Pontifícia Universidade Católica do Rio de Janeiro

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Essays on Asset Allocation Optimization Problems under Uncertainty

TESE DE DOUTORADO

Thesis presented to the Programa de Pós-Graduação em Engenharia Elétrica of the Departamento de Engenharia Elétrica, PUC-Rio as partial fulfillment of the requirements for the degree of Doctor em Engenharia Elétrica.

Advisor: Prof. Cristiano Augusto Coelho Fernandes
Co-Adviser: Prof. Alexandre Street Aguiar

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Abstract

Fernandes, Betina Dodsworth Martins Froment; Fernandes, Cristiano Augusto Coelho (Advisor); Aguiar, Alexandre Street (Co-advisor). **Essays on Asset Allocation Optimization Problems Under Uncertainty**. Rio de Janeiro, 2014. 117p. PhD Thesis – Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

In this thesis we provide two different approaches for determining optimal asset allocation portfolios under uncertainty. We show how uncertainty about expected returns distribution can be incorporated in asset allocation decisions by using the following alternative frameworks: (1) an extension of the Bayesian methodology proposed by Black and Litterman through a dynamic trading strategy built on a learning model based on fundamental analysis; (2) an adaptive dynamic approach, based on robust optimization techniques. This latter approach is presented in two different specifications: an empirical robust loss model and a covariance-based robust loss model based on Bertsimas and Sim approach to model uncertainty sets. To evaluate the importance of the proposed models for distribution uncertainty, the extent of changes in the prior optimal asset allocations of investors who embody uncertainty in their portfolio is examined. The key findings are: (a) it is possible to achieve optimal portfolios less influenced by estimation errors; (b) portfolio strategies of such investors generate statistically higher returns with controlled losses when compared to the classical mean-variance optimized portfolios and selected benchmarks.

Keywords

Distribution uncertainty; dynamic asset allocation; learning algorithms; robust optimization.

Resumo

Fernandes, Betina Dodsworth Martins Froment; Fernandes, Cristiano Augusto Coelho (Orientador); Aguiar, Alexandre Street (Co-orientador). **Ensaios em Problemas de Otimização de Carteiras sob Incerteza**. Rio de Janeiro, 2014. 117p. Tese de Doutorado – Departamento de Engenharia Elétrica, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta tese buscamos fornecer duas diferentes abordagens para a otimização de carteiras de ativos sob incerteza. Demonstramos como a incerteza acerca da distribuição dos retornos esperados pode ser incorporada nas decisões de alocação de ativos, utilizando as seguintes ferramentas: (1) uma extensão da metodologia Bayesiana proposta por Black e Litterman através de uma estratégia de negociação dinâmica construída sobre um modelo de aprendizagem com base na análise fundamentalista, (2) uma abordagem adaptativa baseada em técnicas de otimização robusta. Esta última abordagem é apresentada em duas diferentes especificações: uma modelagem robusta com base em uma análise puramente empírica e uma extensão da modelagem robusta proposta por Bertsimas e Sim em 2004. Para avaliar a importância dos modelos propostos no tratamento da incerteza na distribuição dos retornos examinamos a extensão das mudanças nas carteiras ótimas geradas. As principais conclusões são: (a) é possível obter carteiras ótimas menos influenciadas por erros de estimação, (b) tais carteiras são capazes de gerar retornos estatisticamente superiores com perdas bem controladas, quando comparadas com carteiras ótimas de Markowitz e índices de referência selecionados .

Palavras-chave

Alocação dinâmica de carteiras; algoritmos de aprendizado; otimização robusta.

The mathematical models that we put together act like we have a precise knowledge about the various things that are happening and we don't. So we need to have plans that actually hedge against these various uncertainties.

George Bernard Dantzig

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1
Introduction

1.1 Modern Portfolio Theory and its Extensions

In 1952 Harry Markowitz published the article "Portfolio Selection" which can be considered as the beginning of modern portfolio theory. Portfolio selection is the problem of allocating capital over a number of available assets in order to maximize the return on the investment while minimizing its risk. Since asset future returns are not known at the time of the investment decision, the problem is one of decision-making under uncertainty. Thereby, decisions taken today can only be evaluated at a future time, once the uncertainty regarding the asset returns is revealed.

We can assume that future assets' returns are random variables, which we denote by $R_i, \forall i \in \{1, ..., n\}$.[1] A portfolio is denoted by $x = (x_1, ..., x_n)$ where each x_i corresponds to the fraction of the capital invested in asset i. The values $x_i, \forall i \in \{1, ..., n\}$ are called "portfolio weights", which are the required investment decisions. To represent a portfolio, the weights must satisfy some specified constraints that form a set X of feasible decision vectors. A simple way to define it is by the requirement that the weights are non negative and sum to 1.[2] Formally, we define this set of feasible decision vectors as

$$X = \{(x_1, ..., x_n) \mid \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0,\ \forall i = 1, ..., n\} \tag{1-1}$$

As with the assets' returns, the return of a portfolio is also a random variable expressed by

$$R_p = \sum_{i=1}^{n} x_i R_i \tag{1-2}$$

One important issue arises when the investor must choose among payoff distributions. In attempting to construct a general framework for the decision-making analysis under uncertainty, researchers have sought to establish reasonable criteria

[1] Hereafter, we will assume the notation R_i for arithmetic returns and r_i for geometric (logarithmic) returns.

[2] Which means the investor is fully allocated and that short selling is not allowed.

for the selection of one prospect over another. The first step is to define a preference criterion among random variables and then the investment decisions are taken by solving optimization problems.

Among current research, there are well established approaches that cover mean-risk models and expected utility maximization. In a portfolio context, risk is usually measured by means of a dispersion measure, such as the variance or standard deviation of returns around their expected value.[3] The result of Markowitz portfolio optimization is thus a parabolic efficient frontier, indicating the combinations of assets with the highest expected return given a certain level of risk. Portfolio selection model as proposed by Markowitz (1952) is based on the assumption that investment decisions depend only on the expectation value and covariance structure of asset returns.

However, estimation procedures entail estimation errors which in turn affect the solution of the portfolio selection problem, often resulting in extreme portfolio weights (extreme short selling and large leveraged long positions), unbalanced asset allocations or lack of diversification (see Black and Litterman (1992); Goyal and Welch (1992); Chopra and Ziemba (1993); Bera and Park (2008)). Therefore, although Markowitz framework seems to be very reasonable in theory, it continues to encounter skepticism among practitioners. The lack of confidence of many investment practitioners to mean-variance optimization technology for portfolio selection has motivated the search for new tools to improve it. In this context we present the following works within this thesis developed in the area of portfolio optimization under uncertainty.

1.2 Objective

In real asset allocation problems input data are usually not known exactly. Information used to model a problem is often noisy, incomplete or even incorrect and under these uncertainties, an “optimal" solution can easily be “sub-optimal" or unattainable. In this context, the objective of this thesis is to present different essays in portfolio optimization under uncertainty. We show how uncertainty about expected returns distribution can be incorporated in asset allocation decisions using the following alternative frameworks:

[3] Also known as volatility.

- an extension of the Black and Litterman (BL) model, through a dynamic trading strategy built on a learning model based on fundamental analysis;
- an adaptive dynamic approach, based on robust optimization techniques and polyhedral uncertainty sets.

Accomplishing this allow us to present practical implementations and through this provide a solid intuitive explanation for the workings of the models. In order to limit our scope, we work in discrete time and continually keep our main focus on an intuitive understanding of the asset allocation problem.

1.3 Main Contributions

Our first paper to be presented in Chapter 2 is an extension of the Black-Litterman (BL) model with two major contributions:

- Our first contribution is to present how observed price-earnings ratio and returns can be used to determine a priori estimation of asset expected returns and how this can be integrated into the BL model, regarding investors with different risk profiles.
- Our second contribution is to extend the BL framework to an adaptive optimization model to dynamically update conditional probability distribution of asset returns and mitigate asset allocation instability due to estimation errors.

In the following two papers to be presented in Chapters 3 and 4 respectively, we provide an alternative deterministic methodology to construct uncertainty sets within the framework of robust optimization for linear optimization problems with uncertain parameters. Our approach relies on decision-maker risk tolerance to construct data-driven adaptive polyhedral uncertainty sets from joint dynamics of asset returns. Further, we propose a learning specification algorithm to forecast future returns. This contribution is relevant for it:

- Does not impose any parametric structure to the prediction model. It relies solely on signals extracted from data. The optimal signal is the result of an optimized convex combination of representative signals modeled from adaptive indicators that may vary considering the existing dynamic conditional correlations between assets' returns;

- Considers that optimal portfolio losses are modeled using a robust adaptive approach. Its potential loss is limited by the worst case scenario inside pre-defined adaptive uncertainty sets. We study two cases where the uncertainty sets are defined as:
  - adaptive polyhedral sets described by a list of its vertices, which are set as past assets' returns obtained over moving windows with a length of J-days. This is a purely empirical method to construct an uncertainty set as its information set is limited to past returns;
  - adaptive polyhedral sets described by a historical covariance structure of returns calculated over moving windows. Under this specification, no more than a predetermined number Γ of assets could change simultaneously from a given dynamic estimated nominal value. This method is based on the approach introduced by Bertsimas and Sim (2004) and it is efficient to adjust the robustness of the problem against the level of the conservatism of the solution;
- Incorporates both return predictability and transaction costs, covering all fee structures typically observed on the market to give a more rigorous result for practical purposes. Using financial data from Brazilian asset classes (considering here equities, bonds, currency, commodity and cash), our results suggest that these combined techniques present consistent performance while prevent huge financial losses, especially during crisis periods.

1.4 Outline

The essays in this thesis share the common theme of asset allocation strategies under uncertainty. Each of these examines a distinct, well defined research problem related to the main topic and occupies a separate chapter. In addition, the thesis contains sections on literature review, methodology and case study, which apply to each of these problems within this doctoral study. The remainder of this thesis is organized as follows. Chapter 2 presents an extension of the Black & Litterman model with a dynamic trading strategy built on a learning model based on fundamental analysis, regarding investors with different risk profiles. Chapter 3 provides a robust portfolio optimization problem based on data-driven adaptive polyhedral uncertainty sets. Chapter 4 provides a robust portfolio optimization problem based on Bertsimas and Sim approach to model parameter uncertainty. In Chapter 5 we conclude by summarizing the main results of this thesis in more detail. All the works cited in the thesis and other relevant documents are presented in the Bibliography. The Appendix is included at the end of this document.

Throughout this thesis we will use bold-faced capital letters to indicate matrices, bold-faced lowercase letters to indicate vectors and ordinary letters to indicate scalars. The vector ι refers to the vector of all ones, $\mathbf{0}$ is the vector of all zeros, and $\mathbf{I}$ is an identity matrix.

2
Paper 1: A Dynamic Asset Allocation Model based on the Black-Litterman Approach

In this work we propose a dynamic asset allocation strategy based on the Black-Litterman model. We present how observed price-earnings ratio and returns can be used to determine a priori estimation of assets' expected returns and how this method can be integrated into the Black-Litterman model, regarding investors with different risk profiles. The provided approach dynamically updates the conditional probability distribution of asset returns and mitigates asset allocation instability due to estimation errors. We perform a case study to illustrate that the resulting optimal portfolios outperform traditional mean-variance portfolios.

2.1 Introduction

The portfolio selection model as proposed by Markowitz (1952) is based on the assumption that investment decisions depend only on the expectation value and covariance structure of asset returns. In practice, the sample mean and covariance have been used to implement these portfolios (see Elton and Gruber (1973); Jobson and Korkie (1981b); Jones et al. (1985)).[1] However, estimation procedures entail estimation errors which in turn affect the solution of the portfolio selection problem, often resulting in extreme portfolio weights (extreme short selling and large leveraged long positions), unbalanced asset allocations or lack of diversification and poor out-of-sample performance (see Black and Litterman (1992); Goyal and Welch (1992); Chopra and Ziemba (1993); Siege and Woodgate (2007); Bera and Park (2008)).

Evidence also suggests that optimal portfolios tend to amplify large estimation errors in certain directions.[2] Jobson and Korkie (1981a); Michaud (1989); Best

[1] Furthermore, it leaves little or no room for an investor to use his or her own views on the market when choosing portfolio weights, which would mean there is no reason for an investor to analyze the fundamentals of a company before buying its assets (stocks or bonds).

[2] In fact, if the variance of an asset is significantly underestimated, the optimized portfolio will assign a large weight to it. Similarly, a large weight will be assigned if the mean return of an asset appears to be large as a result of being significantly overestimated. Thus, the risk of the estimated optimal portfolio is typically under-predicted and its return is over-predicted (see Kallberg and Ziemba (1981, 1984); Karoui (2013, 2010)).

and Grauer (1991); Chopra and Ziemba (1993); Britten-Jones (1999), among others, argue that hypersensitivity of the optimal weights in the portfolio follows from the error-maximizing result of mean-variance optimization. Such observations indicate that those inputs need to be estimated very accurately.

Several attempts have been made to reduce the impact of estimation errors in the optimal portfolio composition (see Frost and Savarino (1986, 1988); Michaud (1989); Best and Grauer (1991); Chopra and Ziemba (1993); DeMiguel et al. (2009)). Techniques proposed include introducing weight constraints, Bayesian shrinkage and portfolio re-sampling, to name just a few (see Basak et al. (2009); Jorion (1986); Black and Litterman (1992); Michaud (1989)).[3] A more recent study by Lim et al. (2012) also proposes an interesting framework of relative regret to formulate the portfolio problem.[4]

The integration of quantitative asset allocation models with judgment from portfolio managers and analysts was addressed by Fisher Black and Robert Litterman in the early 1990s through a new mean-variance model based on an Bayesian analytical framework, which could provide more intuitive portfolios by computing a better estimate for the assets' expected returns (see Black and Litterman (1992)). They adopted a practitioner's perspective on model building by considering that the model mathematics should be tractable, the inputs should be intuitive to investment managers and the optimized portfolio should reflect investors' views. Their specification considered two different sources of information on assets' expected returns, which are combined in one simple formula. The first source of information is related to subjective views held by investment managers.[5] The second source of information is obtained quantitatively, from the market implied equilibrium returns. The Black-Litterman (BL) model allows the investor to start with a prior belief about expected returns (subjective views) and to update this prior distribution with market empirical data (model-based estimates, such as CAPM-implied equilibrium returns as an approximation[6]).

[3] Also, it is well known that it is more difficult to estimate mean than covariance of asset returns (see Merton (1980)) and also that errors in estimates of mean have a larger impact on portfolio weights than errors in estimates of covariance (see Jagannathan and Ma (2003)). In order to address this latter problem, recent studies have focused on optimization which relies solely on estimates of covariance and therefore are less vulnerable to estimation errors (see Litterman (1998); Pollak (2012)).

[4] Relative regret evaluates a portfolio by comparing its return to a family of benchmarks.

[5] As those views need not be an exact value of the expected return of an asset, but rather an expression of relative expected returns, this formulation is easier for investors to apply.

[6] Since Black and Litterman (1992) first presented their model, the CAPM has been rejected empirically (Fama and French (1992, 1993)) and several asset pricing models, using a multi-factor

Intuitively, as the posterior expected returns are a combination of the prior investor views with the market equilibrium returns, in the absence of subjective views, the investor should stick to the market via equilibrium views. On the other hand, if the investor has strong views on some asset returns, the portfolio should be tilted to reflect these views combined with the market equilibrium portfolio. Because the market view is always considered, it is less likely to run into unstable or corner solutions. In case the investor holds strong views that dominate the market view, the model allows the results to be significantly adjusted towards these views. Computational experience has shown that the portfolios constructed by this method are more stable and better diversified than those constructed from the conventional mean-variance approach (see Bevan and Winkelmann (1998); Herold (2003); Jones et al. (2007); Becker and Gutler (2010)). Consequently, the BL model has been shown to be appealing among practitioners. A host of US investment firms (Goldman Sachs, JP Morgan, Prudential Financial, BlackRock, Zephyr Analytics) as well as international ones (Vinci Partners, Titan Capital) publishes portfolio recommendations for investor allocations based on the BL model.

The BL model, however does have its shortcomings, as noted by Bertsimas et al. (2012). First, it allows investors to specify their views only on assets' expected returns, but not on their volatility or correlation. Second, it is constructed on a mean-variance approach. Later, the work of Artzner et al. (1999); Rockafellar and Uryasev (2000); Alexander and Baptista (2004) present other risk measures, such as Conditional Value-at-Risk (CVaR) that could be more suitable for measuring risk (see also Browne (2000); Simaan (1997)). Subsequent research has tried to address these inadequacies and further advanced the understanding and implementation of BL framework.

Several authors have presented encouraging results when combining equilibrium returns with trading strategies using the BL framework. They provide evidence that the exhibited optimal portfolios could achieve significant outperformances compared to the classic MV models, according to several important risk-adjusted measures[7] (see Fabozzi et al. (2006); Beach and Orlov (2007); Cheung (2013); Jones et al. (2007); Becker and Gutler (2010)). We mention here a few papers and refer the reader to Walters (2010, 2013) and the references therein for a

approach have been proposed to update the investor prior views. Later, Bartholdy and Peare (2005) compare the performance of these two models for estimates of asset returns and conclude that both models provide similar results.

[7] Such as alpha, Sharpe ratio, Treynor ratio, etc.

more complete survey. Herold (2003) describes an approach in which the BL procedure can be employed with qualitative analysts' forecasts. Fabozzi et al. (2006) incorporate a cross sectional momentum strategy[8] to the BL framework, in which they combine this strategy with market equilibrium to rebalance the portfolio on a monthly basis (see also the work of Koijen et al. (2009) for an interesting application of momentum strategies in asset allocation decisions). Cheung (2013) presents a model that explicitly seeks forward-looking factor views and smoothly blends them in the BL framework to deliver robust allocation to securities. Jones et al. (2007) generate the views to be input into the BL model on the basis of a factor model, in which the view confidences are determined from historical return covariance. Shin et al. (2013) propose the joint use of the views of experts with quantitative data for input in the BL model.

In this work we want to quantify the improvement in portfolio performance of an informed investor who learns from fundamental analysis over an investor who only learns from observed market prices. This extension allows us to examine how the use of different neutral models for learning from prices can affect portfolio performance and establish a framework to test the proposed dynamic learning model (based on fundamental analysis) for their informational content about expected returns.

The remainder of this chapter is organized as follows: Section 2.2 provides a step-by-step derivation of the BL model and extends the model to a dynamic framework; Section 2.3 discusses the proposed trading strategy based on the BL model; Section 2.4 presents a case study applied to the Brazilian market; Section 2.5 concludes this chapter and discusses future research.

2.2 Revisiting Black-Litterman Asset Allocation Model

In this section, we revisit the original Black-Litterman model. For the complete study, see Black and Litterman (1992); He and Litterman (1999). One key contribution of the BL asset allocation model is to assume that the asset's expected return is a random variable itself, instead of a given fixed number as is the case in the Markowitz mean-variance model. The specification that follows relies on this

[8] The idea of a momentum strategy is to buy securities that have performed well and sell the securities that have performed poorly, in the hope that this trend will continue.

assumption.[9]

In this paper we will follow the BL approach proposed by Satchell and Scowcroft (2000) and Christodoulakis and Cass (2002), which is consistent with the definition of the Bayes Theorem. Let us assume that there are n assets in the market, which may include equities, bonds, currencies, etc. Unlike in classical statistics in which the means are considered deterministic (though unobserved), in the BL framework the actual mean in unknown and stochastic, although the covariance matrix of returns is considered fixed and well defined. The model applies the "known covariance unknown mean" Bayesian solution to statistical inference to generate the assets' expected returns suitable for use in the mean-variance Markowitz-type portfolio allocation. In essence, the approach hereafter consists of generating one-step-ahead posterior returns on the assets by means of a precision-matrix-weighted combination of investors' prior views of their future returns with the distribution of their implied excess returns obtained by an equilibrium model (or reverse optimization from the historic covariance and the benchmark index portfolio of securities). Considering that $\mathbf{r} \in \mathbb{R}^n$ is the vector of asset returns with an unknown stochastic mean $\boldsymbol{\mu} \in \mathbb{R}^n$ and a well defined covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ (in particular, non-singular), we have

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \tag{2-1}$$

2.2.1 Investor Prior Views

The first step in the BL approach is to model the investor views. The BL model considers views on expectations. In the normal market (2-1), this corresponds to statements on the variable $\boldsymbol{\mu}$. Furthermore, BL focuses on linear views[10] and allows the investor to express both absolute and relative[11] views. In addition, considering (2-1), the investor must assign levels of confidence to each asset view in the form of confidence intervals. Under the normality assumption, this level of confidence is usually expressed as the standard deviation around the expected return of the view.

This specification has two attractive features concerning the investors' views. The first is to allow investors to express relative views which cannot be expressed in traditional mean-variance portfolio optimization and seems easier for investors

[9] As the mean returns are not observable, one can only infer their probability distribution.

[10] That is, linear combinations of the securities expected returns.

[11] Which corresponds to the following statement: "The investor believes the security A will outperform the security B by $\alpha\%$".

to apply. The second is related to the impact the view will have in the optimal portfolio. When assuming different levels of confidence on the views and because views may be incorrect, the result shows that the confidence level must affect the influence of a particular view in the process. In this sequel we present the model to incorporate the investors' views in the portfolio problem.

Let k be the total number of views, where $k \leq n$, $\mathbf{P}$ be a $k \times n$ matrix of view structure parameters, where each row represents a specific view over the n assets and let $\mathbf{q}$ be a k-vector of the corresponding expected excess returns for each view. The views can be expressed by

$$\mathbf{q} = \mathbf{P}\boldsymbol{\mu} + \varepsilon \tag{2-2}$$

where $\mathbf{q} \in \mathbb{R}^k$ is known, $\mathbf{P} \in \mathbb{R}^{k \times n}$ is known, $\boldsymbol{\mu}$ is the (unknown but required) prior vector of expected return estimates, $\boldsymbol{\varepsilon}$ is the unobservable vector of view estimation errors that is normally distributed as follows

$$\boldsymbol{\varepsilon} \sim N(\mathbf{0}, \boldsymbol{\Omega}) \tag{2-3}$$

where $\boldsymbol{\Omega} \in \mathbb{R}^{k \times k}$ is a diagonal covariance matrix of view estimation errors, which, for simplicity, are considered independent across views.[12] Therefore we can parameterize the prior distribution of expected returns as

$$f(\mathbf{P}\boldsymbol{\mu}) \sim N(\mathbf{q}, \boldsymbol{\Omega}) \tag{2-4}$$

2.2.2 Market Equilibrium Returns

The Black-Litterman model basic assumption is that the expected return of a security should be consistent with market equilibrium unless the investor has a specific view on it. The authors thus propose to extract market implied expected returns using the CAPM equilibrium model and define the asset's equilibrium risk premium as π_i. Assuming that all investors share the same view and at that moment there is only one optimal portfolio, this portfolio is the one that contains all assets proportional to their capitalization weights, that is the market portfolio $\mathbf{x_m}$. The equilibrium risk premiums are such that the demand for these assets exactly equals the outstanding supply (Black (1989)). Assuming the validity of CAPM, it follows

[12] The parameters $\boldsymbol{q}$ and $\boldsymbol{\Omega}$ are called the hyper-parameters. They parameterize the prior probability density function and are specified by the investor.

that

$$\mathbb{E}(r_i) - r_{free} = \beta_i(\mathbb{E}(r_m) - r_{free}), \forall i = 1, 2, ..., n \tag{2-5}$$

where $\mathbb{E}(r_i)$, $\mathbb{E}(r_m)$ and r_{free} correspond respectively to the expected return on asset i, the expected return on the market portfolio and the risk-free rate.[13] Moreover we can express β_i as

$$\beta_i = \frac{cov(r_i, r_m)}{\sigma_m^2}, \forall i = 1, 2, ..., n \tag{2-6}$$

where σ_m^2 is the variance of the market portfolio. Let us denote the return on the market portfolio by

$$r_m = \sum_{j=1}^{n} x_{mj} r_j \tag{2-7}$$

where $\mathbf{x_m} = (x_{m1}, x_{m2}, ..., x_{mn})'$ is the percentage of each asset's market capitalization[14] in a universe of n securities. We can express the asset i equilibrium risk premium as $\pi_i = \mathbb{E}(r_i) - r_{free}$ and it becomes

$$\begin{aligned} \pi_i &= \beta_i(\mathbb{E}(r_m) - r_{free}) && (2\text{-}8) \\ &= \frac{cov(r_i, r_m)}{\sigma_m^2}(\mathbb{E}(r_m) - r_{free}) && (2\text{-}9) \\ &= \frac{\mathbb{E}(r_m) - r_{free}}{\sigma_m^2} \sum_{j=1}^{n} cov(r_i, r_j) x_{mj} \end{aligned}$$

which can be expressed in matrix form as[15]

$$\boldsymbol{\pi} = \delta \boldsymbol{\Sigma} \mathbf{x_m} \tag{2-11}$$

Let us define the market implied risk premium vector as $\boldsymbol{\pi} = (\pi_i, ..., \pi_n)'$ and the average global risk aversion parameter as $\delta = \frac{\mathbb{E}(r_m) - r_{free}}{\sigma_m^2}$.[16] We assume that $\mathbf{x_m}$

[13] Where $r_{free} \in \mathbb{R}_+$.

[14] Or benchmark weights.

[15] The expected returns estimated by the market might also be understood as the result of the optimality conditions of the following Markowitz portfolio problem:

$$\max_{\mathbf{x}} \{\mathbf{x}'\boldsymbol{\mu} + (1 - \mathbf{x}'\iota) r_{free} - \frac{\delta}{2} \mathbf{x}'\Sigma\mathbf{x}\} \tag{2-10}$$

assuming that all investors solve this problem for some specific level of risk and that $\mathbf{x}$ corresponds to the market portfolio $\mathbf{x}_m$.

[16] And also known as the market price of risk. This factor (which is a positive scalar) is based on the formulas in Black (1989).

is a fixed amount and express the covariance matrix of asset returns $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$ by

$$\mathbf{\Sigma} = \begin{pmatrix} cov(r_1, r_1) & \cdots & cov(r_1, r_n) \\ \vdots & \ddots & \vdots \\ cov(r_n, r_1) & \cdots & cov(r_n, r_n) \end{pmatrix} \tag{2-12}$$

These conditions collectively form the CAPM (see Sharpe (1964) for proofs of these results).

The expected returns $\boldsymbol{\mu}$ are considered to be random variables themselves (as the true expected returns $\boldsymbol{\mu}$ of the securities are unknown) with a probability distribution centered at the equilibrium returns and variance proportional to the covariance matrix of the returns. They are assumed to be related to $\boldsymbol{\pi}$ as

$$\boldsymbol{\pi} = \boldsymbol{\mu} + \boldsymbol{\epsilon} \tag{2-13}$$

where $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \tau\mathbf{\Sigma})$. Because the market is not necessarily in equilibrium, the assessment $\boldsymbol{\pi}$ can suffer from errors. The parameter τ is used to specify the relation between the distribution of the asset returns and the distribution of the asset return means. The BL model assumes that the variance in the mean of the return is smaller than the variance in the return itself and therefore τ is an scalar between $(0, 1)$, typically close to zero.[17]Given that the market portfolio is on the efficient frontier (as a consequence of the CAPM) an investor will hold a portfolio consisting of the market portfolio and a risk-free instrument earning the risk-free rate.

The likelihood function of the implied returns from the equilibrium model is thus given by

$$f(\boldsymbol{\pi}|\boldsymbol{\mu}) \sim N(\boldsymbol{\mu}, \tau\mathbf{\Sigma}) \tag{2-14}$$

where $\boldsymbol{\mu}$ is the unobservable mean and $\boldsymbol{\pi}$ and $\mathbf{\Sigma}$ are estimated to encompass all the equilibrium information contained in the distribution.

2.2.3 Bayesian Updating Approach

At this point, the BL model applies Bayes theorem to combine the prior distribution and the likelihood function to create a posterior distribution for the asset's

[17] In fact, it seems reasonable that the elements of $\tau\mathbf{\Sigma}$ should be smaller than those of $\mathbf{\Sigma}$ in a market demonstrating some level of semi-strong form of market efficiency (see Fama (1965)). One can think about $\tau\mathbf{\Sigma}$ as her confidence in estimating the equilibrium expected returns, in which a small value for τ implies a high confidence in the equilibrium estimate.

expected returns $\boldsymbol{\mu}$. It is more natural to think of $\boldsymbol{\pi}$ as the input of the quantitative investor because it depends upon the data. That was the reason why we defined its distribution as the likelihood function (or conditional distribution) and the conjugate prior distribution was represented by the investor's particular views.

It is possible to write the posterior expected return distribution, applying Bayes rule, as

$$f(\boldsymbol{\mu}|\boldsymbol{\pi}) = \frac{f(\boldsymbol{\pi}|\boldsymbol{\mu})f(\boldsymbol{\mu})}{f(\boldsymbol{\pi})} \tag{2-15}$$

where $f(\boldsymbol{\pi}|\boldsymbol{\mu})$ is the conditional probability density function (pdf) of the data equilibrium return, upon the investor's common beliefs, $f(\boldsymbol{\mu})$ is known as the prior pdf that expresses the investor's views and $f(\boldsymbol{\pi})$ represents the marginal pdf of equilibrium returns, a constant that will be absorbed into the integrating constant of $f(\boldsymbol{\mu}|\boldsymbol{\pi})$.

By substituting the distributions (2-4) and (2-14) in (2-15) we obtain the posterior distribution

$$f(\boldsymbol{\mu}|\boldsymbol{\pi}) \sim N(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}^{\mu}_{BL}) \tag{2-16}$$

with the following expression for the posterior mean

$$\boldsymbol{\mu}_{BL} = \left[\ (\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\ \right]^{-1} \left[\ (\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{q}\ \right] \tag{2-17}$$

and for the covariance matrix around the mean

$$\boldsymbol{\Sigma}^{\mu}_{BL} = \left[\ (\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\ \right]^{-1} \tag{2-18}$$

A detailed proof is provided in the Appendix.

The posterior covariance matrix is essentially the uncertainty in the posterior mean estimate about the actual mean and not the covariance of the returns itself. To compute the posterior covariance of returns, it is necessary to add [18] the covariance of the estimate about the mean to the variance of the distribution about the estimate as

$$\boldsymbol{\Sigma}_{BL} \equiv \boldsymbol{\Sigma}^{\mu}_{BL} + \boldsymbol{\Sigma} \tag{2-19}$$

where $\boldsymbol{\Sigma}$ is the known covariance of returns and $\boldsymbol{\Sigma}^{\mu}_{BL}$ is the covariance of the posterior distribution about the true mean.

[18] Given that the estimate error of the mean return is independent of the covariance of the returns around the true mean.

Given the mean $\boldsymbol{\mu}_{BL}$ and the covariance matrix $\boldsymbol{\Sigma}_{BL}$, the optimal portfolio can be calculated by a standard mean-variance optimization method. Assuming a risk aversion parameter δ and ι being a n-vector of all ones, the general maximization problem under the no short selling constraint can be written as

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n} \mathbf{x}'\boldsymbol{\mu}_{BL} - \frac{\delta}{2}\mathbf{x}'\boldsymbol{\Sigma}_{BL}\mathbf{x} \\ s.t. \\ \mathbf{x}'\iota = 1 \\ x_i \geq 0, \forall i \end{aligned} \tag{2-20}$$

2.2.4 A Dynamic Framework for BL Model

Our development so far has considered the restrictive assumption that investors have a single-period horizon. However, more realistically, one might consider the case where investors can make decisions dynamically over time.[19] Allowing investors to rebalance their portfolios inter-temporally can be observed as an expansion of their opportunity sets, thus making more interesting portfolio strategies available relative to static frameworks. However, to induce optimality on rebalancing portfolios it is essential that investors regard their investment opportunity set as time-varying and thereby potentially predictable. To analyze the implications of such investment dynamics we propose a framework that allows for a tractable solution and simple empirical implementation of the technology proposed by Black and Litterman (BL). We will thus extend the BL model specification to a time-varying model, both in the investor views specification and the market equilibrium model.

The investor opportunity set is commonly defined to be the riskless rate of return and the investor probability beliefs about future risky asset returns. Most switching strategies currently in practice adopt some deterministic policy to shift allocation from risky assets to riskless assets.[20] In this paper, we model and test a dynamic asset allocation strategy that uses past information about risky assets to determine the direction and extent of asset class switching (among risky and riskless

[19] The dynamic investment problem for individuals has been the subject of a huge volume of financial theory, starting, in its modern form, with Merton (1969, 1971). They solved the theoretical problem of consumption-portfolio choice in a continuous time context where security prices follow diffusion processes.

[20] Ludvik (1994) suggests to increase allocation in safer assets when the accumulated fund is ahead of some "target". Arts and Vigna (2003) propose a criterion which considers actual realizations of returns on risky assets.

assets).

We extend the basic specification outlined by the BL framework to a more general setting of dynamic asset allocation. For that we need to define the conditional distribution of investors' views with dynamic parameters. Let $\mathbf{P}_t$ be the dynamic matrix of views, $\mathbf{\Omega}_t$ the conditional covariance matrix of views and $\mathbf{q}_t$ the dynamic vector of the expected excess returns for each view. To construct the implied equilibrium returns distribution we need also to define $\boldsymbol{\pi}_t$ as the dynamic implied risk premiums, $\mathbf{\Sigma}_t$ as the conditional historical covariance of returns and τ_t as the parameter to scale the conditional covariance of returns. We can illustrate the relation between both sources of information in the following dynamic framework for the updated mean and covariance matrix

$$\boldsymbol{\mu}_{BL,t} = \left[\ (\tau_t\mathbf{\Sigma}_t)^{-1} + \mathbf{P}_t'\mathbf{\Omega}_t^{-1}\mathbf{P}_t\ \right]^{-1}\left[\ (\tau_t\mathbf{\Sigma}_t)^{-1}\boldsymbol{\pi}_t + \mathbf{P}_t'\mathbf{\Omega}_t^{-1}\mathbf{q_t}\ \right]$$

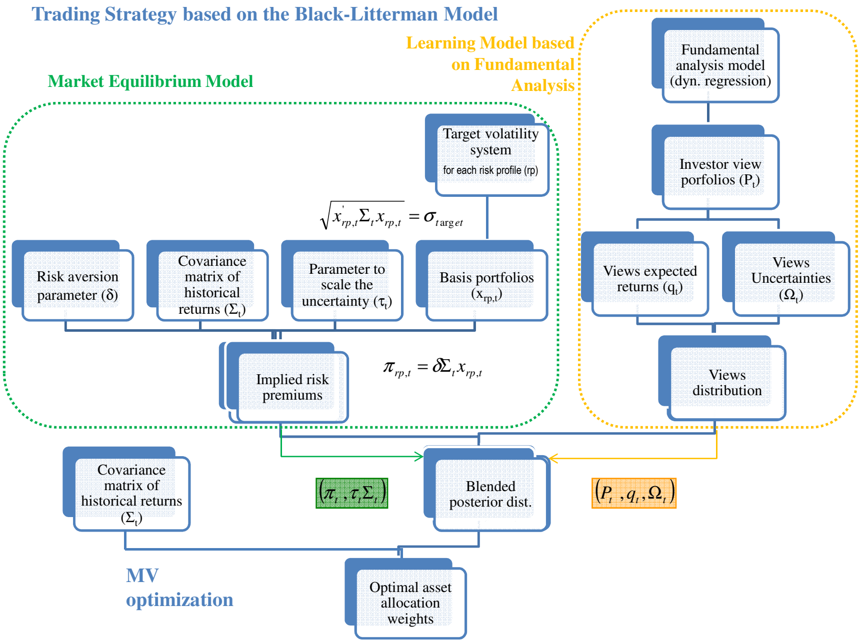
$$\mathbf{\Sigma}_{BL,t} = \left[\ (\tau_t\mathbf{\Sigma}_t)^{-1} + \mathbf{P}_t'\mathbf{\Omega}_t^{-1}\mathbf{P}_t\ \right]^{-1} + \mathbf{\Sigma}_t \tag{2-21}$$

2.3 Trading Strategy based on the Black-Litterman Model (TS-BL)

The proposed trading strategy obtains the optimal asset allocation weights by solving a MV optimization problem. Despite using the same formulation of the MV model, we do not use historical estimates for the asset return distribution. Instead, we use a conditional (dynamically updated) estimate of the covariance matrix and a (blended) posterior distribution for average asset returns, obtained by the combination of estimated views and equilibrium asset returns. We build the views distribution $(\mathbf{P_t}, \mathbf{q_t}, \boldsymbol{\Omega_t})$ by emulating the decision process of fundamental analysts through a (PE-based) dynamic regression model[21] and we extract equilibrium distribution $(\boldsymbol{\pi_t}, \boldsymbol{\Sigma_t}, \tau_t)$ from basis portfolios (for each risk profile) resulting from a target volatility system.

[21] Hereafter let us denote this view model as the "learning model based on fundamental analysis".

Fig. 2.1: TS-BL workflow.



2.3.1 Learning Model based on Fundamental Analysis

One of the most debated questions in recent financial research is whether asset returns or risk premium are predictable.[22] This question is significant for portfolio choice. Economists have different views on whether asset returns are predictable as it is known that given independent identically distributed (iid) returns, investors do not obtain any updated knowledge about the return distribution moments over time. The assumption of iid returns is refuted by empirical literature (see Fama and French (1989); Campbell et al. (1997)) even though there is a relevant ongoing discussion regarding return dynamics (see Cavanagh et al. (1995); Campbell and Cochrane (1999)). Goyal and Welch (2008), among others, argue that the existing empirical models of predicting asset returns do not outperform the simple iid model both in sample and out of sample, and thus, are not useful for investment advice. On the other hand, practitioners and researchers alike have identified several ways to successfully predict future security returns based on historical returns and fundamental data (see Nicholson (1960); Jegadeesh and Titman (1993); Rouwenhorst (1998); Fabozzi et al. (2006); Cochrane (2008)). Following several papers in the

[22] See for example, the July 2008 issue of the Review of Financial Studies.

portfolio choice literature, we will consider that asset returns exhibit some sort of predictability. We assume that there exists a simple environment in which the investor is faced with an investment opportunity set constituted by a risky asset and a risk-free asset. The investor must decide whether to allocate his or her wealth between the risky asset and the risk-free asset. The realization of the risk-free asset return is in general allowed to be stochastic, so it is not necessarily a strictly riskless return.[23] To construct the investor views on the risky asset expected returns (2-2), we assume a dynamic regression model to emulate the decision process of fundamental analysis.[24]

The risky asset expected returns are structured to depend on their past values as well as on historical realizations of other predictive variables. Our approach is to introduce earnings data as an information variable in the dynamic regression model. It seems appropriate to consider earnings data for forecasting returns on equities because earnings are constructed with the objective of helping analysts evaluate a company's fundamental value.[25] Specifically, we decided to use the metric (price-earnings ratio) in our dynamic regression model as this exogenous variable. This metric is based on the implications of the theory of financial markets and the methodology of fundamental analysis.[26]Following our notation for the dynamic regression model, we can describe the investor conditional expected excess returns $(q_{t,i})$ on the risky asset i at time t, based on its past values[27] and price-earnings ratio

[23] This generalization is made in order to mimic the real world where even short-term government bills carry a small degree of risk.

[24] Dynamic regression models have proven to be especially useful for describing the dynamic behavior of economic and financial time series and for forecasting. In addition, forecasts from such models are quite flexible because they can be made conditional on the potential future paths of specified variables in the model.

[25] There is a large amount of literature on the response of securities prices to earnings announcements (see Kormendi and Lipe (1987) for a list of references).

[26] Value investing theory was first derived by teaching classes of Ben Graham and David Dodd at Columbia Business School in 1928 and their 1934 work called Security Analysis (Graham and Dodd (1934)). This theory aims to predict the expected return of a stock by measuring its intrinsic value. They consider that the individual must choose to buy securities whose shares appear undervalued by some form of fundamental analysis, for instance, such securities trade at discounts to book value, have high dividend yields, have low price-earnings multiples or have low price to book ratios. The value investing has proven to be a successful strategy throughout the years (see Fama and French (1992); Barber and Lyon (1997); Dreman and Berry (1995)).

[27] Which means past excess returns.

as

$$er_{t,i} = \phi_{0,t} + \sum_{j=1}^{J} \phi_{j,t} er_{t-j,i} + \sum_{\substack{k_1=1 \\ k_2=1 \\ k_2>k_1}}^{K} \phi_{k_1,t} X_{k1,k2,i} + \varepsilon_t \quad (2\text{-}22)$$

$$q_{t,i} = \mathbb{E}[er_{t,i}]$$

where $er_{t,i}$ stands for the excess return of and asset i at time t and is given by $er_{t,i} = r_{t,i} - r_{free,t}$. We also have as a predictive variable $X_{k1,k2,i}$ which is given by $X_{k1,k2,i} = \frac{PE_{t-k_1,i}}{PE_{t-k_2,i}}$.[28]

To estimate the uncertainty Ω_t associated with the investor view, we assume that conditional variance can be calculated from historical views

$$\Omega_t = Var(\varepsilon_t) \quad (2\text{-}23)$$

And to specify a value for τ_t, the parameter used to scale the investors' uncertainty in their prior estimate of the returns, we will use the intuitive method proposed by He and Litterman (1999) where

$$\tau_t = \frac{\Omega_t}{P_t \Sigma_t P_t'} \quad (2\text{-}24)$$

Unfortunately, the scalar τ_t and the uncertainty in the views Ω_t are the most abstract and difficult parameters to specify in the BL model.[29] The greater the level of confidence in the expressed views, the closer the new return vector will be to the views. If the investor is less confident in the expressed views, the new return vector should be closer to the implied equilibrium return vector.[30]

From the original BL model, the authors only recommend a departure from the equilibrium portfolio if the asset is the subject of a view. And for assets that are the subject of a view, the magnitude of their departure from the equilibrium portfolio is controlled by the ratio of the scalar τ_t to the variance of the error term $\omega_{k,t}$ of the view in question.[31] In fact, the variance of the error term $\omega_{k,t}$ of a view is inversely related to the investor confidence in that particular view. The method proposed

[28] This specification could be extended to a multivariate dynamic regression model.

[29] For a detailed sensitivity analysis on the magnitude of this parameter in the BL model, refer to Fernandes et al. (2012).

[30] The scalar τ_t is more or less inversely proportional to the relative weight given to the implied equilibrium return vector.

[31] Since we consider that those uncertainties are independent across views, the uncertainty matrix

in He and Litterman (1999) calibrates Ω_t given an investor specified confidence level for each view. The investor specifies his or her confidence as a percentage that represents the fraction of the change in returns between 0% confidence and 100% confidence. In fact, there are several different approaches to calibrating the parameter τ_t.[32]

2.3.2 Market Equilibrium Model

We propose an alternative approach to extract market implied expected returns from a proxy of an average portfolio that a typical investor holds, considering his or her risk aversion. Financial services institutions that provide personal financial advice to retail clients are obliged to ensure that the financial products they recommend are suitable to each client's objectives, financial situation and needs. An important part of this assessment is the knowledge of the client's tolerance to risk. Most institutions have developed practices and procedures, which are designed to identify their client's understanding and tolerance for risks. To cite just a few we can mention risk profiling questionnaires, life cycle approach and sensitivity analysis (see White's paper on asset allocation at Barclays[33] and Vanguard investment risk and financial advice[34]).

To design our market equilibrium portfolios, we preserve the CAPM assumption of homogeneous expectations among investors. As such, we assume that in equilibrium, all investors have access to the same information and agree about the risk and expected return of all assets. Let us assume that investors may have several risk profiles and each profile will have a neutral (or basis) asset allocation, which may vary dynamically over time. Let us use this word in the context of Black and Litterman (1992) where the sensible definition of neutral means is the set of ex-

(of variances of the error terms) can be generally defined as

$$\boldsymbol{\Omega} = \begin{pmatrix} \omega_1 & 0 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 & 0 \\ 0 & 0 & \omega_3 & 0 & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & 0 & \omega_k \end{pmatrix} \tag{2-25}$$

where each ω_k is defined as the variance associated to the error term of view k, which should be specified by the investor.

[32] He and Litterman (1999) state they set it equal to 0.05. Satchell and Scowcroft (2000) state many people use a value around 1. Meucci (2010) proposes a formulation of the BL model without this parameter.

[33] Website: http://www.barclayswealth.com.

[34] Website: https://www.vanguard.co.uk.

pected returns that would "clear the market" if all investors shared identical views. We will consider hereafter what happens when we adopt these equilibrium risk premiums[35] as our neutral means when we have no views. Our main assumption is that investors risk profiles will be specified by certain volatility target levels. As so, it is possible to extract the implied allocation as the result of the following dynamic system

$$\sigma_{rp,t} \equiv \sqrt{\mathbf{x}_t' \boldsymbol{\Sigma}_t \mathbf{x}_t} = \sigma_{tg,t} \tag{2-26}$$

where $\sigma_{rp,t}$ corresponds to the portfolio volatility for each risk profile and $\sigma_{tg,t}$ equals the pre-specified volatility target level. Let us assume that $\boldsymbol{\Sigma}_t$ is known at time t. After matching the calculated assets volatility with the portfolio volatility target levels, we arrive at the typical portfolios $\mathbf{x}_{rp,t}$ held by investors, considering their risk profile. By these implied "basis" allocation we can assess the associated implied risk premium $\boldsymbol{\pi}_{rp,t}$ as

$$\boldsymbol{\pi}_{rp,t} = \delta \boldsymbol{\Sigma}_t \mathbf{x}_{rp,t} \tag{2-27}$$

We assume that the average global risk aversion parameter is fixed for all risk profiles. Finally, we have the following expression for the mean and covariance matrix for the dynamic BL model, considering each category risk profile

$$\begin{gathered} \boldsymbol{\mu}_{BL,rp,t} = \left[\ (\tau_t \boldsymbol{\Sigma}_t)^{-1} + \mathbf{P}_t' \boldsymbol{\Omega}_t^{-1} \mathbf{P}_t \ \right]^{-1} \left[\ (\tau_t \boldsymbol{\Sigma}_t)^{-1} \boldsymbol{\pi}_{rp,t} + \mathbf{P}_t' \boldsymbol{\Omega}_t^{-1} \mathbf{q_t} \ \right] \\ \boldsymbol{\Sigma}_{BL,t} = \left[\ (\tau_t \boldsymbol{\Sigma}_t)^{-1} + \mathbf{P}_t' \boldsymbol{\Omega}_t^{-1} \mathbf{P}_t \ \right]^{-1} + \boldsymbol{\Sigma}_t \end{gathered} \tag{2-28}$$

where $\boldsymbol{\Sigma}_t$ is the conditional covariance of returns and $\boldsymbol{\Sigma}_{BL,t}$ is the total conditional covariance of the posterior distribution.

Given the mean $\boldsymbol{\mu}_{BL,rp,t}$ and the covariance matrix $\boldsymbol{\Sigma}_{BL,t}$, the optimal portfolio for each risk profile can be calculated by a standard mean-variance optimization

[35] In the BL framework, the equilibrium concept is useful to provide the investor with a neutral framework which he or she can adjust according to her own views and optimization specification.

method, as described below

$$\begin{aligned}\max_{\mathbf{x}_{rp,t}\in\mathbb{R}^n} \mathbf{x}'_{rp,t}\boldsymbol{\mu}_{BL,rp,t} \\ \text{s.t.} \\ \mathbf{x}'_{rp,t}\boldsymbol{\Sigma}_{BL,t}\mathbf{x}_{rp,t} \leq \sigma^2_{tg,t} \\ \mathbf{x'_{rp,t}}\iota = 1 \\ x_{i,rp,t} \geq 0, \forall i\end{aligned} \tag{2-29}$$

We decided not to allow short selling in our case study because we define it as a typical asset class decision making process where individuals construct a portfolio using one risky asset and one riskless asset.

2.4 Case Study and Application

In this numerical example, we consider daily allocation decisions between the risky asset and the risk-free asset. We consider the Bovespa index[36] as our risky asset and the DI spot rate[37] as our riskless asset. Our dataset comprises daily observations for the Bovespa index and its consolidated price-earnings ratios[38], which were collected from Bloomberg information system.[39] The DI spot rate data were collected from Cetip system.[40] Our full database ranges from June 30, 2004 up to December 28, 2012, comprising 2,102 observations.

We estimate the dynamic regression model for (2-22) based on the following specification:

$$q_{t,i} = \phi_{0,t} + \phi_{1,t}er_{t-1,i} + \phi_{2,t}er_{t-2,i} + \phi_{3,t}er_{t-3,i} + \phi_{4,t}\frac{PE_{t-1,i}}{PE_{t-4,i}} \tag{2-30}$$

where $er_{t-j,i}$ stands for the past excess return of asset i and is given by $er_{t-j,i} =$

[36] The Bovespa index is a gross total return index weighted by traded volume and comprise the most liquid stocks traded on the São Paulo Stock Exchange.

[37] The DI spot rate is the overnight interbank deposit rate. It corresponds to the interest rate at which a depository institution lends immediately available funds (balances within the central bank) to another depository institution overnight. It provides an efficient method whereby banks can access short-term financing from central bank depositories.

[38] The Bovespa index consolidated price-earning ratio is a weighted average of the underlying companies price-earnings.

[39] Website: http://www.bloomberg.com.

[40] Website: http://www.cetip.com.br.

$r_{t-j,i} - r_{free,t}$. The parameters $(\phi_{0,t}, \phi_{1,t}, \phi_{2,t}, \phi_{3,t}, \phi_{4,t})$ are estimated[41] using past data and used to calculate q_t and Ω_t one step ahead. The values for past excess returns $(er_{i,t-j}, \forall j = 1, 2, 3)$ are known, as well as the values for the past consolidated price-earnings ratio for the equity index $(PE_{i,t-j}$, where $j = 1$ and $j = 4)$.[42]

To extract the market implied expected returns for the different risk profile[43] portfolios (defined as conservative, moderate, moderately aggressive, aggressive) we reference this empirical evaluation on the average level for the Brazilian investment fund industry.[44] We assume that $\sigma_{tg,t}$ equals 1% per annum for conservative investors, 4% per annum for moderate investors, 8% per annum for moderately aggressive investors and 12% per annum for aggressive investors.

Figure 2.2 depicts the out-of-sample results for the proposed trading strategy based on the BL model (TS-BL). The graphs on the left side plot the cumulative performance in percentage terms over time for the TS-BL model and compare the results with MV optimal portfolios and a benchmark strategy (buy-and-hold strategy in the DI spot rate). The graphs on the right side plot the risk-return relationship between the TS-BL and the MV optimal portfolios. We plot the average annualized return on the vertical axis versus the annualized volatility for each risk profile on the horizontal axis. In both cases, we plot the graphs to all risk profiles (conservative, moderate, moderately aggressive and aggressive).

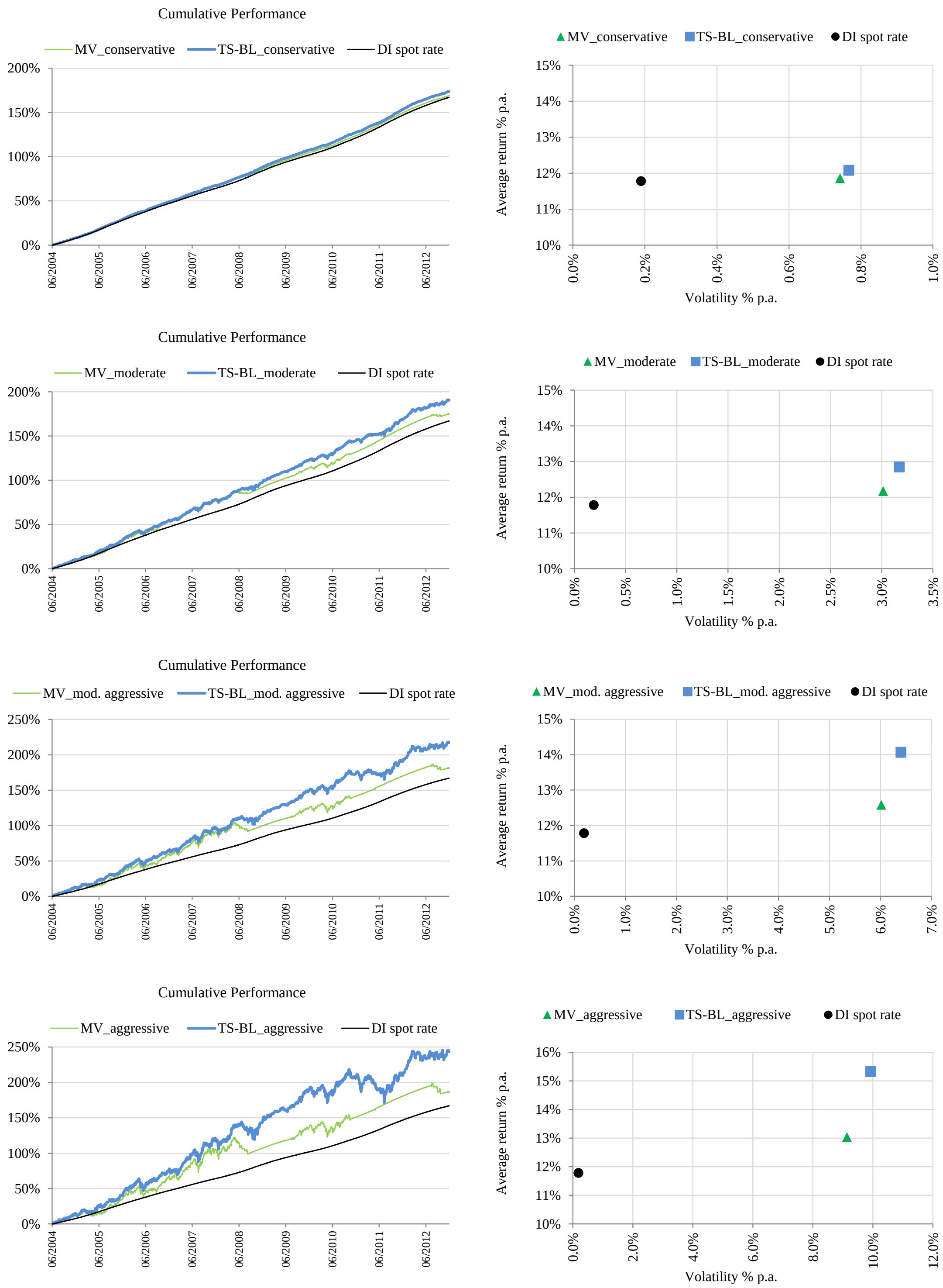
[41] Considering a time period of 252 observations.

[42] The purpose of this simple model is far from being considered the best specification to predict returns. It merely seeks to test the power of our trading strategy based on the BL framework, when we incorporate a different source of information beyond past average returns.

[43] Typically, financial services institutions classify investors ranging from conservative to aggressive profiles. Conservative investors usually want stability and are more concerned with protecting their current investments than increasing the real value of their investments. They usually want their portfolio to provide them with an inflation adjusted income stream to pay their living expenses. Moderate investors are longer term investors who want a relatively stable growth and tolerate some fluctuation. Moderately aggressive investors are long term investors who want good real growth in their capital and for that, accept a fair amount of risk. Finally, aggressive investors are long term investors who want high capital growth. Substantial fluctuations in value are acceptable in exchange for a potentially high long term return.

[44] The Brazilian fund industry has achieved a high degree of maturity, improving its practices, rapidly adapting to the demands of a country that is becoming increasingly stable. Currently there are around 400 fund managers licensed by the CVM (Brazilian Securities Commission) and supervised by ANBIMA (Brazilian Financial and Capital Markets Association) who are responsible for the management of more than 9,000 funds. Total resources invested in Brazilian investment funds reached the R$ 2.0 trillion mark at the end of 2012, a figure equivalent to around 45% of GDP (see Website: http://portal.anbima.com.br).

Fig. 2.2: Optimal portfolios cumulative performance (left side) and risk-return relationship (right side) for several risk profiles.



The optimal portfolios generated by our proposed trading strategy (TS-BL) obtained a superior cumulative performance for all the investors' risk profiles when compared to both MV optimal portfolios and benchmark strategy. Despite the fact that the TS-BL model does not dominate the MV model with respect to both criteria (obtaining a higher return with a lower risk), when increasing the risk tolerance, ranging from conservative to aggressive investors, we can observe an increase in risk[45] (respecting the predefined target levels for each risk profile) followed by an even greater increase in return. We might also analyze in Figure 2.3 the Sharpe index for the MV and TSBL optimal portfolios. One can notice the superiority of the TSBL model in all risk profiles.

Fig. 2.3: Optimal portfolios Sharpe index for several risk profiles.

	Sharpe Index	
Risk Profile	MV	TS-BL
Conservative	0.104	0.390
Moderate	0.130	0.337
Mod. aggressive	0.133	0.356
Aggressive	0.137	0.357

Further, to check whether this evidence is valid for the different market periods over our sample, we consider the fact that the investor holds this strategy for different time intervals and plot in Table 2.1 trailing returns statistics for two different time periods for the proposed model with several risk profiles. The table shows that the proposed TS-BL model presents a higher proportion of positive excess returns when compared to the MV model for all risk profiles. This result is observed for the two analyzed holding periods (6 months and 1 year). We can observe that the TS-BL model for the aggressive profile presents a higher average negative excess return (while still respecting the target volatility imposed by the optimization problem) for both periods.

[45] Measured by volatility.

Tab. 2.1: Comparative table for trailing returns in different time intervals and several risk profiles.

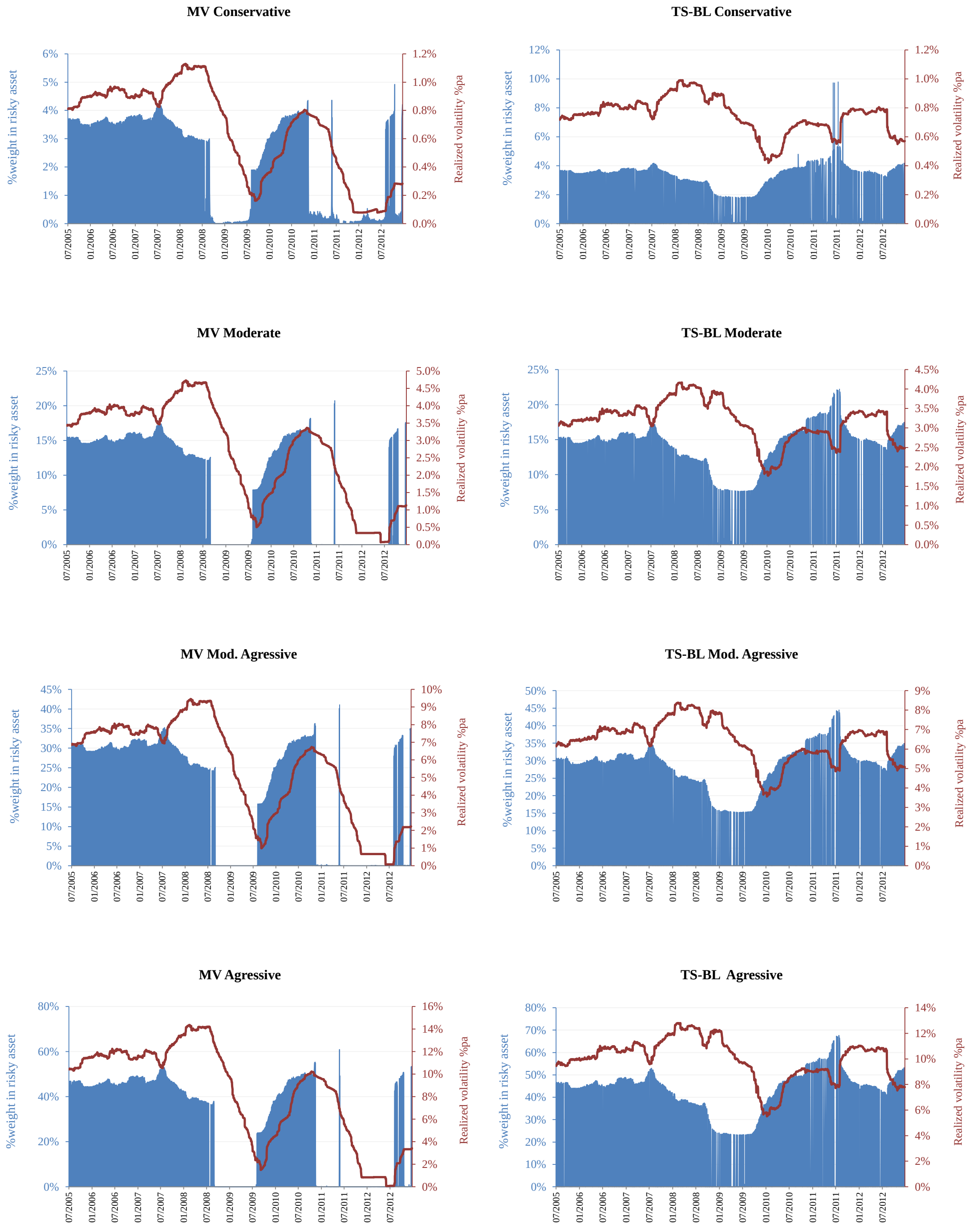
Time interval	Risk Profile	% Proportion of positive excess return (TS-BL returns - MV returns)	Average positive excess returns*	Average negative excess returns*
1 year	Conservative	61%	0,5%	-0,3%
	Moderate	59%	1,7%	-1,3%
	Mod. Aggressive	64%	3,2%	-2,9%
	Aggressive	69%	5,0%	-5,9%
6 months	Conservative	63%	0,3%	-0,2%
	Moderate	63%	1,0%	-1,0%
	Mod. Aggressive	67%	2,2%	-2,4%
	Aggressive	66%	3,5%	-3,9%

excess returns= TS-BL optimal portfolios returns - MV optimal portfolios returns*

To investigate this result, we present Figure 2.4. In this figure we plot the percentage dynamic optimal allocation in the risky asset on the left vertical axis (in blue) against the estimated volatility of the optimal portfolio on the right vertical axis (in red).[46] The graphs on the left side refer to MV optimal portfolios and the graphs on the right refer to TS-BL optimal portfolios. In both cases, we plot the graphs to all risk profiles (conservative, moderate, moderately aggressive and aggressive).

[46] Calculated over a sample of 252 observations.

Fig. 2.4: Optimal allocation in risky asset (in percentage terms) for MV and TS-BL for several risk profiles.



One can observe from Figure 2.4 that both specifications (MV and TS-BL) enhance the allocation in the risky asset, as the investor risk profile moves from conservative to aggressive, respecting the predetermined volatility level set by the optimization problem. However, it is worth mentioning that the MV traditional model, during some crisis periods[47], indicates a total allocation to the riskless asset, which persists for a period of time. This behavior is observed even for aggressive risk profile optimal portfolios.

We understand this out-of-sample behavior for the MV model as the effect of the magnitude of estimation errors. When a crisis scenario occurs, historically based estimates (input in the MV model) exhibit significantly more negative expected returns which persist for a period of time. As a consequence, the MV model does not suggest any allocation in the risky asset. On the other hand, during those same crisis scenarios, our blended posterior expected returns exhibit a different behavior as the TS-BL model reports a modest decrease in the risky asset allocation. One possible explanation is related to the methodology that we use to build this posterior distribution, which combines fundamental analysis priors and equilibrium risk premiums. In this way, our proposed trading strategy seems to help reduce the impact of estimation errors on portfolio weights along time. Furthermore, it is important no notice that the TS-BL model still respects the target volatility imposed by the optimization problem, considering each risk profile (as from Figure 2.4 during those crisis periods the TS-BL model reduces the allocation in the risky asset to a lower level, until risky asset volatility reverts to a lower level).

2.5 Conclusions

In this work we proposed a dynamic asset allocation strategy based on the Black-Litterman model. Our challenge was to apply the BL model considering that the investment strategy should evolve over time in sympathy with the investors risk tolerance and the market environment. We solve an adaptive portfolio choice problem of an investor faced with a time-varying investment opportunity set. We find that it is possible to use the Black-Litterman framework to design more reliable and stable investment strategies. The proposed trading strategy incorporated with neutral equilibrium estimates used in the computation of Black-Litterman implied that expected returns can be used as an effective tool to mitigate allocation instabilities due to estimation errors. The traditional mean-variance model suggests that investors should tilt their allocation from the risky asset towards the risk-free asset

[47] Highlighting the 2008 subprime crisis and the 2011 Europe debt crisis.

during high volatility periods,[48] and this allocation persists for long period of time. This is the result of estimating expected returns based on historical data, which exhibit highly negative estimates during crisis periods. On the other hand, the proposed TSBL model suggests that investors must carry part of their position in the risky asset also through high volatility periods, as long as the risk constraints are satisfied. This is due to the improvement in the expected returns estimate, combining a dynamic trading strategy with equilibrium portfolios. We back-tested the model for different market environments (bull and bear markets) and several investor risk profiles and obtained similar results.

[48] Even for aggressive profile investors.

3
Paper 2: Robust Portfolio Models based on Empirical Adaptive Loss

In this work we provide an approach to model portfolio decisions under uncertainty that relies on decision-maker risk tolerance to construct data-driven adaptive polyhedral uncertainty sets from joint dynamics of asset returns. Further, we propose a learning specification algorithm to forecast future returns. The provided specification dynamically updates conditional probability distribution of asset returns and reduce asset allocation instability due to estimation errors.

3.1 Introduction

Portfolio selection model as proposed by Markowitz (1952) is known as a single-period allocation problem, where the investor is myopic[1] by construction and her goal is to maximize the end of period terminal wealth taking into account her risk aversion. The model is based on the assumption that investment decisions depend only on the expectation value and covariance structure of asset returns.[2] Implementing such portfolios requires the knowledge of both asset expected return and covariance, which is usually considered by classical optimizers (such as mean-variance model) as given with certainty.

However, in real asset allocation problems input data are usually uncertain. In this context, robust optimization (RO) techniques have received significant interest by the investment management community, as they allow portfolio managers to incorporate the uncertainty introduced by estimation error directly into the optimization process. Its goal is to compute solutions with a priori ensured feasibility when the problem parameters are assumed to be unknown but confined within a prescribed uncertainty set. Given optimization problems with uncertain parameters, RO finds the best decision in view of the worst-case parameter values within these uncertainty sets. The uncertainty model is not stochastic, but rather set-based and the decision maker can construct a solution that is optimal for any realization of

[1] Investors are said to be myopic when they take decisions based on a one-period analysis of their investments (see Benartzi and Thaler (1995)).

[2] Both estimates are usually made over the same time horizon.

the uncertainty in a given set. Under RO, modelers agree to accept a suboptimal solution for the nominal values of the data, in order to ensure that the solution remains feasible and near optimal when the data change. In this work we introduce a robust portfolio optimization methodology that models uncertainty by budgeted polyhedral uncertainty sets using data-based information, while retaining the advantage of a linear optimization framework. We define the problem constraints as polyhedral dynamic sets described by a list of its vertices, which are set as past asset returns obtained over moving windows with a length of J-days. Our model specification is designed to be robust to changes in market conditions both in the problem constraints and in the objective function. We also consider the concept of Comprehensive Robustness proposed by Ben-Tal et al. (2006) which aims to control the deterioration in performance when the uncertainties materialize outside of the uncertainty set. Case studies that follow provide some interesting insights about the model behavior in those scenarios. To the best of our knowledge this contribution for portfolio optimization literature is relevant for it provides a tractable approach to dynamic decision-making under uncertainty. Our main contributions are:

(i) We propose a non-parametric prediction model to forecast future returns. It relies solely on signals extracted from data. The optimal signal is the result of an optimized convex combination of representative signals modeled from adaptive indicators that may vary considering the existing dynamic conditional correlations between asset returns.

(ii) Considers that optimal portfolio losses are modeled using a robust adaptive approach. Its potential loss is limited by the worst-case scenario inside predefined polyhedral dynamic uncertainty sets.

(iii) Incorporates transaction costs, covering all fee structures typically observed on the market to give a more rigorous result for practical purposes.

Our model is an adaptive portfolio optimization method in a discrete-time, finite horizon setting which considers risk aversion (in the loss restriction), portfolio constraints and the existence of real transaction costs.

3.1.1 Literature Review in RO applied to Portfolio Problems

Robust optimization (RO) is a powerful modeling methodology that seeks to minimize the negative impact of future events when the values of model parameters are uncertain and their distributions are unknown. We might define a model as robust if it guarantees, with high probability, that the objective function will be achieved while the solution will be feasible for all possible realizations of each uncertain parameter, contained within the bounds of an uncertainty set. RO approach is based on optimizing against the worst-case realization of the uncertainties within a given set, using a min-max objective. Typically, the original uncertain optimization problem is converted into an equivalent deterministic form (its robust counterpart) by duality arguments and then solved using standard optimization algorithms. The robust counterpart of an uncertain mathematical program is a deterministic worst-case formulation in which model parameters are assumed to be uncertain, but confined in a bounded interval known as an uncertainty set.

The notion of uncertainty set for parameters was first studied in the early 1970s, when Soyster (1973) proposed an inexact linear optimization model to construct a solution feasible for all input data belonging to a convex set.[3] Later, the use of uncertainty sets was formally extended to introduce a robust formulation when Ghaoui and Lebret (1997) present a robust least-squares solution to a problem where the parameters are unknown but bounded matrices. They show how this solution can be computed using second order cone programming. This idea was further studied by Ghaoui et al. (1998). They present formulations for robust semidefinite programming and demonstrate sufficient conditions that guarantee the existence of robust solutions. Ben-Tal and Nemirovski (1999, 2000) also provide analysis of RO framework using general convex programming. They firstly pointed out that ellipsoidal uncertainty sets formulate the original uncertain linear programming problems to robust conic quadratic programs that might be efficiently solved by interior point methods. Afterwards they expand the domain to generic convex problems. Both Ben-Tal and Nemirovski (1999, 2000), as well as Ghaoui and Lebret (1997) and Ghaoui et al. (1998) papers pointed out that the approach originally proposed by Soyster (1973) results in robust solutions that exhibit an objective function value much worse than that of the nominal problem as it protects against the worst-case scenario which may not be meaningful for decision-making process. In a less conservative model they proposed to scale down the uncertainty set such that it only includes the "most likely" values of the uncertain parameters. Consequently,

[3] Soyster's model is based on Euclidean distance. Thereat, when the difference between real data and nominal data is relatively large, the feasible region of this model becomes relatively small, which leads the model to be too conservative, and the robust solution will have too much of loss of optimality.

the true value of any uncertain parameter may be outside the bounds of the set, for which the worst-case solution has not accounted; therefore, feasibility is no longer totally guaranteed. And since then, other authors have been working on alternative solutions that could reduce the conservatism of the solution (also known as the price of robustness[4]). Bertsimas and Sim (2004, 2006) proposed an alternative method to control for the level of conservatism of the solution while retaining the advantages of the linear framework introduced by Soyster. Bertsimas and Brown (2009), based on the theory of coherent risk measures, proposed a methodology for constructing uncertainty sets within the framework of RO for linear optimization problems with uncertain parameters. We mention here a few papers and refer the reader to Sim (2004) and the references therein for a more complete survey.

In recent years, robust models have played a major role in portfolio optimization for resolving the sensitivity issue of the classical MV model. Robust portfolios of this kind are relatively insensitive to the distributional input parameters and typically outperform classical Markowitz portfolios (see Ceria and Stubbs (2006)). Goldfarb and Iyengar (2003) present statistical methods for constructing uncertainty sets for factor models of asset returns and show their robust portfolio problem can be reformulated as a second-order cone program. Tutuncu and Koenig (2004) propose a model with box uncertainty sets for mean and covariance and show the arising model can be reduced to a smooth saddle-point problem subject to semidefinite constraints. Ghaoui et al. (2003) show the worst-case VaR (Value-at-Risk) under partial information on the moments can be formulated as a semidefinite program. Fabozzi et al. (2007) and Fabozzi et al. (2010) provide a survey of recent contributions to robust portfolio strategies and cover results derived in terms of both mean-VaR (Value-at-Risk) and mean-CVaR (Conditional Value-at-Risk) risk measures. Bertsimas and Pachamanova (2008) suggest robust linear optimization formulations of the multi-period portfolio optimization problem. More recently, applied papers present interesting empirical results. Guastaroba et al. (2011) investigate robust techniques when applied to a specific portfolio selection problem based on real-life data from London Stock Exchange Market. Gregory et al. (2011) evaluate the cost of robustness for the robust counterpart to the maximum return portfolio optimization problem where the uncertainty of asset returns is modeled by polyhedral uncertainty sets. Ye et al. (2012) present a one-period mean-variance model, where the mean and covariance matrix are assumed to belong to an ex-

[4] Bertsimas and Sim (2004) introduced this concept which considers how "heavily" the objective function value is penalized when we are guarded against constraint violation.

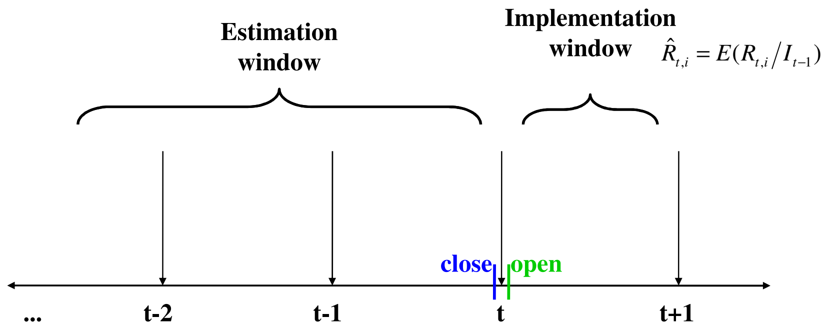
ogenously specified uncertainty set and formulate the problem as a conic program. Jianke et al. (2012) propose a robust counterpart for linear optimization with uncertain data, under a new distance metric, maintaining the problem linearity. Scutella and Recchia (2013) review several mathematical models and related algorithmic approaches to address uncertainty in portfolio asset allocation. Li and Kwon (2013) present an approach to enable investors to seek a robust policy for portfolio selection in the presence of rare but high impact realization of moment uncertainty.

The majority of previous robust portfolio optimization research has stemmed from the work of Ben-Tal and Nemirovski (1998); Ghaoui and Lebret (1997) and Ghaoui et al. (1998) who model unknown parameters by ellipsoidal sets and consider second-order cone programming (see by Lobo and Boyd (2000); Goldfarb and Iyengar (2003); Ghaoui et al. (2003)). In particular, the following work is focused on linear programming and polyhedral uncertainty sets (as of the work of Bertsimas and Sim (2004)).

3.1.2 Notation and Assumptions

In this work we develop an adaptive portfolio selection problem using robust optimization techniques. Hereafter we present the robust optimization framework when the decision-maker must select a strategy before (or without) knowing the exact value taken by the uncertain parameters. We consider a single notation for all models presented. Let $R_{t,i}$ denote the return on asset i between time t and time $t+1$, which is a random variable, and $\hat{R}_{t,i} \equiv \mathbb{E}(R_{t,i} \mid I_{t-1})$ the expected return for asset i between time t and time $t+1$ based on all the information available up to time $t-1$, here represented by set I_{t-1}. We also consider the following diagram to represent the time frame of variables estimation and model implementation.

Fig. 3.1: Time frame diagram for estimation period and implementation period.



We consider daily allocation decisions in all our applications. As such, we assume the allocation in asset i is given by $x_{t,i}$ and is made at the beginning of day t, considering the estimated return $\hat{R}_{t,i}$. Afterwards we test if this allocation was successful in an out-of-sample analysis, comparing this estimated return with the observed return for time period $t+1$.

3.2 Robust Adaptive Portfolio Model - RA

An investment portfolio evolution is driven by how much the individual can save and invest, what she invests in, what return has she earned and how much of that return is surrendered to investment costs (transaction costs[5], administration and incentive fees) and to taxes. Except for asset returns, we have almost complete control for the other factors listed above. The point is: the returns we get on our investments are uncertain. Past performance may be an indicative of the future performance, but is not necessarily a guarantee of future results. This makes modeling asset return dynamics particularly challenging. In this section we describe the discrete-time model[6] we use for asset return dynamics and formulate the optimal investment problem in its terms.

Let us consider a wealth maximization problem subject to an adaptive robust portfolio loss and budget constraints $\forall t$ given by

[5] Such as exchange, settlement, permanence and registration fees, brokerage and slippage costs.
[6] In discrete-time models the asset return dynamics are primarily governed by a rule that dictates how the price or return changes from one period to the following one.

$$\max_{\mathbf{x}\in\mathbb{R}^n} \hat{\mathbf{R}}_t'\mathbf{x_t}, \tag{3-1}$$

s.t.

$$L(\mathbf{R}_t, \mathbf{x_t}) \leq \epsilon_1, \forall \mathbf{R}_t \in \Xi_t \tag{3-2}$$

$$B(\mathbf{R}_{t-1}, \mathbf{x_t}, \mathbf{x_{t-1}}, \mathbf{c_t}) \geq \epsilon_2, \forall i = 1, ..., n \tag{3-3}$$

where $\mathbf{R}_t' = (R_{t,1}, R_{t,2}, ..., R_{t,n})$ is the vector of asset returns, $\mathbf{x_t}' = (x_{t,1}, x_{t,2}, ..., x_{t,n})$ is the vector of decision variables (where each $x_{t,i}$ corresponds to the financial asset allocation in asset i to be executed at the beginning of day t)[7] and n is the number of available assets. The loss function is denoted by $L(.)$ and the budget function is denoted by $B(.)$.

3.2.1 Returns' Adaptive Forecast - $\hat{\mathbf{R}}_t$

The adaptive forecast for future asset return i at time t is given by signal $\hat{R}_{t,i}$. We assume a mixed signals model to predict future asset returns which dynamically selects the signal that performs better, considering an out-of-sample analysis. In this study we propose the use of multiple built-in indicators in the objective function as we believe they can work well to predict the direction and volatility of future prices when combined. We denote those constructed signals by $Sig_{s,t,i}$, in which s stands for signal s, t stands for the day we will implement this signal (as we consider past data up to day $t-1$ to estimate a signal to time t) and i stands for asset i. A detailed formulation for the built-in signals is provided in Appendix. Our mixed signals model is a dynamic linear combination of expected returns given by technical indicators. We blend lagging, leading and volatility signals to construct the signal (return) formula $\forall t$ as

$$\hat{R}_{t,i} \equiv \mathbb{E}(R_{t,i} \mid Sig_{s,t-1,i}) = \sum_{s=1}^{S} \alpha^*_{s,t,i} \hat{Sig}_{s,t,i}, \forall i \tag{3-4}$$

where $\hat{Sig}_{s,t,i}$ correspond to each signal s, estimated using past data up to time t to estimate the signal for time t. S corresponds to the total number of signals used in

[7] We assume the investor allocates all her wealth at each time t. For that, the mixed signals model consider past signals up to time $t-1$ to forecast the return at time t.

our mixed signals model and $\boldsymbol{\alpha}^*_{s,t,i}$ is the optimized vector resulting from

$$\boldsymbol{\alpha}^*_{t,i} = arg \min_{\boldsymbol{\alpha}\in\mathbb{R}^n} \sum_{b=1}^{B} \left[R_{t-b,i} - \sum_{s=1}^{S} \alpha_{s,i} \hat{Sig}_{s,t-b,i} \right]^2$$
$$s.t. \qquad (3\text{-}5)$$
$$\sum_{s=1}^{S} \alpha_{s,i} = 1, \forall i$$

where $\boldsymbol{\alpha^*_{t,i}}' = (\alpha^*_{1,t,i}, \alpha^*_{2,t,i}, ..., \alpha^*_{S,t,i})$ is the vector of optimized weights in each signal and B stands for a robustness parameter defined as a time period during which we must optimize the function above.

From (3-2) we have the loss given by $L(.)$ which is a general loss function that depends on the vector of asset returns $\mathbf{R_t}$ and the decision vector $\mathbf{x_t}$. It is set to be less or equal to a scalar ϵ_1. We propose to model the uncertainty in the problem by restricting the uncertain parameters to belong to polyhedral adaptive uncertainty sets denoted by Ξ_t. Under our formulation, those uncertainty sets are defined as adaptive along time and are constructed considering an empirical approach. We set the past observed returns $\mathbf{R_{t-j}}, \forall j = 1, ..., J$ as points in space and we define the convex polyhedral in terms of a convex hull of this set of points.[8] In the next section we detail the proposed specification.

In addition, we define the budget function $B(.)$ considering the existence of transaction costs.[9] This function depends on the vector of past asset returns $\mathbf{R_{t-1}}$, the decision vector $\mathbf{x_t}$ and also on transaction costs $\mathbf{c_t}$ at time t.

Under this general formulation the decision-maker constructs a solution that is optimal for any realization of the uncertainty in a given set. In the sequel, we outline the proposed approach to construct the uncertainty sets based on available information. We show there are several convenient parameterizations of the uncertainty set that allows the decision-maker a desired level of flexibility in choosing the trade-off between robustness and performance.

[8] As proved by Grunbaum and Ziegler (2003) a bounded convex polytope can be defined as the convex hull of a finite set of points, where the finite set must contain the set of extreme points of the polytope. Such definition is called a vertex representation (V-description). And for a compact convex polytope, the minimal vertex representation is unique and is given by the set of the vertices of the polytope).

[9] A detailed specification for transaction costs is provided in Appendix.

3.2.2 Empirical Robust Loss Constraint - $L(\mathbf{R}_t, \mathbf{x_t}) \leq \epsilon_1$

Many optimization problems involve parameters which are not known in advance but can be estimated. We understand this is generally true, for instance, in asset allocation decisions. Such problems fit perfectly into the framework of RO. In this work we apply RO methods in the context of portfolio asset allocation with focus on the loss specification.

As by (3-2) we assume that we must guarantee that a loss function estimated over the portfolio must be smaller than a predefined limit which would be, in its turn, a function of the investor risk aversion. However, the parameters (specifically, asset returns) are uncertain. We decided to apply robust optimization methods to solve the problem, considering that those uncertain parameters are confined within a prescribed adaptive uncertainty set Ξ_t. Our goal is to find the best decision at any time t in view of the worst-case parameter values within these adaptive uncertainty set.

Let us recover that $\mathbf{R}_t' = (R_{t,1}, R_{t,2}, ..., R_{t,n})$ is the vector of asset returns and $\mathbf{x}_t' = (x_{t,1}, x_{t,2}, ..., x_{t,n})$ is the vector of decision variables. Let us denote by γ (a negative scalar) the parameter that defines a percentage loss of the total wealth at time period $t-1$, denoted by W_{t-1}. We assume the loss constraint (3-2) $\forall t$ is described generally as

$$\mathbf{R}_t'\mathbf{x}_t \geq \gamma W_{t-1}, \forall \mathbf{R}_t \in \Xi_t \tag{3-6}$$

We define the uncertainty set Ξ_t as

$$\Xi_t = \left\{ \mathcal{R}_t \in \mathbb{R}^n \mid \exists \boldsymbol{\zeta} \in \mathcal{Z} : \mathcal{R}_t = \sum_{z=1}^{J} \zeta_z \bar{\mathbf{R}}_z \right\} \tag{3-7}$$

where $\bar{\mathbf{R}}_z$ are return samples and ζ_z is defined in the set

$$\mathcal{Z} = \left\{ \boldsymbol{\zeta} \in [0,1]^J \mid \sum_{z=1}^{J} \zeta_z = 1 \right\} \tag{3-8}$$

The robust loss constraint (3-6) $\forall t$ is equivalent to

$$\left(\sum_{z=1}^{J} \zeta_z \bar{\mathbf{R}}_z \right)' \mathbf{x_t} \geq \gamma W_{t-1}, \forall \zeta_z \in \mathcal{Z} \tag{3-9}$$

which corresponds to

$$\min_{\zeta_z \in \mathcal{Z}} \left(\sum_{z=1}^{J} \zeta_z \bar{\mathbf{R}}_z \right)' \mathbf{x_t} \geq \gamma W_{t-1} \tag{3-10}$$

or rewriting

$$\min_{\zeta_z \in \mathcal{Z}} \sum_{z=1}^{J} \zeta_z \left(\bar{\mathbf{R}}_z' \mathbf{x}_t \right) \geq \gamma W_{t-1} \tag{3-11}$$

For a given allocation $\mathbf{x}_t$, the solution to the minimization problem above would be given by $\zeta_z = 1$ if $\bar{\mathbf{R}}_z' \mathbf{x_t} \leq \bar{\mathbf{R}}_w' \mathbf{x_t}, \forall w \neq z$ or $\zeta_z = 0$, otherwise. So, for a general allocation $\mathbf{x_t}$ the constraint (3-6) would always be on the form

$$\bar{\mathbf{R}}_z' \mathbf{x}_t \geq \gamma W_{t-1}, \text{where } \bar{\mathbf{R}}_z' \mathbf{x_t} \leq \bar{\mathbf{R}}_w' \mathbf{x_t} \tag{3-12}$$

And to guarantee that the robust loss constraint is satisfied for any allocation $\mathbf{x}_t$ it is sufficient to include the J constraints

$$\bar{\mathbf{R}}_z' \mathbf{x}_t \geq \gamma W_{t-1}, \forall z = 1, ..., J \tag{3-13}$$

Hereafter in this work let us assume that the polyhedral adaptive set describing the uncertainty is constructed based on the information available to the decision-maker before the robust optimization approach is implemented (historical data). So, the uncertainty set Ξ_t is estimated adaptively considering past observed returns.

We define the loss function, ensuring that the optimal portfolio does not incur, from time period $t-1$ until time period $t-J$, a daily loss greater than a pre-specified parameter γ that defines a percentage loss of the total wealth at time period $t-1$, denoted by W_{t-1}. Considering J the number of backward days, we introduce a concept of adaptive robustness applied to past performance of the portfolio.[10] This approach enables the model to capture the dynamics of the empirical dependence structure between the assets. Besides that, the robustness seeks to ensure that the model has a risk exposure that is defined according to the investor risk profile (parameter γ).

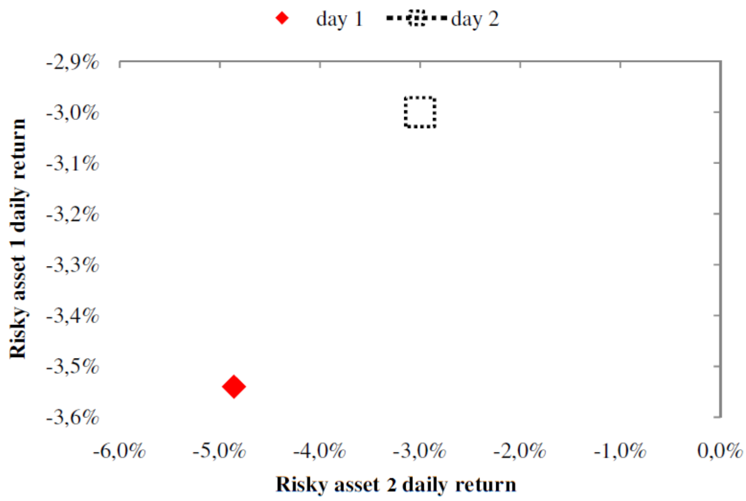
The intuition here is to let the model capture assets dynamic behavior over different market conditions. We expect such adaptive loss constraints to be able

[10] Is is also known as data-driven optimization as it uses observations of random variables as direct inputs to the mathematical programming problems.

to generate, adaptively, different feasible regions[11] for the investor asset allocation decisions.

To illustrate this adaptive loss function, let us consider an intuitive example where we have two risky assets (Asset 1, Asset 2) and one risk-free asset (Asset 3) with the following two-dimensional scatter plot for observed returns in two different days

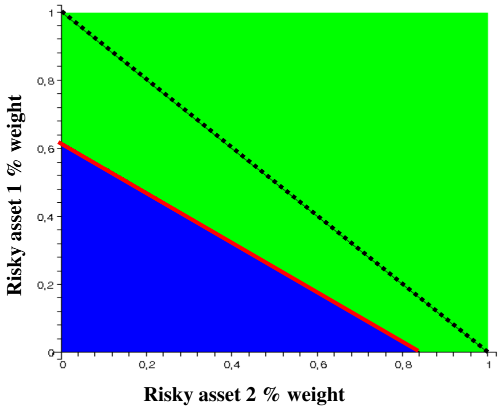
Fig. 3.2: Scatter plot of risky assets daily returns - intuitive example.



Following the proposed loss constraint above, and considering $\gamma = -3\%$, for day 1 we can write the constraint as $-3.54\%x_1 - 4.86\%x_2 + 0.05\%x_3 \geq -3\%$ (in red) and for day 2 we can write the constraint as $-3\%x_1 - 3\%x_2 + 0.05\%x_3 \geq -3\%$ (in black dotted line). We also consider that $x_1 + x_2 + x_3 = 1$. By those equations we can thus construct the following feasible regions for a two-dimensional plot (Asset 1 against Asset 2; the Asset 3 can be understood as being in the third dimension of this plot), depicted in the figure below.

[11] As optimal allocation strategies will vary depending on asset return relationship and its magnitude.

Fig. 3.3: Feasible region (plotted in blue) of the percentage portfolio weights in Asset 1 against Asset 2 - intuitive example.

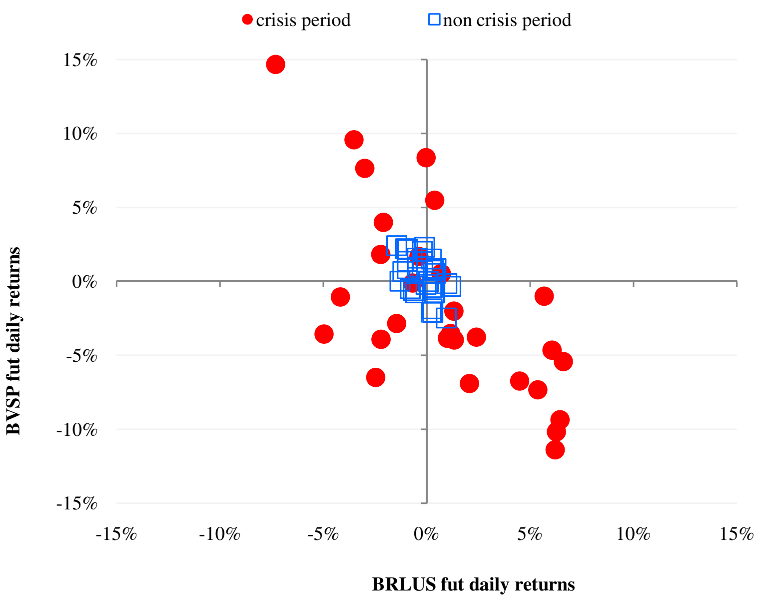


One can notice that the loss constraint built using data in red is active, whereas the loss constraint built using data in black dotted line is not active. The feasible region generated by data in red is smaller, as expected (as the estimated sample returns are negative and with higher magnitude). And this feasible region is adaptive along time, since new constraints are added to the portfolio problem.

Going back to our portfolio problem, as we will see in the case study that follows, we will be able to invest in any of three assets (one risk-free and two risky assets. The loss constraints presented in (3-2) must be able to capture the evolving dynamics between those assets. For that we decided to illustrate how it works during different market conditions (such as non crisis and crisis periods). From Figure 3.4 we can observe the behavior of risky assets daily returns during those periods, which is an input to our portfolio model. Figure 3.4 depicts the past observed returns for two risky assets (BRLUS - an exchange rate and BVSP index - an equity index). We plot in blue squares the observed returns during some non crisis periods and in red dots the observed returns during some crisis periods.[12]

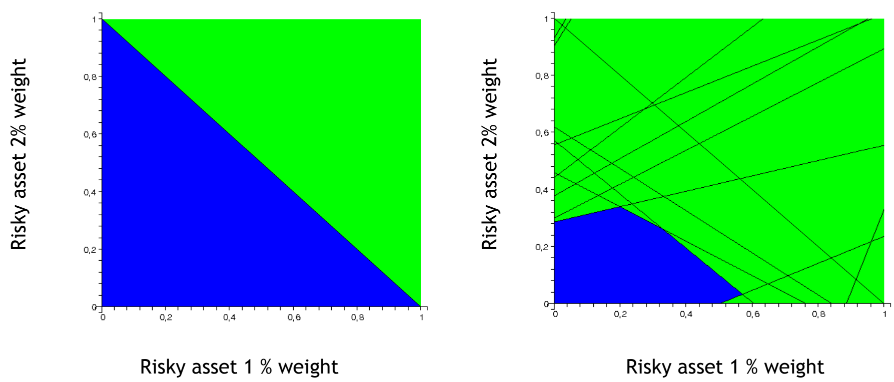
[12] Crisis periods corresponds to high volatility market periods.

Fig. 3.4: Scatter plot of risky assets daily returns: BVSP index versus BRLUS currency, considering different market conditions: non crisis and crisis periods.



Applying the loss constraints as defined in (3-13) we generate the following feasible regions, for non crisis and crisis periods, respectively plotted in the following figure

Fig. 3.5: Feasible regions (plotted in blue) of the percentage portfolio weights in each risky asset (the risk-free asset is plotted in the third dimension of this graph). The first plot is during non crisis period and the second plot is during crisis period.



We can observe the dynamic behavior of the proposed loss constraints from the plots above. During non crisis periods the constraints are not active and the feasible region allows the model to select any allocation (considering that short selling is not allowed) in the risky assets, ranging from 0% up to 100%. However, during crisis period, as the daily observed returns for the risky assets get more extreme (negative returns with higher magnitude), the loss constraints get active and the feasible region for the risky assets is reduced, indicating a higher allocation in the risk-free asset (in the third dimension of the plot). As expected the proposed adaptive empirical loss seems to capture the dynamics of the dependence structure of asset returns. At this point we just want to illustrate what is expected when we define our portfolio model loss constraints as (3-13). In the case study that follows we will analyze the results in more detail.

3.2.3 Budget Constraints

3.2.3.1 An Asset Allocation Problem (RA-AAP)

When we consider an asset allocation problem, we must guarantee the portfolio is fully invested at every time t in any of the available assets and neither leverage nor short selling is allowed. We decide to consider that the strategy does not receive any new investment or withdrawal over time. Further, to consider the impact of transaction costs in the portfolio problem, we specify c_i as total trading costs (given in percentage terms) associated to an order (buy or sell) of a given asset i and u_i corresponds to the asset i financial volume traded (u_i^+ for a buy order and u_i^- for a sell order). In this case, the portfolio problem $\forall t$ becomes

$$\max_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^{n} \hat{R}_{t,i} x_{t,i}, \tag{3-14}$$

s.t.

$$\sum_{i=1}^{N} R_{t-j,i} x_{t,i} \geq \gamma W_{t-1}, \forall j = 1, ..., J \tag{3-15}$$

$$x_{t,i} = x_{t-1,i}(1 + R_{t-1,i}) + (1 - c_i)u_i^+ - (1 + c_i)u_i^-, \forall i \tag{3-16}$$

$$u_i^+ \geq 0, u_i^- \geq 0, \forall i \tag{3-17}$$

$$\sum_{i=1}^{n} u_i^+ - \sum_{i=1}^{n} u_i^- = 0 \tag{3-18}$$

where $x_{t,i} \in \mathbb{R}^+, \forall i$. In the dynamic portfolio problem above we maximize the portfolio daily estimated return (3-14) where each $\hat{R}_{t,i}$ is calculated based on a mixed signals estimation of asset i future return 1-step ahead. In the problem constraints we define the loss restriction in (3-15). The portfolio observed returns for the past

j days (where $j = 1, ...J$) should be greater than a predefined negative parameter γ multiplied by the portfolio wealth for period $t-1$, denoted by W_{t-1}. This loss constraint must be satisfied for all past j days. The following problem constraint (3-16) is necessary to guarantee that the portfolio is fully invested at any time period t. This equation tells us that the financial exposure in each asset i at time t must be equal to its financial exposure at time $t-1$ corrected by its observed return during day $t-1$ (which is given by $R_{t-1,i}$) plus the financial traded volume in each asset i at the begging of day t (which may be a buy order (u_i^+) or a sell order (u_i^-)) minus the financial trading costs spent in each trade (which is given by $c_i(u_i^+ + u_i^-)$). The following equations (3-17) and (3-18) guarantee that the strategy does not receive any new investment or withdrawal over time.

3.2.3.2 A Hedge Fund Problem (RA-HFP)

We might also use the general portfolio problem to model an adaptive hedge fund[13] decision problem, in which the fund manager may use advanced investment strategies, such as leverage or short selling using financial derivatives (futures contracts for instance) to generate high absolute returns with reduced risk, whether in bull or bear markets. We consider a wealth maximization problem in which the objective function is a linear combination of estimated future returns over derivatives ($\hat{R}_{t,i}$) multiplied by its exposures ($w_{t,i}$) plus estimated future returns over cash equivalents ($\hat{R}_{t,c}$) multiplied by its position ($x_{t,c}$).[14] To allow for short selling and leverage positions, we consider the existence of financial derivatives. As a typical hedge fund, this specification allows for exposures in derivatives (denoted by $w_{t,i} \in \mathbb{R}$, where $i = 1, ..., n$[15]) as well as positions in cash equivalents (for simplicity, denoted by a single cash equivalent $x_{t,c} \in \mathbb{R}^+$). The hedge fund problem $\forall t$ is given by

$$\max_{x_c \in \mathbb{R}^+, \mathbf{w} \in \mathbb{R}^n} \left[\hat{R}_{t,c} x_{t,c} + \sum_{i=1}^{n} \hat{R}_{t,i} w_{t,i} \right] \tag{3-19}$$

[13] We will consider that this investment strategy might be called loosely as a hedge fund model since we allow for more flexible investment strategies compared to the prior case study.

[14] Being an investment fund, we consider the portfolio manager maintains the fund liquidity invested in cash equivalents. Since the fund invests in derivatives, we consider that the corresponding margin value blocked is corrected at the DI spot rate, considering that the fund can use government bonds as eligible collateral.

[15] Here n is the number of available derivatives

s.t.

$$\sum_{i=1}^{n} R_{t-j,i} w_{t,i} \geq \gamma W_{t-1}, \forall j = 1, ..., J \quad (3\text{-}20)$$

$$x_{t,c} = W_{t-1} - \sum_{i=2}^{n} c_i (u_i^+ + u_i^-) \quad (3\text{-}21)$$

$$w_{t,i} = w_{t-1,i}(1 + R_{t-1,i}) + u_i^+ - u_i^-, \forall i \quad (3\text{-}22)$$

$$u_i^+ \geq 0, u_i^- \geq 0, \forall i \quad (3\text{-}23)$$

In this hedge fund problem we define the loss restriction as (3-20) since the cash position return contribution is always a positive number (considering positive interest rates in the economy). The derivatives portfolio observed returns for the past j days (where $j = 1, ...J$) should be greater than a predefined negative parameter γ multiplied by the portfolio wealth for period $t-1$, denoted by W_{t-1}. This loss constraint must be satisfied for all past j days. The following constraint (3-21) is necessary to guarantee that the portfolio is fully invested at any time period t. The constraint (3-22) tells us that the financial exposure in each derivative i at time t must be equal to its financial exposure at time $t-1$ corrected by its observed return during day $t-1$ (which is given by $R_{t-1,i}$) plus its financial traded volume at the begging of day t (which may be a buy order (u_i^+) or a sell order (u_i^-)). To model the trading costs (here we consider brokerage costs as well as slippage costs) we consider they are paid using the fund cash equivalent. All exceeding resources at time t are also invested in cash equivalents (3-21). Finally, the equation (3-23) guarantees that the strategy does not receive any new investment or withdrawal over time.

3.3 Case Study

In this section we intend to motivate the reader about the importance to consider an adaptive robust model to generate higher returns while controlling for financial losses. We investigate two different decision problems: the RA-AAP (robust adaptive asset allocation problem with no short selling nor leverage) and the RA-HFP (robust adaptive hedge fund problem with leverage and short selling) and compare the potential of the empirical adaptive robust formulation to generate absolute returns with reduced risk in an out-of-sample evaluation.

3.3.1 Case Study 1: An Asset Allocation Problem (RA-AAP)

Let us consider the example of a dynamic investment strategy, in which the decision maker may decide how much to invest among three[16] Brazilian asset classes, namely:

- Brazilian interbank deposit rate (DI spot rate);
- Bovespa Index (BVSP index)[17];
- Current exchange rate Brazilian Real to US Dollar (BRLUS currency).[18]

Data was obtained from Cetip and Bloomberg databases. We calculate discrete asset returns[19] based on daily price observations during the period of April 5, 2000 up to May 31, 2013, comprising 3,257 observations. We implement a portfolio optimization model considering realistic transaction costs applied to Brazilian securities[20] and a management fee of 1% per annum, calculated on a daily basis over the backtested cumulative returns.[21] The optimization model is implemented in Mosel and solved using Xpress solver. Our objective is to compose a portfolio with the highest daily return subject to a well controlled loss, in order to avoid huge losses during crisis periods. We will adopt as our main benchmarks to compare to our model performance the buy-and-hold strategies in each asset class. And being aware of the well-known difficulties in beating the random walk forecast model (Meese and Rogoff (1983a,b); Kilian and Taylor (2003)), our empirical out-of-sample analysis will also consider this specification as a benchmark to test model predictability.

[16] The resulting dynamic program does not suffer from the curse of dimensionality. As we specified our model as a linear optimization problem (with linear trasaction costs) it is quite simple to solve it for portfolios of multiple assets.

[17] The Bovespa index is a gross total return index weighted by traded volume and is comprised of the most liquid stocks traded on the Sao Paulo Stock Exchange. Data was collected from the IShares Ibovespa Index Fund - BOVA11 series.

[18] The currency spot exchange rate.

[19] as $R_t = \frac{P_t}{P_{t-1}} - 1$.

[20] It is assumed that the trading cost of Brazilian interbank deposit rate (DI spot) is null, but its expected return considers a discount on the bond, a variation equivalent to 99.5% of actual DI spot rate. The transaction costs for other assets consist of: exchange, settlement, registration and permanence fees which are calculated as per BM&FBovespa methodology and brokerage fees by trading volume and slippage costs. Slippage costs are defined as the difference between the expected price of a trade and the price the trade is actually executed at. They usually occur during periods of higher volatility. We decided to estimate future volatility using an EWMA (Exponentially Weighted Moving Average) model for each asset class and evaluate the slippage cost as a percentage of this estimated volatility. The execution cost incurs over the traded volume at each period of time.

[21] An investment fund typically pays its investment manager an annual management fee, which is a percentage of the assets of the fund.

To analyze the empirical results obtained by the different specifications, we will consider a metric of cumulative return given by

$$R_{p,[1,T]} = \prod_{t=1}^{T}(1 + R_{p,t}) - 1 \tag{3-24}$$

where $R_{p,t}$ corresponds to the portfolio observed return between time $t-1$ and time t. We will also consider a metric based on a risk-adjusted return index, which gives some accuracy in measuring non-normality, specifically quantifying the tail risk in the extreme, named the CVaR-Based Sharpe Ratio (see Lin and Ohnishi (2007)):

$$ICVaR_{\alpha}(R_{p,t}) = \frac{1}{\mid CVaR_{\alpha}(R_{p,t}) \mid}\Big(\frac{1}{T}\sum_{t=1}^{T} R_{p,t}\Big) * 100 \tag{3-25}$$

For continuous loss distributions, the CVaR at a given confidence level is the expected loss given that the loss is greater than the VaR at that level. In this work, we calculate the $CVaR_{95\%}(R_{p,t})$ for the 95% confidence level, define it as a negative number (loss) and give it in % per day (%pd).

We evaluate the optimal portfolio against benchmark strategies. We backtest the mixed signal model against the random walk model[22] and buy-and-hold strategies in each asset class (DI spot rate, BRLUS currency and BVSP index). The limit daily loss is set to -3% daily, during a period of J days (where J is assumed to be equal to 45 trading days and B is equal to 20 trading days).

We present in Table 3.1 the comparative results for different time period parameters used in the constructed signals $Sig_{s,t,i}$ that when combined, form our mixed signals model. We use multiple built-in indicators to construct the final signal (return). Each signal considers two different time periods in its specification, denoted by short term time period (K_{ST}) and long term time period (K_{LT}), both expressed in trading days.[23] In the following table we depict the obtained out-of-sample results for the portfolio model specified in (3-14 – 3-18) for some different values of time periods $(K_{ST}, K_{LT}) \in \{(12, 26), (50, 100), (100, 150)\}$ and a sample period of 3,086 daily observations.

[22] We chose to specify the random walk model without a drift component based on the stylized fact that the volatility strongly dominates the mean for in financial time series of returns.

[23] In this thesis we presented all the results implementing both the MSM in the objective function as well as the empirical robust loss constraint. In a future work we distinguish the contribution of the proposed MSM from that of the proposed empirical robust loss specification.

Tab. 3.1: Comparative table varying parameters K_{ST} and K_{LT} for different model specifications and sample period ranging from December 11, 2000 up to May 31, 2013, comprising 3,086 daily observations.

Model Specifications		Cumulative Return	Annualized Average Return	CVaR	I_CVaR
Buy-and-hold strategies	DI spot rate	408,4%	14,2%	0,00%	-
	BRLUS currency	8,7%	2,27%	-2,49%	0,36
	BVSP index	257,1%	15,97%	-4,21%	1,40
Mixed signals model (K_{ST},K_{LT})	(12,26)	530,8%	18,02%	-2,50%	2,63
	(50,100)	**774,0%**	**21,43%**	**-2,65%**	**2,91**
	(100,150)	484,0%	17,54%	-2,68%	2,39
Random walk (no drift)		108,2%	7,19%	-2,03%	1,36

From Table 3.1 one can notice that mixed signals models with empirical robust loss specified in (3-14 – 3-18) present superior cumulative returns for any values of K_{ST} and K_{LT} when compared to both the random walk model and buy-and-hold strategies. We found similar results for both the annualized average return and the CVaR-Based Sharpe Ratio $(I\_CVaR_{95\%}(R_{p,t}))$. By varying the values of parameters K_{ST} and K_{LT} we can generate different signals that when combined generate a different input in the asset i return estimate and thus a different optimal portfolio at each time t. This table depicts that for several values of K_{ST} and K_{LT} our model could produce superior returns and risk-adjusted returns, when compared to selected benchmark strategies.

Figure 3.6 depicts the risk-return relationship among the strategies presented, considering another metric for measuring risk. We plot the average daily return on the vertical axis versus the volatility[24] of daily returns on the horizontal axis. Considering the existence of a risk-free asset (as the DI spot rate, for instance) one can notice that the mixed signals model (with $K_{ST} = 50, K_{LT} = 100$) exhibits the highest Sharpe ratio[25], when compared to both random walk and buy-and-hold risky strategies.

[24] Which is a frequently used risk measure by practitioners and calculated as the standard deviation of daily returns.

[25] This ratio measures the excess return per unit of standard deviation in an investment asset or a trading strategy.

Fig. 3.6: Risk-return relationship for different strategies and sample period ranging from December 11, 2000 up to May 31, 2013, comprising 3,086 daily observations.

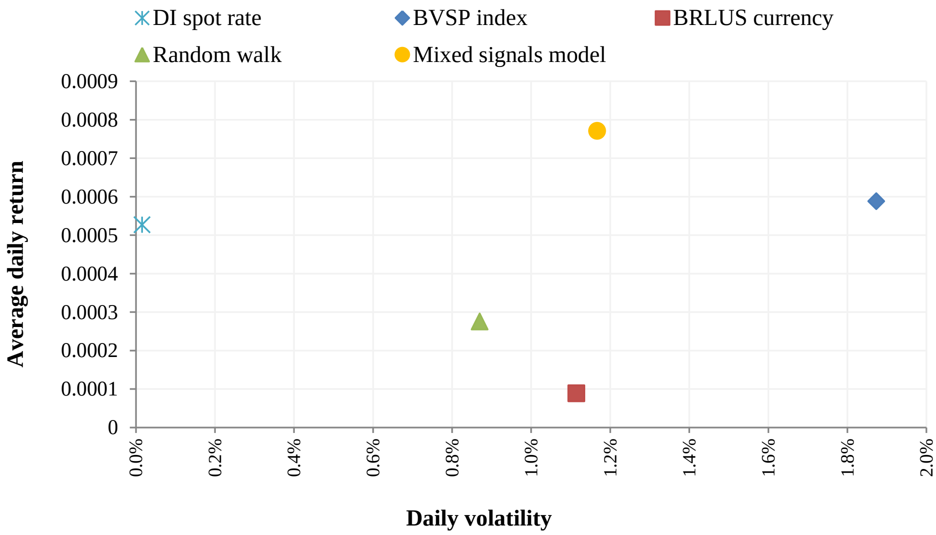
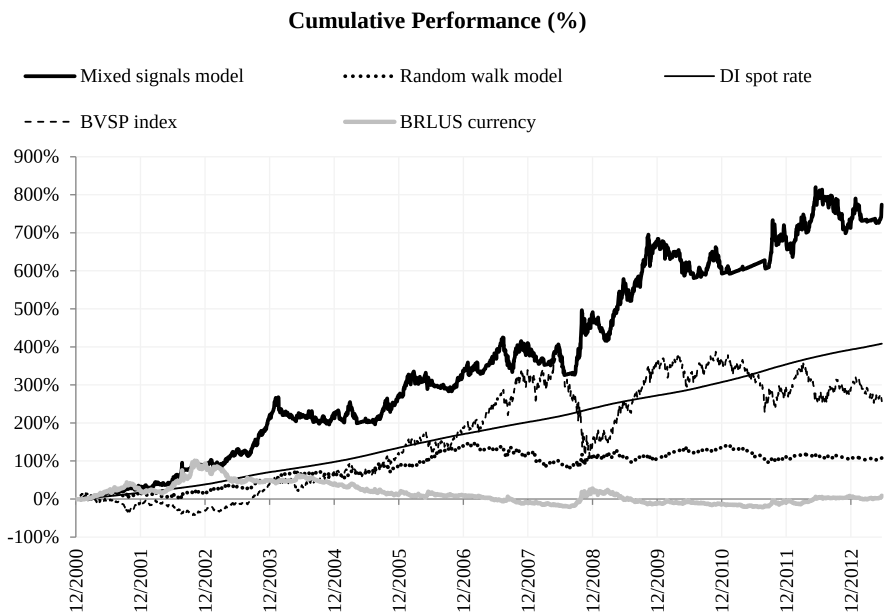


Figure 3.7 depicts the cumulative performance evolution for the mixed signals model with $K_{ST} = 50$ and $K_{LT} = 100$ against the randow walk and buy-and-hold strategies (DI spot rate, BRLUS currency and BVSP index).

Fig. 3.7: Cumulative performance evolution and sample period ranging from December 10, 2000 up to May 31, 2013, comprising 3,087 daily observations.



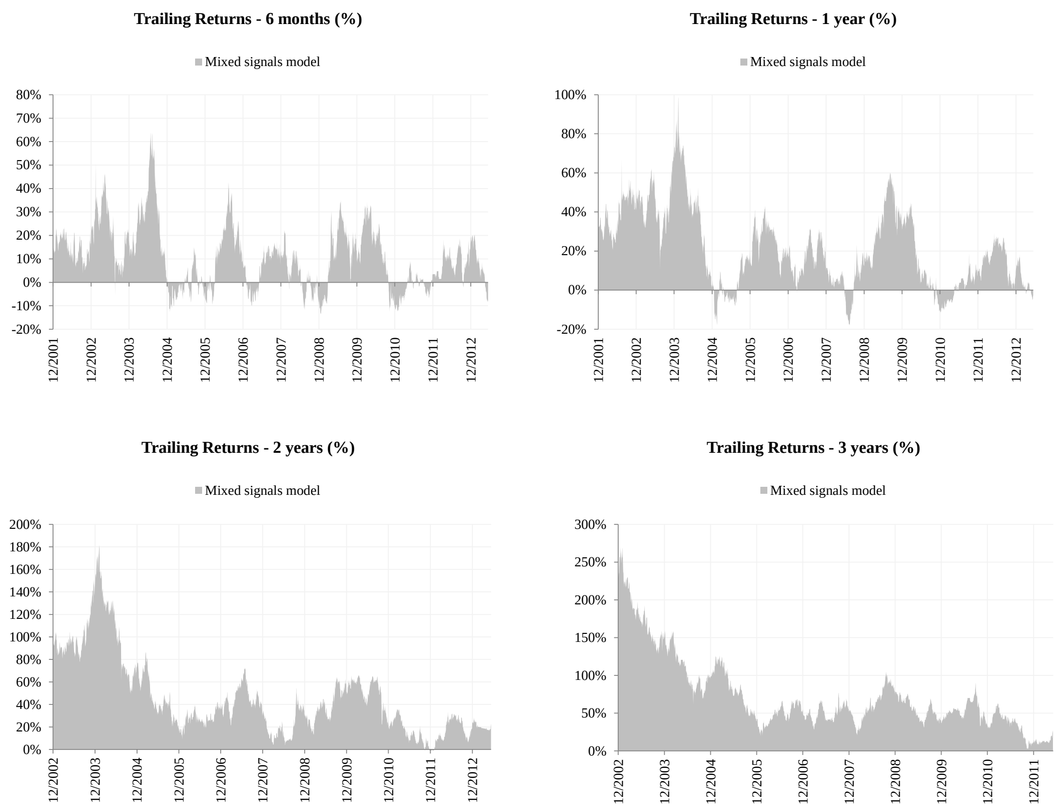
The mixed signals model presents a consistent superior cumulative performance when compared to both buy-and-hold strategies and random walk model.

From Figure 3.7 it is worth mentioning some interesting feature of our model. During some severe crisis (as of 2008 Subprime crisis), as the observed returns for the risky assets get more extreme and their dynamics change, the loss constraints get active and the feasible region for the risky assets is reduced, indicating a higher allocation in the risk-free asset. After some days the model starts to capture the new dynamic of the dependence structure of asset returns and indicates an allocation to risky assets once again. This seems to be a right decision as the model exhibits a positive return in an out-of-sample analysis.

To check whether the superiority of our model is valid for the different market periods over our sample, we consider the investor hold this strategy for different time intervals and plot in Figure 3.8 trailing returns for several time intervals[26] calculated over daily observations. The trailing returns exhibit a higher frequency of positive returns with a greater magnitude. Further, this behavior is even more pronounced with an increase in the time interval considered.

[26] For instance, annual trailing returns correspond to the cumulative returns obtained over a moving window of 252 trading days.

Fig. 3.8: Trailing returns for the mixed signals model considering several time intervals (6 months, 1 year, 2 years, 3 years) and $\gamma = -3\%$ and sample period ranging from December 11, 2000 up to May 31, 2013, comprising 3,086 daily observations.



We can observe a similar result from Table 3.2. This table depicts the proportion of positive excess returns of the mixed signals model compared specifically to a buy-and-hold strategy in the Bovespa index (BVSP index). Despite the average positive excess returns for the proposed model is inferior to the Bovespa index, its average negative excess returns is much smaller, what guarantees a superior cumulative performance and evidences its potential to control for higher losses.

Tab. 3.2: Comparative table for trailing returns (for the mixed signals model and a buy-and-hold strategy in the BVSP index) in different time intervals and sample period ranging from December 11, 2000 up to May 31, 2013, comprising 3,086 daily observations.

	%Proportion of positive excess returns		Average positive excess returns		Average negative excess returns	
Time interval	Mixed signals model	BVSP index	Mixed signals model	BVSP index	Mixed signals model	BVSP index
3 years	72%	58%	36%	70%	-13%	-30%
2 years	62%	56%	29%	47%	-11%	-32%
1 year	58%	53%	18%	30%	-10%	-25%
6 months	56%	49%	12%	19%	-8%	-16%

Figures 3.6, 3.7 and 3.8 suggest that our mixed signals model with empirical robust loss presents a superior performance when compared both to the buy-and-hold strategies and random walk model. This could be explained as follows. Firstly, our asset classes exhibit a clear varying pattern in their relationships, during the time period under analysis. As the relationship between asset classes vary, it is desirable an adaptive model to capture this dynamics. The proposed specification (3-14 – 3-18) considers this dynamic in the decision process, both in the objective function (return forecasts) and in the constraints (returns dependence).

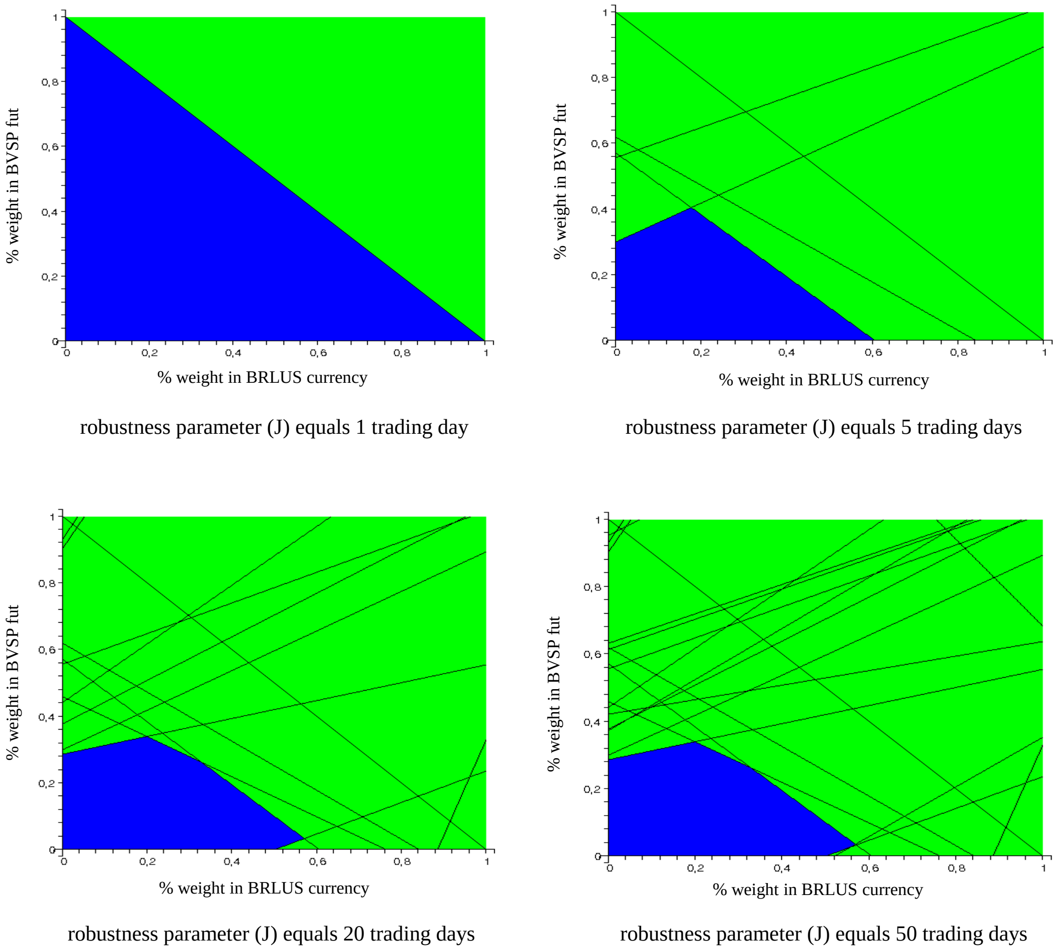
Another important feature of the model is its adaptive robust loss function. The following figures 3.9, 3.10 and 3.11 depict feasible regions for the portfolio weights (%) in each asset class to give further evidence of its behavior during different market conditions. We plot the portfolio weights (%) in BRLUS currency on the horizontal axis versus the portfolio weights (%) in BVSP index on the vertical axis. The feasible regions generated by the problem constraints are plotted in blue. A proper interpretation of these figures enables the reader to better grasp the ability of our model in capturing the evolving dynamics among different asset classes and market conditions.

Figure 3.9 depicts the feasible region of the % weights in the US dollar and the Bovespa index, during a crisis period. More specifically it was generated considering the portfolio allocation in October 2008.[27] By varying the parameter J which controls for the robust loss constraint, we can observe the existence of different feasible regions for the portfolio allocation. In Figure 3.9 we let the parameter J assume the following values $J \in (1, 5, 20, 50)$ trading days. For instance, in our specification when $J = 5$ we have 5 different constraints (one for each day) resulting from equation (3-15) whilst for $J = 50$ we have 50 different constraints, resulting from equation (3-15). One can note that as J assume higher values, the umbrella constraints make the feasible region smaller.[28] As a result, since the investor must be fully allocated, one can notice that the resulting allocation will move towards the risk-free asset, the DI spot rate (which may be understood as being in the third dimension of this two-dimensional plot).

[27] The financial subprime crisis started in earnest after Lehman's failure in mid September 2008 and saw its sharpest decline in October.

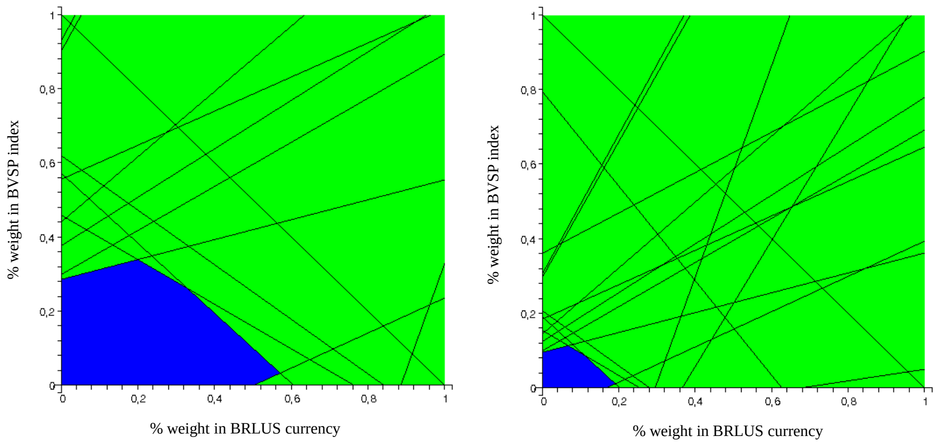
[28] The umbrella constraints are defined in Ardakani and Bouffard (2013) as those constraints which are both necessary and sufficient to describe the feasible set of an optimization problem.

Fig. 3.9: Feasible region of the percentage portfolio weights in each asset class (plotted in blue) for the robust optimization problem considering two risky assets (BRLUS currency and BVSP index) and one risk-free asset (DI spot rate) available for investment - varying the parameter J



In Figure 3.10 we can check the effect of parameter γ in our portfolio problem. One can compare the behavior of feasible regions for risky asset classes, if we let γ take different values. If we first let γ take the value of -3% and then change this value to -1%, one can note that the risky asset classes allocation experiences a sharp decrease. The model solution tells the investor to switch part of her resources to the risk-free asset. Our model seems to be effectively sensitive to the predefined investor risk aversion parameter, namely γ.

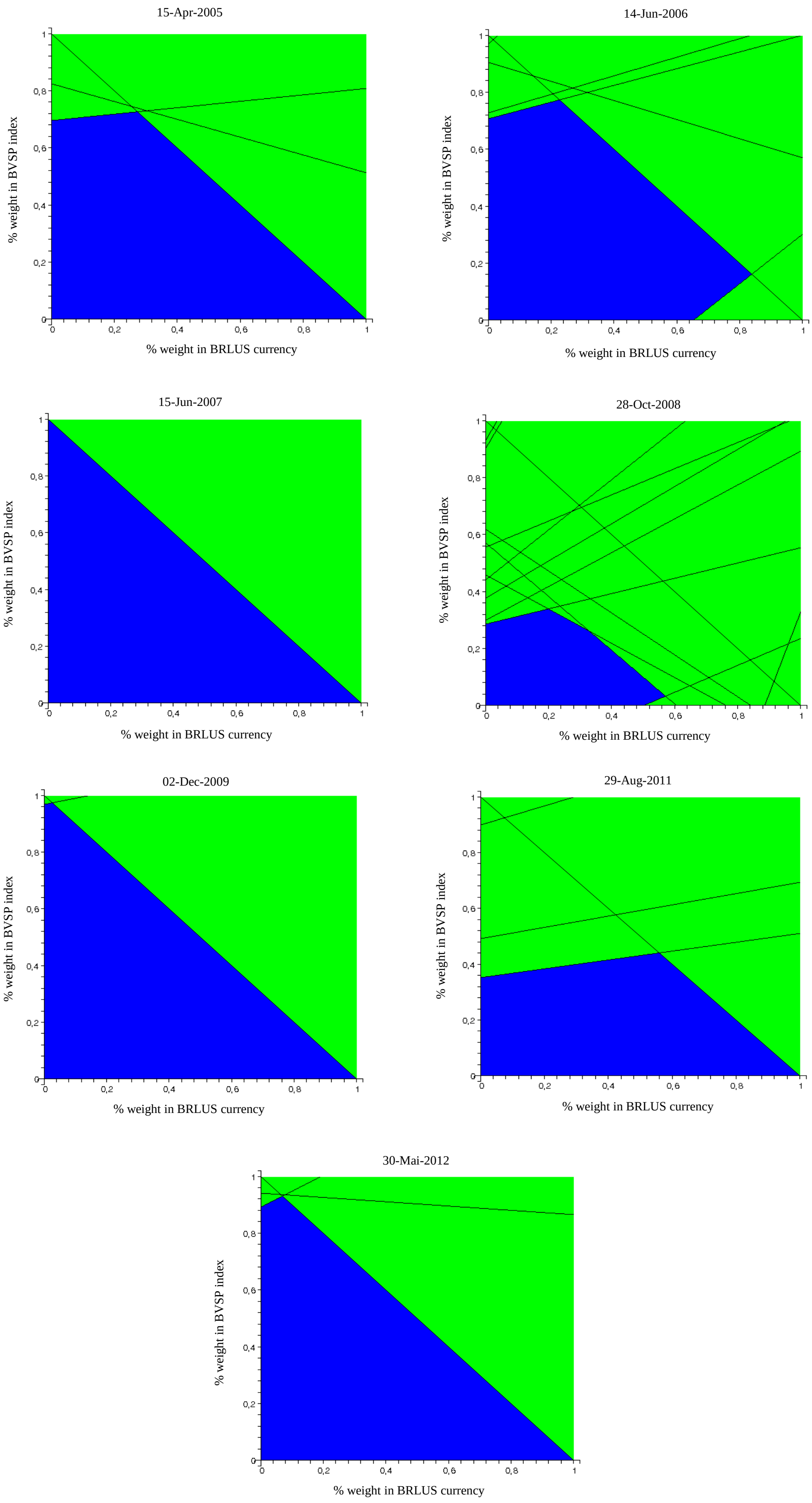
Fig. 3.10: Feasible region of the percentage portfolio weights in each asset class (plotted in blue) for the robust optimization problem considering two risky assets (BRLUS currency and BVSP index) and one risk-free asset (DI spot rate) available for investment - varying the loss parameter γ



loss limit: -3% loss limit: -1%

Figure 3.11 depicts a collection of snapshots for the feasible regions for risky asset classes, considering different time periods, which include bull and bear markets. We can notice that during crisis periods (as of subplots of October 2008 and August 2011) the feasible regions for risky asset classes get tighter as the loss constraints get active and thus we have a move towards an allocation to the risk-free asset.

Fig. 3.11: Feasible region of the percentage portfolio weights in each asset class (plotted in blue) for the robust optimization problem considering two risky assets (BRLUS currency and BVSP index) and one risk-free asset (DI spot rate) available for investment - different market conditions.



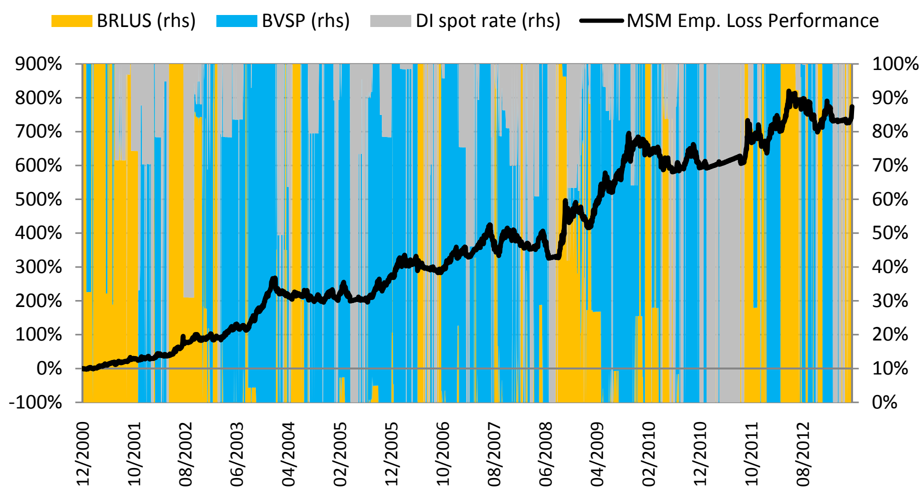
To investigate the decision making process implemented by our model, we recall Figure 3.4 where we plotted the risky asset returns in two specific periods (which we call crisis (October, 28^{th}, 2008) and non crisis periods (June, 15^{th}, 2007).

During the 2008 crisis period, the risky assets exhibited extreme negative returns. Simultaneously, one can observe from Figure 3.11 (in June, 15th, 2007 and October, 28th, 2008 subplots) that the empirical robust loss specification is able to treat this dynamic behavior as market conditions change. As expected, after big negative movements, the feasible region of the % weights in risky assets gets gradually smaller and the optimization model shifts the optimal allocation towards the risk-free asset.

It is well known that robust portfolios do exhibit a non-inferiority property (see Ben-Tal et al. (2006)). Whenever the asset returns are realized within the prescribed uncertainty set, the realized portfolio return will be greater than or equal to the calculated worst-case portfolio return. Nevertheless this property may fail to hold when the asset returns happen to fall outside the uncertainty set. When a rare event (such as a market crash) occurs, the asset returns can materialize far beyond the uncertainty set, and hence the robust portfolio will remain unprotected. A simple way to overcome this problem is to enlarge the uncertainty set to cover also the most extreme events. However, this can lead to robust portfolios that are too conservative and perform poorly under normal market conditions. We decided to consider the existence of a risk-free asset in our portfolio selection problem to check if our model would decide to allocate in this asset when risky assets exhibits returns beyond the predefined uncertainty sets. This pattern could be also verified in Figure 3.7 (during 2008 financial crisis and also during the year of 2011, our optimal portfolio shifts its optimal allocation from risky asset to the risk-free asset).

In the sequel we investigate the cumulative performance of our mixed signals model with robust loss specification by plotting the optimal time-varying portfolio composition in each asset class in Figure 3.12.

Fig. 3.12: Dynamic asset allocation vs cumulative performance evolution (MSM Emp Loss $\gamma = -3\%$) - sample period: Dec 10, 2000 - May 31, 2013: 3,087 daily observations.



One can notice that when both risky assets present a downtrend, the model chooses to be allocated in the risk-free asset (as short selling is not allowed). This is verified specially during the year of 2011 when both risky assets were decreasing in value. The same behavior is observed for other high volatility periods (as of late 2008). This is the effect of using the proposed specification to model the portfolio loss. Another interesting result is noticed when optimal resulting allocation in the BRLUS is increased just after 2008 financial crisis and late 2011. The proposed MSM seems to produce signals which are well captured by the portfolio model. This is also true for periods 2001 and late 2002. On the other hand, from 2004 to 2008, during a downtrend movement in the BRLUS our model does not suggest any relevant allocation in this class. A similar analysis can be made for the BVSP asset class. The model seems to capture the up trend movements, whereas controls for the loss of the strategy. Specifically we can observe that the strategy suggests allocation in equity class during the upward movement from 2003 to mid 2008. During the severe crisis of 2008, the model quickly reverts this exposure and reduces its allocation in equities drastically. A similar behavior is observed during the year of 2011.

Despite being a fairly realistic portfolio optimization algorithm (as it considers real transaction costs and investment fees[29]), this first case study is limited as

[29] We understand that our transaction costs model has its limitations as it considers the trading costs incurring over financial traded volume and not over the number of contracts. Also, we did not

it does not allow for short selling or leverage positions, which are quite common in the Brazilian financial markets. Encouraged by preceding evidences, in the next subsection we investigate a more flexible portfolio experiment in what concerns its investment strategies and instruments.

3.3.2 Case Study 2: A Hedge Fund Problem (RA-HFP)

Allowing for exposure in derivatives (specifically futures contracts) potentially increases the dynamic strategy risk profile as leverage can be enhanced and short positions allowed. We investigate the model behavior in an out-of-sample exercise considering a hedge fund strategy applied to the Brazilian financial market. The manager may decide to invest (long or short) in some different asset classes, represented by the following linear derivatives[30], namely:

- U.S. Dollar Futures Contract (BRLUS fut);[31]
- Ibovespa Futures Contract (BVSP fut);
- Gold Futures Contract (Gold fut);[32]
- Brazilian one-day Interbank Deposit Futures Contract (DI fut);[33]

We also consider the existence of a risk-free asset (cash equivalent), namely the Brazilian interbank deposit rate (DI spot rate). The hedge fund manager might decide whether to invest in any of the derivatives listed above or to just keep the cash invested in the DI spot rate. Data was collected from Cetip and Bloomberg systems. We calculate discrete asset returns based on daily price observations during the same sample period (as of April 5, 2000 up to May 31, 2013, comprising 3,257

carry a study over market depth as we consider a case study in most liquid asset classes in Brazil. The same is true for investment fees, as we decide to consider just the existence of management fee. Portfolio managers may also charge incentive fees on performance over selected benchmarks.

[30] In this case study, we model the portfolio problem to allow trading in futures contracts. For Ibovespa Futures Contract and U.S. Dollar Futures Contract we consider the maturity which was the most liquid for each day and for the Brazilian one-day Interbank Deposit Futures Contract we consider the 1 year maturity.

[31] Exchange rate of Brazilian Reais (BRL) per US Dollars for cash delivery, according to the provisions of Resolution 3265 of 2005 of the National Monetary Council (CMN).

[32] Gold in bars, cast by a refiner and kept in a depository institution, both accredited by BM&FBovespa.

[33] Interest rate effective up to the contract expiration date, defined as the capitalized daily Interbank Deposit (DI) rates verified on the period between the trading day and the last trading day of the contract. It is quoted as effective interest rate per year, based on 252 business days, to three decimal places. In the following study we work with price numbers, as we converted the collected rates.

observations). We implement a hedge fund optimization model considering realistic transaction costs applied to Brazilian derivatives and a management fee of 1% per annum, calculated on a daily basis over the back-tested cumulative returns.[34] The optimization model is implemented in Mosel and solved using Xpress solver.

To line up with Brazilian hedge funds with different risk profiles, we calculate the optimal dynamic allocation for several daily percentage loss parameter values $\gamma = \{-0.10\%, -0.15\%, -0.20\%, -0.30\%, -0.50\%, -1\%\}$. Table 3.3 depicts the annualized values for average return and volatility as well as tail loss measure (CVaR) and maximum exposures (for long and short positions).[35]

[34] Transaction costs for derivatives are calculated as per BM&FBovespa methodology and brokerage fees by trading volume and slippage costs. All execution cost incurs over the traded volume at each period of time. As we consider that the fund might invest in derivatives, we consider that the corresponding margin value blocked is corrected at the DI spot rate. To ensure the optimization of the investors resources, BM&FBOVESPA corrects the margin value during the period that the margin was blocked at a rate close to the DI spot rate (considering that investors can use Brazilian government bonds as eligible collateral). The margin required is the minimum amount the participant must maintain deposited at the clearinghouse to guarantee the settlement of the obligations resulting from the transactions assigned to her. To replicate a typical Brazilian hedge fund strategy, we consider a management fee of 1%per annum.

[35] We let the model vary the exposure in derivatives in the range -100% up to +100% of the funds' net asset value (nav). For the DI future which exhibits a lower volatility level, we let the model vary the exposure from -400% up to +400% of the funds' nav.

Tab. 3.3: Comparative table varying the loss parameter γ.

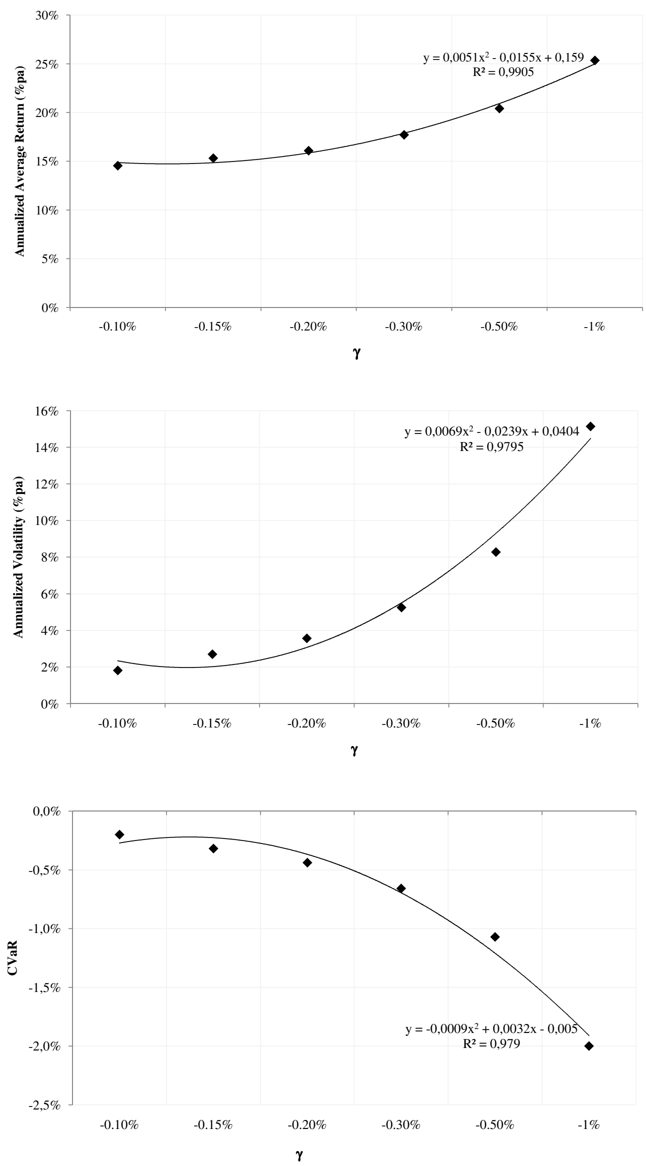
Loss parameter (gamma)	Annualized Average Return %pa	Annualized Volatility %pa	CVaR %pd	Exposure in risky derivatives: Maximum Long positions (% nav)				Maximum Short positions (%nav)			
				BRLUS fut	BVSP fut	Gold fut	DI fut	BRLUS fut	BVSP fut	Gold fut	DI fut
-0.10%	14,54%	1,81%	-0,20%	41%	11%	25%	380%	-32%	-9%	-17%	-366%
-0.15%	15,30%	2,69%	-0,32%	62%	17%	38%	400%	-48%	-13%	-25%	-400%
-0.20%	16,07%	3,57%	-0,44%	83%	23%	51%	400%	-63%	-20%	-33%	-400%
-0.30%	17,69%	5,25%	-0,66%	100%	33%	76%	400%	-95%	-31%	-50%	-400%
-0.50%	20,40%	8,28%	-1,07%	100%	53%	100%	400%	-100%	-53%	-84%	-400%
-1%	25,33%	15,14%	-2,00%	100%	100%	100%	400%	-100%	-100%	-100%	-400%

Comparing the optimal portfolios results[36] one can notice that from both ex-post risk measures (as of volatility and tail loss (CVaR) values), we can verify that varying the loss parameter can effectively control the risk assumed by the strategy. This is also verified by the difference in magnitude between maximum exposures (long or short). Further, for all values of γ the model let the maximum short exposure be smaller or equal to the maximum long exposure. This is in line with our intuition that short positions are riskier than long ones.[37] We complement this analysis by plotting in Figure 3.13 the risk aversion parameter γ by some realized measures, such as average return, volatility and tail loss (CVaR).

[36] We set the parameter J to 45 trading days and robust parameter B to 20 trading days.

[37] The outcome of a short sale is basically the opposite of a regular buy transaction, but the mechanics behind a short sale result in some unique risks. In a short position, losses can be infinite while the upside is limited. When the price moves against the trade, the trade exposure in fact increases in value what enhances the assumed risk.

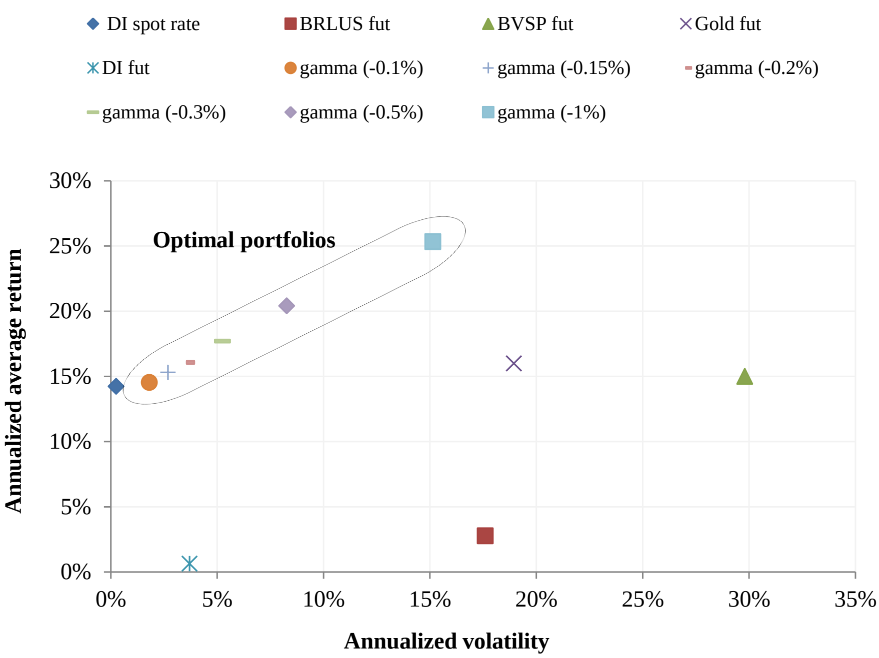
Fig. 3.13: Risk aversion parameter γ versus optimal portfolios' measures.



Those graphs present some evidence that by varying the value of γ in the problem setup, it is possible to obtain optimal portfolios that exhibit more stable risk levels over time. We found direct relationships among the risk aversion parameter γ with optimal portfolios' risk measures (volatility and CVaR). It is possible to obtain any desired level of risk by choosing the appropriate value for γ. Another interesting result is given by the optimal portfolios' average returns and the risk aversion parameter. As expected, we find evidence of a positive risk-return trade-off.[38]

In Figure 3.14 we plot the risk-reward relationship among the optimal portfolios varying the loss parameter. Considering the existence of a risk-free asset (as the DI spot rate, for instance) one can notice that the optimal (mixed signals models) portfolios, with various values for γ, exhibit higher Sharpe ratios when compared to buy-and-hold risky strategies.

Fig. 3.14: Risk-return relationship varying the loss parameter γ.



To further evaluate the model against available benchmarks, we compare our model to the ANBIMA's Hedge Fund Index - IHFA (a hedge fund index based on the evolution of a portfolio composed of selected funds that represent the Brazilian hedge funds sector calculated by ANBIMA - Brazilian Financial and Capital Markets Association).[39]

Figure 3.15 depicts the cumulative performance for two selected optimal portfolios (with $\gamma = (-0.15\%, -0.20\%)$) against the DI spot rate and the IHFA index. We chose those two portfolios based on their risk measures that were very close to that of the IHFA. Our model presents a superior cumulative performance when compared to both the DI spot rate[40] and the industry benchmark - IHFA (hedge fund

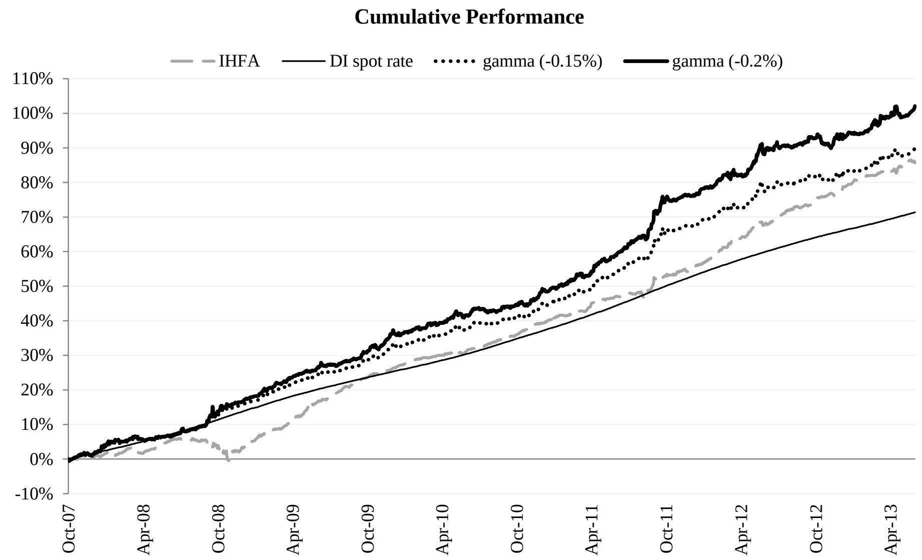
[38] Which is the balance between the desire for the lowest possible risk and the highest possible return.

[39] Website:http://www.andima.com.br/ihfa

[40] Usual benchmark for hedge funds in the Brazilian fund industry.

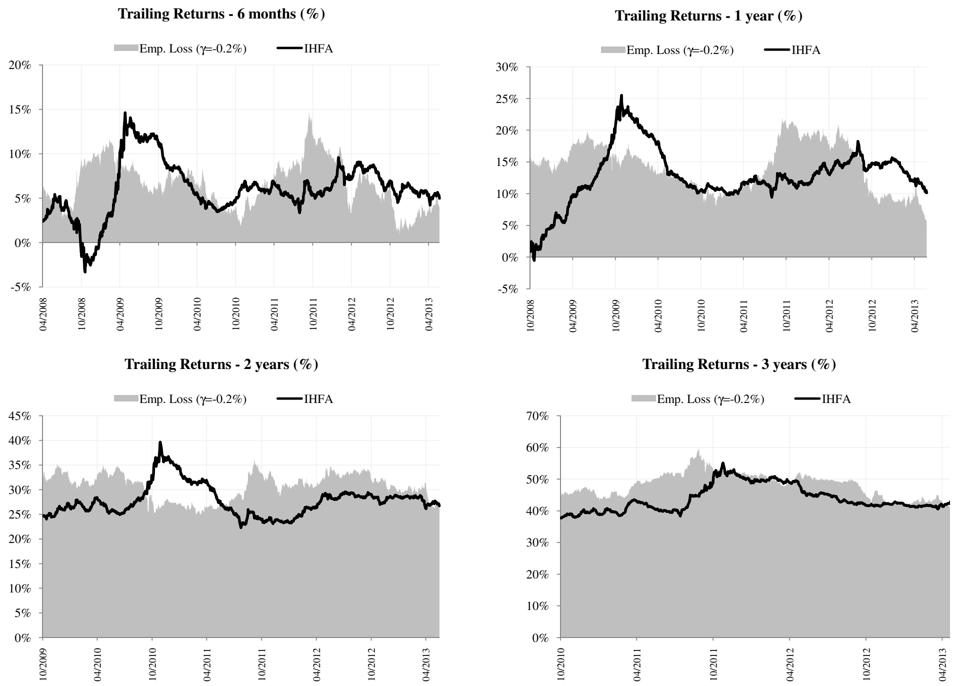
index).

Fig. 3.15: Cumulative performance for selected mixed signals models.



Further, to check whether this evidence is valid for the different market periods over our sample, we consider the investor hold this strategy for different time intervals and plot in Figure 3.16 trailing returns for several time periods for the proposed model with a conservative risk profile (in which $\gamma = -0.20\%$). The plots show an interesting consistency in returns. For any investor that keeps her resources allocated in this strategy for at least 1 year, she receives a minimum return of 8%, what seems in line with a conservative risk profile. Also we can observe from this plot that the proposed model exhibits a higher frequency of positive returns when compared to the IHFA benchmark. And this behavior is even more pronounced with an increase in the time interval considered.

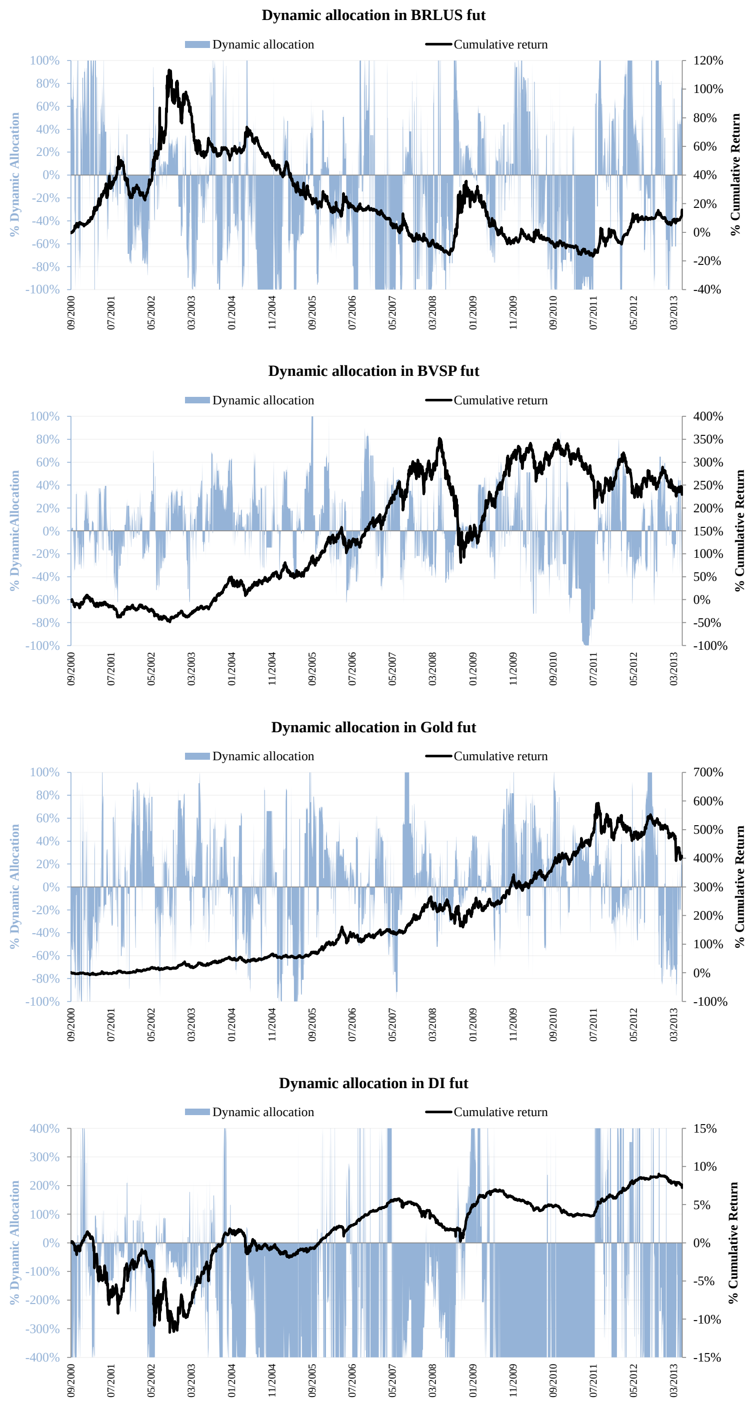
Fig. 3.16: Trailing returns for the mixed signals model vs IHFA considering several time intervals (6 months, 1 year, 2 years, 3 years) and $\gamma = -0.2\%$.



We plot the optimal time-varying portfolio composition[41] in the selected derivatives in Figure 3.17. In this case study, as we allow for leverage and short selling, those graphs get even more interesting to analyze (when compared to Figure 3.12).

[41] For the model with $\gamma = 1\%$.

Fig. 3.17: Dynamic portfolio allocation against cumulative return in selected derivatives.



In the first graph of Figure 3.17 we plot the dynamic optimal allocation in the BRLUS fut on the left vertical axis (plotted in blue) against its cumulative return on the right vertical axis (plotted in black). One can notice that during an upward trend in 2001 the model carries a long position in this derivative. In late 2001 it reverts its exposure and carries a short position until this downtrend is finished. This behavior is also true for the down trend from 2004 to 2008, during which it carries a short position for a long period of time. Later, during the financial crisis of 2008, the model rapidly captures the steep upward trend. More recently, during late 2012 and early 2013, specific up movements were well captured by this mixed signal strategy. In the second graph of Figure 3.17 we plot the dynamic optimal allocation in the BVSP fut on the left vertical axis (plotted in blue) against its cumulative return on the right vertical axis (plotted in black). A similar analysis can be made from its dynamic optimal allocation and ex-post cumulative performance. The model seems to capture the up trend movements, whereas controls for the loss of the strategy. For instance, we can observe that the strategy carries a high short exposure during the year of 2011 position, which seems to be a good decision. In the third graph of Figure 3.17 we plot the dynamic optimal allocation in the Gold fut on the left vertical axis (plotted in blue) against its cumulative return on the right vertical axis (plotted in black). The model seems to capture the up trend movement from late 2008 up to mid 2011; one can argue that Gold fut increased in value during those high volatility periods since it is typically considered a "safe haven"[42]. Further, it is interesting to notice that, on average, the model indicates a lower exposure to BVSP fut and Gold fut when compared to the BRLUS fut. This seems to be in line with the pre-specified risk aversion parameter, as both BVSP fut and Gold fut present a higher volatility level. From the last graph of Figure 3.17 we can investigate the model dynamic allocation in the DI fut.[43] The model indicates, during the year of 2004, a short exposure in this derivative.[44] A short position in the price of a bond is equivalent to a long position in interest rates. And indeed this was a very profitable strategy as interest rates in Brazil experienced an up trend during this period of time. Later, during the 2008 crisis, the model experiences a short period of maximum long exposure in this derivative (which is equivalent to short interest rate). This was also a successful strategy as during this time the government reduced its short term interest rate. A similar behavior is verified for the second semester of 2011. In summary, we found evidence of a dynamic behavior in the selected derivatives. The model seems to capture time-varying patterns both for up trends and down trends, while control for portfolio losses considering each derivative risk

[42] An investment that is expected to retain or even increase its value in times of market turbulence.

[43] We considered the evolution of a buy-and-hold allocation in the bond (its price evolution).

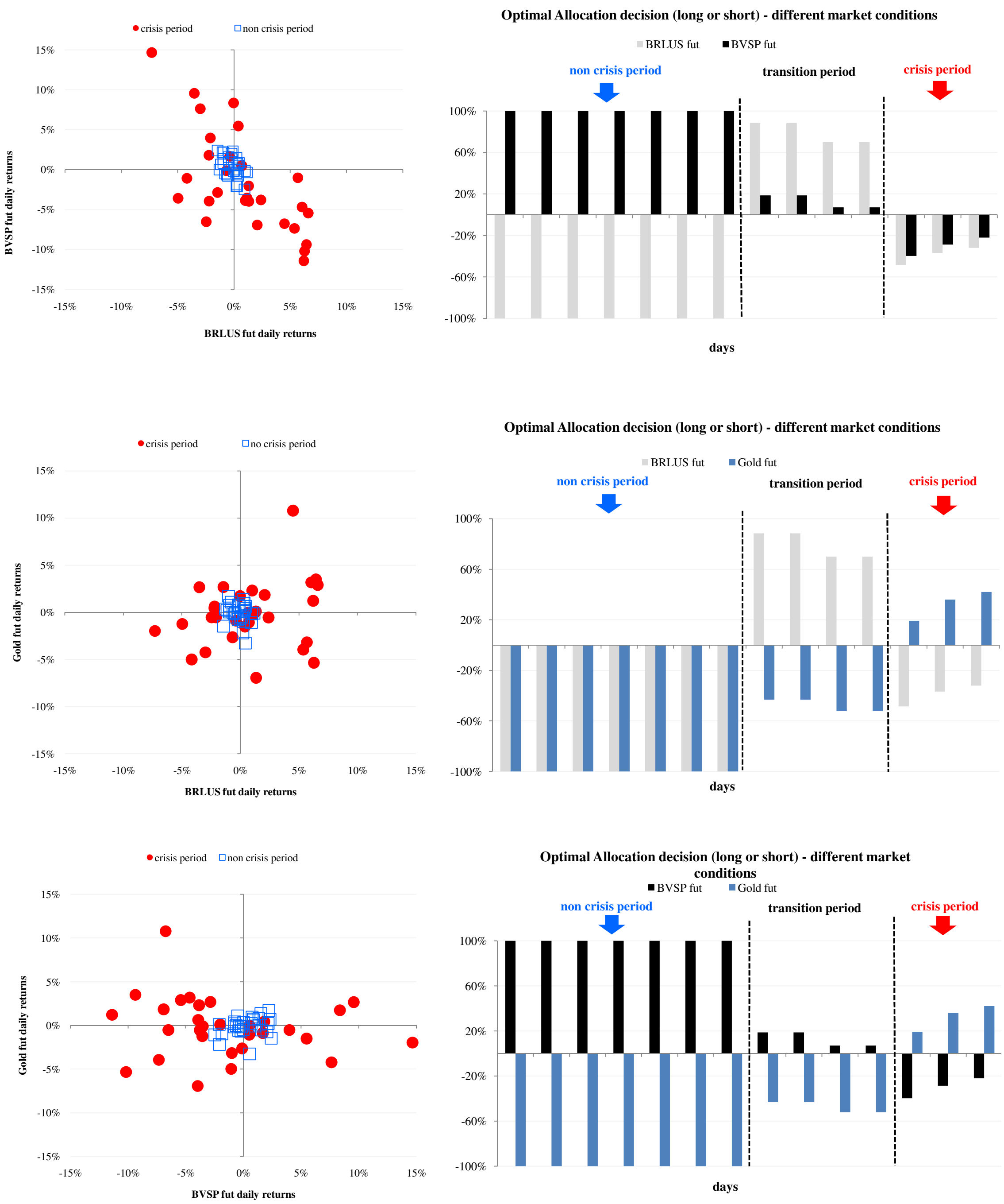
[44] As traded in Brazilian markets, when we carry a long position in the bond it is equivalent to be short in interest rates.

contribution to the portfolio total risk.

To analyze the evolving dynamics among the risky derivatives and the way the model captures and treats those relationships we present some ex-ante and ex-post numbers in Figure 3.18. We illustrate on the left side the risky derivatives daily returns (ex-ante) considering two different market conditions (crisis and non crisis periods)[45] and on the right side the optimal ex-post allocation in each risky asset.

[45] For the crisis period, t refers to October, 28^{th}, 2008 and for the non crisis period, t refers to June, 15^{th}, 2007.

Fig. 3.18: Dynamic portfolio allocation in selected derivatives during crisis and non crisis periods - observed returns as inputs and optimal allocation as results.



The first graph on the left side depicts a scatter plot of the BRLUS fut daily returns on the horizontal axis versus the BVSP fut daily returns on the vertical axis. We plot in blue the daily returns for the non crisis and in red the daily returns for the crisis period. The first graph on the right side plots the optimal allocation decision (ex-post) during a non crisis period, a transition period and a crisis period for both derivatives (BRLUS fut in gray and BVSP fut in black). During non crisis periods, the relationship between them seems to be stable over time. When it comes to transition and crisis period, the optimal allocation decision significantly changes, what suggests the adaptive robust model can capture this varying pattern in the relationship. A similar analysis can be made for the following graphs in Figure 3.18.

The empirical results of the investigation reported in this paper suggest that both adaptive (asset allocation and hedge fund) robust optimization models yield potentially cost effective optimal portfolios across assets and time periods. Evidence suggests that the proposed adaptive robust formulation provides significant risk protection while outperform selected benchmark strategies.

3.4 Concluding Remarks

Robust portfolio optimization has emerged over the last decade as a tractable and insightful approach to decision-making under uncertainty. In this paper we present an adaptive robust asset allocation problem focused on linear programming and polyhedral uncertainty sets (connected to the decision-maker's attitude towards risk). We set the objective function as an adaptive mixed signals strategy and discuss two case studies to investigate the feasibility of the solution. Despite being a simpler formulation[46], the first case study offered some important insights on the use of adaptive strategies and the evolving dynamics among asset classes returns in the Brazilian market. As a further step towards practical implementation in the second case study we allowed for short selling and leverage in selected asset classes which are quite common in hedge fund optimization algorithms. We calibrated the position maximum size in each asset class, varying the risk aversion input parameter γ. This parameter was effective to control for the observed volatility of each strategy. Evidence suggests that it is possible to obtain higher returns when compared to benchmark strategies (naive, buy-an-hold, etc) considering both conditional multivariate probability distribution of returns and transaction costs. Out-of-sample results indicate that the applied RO specification, in particular, could better control

[46] One important common feature of our specifications is that they can be solved with available linear programming software, yet they allow for flexible formulations in which the anticipation of future expected returns, as well as a tolerance level for the error in forecasts, can be explicitly modeled.

the risk and enhance the performance of portfolio optimization methods. Hence we believe both robust adaptive data-driven models provide a interesting alternatives to traditional portfolio problems under uncertainty.

4
Paper 3: A Robust Portfolio Model based on Bertsimas & Sim Approach

4.1 Introduction

Robust optimization models have played a major role in real portfolio problems, providing a powerful tool to solve the sensitivity issue of the classical mean-variance models. Robust optimization (RO) techniques are best applied where parameter values are unknown and their distributions are uncertain since they allow portfolio managers to incorporate estimation errors directly into the portfolio optimization problem. In this framework, the perturbations in the parameters are modeled as unknown, but bounded, and optimization problems are solved assuming worst case behavior of these perturbations.[1] The uncertainty model is not stochastic, but rather set-based and the decision maker can construct a solution that is optimal for the worst-case realization of the uncertainty in a given set.[2]

The earliest studies on robust optimization date back to the early 1970s, when Soyster (1973) proposed the first robust model for linear optimization problems with uncertain data. The resulting model was known to be very conservative in the sense that it gives too much of optimality for the nominal problem in order to ensure robustness. Much later, only in the 1990s, the optimization community revived the interest in robust formulations and a number of important robust formulations and applications followed. Ben-Tal and Nemirovski Ben-Tal and Nemirovski (1999, 2000) as well as Ghaoui and Lebret Ghaoui and Lebret (1997); Ghaoui et al. (1998) provided detailed analysis of the robust optimization framework in linear optimization and general convex programming. Their papers pointed out that the approach originally proposed by Soyster (1973) results in robust solutions that exhibit an objective function value much worse than that of the nominal problem as it protects against the worst-case scenario which may not be meaningful for decision-making process. To address this issue they propose less conservative models by considering uncertain linear problems with ellipsoidal uncertainties, which involve solving the robust counterparts of the nominal problem in the form of conic quadratic prob-

[1] The robustness of a solution is given by its ability to remain feasible with respect to data changes.

[2] Under RO, modelers agree to accept a suboptimal solution for the nominal values of the data, in order to ensure that the solution remains feasible and near optimal when the data change.

lems. However, a practical drawback of their approach is that it leads to nonlinear, although convex, models, which are more demanding computationally than the earlier linear models by Soyster Soyster (1973).

Since then other authors have been working on alternative solutions that could reduce the conservatism of the solution (also known as the price of robustness[3]). Bertsimas and Sim (2004) proposed an alternative method to control for the level of conservatism of the solution while retaining the advantages of the linear framework introduced by Soyster. They designed the problem to protect against the violation of a specified constraint j deterministically, when only a predetermined number Γ_j (also known as budget of uncertainty) of coefficients change. Furthermore, their reasoning guarantees that not only the solution is feasible if less than Γ_j of coefficients change, but also it is still feasible with high probability even if more than Γ_j of coefficients change. In fact, the robust optimization (RO) methodology has been used to various applications (see Ghaoui et al. (2003); Goldfarb and Iyengar (2003); Pinar and Tutuncu (2005); Ben-Tal et al. (2006); Bertsimas and Thiele (2006); Bertsimas and Pachamanova (2008); Bertsimas and Brown (2009); Gregory et al. (2011)). A recent comprehensive survey of theory and applications of RO is discussed in Gabrel et al. (2013).

In this work we provide an extension of Bertsimas and Sim (2004) approach to model uncertainty sets within the framework of robust optimization for linear optimization problems with uncertain parameters. Our model considers adaptive polyhedral uncertainty sets based on two deterministic parameters to control for the level of protection of the solution. The proposed uncertainty sets are defined as dynamic sets described by a historical covariance structure of returns calculated over moving windows, in which no more than a predetermined number Γ of assets could change simultaneously from a given dynamic estimated nominal value. This method seems efficient to adjust the robustness of the problem against the level of the conservatism of the solution. We then evaluate the behavior, robustness and cost of the robust counterpart formulation of a portfolio optimization problem in which unknown parameters are modeled by this predefined polyhedral data-based set. Our model specification is designed to be robust to changes in market conditions both in the problem constraints as in the objective function (we develop a nonparametric adaptive weighted sum method for the objective function of our sample problem

[3] Bertsimas and Sim (2004) introduced this concept which considers how "heavily" the objective function value is penalized when we are guarded against objective under performance or constraint violation.

which uses several standard parameter look-backs for estimation).

4.2 A Robust Adaptive Portfolio Model based on Bertsimas & Sim approach

Bertsimas and Sim (2004) propose a robust formulation that is linear and is able to withstand parameter uncertainty without excessively affecting the objective function and readily extends to discrete optimization problems. They begin considering the following general robust formulation of a discrete linear optimization problem. Let $\mathbf{c}, \mathbf{l}, \mathbf{u}$ be n-vectors, let $\mathbf{A}$ be a $(j \times n)$ matrix and $\mathbf{b}$ be j-vector. The problem is given by

$$\begin{aligned} \max_{\mathbf{x} \in \Re^n} \ & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \\ \mathbf{A}\mathbf{x} & \leq \mathbf{b} \\ \mathbf{l} \leq \mathbf{x} & \leq \mathbf{u} \\ \forall \mathbf{A} & \in \mathbf{\Xi} \end{aligned} \tag{4-1}$$

where $\mathbf{x}$ is the vector of decision variables. Hereafter, it is assumed that data uncertainty only affects the elements in matrix $\mathbf{A}$.

Bertsimas and Sim (2004) introduced a concept of budgeted robust counterpart, relaxing the condition that the optimal solution must be feasible $\forall \mathbf{A} \in \mathbf{\Xi}$ under the assumption that not every parameter would take its worst-case value simultaneously. For that they motivate the formulation as follows. They consider a specific j^{th} row of the uncertain matrix $\mathbf{A}$ with the j^{th} constraint of the nominal problem $\mathbf{a_j'x} \leq b_j$ and let I_j represent the set of coefficients in this row that are subject to parameter uncertainty. For every j, they introduce a parameter Γ_j, which can take any real value in the interval $[0, |I_j|]$. This parameter is used to adjust the robustness of the proposed method against the level of conservatism of the solution (as the value of Γ_j affects the structure of Ξ) . The problem is designed to protect against the violation of a specified constraint j deterministically, when only a predetermined number Γ_j of coefficients change.[4] The solution would be feasible just for some

[4] We can understand $|I_j|$ as the cardinality of set I_j. If Γ_j is integer it is interpreted as the maximum number of parameters that can deviate from their nominal values. When Γ_j is not an integer it indicates that it can vary up to a fraction of a given asset.

$A \in \Xi$ as its goal is to be protected against all cases that up to Γ_j of these coefficients are allowed to change. In this robust optimization framework, the true value a_j of an uncertain parameter, is given by

$$a_j = \bar{a_j} + \hat{a_j} e_j, \forall j \tag{4-2}$$

where $\bar{a_j}$ is a statistical estimate of the expected value of a_j, $\hat{a_j}$ is a statistical estimate of the maximum distance that a_j is likely to deviate from the point estimate $\bar{a_j}$ and e_j is a random variable which is bounded by and symmetrically distributed within the interval $[-1, 1]$. Bertsimas and Sim (2004) modeled uncertainty by budgeted polyhedral uncertainty sets

$$\Xi_\Gamma = \left\{ \mathbf{e} \in \mathbb{R}^n | \sum_j |e_j| \leq \Gamma,\ -1 \leq e_j \leq 1, \forall j \right\} \tag{4-3}$$

Let us present the linear robust counterpart to the portfolio optimization problem, where the robustness is specified in the objective function. The basic portfolio problem is defined as

$$\begin{aligned} \max_{\mathbf{x} \in \Re} \sum_i R_i x_i \\ \text{s.t.} \\ \sum_i x_i = 1 \\ x_i \geq 0, \forall i \end{aligned} \tag{4-4}$$

where the asset returns R_i are uncertain parameters with unknown distributions defined as bounded and symmetric with respect to a point estimate $\bar{R_i}$.[5] Let e_i be a stochastic variable that measures the deviation of parameter R_i from $\bar{R_i}$ and takes values in $[-1, 1]$ being $e_i = \frac{(R_i - \bar{R_i})}{\sigma_i}$, where σ_i is the standard deviation of R_i. Rearranging this equation we can express R_i as

[5] They assume that even though the true distribution of R_i is unknown, historical data can be used to estimate the mean return of asset i.

$$R_i = \bar{R}_i + \sigma_i e_i \tag{4-5}$$

Let $|I|$ be the number of parameters R_i that are uncertain. Bertsimas and Sim (2004) define Γ as a parameter of budget of uncertainty, which is the number of uncertain parameters that take their worst case value $\bar{R}_i - \sigma_i$ for $x_i \geq 0$. Therefore they define $\sum_{i=1}^{n} |e_i| \leq \Gamma$, such that $\Gamma \in (0, |I|)$. Rewriting the portfolio optimization problem substituting (4-5) in (4-4) we get

$$\begin{aligned} \max_{\mathbf{x} \in \Re} \sum_i (\bar{R}_i x_i + \sigma_i e_i x_i) \\ \text{s.t.} \\ \sum_i x_i = 1 \\ x_i \geq 0, \forall i \end{aligned} \tag{4-6}$$

The second term of the objective function can be written as

$$\begin{aligned} -\min_{\mathbf{f}} \sigma_i f_i x_i \\ \text{s.t.} \\ 0 \leq f_i \leq 1 : z_i, \forall i \\ \sum_i f_i \leq \Gamma : l \end{aligned} \tag{4-7}$$

where z_i and l are the dual variables associated to the problem. By duality this corresponds to the problem

$$\begin{aligned} -\max_{z_i, l} (l\Gamma + \sum_i z_i) \\ \text{s.t.} \\ l + z_i \geq \sigma_i x_i, \forall i \\ z_i, x_i \geq 0, \forall i \\ l \geq 0 \end{aligned} \tag{4-8}$$

Substituing this result in (4-6) we can get the following robust optimization problem

$$\begin{aligned} \max_{\mathbf{x}\in\Re} \sum_i \bar{R}_i x_i - \lambda\Gamma - \sum_i z_i \\ \text{s.t.} \\ \sum_i x_i = 1 \\ \lambda + z_i \geq \sigma_i x_i, \forall i \\ \lambda \geq 0 \\ x_i, z_i \geq 0, \forall i \end{aligned} \qquad (4\text{-}9)$$

The robust counterpart of an uncertain optimization problem has as objective to optimize the worst-case performance. Soyster (1973) and Ben-Tal and Nemirovski (1999, 2000) models stipulate that every constraint must be feasible for every uncertain parameter defined within a bounded symmetric set (every uncertain parameter taking its worst case value). Bertsimas and Sim (2004) introduce a model that assumes at most Γ uncertain parameters will take their worst case values and not every parameter. In problem (4-9) one desires the portfolio with the best worst-case return given that Γ asset returns take their worst case values, $\bar{R}_i - \sigma_i, x_i \geq 0$. As a result, this formulation provides optimal portfolios less conservative in the sense that its objective function is not too penalized.

Following the above formulation, we describe the discrete-time model[6] we use for asset return dynamics and formulate the optimal investment problem in its terms.

Let us consider a dynamic wealth maximization problem subject to an adaptive robust portfolio loss and budget constraints $\forall t$ given by

$$\begin{aligned} \max_{\mathbf{x}\in\Re^n} \hat{\mathbf{R}}_t'\mathbf{x_t}, \\ \text{s.t.} \\ L(\mathbf{R}_t, \mathbf{x_t}) \leq \varepsilon_1 \\ B(\mathbf{R}_{t-1}, \mathbf{x_t}, \mathbf{x_{t-1}}, \mathbf{c_t}) \geq \varepsilon_2 \end{aligned} \qquad (4\text{-}10)$$

where n is the number of available assets, $\mathbf{R}_t' = (R_{t,1}, R_{t,2}, ..., R_{t,n})$ is the vector

[6] In discrete time models the asset return dynamics are primarily governed by a rule that dictates how the price or return changes from one period to the following one.

of asset returns, $\mathbf{x_t}' = (x_{t,1}, x_{t,2}, ..., x_{t,n})$ is the vector of decision variables (where each $x_{t,i}$ corresponds to the financial asset allocation in asset i to be executed at the beginning of day t)[7] and n is the number of available assets. The loss constraint function is denoted by $L(.)$ and the budget function is denoted by $B(.)$.

The adaptive forecast for future asset return i at time t is given by signal $\hat{R}_{t,i}$. We assume a mixed signals model to predict future asset returns which dynamically selects the signal that performs better, considering an out-of-sample analysis (as described in (3-4) and (3-5)). A detailed formulation for the built-in signals is provided in Appendix.

In (4-10) the loss function is given by $L(.)$ which is a general loss function that depends on past observed returns and the decision vector $\mathbf{x_t}$ which should be greater than a negative scalar ε_1. Our loss specification is built on Bertsimas and Sim (2004) and uses historical-covariance data to determine the portfolio loss limits whilst considering the investor risk tolerance. Furthermore we define the budget function $B(.)$ considering the existence of transaction costs.[8] This function depends on both past returns $\mathbf{R_{t-1}}$ and decision vector $\mathbf{x_t}$ and also on transaction costs $\mathbf{c_t}$ at time t.

Under this general formulation the decision-maker constructs a solution that is optimal for any realization of the uncertainty in a given set. In the sequel, we outline the proposed approach to construct the uncertainty sets based on Bertsimas and Sim (2004).

4.2.1 Covariance-based Adaptive Robust Loss Function

We propose a covariance-based robust loss function specification to describe the associated polyhedron for the problem constraints. This robust problem is studied with respect to the robust framework of Bertsimas and Sim (2004) to model optimization problems with data uncertainty.

The proposed model assumes uncertainty in the problem constraints. Therefore, it is somehow different from the simple portfolio problem (4-9) which assumes uncertainty in the objective function.

Let us assume there exists only one row j in the uncertain matrix A (see (4-1)). Each row entry $a_i, i \in I$ is assumed to be an asset return which can be

[7] We assume the investor allocates all her wealth at each time t. For that, the mixed signals model consider past signals up to time $t-1$ to forecast the return at time t.

[8] A detailed specification for transaction costs is provided in the Appendix.

modeled as a symmetric and bounded random variable ($R_i, i \in I$) and take values in $[\mu_i - \eta\hat{\sigma}_i e_i, \mu_i + \eta\hat{\sigma}_i e_i]$. The parameter μ_i is the mean or expectation of the distribution and the parameter $\hat{\sigma}_i$ is its estimated standard deviation. We use the parameter η to address one specific aspect of our uncertainty set: its scale. In fact, there are two main aspects of uncertainty sets, named structure and scale. As we will see hereafter, in terms of structure we consider polyhedral uncertainty sets (and by changing Γ will change the number of bounds, which define the polyhedron). And in terms of scale we introduce a new factor named η to redefine our uncertain parameter R_i.[9]

Let us assume the following conditional distribution for $n-$asset returns in period t:

$$\mathbf{R}_t = \boldsymbol{\mu}_t + \eta \mathbf{G}_t \mathbf{e}, \tag{4-11}$$

where $\mathbf{R}_t$ is the $(n \times 1)$ vector of asset returns in period t[10] and $\boldsymbol{\mu}_t$ is the $(n \times 1)$ vector of conditional expected returns estimated for period t. The scalar η which multiplies each element of the triangular matrix $\mathbf{G_t}$ is used to calibrate the tolerance interval for the robust loss constraint. The lower triangular matrix $\mathbf{G_t}$ $(n \times n)$ results from the Cholesky factorization of the conditional covariance matrix of returns estimated for period t ($\boldsymbol{\Sigma}_t = \mathbf{G_t}\mathbf{G_t}'$). The matrix $\mathbf{G_t}$ introduces a conditional dependency in the asset's returns. Finally, let $\mathbf{e}$ denote the $(n \times 1)$ vector of random variables defined in the uncertainty set Ξ_Γ

$$\Xi_\Gamma = \big\{\mathbf{e} \big| \sum_{m=1}^{n} |e_m| \leq \Gamma,\ -1 \leq e_m \leq 1, \forall m \big\} \tag{4-12}$$

We understand it is unlikely that all of the uncertain parameters $R_i, i \in I$ will change at the same time to adversely affect the solution. Therefore, we will use this method to stipulate deterministically how many coefficients could change simultaneously and analyze how we can protect our portfolio results against all cases up to Γ of these coefficients are allowed to change. Note that when $\Gamma = 0$ none of the uncertain parameters take their worst-case value; thus, the budgeted robust counterpart is similar to the non-robust nominal problem and there is no protection against uncertainty. When $\Gamma = |I|$, all of the uncertain parameters take their worst-

[9] While the parameter $\hat{\sigma}_i$ tells us how R_i deviates from the point estimate μ_i the scale factor tell us by how much it deviates.

[10] A random variable.

case value; thus, the solution must be feasible $\forall e \in \mathbf{\Xi_\Gamma}$ and the model converges to the Soyster method, which yield a very conservative solution. Therefore, by allowing $\Gamma \in [0, |I|]$ the decision-maker makes a trade-off between the protection level of the constraint and the degree of conservatism of the solution.

As we want to guarantee that the optimal portfolio returns belong to a given convex uncertainty set Ξ_Γ, we have

$$\mathbf{R}(\mathbf{e})_t'\mathbf{x}_t \geq \gamma W_{t-1}, \forall e \in \Xi_\Gamma \tag{4-13}$$

which is equivalent to

$$\min_{\mathbf{e}\in\Xi_\Gamma} \sum_{i=1}^{n} R(e)_{t,i} x_{t,i} \geq \gamma W_{t-1} \tag{4-14}$$

Let us assume $\mu_{t,i} = \hat{R}_{t,i}$ and that $g_{t,mi}$ is an element of the lower $(n \times n)$ triangular matrix $\mathbf{G_t}$. We arrive at the following nonlinear formulation of the problem

$$\begin{aligned} \max_{\mathbf{x}\in\Re^n} \sum_{i=1}^{n} \hat{R}_{t,i} x_{t,i} \\ \text{s.t.} \\ \sum_{i=1}^{n} \hat{R}_{t,i} x_{t,i} + \eta \min_{\mathbf{e}\in\Xi_\Gamma} \sum_{i=1}^{n} \sum_{m=1}^{n} g_{t,mi} e_m x_{t,i} \geq \gamma W_{t-1} \\ B(\mathbf{R}_{t-1}, \mathbf{x_t}, \mathbf{x_{t-1}}, \mathbf{c_t}) \geq \varepsilon_2 \end{aligned} \tag{4-15}$$

As known from Bertsimas and Sim (2004), the nonlinear formulation of the loss restriction

$$\min_{\mathbf{e}\in\Xi_\Gamma} \sum_{i=1}^{n} \sum_{m=1}^{n} g_{t,mi} e_m x_{t,i}$$

can be written as

$$
\begin{aligned}
-\max_{\mathbf{e}} \sum_{m=1}^{n} \Big| \sum_{i=1}^{n} g_{t,mi} x_{t,i} \Big| f_m \\
\text{s.t.} \\
0 \leq f_m \leq 1 : z_m, \forall m \\
\sum_{m=1}^{n} f_m \leq \Gamma : \lambda
\end{aligned}
\tag{4-16}
$$

where z_m and λ are the dual variables associated with the problem. By duality, (4-16) corresponds to the solution of the following optimization problem

$$
\begin{aligned}
-\min_{\lambda, z_m} (\lambda\Gamma + \sum_{m=1}^{n} z_m) \\
\text{s.t.} \\
\lambda + z_m \geq y_m, \forall m \\
-y_m \leq \sum_{i=1}^{n} g_{t,mi} x_{t,i} \leq y_m, \forall m \\
z_m, y_m \geq 0, \forall m \\
\lambda \geq 0
\end{aligned}
\tag{4-17}
$$

Substituting this result in (4-15) we can get the following optimization problem under the covariance-based adaptive robust loss

$$
\begin{aligned}
\max_{\mathbf{x}\in\Re^n} \sum_{i=1}^{n} \hat{R}_{t,i} x_{t,i} \\
\text{s.t.} \\
\sum_{i=1}^{n} \hat{R}_{t,i} x_{t,i} - \eta\lambda\Gamma - \eta \sum_{m=1}^{n} z_m \geq \gamma W_{t-1} \\
\lambda + z_m \geq y_m, \forall m \\
-y_m \leq \sum_{i=1}^{n} g_{t,mi} x_{t,i} \leq y_m, \forall m \\
z_m, y_m \geq 0, \forall m \\
\lambda \geq 0 \\
B(\mathbf{R}_{t-1}, \mathbf{x_t}, \mathbf{x_{t-1}}, \mathbf{c_t}) \geq \varepsilon_2
\end{aligned}
\tag{4-18}
$$

4.2.2 Budget Constraint

To describe the budget constraint in (4-18) we consider an adaptive hedge fund[11] decision problem, in which the fund manager may use advanced investment strategies, such as leverage or short selling using linear financial derivatives (futures contracts, for instance). We consider a wealth maximization problem in which the objective function is a linear combination of expected returns over derivatives exposure and cash equivalents positions.[12] As a typical hedge fund, this specification allows for exposures in derivatives (denoted by $w_{t,i} \in \mathbb{R}$, where $i = 1, ..., n$[13]) as well as positions in cash equivalents (for simplicity, denoted by a single cash

[11] We will consider that this investment strategy might be called loosely as a hedge fund model since we allow for more flexible investment strategies compared to the prior case study.

[12] Being an investment fund, we consider the portfolio manager maintains the fund liquidity invested in cash equivalents. Since the fund invests in derivatives, we consider that the corresponding margin value blocked is corrected at the DI spot rate, considering that the fund can use government bonds as eligible collateral.

[13] Here n is the number of available derivatives.

equivalent $x_{t,c} \in \mathbb{R}^+$). The hedge fund problem, $\forall t$ is given by

$$\max_{\mathbf{x} \in \Re^n} \left[\hat{R}_{t,c} x_{t,c} + \sum_{i=1}^{n} \hat{R}_{t,i} w_{t,i} \right]$$

$$\text{s.t.}$$

$$\begin{aligned}
\sum_{i=1}^{n} \hat{R}_{t,i} w_{t,i} - \eta\lambda\Gamma - \eta \sum_{m=1}^{n} z_m &\geq \gamma W_{t-1} \\
\lambda + z_m &\geq y_m, \forall m \\
-y_m \leq \sum_{i=1}^{n} g_{t,mi} w_{t,i} &\leq y_m, \forall m \\
z_m, y_m &\geq 0, \forall m \\
\lambda &\geq 0 \\
x_{t,c} &= W_{t-1} - \sum_{i=2}^{n} c_i (u_i^+ + u_i^-) \\
w_{t,i} &= w_{t-1,i}(1 + R_{t-1,i}) + u_i^+ - u_i^-, \forall i \\
u_i^+ &\geq 0, u_i^- \geq 0, \forall i
\end{aligned} \tag{4-19}$$

Since the fund might trade derivatives and cash equivalents, we consider transaction costs are paid using the fund cash equivalent and all exceeding resources at time t are invested in cash equivalents.

4.3 Case Study

In this section we investigate the proposed model (4-19) in an out-of-sample exercise considering a hedge fund strategy applied to the Brazilian financial market. The manager may decide to invest (long or short) in some different asset classes, represented by the following linear derivatives[14], namely:

- U.S. Dollar Futures Contract (BRLUS fut);[15]
- Ibovespa Futures Contract (BVSP fut);

[14] In this case study, either short selling or leverage is allowed. For Ibovespa Futures Contract and U.S. Dollar Futures Contract we consider the maturity which was the most liquid for each day and for the Brazilian one-day Interbank Deposit Futures Contract we consider the 1 year maturity.

[15] Exchange rate of Brazilian Reais (BRL) per US Dollars for cash delivery, according to the provisions of Resolution 3265 of 2005 of the National Monetary Council (CMN).

· Gold Futures Contract (Gold fut);[16]

· Brazilian one-day Interbank Deposit Futures Contract (DI fut);[17]

We also consider the existence of a risk-free asset (cash equivalent), namely the Brazilian interbank deposit rate (DI spot rate). The hedge fund manager may decide whether to invest in any of the derivatives[18] listed above or to just keep the cash invested in the DI spot rate. Data was obtained from Cetip and Bloomberg databases. We calculate discrete asset returns based on daily price observations during the same sample period (as of April 5, 2000 up to May 31, 2013, comprising 3,257 observations). We implement a hedge fund optimization algorithm considering the existence of real transaction costs applied to Brazilian derivatives and a management fee of 1% per annum, calculated on a daily basis over the back-tested cumulative returns.[19] The programming solver Xpress was used to run the optimization problems.

Our objective is to compose a hedge fund portfolio with the highest daily return subject to a controlled loss. To analyze the empirical results obtained by the different specifications we will consider a metric of cumulative return (3-24) and a metric based on a risk-adjusted return index (3-25).

Table 4.1 depicts the annualized values for average return and volatility as well as tail loss measure ($CVaR_\alpha(R_{p,t})$) and maximum exposures (for long and short positions[20]) for all optimal portfolios with a limit daily loss γ set to -0.2% and

[16] Gold in bars, cast by a refiner and kept in a depository institution, both accredited by BM&FBovespa.

[17] Interest rate effective up to the contract expiration date, defined as the capitalized daily Interbank Deposit (DI) rates verified on the period between the trading day and the last trading day of the contract. It is quoted as effective interest rate per year, based on 252 business days, to three decimal places. In the following study we work with price numbers, as we converted the collected rates.

[18] The resulting dynamic program does not suffer from the curse of dimensionality. As we specified our model as a linear optimization problem (with linear transaction costs) it is quite simple to solve it for portfolios of multiple assets.

[19] Transaction costs for derivatives are calculated as per BM&FBovespa methodology and brokerage fees by trading volume and slippage costs. All execution cost incurs over the traded volume at each period of time. As we consider that the fund might invest in derivatives, we consider that the corresponding margin value blocked is corrected at the DI spot rate. To ensure the optimization of the investors resources, BM&FBOVESPA corrects the margin value during the period that the margin was blocked at a rate close to the DI spot rate (considering that investors can use Brazilian government bonds as eligible collateral). The margin required is the minimum amount the participant must maintain deposited at the clearinghouse to guarantee the settlement of the obligations resulting from the transactions assigned to her. To replicate a typical Brazilian hedge fund strategy, we consider a management fee of 1%per annum.

[20] We let the model vary the exposure in derivatives in the range -100% up to +100% of the funds' net asset value (nav). For the DI future which exhibits a lower volatility level, we let the model vary

varying the structure parameter Γ and the scale factor η.

Tab. 4.1: Comparative table varying parameters η and Γ for $\gamma = -0.2\%$.

					Exposure in risky derivatives							
	Annualized Average Return %pa	Annualized Volatility %pa	$CVaR_\alpha$ %pd	$ICVaR_\alpha$	Maximum Long positions (%nav)				Maximum Short positions (%nav)			
					BRLUS fut	BVSP fut	Gold fut	DI fut	BRLUS fut	BVSP fut	Gold fut	DI fut
Loss parameter (γ) equals -0.2%												
Empirical Robust Loss Control												
	16.1%	3.6%	-0.4%	0.135	83%	23%	51%	400%	-63%	-20%	-33%	-400%
Parameterized Robust Loss Control (η =1.0)												
Γ=0	54.6%	51.2%	-7.3%	0.024	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=0.5	57.2%	47.6%	-7.3%	0.025	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=1.0	44.5%	35.3%	-5.4%	0.027	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=1.5	34.1%	23.8%	-3.6%	0.033	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=2.0	28.0%	16.0%	-2.4%	0.042	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=2.5	26.3%	12.8%	-1.9%	0.050	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=3.0	23.8%	11.8%	-1.7%	0.049	100%	100%	99%	400%	-98%	-100%	-100%	-399%
Γ=3.5	23.8%	11.4%	-1.7%	0.051	100%	100%	100%	400%	-94%	-100%	-100%	-400%
Γ=4.0	23.4%	11.3%	-1.7%	0.050	100%	100%	99%	400%	-93%	-100%	-100%	-400%
Parameterized Robust Loss Control (η =2.0)												
Γ=0	54.6%	51.2%	-7.3%	0.024	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=0.5	44.5%	35.3%	-5.4%	0.027	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=1.0	26.5%	14.5%	-2.1%	0.044	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=1.5	21.9%	6.5%	-0.8%	0.096	100%	100%	100%	400%	-100%	-100%	-96%	-400%
Γ=2.0	20.5%	4.2%	-0.5%	0.160	100%	100%	100%	400%	-100%	-100%	-28%	-400%
Γ=2.5	19.9%	3.6%	-0.4%	0.182	100%	100%	100%	400%	-100%	-100%	-27%	-400%
Γ=3.0	19.4%	3.5%	-0.4%	0.180	97%	97%	98%	389%	-96%	-96%	-26%	-388%
Γ=3.5	19.6%	3.5%	-0.4%	0.182	97%	96%	98%	393%	-98%	-98%	-25%	-391%
Γ=4.0	19.5%	3.6%	-0.4%	0.182	97%	95%	98%	391%	-97%	-98%	-25%	-388%
Parameterized Robust Loss Control (η =3.0)												
Γ=0	54.3%	51.2%	-7.3%	0.024	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=0.5	33.3%	23.4%	-3.5%	0.021	100%	100%	100%	400%	-100%	-100%	-100%	-400%
Γ=1.0	21.7%	6.3%	-0.8%	0.037	100%	100%	100%	400%	-100%	-86%	-98%	-400%
Γ=1.5	18.0%	2.8%	-0.3%	0.080	100%	100%	100%	400%	-84%	-25%	-21%	-400%
Γ=2.0	17.0%	2.0%	-0.2%	0.135	100%	100%	100%	400%	-39%	-25%	-16%	-400%
Γ=2.5	17.0%	1.8%	-0.2%	0.158	100%	100%	100%	400%	-31%	-25%	-16%	-400%
Γ=3.0	16.7%	1.7%	-0.2%	0.157	98%	98%	98%	392%	-30%	-24%	-16%	-389%
Γ=3.5	16.8%	1.7%	-0.2%	0.158	97%	97%	99%	394%	-30%	-24%	-16%	-392%
Γ=4.0	16.8%	1.7%	-0.2%	0.158	97%	97%	98%	394%	-30%	-24%	-16%	-391%

Comparing the optimal portfolios results[21] one can notice that from both ex-post risk measures (as of volatility and $CVaR_\alpha(R_{p,t})$ values), we can verify that varying the scale factor $\eta \in (1, 2, 3)$ can effectively control the risk assumed by

the exposure from -400% up to +400% of the funds' nav.

[21] We set the robust parameter B to 20 trading days.

the strategy. As expected, when increasing the value of η (as long as $\Gamma \neq 0$) the optimal portfolio exhibits a lower risk level. This is also verified by the difference in magnitude between maximum exposures (long or short). Further, for all values of η the model let the maximum short exposure be smaller or equal to the maximum long exposure. This is in line with our intuition that short positions are riskier than long ones.[22] On the other hand, when we assume smaller values for η, varying structure parameter Γ has minor effects in controlling the portfolio risk level (even for greater values of Γ).

Furthermore we write in the first line of Table 4.1 the results found in Chapter 3 for the mixed signals model considering an empirical robust loss (for $\gamma = -0.2\%$). We will compare these previous results with the ones found in this Chapter. As one can notice, for $\eta = 1$ we could not achieve optimal portfolios with the same risk level as we found for the mixed signals model with empirical loss. For models with $\eta = 2$ and $\Gamma \geq 2.5$ and also for model $\eta = 3$ and $\Gamma = 1.5$ we could achieve optimal portfolios with similar risk levels, both from volatility and $CVaR_\alpha(R_{p,t})$ measures. Also, one can notice that for this latter loss specification we could obtain optimal portfolios with a higher average return and also with higher values for the risk-adjusted return index named $ICVaR_\alpha(R_{p,t})$.

In Figure 4.1 we plot the risk-reward relationship among the optimal portfolios varying the scale factor η and maintaining $\Gamma = 2.0$. Considering the existence of a risk-free asset (as the DI spot rate, for instance) one can notice that the optimal portfolios, with different values for η, exhibit higher Sharpe ratios when compared to buy-and-hold risky strategies.

[22] The outcome of a short sale is basically the opposite of a regular buy transaction, but the mechanics behind a short sale result in some unique risks. In a short position, losses can be infinite while the upside is limited. When the price moves against the trade, the trade exposure in fact increases in value what enhances the assumed risk.

Fig. 4.1: Risk-return relationship varying the scale factor η for$\gamma = -0.2\%$.

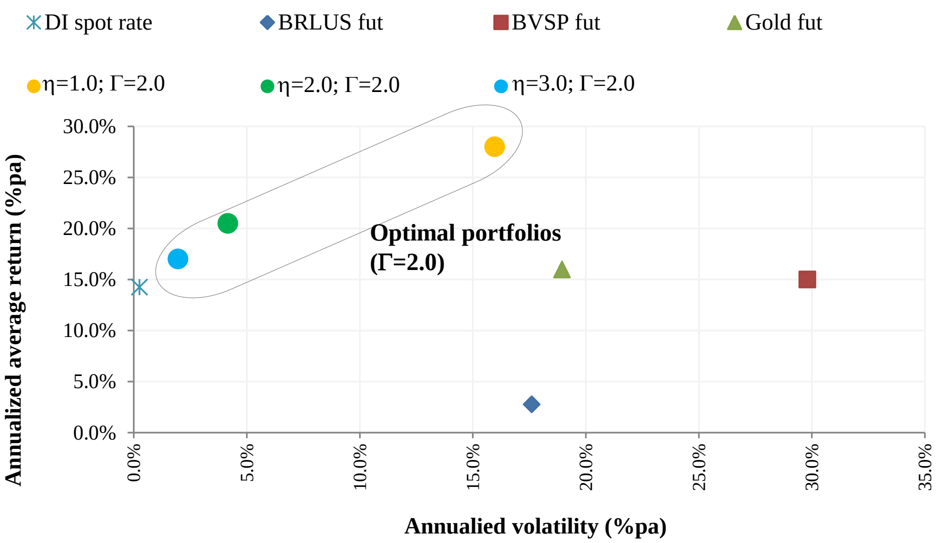
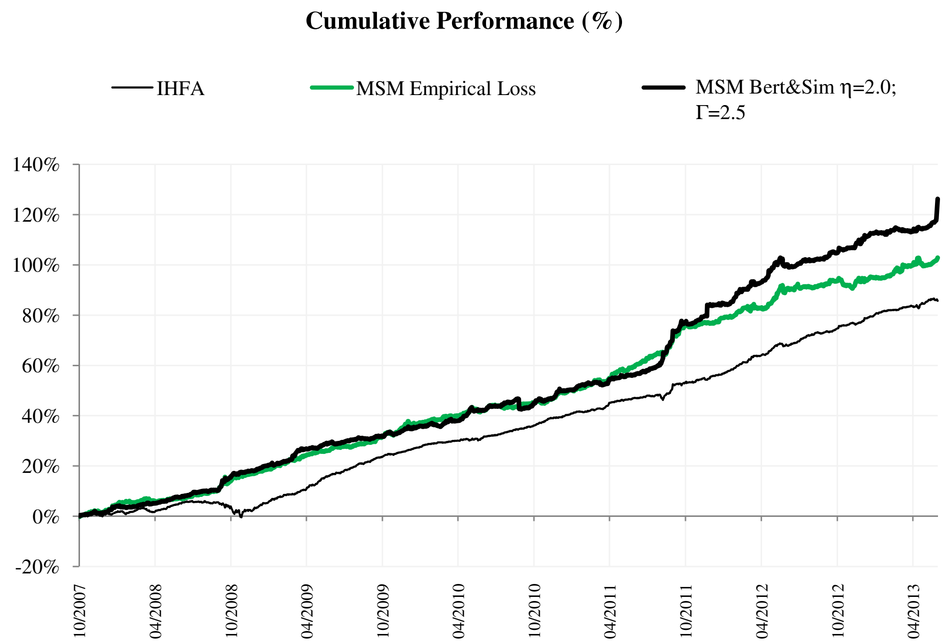


Figure 4.2 depicts the cumulative performance for those selected models (empirical loss and covariance-based loss (adapted from Bertsimas and Sim (2004) approach) with $\eta = 2$, $\Gamma = 2.5$ and $\gamma = -0.2\%$) against the ANBIMA's Hedge Fund Index - IHFA.[23]

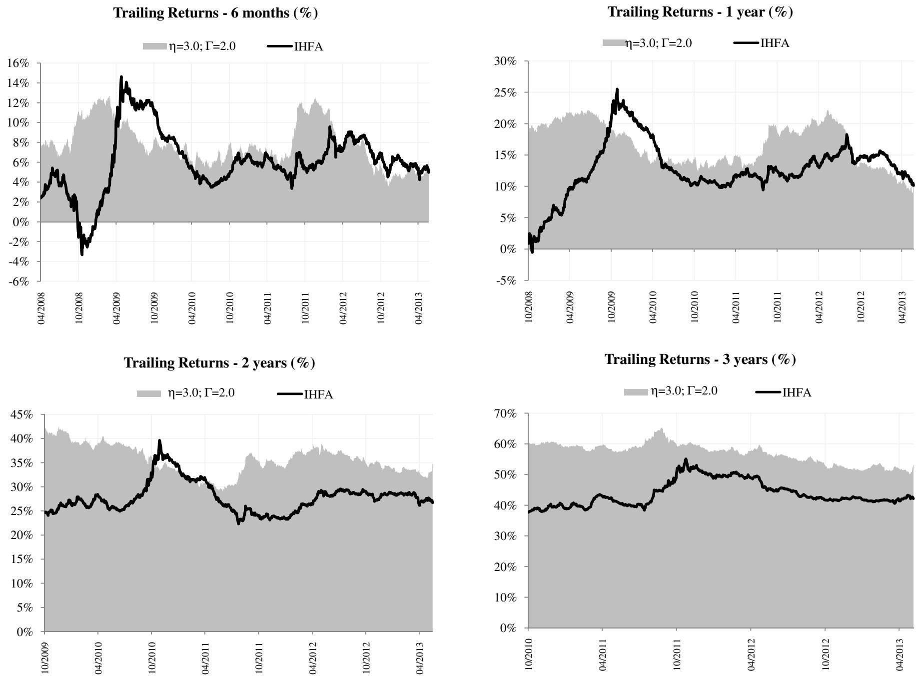
[23] A hedge fund index based on the evolution of a portfolio composed of selected funds that represent the Brazilian hedge funds sector calculated by ANBIMA - Brazilian Financial and Capital Markets Association.

Fig. 4.2: Cumulative performance for the mixed signal models with empirical loss and Bertsimas and Sim (2004) adapted loss with $\eta = 2, \Gamma = 2.5$ *and* $\gamma = -0.2\%$ *against the IHFA.*



Both optimal portfolios present superior cumulative performances when compared to the IHFA. To check whether this evidence is valid for the different market periods over our sample, we consider the investor hold this strategy for different time intervals and plot in Figure 4.3 trailing returns for several time intervals. In this graph we plot MSM with Bertsimas and Sim (2004) adapted loss trailing returns against IHFA trailing returns. One can notice that the model exhibits a higher frequency of positive returns with a greater magnitude. Furthermore, this behavior is even more pronounced with an increase in the time interval considered.

Fig. 4.3: Trailing returns for the mixed signals model based on Bertsimas and Sim (2004) approach, considering several time intervals (6 months, 1 year, 2 years, 3 years) and $\gamma = -0.2\%$.



4.4 Concluding Remarks

In this paper we applied the robust optimization methodology of Bertsimas and Sim (2004) and presented an alternative correlated model, considering the robustness in two levels. We set the objective function of the portfolio problem as an adaptive mixed signals strategy and discuss an application to a hedge fund problem to investigate the feasibility of the solution. The results of the investigation reported in this paper show that robust models yield the most robust and cost effective portfolios. Evidence suggests that it is possible to obtain higher returns when compared to benchmark strategies (buy-and-hold) considering both dynamic correlations and transaction costs. Furthermore, we illustrate interesting results concerning the two-level robustness. As expected, uncertainty sets with a larger range tend to result in higher costs, but increased robustness. And the scale factor η presented a more pronounced effect in the optimal portfolios generated when compared to the structure parameter Γ.

5
Conclusion

This thesis makes some contributions to the area of asset allocation optimization models under uncertainty. We have investigated two standard methods extensively adopted in the asset allocation literature to deal with estimation errors: the Bayesian approach and robust estimation methods.

In the first essay of this thesis presented in Chapter 2 we adopted the Bayesian approach and we address dynamic optimization trading strategies using the BL framework. Our challenge was to present how observed price-earnings ratio and returns can be used to determine a priori estimation of asset expected returns and how this can be integrated into the Black-Litterman model, regarding investors with different risk profiles.

In the following two essays presented in Chapters 3 and 4 respectively we applied robust optimization techniques to solve portfolio problems under uncertainty. We provided two different methodologies to construct uncertainty sets within the framework of robust optimization for linear optimization problems with uncertain parameters. The provided approach considered that optimal portfolio losses were modeled using a robust adaptive function. Its potential loss was limited by the worst-case scenario inside predefined dynamic uncertainty sets. More specifically, in the second essay we modeled uncertainty as polyhedral dynamic sets described by a list of its vertices, which are set as past assets returns obtained over moving windows with a length of J-days. This was an empirical method to construct an uncertainty set as its information set is limited to past returns. In the third essay we modeled uncertainty as polyhedral dynamic sets described by a historical covariance structure of returns calculated over moving windows. Under this latter specification, no more than a predetermined number Γ of assets could change simultaneously from a given dynamic estimated nominal value. This method was based on the approach introduced by Bertsimas and Sim (2004) and it was proved to be efficient to adjust the robustness of the problem against the level of the conservatism of the solution.

While the literature on robust portfolio optimization from operations research is plentiful and enlightening, there is in general a lack of empirical studies on how the methods work with real world applications (as with constraints on positions, leverage, etc). As more studies focus on the empirical aspects, we believe it will be a matter of time before some robust techniques reviewed here will become as indispensable as the classical framework to practitioners.

The essays, therefore, in addition to uncovering new evidence and providing new insight into specific asset allocation issues within the Brazilian financial market, put forward an alternative framework for researchers and specifically practitioners to assess investment outcomes. We hope the key findings that emerge from this thesis would be informative to the existing literature.

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6
Appendix

6.1 Chapter 2

6.1.1 The Proof of BL Formula for the Posterior Distribution of the Expected Returns

To arrive at the formulation for the posterior expected returns and its associated variance, we need to consider Bayes theorem. In the notation we have previously defined, Bayes theorem states that

$$f(\boldsymbol{\mu}|\boldsymbol{\pi}) = \frac{f(\boldsymbol{\pi}|\boldsymbol{\mu})f(\boldsymbol{\mu})}{f(\boldsymbol{\pi})} \tag{6-1}$$

where the terms above have the following interpretation:
- $f(\mathbf{P}\boldsymbol{\mu})$ is the prior pdf that expresses the (prior) views of the investor,
- $f(\boldsymbol{\pi})$ represents the marginal pdf of equilibrium returns (which disappears in the constant of integration),
- $f(\boldsymbol{\pi}|\boldsymbol{\mu})$ is the conditional pdf of the data equilibrium return.

And we can write the pdfs as

$$\begin{aligned} f(\boldsymbol{P\mu}) &= \frac{1}{\sqrt{2\pi^k|\boldsymbol{\Omega}|}} e^{-\frac{1}{2}(\boldsymbol{P\mu}-\boldsymbol{q})'\boldsymbol{\Omega}^{-1}(\mathbf{P}\boldsymbol{\mu}-\boldsymbol{q})} \\ f(\boldsymbol{\pi}|\boldsymbol{\mu}) &= \frac{1}{\sqrt{2\pi^n|\tau\boldsymbol{\Sigma}|}} e^{-\frac{1}{2}(\boldsymbol{\pi}-\boldsymbol{\mu})'\tau\boldsymbol{\Sigma}^{-1}(\boldsymbol{\pi}-\boldsymbol{\mu})} \end{aligned} \tag{6-2}$$

Considering that the numerator of formula (6-1) is proportional to

$$\propto e^{-\frac{1}{2}(\boldsymbol{P\mu}-\boldsymbol{q})'\boldsymbol{\Omega}^{-1}(\mathbf{P}\boldsymbol{\mu}-\boldsymbol{q})-\frac{1}{2}(\boldsymbol{\pi}-\boldsymbol{\mu})'\tau\boldsymbol{\Sigma}^{-1}(\boldsymbol{\pi}-\boldsymbol{\mu})} \tag{6-3}$$

Expanding the terms above, excluding the $-\frac{1}{2}$ and considering that both $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ are symmetric matrices, re-arranging we get

$$\propto \boldsymbol{\mu}' \left[\boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P} + (\tau\boldsymbol{\Sigma}^{-1})\right]\boldsymbol{\mu} - 2\left[\boldsymbol{q}'\boldsymbol{\Omega}^{-1}\boldsymbol{P} + \boldsymbol{\pi}(\tau\boldsymbol{\Sigma}^{-1})\right]\boldsymbol{\mu} + \left[\boldsymbol{q}'\boldsymbol{\Omega}^{-1}\boldsymbol{q} + \boldsymbol{\pi}'(\tau\boldsymbol{\Sigma}^{-1})\boldsymbol{\pi}\right] \tag{6-4}$$

From now on let us assume that

$$\begin{aligned} M_1 &= \left[\boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P} + (\tau\boldsymbol{\Sigma}^{-1})\right] \\ M_2' &= \left[\boldsymbol{q}'\boldsymbol{\Omega}^{-1}\boldsymbol{P} + \boldsymbol{\pi}(\tau\boldsymbol{\Sigma}^{-1})\right] \\ M_3 &= \left[\boldsymbol{q}'\boldsymbol{\Omega}^{-1}\boldsymbol{q} + \boldsymbol{\pi}'(\tau\boldsymbol{\Sigma}^{-1})\boldsymbol{\pi}\right] \end{aligned} \tag{6-5}$$

And we can write the expression (6-4) as

$$\propto \boldsymbol{\mu}'M_1\boldsymbol{\mu} - 2M_2'\boldsymbol{\mu} + M_3 \tag{6-6}$$

Given that $I = M_1^{-1}M_1$ and $M_1 = M_1'$ we can re-write

$$\begin{aligned} &\propto (M_1\boldsymbol{\mu})'M_1^{-1}M_1\boldsymbol{\mu} - 2M_2'M_1^{-1}M_1\boldsymbol{\mu} + M_3 \\ &\propto (M_1\boldsymbol{\mu} - M_2)'M_1^{-1}(M_1\boldsymbol{\mu} - M_2) + [M_3 - M_2'M_1^{-1}M_2] \\ &\propto (\boldsymbol{\mu} - M_1^{-1}M_2)'M_1(\boldsymbol{\mu} - M_1^{-1}M_2) + [M_3 - M_2'M_1^{-1}M_2] \end{aligned} \tag{6-7}$$

Since the last term in brackets is independent of $P\boldsymbol{\mu}$, it disappears in the constant of integration, and we have

$$f(\boldsymbol{\mu}|\boldsymbol{\pi}) \propto e^{-\frac{1}{2}[(\boldsymbol{\mu}-M_1^{-1}M_2)'M_1(\boldsymbol{\mu}-M_1^{-1}M_2)]} \tag{6-8}$$

Substituting the values of equations (6-7) we thus have the following distribution for $\boldsymbol{\mu}|\boldsymbol{\pi}$

$$\boldsymbol{\mu}|\boldsymbol{\pi} \sim N\left(\left[(\tau\boldsymbol{\Sigma}^{-1}) + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right]^{-1}\left[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{q}\right], \left[(\tau\boldsymbol{\Sigma}^{-1}) + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}\right]^{-1}\right) \tag{6-9}$$

6.2 Chapters 3 and 4

6.2.1 The Mixed Signals Model to Predict Future Returns - $\hat{R}_{t,i}$

We assume a mixed signals model for the objective function that could be flexible enough to dynamically select the signal that performs better, considering out-of-sample analysis. We understand that technical indicators can offer a different perspective from which to analyze the price action as they can provide information on the strength and direction of the asset price and/or return. We decided to construct some indicators that complement each other, mixing trend following (lagging indicators), momentum oscillators (leading indicators) and volatility signals. We will then compare the obtained results with a simple random walk model to predict future returns.

Trend followers indicators are based on the analysis of market prices or returns rather than fundamental figures of the companies. This trading strategy tries to take advantage of perceived trends betting that the trend will persist for a period of time. And it is intended to be capable of making profits from both the ups and downs movements in the asset price. Trend follower indicators give the signal only when a trend is already underway and as a consequence the investor always miss a bit of the profit. Momentum oscillators, on the other hand, present buy and sell prompts earlier in time, sometimes even before the trend has started. The investor in this case may well be going against the short term trend but the theory states that it is possible to capture all the benefit of a subsequent change of direction. Furthermore we decided to implement signals based on volatility as we understand that volatility indicators can help investors understand market cycles and improve price prediction.[1] We decided to use those indicators and let our learning algorithm choose dynamically which combination is better adherent to the data.

The first two signals are described as short term and long term simple moving averages[2] as

$$\begin{aligned} \hat{Sig}_{1,t,i} &= \frac{1}{K_{ST}} \sum_{d=1}^{K_{ST}} R_{t-d,i}, \forall i, t \\ \hat{Sig}_{2,t,i} &= \frac{1}{K_{LT}} \sum_{d=1}^{K_{LT}} R_{t-d,i}, \forall i, t \end{aligned} \tag{6-10}$$

where K_{ST} stands for the short term period and K_{LT} stands for the long term period. Those are parameters estimated by an out-of-sample cumulative performance analysis.

However, simple moving average can be disproportionately influenced by old data points and the strength of the dependence among asset returns usually de-

[1] Price movement is usually easier to predict over time than price direction. In fact, volatility signals usually indicates that a movement is about to happen.

[2] A moving average is an indicator that calculates an average price (return) of an asset over a specified number of periods. It filters out random noise and offers a smoother perspective of the price action and helps to define the current direction with a lag. It can be seen as a kind of finite impulse response filter as it smooth out short term fluctuations and highlight longer term trends or cycles.

creases as the separation of observations in time increases. We propose that greater weight should be associated to more recent observations. We thus decided to consider also signals extracted from an exponential moving average, which is a type of infinite impulse response filter that applies weighting factors which decrease exponentially in time. A formulation for signals 3 and 4, considering an exponentially weighted function is given by

$$\hat{Sig}_{3,t,i} = \sum_{d=1}^{K_{ST}} \alpha_d R_{t-d,i}, \forall i, t$$

where

$$\alpha_d = \frac{e^{\frac{-d}{K_{ST}}}}{\sum_{d=1}^{K_{ST}} e^{\frac{-d}{K_{ST}}}}, \ \forall d = 1, ..., K_{ST} \tag{6-11}$$

and

$$\hat{Sig}_{4,t,i} = \sum_{d=1}^{K_{LT}} \alpha_d R_{t-d,i}, \forall i, t$$

where

$$\alpha_d = \frac{e^{\frac{-d}{K_{LT}}}}{\sum_{d=1}^{K_{LT}} e^{\frac{-d}{K_{LT}}}}, \ \forall d = 1, ..., K_{LT} \tag{6-12}$$

And also, to reflect some leading movement in returns, we decided to construct signal 5[3] that is given by

$$\hat{Sig}_{5,t,i} = \frac{\bar{R}^{+}_{t-d,i}}{\bar{R}^{-}_{t-d,i}} - 1 \tag{6-13}$$

[3] Momentum oscillators usually measure the speed and change of price movements. The most common known indicators are Relative Strength Index (RSI) and moving average convergence divergence (MACD). The first technical indicator was developed by J. Welles Wilder in the 70s as a momentum oscillator defined to oscillate between zero and 100 and is considered overbought when above 70 and oversold when below 30. The latter was developed by Gerald Appel in the late 70s, and as although very simple it seems to be a very effective momentum indicator. The MACD turns two trend following indicators into a momentum oscillator by subtracting the longer moving average from the shorter moving average. It fluctuates above and below the zero line as the moving averages converge, cross and diverge.

where $\bar{R}^{+}$ corresponds to the average of all positive returns in the time period K_{mom} and $\bar{R}^{-}$ corresponds to the average of all negative returns in the time period K_{mom}.[4]

Besides those signals, we are also interested in volatility indicators. Thereupon we added a new signal specification that considers past returns mean and volatility to describe future returns. Signals 6 and 7 consider bands around the simple moving average as

$$\hat{Sig}_{6,t,i} = \begin{cases} \hat{\mu}_{t,i} - \varrho * \hat{\sigma}_{t,i} & \text{if } \hat{\mu}_{t,i} \leq 0 \\ \hat{\mu}_{t,i} + \varrho * \hat{\sigma}_{t,i} & \text{if } \hat{\mu}_{t,i} > 0 \end{cases}$$
$$\text{where}$$
$$\hat{\mu}_{t,i} = \frac{1}{K_{ST}} \sum_{d=1}^{K_{ST}} R_{t-d,i}, \tag{6-14}$$
$$\hat{\sigma}_{t,i} = \left[\frac{1}{K_{ST}} \sum_{d=1}^{K_{ST}} \left[R_{t-d,i} - \hat{\mu}_{t,i} \right]^2 \right]^{1/2}, \forall i$$

and ϱ is the number of standard deviations above the means, defined for identifying extreme changes.[5] We define $\hat{Sig}_{7,t,i}$ likewise replacing K_{ST} for K_{LT}.

Finally, to construct signals 8 and 9 we implement the same evaluation as for signals 6 and 7, but in this case, the mean is considered as the exponential moving average formula

$$\hat{Sig}_{8,t,i} = \begin{cases} \hat{\mu}_{t,i} - \varrho * \hat{\sigma}_{t,i} & \text{if } \hat{\mu}_{t,i} \leq 0 \\ \hat{\mu}_{t,i} + \varrho * \hat{\sigma}_{t,i} & \text{if } \hat{\mu}_{t,i} > 0 \end{cases}$$
$$\text{where}$$
$$\hat{\mu}_{t,i} = \sum_{d=1}^{K_{ST}} \frac{e^{\frac{-d}{K_{ST}}}}{\sum_{d=1}^{K_{ST}} e^{\frac{-d}{K_{ST}}}} R_{t-d,i} \tag{6-15}$$
$$\hat{\sigma}_{t,i} = \left[\frac{1}{K_{ST}} \sum_{d=1}^{K_{ST}} \left[R_{t-d,i} - \hat{\mu}_{t,i} \right]^2 \right]^{1/2}, \forall i$$

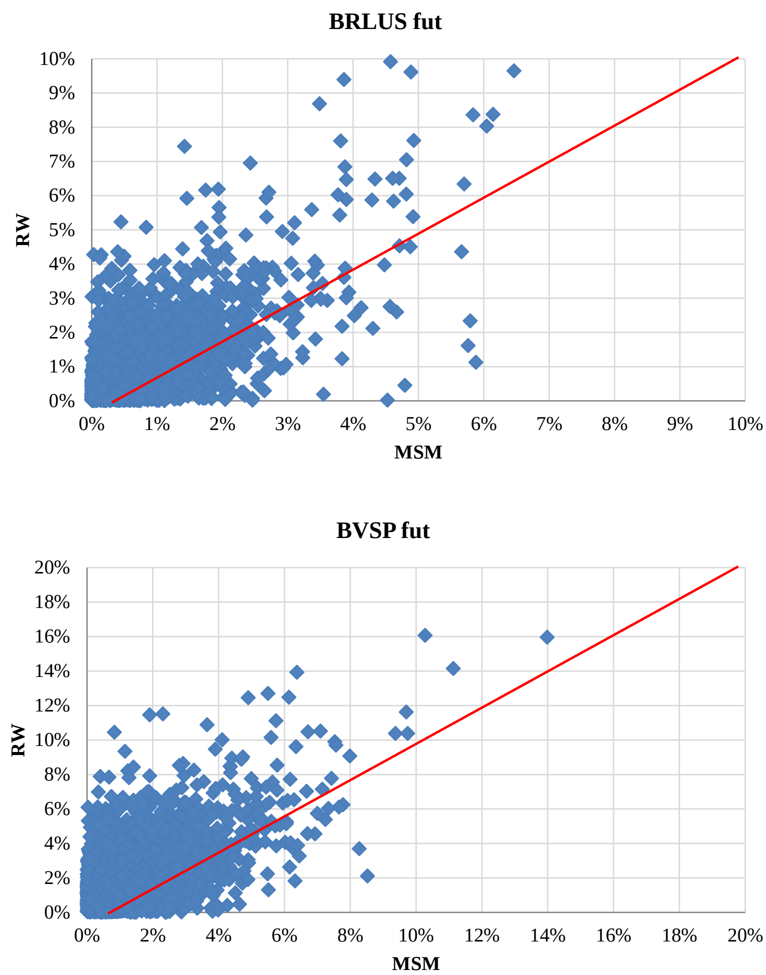
[4] For the RSI calculation, Wilder suggest in his book (see Wilder (1978)) the default value of 14 days. As we are well aware of the over-fitting problem of models that require a huge number of parameters and as this signal is a momentum oscillator as well, we propose to assume the same value for K_{mom}.

[5] In our case study, we set it to two standard deviations above the means. The empirical rule states that about 95.45% of the values lie within 2 standard deviations of the mean in a Gaussian distribution. This approach has been commonly used in recent quantitative trading studies and generally acknowledged as a good method for a significant movement identification cut-off point.

We define $\hat{Sig}_{9,t,i}$ likewise, by replacing K_{ST} for K_{LT}.

To evaluate the performance of the proposed mixed signal model in generating consistent signals to predict future returns we construct Figure 6.1 which depicts a scatter plot of random walk absolute prediction error for the optimized mixed signals model absolute prediction error. As we can see, our proposed model seems to produce smaller errors when compared to the simple random walk model for both risky assets BVSP fut and BRLUS fut.[6]

Fig. 6.1: Returns estimated prediction error 1-step ahead (random walk model $\times$ *mixed signals model)*



We also plot in Table 6.1 the regression statistics for the proposed mixed signal model to predict future returns 1-step ahead in the RA-AAP model.

[6] We choose to plot the mixed signals model considering $K_{ST} = 50, K_{ST} = 100$.

Tab. 6.1: Mixed signal model statistics

MSM Statistics	Coeff	t- Stat	P-Value	R-Square
DI Spot rate	1.0936	326.85	0.00%	97.15%
BRLUS currrency	0.9728	16.60	0.00%	8.09%
BVSP index	0.9097	13.65	0.00%	5.61%

6.2.2 Function Specification for Transaction Costs

Under our formulation, we assume the following model for the total transaction costs, where transaction costs ϕ_i are defined, by assumption, as convex on the financial volume of each asset u_i as

$$\phi_i(u_i) = \begin{cases} c_i^+ u_i & \text{if } u_i \geq 0 \\ -c_i^- u_i & \text{if } u_i \leq 0 \end{cases} \tag{6-16}$$

where c_i^+ and c_i^- are the transaction costs (in percentage %) associated to an order (buy or sell) of a given asset i.[7]

We will introduce the following notation to express transaction costs in our optimization problem. Let us consider new variables $u^+, u^- \in \Re^n$. We can express the total transaction executed on asset i as

$$\begin{aligned} u_i &= u_i^+ - u_i^-, \\ u_i^+ &\geq 0, u_i^- \geq 0 \end{aligned} \tag{6-17}$$

And the transaction cost function ϕ_i can be represented by

$$\phi_i = c_i^+ u_i^+ + c_i^- u_i^- \tag{6-18}$$

[7] In practice, transaction costs are not a convex function of the financial trading volume. If we disregard the slippage cost, they are closer to concave functions (we have for instance the charge of a flat fee for small transaction volumes, but above certain transacted amount, the fee is a decreasing function of the financial volume). However, depending on the market impact and liquidity, this concave effect may be counteracted. Therefore, we decided to keep our cost function linear and be more conservative in our analysis.

6.2.2.1 *Realistic Transaction Costs*

In all our case studies we considered the BMF&Bovespa, brokerage and slippage costs. We calculated the BMF&Bovespa costs considering a R$50mm nav fund. Those costs refer to trading, settlement, registration, custody, and clearing:

- Exchange fee refers to the trading service
- Settlement fee to cover expenses incurred by the Clearinghouse
- Permanence fee to keep track of positions
- Registration fee refers to the registration service by the Clearing

Furthermore, we consider brokerage fees by trading volume with a devolution of 99% for BRLUS contracts and 95% otherwise. To calculate slippage costs we estimate future volatility (3-month period) using an EWMA model for each asset class and evaluate the slippage cost as a function of this estimated volatility. Figure 6.2 depicts those transaction costs % over financial traded volume along time.

Fig. 6.2: Transaction costs evolution for some asset classes.