Bibliography


A Quick Survey of the Finite Element Method

In this appendix we present the very basics of Finite Element Theory. The literature in the field is vast and we suggest [8] and [5] for those seeking more details on the subject. Let us start by motivating the method in its simplest application.

A.1 Variational Formulation

The classical formulation of the Poisson Equation with Dirichlet boundary conditions is the problem of finding a function \( u \in C^2(\overline{\Omega}) \) satisfying

\[
-\Delta u(x) = g(x), \quad x \in \Omega, \quad u(x) = 0, \quad x \in \partial \Omega.
\]  

(A-1)

Multiplying equation (A-1) above by a function \( v \in C_0^\infty(\Omega) \) and integrating over the domain we obtain

\[
\int_{\Omega} -v(x) \Delta u(x) = \int_{\Omega} g(x)v(x).
\]  

(A-2)

Integrating by parts (i.e., using Green’s first identity), the compact support of \( v \) annihilates the boundary terms and we obtain

\[
\int_{\Omega} \nabla v(x) \cdot \nabla u(x) = \int_{\Omega} g(x)v(x), \quad v \in C_0^\infty(\Omega).
\]  

(A-3)

The equation above makes sense not only for functions \( v \in C_0^\infty(\Omega) \), but for the broader class of functions in \( V = H_0^1(\Omega) \). This is the variational formulation of Poisson’s problem:

\[
\int_{\Omega} \nabla v(x) \cdot \nabla u(x) = \int_{\Omega} g(x)v(x), \quad \forall v \in V.
\]  

(A-4)

In fact, (A-4) is equivalent to the original equation (A-1). We no longer need additional information on the boundary. This is already built-in in the choice of the space \( V = H_0^1(\Omega) \).

That (A-4) has a unique solution \( u \in H_0^1(\Omega) \) is a straightforward consequence of Riesz Representation Theorem applied to the Hilbert space \( V \). The Finite Element Method begins by replacing the infinite-dimensional space \( V \) above by a finite dimensional subspace \( V_h \). The functions of \( V_h \) are called
finite elements. In order to still have functions vanishing in the boundary, we require that each element also have this property. Once in $X_h$ (A-4) reduces to a finite-dimensional linear system. Indeed, if $\{\psi_1, \ldots, \psi_N\}$ is a basis of $X_h$, it is sufficient that (A-4) is satisfied for each of the $\psi_i$. Also expanding $u(x) = \sum_j u_j \psi_j(x)$ we obtain the equivalent set of $N = \dim V$ equations:

$$
\left( \sum_j \int_\Omega \nabla \psi_i(x) \cdot \nabla \psi_j(x) \right) u_j = \int_\Omega g(x) \psi_i(x), \quad i = 1, \ldots, N. \tag{A-5}
$$

It is easy to see that this is an $N \times N$ system

$$
K u = \hat{g} \tag{A-6}
$$

where the stiffness matrix $K$ is given by $K_{ij} = \int_\Omega \nabla \psi_i(x) \cdot \nabla \psi_j(x)$ and $\hat{g}_i = \int_\Omega g(x) \psi_i(x)$.

### A.2 Triangulation and $P_1$ Elements

Regardless of the element space or even the choice of basis, the stiffness matrix is always a positive definite matrix. Indeed, for $u \neq 0$,

$$
\langle Ku, u \rangle = \sum_{i,j} \int_\Omega \nabla \psi_i(x) \cdot \nabla \psi_j(x) u_i u_j = \sum_{i,j} \int_\Omega \nabla (u_i \psi_i(x)) \cdot \nabla (u_j \psi_j(x))
$$

$$
= \int_\Omega \sum_i \nabla (u_i \psi_i(x)) \cdot \sum_j \nabla (u_j \psi_j(x)) = \int_\Omega \nabla u(x) \cdot \nabla u(x) > 0.
$$

On the other hand, the sparsity pattern of $K$ depends on the choice of basis for $X_h$. It is desirable to have the support of the $\psi_i$’s overlapping as little as possible (this is not the only possibility, but it is the one we pursue here; an alternative would be spectral elements).

Let us describe briefly a way of designing the finite element used in this work. We want to take $V_h$ consisting of continuous, piecewise linear functions. To allow for interpolation, we split the (rectangular) domain into triangles, as on the left of Figure A.1. Here, instead, we consider the more regular triangulation given by the figure on the right.

A function $f \in X_h$ can be described by its values on each vertex $\nu_i$ of the triangulation. This space is called in the literature $P_1 P_1$. Keeping in mind sparsity, we choose as a basis of $X_h$ the nodal functions $\psi_i$ defined by

$$
\psi_i(\nu_j) = \delta_{ij}, \quad i, j = 1, \ldots, \dim X_h. \tag{A-7}
$$

It is clear that the support of $\psi_i$, naturally associated with vertex $\nu_i$, will overlap at most that of the nodal functions corresponding to neighboring
vertices. Figure A.2 shows the graph of a typical $\psi_i$ in a regular mesh. Since the

functions are piecewise linear, the integral needed to compute $K$, at least on each triangle, is the integral of a constant. That is a good motivation to range over triangles as we assemble $K$, instead of doing it scanning all possible indices $(i, j)$. That is, denoting by $T$ a generic triangle in the underlying triangulation of $\Omega$, we have

$$K_{ij} = \int_{\Omega} \nabla \psi_i(x) \cdot \nabla \psi_j(x) = \sum_T \int_T \nabla \psi_i(x) \cdot \nabla \psi_j(x).$$

What we do then is start with a zero matrix, range over all triangles computing a local $3 \times 3$ stiffness matrix and add this contribution to the actual $K$.

### A.3 Error Bounds, Convergence

We are interested in having control over the error incurred when we replace our original problem in $V$ by a finite-dimensional version $X_h$. The subscript $h$ in $X_h$ refers to a typical triangle size in a triangulation. A first
remark is that the solution $u_h$ of the linear system (A-6) is the closest we can get to the true solution $u$, measured in the $H^1$-seminorm — this is Céa’s Lemma.

**Lemma 2** (Céa’s Lemma). *For any finite element $v_h$ function we have*

$$|u - u_h|_1 \leq |u - v_h|_1. \quad (A-8)$$

We then study the case where $v_h$ above is the interpolation $\Pi_h u$ of the solution $u$ in the space $X_h$. The idea is similar to a Taylor expansion. If our solution $u$ is regular enough, it is possible to estimate the difference $u - \Pi_h u$ in terms of its higher derivatives and obtain a bound of the form

$$|u - \Pi_h u|_1 \leq Ch. \quad (A-9)$$

Notice that we do not need to know the solution $u$ a priori, only some estimate on its higher order derivatives. Its regularity is also a consequence of domain regularity, which we do have in the rectangular case. The constant in (A-9) is also uniform provided for instance if we keep refining a given triangulation in a way that the triangles do not get too deformed. This provides a check for the computations we performed — halving triangles should halve the error.