac{-L^*Y}{2m}}\right)\right].$$

Therefore, $G''(Y)$ evaluated at the point $Y = y_0$ results in

$$G''(y_0) = \frac{(L^* - U^*)}{2ma}\left(\frac{2m(b+1)\ln(U^*/L^*)}{U^* - L^*}\right)^b \left(\frac{L^{*\frac{U^*(b+1)}{U^*-L^*}}}{U^{*\frac{L^*(b+1)}{U^*-L^*}}}\right)$$

Note that $a > 0$, $b + 1 > \frac{1}{2}$, $U^* - L^* > 0$ and $\ln(U^*/L^*) > 0$. Thus, since $\frac{(L^*-U^*)}{2ma} < 0$, $\left(\frac{2m(b+1)\ln(U^*/L^*)}{U^*-L^*}\right)^b > 0$ and $\left(\frac{L^{*\frac{U^*(b+1)}{U^*-L^*}}}{U^{*\frac{L^*(b+1)}{U^*-L^*}}}\right) > 0$, we have $G''(y_0) < 0$. Accordingly, $G(Y)$ is concave with a unique maximum at y_0.

<u>Note 1</u>. The coverage $G(Y)$ is a non-monotonic concave function with the maximum point at $(y_0,\ G(y_0))$. In other words, $G(Y)$ is increasing over the interval $(0,\ y_0)$ and decreasing over $(y_0,\ \infty)$ because the coverage $G(Y)$ has only one stationary point $(y_0,\ G(y_0))$, that is, $G'(y_0) = 0$, and $G''(y_0) < 0$, where

$$y_0 = m(n-1)\ln(U^*/L^*)/(U^* - L^*) \text{ and}$$

$$\begin{aligned} G(y_0) &= Max(G(Y)) \\ &= F_{\chi^2_{n-1}}\left(\frac{\ln(U^*/L^*)}{(U^* - L^*)}\chi^2_{n-1,1-\frac{\beta^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{\ln(U^*/L^*)}{(U^* - L^*)}\chi^2_{n-1,\frac{\beta^*}{2}}\right). \end{aligned}$$

Since $\gamma \neq 0$ (specifically, $0 < \gamma < 1$), the specified proportion $1 - \beta$ must be smaller than the $Max(G(Y))$, that is, $1 - \beta < G(y_0)$.

<u>Note 2</u>. Given the values of m, n, β and γ, the actual coverage $G(Y)$ is used to find the value of β^* by a search method. Since the actual coverage $G(Y)$ is a concave function of Y, the coverage probability $P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$ is equivalent to $P_Y(y_1 \leq Y \leq y_2)$, as shown in Equation (23):

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) = F_{\chi^2_{m(n-1)}}(y_2) - F_{\chi^2_{m(n-1)}}(y_1),$$

where y_1 and y_2 $(y_1 < y_2)$ are the solutions of $G(Y;\beta^*,m,n) = 1-\beta$ (see Equation 21). Therefore, the tolerance interval from Equation (22) given earlier (subchapter 3.1) as follows

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) = \gamma,$$

can be rewritten as $F_{\chi^2_{m(n-1)}}(y_2\ ) - F_{\chi^2_{m(n-1)}}(y_1) = \gamma$. Accordingly, β^* is found by solving (see Equation 24) a system of three nonlinear equations for β^*, y_1 and y_2:

$$\begin{cases} F_{\chi^2_{n-1}}\left(\frac{y_1}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{y_1}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}}\right) = 1-\beta \\ F_{\chi^2_{n-1}}\left(\frac{y_2}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}}\right) - F_{\chi^2_{n-1}}\left(\frac{y_2}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}}\right) = 1-\beta \\ F_{\chi^2_{m(n-1)}}(y_2\ ) - F_{\chi^2_{m(n-1)}}(y_1) = \gamma \end{cases}$$

using a search method, where $0 < y_1 < y_0 < y_2 < \infty$ and $1-\beta < Max(G(Y))$. Note that

a) Since $P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) = \gamma$ and $0 < \gamma < 1$, we have

   $0 < P(G(Y) \geq 1-\beta) < 1$. Thus, $0 < 1-\beta < Max\big(G(Y)\big)$.

b) The actual coverage $G(Y)$ attains the minimum specified proportion (i.e., $G(Y) = 1-\beta$) at:
   - The minimum value of the ratio of the estimated variance $Min(S_p^2/\sigma^2)$ or Y=y_1 (Underestimation of σ^2: $S_p^2/\sigma^2 < 1$), and at
   - The maximum value of the ratio of the estimated variance $Max(S_p^2/\sigma^2)$ or Y=y_2 (Overestimation of σ^2: $S_p^2/\sigma^2 > 1$)

Appendix B – Tables of two-sided tolerance factors for S^2

Table B.1 - The exact $(1-\beta=0.90, \gamma)$ two-sided lower and upper tolerance factors for S^2 based on n observations (L^* and U^*, respectively) using m subgroups each of size n to estimate σ^2

		$1-\beta=0.90$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
5	2	0.9929	2E-05	8.5015	0.9988	5E-07	11.8291	≈1	1E-12	24.4052
	3	0.9838	0.0081	4.8156	0.9944	0.0028	5.8720	0.9998	0.0001	9.0020
	4	0.9783	0.0405	3.7220	0.9907	0.0228	4.3299	0.9991	0.0048	5.9793
	5	0.9745	0.0844	3.1794	0.9879	0.0571	3.6063	0.9983	0.0208	4.7111
	6	0.9718	0.1289	2.8491	0.9857	0.0959	3.1794	0.9976	0.0457	4.0096
	7	0.9696	0.1701	2.6245	0.9840	0.1339	2.8947	0.9968	0.0745	3.5613
	8	0.9680	0.2071	2.4609	0.9825	0.1694	2.6896	0.9962	0.1041	3.2482
	9	0.9667	0.2400	2.3356	0.9813	0.2018	2.5340	0.9956	0.1329	3.0160
	10	0.9657	0.2694	2.2363	0.9803	0.2312	2.4113	0.9951	0.1602	2.8362
	15	0.9628	0.3777	1.9366	0.9768	0.3429	2.0475	0.9931	0.2713	2.3190
	20	0.9614	0.4481	1.7800	0.9750	0.4167	1.8628	0.9918	0.3496	2.0652
	25	0.9607	0.4984	1.6807	0.9738	0.4697	1.7479	0.9908	0.4075	1.9107
10	2	0.9701	0.0004	5.9236	0.9843	0.0001	7.0709	0.9978	2E-06	10.6115
	3	0.9601	0.0201	3.9149	0.9748	0.0127	4.3749	0.9927	0.0037	5.6119
	4	0.9548	0.0671	3.1900	0.9695	0.0511	3.4767	0.9890	0.0255	4.2108
	5	0.9513	0.1193	2.8018	0.9660	0.0984	3.0115	0.9863	0.0610	3.5349
	6	0.9489	0.1679	2.5553	0.9634	0.1446	2.7215	0.9842	0.1002	3.1305
	7	0.9470	0.2110	2.3828	0.9614	0.1865	2.5209	0.9825	0.1383	2.8584
	8	0.9456	0.2487	2.2543	0.9598	0.2238	2.3727	0.9811	0.1737	2.6611
	9	0.9445	0.2817	2.1542	0.9585	0.2570	2.2580	0.9800	0.2059	2.5107
	10	0.9436	0.3108	2.0736	0.9574	0.2865	2.1662	0.9790	0.2352	2.3916
	15	0.9409	0.4169	1.8247	0.9539	0.3951	1.8856	0.9755	0.3467	2.0348
	20	0.9396	0.4848	1.6913	0.9521	0.4652	1.7377	0.9734	0.4209	1.8513
	25	0.9389	0.5330	1.6055	0.9510	0.5151	1.6437	0.9720	0.4744	1.7365
15	2	0.9563	0.0008	5.2571	0.9705	0.0003	5.9474	0.9901	4E-05	7.8966
	3	0.9477	0.0265	3.6432	0.9612	0.0196	3.9428	0.9820	0.0090	4.7117
	4	0.9431	0.0788	3.0209	0.9562	0.0656	3.2125	0.9773	0.0418	3.6887
	5	0.9401	0.1337	2.6786	0.9529	0.1173	2.8205	0.9740	0.0853	3.1677
	6	0.9379	0.1835	2.4577	0.9505	0.1656	2.5711	0.9716	0.1292	2.8461
	7	0.9363	0.2269	2.3014	0.9486	0.2084	2.3963	0.9697	0.1700	2.6252
	8	0.9350	0.2647	2.1840	0.9472	0.2461	2.2658	0.9682	0.2067	2.4628
	9	0.9340	0.2977	2.0920	0.9460	0.2793	2.1640	0.9669	0.2396	2.3375
	10	0.9332	0.3267	2.0175	0.9450	0.3086	2.0819	0.9658	0.2691	2.2374
	15	0.9307	0.4317	1.7853	0.9417	0.4156	1.8282	0.9622	0.3790	1.9327
	20	0.9295	0.4986	1.6597	0.9400	0.4842	1.6926	0.9600	0.4510	1.7729
	25	0.9288	0.5459	1.5786	0.9390	0.5328	1.6057	0.9585	0.5024	1.6718
20	2	0.9477	0.0011	4.9468	0.9608	0.0006	5.4481	0.9816	0.0001	6.7863
	3	0.9402	0.0304	3.5093	0.9523	0.0242	3.7353	0.9728	0.0137	4.2974
	4	0.9361	0.0854	2.9360	0.9476	0.0743	3.0823	0.9678	0.0530	3.4379
	5	0.9334	0.1416	2.6162	0.9446	0.1281	2.7251	0.9645	0.1007	2.9872
	6	0.9315	0.1919	2.4081	0.9424	0.1773	2.4954	0.9621	0.1468	2.7043
	7	0.9301	0.2356	2.2599	0.9407	0.2206	2.3331	0.9602	0.1887	2.5079
	8	0.9289	0.2734	2.1481	0.9394	0.2583	2.2114	0.9587	0.2259	2.3622
	9	0.9280	0.3063	2.0602	0.9383	0.2914	2.1160	0.9575	0.2590	2.2492
	10	0.9273	0.3352	1.9888	0.9373	0.3206	2.0388	0.9564	0.2884	2.1584
	15	0.9250	0.4395	1.7651	0.9344	0.4266	1.7987	0.9528	0.3971	1.8796
	20	0.9238	0.5059	1.6435	0.9327	0.4944	1.6693	0.9507	0.4676	1.7318
	25	0.9231	0.5528	1.5649	0.9317	0.5423	1.5861	0.9493	0.5179	1.6377

Table B.1 - (Continued)

		$1-\beta=0.90$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
25	2	0.9419	0.0013	4.7646	0.9539	0.0008	5.1632	0.9743	0.0003	6.1870
	3	0.9351	0.0330	3.4285	0.9460	0.0274	3.6121	0.9654	0.0174	4.0585
	4	0.9314	0.0897	2.8843	0.9417	0.0800	3.0039	0.9606	0.0610	3.2899
	5	0.9290	0.1467	2.5780	0.9389	0.1351	2.6673	0.9573	0.1111	2.8793
	6	0.9273	0.1973	2.3776	0.9369	0.1849	2.4493	0.9550	0.1586	2.6190
	7	0.9260	0.2410	2.2345	0.9353	0.2283	2.2947	0.9532	0.2010	2.4369
	8	0.9249	0.2788	2.1261	0.9341	0.2661	2.1782	0.9517	0.2385	2.3011
	9	0.9241	0.3116	2.0407	0.9331	0.2991	2.0867	0.9505	0.2716	2.1954
	10	0.9234	0.3404	1.9712	0.9322	0.3282	2.0125	0.9495	0.3010	2.1102
	15	0.9212	0.4444	1.7528	0.9294	0.4335	1.7806	0.9460	0.4087	1.8470
	20	0.9201	0.5104	1.6337	0.9279	0.5007	1.6550	0.9440	0.4783	1.7064
	25	0.9194	0.5569	1.5565	0.9269	0.5482	1.5740	0.9426	0.5277	1.6166
30	2	0.9377	0.0015	4.6434	0.9486	0.0010	4.9775	0.9682	0.0004	5.8115
	3	0.9315	0.0349	3.3737	0.9414	0.0298	3.5297	0.9596	0.0204	3.9024
	4	0.9281	0.0928	2.8491	0.9374	0.0842	2.9511	0.9549	0.0669	3.1917
	5	0.9259	0.1503	2.5520	0.9348	0.1401	2.6282	0.9518	0.1187	2.8072
	6	0.9243	0.2011	2.3569	0.9329	0.1902	2.4181	0.9495	0.1670	2.5617
	7	0.9231	0.2448	2.2171	0.9314	0.2338	2.2685	0.9478	0.2098	2.3890
	8	0.9221	0.2826	2.1111	0.9303	0.2715	2.1556	0.9464	0.2474	2.2599
	9	0.9213	0.3154	2.0274	0.9293	0.3045	2.0667	0.9452	0.2805	2.1590
	10	0.9207	0.3441	1.9593	0.9285	0.3334	1.9946	0.9442	0.3098	2.0776
	15	0.9186	0.4477	1.7445	0.9259	0.4383	1.7683	0.9409	0.4168	1.8248
	20	0.9175	0.5134	1.6270	0.9244	0.5051	1.6453	0.9390	0.4857	1.6892
	25	0.9168	0.5598	1.5508	0.9235	0.5523	1.5659	0.9377	0.5346	1.6022
50	2	0.9279	0.0020	4.3957	0.9364	0.0016	4.6080	0.9523	0.0009	5.1044
	3	0.9232	0.0392	3.2595	0.9306	0.0353	3.3611	0.9450	0.0279	3.5933
	4	0.9205	0.0996	2.7753	0.9274	0.0934	2.8420	0.9409	0.0808	2.9940
	5	0.9188	0.1581	2.4973	0.9253	0.1510	2.5472	0.9383	0.1359	2.6609
	6	0.9175	0.2092	2.3132	0.9237	0.2017	2.3533	0.9363	0.1856	2.4448
	7	0.9166	0.2530	2.1807	0.9226	0.2455	2.2143	0.9348	0.2290	2.2912
	8	0.9158	0.2907	2.0797	0.9216	0.2832	2.1088	0.9336	0.2668	2.1753
	9	0.9152	0.3233	1.9997	0.9209	0.3159	2.0254	0.9326	0.2998	2.0843
	10	0.9146	0.3518	1.9344	0.9202	0.3447	1.9574	0.9318	0.3288	2.0104
	15	0.9129	0.4546	1.7274	0.9181	0.4483	1.7430	0.9290	0.4341	1.7791
	20	0.9120	0.5197	1.6135	0.9169	0.5141	1.6255	0.9274	0.5014	1.6535
	25	0.9114	0.5656	1.5393	0.9160	0.5606	1.5492	0.9263	0.5490	1.5725
75	2	0.9221	0.0024	4.2636	0.9288	0.0020	4.4169	0.9418	0.0013	4.7612
	3	0.9183	0.0417	3.1973	0.9241	0.0387	3.2714	0.9356	0.0327	3.4359
	4	0.9161	0.1035	2.7349	0.9214	0.0988	2.7836	0.9321	0.0891	2.8917
	5	0.9146	0.1626	2.4674	0.9196	0.1572	2.5038	0.9298	0.1458	2.5847
	6	0.9136	0.2138	2.2894	0.9184	0.2082	2.3186	0.9281	0.1962	2.3836
	7	0.9128	0.2576	2.1608	0.9174	0.2520	2.1852	0.9269	0.2399	2.2399
	8	0.9122	0.2952	2.0626	0.9166	0.2896	2.0837	0.9258	0.2776	2.1310
	9	0.9117	0.3277	1.9847	0.9160	0.3223	2.0032	0.9250	0.3104	2.0451
	10	0.9112	0.3561	1.9209	0.9154	0.3508	1.9376	0.9243	0.3393	1.9752
	15	0.9098	0.4583	1.7183	0.9136	0.4537	1.7295	0.9219	0.4435	1.7551
	20	0.9090	0.5230	1.6064	0.9126	0.5190	1.6150	0.9205	0.5098	1.6348
	25	0.9084	0.5687	1.5334	0.9119	0.5650	1.5405	0.9196	0.5567	1.5569

Table B.1 - (Continued)

		$1-\beta=0.90$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
100	2	0.9188	0.0026	4.1924	0.9245	0.0022	4.3158	0.9356	0.0016	4.5867
	3	0.9154	0.0432	3.1636	0.9204	0.0406	3.2235	0.9301	0.0356	3.3540
	4	0.9136	0.1057	2.7130	0.9180	0.1018	2.7523	0.9270	0.0938	2.8382
	5	0.9123	0.1650	2.4512	0.9165	0.1606	2.4805	0.9250	0.1513	2.5447
	6	0.9114	0.2164	2.2765	0.9154	0.2118	2.2999	0.9235	0.2020	2.3515
	7	0.9107	0.2601	2.1501	0.9145	0.2556	2.1696	0.9223	0.2458	2.2129
	8	0.9102	0.2976	2.0534	0.9138	0.2931	2.0703	0.9214	0.2835	2.1077
	9	0.9097	0.3301	1.9766	0.9133	0.3257	1.9914	0.9207	0.3162	2.0245
	10	0.9093	0.3584	1.9137	0.9128	0.3542	1.9270	0.9200	0.3449	1.9567
	15	0.9081	0.4603	1.7135	0.9112	0.4566	1.7224	0.9179	0.4485	1.7425
	20	0.9074	0.5247	1.6027	0.9103	0.5215	1.6095	0.9167	0.5143	1.6251
	25	0.9069	0.5702	1.5304	0.9097	0.5674	1.5359	0.9159	0.5608	1.5488
200	2	0.9128	0.0030	4.0714	0.9166	0.0027	4.1475	0.9241	0.0023	4.3073
	3	0.9104	0.0458	3.1058	0.9137	0.0441	3.1428	0.9201	0.0408	3.2206
	4	0.9091	0.1096	2.6756	0.9120	0.1071	2.6998	0.9179	0.1019	2.7508
	5	0.9082	0.1693	2.4235	0.9109	0.1665	2.4414	0.9163	0.1608	2.4795
	6	0.9075	0.2208	2.2546	0.9101	0.2179	2.2688	0.9152	0.2119	2.2992
	7	0.9070	0.2645	2.1318	0.9094	0.2616	2.1437	0.9144	0.2557	2.1691
	8	0.9067	0.3018	2.0378	0.9090	0.2991	2.0480	0.9137	0.2933	2.0698
	9	0.9063	0.3341	1.9629	0.9086	0.3315	1.9718	0.9132	0.3258	1.9911
	10	0.9061	0.3624	1.9015	0.9082	0.3598	1.9095	0.9127	0.3543	1.9267
	15	0.9052	0.4636	1.7055	0.9071	0.4614	1.7107	0.9112	0.4567	1.7223
	20	0.9047	0.5276	1.5966	0.9064	0.5257	1.6006	0.9103	0.5216	1.6094
	25	0.9043	0.5728	1.5254	0.9060	0.5711	1.5286	0.9096	0.5674	1.5359
250	2	0.9113	0.0031	4.0432	0.9147	0.0029	4.1090	0.9213	0.0024	4.2454
	3	0.9092	0.0465	3.0923	0.9121	0.0450	3.1243	0.9177	0.0420	3.1907
	4	0.9080	0.1105	2.6668	0.9105	0.1083	2.6877	0.9157	0.1039	2.7312
	5	0.9072	0.1704	2.4171	0.9095	0.1679	2.4325	0.9143	0.1629	2.4649
	6	0.9066	0.2218	2.2495	0.9088	0.2193	2.2617	0.9133	0.2142	2.2876
	7	0.9062	0.2655	2.1276	0.9083	0.2630	2.1378	0.9125	0.2580	2.1594
	8	0.9058	0.3028	2.0342	0.9078	0.3005	2.0429	0.9119	0.2955	2.0614
	9	0.9055	0.3351	1.9597	0.9075	0.3328	1.9674	0.9114	0.3280	1.9837
	10	0.9053	0.3633	1.8987	0.9072	0.3611	1.9055	0.9110	0.3564	1.9201
	15	0.9045	0.4643	1.7037	0.9061	0.4625	1.7081	0.9096	0.4585	1.7178
	20	0.9040	0.5283	1.5953	0.9055	0.5267	1.5986	0.9088	0.5232	1.6060
	25	0.9037	0.5734	1.5243	0.9051	0.5720	1.5270	0.9082	0.5688	1.5331
∞	2	0.9000	0.0039	3.8415	0.9000	0.0039	3.8415	0.9000	0.0039	3.8415
	3	0.9000	0.0513	2.9957	0.9000	0.0513	2.9957	0.9000	0.0513	2.9957
	4	0.9000	0.1173	2.6049	0.9000	0.1173	2.6049	0.9000	0.1173	2.6049
	5	0.9000	0.1777	2.3719	0.9000	0.1777	2.3719	0.9000	0.1777	2.3719
	6	0.9000	0.2291	2.2141	0.9000	0.2291	2.2141	0.9000	0.2291	2.2141
	7	0.9000	0.2726	2.0986	0.9000	0.2726	2.0986	0.9000	0.2726	2.0986
	8	0.9000	0.3096	2.0096	0.9000	0.3096	2.0096	0.9000	0.3096	2.0096
	9	0.9000	0.3416	1.9384	0.9000	0.3416	1.9384	0.9000	0.3416	1.9384
	10	0.9000	0.3695	1.8799	0.9000	0.3695	1.8799	0.9000	0.3695	1.8799
	15	0.9000	0.4693	1.6918	0.9000	0.4693	1.6918	0.9000	0.4693	1.6918
	20	0.9000	0.5325	1.5865	0.9000	0.5325	1.5865	0.9000	0.5325	1.5865
	25	0.9000	0.5770	1.5173	0.9000	0.5770	1.5173	0.9000	0.5770	1.5173

Nota that 2E-05 means $2 * 10^{-5}$.

Table B.2 - The exact $(1-\beta=0.95,\gamma)$ two-sided lower and upper tolerance factors for S^2 based on n observations (L^* and U^*, respectively) using m subgroups each of size n to estimate σ^2

		$1-\beta=0.95$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
5	2	0.9989	5E-07	11.9599	0.9999	3E-09	16.7709	≈1	2E-17	34.6516
	3	0.9959	0.0020	6.1986	0.9990	0.0005	7.6127	≈1	8E-06	11.7104
	4	0.9937	0.0175	4.6079	0.9979	0.0083	5.3922	0.9999	0.0011	7.4725
	5	0.9920	0.0460	3.8438	0.9969	0.0281	4.3821	0.9997	0.0080	5.7435
	6	0.9908	0.0796	3.3874	0.9961	0.0552	3.7977	0.9996	0.0224	4.8040
	7	0.9897	0.1137	3.0807	0.9954	0.0849	3.4130	0.9994	0.0421	4.2110
	8	0.9889	0.1461	2.8589	0.9949	0.1145	3.1387	0.9992	0.0644	3.8008
	9	0.9882	0.1761	2.6903	0.9944	0.1428	2.9321	0.9990	0.0876	3.4990
	10	0.9877	0.2037	2.5572	0.9939	0.1695	2.7702	0.9989	0.1107	3.2667
	15	0.9860	0.3105	2.1618	0.9923	0.2770	2.2949	0.9982	0.2124	2.6066
	20	0.9852	0.3830	1.9596	0.9914	0.3521	2.0570	0.9978	0.2891	2.2876
	25	0.9847	0.4360	1.8329	0.9908	0.4075	1.9107	0.9974	0.3478	2.0951
10	2	0.9912	3E-05	8.1141	0.9966	5E-06	9.8414	0.9998	2E-08	15.0232
	3	0.9860	0.0070	4.9619	0.9926	0.0037	5.5974	0.9986	0.0007	7.2672
	4	0.9831	0.0341	3.9040	0.9901	0.0238	4.2856	0.9974	0.0096	5.2398
	5	0.9812	0.0719	3.3551	0.9883	0.0560	3.6282	0.9964	0.0305	4.2929
	6	0.9798	0.1113	3.0129	0.9870	0.0920	3.2264	0.9956	0.0583	3.7375
	7	0.9788	0.1487	2.7764	0.9860	0.1273	2.9523	0.9949	0.0882	3.3689
	8	0.9779	0.1829	2.6019	0.9852	0.1604	2.7517	0.9943	0.1179	3.1045
	9	0.9772	0.2139	2.4670	0.9845	0.1909	2.5976	0.9939	0.1463	2.9045
	10	0.9767	0.2419	2.3591	0.9839	0.2187	2.4750	0.9934	0.1729	2.7472
	15	0.9750	0.3481	2.0302	0.9820	0.3262	2.1050	0.9919	0.2800	2.2821
	20	0.9742	0.4188	1.8571	0.9809	0.3988	1.9131	0.9909	0.3553	2.0466
	25	0.9737	0.4700	1.7471	0.9802	0.4517	1.7925	0.9902	0.4110	1.9008
15	2	0.9846	0.0001	7.1060	0.9914	3E-05	8.1502	0.9982	1E-06	11.0691
	3	0.9796	0.0103	4.5835	0.9865	0.0068	5.0007	0.9953	0.0023	6.0575
	4	0.9769	0.0423	3.6765	0.9838	0.0331	3.9342	0.9933	0.0183	4.5644
	5	0.9751	0.0834	3.1927	0.9820	0.0703	3.3797	0.9918	0.0466	3.8288
	6	0.9738	0.1247	2.8862	0.9807	0.1092	3.0337	0.9907	0.0799	3.3839
	7	0.9728	0.1631	2.6721	0.9796	0.1463	2.7943	0.9898	0.1135	3.0826
	8	0.9721	0.1978	2.5128	0.9788	0.1804	2.6174	0.9890	0.1455	2.8633
	9	0.9714	0.2291	2.3889	0.9781	0.2114	2.4805	0.9884	0.1753	2.6956
	10	0.9709	0.2571	2.2893	0.9775	0.2395	2.3709	0.9879	0.2028	2.5625
	15	0.9694	0.3628	1.9827	0.9756	0.3463	2.0360	0.9860	0.3104	2.1625
	20	0.9686	0.4328	1.8196	0.9746	0.4178	1.8599	0.9849	0.3842	1.9560
	25	0.9681	0.4833	1.7155	0.9739	0.4696	1.7482	0.9841	0.4384	1.8267
20	2	0.9800	0.0002	6.6385	0.9869	0.0001	7.3946	0.9957	7E-06	9.4149
	3	0.9754	0.0124	4.3968	0.9820	0.0090	4.7120	0.9918	0.0041	5.4909
	4	0.9729	0.0471	3.5618	0.9794	0.0391	3.7593	0.9894	0.0249	4.2344
	5	0.9713	0.0899	3.1100	0.9776	0.0788	3.2542	0.9877	0.0576	3.5968
	6	0.9701	0.1321	2.8213	0.9763	0.1193	2.9355	0.9864	0.0938	3.2045
	7	0.9692	0.1710	2.6185	0.9753	0.1572	2.7133	0.9855	0.1291	2.9359
	8	0.9685	0.2060	2.4669	0.9746	0.1917	2.5483	0.9847	0.1622	2.7388
	9	0.9679	0.2373	2.3486	0.9739	0.2229	2.4200	0.9840	0.1927	2.5870
	10	0.9675	0.2654	2.2532	0.9734	0.2510	2.3169	0.9834	0.2205	2.4660
	15	0.9660	0.3707	1.9579	0.9715	0.3574	1.9999	0.9815	0.3278	2.0991
	20	0.9652	0.4403	1.8000	0.9705	0.4282	1.8318	0.9803	0.4007	1.9076
	25	0.9647	0.4905	1.6989	0.9699	0.4793	1.7249	0.9795	0.4539	1.7869

Table B.2 - (Continued)

		$1-\beta=0.95$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
25	2	0.9767	0.0002	6.3656	0.9834	0.0001	6.9643	0.9929	2E-05	8.5124
	3	0.9724	0.0139	4.2840	0.9787	0.0107	4.5401	0.9885	0.0058	5.1611
	4	0.9701	0.0504	3.4918	0.9761	0.0432	3.6535	0.9860	0.0300	4.0376
	5	0.9687	0.0942	3.0592	0.9744	0.0845	3.1778	0.9843	0.0655	3.4566
	6	0.9676	0.1370	2.7814	0.9732	0.1259	2.8754	0.9830	0.1034	3.0953
	7	0.9668	0.1761	2.5854	0.9723	0.1643	2.6636	0.9820	0.1398	2.8461
	8	0.9661	0.2112	2.4386	0.9715	0.1991	2.5057	0.9812	0.1735	2.6622
	9	0.9656	0.2426	2.3237	0.9709	0.2303	2.3827	0.9805	0.2042	2.5201
	10	0.9651	0.2706	2.2309	0.9704	0.2584	2.2836	0.9799	0.2322	2.4064
	15	0.9637	0.3757	1.9427	0.9687	0.3644	1.9776	0.9780	0.3393	2.0596
	20	0.9630	0.4450	1.7880	0.9677	0.4347	1.8145	0.9768	0.4115	1.8773
	25	0.9625	0.4949	1.6888	0.9671	0.4855	1.7104	0.9760	0.4640	1.7619
30	2	0.9742	0.0003	6.1849	0.9806	0.0001	6.6847	0.9904	4E-05	7.9446
	3	0.9702	0.0150	4.2078	0.9761	0.0120	4.4252	0.9858	0.0071	4.9445
	4	0.9681	0.0527	3.4442	0.9736	0.0462	3.5822	0.9832	0.0340	3.9063
	5	0.9667	0.0973	3.0246	0.9721	0.0886	3.1259	0.9814	0.0714	3.3623
	6	0.9657	0.1404	2.7541	0.9709	0.1306	2.8345	0.9802	0.1104	3.0214
	7	0.9650	0.1797	2.5628	0.9700	0.1693	2.6298	0.9792	0.1475	2.7851
	8	0.9644	0.2148	2.4192	0.9693	0.2042	2.4767	0.9784	0.1816	2.6101
	9	0.9639	0.2462	2.3067	0.9687	0.2355	2.3572	0.9777	0.2125	2.4745
	10	0.9634	0.2743	2.2157	0.9682	0.2636	2.2608	0.9772	0.2406	2.3657
	15	0.9621	0.3791	1.9323	0.9666	0.3693	1.9623	0.9753	0.3474	2.0325
	20	0.9614	0.4482	1.7798	0.9657	0.4393	1.8026	0.9741	0.4190	1.8565
	25	0.9609	0.4979	1.6819	0.9651	0.4898	1.7005	0.9733	0.4710	1.7447
50	2	0.9683	0.0004	5.8188	0.9735	0.0003	6.1324	0.9825	0.0001	6.8756
	3	0.9651	0.0176	4.0493	0.9697	0.0153	4.1903	0.9781	0.0110	4.5139
	4	0.9634	0.0579	3.3444	0.9677	0.0532	3.4346	0.9757	0.0438	3.6402
	5	0.9623	0.1040	2.9517	0.9664	0.0978	3.0182	0.9741	0.0852	3.1693
	6	0.9615	0.1478	2.6966	0.9654	0.1410	2.7494	0.9729	0.1266	2.8694
	7	0.9609	0.1874	2.5152	0.9647	0.1803	2.5591	0.9720	0.1649	2.6591
	8	0.9604	0.2227	2.3785	0.9641	0.2154	2.4162	0.9713	0.1997	2.5021
	9	0.9600	0.2541	2.2709	0.9636	0.2468	2.3041	0.9707	0.2309	2.3797
	10	0.9597	0.2821	2.1837	0.9632	0.2748	2.2134	0.9702	0.2590	2.2810
	15	0.9585	0.3864	1.9109	0.9618	0.3797	1.9305	0.9685	0.3649	1.9759
	20	0.9579	0.4549	1.7630	0.9610	0.4489	1.7780	0.9675	0.4353	1.8129
	25	0.9575	0.5042	1.6677	0.9605	0.4987	1.6800	0.9668	0.4862	1.7087
75	2	0.9646	0.0005	5.6255	0.9688	0.0004	5.8498	0.9767	0.0002	6.3606
	3	0.9620	0.0192	3.9634	0.9657	0.0173	4.0658	0.9727	0.0137	4.2944
	4	0.9606	0.0610	3.2899	0.9640	0.0573	3.3557	0.9706	0.0499	3.5019
	5	0.9597	0.1078	2.9120	0.9629	0.1032	2.9604	0.9691	0.0935	3.0681
	6	0.9590	0.1521	2.6652	0.9620	0.1469	2.7036	0.9681	0.1360	2.7893
	7	0.9585	0.1918	2.4893	0.9614	0.1865	2.5212	0.9673	0.1750	2.5925
	8	0.9581	0.2271	2.3563	0.9609	0.2217	2.3836	0.9667	0.2100	2.4449
	9	0.9577	0.2585	2.2515	0.9605	0.2531	2.2755	0.9661	0.2413	2.3294
	10	0.9574	0.2864	2.1664	0.9602	0.2811	2.1879	0.9657	0.2694	2.2361
	15	0.9565	0.3903	1.8994	0.9590	0.3854	1.9136	0.9642	0.3747	1.9458
	20	0.9560	0.4585	1.7542	0.9583	0.4541	1.7650	0.9633	0.4443	1.7898
	25	0.9556	0.5075	1.6604	0.9579	0.5035	1.6692	0.9627	0.4945	1.6896

Table B.2 - (Continued)

		$1-\beta=0.95$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
100	2	0.9624	0.0006	5.5221	0.9661	0.0005	5.7017	0.9730	0.0003	6.1007
	3	0.9602	0.0201	3.9169	0.9633	0.0185	3.9994	0.9694	0.0154	4.1803
	4	0.9590	0.0627	3.2604	0.9618	0.0596	3.3134	0.9675	0.0534	3.4295
	5	0.9581	0.1100	2.8904	0.9608	0.1061	2.9294	0.9662	0.0981	3.0149
	6	0.9576	0.1544	2.6482	0.9601	0.1502	2.6791	0.9652	0.1413	2.7471
	7	0.9571	0.1942	2.4752	0.9596	0.1899	2.5008	0.9645	0.1805	2.5574
	8	0.9568	0.2296	2.3443	0.9591	0.2251	2.3662	0.9639	0.2157	2.4148
	9	0.9565	0.2609	2.2410	0.9588	0.2565	2.2602	0.9635	0.2470	2.3029
	10	0.9562	0.2888	2.1571	0.9585	0.2844	2.1742	0.9631	0.2751	2.2124
	15	0.9554	0.3924	1.8933	0.9574	0.3885	1.9046	0.9617	0.3799	1.9300
	20	0.9549	0.4604	1.7495	0.9568	0.4569	1.7581	0.9609	0.4491	1.7776
	25	0.9546	0.5092	1.6565	0.9564	0.5061	1.6635	0.9604	0.4989	1.6795
200	2	0.9585	0.0007	5.3480	0.9610	0.0006	5.4574	0.9659	0.0005	5.6892
	3	0.9569	0.0218	3.8378	0.9590	0.0207	3.8885	0.9632	0.0186	3.9954
	4	0.9560	0.0658	3.2102	0.9579	0.0638	3.2426	0.9617	0.0598	3.3113
	5	0.9554	0.1138	2.8537	0.9572	0.1113	2.8775	0.9608	0.1063	2.9280
	6	0.9550	0.1585	2.6194	0.9567	0.1558	2.6381	0.9600	0.1503	2.6782
	7	0.9547	0.1984	2.4514	0.9563	0.1957	2.4669	0.9595	0.1900	2.5001
	8	0.9544	0.2337	2.3240	0.9560	0.2310	2.3372	0.9591	0.2253	2.3656
	9	0.9542	0.2650	2.2234	0.9557	0.2623	2.2349	0.9587	0.2566	2.2598
	10	0.9541	0.2928	2.1414	0.9555	0.2901	2.1517	0.9584	0.2845	2.1738
	15	0.9535	0.3959	1.8831	0.9547	0.3936	1.8897	0.9574	0.3886	1.9044
	20	0.9531	0.4635	1.7418	0.9543	0.4614	1.7469	0.9568	0.4569	1.7580
	25	0.9529	0.5121	1.6503	0.9540	0.5102	1.6544	0.9564	0.5061	1.6634
250	2	0.9575	0.0007	5.3078	0.9598	0.0006	5.4019	0.9641	0.0005	5.5991
	3	0.9561	0.0222	3.8194	0.9580	0.0212	3.8631	0.9617	0.0194	3.9543
	4	0.9553	0.0665	3.1985	0.9570	0.0648	3.2264	0.9603	0.0613	3.2849
	5	0.9548	0.1147	2.8452	0.9563	0.1125	2.8657	0.9594	0.1082	2.9087
	6	0.9544	0.1594	2.6127	0.9559	0.1571	2.6288	0.9588	0.1524	2.6628
	7	0.9541	0.1994	2.4459	0.9555	0.1970	2.4592	0.9583	0.1922	2.4874
	8	0.9539	0.2347	2.3193	0.9552	0.2323	2.3306	0.9579	0.2274	2.3547
	9	0.9537	0.2659	2.2193	0.9550	0.2636	2.2292	0.9576	0.2588	2.2502
	10	0.9536	0.2937	2.1378	0.9548	0.2915	2.1466	0.9573	0.2867	2.1653
	15	0.9530	0.3967	1.8808	0.9541	0.3948	1.8864	0.9564	0.3905	1.8987
	20	0.9527	0.4642	1.7401	0.9537	0.4625	1.7444	0.9558	0.4587	1.7537
	25	0.9525	0.5127	1.6489	0.9534	0.5111	1.6523	0.9555	0.5077	1.6599
∞	2	0.9500	0.0010	5.0239	0.9500	0.0010	5.0239	0.9500	0.0010	5.0239
	3	0.9500	0.0253	3.6889	0.9500	0.0253	3.6889	0.9500	0.0253	3.6889
	4	0.9500	0.0719	3.1161	0.9500	0.0719	3.1161	0.9500	0.0719	3.1161
	5	0.9500	0.1211	2.7858	0.9500	0.1211	2.7858	0.9500	0.1211	2.7858
	6	0.9500	0.1662	2.5665	0.9500	0.1662	2.5665	0.9500	0.1662	2.5665
	7	0.9500	0.2062	2.4082	0.9500	0.2062	2.4082	0.9500	0.2062	2.4082
	8	0.9500	0.2414	2.2875	0.9500	0.2414	2.2875	0.9500	0.2414	2.2875
	9	0.9500	0.2725	2.1918	0.9500	0.2725	2.1918	0.9500	0.2725	2.1918
	10	0.9500	0.3000	2.1136	0.9500	0.3000	2.1136	0.9500	0.3000	2.1136
	15	0.9500	0.4021	1.8656	0.9500	0.4021	1.8656	0.9500	0.4021	1.8656
	20	0.9500	0.4688	1.7291	0.9500	0.4688	1.7291	0.9500	0.4688	1.7291
	25	0.9500	0.5167	1.6402	0.9500	0.5167	1.6402	0.9500	0.5167	1.6402

Nota that 2E-05 means $2*10^{-5}$.

Table B.3 - The exact ($1-\beta=0.99, \gamma$) two-sided lower and upper tolerance factors for S^2 based on n observations (L^* and U^*, respectively) using m subgroups each of size n to estimate σ^2

		$1-\beta=0.99$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
5	2	≈1	5E-11	20.6031	≈1	9E-15	28.9613	≈1	2E-28	59.8544
	3	0.9998	0.0001	9.4733	≈1	8E-06	11.6882	≈1	2E-08	18.0015
	4	0.9997	0.0025	6.6465	0.9999	0.0008	7.8139	≈1	4E-05	10.8473
	5	0.9995	0.0116	5.3448	0.9999	0.0057	6.1198	≈1	0.0009	8.0364
	6	0.9993	0.0272	4.5882	0.9998	0.0162	5.1646	≈1	0.0046	6.5457
	7	0.9992	0.0468	4.0893	0.9997	0.0312	4.5477	≈1	0.0120	5.6215
	8	0.9990	0.0683	3.7334	0.9997	0.0490	4.1141	≈1	0.0227	4.9910
	9	0.9989	0.0902	3.4654	0.9996	0.0681	3.7913	≈1	0.0359	4.5322
	10	0.9988	0.1118	3.2555	0.9996	0.0876	3.5407	≈1	0.0505	4.1825
	15	0.9985	0.2063	2.6412	0.9994	0.1777	2.8178	0.9999	0.1281	3.2065
	20	0.9983	0.2776	2.3356	0.9992	0.2492	2.4631	0.9999	0.1965	2.7460
	25	0.9982	0.3324	2.1482	0.9991	0.3054	2.2480	0.9998	0.2528	2.4724
10	2	0.9996	7E-08	13.6929	0.9999	3E-09	16.8501	≈1	2E-13	25.9358
	3	0.9988	0.0006	7.4608	0.9996	0.0002	8.5084	≈1	1E-05	11.1514
	4	0.9984	0.0071	5.5580	0.9993	0.0040	6.1550	0.9999	0.0010	7.5891
	5	0.9980	0.0228	4.6137	0.9990	0.0156	5.0268	0.9998	0.0064	5.9927
	6	0.9977	0.0444	4.0409	0.9988	0.0335	4.3563	0.9998	0.0175	5.0810
	7	0.9975	0.0688	3.6525	0.9987	0.0549	3.9078	0.9997	0.0330	4.4875
	8	0.9973	0.0936	3.3698	0.9985	0.0777	3.5847	0.9996	0.0510	4.0681
	9	0.9972	0.1180	3.1536	0.9984	0.1005	3.3394	0.9996	0.0703	3.7547
	10	0.9970	0.1413	2.9822	0.9983	0.1228	3.1461	0.9995	0.0899	3.5106
	15	0.9966	0.2386	2.4694	0.9979	0.2185	2.5729	0.9993	0.1799	2.8028
	20	0.9964	0.3097	2.2068	0.9977	0.2904	2.2826	0.9992	0.2513	2.4532
	25	0.9962	0.3637	2.0429	0.9975	0.3454	2.1030	0.9991	0.3075	2.2399
15	2	0.9988	6E-07	11.7886	0.9996	7E-08	13.7603	≈1	3E-10	19.0356
	3	0.9978	0.0011	6.8203	0.9989	0.0005	7.5256	0.9998	0.0001	9.2487
	4	0.9972	0.0101	5.1927	0.9985	0.0068	5.6059	0.9996	0.0027	6.5806
	5	0.9968	0.0287	4.3613	0.9981	0.0220	4.6514	0.9994	0.0119	5.3229
	6	0.9965	0.0528	3.8486	0.9979	0.0432	4.0721	0.9993	0.0273	4.5832
	7	0.9963	0.0788	3.4971	0.9977	0.0672	3.6792	0.9992	0.0467	4.0921
	8	0.9961	0.1049	3.2392	0.9975	0.0918	3.3931	0.9991	0.0679	3.7401
	9	0.9960	0.1301	3.0407	0.9973	0.1159	3.1743	0.9990	0.0895	3.4742
	10	0.9958	0.1539	2.8827	0.9972	0.1390	3.0008	0.9989	0.1109	3.2653
	15	0.9954	0.2520	2.4052	0.9968	0.2364	2.4806	0.9986	0.2048	2.6498
	20	0.9952	0.3230	2.1577	0.9965	0.3080	2.2134	0.9984	0.2766	2.3399
	25	0.9950	0.3766	2.0023	0.9963	0.3625	2.0467	0.9982	0.3322	2.1488
20	2	0.9981	2E-06	10.8919	0.9991	3E-07	12.3385	0.9999	6E-09	16.0812
	3	0.9970	0.0015	6.5001	0.9982	0.0009	7.0391	0.9995	0.0002	8.3351
	4	0.9964	0.0120	5.0060	0.9977	0.0089	5.3265	0.9992	0.0043	6.0758
	5	0.9960	0.0323	4.2308	0.9973	0.0263	4.4575	0.9990	0.0163	4.9799
	6	0.9957	0.0577	3.7484	0.9971	0.0493	3.9239	0.9988	0.0345	4.3245
	7	0.9955	0.0847	3.4157	0.9968	0.0746	3.5591	0.9986	0.0561	3.8845
	8	0.9953	0.1113	3.1706	0.9967	0.1002	3.2921	0.9985	0.0789	3.5665
	9	0.9951	0.1369	2.9813	0.9965	0.1250	3.0869	0.9983	0.1018	3.3248
	10	0.9950	0.1610	2.8302	0.9964	0.1486	2.9237	0.9982	0.1241	3.1339
	15	0.9946	0.2595	2.3710	0.9959	0.2466	2.4310	0.9979	0.2197	2.5665
	20	0.9944	0.3303	2.1314	0.9957	0.3180	2.1760	0.9976	0.2916	2.2778
	25	0.9942	0.3837	1.9805	0.9955	0.3721	2.0162	0.9975	0.3468	2.0985

Table B.3 - (Continued)

		$1-\beta=0.99$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
25	2	0.9974	3E-06	10.3668	0.9986	7E-07	11.5174	0.9997	3E-08	14.4320
	3	0.9963	0.0018	6.3059	0.9976	0.0012	6.7461	0.9992	0.0004	7.7929
	4	0.9958	0.0134	4.8913	0.9971	0.0104	5.1555	0.9988	0.0058	5.7691
	5	0.9954	0.0348	4.1501	0.9967	0.0293	4.3378	0.9985	0.0197	4.7687
	6	0.9951	0.0610	3.6862	0.9964	0.0536	3.8319	0.9982	0.0398	4.1639
	7	0.9949	0.0885	3.3651	0.9962	0.0797	3.4844	0.9981	0.0629	3.7549
	8	0.9947	0.1156	3.1278	0.9960	0.1058	3.2290	0.9979	0.0868	3.4577
	9	0.9946	0.1414	2.9443	0.9959	0.1310	3.0322	0.9978	0.1104	3.2308
	10	0.9944	0.1656	2.7974	0.9957	0.1549	2.8753	0.9977	0.1332	3.0511
	15	0.9940	0.2643	2.3496	0.9953	0.2532	2.3998	0.9972	0.2298	2.5136
	20	0.9938	0.3350	2.1150	0.9950	0.3244	2.1524	0.9970	0.3016	2.2380
	25	0.9937	0.3882	1.9668	0.9948	0.3783	1.9969	0.9968	0.3564	2.0663
30	2	0.9969	4E-06	10.0198	0.9982	1E-06	10.9806	0.9995	1E-07	13.3760
	3	0.9958	0.0021	6.1745	0.9971	0.0014	6.5490	0.9988	0.0006	7.4316
	4	0.9953	0.0144	4.8130	0.9966	0.0116	5.0392	0.9984	0.0071	5.5616
	5	0.9949	0.0366	4.0948	0.9962	0.0316	4.2560	0.9980	0.0225	4.6247
	6	0.9946	0.0634	3.6435	0.9959	0.0567	3.7689	0.9978	0.0439	4.0538
	7	0.9944	0.0913	3.3303	0.9957	0.0834	3.4330	0.9976	0.0680	3.6656
	8	0.9943	0.1186	3.0984	0.9955	0.1099	3.1855	0.9974	0.0926	3.3825
	9	0.9941	0.1445	2.9187	0.9953	0.1353	2.9946	0.9973	0.1168	3.1658
	10	0.9940	0.1688	2.7748	0.9952	0.1594	2.8420	0.9971	0.1399	2.9937
	15	0.9936	0.2676	2.3348	0.9948	0.2579	2.3783	0.9967	0.2372	2.4767
	20	0.9934	0.3383	2.1037	0.9945	0.3290	2.1361	0.9965	0.3088	2.2103
	25	0.9933	0.3914	1.9574	0.9944	0.3827	1.9835	0.9963	0.3634	2.0437
50	2	0.9955	8E-06	9.3212	0.9967	4E-06	9.9191	0.9985	9E-07	11.3474
	3	0.9945	0.0027	5.9016	0.9957	0.0021	6.1443	0.9975	0.0012	6.7013
	4	0.9940	0.0169	4.6488	0.9952	0.0146	4.7973	0.9970	0.0106	5.1338
	5	0.9937	0.0407	3.9782	0.9948	0.0369	4.0847	0.9966	0.0297	4.3246
	6	0.9935	0.0687	3.5533	0.9945	0.0638	3.6362	0.9963	0.0540	3.8228
	7	0.9933	0.0974	3.2567	0.9943	0.0917	3.3247	0.9961	0.0802	3.4775
	8	0.9932	0.1252	3.0361	0.9942	0.1190	3.0939	0.9959	0.1063	3.2236
	9	0.9931	0.1514	2.8647	0.9940	0.1450	2.9149	0.9958	0.1315	3.0279
	10	0.9930	0.1759	2.7269	0.9939	0.1693	2.7715	0.9957	0.1553	2.8717
	15	0.9926	0.2749	2.3038	0.9935	0.2682	2.3326	0.9952	0.2536	2.3978
	20	0.9924	0.3452	2.0799	0.9933	0.3389	2.1016	0.9950	0.3249	2.1508
	25	0.9923	0.3981	1.9378	0.9932	0.3921	1.9553	0.9948	0.3788	1.9953
75	2	0.9945	1E-05	8.9574	0.9956	8E-06	9.3799	0.9974	3E-06	10.3572
	3	0.9937	0.0032	5.7547	0.9947	0.0027	5.9300	0.9964	0.0018	6.3237
	4	0.9932	0.0183	4.5595	0.9942	0.0166	4.6674	0.9959	0.0132	4.9078
	5	0.9930	0.0431	3.9146	0.9939	0.0402	3.9921	0.9955	0.0343	4.1643
	6	0.9928	0.0718	3.5040	0.9936	0.0680	3.5644	0.9952	0.0603	3.6986
	7	0.9926	0.1009	3.2164	0.9935	0.0966	3.2659	0.9950	0.0877	3.3760
	8	0.9925	0.1289	3.0020	0.9933	0.1243	3.0440	0.9948	0.1146	3.1376
	9	0.9924	0.1554	2.8351	0.9932	0.1505	2.8716	0.9947	0.1403	2.9531
	10	0.9923	0.1800	2.7008	0.9931	0.1750	2.7331	0.9946	0.1645	2.8054
	15	0.9920	0.2789	2.2869	0.9928	0.2739	2.3078	0.9942	0.2631	2.3548
	20	0.9919	0.3490	2.0672	0.9926	0.3444	2.0829	0.9939	0.3341	2.1183
	25	0.9918	0.4017	1.9274	0.9924	0.3973	1.9400	0.9938	0.3875	1.9689

Table B.3 - (Continued)

		$1-\beta=0.99$								
		$\gamma=0.90$			$\gamma=0.95$			$\gamma=0.99$		
m	n	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*	$1-\beta^*$	L^*	U^*
100	2	0.9939	1E-05	8.7653	0.9949	1E-05	9.1003	0.9966	4E-06	9.8583
	3	0.9931	0.0034	5.6756	0.9940	0.0030	5.8163	0.9956	0.0022	6.1272
	4	0.9928	0.0192	4.5112	0.9936	0.0177	4.5980	0.9951	0.0148	4.7888
	5	0.9925	0.0445	3.8801	0.9933	0.0421	3.9424	0.9948	0.0371	4.0794
	6	0.9924	0.0735	3.4772	0.9931	0.0704	3.5258	0.9945	0.0640	3.6326
	7	0.9922	0.1028	3.1945	0.9929	0.0993	3.2343	0.9943	0.0919	3.3220
	8	0.9921	0.1310	2.9836	0.9928	0.1272	3.0173	0.9941	0.1193	3.0917
	9	0.9920	0.1575	2.8191	0.9927	0.1536	2.8484	0.9940	0.1452	2.9132
	10	0.9919	0.1822	2.6867	0.9926	0.1781	2.7126	0.9939	0.1695	2.7700
	15	0.9917	0.2810	2.2779	0.9923	0.2770	2.2946	0.9935	0.2683	2.3319
	20	0.9915	0.3511	2.0604	0.9921	0.3473	2.0729	0.9933	0.3391	2.1010
	25	0.9914	0.4036	1.9219	0.9920	0.4001	1.9320	0.9931	0.3923	1.9548
200	2	0.9927	2E-05	8.4468	0.9934	2E-05	8.6460	0.9948	1E-05	9.0768
	3	0.9922	0.0039	5.5423	0.9928	0.0036	5.6275	0.9940	0.0030	5.8094
	4	0.9919	0.0207	4.4294	0.9925	0.0197	4.4821	0.9936	0.0177	4.5945
	5	0.9917	0.0470	3.8217	0.9923	0.0454	3.8595	0.9933	0.0421	3.9403
	6	0.9916	0.0765	3.4319	0.9921	0.0746	3.4614	0.9931	0.0705	3.5243
	7	0.9915	0.1062	3.1576	0.9920	0.1040	3.1817	0.9929	0.0994	3.2332
	8	0.9914	0.1346	2.9524	0.9919	0.1323	2.9727	0.9928	0.1273	3.0164
	9	0.9913	0.1612	2.7921	0.9918	0.1588	2.8097	0.9927	0.1537	2.8477
	10	0.9913	0.1859	2.6629	0.9917	0.1835	2.6784	0.9926	0.1782	2.7120
	15	0.9911	0.2847	2.2628	0.9915	0.2823	2.2727	0.9923	0.2771	2.2943
	20	0.9910	0.3545	2.0493	0.9914	0.3522	2.0566	0.9921	0.3474	2.0727
	25	0.9909	0.4067	1.9129	0.9913	0.4047	1.9188	0.9920	0.4001	1.9318
250	2	0.9924	2E-05	8.3742	0.9931	2E-05	8.5445	0.9943	1E-05	8.9082
	3	0.9919	0.0040	5.5116	0.9925	0.0038	5.5847	0.9936	0.0032	5.7392
	4	0.9917	0.0211	4.4105	0.9922	0.0202	4.4557	0.9932	0.0185	4.5513
	5	0.9915	0.0475	3.8082	0.9920	0.0461	3.8406	0.9929	0.0433	3.9093
	6	0.9914	0.0773	3.4215	0.9918	0.0755	3.4467	0.9927	0.0720	3.5002
	7	0.9913	0.1070	3.1491	0.9917	0.1051	3.1697	0.9926	0.1011	3.2135
	8	0.9912	0.1355	2.9453	0.9916	0.1334	2.9626	0.9925	0.1292	2.9997
	9	0.9912	0.1621	2.7860	0.9916	0.1600	2.8010	0.9924	0.1556	2.8331
	10	0.9911	0.1868	2.6574	0.9915	0.1847	2.6707	0.9923	0.1802	2.6991
	15	0.9910	0.2855	2.2594	0.9913	0.2835	2.2678	0.9920	0.2791	2.2860
	20	0.9909	0.3552	2.0467	0.9912	0.3534	2.0529	0.9918	0.3493	2.0665
	25	0.9908	0.4074	1.9109	0.9911	0.4057	1.9158	0.9917	0.4019	1.9268
∞	2	0.9900	4E-05	7.8794	0.9900	4E-05	7.8794	0.9900	4E-05	7.8794
	3	0.9900	0.0050	5.2983	0.9900	0.0050	5.2983	0.9900	0.0050	5.2983
	4	0.9900	0.0239	4.2794	0.9900	0.0239	4.2794	0.9900	0.0239	4.2794
	5	0.9900	0.0517	3.7151	0.9900	0.0517	3.7151	0.9900	0.0517	3.7151
	6	0.9900	0.0823	3.3499	0.9900	0.0823	3.3499	0.9900	0.0823	3.3499
	7	0.9900	0.1126	3.0913	0.9900	0.1126	3.0913	0.9900	0.1126	3.0913
	8	0.9900	0.1413	2.8968	0.9900	0.1413	2.8968	0.9900	0.1413	2.8968
	9	0.9900	0.1681	2.7444	0.9900	0.1681	2.7444	0.9900	0.1681	2.7444
	10	0.9900	0.1928	2.6210	0.9900	0.1928	2.6210	0.9900	0.1928	2.6210
	15	0.9900	0.2910	2.2371	0.9900	0.2910	2.2371	0.9900	0.2910	2.2371
	20	0.9900	0.3602	2.0306	0.9900	0.3602	2.0306	0.9900	0.3602	2.0306
	25	0.9900	0.4119	1.8983	0.9900	0.4119	1.8983	0.9900	0.4119	1.8983

Nota that 2E-05 means $2 * 10^{-5}$.

Appendix C – Proof of Equation (26): approximate two-sided tolerance limits for sample variances based on the CE method

In this Appendix C, the proof of Equation (26), which is required to find the value of β^*_{CE}, is provided so that the approximate two-sided tolerance factors for sample variances based on the CE method can be obtained.

To obtain Equation (26), we need to demonstrate the approximation of the coverage probability shown in Equation (25):

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) \cong 1 - F_{\chi^2_{m(n-1)}}\left(R(Y;\beta^*,m,n,1-\beta)\right).$$

First, from Equation (22), we have the coverage probability:

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$$
$$= P_Y\left(P_W\left(\frac{Y}{m(n-1)}\chi^2_{n-1,\frac{\beta^*}{2}} \leq W \leq \frac{Y}{m(n-1)}\chi^2_{n-1,1-\frac{\beta^*}{2}} \mid Y\right) \geq 1-\beta\right),$$

where recall that $W = (n-1)S^2/\sigma^2$ and $Y = m(n-1)\,S_p^2/\sigma^2$ follow chi-square distributions with $(n-1)$ and $m(n-1)$ df, respectively. The WH cube root approximation is used to transform the chi-squared random variable W into $WT = \sqrt[3]{\frac{W}{n-1}}$ that follows a normal distribution with mean $1-d = 1-\frac{2}{9(n-1)}$ and variance $d = \frac{2}{9(n-1)}$. Thus, the approximation of the coverage probability is given by

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$$
$$\cong P_Y\left(P_{WT}\left(\sqrt[3]{\frac{Y}{m(n-1)^2}\chi^2_{n-1,\frac{\beta^*}{2}}} \leq WT \leq \sqrt[3]{\frac{Y}{m(n-1)^2}\chi^2_{n-1,1-\frac{\beta^*}{2}}} \mid Y\right) \geq 1-\beta\right).$$

Then, $WT \sim N(1-d,d)$ is standardized so that $Z = \frac{WT-(1-d)}{\sqrt{d}} \sim N(0,1)$. We get

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$$
$$\cong P_Y\left(P_Z\left(\frac{\sqrt[3]{\frac{Y}{m(n-1)^2}\chi^2_{n-1,\frac{\beta^*}{2}}}-(1-d)}{\sqrt{d}} \leq Z \leq \frac{\sqrt[3]{\frac{Y}{m(n-1)^2}\chi^2_{n-1,1-\frac{\beta^*}{2}}}-(1-d)}{\sqrt{d}} \mid Y\right) \geq 1-\beta\right).$$

From this last equation, let A and B denote functions of the chi-squared random variable Y, given by $A(Y) = \frac{\sqrt[3]{\frac{Y}{m(n-1)^2}\chi^2_{n-1,1-\frac{\beta^*}{2}}}-(1-d)}{\sqrt{d}}$ and $B(Y) = \frac{\sqrt[3]{\frac{Y}{m(n-1)^2}\chi^2_{n-1,\frac{\beta^*}{2}}}-(1-d)}{\sqrt{d}}$.

Note that, given any realization of Y $(y > 0)$ and the fact that $0 < \beta^* < 1$, $A(Y) > B(Y)$. Thus, we have

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) \cong P_Y(P_Z(B(Y) \leq Z \leq A(Y)|\, Y) \geq 1-\beta).$$

Next, we can manipulate this last expression, namely, subtracting $\left(\frac{A(Y)+B(Y)}{2}\right)$ and then squaring each end of Z so that we obtain:

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$$
$$\cong P\left(P\left(\left(Z-\frac{A(Y)+B(Y)}{2}\right)^2 \leq \left(\frac{A(Y)-B(Y)}{2}\right)^2 \mid Y\right) \geq 1-\beta\right).$$

Let denote $J = \left(Z-\frac{A(Y)+B(Y)}{2}\right)^2$ the random variable that follows a non-central chi-square distribution with 1 degree of freedom and non-centrality parameter $\left(\frac{A(Y)+B(Y)}{2}\right)^2$. Thus, we have

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) \cong P\left(P\left(J \leq \left(\frac{A(Y)-B(Y)}{2}\right)^2 \mid Y\right) \geq 1-\beta\right)$$
$$\cong P\left(\chi^2_{1,1-\beta}\left(\left(\frac{A(Y)+B(Y)}{2}\right)^2\right) \leq \left(\frac{A(Y)-B(Y)}{2}\right)^2 \mid Y\right),$$

where $\chi^2_{1,1-\beta}\left(\left(\frac{A(Y)+B(Y)}{2}\right)^2\right)$ denotes the $(1-\beta)$-quantile of a non-central chi-square distribution with 1 df and non-centrality parameter $\left(\frac{A(Y)+B(Y)}{2}\right)^2$.

Since $\left(\frac{A(Y)-B(Y)}{2}\right)^2 = \frac{9}{8\left(\sqrt[3]{m^2(n-1)}\right)}Y^{\frac{2}{3}}\left(\sqrt[3]{\chi^2_{n-1,1-\frac{\beta^*}{2}}}-\sqrt[3]{\chi^2_{n-1,\frac{\beta^*}{2}}}\right)^2$, we obtain:

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta)$$
$$\cong P\left(\chi^2_{1,1-\beta}\left(\left(\frac{A(Y)+B(Y)}{2}\right)^2\right) \leq \frac{9}{8\left(\sqrt[3]{m^2(n-1)}\right)}Y^{\frac{2}{3}}\left(\sqrt[3]{\chi^2_{n-1,1-\frac{\beta^*}{2}}}-\sqrt[3]{\chi^2_{n-1,\frac{\beta^*}{2}}}\right)^2 \mid Y\right),$$

Hence, the approximation of the coverage probability (shown in Equation 25) is given by

$$P_Y(G(Y;\beta^*,m,n) \geq 1-\beta) \cong 1 - F_{\chi^2_{m(n-1)}}\big(R(Y;\beta^*,m,n,1-\beta)\big),$$

where $R(Y;\beta^*,m,n,1-\beta) = \dfrac{16\left(\sqrt{2m^2(n-1)}\right)\left(\sqrt{\chi^2_{1,1-\beta}\left(\left(\frac{A(Y)+B(Y)}{2}\right)^2\right)}\right)^3}{27\left(\sqrt[3]{\chi^2_{n-1,1-\frac{\beta^*}{2}}}-\sqrt[3]{\chi^2_{n-1,\frac{\beta^*}{2}}}\right)^3}$

Finally, as explained in Subchapter 3.2.1., using the conditional expectation in the left side of Equation (25), Equation (26) is obtained.

Appendix D – Proof of Equation (29): approximate two-sided tolerance limits for sample variances based on the KMM method

In this Appendix D, the derivations are made to obtain directly the formulas of the approximate tolerance limits based on the KMM method rather than the formulas to find the exact or approximate values of β^* (Appendixes A and C, respectively, for the exact and the CE methods).

Similarly to Equation (1), the approximate $(1-\beta, \gamma)$ two-sided tolerance interval for sample variances based on the KMM method, which is obtained in three steps, can be defined by

$$P_{S_1^2,S_2^2,\dots,S_m^2}\left(P_{S^2}\left(\hat{S}^2_{L_{KMM}} \leq S^2 \leq \hat{S}^2_{U_{KMM}} \mid S_1^2, S_2^2, \dots, S_m^2\right) \geq 1-\beta\right) = \gamma,$$

where $\hat{S}^2_{L_{KMM}}$ and $\hat{S}^2_{U_{KMM}}$ are the approximate lower and upper tolerance limits, respectively.

First step: We obtain m sample variances $\{S_1^2, S_2^2, \dots, S_m^2\}$ from the Phase I reference data (m random samples (subgroups) each of size n). The i-*th* Phase I sample variance is given by $S_i^2 = \frac{1}{n-1}\sum_{j=1}^{n}\left(X_{ij}-\bar{X}_i\right)^2$, where $\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ is the ith Phase I sample mean and X_{ij} is the j-*th* observation of the i-*th* Phase I sample ($X_{ij} \sim N(\mu, \sigma^2)$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$). Since each S_i^2 follows a gamma distribution with shape parameter $\left(\frac{n-1}{2}\right)$ and scale parameter $2\sigma^2/(n-1)$, a sample of m gamma observations $\{S_1^2, S_2^2, \dots, S_m^2\}$ are generated from the Phase I reference data.

As used by Krishnamoorthy et al. (2008) in their proposed Normal-Based Method, we apply the WH transformation on each one of the m Phase I sample variances $\{S_1^2, S_2^2, \dots, S_m^2\}$, namely, the cube-root of each S_i^2: $T_i = (S_i^2)^{\frac{1}{3}}$, so that we get a sample of m normal observations:

$T_1 = (S_1^2)^{\frac{1}{3}}$, $T_2 = (S_2^2)^{\frac{1}{3}}$, …, $T_m = (S_m^2)^{\frac{1}{3}}$, where each T_i follows a normal distribution ($i = 1, 2, \dots, m$).

Second step: Given a sample of normal observations obtained in the First step $\{T_1^2, T_2^2, \ldots, T_m^2\}$ and the corresponding mean and standard deviation sample (denoted as $\bar{T}$ and S_T, respectively), we can deal with the widely used tolerance interval for a normal distribution, as considered in Krishnamoorthy et al. (2008). It is well known that the exact two-sided "k-sigma" tolerance interval for the normal distribution of T, tolerance factor of which is k, is given by

$$P_{\bar{T},S_T}(P_T(\bar{T} - k\, S_T \leq T \leq \bar{T} + k\, S_T \mid \bar{T}, S_T) \geq 1 - \beta) = \gamma,$$

Third step: The normal random variable T is back transformed into the gamma random variable S^2 (i.e., $(T)^3 = S^2$). Thus, we have the approximate two-sided tolerance interval for sample variances, which follow a gamma distribution

$$P_{\bar{T},S_T}(P_{S^2}([\bar{T} - k\, S_T]^3 \leq S^2 \leq [\bar{T} + k\, S_T]^3 \mid \bar{T}, S_T) \geq 1 - \beta) = \gamma.$$

Thus, the approximate tolerance limits for sample variances based on the KMM method are given by

$$\hat{S}^2_{L_{KMM}} = [\bar{T} - k\, S_T]^3 \text{ and } \hat{S}^2_{U_{KMM}} = [\bar{T} + k\, S_T]^3,$$

Appendix E - Proof of the obtained cdf of $CRL_{q,one}$ of the one-sided S^2 chart (Equation 38)

First, the probability mas function (pmf) of the $CRL_{q,one}$ (denoted as $f_{CRL_{q,one}}$) is obtained.

i. From Equation (16), let d and b be defined as:

$$d = F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha}\right)$$

$$b = ln(1-q)/ln\left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha}\right)\right) = ln(1-q)/ln(d)$$

Thus,

$$CRL_{q,one} = \lceil b \rceil$$

$$= \left\lceil ln(1-q)/ln\left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha}\right)\right)\right\rceil = \lceil ln(1-q)/ln(d) \rceil$$

ii. Note that $b > 0$ and $\lceil b \rceil = CRL_{q,one} = \{1, 2, 3, \dots\}$. Using the definition of ceiling function, we have the following relation between $\lceil b \rceil$ and b (also, see the graph shown in Figure E):

$$\lceil b \rceil - 1 < b \leq \lceil b \rceil, \text{ where } b > 0 \text{ and } \lceil b \rceil = CRL_{q,one} = \{1, 2, 3, \dots\}$$

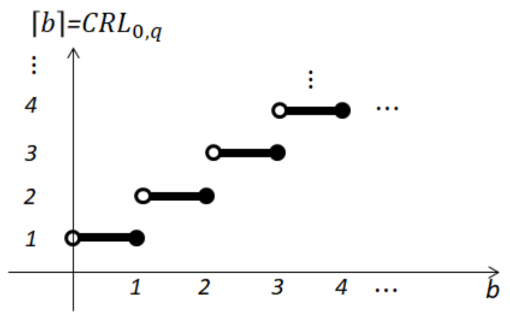

Figure E. Relation between $\lceil \boldsymbol{b} \rceil$ ($= \boldsymbol{CRL_{q,one}(\rho^2)}$) and $\boldsymbol{b}$: $\lceil \boldsymbol{b} \rceil - \mathbf{1} < \boldsymbol{b} \leq \lceil \boldsymbol{b} \rceil$

iii. Next, using the relation between $\lceil b \rceil$ and b indicated in (ii), if $\lceil b \rceil = \tau$ (where $\tau = \{1, 2, 3, \dots\}$), then $\tau - 1 < b \leq \tau$. Hence, the pmf of the $CRL_{q,one}$ can be defined as follow

$f_{CRL_{q,one}}(\tau) = P\big(CRL_{q,one}(\rho^2) = \tau\big) = P(\lceil b \rceil = \tau) = P(\tau - 1 < b \leq \tau)$,

where: $\tau = \{1, 2, 3, \ldots\}$

iv. Then, using the expression of b from (i), the pmf of CRL_q is derived as follows:

$$f_{CRL_q}(\tau) = P(\tau - 1 < b \leq \tau) = P\left(\tau - 1 < \frac{ln(1-q)}{ln(d)} \leq \tau\right)$$
$$= P\big(ln(1-q)^{1/(\tau-1)} < ln(d) \leq ln(1-q)^{1/(\tau)}\big)$$
$$= P\big((1-q)^{1/(\tau-1)} < d \leq (1-q)^{1/\tau}\big)$$
$$= P\big(d > (1-q)^{1/(\tau-1)}\big) - P\big(d > (1-q)^{1/\tau}\big)$$

v. Thus, using the equation of d from (i) and considering $(1-q)^{1/(\tau-1)} = M$ and $(1-q)^{1/\tau} = N$, the pmf of $CRL_{q,one}$ becomes:

$$f_{CRL_{q,one}}(\tau) = P\left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha}\right) > M\right)$$
$$- P\left(F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha}\right) > N\right)$$

When the values of the cdf of a chi-squared variable with $n-1$ df meet $F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha}\right) > M$, the corresponding quantiles of this distribution meet the next relation $\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha} > \chi^2_{n-1,\,M}$. The relation $F_{\chi^2_{n-1}}\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha}\right) > N$ must be treated in a similar way, yielding:

$$f_{CRL_{q,one}}(\tau) = P\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha} > \chi^2_{n-1,\,M}\right)$$
$$- P\left(\frac{Y}{\rho^2 m(n-1)}\chi^2_{n-1,\,1-\alpha} > \chi^2_{n-1,\,N}\right)$$

$$f_{CRL_{q,one}}(\tau) = P\left(Y > \frac{\rho^2 m(n-1)}{\chi^2_{n-1,\,1-\alpha}}\chi^2_{n-1,\,M}\right) - P\left(Y > \frac{\rho^2 m(n-1)}{\chi^2_{n-1,\,1-\alpha}}\chi^2_{n-1,\,N}\right)$$

vi. Finally, because Y follows a chi-square distribution with $m(n-1)$ df, we can find the pmf of $CRL_{q,one}(\rho^2)$:

$$\boldsymbol{f_{CRL_{q,one}}(\tau)}$$
$$\boldsymbol{= F_{\chi^2_{m(n-1)}}\left(\frac{\rho^2 m(n-1)}{\chi^2_{n-1,\,1-\alpha}}\chi^2_{n-1,\,(1-q)^{1/\tau}}\right) - F_{\chi^2_{m(n-1)}}\left(\frac{\rho^2 m(n-1)}{\chi^2_{n-1,\,1-\alpha}}\chi^2_{n-1,\,(1-q)^{1/(\tau-1)}}\right),}$$
$$\boldsymbol{\tau = \{1, 2, 3, \ldots\}}$$

where, in general, $\chi^2_{n-1,p}$ denotes the p-quantile of the distribution of a chi-squared random variable with $n-1$ df (i.e., $F^{-1}_{\chi^2_{n-1}}(p)$) and $F_{\chi^2_{m(n-1)}}$ denotes the cumulative distribution function (cdf) of a chi-squared random variable with $m(n-1)$ df. Note that the pmf of $CRL_{q,one}$ depends on m, n and α values. It is worth to note that since the $CRL_{q,one}(\rho^2)$ is a discrete random variable (positive integer values), its pmf is zero for all non-integer values of $CRL_{q,one}$.

Finally, the cumulative distribution function (cdf) of the $CRL_{q,one}$ (denoted as $F_{CRL_{q,one}}$) is obtained.

i. Given that the CRL_q is a discrete random variable, we can define its cdf as follow (see also Figure E):

$t < 1$, $F_{CRL_{q,one}}(t) = P\big(CRL_{q,one}(\rho^2) \le t\big) = P\big(CRL_q \le 0\big) = 0$

$j \le t < j+1$ (where $j = \{1, 2, 3, \dots\}$),

$$F_{CRL_q}(t) = P\big(CRL_{q,one}(\rho^2) \le t\big) = P\big(CRL_{q,one}(\rho^2) \le j\big)$$

$$= \sum_{i=1}^{j} P\big(CRL_{q,one}(\rho^2) = i\big)$$

ii. The pmf of $CRL_{q,one}(\gamma^2)$ can be expressed as a function w of τ :

$$f_{CRL_{q,one}(\rho^2)}(\tau)$$

$$= F_{\chi^2_{m(n-1)}}\left(\frac{\rho^2 m(n-1)}{\chi^2_{n-1,\,1-\alpha}}\chi^2_{n-1,\,(1-q)^{1/\tau}}\right) - F_{\chi^2_{m(n-1)}}\left(\frac{\rho^2 m(n-1)}{\chi^2_{n-1,\,1-\alpha}}\chi^2_{n-1,\,(1-q)^{1/(\tau-1)}}\right).$$

$$= w(\tau) - w(\tau-1)$$

and, if $\tau = 0$, $w(0) = 0$.

iii. Then, the cdf of $CRL_{q,one}(\rho^2)$ (Equation 38) when $t \ge 1$, $j \le t < j+1$ and $j = \{1,2,3,\dots\}$ (see part (ii.)) can be calculated as:

of the one-sided S^2 chart (using the pmf of the $CRL_{q,one}$)

$$F_{CRL_{q,one}}(t) = P\big(CRL_{q,one}(\rho^2) \le t\big) = P\big(CRL_{q,one}(\rho^2) \le j\big)$$

$$= P\big(CRL_{q,one} = 1\big) + P\big(CRL_{q,one} = 2\big) + P\big(CRL_{q,one} = 3\big) + \cdots + P\big(CRL_{q,one} = j\big)$$

$$= f_{CRL_{q,one}}(1) + f_{CRL_{q,one}}(2) + f_{CRL_{q,one}}(3) + \cdots + f_{CRL_{q,one}}(j)$$

$$= \big(w(1) - w(0)\big) + \big(w(2) - w(1)\big) + \big(w(3) - w(2)\big) + \cdots + \big(w(j) - w(j-1)\big)$$

$$= w(j) - w(0) = w(j)$$

$$= F_{\chi^2_{m(n-1)}}\left(\frac{\rho^2 m(n-1)}{\chi^2_{n-1,\,1-\alpha}}\chi^2_{n-1,\,(1-q)^{1/j}}\right)$$

iv. Thus, the cdf of the $CRL_{q,one}(\rho^2)$ is defined as:

$$F_{CRL_{q,one}}(t) = P\big(CRL_{q,one}(\rho^2) \le t\big)$$

$$= \begin{cases} 0, & t < 1 \\ F_{\chi^2_{m(n-1)}}\left(\dfrac{\rho^2 m(n-1)\chi^2_{n-1,\,(1-q)^{1/j}}}{\chi^2_{n-1,\,1-\alpha}}\right), & t \ge 1,\ j \le t < j+1 \ \ and \ \ j = \{1,2,3,\ldots\} \end{cases}$$

Substituting $\lfloor t \rfloor$ for j (where $\lfloor t \rfloor$ denotes the largest integer less or equal to t), the equation above can be rewritten as

$$\boldsymbol{F_{CRL_{q,one}}(t) = P\big(CRL_{q,one}(\rho^2) \le t\big) = \begin{cases} 0, & t < 1 \\ F_{\chi^2_{m(n-1)}}\left(\dfrac{\rho^2 m(n-1)\chi^2_{n-1,\,(1-q)^{1/\lfloor t \rfloor}}}{\chi^2_{n-1,\,1-\alpha}}\right), & t \ge 1 \end{cases}}$$

Appendix F – R codes

In this part, the R codes for computing the exact and approximate (based on the CE and KMM methods) two-sided tolerance factors for the sample variance are provided.

R CODES FOR COMPUTING THE EXACT TOLERANCE FACTORS

```
########## (1) Search method (secant method)  ##########
### Secant method (increasing) ###
secantc <- function (fun, x0, x1, tol=1e-10, niter=10000000){
  for ( i in 1:niter ) {
    funx1 <- fun(x1)
    funx0 <- fun(x0)
    x2 <- ( (x0*funx1) - (x1*funx0) )/( funx1 - funx0 )
    funx2 <- fun(x2)
    if (abs(funx2) < tol) {
      return(x2)
    }
    if (funx2 < 0)
      x0 <- x2     ####convex function
    else
      x1 <- x2     ####concave function
  }
  stop ("exceeded allowed number of interactions")
}
### Secant method (decreasing) ###
secant <- function (fun, x0, x1, tol=1e-10, niter=100000){
  for (i in 1:niter ) {
    funx1 <- fun(x1)
    funx0 <- fun(x0)
```

```
    x2 <- ( (x0*funx1) - (x1*funx0) )/( funx1 - funx0 )
    funx2 <- fun(x2)
    if (abs(funx2) < tol) {
      return(x2)
    }
    if (funx2 < 0)
      x1 <- x2   ####convex function
    else
      x0 <- x2   ####concave function
  }
  stop("exceeded allowed number of interactions")
}
########## (2) confidence of the TI [gamma_val=(P(G(Y)>=t)] ##########
gamma_val <- function (t,beta_star,m,n) {
  Y0<-m*((n-1)^2)*log((qchisq(1-(beta_star/2),n-1))/(qchisq((beta_star/2),n-1)))/(qchisq(1-(beta_star/2),n-1)-(qchisq((beta_star/2),n-1)))
  max_G<-pchisq((Y0/(m*(n-1)))*qchisq(1-(beta_star/2),n-1),n-1)-pchisq((Y0/(m*(n-1)))*qchisq(beta_star/2,n-1),n-1)
  G_function <- function (Y) {
    a <-pchisq((Y/(m*(n-1)))*qchisq(1-(beta_star/2),n-1),n-1)-pchisq((Y/(m*(n-1)))*qchisq(beta_star/2,n-1),n-1)
    return(a)
  }
  G_functionsec <- function (g_func) {
    k <- G_function(g_func)-t
    return(k)
  }
  if (t<max_G) {
    y1 <- secantc(G_functionsec,1.0e-50,Y0)
    y2 <- secant(G_functionsec,Y0,1.0e+50)
    c <- pchisq(y2,m*(n-1))-pchisq(y1,m*(n-1))
  }
  else {c <- 0}
```

```
  return(c)
}
##########  (3) Find the beta* (beta_star) ##########

find_beta_star<- function(m,n,beta,nom_gamma){
  G_quantile<- function(m,n,beta_star,nom_gamma){
    gamma_valsec <- function (s) {
      a <- gamma_val(s,beta_star,m,n)-nom_gamma
      return (a)
    }
    d<-secantc(gamma_valsec,1-beta,1)
    d
  }
  beta_ref<-2*(1-pchisq(m*(n-1)*qchisq(1-beta,n-1)/qchisq(1-nom_gamma,m*(n-
1)),n-1))
  if(beta_ref<beta){
    ref_Uval<-beta_ref
  }
  if(beta_ref>=beta){
    ref_Uval<-beta
  }
  if(beta_ref-(beta/5)>0){
    ref_Lval<-ref_Uval-(beta/5)
  }
  if(beta_ref-(beta/5)<=0){
    ref_Lval<-0.00001
  }
  beta_adj <- function (beta_star) {
    b <- G_quantile(m,n,beta_star,nom_gamma)-(1-beta)
    return (b)
  }
  resp<-secantc(beta_adj,ref_Lval,ref_Uval)
```

```
  Ltwo<-qchisq(resp/2,n-1)/(n-1)
  Utwo<-qchisq(1-(resp/2),n-1)/(n-1)
  vect_resp<-c(1-resp,Ltwo,Utwo)
  return(vect_resp)
}
########## INSTRUCTIONS  ##########
##### First step: Run the Code (1)-(3) provided above #######
##### Second step: Use the function "find_beta_star(m,n,beta,nom_gamma)" from Code (3) called "Find the beta*two", insert the values of m, n, beta and nom_gamma #######
##### Third step: The output of the Second step is the vector (1-beta*two,L*two,U*two), that is, the 1-beta star, and the lower and the upper tolerance factors  #######
##### EXAMPLE: If we consider find_beta_star(m=25,n=5,beta=0.05,nom_gamma=0.95), we obtain: (1-beta*two=0.97444508,L*two=0.08452931,U*two=3.17776208)  ##########
##### NOTE: If we study a distribution of the sample variance (which comes from a sample of size n) using a single sample of N observations to estimate sigma^2, we should consider m=(N-1)/(n-1) in our code #######
```

R CODES FOR COMPUTING THE APPROXIMATE TOLERANCE FACTORS BASED ON THE CE METHOD

```
########## (1) Search method (secant method)  ##########
secant.method <- function(f, x0, x1, tol=1e-10, n_int=10000000){   ###100000 ### tol=1e-10
  for ( i in 1:n_int ) {
    fx1 <- f(x1)
    fx0 <- f(x0)
    x2 <- ( (x0*fx1) - (x1*fx0) )/( fx1 - fx0 )
    fx2 <- f(x2)
    if (abs(fx2) < tol) {
      return(x2)  ##### the search value will be the first one that is less than tol value (it can be + or -)
    }
    if (fx2 < 0)
      x1 <- x2     ####convex function
```

```
    else
      x0 <- x2     ####concave function
  }
  stop("out of the specified n_int")
}
######## (2) using Wilson HIlferton approximation ##############
WH_gamma_approx<-function(beta,beta_star,m,n) {
  EXPRESS<-function(u){
    U_two<-qchisq(1-beta_star/2,n-1)/(n-1)
    L_two<-qchisq(beta_star/2,n-1)/(n-1)
    A<-(((U_two*qchisq(u,m*(n-1))/(m*(n-1)))^(1/3))-(1-(2/(9*(n-1)))))/((2/(9*(n-
1)))^0.5)
    B<-(((L_two*qchisq(u,m*(n-1))/(m*(n-1)))^(1/3))-(1-(2/(9*(n-1)))))/((2/(9*(n-
1)))^0.5)
    a_b<-(((U_two*(n-1))^(1/3))-((L_two*(n-1))^(1/3)))^3
    cte<-(((n-1)^0.5)*m*16*(2^0.5))/27
    invchi<-qchisq(p=1-beta,df=1,ncp=((A+B)/2)^2)
    expr<-pchisq(cte*(invchi^(3/2))/a_b,(m*(n-1)))
    return(expr)
  }
  integrate_value<-integrate(EXPRESS,lower=0,upper=1, rel.tol = 1e-10)  ### 1e-
5 (for n=4, m=50,75, gamma=0.90 and 1-beta=0.90) ## rel.tol =
.Machine$double.eps^0.13
  d<-integrate_value$value
  return(1-d)
}
######## (3) Finding the beta_star using the CE method #####
find1_beta_star<- function(m,n,gamma,beta){
  find2_beta_star <- function (beta_star) {  ###beta=beta*_two
    b <- WH_gamma_approx(beta,beta_star,m,n)-(gamma)   ##using Wilson
HIlferton approximation 1 ####   #((1+0.2)*0.0027)###beta_tol=beta; p=1-
gamma
    return (b)
  }
```

```
  beta_ref<-2*(1-pchisq(m*(n-1)*qchisq(1-beta,n-1)/qchisq(1-gamma,m*(n-1)),n-
1))
  if(beta_ref<beta){
    ref_Uval<-beta_ref
  }
  if(beta_ref>=beta){
    ref_Uval<-beta
  }
  if(ref_Uval/100-(beta/200)>0){  ###### if(ref_Uval/10-(beta/20)>0)
    ref_Lval<-ref_Uval/100-(beta/200)   #####  ref_Lval<-ref_Uval/10-(beta/20)
  }
  if(ref_Uval/100-(beta/200)<=0){   ######   if(ref_Uval/10-(beta/20)<=0)
    ref_Lval<-1.47e-08  ## 0.000001
  }
  resp<-secant.method(find2_beta_star,ref_Lval,ref_Uval) ### 1e-04,0.01
###0.0009,0.002 ##0.000003572776
##secant.method(find2_beta_star,ref_Lval,ref_Uval)
  L_adj<-qchisq(resp/2,n-1)/(n-1)
  U_adj<-qchisq(1-(resp/2),n-1)/(n-1)
  answ<-c(1-resp,L_adj,U_adj)
  return(answ)
}
##########  INSTRUCTIONS  ##########
##### First step: Run the Code (1)-(3) provided above #######
##### Second step: Use the function "find1_beta_star(m,n,gamma,beta)" from
Code (3) called "Finding the beta_star using the CE method", insert the values of
m, n, gamma and beta #######
##### Third step: The output of the Second step is the vector (1-
beta*two,L*two,U*two), that is, the beta star, and the lower and the upper
tolerance factors based on the CE method  #######
##### EXAMPLE: If we consider
find1_beta_star(m=25,n=5,gamma=0.95,beta=0.05), we obtain: (1-
beta*two=0.97466208,L*two=0.08414861,U*two=3.18269628)  ##########
```

R CODES FOR COMPUTING THE APPROXIMATE TOLERANCE FACTORS BASED ON THE KMM METHOD

```
######  Simulated tolerance factors based on the KMM method ##########

> install.packages("tolerance")

> library(tolerance)


KMM_method <- function (alpha,gamma,m,n,SIMrep) {
  k<-K.factor(n=m, f = NULL, alpha = 1-gamma, P = 1-alpha, side = 2, method =
"EXACT", m = 50)
  L_KMM<-numeric(SIMrep)
  U_KMM<-numeric(SIMrep)
  count<-0
  for (i in 1:SIMrep) {
    w_values<-rchisq(n=m,df=n-1)
    w_cube_root<-w_values^(1/3)
    if(((mean(w_cube_root)-(k*sd(w_cube_root)))^3)/mean(w_values)<0){
      count<-count+1
    }
    L_KMM[i]<-max(((mean(w_cube_root)-
(k*sd(w_cube_root)))^3)/mean(w_values),0)
    U_KMM[i]<-((mean(w_cube_root)+(k*sd(w_cube_root)))^3)/mean(w_values)
  }
  lower_KMM<-mean(L_KMM)
  upper_KMM<-mean(U_KMM)
  one_minus_alphaKMM<-pchisq(upper_KMM*(n-1),n-1)-
pchisq(lower_KMM*(n-1),n-1)
  return(c(one_minus_alphaKMM,lower_KMM,upper_KMM,count))
}
```