

## 7.

### Variable-gain output feedback control

#### 7.1. Introduction

In designing control laws, the usual first step is to describe the plant at a given operating point and next to develop them with a satisfactory performance for that plant model. In this case, an important property of a satisfactory control law is that its performance should not deteriorate when “small” variations in the operating conditions occur. The control law should be robust, or insensitive to small variations in the (usually unmeasured) parameters describing the operation conditions.

However, the design of most control systems of practical interest further requires control laws which maintain high performance in the presence of large changes in the operating point parameters. When dealing with large variations in the plant model parameters, a single constant-gain control law, no matter how robust, cannot attain the level of performance of a control law with variable gains. Thus, variable-gain control laws provide a class of controllers which can maintain high performance over a wide range of operating conditions. In the next section, an optimal control problem for the design of variable-gain output feedback control laws is developed, according to Halyo [ 18 ]. Standard or conventional (constant) gain scheduling techniques have provided a method of designing variable-gain control systems which can accommodate significant variations in the plant operating point parameters while continuing to make use of the accumulated knowledge and experience in the design of linear systems. The ability to use the well-established theory and accumulated experience in the design of linear systems, while extending its use and applicability to control nonlinear systems makes the concept of variable-gain control laws highly attractive. The conventional approach to the design of control systems using gain scheduling consists of the following steps.

1. A small number of operating points covering the operating range is selected, the method of selection being largely arbitrary.

2. For each operating condition selected, a satisfactory linear design with the same basic structure is obtained using any available design technique.
3. Using interpolation techniques, the control system gains are expressed as a linear function (i.e., a schedule) of the parameters describing the operating conditions, so that the schedule evaluated at any of the selected operating conditions is as “close” as possible to the design made in Step 2 for that operating condition.

While the approach outlined above may result in satisfactory designs of variable-gain control systems (usually after a considerable amount of trial-and-error), some of the problems encountered in this approach are immediately apparent from the steps outlined. Some of these problems are given below: a) The selection of the points covering the operating range is quite arbitrary. Different selections can result in significantly different gain-scheduled control laws with differing performance characteristics. In general, the selected points are represented in the gain-schedule, while other points are not represented at all. b) The amount of effort involved in designing a satisfactory system for each operating point can be considerable. Particularly, if many points have to be selected to cover the operating range, this effort may become impractical. c) No matter what interpolation technique is used in Step 3, the gain schedule will not be the same as the system designed meticulously in Step 2 for the selected operating points, with the possible exception of unusual situations. Thus, even at the operating points selected, the gain-scheduled control system may not have a satisfactory performance. In fact, the gain-scheduled system may even be unstable at one or more of the operating points selected. d) There is no guarantee that the conventional gain scheduling approach outlined will result in a satisfactory or even stable gain-scheduled control law over the operating range even when such a system exists. e) The three basic steps outlined lack a control theory framework which provides insight into the solution of the problem resulting in an integrated variable-gain control law. In other words, each step is largely independent of the others; e.g., the systems designed for the particular operating conditions selected may be excellent designs for those conditions, but may collectively result in a very poor fit or an unacceptable gain-schedule. Conventional gain scheduling is essentially a curve fitting or interpolation problem rather than a control law design problem.

## 7.2. Formulation

In this section, an optimal control problem will be formulated to design variable-gain output feedback laws. The objective is to develop a control theory framework to overcome the limitations of constant gain output feedback control. Consider a non-linear differential system  $\mathcal{S}$  represented by its state equations, linearized at a given operating condition defined by the parameter vector  $\mathbf{p}$ .

$$\dot{\mathbf{x}}(t, \mathbf{p}) = \mathbf{A}(\mathbf{p})\mathbf{x}(t, \mathbf{p}) + \mathbf{B}(\mathbf{p})\mathbf{u}(t, \mathbf{p}) \quad (7.1)$$

$$\mathbf{y}(t, \mathbf{p}) = \mathbf{C}(\mathbf{p})\mathbf{x}(t, \mathbf{p}) \quad (7.2)$$

where

$\mathbf{p}$  is a  $q$  – vector of parameters describing the system operating points;

$\mathbf{x}(t, \mathbf{p})$  is an  $n$ -vector describing the plant state perturbation at time  $t$  for the given operating condition  $\mathbf{p}$ .

$\mathbf{y}(t, \mathbf{p})$  is an  $m$ -vector describing the measurements or the feedback variables at time  $t$ .

A sample-data control approach to the problem is adopted. Thus, the control commands  $\mathbf{u}(t, \mathbf{p})$  will be assumed to remain constant over the sampling interval with duration of seconds,

$$\mathbf{u}(t, \mathbf{p}) = \mathbf{u}(k, \mathbf{p}) \quad \forall (kT, (k+1)T), \quad k \dots, -1, 0, 1, \dots$$

$$\xrightarrow{\mathbf{u}} [\text{Sample and Hold}] \longrightarrow [\text{System defined by (7.1), (7.2)}] \xrightarrow{\mathbf{y}} \quad (7.3)$$

Figure 7-1 - The input is sampled and held constant over the intervals  $(kT, (k+1)T)$  by the sample-and-hold box before being applied to the system  $\mathcal{S}$ .

In Figure 7-1, this constraint is represented by the insertion of the sample-and-hold box between the input  $\mathbf{u}(t)$  and the system  $\mathcal{S}$ . By definition of the sample-and-hold box, if  $\mathbf{u}(t)$  is its input, then its output is the piecewise constant function  $\mathbf{u}_0(t)$  specified by:

$$\mathbf{u}_0(t) = \mathbf{u}(kt), \quad kT \leq t \leq (k+1)T, \quad \text{for all integers } k.$$

The input is sampled every  $T$  second and  $T$  is referred to as sampling period.

A continuous system described by (8.1), (8.2) and (8.3) can now be described by the discrete system

$$\mathbf{x}(k+1, \mathbf{p}) = \Phi(\mathbf{p})\mathbf{x}(k, \mathbf{p}) + \Gamma(\mathbf{p})\mathbf{u}(k, \mathbf{p}) \quad (7.4)$$

$$\mathbf{y}(k, \mathbf{p}) = \mathbf{C}(\mathbf{p})\mathbf{x}(k, \mathbf{p}) \quad (7.5)$$

where

$\mathbf{x}(k, \mathbf{p})$  is the continuous state at the sampling instant  $kT$ , i.e.,

$$\mathbf{x}(k, \mathbf{p}) = \mathbf{x}(kT, \mathbf{p}) \quad (7.6)$$

and  $\Phi(\mathbf{p}), \Gamma(\mathbf{p}), \mathbf{C}(\mathbf{p})$  are the discrete versions of the continuous plant dynamics in (7.1) and (7.2). Note that if the control conforms to (7.3), then the discrete system (7.4) describes the continuous state at the sampling instants,  $kT$ , with no approximation, as indicated by (7.6).

The control law structure of interest is the class of variable-gain output feedback controllers which may be described by

$$\mathbf{u}(k, \mathbf{p}) = \mathbf{K}(\mathbf{p}) \mathbf{y}(k, \mathbf{p}) \quad (7.7)$$

Where  $\mathbf{y}(k, \mathbf{p})$  given by (7.5) represents the variables which have been

selected for feedback purposes :  $\mathbf{y} = \begin{bmatrix} \dot{h}/B \\ \dot{\alpha} \\ h/B \\ \alpha \\ \delta_1 \\ \delta_2 \end{bmatrix}$ . The pitch velocity of the winglets

was not selected for feedback purposes, see equation (6.3) and  $\mathbf{K}(\mathbf{p})$  is the variable gain for the operating point, This controller is a simple extension of the constant gain output feedback case treated previously in Chapter 6.

At a given operating point,  $\mathbf{p}$ , the control structure (7.7) corresponds to a discrete feedback formulation. However, in the current formulation, the quantity to be optimized is not a single control gain matrix for a particular operating point, but rather the global control law defined by the collection of control gains over the total operating range,  $\{\mathbf{K}(\mathbf{p}), \mathbf{p} \in \mathcal{R}\}$ , where  $\mathcal{R}$  is the collection of all operating points  $\mathbf{p}$  of interest for the design under consideration. Therefore, the relationship of the control gains at different operating points must be an integral part of the optimal control problem under formulation.

Consider a linear functional relationship between the gain and the operating point parameters, i.e.,

$$\mathbf{K}(\mathbf{p}) = \mathbf{K}_0 + \sum_{j=1}^q p_j \mathbf{K}_j, \quad \mathbf{p} \in \mathcal{R} \quad (7.8)$$

where coefficient matrices  $\mathbf{K}_j = \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_q$  determine the variable-gain control law over the total operating range.

It should be noted that, since it is not feasible to compute the gain  $K(p)$  at every point of  $\mathcal{R}$ , it is necessary to select some type of functional relationship between the gain and the parameters  $p_j$  whether linear or nonlinear. While the relation in (7.8) is linear in the operating point parameters  $p_j$ , these parameters can be selected as nonlinear functions of the physical variables of the plant. Thus, the linearity constraint in (7.8) is considerably less restrictive than is at first apparent, but provides a structure which can be handled more easily in both analytical development and actual implementation. A controller structure which can handle nonlinear relationships between the control gain and the parameter vector  $p_j$  will be presented below.

The significant difference between the conventional gain scheduling approach and the variable-gain output feedback approach formulated here is that the latter approach does not design local control laws, but rather determines the global control law  $\{K(p), p \in \mathcal{R}\}$  by optimizing a control objective over many operating points. If some operating points have less stability margin than others, the optimization automatically places greater emphasis on those operating conditions. Nonlinear relationships between the control gain and the operating condition parameters can be treated in the following manner.

Let  $G_i(p)$  be a  $m \times m$  matrix of real functions of  $p$  for each  $i = 1, 2, \dots, q$ . As in (7.8),  $K_j$  is a  $r \times m$  matrix for  $i = 1, 2, \dots, q$ .

Now consider the relation

$$K(p) = K_0 + \sum_{i=1}^q K_i G_i(p), \quad p \in \mathcal{R} \quad (7.9)$$

The elements of the matrices  $G_i(p)$  are arbitrarily selected integrable functions of  $p$  which may contain nonlinearities in terms of  $p$ . This relation is referred to as a separable form, since the operating point parameter  $p$  and the gain coefficient matrices  $K_i$  are separately expressed.

It should be noted that the linear relationship in (7.8) is a special case of the separable form in (7.9). The following selection of  $G_i(p)$  in (7.9) results in (7.10):

$$G_i(p) = p_i I, \quad 1 \leq i \leq q \quad (7.10)$$

Thus, the variable-gain output feedback control law structure will be defined by (7.7) and (7.9), resulting in a feedback law of the form

$$u(k, p) = - \left[ K_0 + \sum_{i=1}^q K_i G_i(p) \right] \cdot y(k, p), \quad p \in \mathcal{R} \quad (7.11)$$

Following the optimal quadratic sampled data formulation, consider a cost or objective function of a form similar to equation (6.5)

$$J_{lc}(\mathbf{K}(p), p) = \lim_{t_f \rightarrow \infty} \frac{1}{t_f} \int_0^{t_f} E \{ \mathbf{x}(t, p)' \mathbf{Q}_c(p) \mathbf{x}(t, p) + \mathbf{u}(t, p)' \mathbf{R}_c(p) \mathbf{u}(t, p) \} dt \quad (7.12)$$

where  $E$  denotes the expectation operator and  $J_{lc}$  is referred to as the local continuous objective function, as it expresses the objective at the operating point  $p$ . It can be shown that the local continuous objective function  $J_{lc}$  is equivalent to a local objective function for the discrete optimization problem of the form:

$$J_l(\mathbf{K}(p), p) = \lim_{N_f \rightarrow \infty} \frac{1}{2(N+1)} \sum_{k=0}^N E \{ \mathbf{x}(k+1, p)' \mathbf{Q}(p) \mathbf{x}(k+1, p) + \mathbf{u}(k, p)' \mathbf{R}(p) \mathbf{u}(k, p) \} \quad (7.13)$$

Both  $\mathbf{Q}(p)$ ,  $\mathbf{R}(p)$  are diagonal matrices independent of each other. In other words, cross terms between state and control are not included in the objective functions (7.11) and (7.12).

The operating range is defined by the constraints

$$a_i \leq p_i \leq b_i, \quad 1 \leq i \leq q, \quad p_i \in \mathcal{R} \quad (7.14)$$

For notational convenience, (7.14) is expressed as:

$$a \leq p \leq b \quad (7.15)$$

In this formulation, the quantity to be optimized is not a single control gain for a particular operating point, but rather a global control law defined by the set of control gains over the total operating range  $\{\mathbf{K}(p), p \in \mathcal{R}\}$ . Similarly, the objective or cost function for the design problem is not a local objective corresponding to the system performance at a single operating point, but a global objective with performance specifications over the complete operating range  $a_i \leq p_i \leq b_i$ . Thus, the cost function for the design problem can be selected as:

$$\bar{J} = \int_{a_1}^{b_1} \dots \int_{a_q}^{b_q} f(p) J_l[\mathbf{K}(p), p] dp_q \dots dp_1 = \int_a^b f(p) J_l[\mathbf{K}(p), p] dp \quad (7.16)$$

where  $f(p)$  is a non-negative scalar function of  $p$  selected by the designer to allow greater weighting of certain regions of the operating range over other regions. The shorthand notation on the right-hand-side of (7.16) uses a single integral sign from  $a$  to  $b$  to denote  $q$  scalar integrals. In general, the operating

range  $\mathcal{R}$  need not be of the form of (7.15). Thus, in general, the desired global objective function for the variable-gain output feedback problem is

$$\bar{J}(K) = \int_{\mathcal{R}} f(p) J_l [K(p), p] dp \quad (7.17)$$

where the local cost functions  $J_l [K(p), p]$  are weighted over the operating range, and

$$K = [K_0 \ K_1 \ \dots \ K_q] \quad (7.18)$$

The notation on the left-hand-side of (7.17) simply recognizes that for the feedback structure selected, the global cost is determined by the choice of  $K$ . A discrete version of the global cost  $\bar{J}$  can be expressed as

$$\bar{J}(K) = \sum_{j=1}^q f_j J_l [K(p_j), p_j] \quad (7.19)$$

Where  $J_l$  is defined by (7.13),  $f_j$  is the discrete weighting for the  $j^{\text{th}}$  operating point and  $[p_j, j = 1, 2, \dots, q]$  correspond to the plant operation points of interest for the design.

The variable-gain output feedback control law design can now be posed as an optimal control problem. The optimization consists of finding a variable-gains control law,  $K(p)$  over the design operating range  $\mathcal{R}$ , which minimizes the cost function  $J_l$  in (7.12) or the discrete version  $J(K)$  in (7.19), subject to the constraints of (7.4) to (7.7) and (7.9).

The optimal control approach taken here puts the variable-gain design problem into a theoretical setting. The desired characteristics of the controller are specified from the outset. No interpolation or curve fitting of local control laws arises in this approach. Since instability at an arbitrary operating point would result in an infinite cost locally and globally for most practical systems, the optimal variable-gain control law will stabilize all the operating points considered in the design.

### 7.3. Optimization of the variable-gain output feedback control problem

In this section, the necessary conditions for optimization of the variable-gain output feedback control problem posed in the previous section will be derived. Rather than using the Lagrangian approach and differentiating the augmented cost function to obtain the necessary conditions, a different path is followed, in

which the incremental cost is obtained. This was the approach of Levine and Athans [ 41 ], as applied to the standard output feedback problem in Chapter 6.

As already stated in Chapter 6, the optimization consists of finding a variable-gain control law,  $\mathbf{K}(p)$  over the design operating range  $\mathcal{R}$  which minimizes the cost function of the discrete version  $J(\mathbf{K})$  in (7.19) , subject to the constraints of (7.4) to (7.7) and (7.9), i.e.,

$$\left. \frac{\partial J}{\partial \mathbf{K}} \right|_{\mathbf{K}^*} = 0 \quad (7.19)$$

The necessary derivative will be calculated and evaluated. A key lemma in this calculation is due to Kleinman [ 27].

Lemma: Let  $f(\mathbf{X})$  be a trace function. Then, if one can write:

$$\lim_{\varepsilon \rightarrow 0} (f(\mathbf{X} + \varepsilon \Delta \mathbf{X}) - f(\mathbf{X})) = \varepsilon \operatorname{tr} [\mathbf{M}(\mathbf{X}) \Delta \mathbf{X}] \quad (7.19)$$

where  $\mathbf{M}(\mathbf{X})$  is a  $n \times r$  matrix,  $\mathbf{X}$  is a  $r \times n$  matrix, then

$$\frac{\partial f(\mathbf{X})}{\partial \mathbf{X}} = \mathbf{M}(\mathbf{X})' \quad (7.20)$$

where  $\mathbf{M}(\mathbf{X})'$  is the transpose of  $\mathbf{M}(\mathbf{X})$ .

By definition,  $f(\bullet)$  is a trace function of the matrix  $\mathbf{X}$  if  $f(\mathbf{X})$  is of the form:

$$f(\mathbf{X}) = \operatorname{tr} [\mathbf{F}(\mathbf{X})] \quad (7.21)$$

Where  $\mathbf{F}(\bullet)$  is a continuously differentiable mapping from the space of  $r \times n$  matrices into the space of  $n \times n$  matrices.

Another useful fact in evaluating the derivative  $\partial J / \partial \mathbf{K}$  can be found in Bellman [ 3 ]. That is, if

$$\mathbf{F}(\mathbf{X}) = \exp [(\mathbf{A} + \mathbf{B}\mathbf{X})t] = e^{[(\mathbf{A} + \mathbf{B}\mathbf{X})t]} \quad (7.22)$$

Then, to the first order in  $\varepsilon$  :

$$\begin{aligned} & \mathbf{F}(\mathbf{X} + \varepsilon \Delta \mathbf{X}) = \\ & \exp [(\mathbf{A} + \mathbf{B}\mathbf{X})t] + \varepsilon \int_0^t \exp[(\mathbf{A} + \mathbf{B}\mathbf{X})(t - \sigma)] \mathbf{B} \Delta \mathbf{X} \exp[(\mathbf{A} + \mathbf{B}\mathbf{X}) \sigma] d\sigma \end{aligned} \quad (7.23)$$

The argument used to prove (7.23) can be used to prove that, to the first order in  $\varepsilon$  ,

$$\begin{aligned} & \mathbf{F}(\mathbf{X} + \varepsilon \Delta \mathbf{X}) = \exp [(\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C})t] \\ & - \varepsilon \int_0^t \exp[(\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C})(t - \sigma)] \mathbf{B} \Delta \mathbf{K} \mathbf{C} \exp[(\mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C}) \sigma] d\sigma \end{aligned} \quad (7.24)$$



The calculation of  $\partial J / \partial K$  can proceed as follows. Starting with the performance criterion for the standard output feedback gains

$$J = \mathbf{x}(0)' \left[ \frac{1}{2} \int_0^\infty [\Phi(t, 0)' (\mathbf{Q} + \mathbf{C}' \mathbf{K}' \mathbf{R} \mathbf{K} \mathbf{C}) \Phi(t, 0)] dt \right] \mathbf{x}(0) \quad (6.10)$$

or

$$J(\mathbf{K}) =$$

$$\frac{1}{2} \text{tr} \left[ \int_0^\infty \exp [(\mathbf{A} - \mathbf{B} \mathbf{K} \mathbf{C})' t] (\mathbf{Q} + \mathbf{C}' \mathbf{K}' \mathbf{R} \mathbf{K} \mathbf{C}) \exp ( [\mathbf{A} - \mathbf{B} \mathbf{K} \mathbf{C}] t ) dt \right] \quad (7.25)$$

$$J(\mathbf{K} + \varepsilon \Delta \mathbf{K}) = \frac{1}{2} \text{tr} \left\{ \int_0^\infty \exp [(\mathbf{A}_c - \varepsilon \mathbf{B} \Delta \mathbf{K} \mathbf{C})' t] [\mathbf{Q} + \mathbf{C}' (\mathbf{K}' + \varepsilon \Delta \mathbf{K}') \mathbf{R} (\mathbf{K} + \varepsilon \Delta \mathbf{K}) \mathbf{C}] \exp [(\mathbf{A}_c - \varepsilon \mathbf{B} \Delta \mathbf{K} \mathbf{C}) t] dt \right\} \quad (7.26)$$

It is assumed that a  $\mathbf{K}$  which stabilizes  $\mathbf{A}_c$  exists, where

$$\mathbf{A}_c = [\mathbf{A} - \mathbf{B} \mathbf{K} \mathbf{C}] \quad (7.27)$$

Following the manipulations in Levine and Athans [ 41 ], the following equation is obtained:

$$\begin{aligned} \partial J / \partial \mathbf{K} = & \int_0^\infty \{ \mathbf{R} \mathbf{K} \mathbf{C} \exp [\mathbf{A}_c t] \exp [\mathbf{A}_c' t] \mathbf{C}' \} dt - \\ & \int_0^\infty \left\{ \int_0^t \{ \mathbf{B}' \exp [\mathbf{A}_c' (t - \sigma)] (\mathbf{Q} + \mathbf{C}' \mathbf{K}' \mathbf{R} \mathbf{K} \mathbf{C}) \exp [\mathbf{A}_c t] \exp [\mathbf{A}_c' \sigma] \mathbf{C}' \} d\sigma \right\} dt \end{aligned} \quad (7.28)$$

It can be shown [ 41 ] that

$$\begin{aligned} & - \int_0^\infty \left\{ \int_0^t \{ \mathbf{B}' \exp [\mathbf{A}_c' (t - \sigma)] (\mathbf{Q} + \mathbf{C}' \mathbf{K}' \mathbf{R} \mathbf{K} \mathbf{C}) \exp [\mathbf{A}_c t] \exp [\mathbf{A}_c' \sigma] \mathbf{C}' \} d\sigma \right\} dt = \\ & - \int_0^\infty \{ \mathbf{B}' \exp [\mathbf{A}_c' \tau] (\mathbf{Q} + \mathbf{C}' \mathbf{K}' \mathbf{R} \mathbf{K} \mathbf{C}) \exp [\mathbf{A}_c \tau] \} d\tau \cdot \int_0^\infty \{ \exp [\mathbf{A}_c \sigma] \exp [\mathbf{A}_c' \sigma] \mathbf{C}' \} d\sigma \end{aligned} \quad (7.29)$$

And (7.28) can be rewritten as:

$$\begin{aligned} \partial J / \partial \mathbf{K} = & \mathbf{R} \mathbf{K} \mathbf{C} \left\{ \int_0^\infty \exp [\mathbf{A}_c t] \exp [\mathbf{A}_c' t] dt \right\} \mathbf{C}' - \\ & - \mathbf{B}' \left\{ \int_0^\infty \exp [\mathbf{A}_c' \tau] (\mathbf{Q} + \mathbf{C}' \mathbf{K}' \mathbf{R} \mathbf{K} \mathbf{C}) \exp [\mathbf{A}_c \tau] d\tau \right\} \cdot \\ & \cdot \left\{ \int_0^\infty \exp [\mathbf{A}_c \sigma] \exp [\mathbf{A}_c' \sigma] d\sigma \right\} \mathbf{C}' \end{aligned} \quad (7.30)$$

Setting

$$\partial J / \partial \mathbf{K} |_{\mathbf{K}^*} = 0, \quad (7.31)$$

$$\mathbf{L}^* = \int_0^\infty \{ \exp [\mathbf{A}_c^* \sigma] \exp [\mathbf{A}_c^{*'} \sigma] \mathbf{C}' \} d\sigma, \quad (7.32)$$

$$\mathbf{P}^* = \int_0^\infty \{ \exp [\mathbf{A}_c^{*'} \tau] (\mathbf{Q} + \mathbf{C}' \mathbf{K}^{*'} \mathbf{R} \mathbf{K}^*) \exp [\mathbf{A}_c^* \tau] \} d\tau \quad (7.33)$$

And substituting (7.31) and (7.32) in (7.30), results in:

$$\partial J / \partial \mathbf{K} |_{\mathbf{K}^*} = \mathbf{R} \mathbf{K}^* \mathbf{C} \mathbf{L}^* \mathbf{C}' - \mathbf{B}' \mathbf{P}^* \mathbf{L}^* \mathbf{C}' = \mathbf{0} \quad (7.34)$$

$$\mathbf{K}^* \mathbf{C} \mathbf{L}^* \mathbf{C}' = \mathbf{R}^{-1} \mathbf{B}' \mathbf{P}^* \mathbf{L}^* \mathbf{C}' \quad (7.35)$$

And finally,

$$\mathbf{K}^* = \mathbf{R}^{-1} \mathbf{B}' \mathbf{P}^* \mathbf{L}^* \mathbf{C}' [\mathbf{C} \mathbf{L}^* \mathbf{C}']^{-1} \quad (7.36)$$

where

$$\mathbf{A}_c^* = [\mathbf{A} - \mathbf{B} \mathbf{K}^* \mathbf{C}] \quad (7.37)$$

The expressions with the signal (\*) refer to variables containing that  $\mathbf{K}^*$  which minimizes the performance criterion  $J(\mathbf{K})$ .

Assuming that  $\mathbf{P}^*$ ,  $\mathbf{L}^*$ ,  $\mathbf{K}^*$  exist such that  $\mathbf{A}_c^*$  as defined in (7.37) is stable, assuming  $\mathbf{P}^*$ ,  $\mathbf{L}^*$  are solutions of (7.36), (7.32) and (7.33), then  $\mathbf{P}^*$ ,  $\mathbf{L}^*$ ,  $\mathbf{K}^*$  are solutions of the following algebraic equations:

$$\mathbf{P}^* [\mathbf{A} - \mathbf{B} \mathbf{K}^* \mathbf{C}] + [\mathbf{A} - \mathbf{B} \mathbf{K}^* \mathbf{C}]' \mathbf{P}^* + (\mathbf{Q} + \mathbf{C}' \mathbf{K}^{*'} \mathbf{R} \mathbf{K}^* \mathbf{C}) = \mathbf{0} \quad (7.38)$$

$$\mathbf{L}^* [\mathbf{A} - \mathbf{B} \mathbf{K}^* \mathbf{C}]' + [\mathbf{A} - \mathbf{B} \mathbf{K}^* \mathbf{C}] \mathbf{L}^* + \mathbf{I} = \mathbf{0} \quad (7.39)$$

$$\mathbf{K}^* = \mathbf{R}^{-1} \mathbf{B}' \mathbf{P}^* \mathbf{L}^* \mathbf{C}' [\mathbf{C} \mathbf{L}^* \mathbf{C}']^{-1} \quad (7.40)$$

Considering now a linear functional relationship between the gains and a certain operating point parameter,  $p$ , and the initial gains :

$$\mathbf{K}(p) = \mathbf{K}_0 + \sum_{i=1}^q \mathbf{K}_i \mathbf{G}_i(p), \quad p \in \mathcal{R} \quad (7.9)$$

$$\mathbf{G}_i(p) = p_i \mathbf{I}, \quad 1 \leq i \leq q \quad (7.10)$$

$$\mathbf{K}(p) = \mathbf{K}_0 + \sum_{i=1}^q p_i \mathbf{K}_i, \quad p \in \mathcal{R} \quad (7.41)$$

$$\mathbf{A}_c(p) = [\mathbf{A}(p) - \mathbf{B} \mathbf{K}(p) \mathbf{C}] = [\mathbf{A}(p) - \mathbf{B} \mathbf{K}_0 \mathbf{G}(p) \mathbf{C}] \quad (7.42)$$

And also considering that

$$\bar{\mathbf{C}}(p) = \mathbf{G}(p) \mathbf{C} \quad (7.43)$$

$$\mathbf{A}_c(p) = [\mathbf{A}(p) - \mathbf{B} \mathbf{K}_0 \bar{\mathbf{C}}(p)] \quad (7.44)$$

In order to satisfy (7.41), the matrices  $\mathbf{K}_0$  and  $\mathbf{G}(p)$  are rearranged as:

$$\mathbf{K}_0 = [ [\mathbf{K}_I] [\mathbf{K}_0] ] \quad (7.45)$$

$$\mathbf{K}_I = \begin{bmatrix} K_I(1,1) & K_I(1,2) & K_I(1,3) & K_I(1,4) & K_I(1,5) & K_I(1,6) \\ K_I(2,1) & K_I(2,2) & K_I(2,3) & K_I(2,4) & K_I(2,5) & K_I(2,6) \end{bmatrix}$$

$$\mathbf{K}_0 = \begin{bmatrix} K_0(1,1) & K_0(1,2) & K_0(1,3) & K_0(1,4) & K_0(1,5) & K_0(1,6) \\ K_0(2,1) & K_0(2,2) & K_0(2,3) & K_0(2,4) & K_0(2,5) & K_0(2,6) \end{bmatrix}$$

$$G(p) = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.46)$$

$$\bar{\mathbf{C}}(p) = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.47)$$

Hence, the equations (7.38), (7.39) and (7.40), can be extended to the variable-gains design and expressed as shown below, for a certain operating point  $p_j$ ,  $j = 1, 2, \dots, q$  and an initial  $\mathbf{K}_0$ .

$$\begin{aligned} & \mathbf{P}_j [\mathbf{A}(p) - \mathbf{B}\mathbf{K}_0 \bar{\mathbf{C}}(p)] + [\mathbf{A}(p) - \mathbf{B}\mathbf{K}_0 \bar{\mathbf{C}}(p)]' \mathbf{P}_j + \\ & + \left( \mathbf{Q}(p) + \bar{\mathbf{C}}(p)' \mathbf{K}_0' \mathbf{R}(p) \mathbf{K}_0 \bar{\mathbf{C}}(p) \right) = \mathbf{0} \end{aligned} \quad (7.48)$$

$$\mathbf{L}_j [\mathbf{A}(p) - \mathbf{B}\mathbf{K}_0 \bar{\mathbf{C}}(p)]' + [\mathbf{A}(p) - \mathbf{B}\mathbf{K}_0 \bar{\mathbf{C}}(p)]' \mathbf{L}_j = \mathbf{0} \quad (7.49)$$

$$\mathbf{K}^* = [(\mathbf{f}_{w-p}) \mathbf{R}(p)]^{-1} [(\mathbf{f}_{w-p}) \mathbf{B}' \mathbf{P}_j \mathbf{L}_j \bar{\mathbf{C}}(p)'] [\bar{\mathbf{C}}(p) \mathbf{L}_j \bar{\mathbf{C}}(p)']^{-1} \quad (7.50)$$

$$\mathbf{K}^* = \mathbf{H2}_j \cdot [\mathbf{H1}_j]^{-1} \quad (7.51)$$

where

$$\mathbf{H2}_j = (\mathbf{f}_{w-p}) \mathbf{B}' \mathbf{P}_j \mathbf{L}_j \bar{\mathbf{C}}(p)' \quad (7.52)$$

$$\mathbf{H1}_j = (fw\_p) \mathbf{R}(p) [\bar{\mathbf{C}}(p) \mathbf{L}_j \bar{\mathbf{C}}(p)'] \quad (7.53)$$

where  $(fw\_p)$  is the weight on the operation point  $p_i$ , which corresponds to the wind velocity.

Other equation of interest is the cost function associated to the operating point  $p_j$ ,  $j = 1, 2, \dots, q$

$$J_j = \text{trace}[(fw\_p) \mathbf{L}_j] \quad (7.54)$$

It is useful to write down the equations in explicit form as:

$$\mathbf{B} = \begin{bmatrix} \mathbf{M}_s^{-1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_\delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_\delta^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}' \quad (6.2)$$

$\mathbf{A}(p)$  is the state matrix of the open loop system, and function of  $p_j$  alone.

$$\mathbf{Q}(p) = \text{diag} [mB \quad I_\alpha \quad 0 \quad 0 \quad (q\_p \cdot k_h B) \quad k_\alpha \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \quad (7.53)$$

where  $q\_p$  is the weight on the displacement of the heaving mode, and

$$\mathbf{R}(p) = (fr\_p) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (7.55)$$

where  $(fr\_p)$  is the weight on the control signals.

#### 7.4. Program fluxogram

The solution of the optimal control problem to design variable-gains feedback control laws is obtained in an iterative way by selecting an initial stable gain  $\mathbf{K}_0$  solving two Lyapunov-type equations

$$\mathbf{H2} = \sum_1^q (fw\_p) \mathbf{B}' \mathbf{P}_j \mathbf{L}_j \bar{\mathbf{C}}(p)' \quad (7.56)$$

$$\mathbf{H1} = \sum_1^q (fw\_p) \mathbf{R}(p) [\bar{\mathbf{C}}(p) \mathbf{L}_j \bar{\mathbf{C}}(p)'] \quad (7.57)$$

for each  $p_j$ ,  $j = 1, 2, \dots, q$  and determining a new  $\mathbf{K}_0$  after a summation procedure over the  $q$  operating points which results in a better approximation. The variable-gains output feedback algorithm is presented in Halyo et al. [ 17].

Finally,

$$\mathbf{K}_0^{\text{new}} = \mathbf{H2} \cdot [\mathbf{H1}]^{-1} \quad (7.58)$$

and the cost function corresponding to the  $q$  operation points is:

$$J(\mathbf{K}_0) = \sum_1^q \text{trace} [(fw\_p) \mathbf{L}_j] \quad (7.59)$$

The newly found  $\mathbf{K}_0^{\text{new}}$  is better than the initial  $\mathbf{K}_0$ .

The best  $\mathbf{K}_0$  is the one which makes the cost function  $J(\mathbf{K}_0)$  and is obtained in the iterative way by the following steps:

Step 1:

Define the initial values and the tolerance settings.

Define weight laws as functions of  $p$ .

Define a set of  $q$  velocities  $p_j$ ,  $j = 1, 2, \dots, q$ .

Select an initial stable gain  $\mathbf{K}_0$ .

Step 2

Write a loop  $j = 1, 2, \dots, q$

Solve the Lyapunov type equations (7.48) and (7.49) for each  $j$ .

Compute  $\mathbf{H2}_j$  and  $\mathbf{H1}_j$  according to equations (7.52) and (7.53).

Compute  $J_j$  according to equation (7.54).

If  $\mathbf{P}_j$  or  $\mathbf{L}_j$  is non-negative definite go to step 5, else continue.

end

Step 3

Compute  $\mathbf{H2}$  and  $\mathbf{H1}$  according to equations (7.56) and (7.57).

Compute  $J(\mathbf{K}_0)$  according to equation (7.59).

Determine  $\mathbf{K}_0^{\text{new}} = \mathbf{H2} \cdot [\mathbf{H1}]^{-1}$ .

Step 4

Compute the costs  $J(\mathbf{K}_0)$  and  $J(\mathbf{K}_0^{\text{new}})$

If  $J(\mathbf{K}_0) > J(\mathbf{K}_0^{\text{new}})$  go to Step 5, else go to Step 6.

Step 5

Reduce step:  $\alpha_i = \alpha_i / z$ ,  $z > 1$ .

If  $||J(\mathbf{K}_0) - J(\mathbf{K}_0^{\text{new}})|| > dJ_{\text{tol}}$  go to STEP 5, else go to STEP 6.

Step 6

Check convergence criteria: If  $||J_j - J_{j+1}|| > dJ_{\text{tol}}$  go to STEP 2, else stop.

$dJ_{\text{tol}}$  is the tolerance set for the difference between cost functions of two

successive iterations. It may be 10, 1,  $10^{-1}$ , etc, depending on the desired accuracy.

Note that a systematic way to obtain the optimal  $K_0^{\text{opt}}$  starting with a non-optimal matrix which results in positive definite matrices  $P_j$  and  $L_j$  is shown in item 8.3.2.

### 7.5. Application of the variable-gains concept to the aerodynamic control of a deck wings system

Following Wilde and Fujino [ 96 ], the design methodology and algorithm for determining variable gains is applied to the active control of the bridge deck by two control surfaces, as described by equations (7.1) and (7.2).

The parameter vector  $p$  is assumed to be represented solely by the wind velocity  $U$ . Thus, in equation (7.41),

$$K(p) = K_0 + \sum_{i=1}^q p_i K_i, \quad p \in \mathcal{R} \quad (7.41)$$

the variable gain is simplified to:

$$K(U) = K_0 + U K_1, \quad U \in \mathcal{R} \quad (7.60)$$

The operating range is selected as:  $9 \text{ m/s} \leq U \leq 21 \text{ m/s}$

The set of operating points is selected as  $p = p(j)$ ,  $j = 1, 2, 3, 4, 5$ ;  $p(j) = 9, 12, 15, 18, 21$ . The variable-gains procedure offers the possibility to assign different control strategies in different operating regions. Theoretically, the selection of the weighting function  $f(U)$  and weighting matrices  $Q(U)$ ,  $R(U)$  allows placing the closed-loop poles in the desired location. However, the relationship between  $f(U)$ ,  $Q(U)$ ,  $R(U)$  and the location of poles is not explicitly known and choice of weighting factors can be done only by trial and error.

The weighting matrices  $Q(U)$ ,  $R(U)$  and function  $f(U)$  are experimented over the design operating range pursuing the following criteria:

In the low wind range the control action increases the system damping to prevent build up of the vibration amplitudes

In the high wind range the control action is directed to drive the natural frequencies of pitching and heaving modes away to each other to reduce coupling between them.

The weighting function  $f(U)$  was chosen as a third degree polynomial, and is obtained by means of the Matlab command  $\text{poly\_fw}=\text{polyfit}(U\_p, fw\_p, 3)$ , where  $U\_p$  is the set of the operating points  $p(j)$ .  $fw\_p$  is the set of weighting ordinates corresponding to the set of operating points. The number 3 represents a third degree polynomial. The trial  $fw\_p = \text{function of } U\_p$  was selected as  $fw\_p = \{100, 300, 170, 80, 100\}$ .

The matrix  $\mathbf{R}(U)$  associated to the control signal  $\mathbf{u}$  is selected in a way that the weights in both wings are the same, i.e.:  $\mathbf{R}(U) = f_r(U) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . The weighting function  $f_r(U)$  was chosen as a second degree polynomial, and is obtained by means of the Matlab command  $\text{poly\_fr}=\text{polyfit}(U\_p, fr\_p, 2)$ , where  $U\_p$  is the set of operating points  $p(j)$ .  $fr\_p$  is the set of weighting ordinates corresponding to the set of operating points and 2 represents a second degree polynomial. The trial  $fr\_p = \text{function of } U\_p$  was selected as  $fr\_p = \{50, 100, 300, 500, 925\}$ .

The weighting function  $f_q(U)$  ( $k_h B$ ) is assigned to the term of the matrix  $\mathbf{Q}$  which will be multiplied by the displacement term  $h / B$  of the state vector.

$$\mathbf{Q} = \text{diag. matrix} [mB \quad I_\alpha \quad I_\delta \quad I_\delta \quad f_q(U)(k_h B) \quad k_\alpha \quad k_w B_w \quad k_w B_w]$$

The variables are the same as the ones shown in Chapter 6 for the constant gain, and therefore

$$\mathbf{Q} = \text{diag. matrix} [0.0559 \quad 0.00191 \quad 0 \quad 0 \quad (f_q(U) 3.4707) \quad 1.21462 \quad 0 \quad 0]$$

As already mentioned in Chapter 6, another choice for  $\mathbf{Q}$  would be to assign the weight  $f_q(U)$  to the pitching mode exclusively. Then, matrix  $\mathbf{Q}$  would be selected as:

$$\mathbf{Q} = \text{diag. matrix} [0.0559 \quad 0.00191 \quad 0 \quad 0 \quad 3.4707 \quad (f_q(U) 1.21462) \quad 0 \quad 0]$$

The weighting function  $f_q(U)$  was chosen as a third degree polynomial and is obtained by means of the Matlab command  $\text{poly\_qp}=\text{polyfit}(U\_p, q\_p, 3)$ , where  $U\_p$  is the set of operating points  $p(j)$ .  $q\_p$  is the set of weighting ordinates corresponding to the set of operating points and 3 represents a third degree polynomial. The trial  $q\_p = \text{function of } U\_p$  was selected as  $q\_p = \{1, 5, 10, 10, 68\}$ . The plot of all polynomials are shown in Figure 7-2.

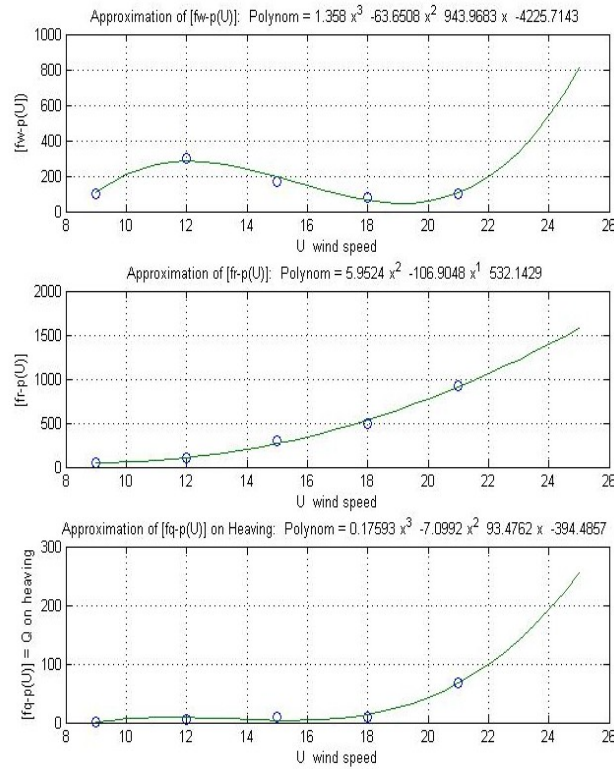


Figure 7-2 - Approximation polylines

Upper curve: Plot of the weighting function  $fw_p(U)$  by means of the Matlab command `poly_fw=polyfit(U_p, fw_p,3)`

Middle curve: Plot of the weighting function  $fr_p(U)$  by means of the Matlab command `poly_fr=polyfit(U_p, fr_p,2)`

Lower curve: Plot of the weighting function  $q_p(U)$  by means of the Matlab command `poly_qp=polyfit(U_p, q_p,3)`

For the first iteration, the initial stable gain was selected as the matrices occurring in Wilde and Fujino [ 96].

$$\mathbf{K}_0 = [ \mathbf{K_I} \quad \mathbf{K_0} ] \quad (7.45)$$

$$\mathbf{K_I} = \begin{bmatrix} 2.279E-4 & 0.0138 & -0.0169 & 0.3123 & 0.2405 & 0.0335 \\ 0.0018 & -0.0052 & -0.0077 & -0.1451 & -0.0452 & -0.0281 \end{bmatrix}$$

$$\mathbf{K_0} = \begin{bmatrix} -0.0122 & -0.0799 & -0.0440 & -2.9654 & -2.5333 & -0.5372 \\ 0.0087 & 0.0021 & 0.3097 & 1.3894 & 0.3455 & 0.4342 \end{bmatrix}$$

After 104 iterations, and taking into account the selected weightings  $\mathbf{R}(U)$ ,  $\mathbf{Q}(U)$  and  $f(U)$ , the following gains result :

$$\mathbf{K_I} = \begin{bmatrix} -0.0030057 & 0.0093008 & -0.025339 & 0.39971 & 0.25147 & 0.024166 \\ 0.038576 & -0.011037 & 0.054865 & -3.9167 & -2.4531 & -0.43483 \end{bmatrix}$$

$$\mathbf{K_0} = \begin{bmatrix} 0.000283 & -0.004126 & 0.0055029 & -0.16693 & -0.044455 & 0.005809 \\ -0.026579 & -0.022619 & 0.13298 & 1.512 & 0.15235 & -0.03071 \end{bmatrix}$$



## 7.6. Frequencies and damping factors of the variable gains control

The variation of damping factors and natural frequencies of the closed-loop system with variable gain control versus wind velocity are shown in Figure 7-3. The two damping ratio curves are recognizable by the fact that at  $U=0$  m/s, the logarithmic decrement  $\delta_h = 0.007$  is bigger than  $\delta_\alpha = 0.006$ . From  $U=0$  to  $U=25$  m/s the curves can be traced. Curves other than pitching and heaving are meaningless. The frequency curves can be recognized in Figure 7-3 if damping factors and frequencies are referenced to a certain velocity. The MATLAB Program "damp" provided eigenvalues, damping factors and frequencies corresponding to the state matrix  $A_c$ , calculated for  $U=15.1$  m/s, chosen at random. The main results in Table 7-1 are depicted in bold types and highlighted in Figure 7-3 for convenience. Another way to recognize the frequency curves is the fact that both frequencies are known at  $t=0$ . Figure 7-3 shows that the system is stable for all wind velocities between 9 and 25 m/s, because no damping factor is negative in this interval. The added aerodynamic damping in the region where wind velocity surpasses the critical one keeps the modal damping ratios of both structural modes above 18%. The same conclusion can be drawn by consulting Figure 7-4 and Figure 7-5. Plots of the root loci are shown below from 9 to 23 m/s in intervals of 1.0 m/s. All step plots are put together in Figure 7-5. This plot shows that the system is stable because its poles are in the left-hand side of the s-plane and therefore the system is stable in this range of velocities.

	Eigenvalue	Damping	Frequency (rad/s)
<b>Heaving</b>	-2.80E+00	1.00E+00	2.80E+00
	<b>-2.91e+000 + 1.54e+001i</b>	<b>1.86E-01</b>	<b>1.57E+01</b>
	-2.91e+000 - 1.54e+001i	1.86E-01	1.57E+01
	-1.64e+001 + 6.12e-001i	9.99E-01	1.64E+01
	-1.64e+001 - 6.12e-001i	9.99E-01	1.64E+01
	-1.77E+01	1.00E+00	1.77E+01
<b>Pitching</b>	-1.77e+001 + 2.94e+001i	5.17E-01	3.43E+01
	-1.77e+001 - 2.94e+001i	5.17E-01	3.43E+01
	<b>-1.83e+001 + 4.49e+001i</b>	<b>3.77E-01</b>	<b>4.85E+01</b>
	-1.83e+001 - 4.49e+001i	3.77E-01	4.85E+01
	-3.55e+001 + 1.58e+001i	9.14E-01	3.88E+01
	-3.55e+001 - 1.58e+001i	9.14E-01	3.88E+01
	-9.48E+01	1.00E+00	9.48E+01
	-9.75E+01	1.00E+00	9.75E+01

Table 7-1 - Table of eigenvalues, damping factors and frequencies of the state matrix for  $U = 15.1$  m/s.

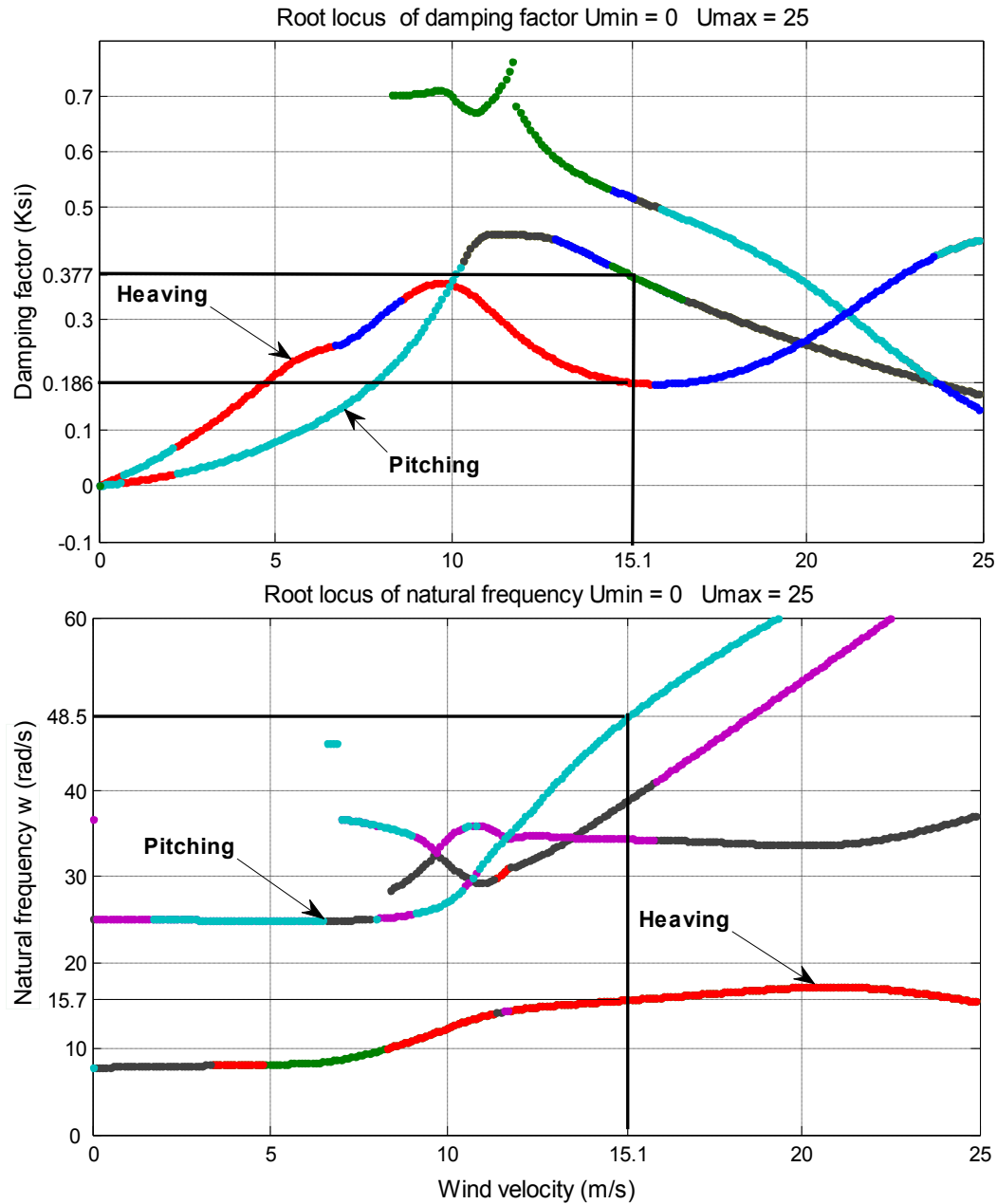


Figure 7-3 - Variation of damping factors and natural frequencies of the closed loop system with variable-gain control.

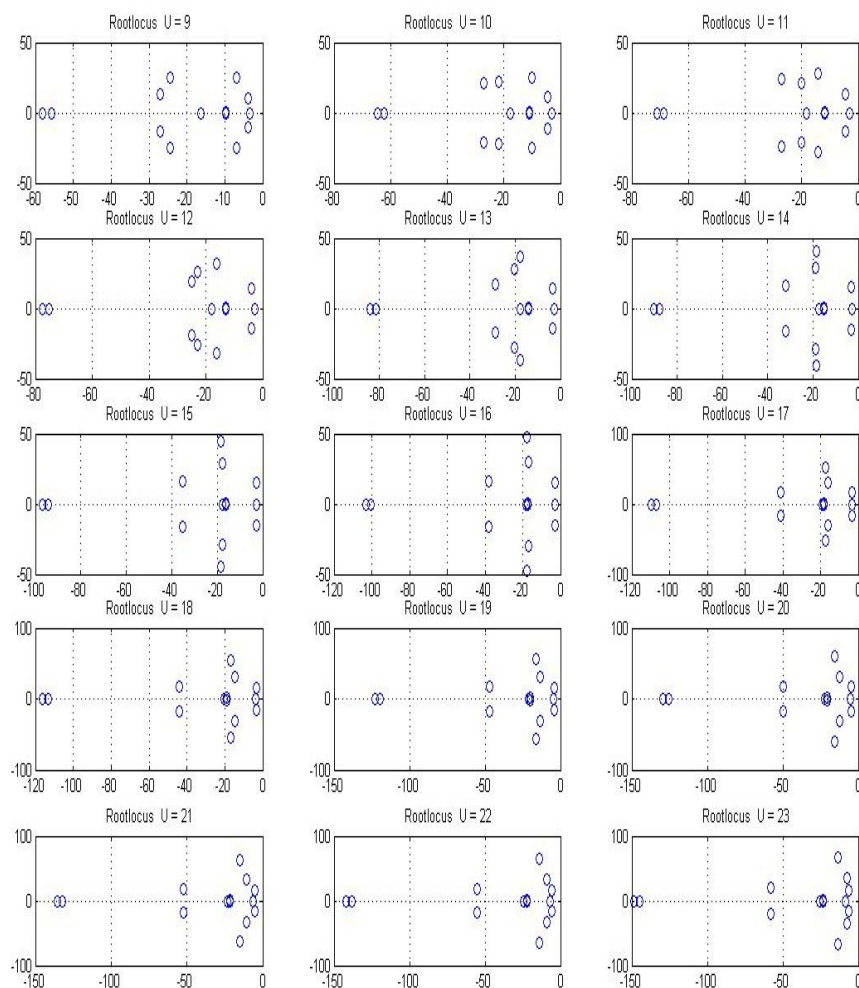


Figure 7-4- Plots of the rootloci from  $U=9$  to  $23$  in steps of  $1$  m/s.

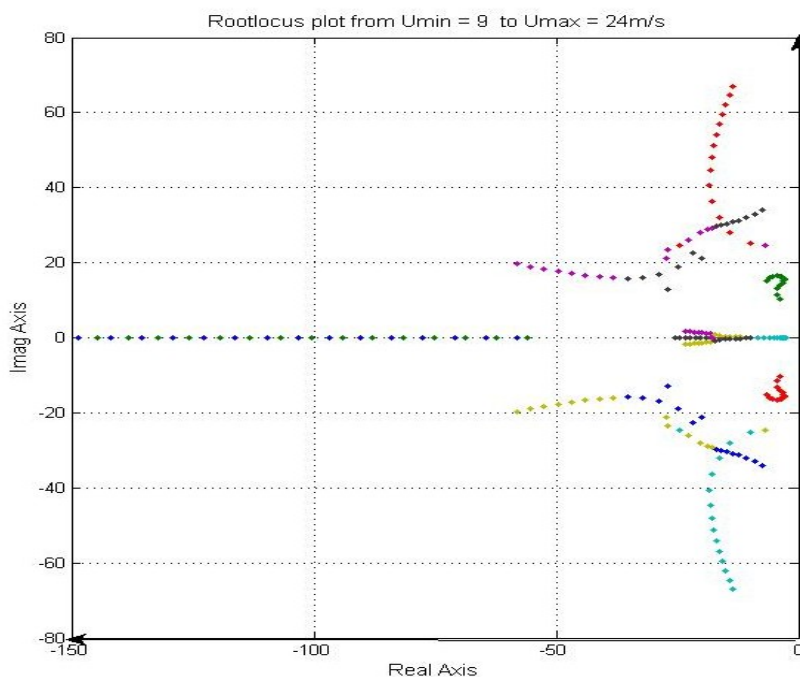


Figure 7-5 - Root loci plot of all steps put together, from  $U=9$  to  $23$  m/s.

### 7.7. Impulse responses of the controlled system for variable gains

The impulse considered is the angular velocity  $d\alpha/dt = (\int F dt)/I_\alpha = 1/I_\alpha$ , which is the same as a rotation  $[\mathbf{M}(2,2)]^{-1}$  applied to the bridge deck:

$$\mathbf{x}_0 = \begin{bmatrix} 0 & \frac{1}{I_\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [0 \ 516.93 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$$

The results of the variable control are shown in two groups, the first depicting displacement amplitudes and the second velocity amplitudes, to facilitate comparison. The amplitude of both displacements and velocities tend to zero within one second after the impulse, but in relative terms the amplitudes are bigger for higher wind velocities.

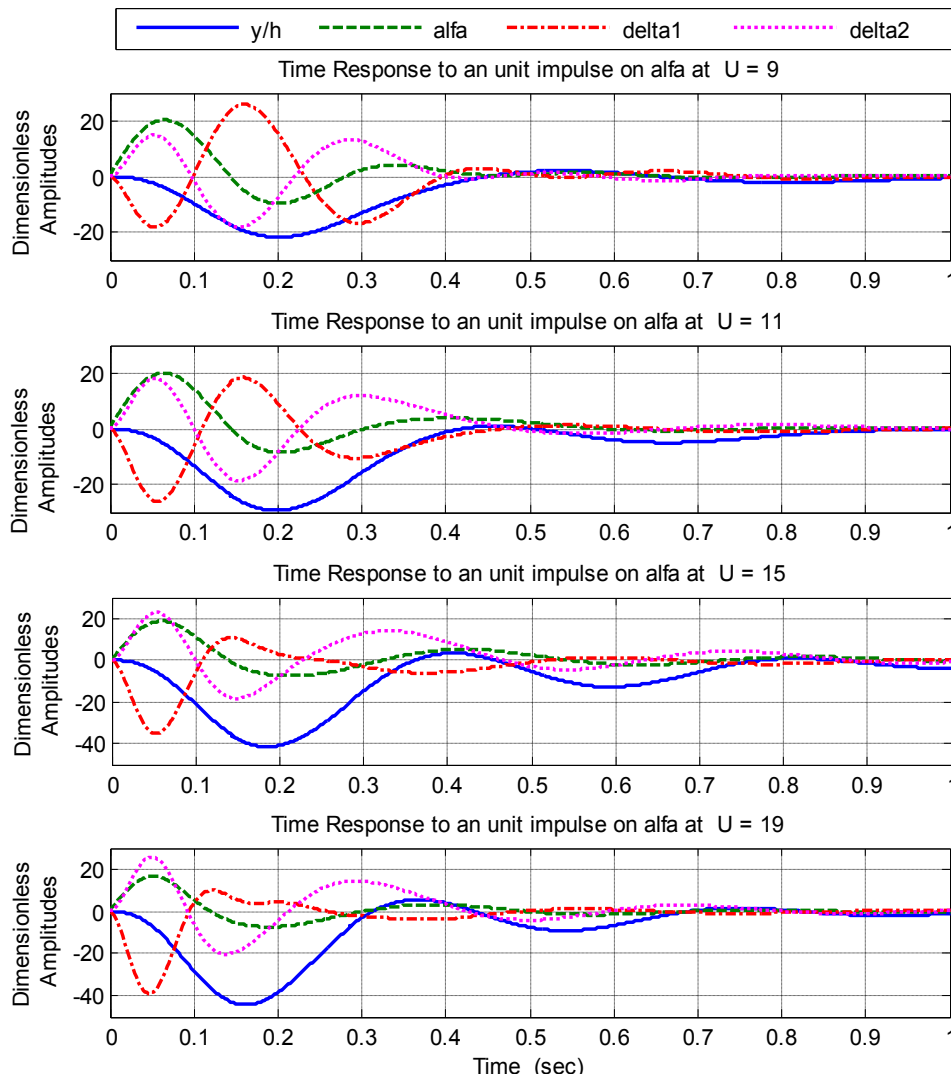


Figure 7-6 - Responses  $h/B$ ,  $\alpha$ ,  $\delta_1$ ,  $\delta_2$  of the closed-loop system to an impulse  $X_0$  at wind velocities ranging from 9 to 21 m/s, for variable gains.

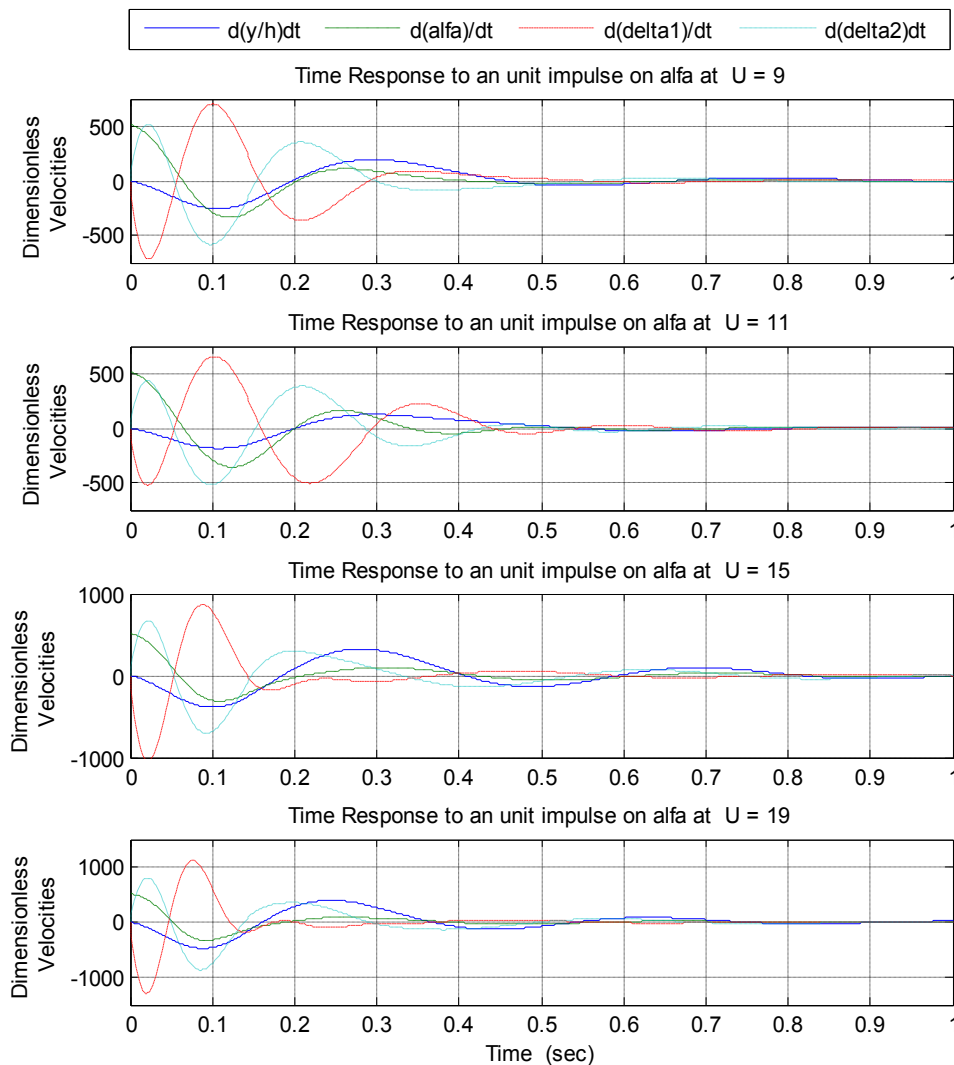


Figure 7-7- Responses  $d(h/B)/dt$ ,  $d\alpha/dt$ ,  $d\delta_1/dt$ ,  $d\delta_2/dt$  of the closed-loop system to an impulse  $X_0$  at wind velocities ranging from 9 to 21 m/s, for variable gains.

The performance of the static and variable gains can be compared by confronting amplitude responses of the deck and wing structure when subjected to a unit impulse while the wind blows at different velocities.

As an example, results obtained for the static gain of 19m/s and wind velocity equal to 9m/s are compared with results obtained for variable gains.

Figure 7-8 shows impulse responses for a static gain of 19m/s at  $U=9$  m/s, while Figure 7-6 and Figure 7-7 show impulse responses  $h/B$ ,  $\alpha$ ,  $\delta_1$ ,  $\delta_2$  and  $d(h/B)/dt$ ,  $d\alpha/dt$ ,  $d\delta_1/dt$ ,  $d\delta_2/dt$  of the closed-loop system for variable gains in the domain  $9 \leq U \leq 21$  m/s.

As the plots show for static gains of 19m/s and  $U=9$  m/s, the leading wing rotates twice than necessary (-1300 to 1000 1/s) in comparison with variable control gains (-750 to 750 1/s) while the amplitude of the leading wing rotations is also bigger (-40 to 20 compared to -20 to 25).

Variation of the amplitudes of wing rotations and corresponding velocities was not possible with constant output feedback gains, causing detrimental effects in the system performance.

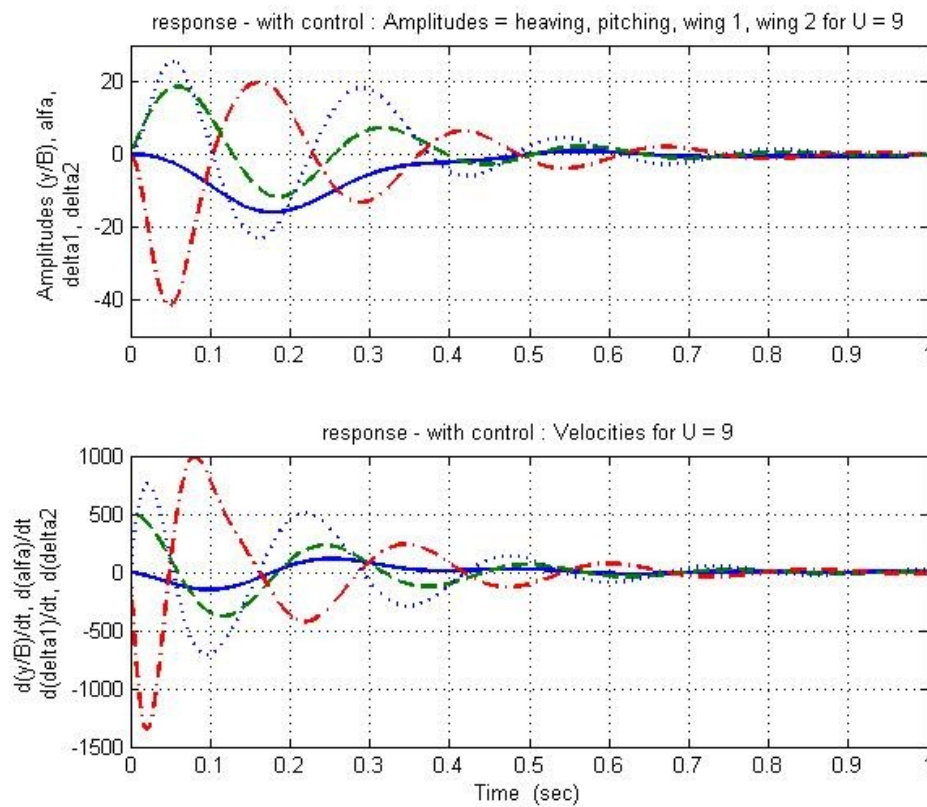


Figure 7-8 - Impulse responses of the closed loop system with static gains for  $U_g = 19$ ms at 9 ms.