3
The Problem

3.1
Some basic geometry

We present an abstract setup for a global Lyapunov-Schmidt decomposition for the nonlinear operator to be defined in the next section.

Let $X$ and $Y$ be Banach spaces which are split as direct sums of horizontal and vertical subspaces, $X = W_X \oplus V_X$ and $Y = W_Y \oplus V_Y$. Here $W_X$ and $W_Y$ are closed subspaces and $k = \dim V_X = \dim V_Y < \infty$. There are unique projections $P_X : X \to W_X$ and $P_Y : Y \to W_Y$ with kernels $V_X$ and $V_Y$ respectively, and complementary projections $Q_X : X \to V_X$ and $Q_Y : Y \to V_Y$ given by $Q_X = I - P_Y$ and $Q_Y = I - P_X$. Sets of the form $x + W_X$ (resp. $y + W_Y$) or $x + V_X$ (resp. $y + V_Y$) will be denoted by horizontal and vertical affine subspaces. The height of a horizontal affine subspace $v + W_X$ (resp. $v + W_Y$) is $v \in V_X$ (resp. $v \in V_Y$). In the definitions below, $F$ is a $C^1$ operator from $X$ to $Y$, not necessarily linear.

Definition 1. A fiber through a point $x \in X$ is the set $F^{-1}(y + V_Y)$, where $y = F(x)$.

That is, a fiber is the inverse image of a vertical line. Fibers were used in [4] and [18] to provide very geometric proofs of results of Ambrosetti-Prodi type. They were also considered in the study of first order periodic differential equations in [14].

Definition 2. Given an arbitrary $v \in V_X$, we define the projected restriction operator $F_v : W_X \to W_Y$ by $F_v(w) = PF(v + w)$.

The working hypothesis on $F$ is very stringent: we assume that $F$ is a $C^1$ map for which $F_v : W_X \to W_Y$ is a diffeomorphism for any $v$. Thus, horizontal affine subspaces are sent injectively by $F$ to their images, which are graphs of functions from $W_Y$ to $V_Y$. For brevity, we then say that $F$ is flat. Clearly, the definition depends on the decompositions of $X$ and $Y$, but we will not mention them in order to simplify notation. There is a global form of operators for which $F_v$ is as above.
Proposition 2. Let $F : X \to Y$ be flat. Then the function

$$\Phi : \tilde{X} = W_Y \oplus V_X \to W_X \oplus V_X, \quad \Phi(z, v) = ((F_v)^{-1}(z), v)$$

is a $C^1$ diffeomorphism such that $\tilde{F} = F \circ \Phi : \tilde{X} \to Y$ becomes $\tilde{F}(z, v) = (z, \phi(z, v))$ for a $C^1$ function $\phi : \tilde{Y} \to V_Y$.

Proof. We denote by $\partial_w F_v, \partial_v F_v$ the partial derivatives of the map $(w, v) \mapsto PF(w, v)$. Analyzing the diagram below,

we see that $\Phi = \xi^{-1}, \phi = Q_v F \xi^{-1}$. The function $\xi$ is one-to-one and onto and its derivative, in block-matrix notation, is

$$\xi'(w, v) = \begin{bmatrix} \partial_w F_v(w) & \partial_v F_v(w) \\ 0 & I \end{bmatrix} = \begin{bmatrix} F_v'(w) & \partial_v F_v(w) \\ 0 & I \end{bmatrix}.$$

Applying the inverse function theorem ($F_v'$ is invertible and thus also $\xi'$), we see that $\xi$ is a global diffeomorphism. \hfill $\Box$

Not only do fibers stretch out indefinitely, but they do so in a smooth way.

Proposition 3. Let $F : X \to Y$ be flat. Then each fiber $\alpha$ is a $C^1$ surface of dimension $k = \dim V_X$, which intersects each horizontal affine subspace exactly once, always transversally. The height map $x \mapsto Q_x x$ is a diffeomorphism between $\alpha$ and $V_X$.

The fact that $\alpha$ and a horizontal affine subspace $x + W_X$ meet transversally at a point $x$ means that $X$ is a direct sum of the tangent space of $\alpha$ at $x$ and $W_X$. According to the proposition, the horizontal subspace parametrizes (bijectively) the set of fibers, and the vertical subspace is a parametrization of each fiber. Also, horizontal affine subspaces are sent injectively by $F$ to their images, which are graphs of functions from $W_Y$ to $V_Y$. On the other hand, fibers are not taken injectively (nor subjectively!) to vertical subspaces necessarily. In particular, the given hypothesis are not enough to imply the properness of the map $F : X \to Y$.

Proof. We use the change of variables $\Phi(z, v) = (F_v^{-1}(z), v)$ defined in the previous proposition. This map, from the domain of $\tilde{F}$ to the domain of $F,$
clearly takes each vertical affine subspace in $\tilde{X}$ to a fiber of $F$ diffeomorphically and so that that heights are preserved. Every statement about fibers now follows from its analogous counterpart for vertical affine subspaces in $\tilde{X}$.

We now consider the effect of flatness on the linearizations.

**Corollary 1.** Let $F : X \to Y$ be flat. The Jacobian $F'(x) : X \to Y$ is a Fredholm operator of index zero at $x \in X$. The restriction of $F'(x)$ to $W_x$ is an isomorphism between $W_x$ and its (closed) range, which is transversal to $V_Y$. If $x_c$ is a critical point of $F$ contained in the fiber $\alpha$, then $\text{Ker}(F'(x_c)) \subset T_{x_c}\alpha$.

**Proof.** By flatness, the derivative $P_yF'(x) : W_x \to W_Y$ is a linear isomorphism, hence a Fredholm operator of index 0. Thus the map $T : W_x \oplus V_x \to W_Y \oplus V_Y$ given by $T(w, v) = (P_yF'(x), 0)$ is also Fredholm of index zero. The same is true for $F'(x) : X \to H^{-1}(\Omega)$, since $F'(x) - T$ is the finite range operator $w + v \mapsto QDF(x)w + F'(x)V$.

Transversality of $F'(x)W_x$ and $V_Y$ follows from the fact that $F'(x) : W_x \to F'(x)W_x$ must be injective, with closed range.

At a critical point $x_c \in \alpha$, use the transversality of the intersection of $\alpha$ and $(x_c + W_x)$ proved in the previous proposition to split $X = W_x \oplus T_{x_c}\alpha$. Now combine $Y = F'(x)W_x \oplus V_Y$ with the fact that $F'(x) : W_x \to F'(x)W_x$ is an isomorphism and $F'(x)T_{x_c}\alpha \subset V_Y$ to conclude that $\text{Ker}(F'(x_c)) \subset T_{x_c}\alpha$.

### 3.2 The Nonlinear Operator

For this section we set $X = H^1_0(\Omega)$, $Y = H^{-1}(\Omega)$. The corresponding projections will be denoted $P_1$, $Q_1$, $P_{-1}$, $Q_{-1}$. The norm used in $H^1_0(\Omega)$ will be $\|u\| = \|u\|_1 = \langle u, u \rangle^{\frac{1}{2}}_1$. Notice that this is equivalent to the full $H^1$ norm, by Friedrich’s inequality. We will use often this result. For the expansion of $u \in H^1_0(\Omega)$ we use the notation $u(x) = \sum u_i \varphi_i(x)$, with $u_i = \langle u, \varphi_i \rangle_{1} / \langle \varphi_i, \varphi_i \rangle_{1}$. We have $H^1_0(\Omega) \simeq H^{-1}(\Omega)$ via $\langle \tilde{u}, \cdot \rangle = \langle u, \cdot \rangle_1$, where we denote with a tilde the functional induced by an element of $H^1_0(\Omega)$. From Hilbert space theory we also know that

$$\|\tilde{u}\|_{-1} = \|u\|_1 \quad \text{and} \quad \tilde{u} \overset{H^{-1}}{\to} \tilde{u} \iff u \overset{H^1_0}{\to} u.$$  

For a $C^1$ function $f$ of bounded derivative, we define $F : X \to Y$ by

$$F(u) = -\Delta u - f(u). \quad (3.1)$$
The Laplacian above is understood as the weak Laplacian, acting as $u \mapsto \langle u, \cdot \rangle$, and $f(u)$ is the functional associated to the $L^2(\Omega)$ function

$$f(u) : z \mapsto \langle f(u), z \rangle_0.$$  

We wish now to split our spaces in direct sums of a certain finite-dimensional space and its orthogonal complement. Denote the eigenvalues of $-\Delta : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$ by $0 < \lambda_1 < \lambda_2 \leq \ldots$ with corresponding eigenvectors $\varphi_i$. The eigenvectors may be taken to be orthogonal functions (in the occasional situation of multiplicity) and orthogonality holds simultaneously for all considered Sobolev spaces.

**Definition 3.** A set $J$ of indices is said to be complete if $j \in J$ whenever $\lambda_i = \lambda_j$ and $i \in J$.

That is, if a complete set includes an index of a multiple eigenvalue, then it contains all indices associated with it.

For $J$ a given finite complete set of indices, define the spaces $V_i = \text{Span}\{\varphi_j, \ j \in J\}$ and $V_{-1} = \text{Span}\{\tilde{\varphi}_j, \ j \in J\}$. Since each space is closed (they are finite-dimensional), we can split the whole spaces as

$$X = W_1 \oplus V_1, \quad Y = W_{-1} \oplus V_{-1}, \quad \text{where} \quad W_1 = V_1^\perp, \quad W_{-1} = V_{-1}^\perp.$$  

Recall that the inner product in $Y$ is $\langle \tilde{u}, \tilde{v} \rangle_{-1} = \langle u, v \rangle_1$.

**Proposition 4.** The correspondence $\tilde{u} \leftrightarrow u$ is also a bijection between $W_1$ and $W_{-1}$. Moreover, $W_1 = \{w \in X : \forall \tilde{v} \in V_{-1}, \langle \tilde{v}, w \rangle = 0\}$ and $\Delta W_1 = W_{-1}$.

**Proof.** This follows directly from $\langle \tilde{v}, w \rangle = \langle v, w \rangle_1 = \langle \tilde{v}, \tilde{w} \rangle_{-1}$ and the fact that $\langle \Delta w, \tilde{v} \rangle_{-1} = \langle w, v \rangle_1$. $\square$

**Definition 4.** Given a complete set $J$, a $C^1$ function $f : \mathbb{R} \rightarrow \mathbb{R}$ interacts with $J$ if

1. the only eigenvalues $\lambda_i$ in the image of $f'$ are labeled by indices in $J$,

2. there are no eigenvalues in the boundary of the image of $f'$.

As in the abstract case, we define the orthogonal projection $P_{-1} : Y \rightarrow W_{-1}$, defined by $\langle P_{-1} \tilde{u}, \tilde{w} \rangle_{-1} = \langle \tilde{u}, \tilde{w} \rangle_{-1} = \langle u, w \rangle_1$ for each $\tilde{w} \in W_{-1}$. For a given $v \in V_1$, we define the restricted projection $F_v : W_1 \rightarrow W_{-1}$ by

$$F_v(w) = P_{-1} F(v + w), \quad (3-2)$$

for each $w \in W_1$. The projections $P_{-1}$ and $F_v$ respect the orthogonality and the eigenvalue structure of the original operator.
Our next goal is to prove that if $f$ interacts with a complete set $J$, then the function $F$ is flat with respect to the decompositions induced by $J$. The first step is the local version of this property.

**Proposition 5.** Let $J = \{l + 1 \leq \ldots \leq r - 1\}$ be a complete set and $f : \mathbb{R} \to \mathbb{R}$ a $C^1$ function interacting with $J$, then the derivatives of the restricted projection $F_\nu$ are uniformly bounded below. More precisely, there exists $C > 0$ such that

$$\forall v \in V_i \forall w \in W_i \forall h \in W_i, \quad \|F'_\nu(w)h\|_{-1} \geq C\|h\|_1. \quad (3-3)$$

Also, all derivatives of $F_\nu$ are invertible.

This estimate, for the case $J = \{1\}$, i.e., when the nonlinearity $f$ interacts only with the first eigenvalue $\lambda_1$, has been extensively used ([1], [4]). It is also used in [9] in the case $J = \{1, \ldots, r\}$.

**Proof.** From Proposition 1, each restricted projection $F_\nu : W_i \to W_{-1}$ is $C^1$ with derivative $F'_\nu(w) : W_i \to W_{-1}$ given by $F'_\nu(w)h = -\Delta h - f'(w)h$. Let $\text{Ran} f' = [a, b]$, so that $\lambda_l < a < \lambda_{l+1}$ and $\lambda_{r-1} < b < \lambda_r$. Let $h \in W_i$ be of unit norm and let $\bar{h}^o$ be the functional $\langle \bar{h}^o, \cdot \rangle = \langle h, \cdot \rangle_0$ and $\gamma = (a + b)/2$. Adding and subtracting $\gamma \bar{h}^o$ and setting $u = w + v$ we have

$$\|F'_\nu(w)h\|_{-1} = \|P_{-1}(\Delta h - \gamma \bar{h}^o) - P_{-1}(f'(u)h - \gamma \bar{h}^o)\|_{-1}$$
$$\geq \|P_{-1}(\Delta h - \gamma \bar{h}^o)\|_{-1} - \|P_{-1}(f'(u)h - \gamma \bar{h}^o)\|_{-1}$$
$$\geq \|A\|_{-1} - \|B\|_{-1}. \quad (3-4)$$

In what follows, we will write $z$ for an arbitrary element of $H^1_0(\Omega)$, and $w$ for one in $W_i$. Let us start with a bound for $\|B\|_{-1}$.

$$\|B\|_{-1} = \sup_{|z| = 1} \langle P_{-1}(f'(u)h - \gamma \bar{h}^o), z \rangle = \sup_{|u| = 1} \langle f'(u)h - \gamma \bar{h}^o, w \rangle$$
$$= \sup_{|u| = 1} \langle f'(u) - \gamma \rangle h, w \rangle_0 \leq \|f' - \gamma\|_\infty \sup_{|u| = 1} \langle |h|, |w| \rangle_0.$$

By Cauchy-Schwartz, the supremum above is realized when $|w|$ is a scalar multiple of $|h|$, which is achieved by $w = \rho h$. Since $h$ is assumed unitary, $\rho = 1$ and, defining $c = \|f' - \gamma\|_\infty$,

$$\|B\|_{-1} \leq c \langle |h|, |h| \rangle_0 = c \langle |h|_0^2 \rangle = \sum_{i \in J} c h_i^2 |\varphi_i|^2 = \sum_{i \in J} (c/\lambda_i) h_i^2 |\varphi_i|^2. \quad (3-5)$$

We will use now the decomposition $W_i = W_+ \oplus W_-$. The spaces are given by $W_- = \{u : u = \sum_{i \leq l} u_i \varphi_i\}$, $W_+ = \{u : u = \sum_{i \geq r} u_i \varphi_i\}$ and are orthogonal both in $\langle , \rangle_0$ and $\langle , \rangle_1$. 


To estimate $\|\tilde{A}\|_{-1}$, start with

$$
\|\tilde{A}\|_{-1} = \sup_{|z|=1} \langle P_{-1}(-\Delta h - \gamma \tilde{h}^0), z \rangle = \sup_{|w|=1} \langle -\Delta h - \gamma \tilde{h}^0, w \rangle
$$

$$
= \sup_{|w|=1} (\langle h, w \rangle_1 - \gamma \langle h, w \rangle_0).
$$

Now we choose $w = h_+ - h_-$ above, noting that it also has unit norm.

$$
\|\tilde{A}\|_{-1} \geq \langle h, h_+ - h_- \rangle_1 - \gamma \langle h, h_+ - h_- \rangle_0
$$

$$
= (|h_+|^2 - \gamma |h_+|^2_0) + (\gamma |h_-|^2_0 - |h_-|^2)
$$

$$
= \sum_{i \geq r} h_i^2(|\varphi_i|^2_1 - \gamma |\varphi_i|^2_0) + \sum_{i \leq \ell} h_i^2(\gamma |\varphi_i|^2_0 - |\varphi_i|^2_1)
$$

$$
= \sum_{i \geq r} (1 - \gamma/\lambda_i) h_i^2 |\varphi_i|^2_1 + \sum_{i \leq \ell} (\gamma/\lambda_i - 1) h_i^2 |\varphi_i|^2_1.
$$

That the coefficients above are all positive follows from the completeness of the set $J$. We have then

$$
\|\tilde{A}\|_{-1} \geq \sum_{i \notin J} |1 - \gamma/\lambda_i| h_i^2 |\varphi_i|^2_1 = \sum_{i \notin J} (C_i/\lambda_i) h_i^2 |\varphi_i|^2_1. \tag{3-6}
$$

Combining equations (3-4), (3-5) and (3-6) we get

$$
\|F'_v(w)h\|_{-1} \geq \sum_{i \notin J} (C_i - c)/\lambda_i h^2_i |\varphi_i|^2_1
$$

$$
\geq \left( \inf_{i \notin J} (C_i - c)/\lambda_i \right) \sum_{i \notin J} h^2_i |\varphi_i|^2_1
$$

$$
= \left( \inf_{i \notin J} (C_i - c)/\lambda_i \right) |h|^2_1
$$

$$
= C|h|^2_1 = C,
$$

which establishes (3-3), since $C \geq \min\{1 - b/\lambda_{r+1}, a/\lambda_{l-1} - 1\} > 0$. In particular, the derivative of $F'_v(w)$ is always injective. To prove invertibility, we write

$$
F'_v(w)h = P_{-1} \circ F'(v + w) \circ \iota_h,
$$

where $\iota$ denotes the inclusion from $W_1$ into $H^1_0(\Omega)$ and notice that the composition of these three Fredholm operators is also Fredholm, with index given by the sum of the individual indices, namely, zero.

The following result is a global inversion theorem, first obtained by Hadamard in the finite-dimensional case [3].
Lemma 1. Let $\Phi : X \to Y$ be a $C^1$ map between Banach spaces $X$ and $Y$ such that $\Phi'(u)$ is invertible for each $u$. Suppose there exists $C > 0$ such that
\[ \forall u, h, \quad \|\Phi'(u)h\| \geq C\|h\|. \] (3-7)
Then $\Phi$ is a global $C^1$-diffeomorphism.

Theorem 1. Let $J$ be a complete set and $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function interacting with $J$. Then each restricted projection $F_v : W_1 \to W_{-1}$ is a $C^1$ diffeomorphism. Thus $F : H^1_0(\Omega) \to H^{-1}(\Omega)$ is flat.

Proof. Simply combine Proposition 5 and the lemma above.

In the case where $f$ does not interact with the spectrum, the full operator is a diffeomorphism. This result was fully obtained by [11], after an initial version of it by [12], and follows from a simple adaptation of the proof of the above theorem.

When $f$ interacts with a set $J$ containing a single element $j$, then only the $j$-th eigenvalue of the Jacobian $F'$ may become zero.

Proposition 6. If $f \in C^1$, $f'(\mathbb{R}) = [a, b]$, with $\lambda_{k-1} < a < b < \lambda_{k+1}$ and $u_c$ is a critical point of $F$, then the only zero eigenvalue of $F'(u_c)$ is the $k$-th one. In particular, it is simple.

An analogous result holds for a general complete set $J$: the only zero eigenvalues of $F'$ are labelled by indices in $J$.

Proof. By the Fredholm property, we must have a nonzero $\xi$ with $F'(u_c)\xi = -\Delta \xi - f'(u_c)\xi = 0$. In other words, 1 is an eigenvalue of the generalized problem $-\Delta u = \mu f'(u_c)u$, which we write as $\mu_j(f'(u_c)) = 1$ for some $j$. An application of a comparison theorem yields then
\[ \lambda_{k-1} < f'(u_c) < \lambda_{k+1} \Rightarrow \mu_j(\lambda_{k-1}) > \mu_j(f'(u_c)) > \mu_j(\lambda_{k+1}), \] (3-8)
or, since $\mu_j(\lambda) = \lambda_j/\lambda$ for constant $\lambda$,
\[ \lambda_j/\lambda_{k-1} > 1 > \lambda_j/\lambda_{k+1} \Rightarrow \lambda_{k-1} < \lambda_j < \lambda_{k+1} \Rightarrow j = k. \] (3-9)

The bound in Proposition 5 allows to make precise the idea that fibers are uniformly steep and images under $F$ of horizontal affine subspaces are uniformly flat.

Proposition 7. Let $J$ be a complete set of indices with $|J| = k$ and $f : \mathbb{R} \to \mathbb{R}$ a $C^1$ function interacting with $J$. Let $u(t) = w(t) + \sum_{j \in J} t_j \phi_j$ be a
parametrization of a fiber $\alpha$, where $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$ and $w(t) \in W_i$. Then there exists a positive constant $C$, independent of $t$, such that

$$\|\nabla_t w(t)\|_1 \leq C \sum_{j \in J} \|\varphi_j\|_1.$$ 

In particular, there exist positive constants $A$, $B$, independent of $t$, such that

$$\|w(t)\|_1 \leq A + B\|t\|.$$ 

Let $W_u \subset H^{-1}(\Omega)$ be the image under $F$ of an horizontal affine subspace $u + W_1$, passing by $u \in H_0^1(\Omega)$. Then the angle between a vector in the tangent space $T_{F(u)}W_u$ at a point $F(u) \in W_u$ and its orthogonal projection in $W_{-1}$ is uniformly bounded above by a constant less than $\pi/2$ for all $u \in H_0^1(\Omega)$.

Proof. Fibers are inverses under $F$ of vertical affine subspaces in $H^{-1}(\Omega)$. Thus $PF(u(t)) = \text{const.}$ and, taking derivatives,

$$(PF)'(u(t)) \partial_j u(t) = PF'(u(t)) \partial_j u(t) = 0.$$ (3-10)

Write $u(t) = w(t) + v(t)$ and expand $v(t) = \sum_{j \in J} t_j \varphi_j$, so that $\partial_j u(t) = \partial_j w(t) + \varphi_j$. For $h \in W_1$, we have $PF'(u(t))h = F'_v(w(t))h$ and thus, setting $h = \partial_j w(t)$,

$$F'_v(w(t))\partial_j w(t) = PF'(u(t))\partial_j w(t) = -PF'(u(t))\varphi_j.$$ 

Using first the lower bound (3-3) and then the boundedness of $F'$,

$$C_1 \|\partial_j w(t)\|_1 \leq \|F'_v(w(t))\partial_j w(t)\|_{-1} = \|PF'(u(t))\varphi_j\|_{-1} \leq C_2 \|\varphi_j\|_1,$$

for some positive constant $C_2$. Thus $\|\nabla_t w(t)\|_1 \leq C \sum_{j \in J} \|\varphi_j\|_1$. A bound of the form $\|w(t)\|_1 \leq A + B\|t\|$ is now immediate.

To see that the tangent space $T_{F(u)}W_u$ is bounded away from the vertical subspace, consider the sequence of simple estimates

$$C_1 \|h\|_1 \leq \|PF'(u)h\|_{-1} \leq \|F'(u)h\|_{-1} \leq C_3 \|h\|_1.$$ 

The cosine between a vector $F'(u)h \in T_{F(u)}W_u$ and the horizontal subspace $W_{-1}$ is given by the quotient $\|PF'(u)h\|_{-1}/\|F'(u)h\|_{-1}$, which is bounded from below by $C_1/C_3$. 

The result may be interpreted as a source of stability for the numerics described in the next sections. We indicate a first application immediately. From Theorem 1, the function $F : V_1 \oplus W_1 \to V_{-1} \oplus W_{-1}$ admits a global
Lyapunov-Schmidt decomposition, where $V_i$ is generated by the eigenvectors $\varphi_j, j \in J$.

When performing numerics, however, we do not work with $\varphi_j$ — indeed, a general domain $\Omega$ does not allow for a formula for the eigenvectors. Even when this happens, as for rectangles, we must still consider the fact that the computations are performed on a finite dimensional subspace. In our case (see Section 4.2), we are using finite elements of the standard type $P_1$, generating an approximation $X_h$ to the domain $H^1_0(\Omega)$ and counter-domain $H^{-1}(\Omega)$. Since $\varphi_j \notin X_h$, we have to consider approximations $\varphi_j^h \in X_h$.

An $\epsilon$-tilted Lyapunov-Schmidt decomposition of $F$ is a pair of splittings $F : \tilde{V}_X \oplus \tilde{W}_X \to \tilde{V}_Y \oplus \tilde{W}_Y$, for which $F$ admits a global Lyapunov-Schmidt decomposition and the four subspaces $\tilde{V}_X$, $\tilde{W}_X$, $\tilde{V}_Y$ and $\tilde{W}_Y$ are $\epsilon$-close to their untilted counterparts. Here, one may take the distance between two subspaces as the maximal angle between them.

**Corollary 2.** For $\epsilon$ sufficiently small and subspaces $\tilde{V}_X$, $\tilde{W}_X$, $\tilde{V}_Y$ and $\tilde{W}_Y$ $\epsilon$-close to their untilted counterparts, the splittings $F : \tilde{V}_X \oplus \tilde{W}_X \to \tilde{V}_Y \oplus \tilde{W}_Y$ induce a tilted Lyapunov-Schmidt decomposition of $F$.

**Proof.** This is an immediate consequence of the above proposition. $\square$

The results in this section considered the nonlinear operator $F$ as acting between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$. Analogous results (stable global Lyapunov-Schmidt decomposition, boundedness and coercivity estimates, uniform flatness and steepness) also hold for $F : H^2_0 \to L^2$. 