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Decomposition and relaxation algorithms for nonconvex mixed integer quadratically constrained quadratic programming problems

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Thesis presented to the Programa de Pós–graduação em Engenharia de Produção of PUC-Rio in partial fulfillment of the requirements for the degree of Doutor em Engenharia de Produção.

Advisor : Prof. Silvio Hamacher
Co-advisor: Prof. Fabricio Oliveira

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Abstract

Andrade, Tiago Coutinho Carneiro de Andrade; Hamacher, Silvio (Advisor); Oliveira, Fabricio (Co-Advisor). **Decomposition and relaxation algorithms for nonconvex mixed integer quadratically constrained quadratic programming problems**. Rio de Janeiro, 2018. 80p. Tese de doutorado – Departamento de Engenharia Industrial, Pontifícia Universidade Católica do Rio de Janeiro.

This thesis investigates and develops algorithms based on Lagrangian relaxation and normalized multiparametric disaggregation technique to solve nonconvex mixed-integer quadratically constrained quadratic programming. First, relaxations for quadratic programming and related problem classes are reviewed. Then, the normalized multiparametric disaggregation technique is improved to a reformulated version, in which the size of the generated subproblems are reduced in the number of binary variables. Furthermore, issues related to the use of the Lagrangian relaxation to solve nonconvex problems are addressed by replacing the dual subproblems with convex relaxations. This method is compared to commercial and open source off-the-shelf global solvers using randomly generated instances. The proposed method converged in 35 of 36 instances, while Baron, the benchmark solver that obtained the best results only converged in 4 of 36. Additionally, even for the one instance the methods did not converge, it achieved relative gaps below 1% in all instances, while Baron achieved relative gaps between 10% and 30% in most of them.

Keywords

Quadratically constrained quadratic programming; Mixed-integer programming; Decomposition; Convex relaxation; Lagrangian relaxation; Normalized multiparametric disaggregation technique.

Resumo

Andrade, Tiago Coutinho Carneiro de Andrade; Hamacher, Silvio; Oliveira, Fabricio. **Algoritmos baseados em decomposição e relaxação para problemas de programação inteira mista quadrática com restrições quadráticas não convexa**. Rio de Janeiro, 2018. 80p. Tese de Doutorado – Departamento de Engenharia Industrial, Pontifícia Universidade Católica do Rio de Janeiro.

Esta tese investiga e desenvolve algoritmos baseados em relaxação Lagrangiana e técnica de desagregação multiparamétrica normalizada para resolver problemas não convexos de programação inteira-mista quadrática com restrições quadráticas. Primeiro, é realizada uma revisão de técnias de relaxação para este tipo de problema e subclasses do mesmo. Num segundo momento, a técnica de desagregação multiparamétrica normalizada é aprimorada para sua versão reformulada onde o tamanho dos subproblemas a serem resolvidos tem seu tamanho reduzido, em particular no número de variáveis binárias geradas. Ademais, dificuldas em aplicar a relaxação Lagrangiana a problemas não convexos são discutidos e como podem ser solucionados caso o subproblema dual seja substituído por uma relaxação não convexa do mesmo. Este método Lagrangiano modificado é comparado com resolvedores globais comerciais e resolvedores de código livre. O método proposto convergiu em 35 das 36 instâncias testadas, enquanto o Baron, um dos resolvedores que obteve os melhores resultados, conseguiu convergir apenas para 4 das 36 instâncias. Adicionalmente, mesmo para a única instância que nosso método não conseguiu resolver, ele obteve um gap relativo de menos de 1%, enquanto o Baron atingiu um gap entre 10% e 30% para a maioria das instâncias que o mesmo não convergiu.

Palavras-chave

Programação quadrática com restrições quadráticas; Programação inteira-mista; Decomposição; Relaxação convexa; Relaxação Lagrangiana; Técnica de desagregação multiparamétrica normalizada.

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BD	– Benders decomposition
BnB	– branch and bound
DWD	– Dantiz-Wolfe decomposition
GBD	– generalized Benders decomposition
LD	– Lagrangian decomposition
LDP	– Lagrangian dual problem
LDS	– Lagrangian dual subproblem
LB	– lower bound
LP	– linear programming
LR	– Lagrangian relaxation
MDT	– multiparametric disaggregation technique
MINLP	– mixed-integer nonlinear programming
MIQCQP	– mixed-integer quadratically constrained quadratic programming
MIP	– mixed-integer (linear) programming
NGBD	– nonconvex generalized Benders decomposition
NLP	– nonlinear programming
NMDT	– normalized multiparametric disaggregation technique
p-LD	– p-Lagrangian decomposition
p-LDP	– p-Lagrangian dual problem
p-LDS	– p-Lagrangian dual subproblem
p-LR	– p-Lagrangian relaxation
QCQP	– quadratically constrained quadratic programming
QP	– quadratic programming
RHS	– right hand side
RLT	– reformulation linearizion techineque
RMDT	– reformulated multiparametric disaggregation technique
ROPP	– refinery operational planning problem
SDP	– semidefinite programming
SOCP	– second order conic programming
SP	– stochastic programming
UB	– upper bound

There once lived a man
who learned how to slay dragons
and gave all he possessed to mastering the art.

After three years
he was fully prepared but,
alas, he found no opportunity
to practice his skills.

Dschuang Dsi.

As a result he began
to teach how to slay dragons.

René Thom.

1 Introduction

The main objective of this thesis is to develop and investigate methods to solve nonconvex quadratically constrained quadratic programming, possibly with integer variables, ((MI)QCQP) problems.

The MIQCQP is a very general class of mathematical programming problems. Such as (mixed integer) linear programming, convex (mixed integer) quadratically constrained quadratic programming, quadratic programming, and convex quadratic programming, polynomial programming, semidefinite programming and conic programming can be reduced to MIQCQP. Additionally, Taylor theorem tells that all analytical functions can be approximated using polynomials, and it is known that any polynomial programming can be reformulated by MIQCQP [1].

Moreover, (MI)QCQP is a natural way to model many important processes in areas such as heat integration networks, separation systems, reactor networks, batch processes, pooling problems and refinery operations planning problem [2, 3, 4, 5, 6].

Although the techniques developed here are general and intentionally build to be applicable to any MIQCQP problem or to a general subclass. The problem that inspired the initial research was the Refinery Operations Planning Problem (ROPP). ROPP is an important problem in the oil and gas industry and has been aiding by use of optimization tools practically since the development of simplex Method and the birth of Linear Programming.

We first review techniques used to solver MIQCQP in the literature. Then, we investigate one promising technique in the state-of-art and improve on it by controlling the sizes of subjacent subproblems being solved. Finally, we investigate how can a MIQCQP with a special block angular structure — as is the case of stochastic programming — can be decomposed efficiently based on a original algorithm that is build on top of a modified Lagrangian relaxation.

1.1 Objectives

The main objective of the thesis can be broken down into two secondary objectives. First, we explore how (MI)QCQP problems can be relaxed, i.e., using piecewise convex (linear) relaxations to solve a nonconvex MIQCQP. The main relaxation technique is an improved version of normalized multiparametric disaggregation technique. Second, we investigate how Lagrangian relaxation can be used to solve (MI)QCQP problems exploiting special decomposable structure if it exists. Additionally, we discuss issues that arise while solving a nonconvex problem with Lagrangian relaxation and how they can be resolved combining this approach with another relaxation, namely, a reformulation of the normalized multiparametric disaggregation technique.

Other minor objetive of the thesis is to prove theorical properties of the methods developed here and to compare our proposed methods with state-of-art algorithms for nonconvex (MI)QCQP.

1.2 Thesis structure

The first chapter discusses the background necessary for the remaining of the Thesis. Chapter 2 presents the formulation for the MIQCQP and how it can be solved. Most methods rely on relaxations for the problem. This chapter provides an overview of the main relaxations for this type of mathematical programming problem. Section 2.1 derives the McCormick envelopes, a linear relaxation for the product of two continuous variables. This chapter is partially based on Andrade et al [6] published at Industrial & Engineering Chemistry Research and on a submitted paper to Journal of Global Optimization.

Chapter 3 presents piecewise relaxations for the MIQCQP. This family of relaxations has one advantage over linear relaxations such as the McCormick envelopes, it can provide arbitrarily tight relaxations. On the other hand, binary variables are added to the problem, resulting in a harder problem to be solved. First, a relaxation called normalized multiparametric disaggregation technique (NMDT) is presented. Then, it is shown how the model size can be improved through reformulations resulting in the reformulated normalized multiparametric disaggregation technique (RNMDT). This chapter is based on an accepted paper to Journal of Global Optimization.

Chapter 4 investigate how MIQCQP problems with special separable structure can be decomposed into smaller problems and solved separably. First, the procedure is done using a classic relaxation named Lagrangian relaxation (LR). Then, it is shown the issues that appear when this technique is applied to nonconvex problems. Furthermore, a new relaxation based on both LR and RNMDT is presented. This chapter is based on a yet unpublished paper that we intent to submit to Mathematical Programming.

Last, Chapter 5 presents conclusions and discuss further research possibilities. Appendixes are presented to provide additional information to the main text. Appendix A presents a table with the complete results for the continuous instances used in Chapter 3. Appendix B complements the analysis from Chapter 3.

2
Mixed-integer quadratically constrained quadratic programming

In this chapter, the following general nonconvex (mixed-integer) quadratically constrained quadratic programming ((MI)QCQP) problems with box constraints are considered.

$$\min\ x^T Q_0 x + f_0(x,y) \tag{2-1}$$

s.t.:

$$x^T Q_r x + f_r(x,y) \leq 0, \quad ,\forall r \in I_{1,m} \tag{2-2}$$

$$x_i \in [X_i^L, X_i^U] \quad ,\forall i \in I_{1,n_1} \tag{2-3}$$

$$y_i \in \{Y_i^L, \ldots, Y_i^U\} \quad ,\forall i \in I_{1,n_2}, \tag{2-4}$$

where $I_{a,b} = \{a, \ldots, b\}$ is the subset of integers between a and b (inclusive), for all $r \in I_{0,m}, Q_r$ is a symmetric matrix, $f_0 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is a linear function, and for all $r \in I_{1,m}$, $f_r : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ are affine functions. The variable x can assume any value between its bounds X^L and X^U, and y can assume any integer value between Y^L and Y^U. One implicit assumption in formulation (2-1)–(2-4) is that all variables that appear in product terms are continuous, as a product term containing at least one integer variable can be trivially linearized. If $n_2 = 0$, the problem is reduced to a nonconvex QCQP problem.

An (MI)QCQP problem is called convex if its continuous relaxation is convex regardless of the nonconvexity introduced by the integrality constraints of the decision variables. This problem is convex if Q_r is positive semi-definite for all $r \in I_{0,m}$, and nonconvex otherwise. In this study, the latter case is considered, i.e., when Q_r is not positive semi-definite.

The (MI)QCQP problem with box constraints is known to be NP-hard [7], even without quadratic constraints. It should be noted that if the box constraints are removed, the problem is undecidable [8]. A detailed definition and implications of undecidability and NP-hardness can be found in [9].

It is known that MIQCQP problems are equivalent to QCQP problems, as any integer variable can be defined as a sum of binary variables, and the constraint $y = y^2$ can be added to represent the integrality condition $y \in \{0, 1\}$. Although this transformation is possible, it is usually undesirable because it generally results in more computationally difficult nonconvex problems.

As (MI)QCQP problems and their variants are difficult to solve, many alternative solution methods have been proposed. They can be classified into exact methods, such as spatial Branch-and-Bound (BnB), and heuristic methods. The former can ensure that a globally optimal solution will be achieved, whereas the latter can only ensure local optimality of the solutions. Furthermore, nearly all methods involve relaxation techniques.

A commonly used exact algorithm for solving MIQCQP problems is BnB and its variants. If the problem is convex, BnB obtains bounds and eventually globally optimal solutions by relaxations of the integrality constraints. If the problem is nonconvex, spatial BnB is typically used, and the nonconvexity that arises from nonlinearity must also be relaxed via convex relaxations. In spatial BnB, both integer and continuous variables are usually branched by partitioning the feasible region into hyper-rectangles within the search space. Other forms of branching using different types of polyhedra have also been employed [10]. In general it is recommended that a domain reduction step be performed to accelerate BnB [11, 12]. A survey on BnB applied to nonconvex problems may be found in [13].

The relaxations for MIQCQP problems can be classified into five categories. The first category consists of linear relaxations whereby the problem is relaxed to a mixed-integer programming (MIP) problem without auxiliary integer variables. A classic approach of this type relies on McCormick envelopes [14], where bounded auxiliary variables representing the product of two variables are added to the problem. If the variables that appear in the product assume their bounds, the auxiliary variable will assume the value of the product; therefore, the relaxation will then be exact. Al-Khayyal [15] showed that McCormick envelopes represent the convex and concave envelopes of the function $f(x, y) = xy$ defined in a rectangle, that is, we assume $f : [X^L, X^U] \times [Y^L, Y^U] \to \mathbb{R}$. Bao et al. [16] proposed a tighter relaxation, namely polyhedral multiterm relaxation, obtained by determining the convex envelope of the sum of the quadratic terms. Sherali and Adams [17] proposed the reformulation-linearization technique (RLT), which is a systematic approach for generating valid constraints to a problem using linear equations and inequalities including the bounding constraints, and thus strengthening its linear relaxation. In particular, McCormick envelopes can be derived using RLT.

The second category uses convex relaxations. The resulting relaxed problem is still nonlinear, possibly with integer variables, but its continuous relaxation is convex. This can be achieved by adding convex terms with sufficiently large coefficients [18] or by decomposing the quadratic matrices into a sum of positive and negative matrices and then linearizing the second term only. This method is known as the difference of convex functions (DC) approach. Fampa et al. [19] proposed several approaches for decomposing the quadratic functions as sums of a convex and a concave function. Another possible strategy is to determine envelopes for the quadratic terms over regions other than rectangles [10].

The third category uses Lagrangian relaxations and Lagrangian bounds [20, 21, 22]. Although the relaxed problem is convex, the resulting Lagrangian relaxation subproblem is usually nonconvex and as difficult to solve as the original problem. Augmented Lagrangian relaxation [23, 24] can be used to obtain a tighter relaxation with convex subproblems, however, these methods introduce nonlinear terms to the dual problem. Moreover, both traditional Lagrangian relaxation and the augmented version usually result in nonsmooth problems.

The fourth category is based on conic programming, which can be considered a generalization of linear programming. The most common approach is semidefinite programming (SDP) [25, 26, 27]. Another common approach is second-order conic programming (SOCP). Sherali and Fraticelli [28] proposed a cutting plane generation method using SDP. Anstreicher [29] compared SDP and RLT relaxations and proposed an integrated approach for using SDP to tighten RLT relaxations. Linderoth [10] showed that the relaxation of quadratic terms over triangular regions of the form

$$\{(x, y, w) | w = xy, X^L \leq x \leq X^U, Y^L \leq y \leq Y^U, x + y \leq c\}$$

can be formulated using SOCP. More recently, Bomze et al. [30] and Bomze [31] proposed copositive programming (a subclass of conic programming) for solving quadratic problems. The resulting conic programming is nonconvex; however, it provides a tighter relaxation.

The fifth category is based on partitioning the solution space and relaxing each partition independently. The partitions can be obtained by adding binary variables or using disjunctive programming [32]. The most traditional approach is based on piecewise McCormick envelopes [33, 34, 35, 36]. A recent alternative method, which also relies on convex envelopes, is the nominalized multiparametric disaggregation technique (NMDT) proposed by Castro [37]. A similiar approach was propsed by Gupte et al. [38]. NMDT and its variations will be reviewed in more detail in Chapter 3.

A relaxation method of the fifth class was chosen to be used and improved in this study in Chapter 3 because this class can generate arbitrarily tight relaxations without using spatial BnB. Moreover, these relaxations yield problems that can be solved using off-the-shelf MIP solvers, such as CPLEX [39], GUROBI [40], and XPRESS [41], which are known to be reliable and efficient. The next section is dedicated to review in more detail the McCormick envelopes, that is a basic relaxation that appear in other relaxations to the MIQCQP, including to the relaxations that we will use in the next chapters.

2.1 McCormick envelopes

In this section, the McCormick envelopes are derived and applied to the (MI)QCQP formulation given in Chapter 2. Then, an heuristic based on [6] is construct and later on tested in computational experiments.

2.1.1 Formulation

Consider the following set:

$$\{(x, y, w) | X^L \leq x \leq X^U, Y^L \leq y \leq Y^U, w = xy\} \tag{2-5}$$

First, let two functions, $h_1 : [X^L, X^U] \to \mathbb{R}$ and $h_2 : [Y^L, Y^U] \to \mathbb{R}$ to be defined as $h_1(x) = x - X^L$ and $h_2(y) = y - Y^L$. Those functions are nonnegative by construction. Now, let the function $w_{1,2} : [X^L, X^U] \times [Y^L, Y^U]$ be the defined as the product of functions h_1 and h_2. Since it is defined at every point as the product of two nonnegative numbers, $w_{1,2} \geq 0$. Thus, we have the relation given by Proposition 1.

Proposition 1 *The following fours inequalities are valid:*
$xy \geq xY^L + X^L y - X^L Y^L$
$xy \geq xY^U + X^L y - X^U Y^U$
$xy \leq xY^U + X^L y - X^L Y^U$
$xy \leq xY^L + X^U y - X^U Y^L$

Proof. The first inequality can be proved as follows: $w_{1,2} = (x - X^L)(y - Y^L) = xy - xY^L - X^L y + X^L Y^L \geq 0$
The other three inequality proofs are analogous. ■

Those inequalities, proposed by McCormick [14], can be used to define set (2-6) that serves as a relaxation to the set (2-5). Additionally, those inequalities are used to define the convex and concave envelopes to the function $f : [X^L, X^U] \times [Y^L, Y^U] \to \mathbb{R}$ where $f(x, y) = xy$ as can be visualized in Figure 2.1. Thus, they are named McCormick envelopes.

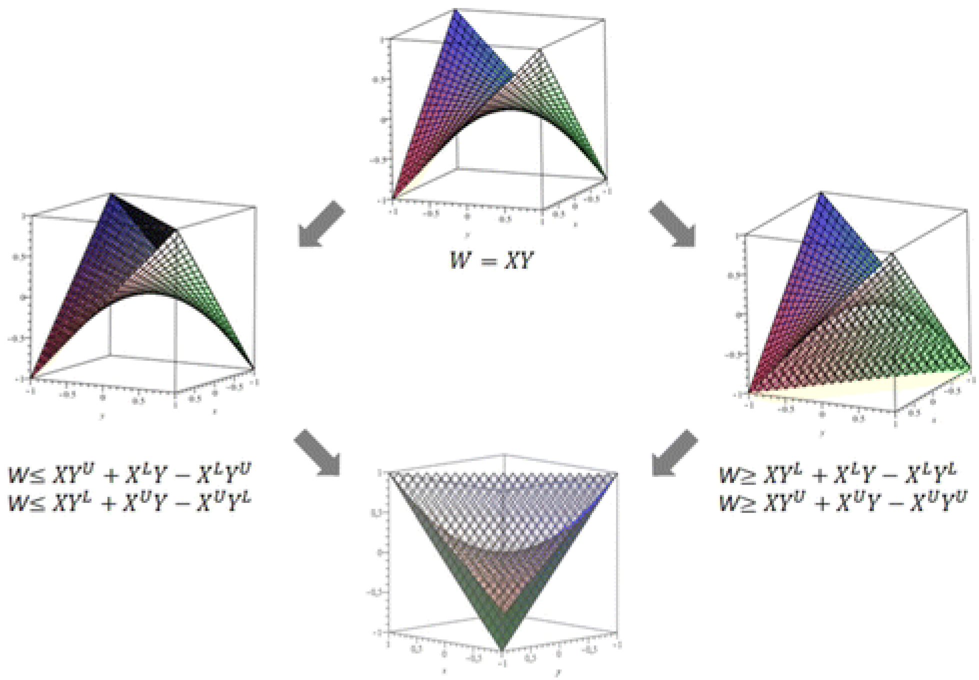


Figure 2.1: McCormick envelopes

$$\begin{aligned}\{(x, y, w)|X^L \leq x \leq X^U, Y^L \leq y \leq Y^U, w \geq xY^L + X^L y - X^L Y^L,\\ w \geq xY^U + X^L y - X^U Y^U, w \leq xY^U + X^L y - X^L Y^U,\\ w \leq xY^L + X^U y - X^U Y^L\}\end{aligned} \tag{2-6}$$

This new Set (2-6) has important relations to the Set (2-5). The former is a superset for the later, and is its convex hull. Furthermore, if $\forall x \in \{X^L, X^U\}$ and $\forall y \in \{Y^L, Y^U\}$ and $(x, y, w) \in$ Set (2-6), then $w = xy$.

Proposition 2 *Set* (2-5) $\subset$ *Set* (2-6)

Proof. Set (2-5) $= \{(x, y, w)|X^L \leq x \leq X^U, Y^L \leq y \leq Y^U, w \geq xY^L + X^L y - X^L Y^L, w \geq xY^U + X^L U - X^U Y^U, w \leq xY^U + X^L y - X^L Y^U, w \leq xY^L + X^U y - X^U Y^L, w = xy\} \subset$ Set (2-6) ■

Proposition 3 *Set* (2-6) *is the convex hull of Set* (2-5)

Proof. Let A, B be sets. For set A be the convex hull of Set B, three conditions must be met. i) $B \subset A$; ii) A must be convex; iii) There cannot exist a convex Set C such that $B \subset C$ and $A \not\subset C$.

Proposition 2 gives condition (i); since Set (2-6) is an intersection of semispaces, it is convex, thus satisfying condition (ii); let C be a convex set such that Set (2-5) $\subset$ C, then, the four points $(X^LY^L, X^LY^L), (X^LY^U, X^LY^U), (X^UY^L, X^UY^L), (X^UY^U, X^UY^U)$ belong to C. Since C is convex, all points in the polytope with these four points as vertices also belongs to C, thus, Set (2-6) $\subset C$, which satisfies condition (iii) and completes the proof. ■

Since the McCormick envelopes provide the convex hull to a product between two variables limited in a rectangular region and it can be represented using a polyhedral formulation (Proposition 3), it can be used to generate linear relaxations to the (MI)QCQP directly or to be incorporated in tighter relaxations. This second approach will be done in Chapter 3 where piecewise relaxations are created using additional binary variables to reduce the bound of continuous variables that appear in products, and then relax the products using the McCormick envelopes.

3
Piecewise relaxation

In this chapter, the mathematical background of NMDT is reviewed, and the related notation is introduced. Moreover, an initial formulation of NMDT is presented that will be central to this study.

NMDT appears as a natural progression of relaxations that have recently been used for solving either (MI)QCQP problems or certain subclasses of these problems such as bilinear programming problems. The ideas that led to the development of NMDT are reviewed below.

Li and Chang [42] proposed an approximation to the quadratic problem using a binary expansion of all variables. Based on this idea, Teles et al. [43] proposed the multiparametric disaggregation technique (MDT) as an approximation to polynomial programming.

Kolodziej et al. [44] proposed a relaxation for QCQP problems based on MDT by performing a decimal expansion on a subset of the variables and by including additional continuous variables with arbitrarily tight bounds. The products of binary variables and continuous variables were linearized exactly, and the products of two continuous variables were relaxed using McCormick envelopes. Additionally, they showed that their formulation can be obtained using disjunctive programming.

Later, Castro [37] proposed the normalized multiparametric disaggregation technique (NMDT) and showed that it is advantageous to normalize the variables before performing the decimal expansion, as the number of partitions for all variables is more controllable.

In the remainder of this section, the formulation of NMDT is presented. Given a (MI)QCQP problem, let the set of the indexes that appear in a quadratic term be defined as $QT = \{(i, j) \in I^2_{1,n_1} | j \geq i, \exists r \in I_{0,m}, |Q_{r,i,j}| > 0\}$, and the set of indexed of variables that will be discretized as $DS = \{j \in I_{1,n_1} | \exists i \in I_{1,n_1}, (i, j) \in QT\}$.

The variables x_j for all $j \in DS$ are normalized as follows:

$$x_j = (X^U_j - X^L_j)\lambda_j + X^L_j \quad , \forall j \in DS. \tag{3-1}$$

$\lambda_j \in [0, 1]$ is discretized in partitions of size 10^p each, where p corresponds to a precision factor. The variables $\Delta\lambda_j$ are added to allow λ_j to attain all values in the interval $[0, 1]$. Thus,

$$\lambda_j = \sum_{k \in I_{0,9}, l \in I_{p,-1}} k10^l z_{j,k,l} + \Delta\lambda_j, \quad \forall j \in DS \tag{3-2}$$

$$0 \leq \Delta\lambda_j \leq 10^p, \quad \forall j \in DS. \tag{3-3}$$

The following relations are obtained by multiplying both sides of (3-1) and (3-2) by x_i for all $i \in I_{1,n}$.

$$x_i x_j = (X_j^U - X_j^L) x_i \lambda_j + x_i X_j^L, \quad \forall i, j \in QT \tag{3-4}$$

$$x_i \lambda_j = \sum_{k \in I_{0,9}, l \in I_{p,-1}} k10^l x_i z_{j,k,l} + x_i \Delta\lambda_j, \quad \forall i, j \in QT. \tag{3-5}$$

Subsequently, the auxiliary variables $w_{i,j}$, $\hat{x}_{i,j,k,l}$, $v_{i,j}$, and $\Delta v_{i,j}$ are included to represent the products $x_i x_j$, $x_i z_{j,k,l}$, $x_i \lambda_j$, and $x_i \Delta\lambda_j$, respectively. Using these auxiliary variables, we obtain

$$w_{i,j} = (X_j^U - X_j^L) v_{i,j} + x_i X_j^L, \quad \forall i, j \in QT \tag{3-6}$$

$$v_{i,j} = \sum_{k \in I_{0,9}, l \in I_{p,-1}} k10^l \hat{x}_{i,j,k,l} + \Delta v_{i,j}, \quad \forall i, j \in QT. \tag{3-7}$$

Constraints (3-8)–(3-9) are known as the McCormick envelopes and provide a relaxation of the product of two continuous variables. The product of binary and continuous variables is exactly linearized by constraints (3-10)–(3-12).

$$X_i^L \Delta\lambda_j \leq \Delta v_{i,j} \leq X_i^U \Delta\lambda_j, \quad \forall i, j \in QT \tag{3-8}$$

$$10^p (x_i - X_i^U) + X_i^U \Delta\lambda_j \leq \Delta v_{i,j} \leq 10^p (x_i - X_i^L) + X_i^L \Delta\lambda_j, \quad \forall i, j \in QT \tag{3-9}$$

$$\sum_{k \in I_{0,9}} z_{j,k,l} = 1, \quad \forall j \in DS, l \in I_{p,-1} \tag{3-10}$$

$$\sum_{k \in I_{0,9}} \hat{x}_{i,j,k,l} = x_i, \quad \forall i, j \in QT, l \in I_{p,-1} \tag{3-11}$$

$$X_i^L z_{j,k,l} \leq \hat{x}_{i,j,k,l} \leq X_i^U z_{j,k,l}, \quad \forall i, j, k, l. \tag{3-12}$$

Furthermore, using the variable $w_{i,j}$, the objective function (2-1) and the original constraints (2-2) are replaced by Equations (3-13) and (3-14), respectively.

$$\min \sum_{i|(i,i)\in QT} Q_{0,i,i}w_{i,i} + 2 \sum_{(i,j)\in QT|j>i} Q_{0,i,j}w_{i,j} + f_0(x,y) \tag{3-13}$$

$$\sum_{i|(i,i)\in QT} Q_{r,i,i}w_{i,i} + 2 \sum_{(i,j)\in QT|j>i} Q_{r,i,j}w_{i,j} + f_r(x,y) \leq 0, \quad \forall r \in I_{1,m}. \tag{3-14}$$

We need to define one additional constraint that will serve the purpose of simplifying the technical results stated later on. Constraint (3-15) represents an alternative nonlinear definition of the variable Δv.

$$\Delta v_{i,j} = x_i \Delta\lambda_j \tag{3-15}$$

Definition 1 *For every p, $EQUIV_p$ is defined as the problem of minimizing the objective function* (3-13)*, subject to the constraints* (3-1)–(3-3)*,* (3-6)*,* (3-7)*,* (3-10)–(3-12)*,* (3-14)*, and* (3-15)*.*

Definition 2 *For every p, the set FS-$EQUIV_p$ is defined as the feasible set of problem $EQUIV_p$. That is, $(x,y,w,z,v,\hat{x},\lambda,\Delta\lambda,\Delta v) \in$ FS-$EQUIV_p$ if and only if it satisfies constraints* (3-1)–(3-3)*,* (3-6)*,* (3-7)*,* (3-10)–(3-12)*,* (3-14)*, and* (3-15)*.*

Lemma 1 *For all $p \leq 0$, $EQUIV_p$ is equivalent to the original problem* (2-1)–(2-4)*.*

Lemma 1 is trivial, as all additional constraints (and associated variables) are redundant and the linearizations are exact. Problem EQUIV$_p$ is useful as an intermediate step in proving that NMDT$_p$ is a relaxation of the original (MI)QCQP problem.

Definition 3 *For every p, $NMDT_p$ is defined as the problem of minimizing the objective function* (3-13) *subject to the constraints* (3-1)–(3-3)*,* (3-6)–(3-12)*, and* (3-14)*.*

Definition 4 *For every p, the set FS-$NMDT_p$ is defined as the feasible set of problem $NMDT_p$. That is, $(x,y,w,z,v,\hat{x},\lambda,\Delta\lambda,\Delta v) \in$ FS-$NMDT_p$ if and only if it satisfies constraints* (3-1)–(3-3)*,* (3-6)–(3-12)*, and* (3-14)*.*

Proposition 4 *$NMDT_p$ is a relaxation of $EQUIV_p$ for every $p \leq 0$.*

Proof. Both problems have the same objective function. Thus, NMTD_p will be a relaxation of EQUIV_p if FS-$\mathrm{NMDT}_p \supseteq$ FS-EQUIV_p. The constraints that are used to define both feasible sets are nearly the same. The only difference is that FS-NMDT_p has constraints (3-8) and (3-9) instead of (3-15). As the former are the McCormick envelopes of the product that appears in the latter, it follows that constraints (3-8) and (3-9) are implied from constraint (3-15), whereas the converse is not true. Therefore, FS-$\mathrm{NMDT}_p \supseteq$ FS-EQUIV_p, and the proposition follows. ■

Proposition 5 *$NMDT_p$ is a relaxation of the original (MI)QCQP problem for every $p \leq 0$.*

Proof. The result follows directly from Lemma 1 and Proposition 4. ■

Theorem 1 *For any pair (p_1, p_2) with $p_1 < p_2 \leq 0$, $NMDT_{p_2}$ is a relaxation of $NMDT_{p_1}$.*

Proof. It should be noted that NMDT_{p_1} has more variables than NMDT_{p_2}. Thus, the feasible sets FS-NMDT_{p_1} and FS-NMDT_{p_2} cannot be compared directly, as they have different dimensions. To allow such a comparison, a mapping M : FS-$\mathrm{NMDT}_{p_1} \rightarrow$ FS-NMDT_{p_2} is constructed so that every element $(x, y, w, z, v, \hat{x}, \lambda, \Delta\lambda, \Delta v) \in$ FS-NMDT_{p_1} evaluated in the objective function of NMDT_{p_1} is equal to $M(x, y, w, z, v, \hat{x}, \lambda, \Delta\lambda, \Delta v)$ evaluated in the objective function of NMDT_{p_2}. Let M be defined as

$$
\begin{aligned}
&x_i^{NMDT_{p_2}} = x_i^{NMDT_{p_1}}, \quad \forall i \in I_{1,n_1} \\
&y_i^{NMDT_{p_2}} = y_i^{NMDT_{p_1}}, \quad \forall i \in I_{1,n_2} \\
&w_i^{NMDT_{p_2}} = w_i^{NMDT_{p_1}}, \quad \forall i, j \in QT \\
&z_{j,k,l}^{NMDT_{p_2}} = z_{j,k,l}^{NMDT_{p_1}}, \quad \forall j \in DS, k \in I_{0,9}, l \in I_{p_2,0} \\
&v_{i,j}^{NMDT_{p_2}} = v_{i,j}^{NMDT_{p_1}}, \quad \forall i, j \in QT \\
&\hat{x}_{i,j,l}^{NMDT_{p_2}} = \hat{x}_{i,j,l}^{NMDT_{p_1}}, \quad \forall i, j \in QT, k \in I_{0,9}, l \in I_{p_2,0} \\
&\lambda_j^{NMDT_{p_2}} = \lambda_j^{NMDT_{p_1}}, \quad \forall j \in DS \\
&\Delta\lambda_j^{NMDT_{p_2}} = \Delta\lambda_j^{NMDT_{p_1}} + \sum_{l \in I_{p_1, p_2 - 1}} k10^l z_{j,k,l}^{NMDT_{p_1}}, \quad \forall j \in DS \\
&\Delta v_{i,j}^{NMDT_{p_2}} = \Delta v_{i,j}^{NMDT_{p_1}} + \sum_{l \in I_{p_1, p_2 - 1}} k10^l \hat{x}_{i,j,l}^{NMDT_{p_1}}, \quad \forall i, j \in QT
\end{aligned}
$$

It is straightforward to verify that the image of this mapping is in the feasibility set FS-NMDT_{p_2}, completing the proof. ■

Theorem 2 *For any pair (p_1, p_2) with $p_1 < p_2 \leq 0$, $NMDT_{p_1}$ is a tighter (or equal) relaxation of the original (MI)QCQP problem than $NMDT_{p_2}$.*

Proof. By Proposition 5, both problems are relaxations of the original problem. By Theorem 1, NMDT_{p_2} is a relaxation of NMDT_{p_1}, it follows that NMDT_{p_1} is a tighter relaxation of the original problem than NMDT_{p_2}. ■

3.1 Algorithm

Algorithm 1 was originally proposed in Castro [37] for solving an (MI)QCQP problem using NMDT. It is similar to the algorithm in Kolodziej et al. [44].

Algorithm 1 Algorithm to NMDT

Step 0. Choose $p = 0$ and let $UB = +\infty$ and $iteration = 0$.
Step 1. $iteration = iteration + 1$.
Step 2. Solve relaxation $NMDT_p$, obtaining LB and (x^R, y^R).
Step 3. Solve original problem with a local solver with initial solution (x^R, y^R) and fix integer variables at y^R. If a new best solution is found, store the incumbent solution (x^*, y^*) and update UB.
Step 4. If one of the stopping criteria (discussed below) is met, stop. Otherwise, set $p = p - 1$ and return to Step 1.

The principle of this algorithm is to tighten the relaxation as the iterations progress by decreasing the parameter p, thus gradually increasing the lower bound (LB). Feasible solutions are obtained using local methods with warm starts and fixing integer variables, which in turn provides upper bounds (UB). An incumbent solution is the best feasible solution obtained during the execution of the algorithm. Common stop criteria are the maximum number of iterations, the maximum time elapsed, and the relative or absolute gap with respect to a certain threshold.

If the feasible space of the original (MI)QCQP problem is not empty, the algorithm converges to the optimal value of the (MI)QCQP problem since $\lim_{p \to -\infty} \Delta\lambda_j = 0$ and, if $\Delta\lambda_J = 0$, then $w_{i,j} = x_i x_j$. Thus, the relaxed solution is feasible for the original problem and $UB = LB$.

Although convergence is only asymptotically guaranteed, it is often observed (as will also be seen in the computational experiments presented later) that feasible solutions are obtained within a few iterations of the algorithm. Furthermore, the lower bound in each iteration is at least as good as the bound in the previous iteration, as stated in the following theorem.

Theorem 3 *The sequence of lower bounds generated by Algorithm 1 is monotonic.*

Proof. As in each iteration, the value of p is decreased, the result follows from Theorem 2. ■

3.2 Reformulated Normalized Multiparametric Disaggregation

Herein, the main contributions of this study are presented. It is first shown that a binary expansion is preferable to a decimal expansion in NMDT. Subsequently, a reformulation of the problem is presented in which the number of variables (both binary and continuous) and constraints are reduced. Finally, an alternative algorithm is developed.

3.2.1 Reformulation using binary expansion

The first reformulation consists in changing from 10 to 2 the numerical base that is used for representing the continuous variables. It should be noted that this idea is not new and has already been successfully applied to other techniques related to RNMDT [45]. Nevertheless, Castro [37] used decimal representation for the NMDT. Despite a brief mention in that other bases may be chosen, to the best of our knowledge [37], base-2 (or binary) expansions have not been applied in this context. Other key difference between the base-2 expansion used in Teles et al. [45] and the proposed approach is that, while the former uses base-2 for the MDT only as a means to reduce the total of auxiliary variables while trying to maintain the same precision level, our focus is to use the base-2 formulation to control how the model grows as the precision level increase between iterations.

The formulation using a binary expansion (i.e., a representation in which each variable is replaced by a base-2 expansion) instead of the traditional decimal expansion is obtained by modifying constraints (3-2), (3-3), (3-7), (3-9), (3-10), and (3-11). The new constraints are obtained by replacing the number 10 by 2 and 9 by 1, respectively, wherever they appear in these constraints. This procedure results in the new constraints (3-16)–(3-21).

$$\lambda_j = \sum_{k \in I_{0,1}, l \in Ip,-1} 2^l k z_{j,k,l} + \Delta\lambda_j \quad , \forall j \in DS \tag{3-16}$$

$$0 \leq \Delta\lambda_j \leq 2^p \quad , \forall j \in DS \tag{3-17}$$

$$v_{i,j} = \sum_{k \in I_{0,1}, l \in Ip,-1} 2^l k \hat{x}_{i,j,k,l} + \Delta v_{i,j} \quad , \forall i,j \in QT \tag{3-18}$$

$$2^p(x_i - X_i^U) + X_i^U \Delta\lambda_j \leq \Delta v_{i,j} \leq 2^p(x_i - X_i^L) + X_i^L \Delta\lambda_j \quad , \forall i,j \in QT \tag{3-19}$$

$$\sum_{k \in I_{0,1}} z_{j,k,l} = 1 \quad , \forall j \in DS, l \in I_{p,-1} \tag{3-20}$$

$$\sum_{k \in I_{0,1}} \hat{x}_{i,j,k,l} = x_i \quad , \forall i,j \in QT \tag{3-21}$$

Despite its simplicity, this reformulation allows a significant reduction in the number of auxiliary binary variables required in the variable expansion for a given precision 10^p. The following propositions allow the comparison of the total number of binary variables required in the base-10 and the base-2 expansions.

Proposition 6 *The number of auxiliary binary variables z for NMDT in base 10 is $10(-p)|DS|$ for a given value of the parameter $p < 0$, where $|DS|$ is the cardinality of the set QT, i.e., the number of quadratic terms in the original (MI)QCQP problem.*

Proof. For every $j \in DS$, $\sum_{k \in I_{0,9}} \sum_{l \in I_{p,-1}} 1$ binary variables are added to the problem. Therefore, the number of added auxiliary binary variables z is

$$|DS| \sum_{k \in I_{0,9}} \sum_{l \in I_{p,-1}} = |DS| \times |I_{0,9}| \times |I_{p,-1}| = 10(-p)|DS|.$$

■

Proposition 7 *The number of auxiliary binary variables z for NMDT in base 2 is $2(-p)|DS|$ for a given value $p < 0$.*

Proof. For every $j \in DS$, $\sum_{k \in I_{0,1}} \sum_{l \in I_{p,-1}} 1$ binary variables are added to the problem. Therefore, the number of added auxiliary binary variables z is

$$|DS| \sum_{k \in I_{0,1}} \sum_{l \in I_{p,-1}} = |DS| \times |I_{0,1}| \times |I_{p,-1}| = 2(-p)|DS|.$$

■

Figures 3.1 and 3.2 illustrate the discretization of λ using decimal and binary base, respectively. Even though for a given p, the binary expansion

Figure 3.1: Discretization using decimal expansion

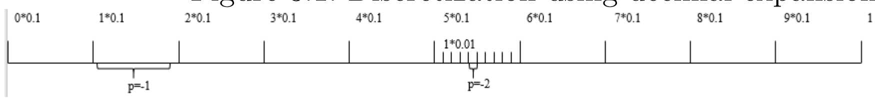
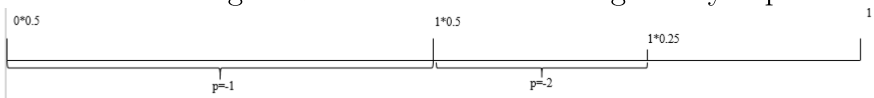


Figure 3.2: Discretization using binary expansion



provides less precision than the decimal expansion, it also requires fewer binary variables. Alternatively, for a given desired precision, fewer binary variables are required, as will be further discussed in Section 3.2.4.

3.2.2
Eliminating redundant variables and constraints

The original formulation of NMDT presents redundancy in both variables and constraints. Therefore, the first step of the reformulation process is to eliminate these redundant terms. The variables λ and v can be eliminated by replacing them in every constraint they appear with the form given by constraints (3-16) and (3-18), respectively.

The second step consists of replacing $z_{j,0,l}$ with $1 - z_{j,1,l}$ for all $j \in DS, l \in I_{p,-1}$. This renders equation (3-20) redundant. Similarly, variable $\hat{x}_{i,j,0,l}$ is replaced by $x_i - \hat{x}_{i,j,0,l}$, thus rendering equation (3-21) redundant.

The last two steps do not involve elimination of constraints, but rather rearrangement of variable labels and indices to accomodate the previous simplifications. First, index k can be dropped, as it refers to the singleton set ($\{1\}$). It should be noted that $k = 0$ can be disregarded (see, for example, constraint (3-16)), as it only adds variables with null coefficient to the summation. The last step consists of replacing $\Delta\lambda$ and Δv with Δx and Δw, respectively, as λ and v no longer exist.

The simplified model (3-22)-(3-33) is hereinafter referred to as reformulated normalized multiparametric disaggregation technique (RNMDT). For every $p \leq 0$ it is denoted as RNMDT$_p$, following the notation used for the previous models.

$$\min \sum_{i|(i,i)\in QT} Q_{0,i,i}w_{i,i} + 2 \sum_{(i,j)\in QT|j>i} Q_{0,i,j}w_{i,j} + f_0(x,y) \tag{3-22}$$

$$\text{s.t.:} \sum_{i|(i,i)\in QT} Q_{r,i,i}w_{i,i} + 2 \sum_{(i,j)\in QT|j>i} Q_{r,i,j}w_{i,j} + f_r(x,y) \leq 0, \quad \forall r \in I_{1,m} \tag{3-23}$$

$$x_j = (X_j^U - X_j^L)(\sum_{l\in I_{p,-1}} 2^l z_{j,l} + \Delta x_j), \quad \forall j \in DS \tag{3-24}$$

$$w_{i,j} = (X_j^U - X_j^L)(\sum_{l\in I_{p,-1}} 2^l \hat{x}_{i,j,l} + \Delta w_{i,j}), \quad \forall i \in I_{1,n}, j \in I_{1,n}|(i,j) \in QT \tag{3-25}$$

$$0 \leq \Delta\lambda_j \leq 2^p, \quad \forall j \in DS \tag{3-26}$$

$$2^p(x_i - X_i^U) + X_i^U \Delta x_j \leq \Delta w_{i,j} \leq 2^p(x_i - X_i^L) + X_i^L \Delta x_j, \quad \forall i,j|(i,j) \in QT \tag{3-27}$$

$$x_i^L \Delta x_j \leq \Delta w_{i,j} \leq x_i^U \Delta x_j, \quad \forall i,j|(i,j) \in QT \tag{3-28}$$

$$X_i^L z_{j,l} \leq \hat{x}_{i,j,l} \leq X_i^U z_{j,l}, \quad \forall i,j,l \in QT \times I_{0,p} \tag{3-29}$$

$$X_i^L(1 - z_{j,l}) \leq x_i - \hat{x}_{i,j,l} \leq X_i^U(1 - z_{j,l}), \quad \forall i,j,l \in QT \times I_{0,p} \tag{3-30}$$

$$x_i \in [X_i^L, X_i^U], \quad \forall i \in I_{1,n_1} \tag{3-31}$$

$$y_i \in \{Y_i^L, \ldots, Y_i^U\}, \quad \forall i \in I_{1,n_2} \tag{3-32}$$

$$z_{j,l} \in \{0,1\}, \quad \forall j,l \in DS \times I_{0,p}. \tag{3-33}$$

The following technical results concern the reduction in the number of binary variables necessary for representing the expansions, to a given precision 10^p, after performing the proposed reformulations. The total reduction in the number of binary variables is such that only one tenth of the original number of binary variables is required when combining the proposed reformulation and the change of base.

Proposition 8 *The number of auxiliary binary variables z for RNMDT is $(-p)|DS|$ for a given parameter $p < 0$.*

Proof. For every $j \in DS$, $\sum_{l\in I_{p,-1}} 1$ binary variables are added to the model. Therefore, the number of added binary variables z is $|DS| \times \sum_{l\in I_{p,-1}} 1 = |DS| \times |I_{p,-1}| = (-p)|DS|$. ■

Theorem 4 *For every $p \leq 0$, the number of auxiliary binary variables z for RNMDT$_p$ is one tenth of the number of binary variables of the problem NMDT$_p$.*

Proof. The proof follows from Propositions 6 and 8. ■

Moreover, most of the technical results previously presented concerning the reformulation and the change of base still hold. They are reproduced for the sake of completeness.

Proposition 9 *$RNMDT_p$ problem is a relaxation of the original (MI)QCQP problem of every $p \leq 0$.*

Proof. The proof is analogous to that of Proposition 5. ■

Theorem 5 *For any pair of (p_1, p_2) with $p_1 < p_2 \leq 0$, $RNMDT_{p_2}$ is a relaxation of $NMDT_{p_1}$.*

Proof. The proof is analogous to that of Theorem 1. ■

Theorem 6 *For any pair of (p_1, p_2) with $p_1 < p_2 \leq 0$, $RNMDT_{p_1}$ is a tighter (or equal) relaxation of the original (MI)QCQP problem than $RNMDT_{p_2}$.*

Proof. The proof is analogous to that of Theorem 2. ■

3.2.3 Dynamic-precision RNMDT algorithm

One disadvantage of Algorithm 1 is that all discretized variables are expanded using the same number of partitions (or the same precision in the MDT case), which can result in a rapid increase in the number of binary variables that are added to the problem. In this section, an alternative algorithm is proposed for solving the (MI)QCQP problem using RNMDT, whereby the number of partitions is increased only for the variables that will potentially improve (i.e., tighten) the relaxation. Initially, the single precision parameter p is replaced with a parameter vector p_j for all $j \in DS$, where each entry represents the number of partitions that will be used to expand the variable x_j for all $j \in DS$. The variables that will have their precision increased are then chosen dynamically in each iteration.

This procedure is summarized in Algorithm 2. In each iteration, the variables for which the number of partitions will be increased are chosen by ranking them using the function f_{rank} given in (3-34). The first term of this function represents the absolute error of the relaxation for the pure quadratic terms in which a given variable is present. The second term is the error in the bilinear terms in which the variable appears. The first N_1 variables with the largest function value are selected and their precision is increased, i.e., p_j is reduced by one unit. For every N_2 iterations, each p_j for all j, is reduced by one unit to ensure convergence (i.e., to ensure that for every $j \in DS$, $p_j \rightarrow -\infty$; therefore, $w_{i,j} \rightarrow x_i x_j$).

$$f_{\mathrm{rank}}(j) = \sum_{r} |Q_{r,j}(w_{j,j} - x_j^2)| + 2 \sum_{((r,i)|i>j|(i,j)\in QT)} |Q_{r,i,j}(w_{i,j} - x_i x_j)| \quad (3\text{-}34)$$

Algorithm 2 Dynamic-precision RNMDT algorithm

Step 0. For all $j \in DS$, set $p_j = 0$ and let $UB = +\infty$ and $iteration = 0$.
Step 1. $iteration = iteration + 1$.
Step 2. Solve relaxation and obtain LB and point (x^R, y^R).
Step 3. Solve original problem with a local solver with initial solution (x^R, y^R) and fixing integer variables at y^R. If a new best solution is found, save the incumbent solution (x^*, y^*) and update UB.
Step 4. If some of the stopping criteria is met, stop. Otherwise continue.
if $iteration + 1$ is not a multiple of N_2 **then**
  Step 5. Rank j using f_{rank}, and for the first N_1 indexes j ranked by f_{rank}, set $p_j = p_j - 1$. return to step 1.
else
  Step 5. For all j, set $p_j = p_j - 1$. return to Step 1.
end if

Theorem 7 *The sequence of lower bounds generated by Algorithm 2 is monotonic.*

Proof. As the parameter p is point-wise decreased in each iteration, monotonicity follows from Theorem 6. ■

3.2.4 Discussions

It should be noted that the proposed changes aim at reducing the total number of binary variables required for obtaining the relaxation at each iteration. In that sense, the change of base reduces the number of binary variables necessary for expanding the continuous variables, the elimination of redundant variables and constraints reduces the overall model size. The proposed algorithm controls the increase in the model size between iterations.

Table 3.1 shows the precision and the number of additional binary variables for each choice of the parameter p. The first column represents different choices of p. The remaining columns are grouped in pairs. The first column for each pair represents the precision for the chosen p, i.e., the tightness of the bounds of $\Delta\lambda$ or Δx depending on whether the model is NMDT or RDNMT, respectively. The second column represents the number of auxiliary variables z that are added for each continuous variable that is discretized. There are three pairs of columns, the first is for NDMT using base 10, the second for NMDT using base 2, and the last for RNMDT.

Table 3.1: Precision x number of binary variables

	NMDT - base 10		NMDT - base 2		RNMDT	
p	precision	binary variables	precision	binary variables	precision	binary variables
0	1.00E+00	0	1.00E+00	0	1.00E+00	0
-1	1.00E-01	10	5.00E-01	2	5.00E-01	1
-2	1.00E-02	20	2.50E-01	4	2.50E-01	2
-3	1.00E-03	30	1.25E-01	6	1.25E-01	3
-4	1.00E-04	40	6.25E-02	8	6.25E-02	4
-5	1.00E-05	50	3.13E-02	10	3.13E-02	5
-6	1.00E-06	60	1.56E-02	12	1.56E-02	6
-7	1.00E-07	70	7.81E-03	14	7.81E-03	7
-8	1.00E-08	80	3.91E-03	16	3.91E-03	8
-9	1.00E-09	90	1.95E-03	18	1.95E-03	9
-10	1.00E-10	100	9.77E-04	20	9.77E-04	10

It is easily seen that the binary expansion has two major advantages compared to the decimal expansion; namely, it allows more control over accuracy and generates fewer binary variables for each chosen accuracy. As an illustrative example, if NMDT in base 10 is chosen, ten binary variables are necessary for a precision of 10^{-1}, whereas for RNMDT, the same number of variables results in a precision of $9.77E-04$.

Table 3.2 shows the model complexity before and after the elimination of redundant variables and constraints. It is noticeable that the number of additional binary variables (represented by variable z) is reduced by a factor of 5, owing to the base change, and by half after the redundancy elimination. Clearly, the remainder of the model size decreases as well. However, it should be noted that comparing the model sizes for the same parameter p value can be misleading, since the same value of p leads to different precisions in the different formulations. Nevertheless, if one compares the formulations for a given precision level, the reduction in the number of auxiliary binary variables from base 10 to the RNMDT formulation is approximately by a factor of 3, as can be observed in Table 3.1.

As the Algorithm 2 (dynamic-precision algorithm) may require different number of iterations from Algorithm 1, and each iteration may have different computational cost, their efficiency is not directly comparable by theoretical analysis. However, the advantage of the dynamic-precision RNMDT algorithm will become clear in the next section in which computational experiments are presented

3.3 Computational experiments

In this section, the results obtained using the proposed relaxation and algorithm are presented. The QCQP problem instances were obtained from the literature and we also consider some randomly generated MIQCQP problem

Table 3.2: Model Complexity

	NMDT - base 10	**NMDT - base 2**	**RNMDT**
x	n_1	n_1	n_1
y	n_2	n_2	n_2
w	$\|QT\|$	$\|QT\|$	$\|QT\|$
z	$10(-p)\|DS\|$	$2(-p)\|DS\|$	$(-p)\|DS\|$
λ	$\|DS\|$	$\|DS\|$	0
$\Delta x/\Delta\lambda$	$\|DS\|$	$\|DS\|$	$\|DS\|$
v	$\|QT\|$	$\|QT\|$	0
$\hat{x}$	$10(-p)\|QT\|$	$2(-p)\|QT\|$	$(-p)\|QT\|$
$\Delta v/\Delta w$	$\|QT\|$	$\|QT\|$	$\|QT\|$
binary variables	$10(-p)\|DS\|$	$2(-p)\|DS\|$	$(-p)\|DS\|$
integer variables	n_2	n_2	n_2
continuous variables	$n_1 + (10(-p) + 3)\|QT\| + 2\|DS\|$	$n_1 + (2(-p) + 3)\|QT\| + 2\|DS\|$	$n_1 + ((-p) + 2)\|QT\| + \|DS\|$
constraints	$m + (2 + (-p))DS + (4 + 11(-p))\|QT\|$	$m + (2 + (-p))\|DS\| + (4 + 3(-p))\|QT\|$	$m + (-p)\|DS\| + (3 + 2(-p))\|QT\|$

instances. All instances were solved by four methods: i) Algorithm 1 and NMDT in base 10, ii) Algorithm 1 and NMDT in base 2, iii) Algorithm 1 and RNMDT, and iv) Algorithm 2 and RNMDT.

The algorithms were implemented in GAMS on an Intel i7-3612QM with 8GB. The LP/MIP solver was CPLEX 12.6, and the local nonlinear solver was CONOPT 3.17. For the dynamic-precision RNMDT algorithm, N_1 and N_2 were set to 3 and 10, respectively. These values were selected based on early experiments that will be discussed next. A time limit of 1000 seconds and an absolute gap $|UB - LB|$ (optimality tolerance) of 0.001 were set as stopping criteria.

3.3.1 Literature Instances

These instances were originally presented in [16]. They were provided by the Optimization Firm, which is responsible for the development of the BARON solver [46].

All instances are QCQP minimization problems. All variables are continuous except for the auxiliary z variables in the relaxation, which are discrete.

There are 135 instances, and all are nonconvex. The number of variables ranged from 10 to 50, and the number of constraints from 10 to 100. The density of the quadratic matrices Q was 25%, 50%, or 100%. The linear part was 100% dense for all problems. The coefficients from both the quadratic and the linear terms were chosen to be randomly generated numbers chosen uniformly between 0 and 1. Table 3.3 classifies instances according to problem size and density.

Table 3.3: Instances sizes

	variables	**constraints**	**density (%)**	**number of instances**
	8 to 10	8 to 40	25	18
small	10 to 20	10 to 40	50	18
	10 to 20	10 to 40	100	18
	28 to 40	28 to 80	25	18
medium	30 to 40	30 to 80	50	18
	30 to 40	30 to 80	100	18
	48	48 to 96	25	9
large	50	50 to 100	50	9
	50	50 to 100	100	9

3.3.1.1 Parameterizing the dynamic-precision algorithm

Algorithm 2 requires the parameters N_1 and N_2 to be set beforehand. To select these values, a representative instance was chosen from the group of large instances and solved with a time limit of 48 hours using Algorithm 1 (classic algorithm) and Algorithm 2 (Dynamic-precision algorithm) setting the parameter pair (N_1, N_2) to $(1, 20)$, $(3, 10)$, $(5, 10)$, and $(10, 5)$. The results are shown in Figure 3.3.1.1 using log-transformation on the time axis. In this figure, each dot represents an iteration and each line a different setting.

As can be seen in Figure 3.3.1.1, Algorithm 2 was more efficient than Algorithm 1, as the latter required considerably more time to complete iteration 2, primarily owing to the number of binary variables added in the relaxation. If a significantly small number of continuous variables have their discretization refined per iteration, then several consecutive iterations with little or no improvement may be observed. For example, this can be seen in the setting $(1, 20)$. In contrast, if a considerably large number of variables are expanded the algorithm requires a larger amount of time to complete the first iterations, as can be seen, for example, in the setting $(10, 5)$. Figure 3.3.1.1

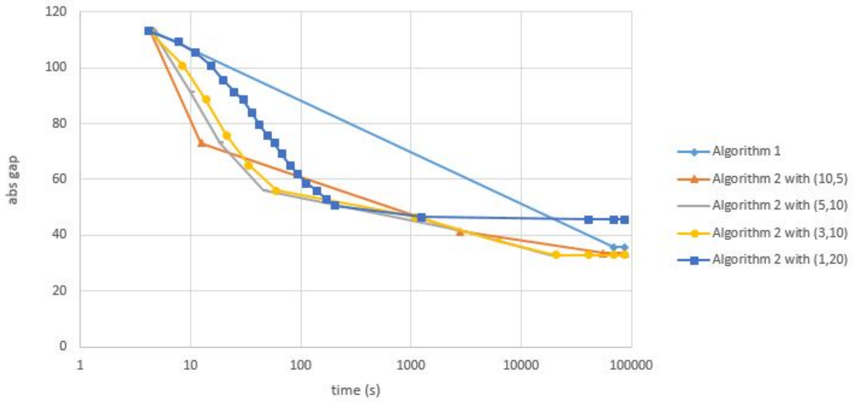


Figure 3.3: Setting parameter N_1 and N_2 (logarithm)

shows that the two intermediate settings are nearly equivalent; however, the setting $(3, 10)$ was chosen as a conservative option in terms of problem growth (as it expands fewer variables per iteration). This experiment showed that, in this case, the performance of the dynamic-precision algorithm for the given parameters was, to a certain degree, robust. A similar behavior was also observed in preliminary experiments with other instances.

3.3.1.2
Numerical results

Given the choice of parameters $(N_1, N_2) = (3, 10)$ for the dynamic-precision algorithm, all four methods were used for solving the 135 instances. In these experiments, the same optimality tolerance of 0.001 was used; however, the time limit was reduced to 1000 seconds. One can conclude from the numerical results, the three proposed improvements surpassed, in terms of performance, the relaxation and the algorithm in Castro [37]. The instances solved are summarized in Table 3.4.

NMDT using base 2 and RNMDT solved 18 additional instances when compared with NMDT using base 10. In particular, they solved seven additional small instances with 100% density. RNMDT with the dynamic-precision algorithm solved two additional instances compared with RNMDT combined with the classic algorithm, namely, one large instance with 25% density and one medium instance with 50% density. To compare the performance of the methods in the instances for which the optimality gap was not closed, Table 3.5 presents the average relative gaps after termination of the algorithm due to the time limit criterion.

The proposed improvements over the NMDT formulation and the algorithm presented in Castro [37] were both successful in terms of the number of instances solved and also the quality of the bounds obtained for the instances

Table 3.4: Solved instances

size	density (%)	total instances	NMDT + base 10	NMDT + base 2	RNMDT + Algorithm 1	RNMDT + Algorithm 2
small	25	18	18	18	18	18
	50	18	18	18	18	18
	100	18	9	16	16	16
	average	54	45	52	52	52
medium	25	18	18	18	18	18
	50	18	9	9	9	10
	100	18	-	-	-	-
	average	54	27	27	27	28
large	25	9	1	5	5	6
	50	9	-	-	-	-
	100	9	-	-	-	-
	average	27	1	5	5	6
	average	135	66	84	84	86

that could not be solved to optimality. The single most significant improvement was due to the change in the base of the expansion from 10 to 2 which reducing the average relative gap by nearly half. Furthermore, additional gains were obtained, albeit to a lesser degree, using the reformulation and Algorithm 2. The reformulation and proposed algorithm were significantly more successful for the larger and denser instances, as these instances had many quadratic terms and thus, many variables are needed to be expanded. Complete results for these instances are presented in Appendix A.

3.3.2 Generated instances

Six MIQCQP instances with 100% density were generated to test the proposed reformulations and algorithms in the mixed-integer case. These instances are available from the authors upon request. As MIQCQP problems are typically more computationally demanding, the time limit was increased to 7200 seconds.

Table 3.6 shows the model sizes for the six instances. Table **??** shows the relative gap achieved for these instances. The average relative gap for NMDT in base 10, NMDT in base 2, RNMDT, and RNDMT with the dynamic-precision algorithm is 163.4%, 121.7%, 124.7%, and 111.3%, respectively. It is clear that Algorithm 2 exhibited the best performance for these instances as well. Moreover, the three methods outperform the formulation and the algorithm in [37], as the lower bound was improved in all cases. It should be noted that the

Table 3.5: Relative gaps

size	density (%)	NMDT + base 10	NMDT + base 2	RNMDT + Algorithm 1	RNMDT + Algorithm 2
small	25	0.0%	0.0%	0.0%	0.0%
	50	0.0%	0.0%	0.0%	0.0%
	100	14.5%	0.0%	0.0%	0.0%
	total	4.8%	0.0%	0.0%	0.0%
medium	25	1.9%	0.0%	0.0%	0.0%
	50	30.3%	3.4%	3.5%	3.7%
	100	112.0%	65.8%	65.7%	53.3%
	total	48.1%	23.1%	23.1%	19.0%
large	25	23.8%	2.0%	2.2%	2.4%
	50	98.0%	56.4%	53.5%	49.4%
	100	185.8%	155.2%	152.2%	115.6%
	total	102.5%	71.2%	69.3%	55.8%
total		41.7%	23.5%	23.1%	18.8%

Table 3.6: Mixed-integer instances

instance	continuous variables	integer variables	constraints	density (%)
1	20	10	20	100
2	30	10	30	100
3	30	30	30	100
4	30	30	60	100
5	50	30	100	100
6	100	100	100	100

reformulated NMDT without the algorithm presented worse average relative gap than the NMDT without the reformulation, which was primarily observed in a single instance (instance 6).

3.3.3 Comparison with open-source solver

In the experiments that we presented so far, we compared our improvements with the approach proposed in Castro [37]. Next, we present results obtained from comparing the RNMDT with Algorithm 2, the best performing of the four configurations tested, with Couenne [47], a state-of-art open-source global solver for MINLP (MIQCQP inclusive) made available by the COIN-OR [48] initiative. Couenne relies on convex over and under envelopes and spatial BnB.

Table 3.8 details the size of generated instances. Note that, as is the case for the previously subsections, these are fully dense instances, e.g., an instance with 50 constraints and 50 variables has $(50 \times 49/2 + 50) \times (50 + 1) = 65025$

Table 3.7: Relative gaps for generated instances

instance	**NMDT + base 10**	**NMDT + base 2**	**RNDMT + Algorithm 1**	**RNDMT + Algorithm 2**
1	15.9%	0.1%	0.1%	0.1%
2	114.7%	51.1%	47.2%	54.1%
3	58.2%	20.1%	18.6%	18.4%
4	89.0%	50.3%	50.8%	48.4%
5	340.9%	258.8%	257.5%	222.5%
6	361.6%	349.8%	373.4%	324.4%
total	**163.4%**	**121.7%**	**124.6%**	**111.3%**

Table 3.8: Instances size - comparison with open-source solver

instance	**continuous variable**	**constraints**	**integer variables**
1	50	50	0
2	50	50	10
3	50	50	50
4	60	60	50
5	60	60	60
6	70	70	50
7	100	100	0
8	100	100	10

bilinear terms (that is, nonzero entries in the Hessian matrices). We opted for this setting so that we could asses the performance of the algorithm under the most challenging instances possible using a similar number of variables and constraints of those instances available in the literature. Nevertheless, we highlight that practical problems of that nature are typically much sparser, meaning that larger instances could potentially be solved, if those instances were available. Considering the computational platform used, we were not able to solve instances larger than instance 8 in Table 3.8 due to memory shortage caused by the size of the dense Hessian matrix.

Table 3.9 shows the results in terms of relative gaps for both RNMDT with Algorithm 2 and for Couenne. All experiments were terminated due to the time limit of 3600s. RNMDT was not able to find a solution for instance 5, even though we observed that the upper bound reported by RNMDT was 16% better than that reported by Couenne. In Section 3.3.4 we present the performance profiles for these results.

Table 3.9: Results - relative gap - comparison with open-source solver

instance	Couenne	RNMDT + Algorithm 2
1	315%	262%
2	294%	234%
3	229%	167%
4	305%	278%
5	267%	211%
6	302%	265%
7	530%	464%
8	554%	465%

3.3.4 Performance profiles

To provide a structured comparison between the configurations being compared, performance profiles based on Dolan and More [49] are presented. Let $t_{s,ip}$ be the time taken by a given solver or algorithm $s \in S$ to solve the instance problem $ip \in IP$. Let $r_{s,ip}$ be defined as follows $r_{s,ip} = t_{s,ip}/\min\{t_{s,ip} : s \in S\}$ where S is the set of all solvers and algorithm that are being compared in the experiment. Let the time performance profile $\rho_t(\tau)$ be defined as $\rho_t(\tau) = |\{ip \in IP : r_{s,ip} \leq \tau\}|/|IP|$, where IP is the set of all instance problems of the experiment and $|\cdot|$ denotes the cardinality of $\cdot$. Similarly, let $g_{s,ip}$ be the relative gap achieved by the solver or algorithm s for the instance problem ip. Let the relative gap performance profile $\rho_g(\tau)$ be defined as $\rho_g(\tau) = |\{ip \in IP : g_{s,ip} \leq \tau\}|/|IP|$. Figure 3.4 and 3.5 presents the time and gap performance profile, respectively, for the computational experiments performed in sections 3.3.1 and 3.3.2 combined, while Figure 3.6 presentes the gap profile for the instances used in section 3.3.3.

The NMDT with basis 2 and the RNMDT with Algorithm 1 presented similar performance profiles, and both were faster and achieved better bounds then NMDT with basis 10. The RNMDT with Algorithm 2 was slower then with Algorithm 1, since it requires more iterations for the instances that both could solve, which explains the behavior depicted on the beginning (left-hand side) of the time performance profile. However, the gap performance profile shows its superior performance in terms of reaching smaller optimality gaps. An alternative analysis using box-plot is present in Appendix B.

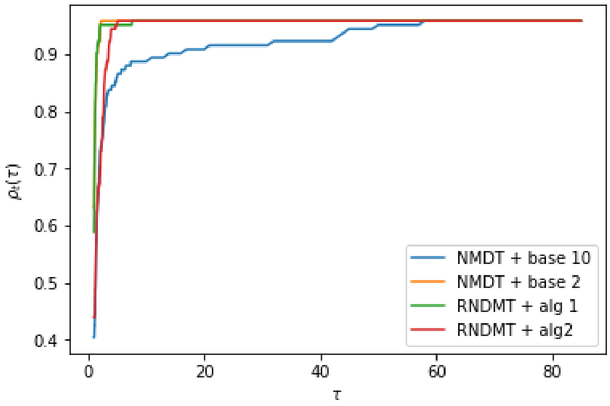


Figure 3.4: time performance profile - sections 3.3.1 and 3.3.2

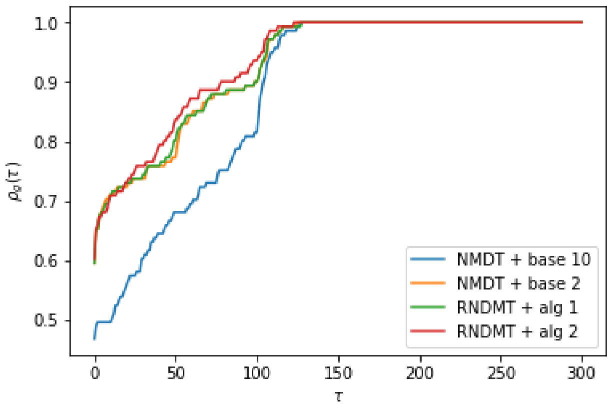


Figure 3.5: relative gap performance profile - sections 3.3.1 and 3.3.2

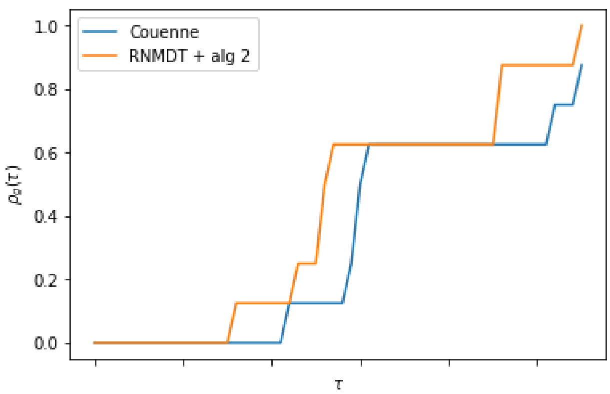


Figure 3.6: relative gap performance profile - section 3.3.3

3.4 Discussion

Three key improvements to the NMDT were proposed. Namely, the replacement of decimal expansion with binary expansion, the reduction of model size, thus eliminating redundant variables and constraints in the formulation, and a new algorithm for solving (MI)QCQP problems using this relaxation that allows the control of the number of binary variables added per iteration.

Instances from the literature and also a set of randomly generated instances were used to assess the performance of the reformulations and the new algorithm. The results showed that the reformulation is easier to solve than the formulation available in the literature, thus providing better bounds at the same computational cost and achieving global optimality for more instances. The proposed algorithm appears to be particularly useful in the presence of many quadratic terms, as in the case of high-density problems. Despite having more parameters to configure, preliminary experiments suggest that its performance is robust for different parameter settings.

The proposed method (RNMDT + Algorithm 2) also showed good results when compared to the state-of-art (open-source) global solver Couenne. Future work include to incorporate cuts and other primal heuristics in our method to increase performance (such as those available in global solvers such as Couenne), and to compare with other global solvers using instances derived from real-world problems.

4
Decomposition

In this chapter, we address nonconvex (mixed-integer) quadratically constrained quadratic programs ((MI)QCQP) with box constraints and decomposable structure, which is a special case of the Problem (2-1)-(2-4) described in Chapter 2 and can be generally represented as:

$$f^* = \max \sum_{s \in S} (x_s^T Q_0 x_s + f_0(x_s, y_s)) \tag{4-1}$$

s.t.:

$$x_s^T Q_r x_s + f_r(x_s, y_s) \leq 0 \qquad \forall s \in S, \forall r \in \mathcal{C}_s \tag{4-2}$$

$$x_{s,i} \in [X_{s,i}^L, X_{s,i}^U] \qquad \forall s \in S, \forall i \in \mathcal{VC}_s \tag{4-3}$$

$$y_{s,i} \in \{Y_{s,i}^L, \ldots, Y_{s,i}^U\} \qquad \forall s \in S, \forall i \in \mathcal{VI}_s \tag{4-4}$$

$$\sum_{s \in S} (A_s^1 x_s + B_s^1 y_s) = b^1 \tag{4-5}$$

$$\sum_{s \in S} (A_s^2 x_s + B_s^2 y_s) \geq b^2 \tag{4-6}$$

Taking $I_{a,b} = \{a, \ldots, b\}$ is the subset of integers ranging from a and b (inclusive), $m = \sum_{s \in S} |C_s|$, $\forall r \in I_{0,m}, Q_r$ is a symmetric matrix, f_0 is a linear function and $\forall r \in I_{1,m}, f_r$ is an affine function. Variable x can assume any continuous value between its bounds, X^L and X^U and variable y can assume any integer value between Y^L and Y^U, A_s and B_s are matrices of adequate size, and b is a vector.

Moreover, $s \in S$ is an index for separable subproblems; $\mathcal{C}_s$, $\mathcal{VC}_s$ and $\mathcal{VI}_s$ are the sets for the indexes for constraints, continuous variables and integer variables sets, respectively, for each subproblem $s \in S$. We assume that for each two subproblems $s_1, s_2 \in S^2$ such that $s_1 \neq s_2$, it follows that $\mathcal{C}_{s_1} \cap \mathcal{C}_{s_2} = \emptyset$, $\mathcal{VC}_{s_1} \cap \mathcal{VC}_{s_2} = \emptyset$ and $\mathcal{VI}_{s_1} \cap \mathcal{VI}_{s_2} = \emptyset$, i.e., each subproblem has its own variables and constraints. The only constraints that have variables from different subproblems are constraints (4-5) and (4-6), which are refereed to as linking or complicating constraints. If it was not for these constraints, each subproblem could be solved independently. The linking constraints are represented as equalities (eq. (4-5)), and linear inequalities (eq. (4-6)). It is important to highlight that, although we use the word subproblem to call each block structure of the problem, rigorous, they become subproblem only after a decomposition is applied and each of the subproblems is solved separably.

The (MI)QCQP is a NP-Hard problem, i.e., it is at least as hard to solve as the most difficult decision problem. A common approach to solving larger instances in mathematical programming programs is decomposition, i.e., to split the problem into several smaller problems that are more tractable and can be solved separately or even in parallel.

In linear programming, the three most common decompositions frameworks are Dantzig-Wolf decomposition (DWD) [50], Benders decomposition (BD) [51], and Lagrangian relaxation (LR) [52]. It should be noted that LR is not limited to be used in a decomposition framework, but it can be used to relax complicating constraints, thus, allowing the problem to be separated in independent subproblems. Lagrangian decomposition (LD) [53, 54] can be viewed as a special case of Lagrangian relaxation decomposition strategies. First, a complicated variable is cloned, and then, a complicated constraint linking the cloned variables are relaxed using Lagrangian duality theory. The BD can be stated as the dual of DWD. If the LD is solved using the cutting-planes algorithm it can be viewed as a BD. Which of the three approaches will be the most appropriated will depend on the structure of the problem, e.g., whether the problem has complicating constraints or variables.

Furthermore, the BD in its classic form (and DWD) cannot be applied to nonlinear programming problems. Geoffrion [55] proposed the generalized Benders decomposition (GBD) based on BD to decompose convex nonlinear programming. Later, Li, Tomasgard, and Barton [56] improved the GBD through the nonconvex generalized Benders decomposition (NGBD) to decompose nonconvex nonlinear programming. Even though the LR and LD can be applied, in principle, directly to a nonconvex problem, nonconvex subproblems must be solved to global optimality as will be seen in Section 4.1, which can be challenging since they belong to NP-Hard class.

This chapter proposes a class of dual functions based on LR and a relaxation of the subproblems that will be performed using the reformulated normalized multiparametric disaggregation technique (RNMDT) presented in Chapter 3. The idea of relaxing the subproblems is inspired by the approach used to expand the GBD to the NGBD. This new dual allows one to use the same decomposition strategies as the classic LR but with easier subproblems to be solved.

4.1 Lagrangian relaxation

Lagrangian relaxation is a common technique to solve or to obtain bounds for nonlinear convex programming problems that relies on relaxing constraints and adding terms to the objective function. However, its use presents some issues when applied to nonconvex problems as is showed in this section.

Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^{m_1}$ and $h : \mathbb{R}^n \to \mathbb{R}^{m_2}$ be twice continuously differentiable functions. Suppose that we are interested in solving the Problem (4-7), also called the primal problem.

$$\begin{aligned} f^* = \max_x \quad & f(x) \\ \text{s.t.} \quad & g(x) \leq 0 \\ & h(x) = 0 \end{aligned} \tag{4-7}$$

Function $L : \mathbb{R}^n \times \mathbb{R}_+^{m^1} \times \mathbb{R}^{m^2} \to \mathbb{R}$ can be defined by $L(x, \mu, \lambda) = f(x) - \mu^T g(x) + \lambda^T h(x)$ where $(\mu, \lambda) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m^2}$ is called Lagrangian multipliers or dual variables and the function is called Lagrangian function. It is common to say that the constraints g and h are relaxed.

If the multipliers μ and λ are fixed, a dual function $\phi : \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ can be defined as the maximum of Problem 4-8. This function is called the (Lagrangian) dual function.

$$\phi(\mu, \lambda) = \max_x \quad f(x) - \mu^T g(x) + \lambda^T h(x) \tag{4-8}$$

This function ϕ, defined by an optimization problem, will hereinafter be called the Lagrangian subproblem (LDS) or dual subproblem. In what follows, we present some important properties of the dual function ϕ that will be used later on.

Proposition 10 *ϕ is a convex function.*

Proof. A function is convex if and only if its epigraph is convex. Its epigraph is the intercession of the epigraphs of each affine function $f(x) - \mu^T g(x) + \lambda^T h(x)$. Since any affine function is convex, so are their epigraphs. An arbitrary intersection of convex sets is convex as well, thus, ϕ is convex. ■

Proposition 11 *ϕ is a lower semicontinuous function.*

Proof. A function is lower semicontinuous if and only if its epigraph is closed. Its epigraph is the intercession of the epigraphs of each affine function $f(x) - \mu^T g(x) + \lambda^T h(x)$ that is convex. Affine function epigraphs are closed and an arbitrary intersection of closed sets is also closed. Therefore, ϕ is lower semicontinuous. ■

Even though the function ϕ and the LD problem are always convex, the dual subproblem is nonconvex if f is nonconcave, g is nonconvex, or h is not affine. Thus, LDS may be as hard to solve to global optimality as the primal problem. This will hereinafter be called issue 1. The LDS feasible solutions and optimal value have a relation with the primal feasible solutions and optimal value respectively as stated in the following propositions.

Proposition 12 (Weak Duality 1) $\phi(\mu, \lambda) \geq f(x), \;\forall \mu \geq 0, \lambda \in \mathbb{R}^{m_2} \;\forall x \in \{x \in \mathbb{R}^n | g(x) \leq 0, h(x) = 0\}$

Proof. $\phi(\mu) = \max_x f(x) - \mu^T g(x) + \lambda^T h(x) \geq \max_{x \in \{\mathbb{R}^n | g(x) \leq 0, h(x)=0\}} f(x) - \mu^T g(x) + \lambda^T h(x) \geq \max_{x \in \{\mathbb{R}^n | g(x) \leq 0, h(x)=0\}} f(x) \geq f(x)$ ■

For every chosen μ and λ, the Lagrangian dual function ϕ provides an upper bound for the primal problem. Our objective is to obtain the best possible bound. Thus, we are interested in solving the problem defined in Problem (4-9) that is called (classic) Lagrangian Dual problem (LDP).

$$\begin{aligned} \phi^* = \min_{\mu, \lambda} \quad & \phi(\mu, \lambda) \\ \text{s.t.} \quad & \mu \geq 0 \end{aligned} \tag{4-9}$$

Remark 1 *Although the relaxation in Chapter 3 was for a minimization problem. It is straight forward to prove the equivalent theorems and propositions for a maximization problem. In the present chapter we define a primal maximization problem, so we obtain a minimization dual problem 4-9.*

Proposition 13 (Weak Duality 2) $\phi^* \geq f^*$ *where* ϕ^* *and* f^* *are the optimal values of the dual and primal problems respectively.*

Proof. It is a direct consequence of Proposition 12. ■

Proposition 14 (Strong Duality) *If f and g are convex functions and the primal problem satisfies the Slater constraint qualification, then* $f^* = \phi^*$

For the definition of Slater constraint qualification see Boyd [57]. For the proof of this proposition and alternative versions that use others constraints qualifiers other than the Slater condition, see Robinson [58]. For a complete review of Lagrangian duality theory and the proofs of the above propositions, see Ruszczynki [59].

The convexity hypothesis of Proposition 14 may be too strong for problems that arise naturally in many areas, therefore a duality gap may exist for these problems. This will hereinafter be called issue 2. Nevertheless, the Lagrangian dual problem can still be used to obtain bounds for the problem using weak duality. In some contexts that the strong duality is not valid, as in integer programming, the dual problem is usually called the Lagrangian relaxation.

Even if we are willing to accept to use a nonzero duality gap and the original problem is feasible and bounded, the Lagrangian dual problem can be unbounded, i.e., $\phi^* = +\infty$ (issue 3).

If the primal problem is nonconvex, we have three main issues to address when using the Lagrangian duality, i.e., we must solve nonconvex subproblems (issue 1), only the weak duality is valid (issue 2) and the generated dual bound can be $+\infty$ (issue 3).

The second and third issues are traditionally addressed by modifying the Lagrangian function adding penalty terms. Linear penalty expressions were introduced by Pietrzykowski [60] and Zangwill [61]. Quadratic penalty expressions by Courant [62]. Penalty expressions with both linear and quadratic pieces were proposed by Rockafellar [23]. A generalized approach called augmented Lagrangian was presented by Rockafellar and Wets [63, 64, 65]. Two disadvantages of the augmented Lagrangian is that it usually ruins the problem separable structure when it exists, thus, compromising decomposition strategies [66] and it also adds nonlinearities to the problem.

Another common approach to address the third issue is to relax only a subset of the constraints in a way that the Lagrangian subproblem is always bounded, e.g., not relaxing box constraints if they exist. If only a subset of the constraints is relaxed, this is called partial Lagrangian relaxation or semi Lagrangian relaxation. One reason to do so is to generate fewer Lagrangian multipliers, thus it might be easier to find their optimal value and to obtain the optimal dual bound. In particular, if the subproblem is always bounded, which is the case when the box constraints are not relaxed, the dual bound will always be finite and the issue 3 will be resolved.

We will concentrate on addressing issues 1 and 3 for the nonconvex (MI)QCQP. Additionally, the (MI)QCQP is an undecidable problem if the variables are unbounded [8] and NP-Hard otherwise, thus, in this work, we will assume that all variables have known bounds. The extension of the framework presented in this study to solve the second issue is not within the scope of this study and will be the topic of a future research.

The new class of dual problems we develop in this chapter addresses the issue 3, replacing the nonconvex (MI)QCQP subproblems with MIP relaxations, which also may be intractable due to being NP-Hard. However, there are more reliable techniques that have been showed to be efficient in many real cases, e.g., Branch and Cut and widely available off-the-shelf commercial solvers such as GUROBI [40] and CPLEX [39]. These relaxations are obtained using the reformulated normalized multiparametric disaggregation (RNMD) developed in Chapter 3. The MIP relaxations are bounded, solving the first issue as well.

A possible approach to solve decomposable QCQP problems is to relax at least all quadratic constraints, thus, obtaining a subproblem that is equivalent to a SDP problem that can be solved to global optimality using interior point methods [67]. On the other hand, the formulation of the SDP that is equivalent to the LR is not trivial to find, especially if not all constraints are relaxed. Another issue with this approach is that we may generate large dual bounds that could be reduced if an alternative subset of constraints were to be relaxed. Dentcheva and Römisch [68] give a framework to estimate how large the duality gap will be if a subset of constraints is relaxed, then they apply this methodology to decide which decomposition classic forms of the nonconvex stochastic programming constraints generates the smallest duality gap. In this study, only constraints that are identify as complicating constraints ((4-5) and (4-6)) will be relaxed. The resulting dual subproblem is represented in Problem (4-10)-(4-13).

$$\phi(\mu,\lambda) = \max \sum_{s\in S}(x_s^T Q_0 x_s + f_0(x_s,y_s)) - \mu^T(A_s^1 x_s + B_s^1 y_s) + \lambda^T(A_s^2 x_s + B_s^2 y_s) \tag{4-10}$$

s.t.:

$$x_s^T Q_r x_s + f_r(x_s,y_s) \le 0 \quad ,\forall s\in S, \forall r \in \mathcal{C}_s \tag{4-11}$$

$$x_{s,i} \in [X_{s,i}^L, X_{s,i}^U] \quad ,\forall s\in S, \forall i \in \mathcal{VC}_s \tag{4-12}$$

$$y_{s,i} \in \{Y_{s,i}^L, \dots, Y_{s,i}^U\} \quad ,\forall s\in S, \forall i \in \mathcal{VI}_s \tag{4-13}$$

4.1.1 Motivating example

In this subsection, a motivating example is developed with the purpose of showing that even in simple cases, the Lagrangian dual problem can fail to obtain a valid bound if its LDS's solution is not global optimum. Consider problem (4-14).

$$\begin{aligned} \max_x\ & x_1 x_2 \\ \text{s.t. } & x_1 + 2x_2 \le 1 \\ & x \in [0,1]^2 \end{aligned} \tag{4-14}$$

For a fixed nonnegative multiplier μ, the DSP for 4-14 is:

$$\begin{aligned} &\max_x \; x_1 x_2 - \mu(x_1 + 2x_2 - 1) \\ &\text{s.t. } x \in [0,1]^2 \end{aligned} \tag{4-15}$$

This problem and its LDS version were implemented at GAMS 23.7 [69]. Solving Problem (4-14) with the global solver BARON 9.3, we obtain the optimal value 0.125. Since the LD and this problem have a (weak) dual relationship, Problem (4-15) should provide an upper bound, a value greater or equal than 0.125, as its optimal value for all fixed nonnegative multiplier. However, if we solve it using the local solver 3 [70], it returns a solution that is zero for all variables, corresponding to a zero objective function value, and thus, is not a valid upper bound, since the feasible solution $(x_1, x_2) = (\frac{1}{2}, \frac{1}{4})$ has a greater value than that.

It is worth highlighting that, for this particular problem, we could improve the solution by using another local solver that uses alternative methods, providing a warm start or even by applying a multi-start strategy. However, the point that has been made here remains. If one cannot guarantee that the solutions obtained for the Lagrangian dual subproblems are global maxima, one cannot be sure about the validity of the resulting bounds which, in turn, compromises the efficacy and efficiency of solutions methods that rely on this type of relaxation.

To obtain a bound, it suffices that the dual subproblem is solved for only one fixed multiplier, not necessarily the optimal one. The Lagrangian duality theory states that its (global) optimal solution approaches a valid bound for the primal problem, but not a local solution.

After our method is presented, we will show how to address the issue of generating invalid bound in Section 4.2.1.

4.1.2 Multipliers update

The dual problem is, in general, nonsmooth, i.e., not differentiable. Nevertheless, it is convex, and there is an extensive literature on how to solve this type of problem by updating the Lagrangian multipliers. Held et al. [71] proposed the classical approach which becomes known as the subgradient method. Improvements of this method were proposed by Camerini et al. [72] and Fisher [52]. An alternative method that presents better convergence properties is the cutting plane method by Goldstein [73] and Kelley [74]. An improvement of this method is presented by Marsten et al. [75]. Other methods include the Volume algorithm [76] and bundle method [77, 78], that is more stabilized than cutting-planes methods. In this study, we use the bundle method, since preliminary experiments showed that it has a good trade-off between easiness to implement and convergence rate.

4.2
p-Lagrangian relaxation

Using the RNMDT relaxation, one can construct problems that are solvable and have a weak dual relation to the mixed-integer quadratically constrained quadratic program (MIQCQP). This is an important feature of the relaxation, as it implies that one can apply a Lagrangian relaxation to the MIQCQP and then relax the dual subproblem (LDS) (Problem 4-8) using RNMD with arbitrary precision, changing the value of p. Hence, for each fixed p, we obtain a relaxed version of LDS and a dual problem associated with it, which we will call the p-Lagrangian dual subproblem (p-LDS) and p-Lagrangian dual problem (p-LDP). Alternatively, one can apply the RNMDT to relax the primal problem and then apply the Lagrangian relaxation to relax Constraint (4-5) and (4-6) and get the same dual function. The procedure to relax one or more constraints with this method is called p-Lagrangian relaxation (p-LR) and the procedure to relax complicated constraints to solve subproblems independently is called p-Lagrangian decomposition (p-LD).

The parameter p controls the degree of relaxation of the dual subproblem. Thus, the possibility of controlling how tight the dual bound becomes is one of the interesting properties of the p-LDP, i.e., if we solve two p-LDP setting two different values for p, the one with the smaller parameter p will provide a better or equal bound compared with the one with the larger p.

This relaxation of the dual subproblem is equivalent to replacing the dual function ϕ with an over-estimator $\hat{\phi}_p$, obtaining a dual function class, one dual function for each $p \in \mathbb{Z}_-$. The functions $\hat{\phi}_p$ of this class have the following properties:

Proposition 15 $\hat{\phi}_p \geq \phi, \ \forall \ p \leq 0$

Proof. Consequence of Proposition 5. ■

Proposition 16 (Weak duality for p-LD) $\hat{\phi}_p \geq f^*, \ \forall \ p \leq 0$ *where* f^* *is the primal optimal value*

Proof. It is known by weak duality that $\phi \geq f^*$. Proposition 15 gives that $\hat{\phi}_p \geq \phi$. It follows by transitivity that $\hat{\phi}_p \geq f^*$. ■

Proposition 17 $\hat{\phi}_p$ *is a polyhedral function (continuous piecewise linear)* $\forall p \leq 0$

Proof. The function is defined at every point as the optimal value of an optimization problem, where the function is defined as the pointwise supremum of affine functions. Since the optimization problem in question is a MIP, there are only a finite number of relevant optimal values, i.e., the vertices of the convex hull of the problem. Thus, the function is equivalent to a pointwise maximum of a finite number of affine functions. Therefore, $\hat{\phi}_p$ is polyhedral. ■

Proposition 18 $\hat{\phi}_p$ *is convex* $\forall p \leq 0$

Proof. Analogous to proposition 10. ■

Proposition 19 $\hat{\phi}_{p_2} \geq \hat{\phi}_{p_1}$ $\forall p_1 \leq p_2 \leq 0$

Proof. Consequence of Theorem 1 ■

Proposition 20 $\inf_{\mu \geq 0, \lambda} \hat{\phi}_{p_2}(\mu, \lambda) \geq \inf_{\mu \geq 0, \lambda} \hat{\phi}_{p_1}(\mu, \lambda) \geq \inf_{\mu \geq 0} \phi(\mu, \lambda)$, $\forall p_1 \leq p_2 \leq 0$

Proof. It is a consequence of Propositions 15 and 19. ■

Proposition 21 $\{\hat{\phi}_{p=-n}\}_{n \in \mathbb{N}}$ *is a monotonic decreasing sequence.*

Proof. It follows from Proposition 20 ■

The following theorems are about the convergence of the sequence of the dual functions generated by the proposed methodology of this chapter, to the classic (partial) dual function. First pointwise convergence is proved in Theorem 8. Then, in Theorem 9, it is proved a relation between monotonic and uniform convergence for monotonic sequences, This is a less restrictive version of the Dini's Theorem where only semicontinuity is required instead of continuity. Finally, uniform convergence is proved when the dual functions are restricted to an arbitrary compact. For a review of concepts including types of convergence, upper and lower semicontinuity, open sets, compactness and Dini's Theorem, the reader should refer to Rudin [79].

Theorem 8 $\{\hat{\phi}_{p=-n}\}_{n \in \mathbb{N}}$ *converges pointwise to* ϕ

Proof. For every $\mu \geq 0, \lambda \in \mathbb{R}^{m_2}$. If $p \to -\infty$, then $\forall j \in DS, \Delta x_j = 0, \forall i$ that $(i, j) \in QT, \Delta w_{i,j} = 0$ and $w_{i,j} = 0$. Thus, the optimal solution of the subproblem defining $\{\hat{\phi}_{p=-n}\}_{n \in \mathbb{N}}(\mu, \lambda)$ becomes feasible for the dual subproblem of $\phi(\mu, \lambda)$ when $p \to -\infty$. Hence, $\mu \geq 0, \lim_{p \to -\infty} \hat{\phi}_p(\mu, \lambda) \leq \phi(\mu, \lambda)$. Therefore, the pointwise convergence follows using Proposition 15. ■

Theorem 9 *Let $\{f_n\}_{n\in\mathbb{N}}$ be a monotonic decreasing sequence of real upper semicontinuous functions defined over a compact set K that converges pointwise to a lower semicontinuous function f, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f.*

Proof. Let the function g_n be defined as $g_n = f_n - f$. It follows by its definition that $g_n \geq 0$ and g_n is upper semicontinuous for all $n \in \mathbb{N}$ and the sequence $\{g_n\}_{n\in\mathbb{N}}$ is monotonic decreasing.

For any $\epsilon > 0$, let E_n be the set of those $x \in K$ such that $g_n(x) < \epsilon$. Since g_n is upper semicontinuous, E_n is open. Since $\{g_n\}_{n\in\mathbb{N}}$ is monotonic decreasing, the sets E_n are ascending sets, i.e., $E_n \subset E_{n+1}$.

Since f_n converges pointwise to f, the sets $\{E_n\}_{n\in\mathbb{N}}$ forms an open cover to the domain K. By compactness, there exists a finite open subcover $\{E_n\}_{n\leq N}$ for K. Since the covers sets are ascending, $\cup_{n\leq N}\{E_n\} = E_N$. Thus $K \subset E_N$. Therefore if follows that for all $n \geq N$, for all $x \in K, |f_n(x) - f(x)| < \epsilon$ and $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f. ■

Theorem 10 *$\{\hat{\phi}_{-n}\big|_K\}_{n\in\mathbb{N}}$ converges uniformly to $\phi\big|_K$ where K is a compact set and $\hat{\phi}_{-n}\big|_K$ and $\phi\big|_K$ are the dual functions with their domain restricted to the compact K.*

Proof. i) $\{\hat{\phi}_{-n}\big|_K\}_{n\in\mathbb{N}}$ is a monotone sequence as stated in Proposition 21; ii) the sequence $\{\hat{\phi}_{-n}\big|_K\}_{n\in\mathbb{N}}$ converge pointwise as stated by Theorem 8; iii) each $\hat{\phi}_{-n}\big|_K$ is a polyhedral function, thus continuous, and in particular, upper semiconstinuous as stated in Proposition 17; iv) $\phi\big|_K$ is lower semicontinuous as stated in Proposition 11; v) Therefore, the convergence uniformly follows from Theorem 9. ■

Corollary 1 *The dual function ϕ is a continuous function.*

Proof. For any compact K, a sequence of continuous functions $(\{\hat{\phi}_{-n}\big|_K\}_{n\in\mathbb{N}})$ converges uniformly to $\phi\big|_K$, so the limit is also continuous. Any $(\mu, \lambda) \in \mathbb{R}_+^{m_1} \times \mathbb{R}^{m_2}$ with $\mu > 0$ belongs to a open set inside a compact set, so the function ϕ is continuous in every point, therefore ϕ is continuous. The case where $\mu = 0$ is also continuous since (μ, λ) belongs to the intersection of a compact with $[0, \infty)^{m_1} \times \mathbb{R}^{m_2}$. The case where some of the μ are null and other are positive, is analogous. Therefore, completing the proof. ■

4.2.1 Motivating example: part 2

Applying the $p-$LR to the motivating problem presented in Section 4.1.1 we obtain the Problem 4-16.

$$\begin{aligned}
\max_x \ & w_{1,2} - \mu(x_1 + 2x_2 - 1) \\
\text{s.t. } & w_{1,2} = 2^{-1}\hat{x}_{1,2,-1} + v_{1,2} \\
& x_2 = 2^{-1}z_{2,-1} + \Delta x_2 \\
& 0 \leq \hat{x}_{1,2,-1} \leq z_{2,-1} \\
& 0 \leq x_1 - \hat{x}_{1,2,-1} \leq 1 - z_{2,-1} \\
& 2^{-1}(x_1 - 1) + \Delta x_2 \leq v_{1,2} \leq 2^{-1}x_1 \\
& 0 \leq v_{1,2} \leq \Delta x_2 \\
& z_{2,-1} \in \{0,1\} \\
& x \in [0,1]^2 \\
& \Delta x_2 \in [0, 2^{-1}]
\end{aligned} \tag{4-16}$$

Here, the x_2 variable was discretized with a precision $p = -1$ with the Lagrangian multiplier μ fixed at zero. Solving the resulting problem with CPLEX 12.3 [39], we obtain the dual bound $\hat{\phi}_{-1}(0) = 1$, that is valid since it is greater than 0.125. Of course, that is not a tight bound, but that was obtained using only one multiplier value as an example. This could be strengthened by applying a subgradient subroutine to choose a better multiplier (the optimum is $\mu = 0.25$) or by increasing the used precision, lowering parameter p, but that is not the purpose of this example.

If one is interested in obtaining a tighter bound, it is necessary to choose more carefully the parameter p and the Lagrangian multipliers values. If p and μ are set respectively to -2 and 0.25, the dual bound obtained is $\hat{\phi}_{-2}(0.25) = 0.125$, the same as the primal optimal value. On the next section, it is presented how one can choose these values.

4.3 Algorithm

An algorithm to solve a nonconvex problem, as it is the case of the (MI)QCQP usually relies on finding dual and primal bounds. Dual bounds are upper (lower) bounds for maximization (minimization) problems. Primal bounds are the objective value for feasible solutions and provide lower (upper) bounds. To obtain tight dual bounds using p-LD, one shall choose good values for the precision parameter p and for the Lagrangian multipliers. On the other hand, to find a primal bound, one shall obtain a feasible solution. For the first part, the proposed algorithm in this section is inspired by Algorithm 1 and in methods for solving nonsmooth problems used to solve the Lagrangian dual discussed in Section 4.1.2, such as the bundle method. For the latter, the integer variables are fixed and a nonlinear local solver is used with a warm start to solve problem (4-1)-(4-6).

Algorithm 3 p-LD algorithm

Step 0. Set $p = 0$ and starting lagrangian multiplier

Step 1. Compute a $p-$Lagrangian dual bound using the Lagrangian multiplier and obtain a feasible solution using a Lagrangian heuristic

Step 2. Update the Lagrangian multipliers using a nonsmooth optimization algorithm step

Step 3. If a stop condition of type 1 is met, set $p = p - 1$

Step 4. If a stop condition of type 2 is met, stop. Otherwise, return to Step 1

Some of the steps in Algorithm 3 must be clarified. A Lagrangian heuristic in Step 1 can be any method to generate feasible solutions. In this study, a local solver will be used to solve the original problem having the dual solution as a warm start, fixing any integer variables if they exist, and also fixing complicated variables that allow the primal problem to be solved decomposability. Nonsmooth optimization algorithm step can be any algorithm described in Section 4.1.2. A stop condition of type 1 is a stop criterion to the dual problem. They are discussed in the references provided in Section 4.1.2. A stop condition of type 2 is a condition to stop the whole algorithm, e.g., time limit, iteration limit, or a threshold relative or absolute gap using primal and dual bounds.

4.4 Numerical experiments

Random instances of the form (4-1)-(4-6) were generated, where only the right-hand side (RHS) implicit defined at f_r functions varies in each subproblem. The linking constraints are on the form $x_{i,s} = x_{i,s+1}$ for each i in a subset of a and for each $s \in S - \{s^{last}\}$ assuming that S is a finite ordered set and s^{last} is the last element of this set. This form is commonly used in the stochastic programming (SP) with fixed recourse, where each subproblem is represented by a scenario and the variables that must be equal among all scenarios - first stage variables in the two-stage stochastic programming case - are duplicated and the linking constraints are added to the problem dictating that duplicated variables values must be the same, also

Table 4.1: Instances sizes

	continuous				mixed-integer			
instance	# subprob.	# c. var.	# constrs.	# i. var.	# subprob.	# c. var.	# constrs.	# i. var.
1	2	21	41	0	2	21	41	8
2	3	31	61	0	3	31	61	12
3	4	41	81	0	4	41	81	16
4	5	51	101	0	5	51	101	20
5	6	61	121	0	6	61	121	24
6	7	71	141	0	7	71	141	28
7	8	81	161	0	8	81	161	32
8	9	91	181	0	9	91	181	36
9	10	101	201	0	10	101	201	40
10	20	201	401	0	20	201	401	80
11	30	301	601	0	30	301	601	120
12	40	401	801	0	40	401	801	160
13	50	501	1001	0	50	501	1001	200
14	60	601	1201	0	60	601	1201	240
15	70	701	1401	0	70	701	1401	280
16	80	801	1601	0	80	801	1601	320
17	90	901	1801	0	90	901	1801	260
18	100	1001	2001	0	100	1001	2001	400

know as nonanticipativity constraints. This form for the SP was proposed by Ruszczynski [59].

Instances with 2 to 10 subproblems - with a step of 1 - were generated with subproblems with 10 continuous variables and 20 constraints. Instances from 20 to 100 subproblems with a step of 10 were generated. Each of them was solved using Algorithm 3 with the Bundle method to update the multipliers. The quality and time of solution were compared with RNMDT with Algorithm 2 and with the global solvers Lindo Global [80], Scip [81], Couenne [47], Antigone [82] and Baron [46]. Since only Couenne is open Source and the other solvers are commercial and requires a license, we used the Neos Servers [83, 84, 85] to run them. As the computers used by Neos Servers have better processors (Xeons) and more memory than the computer that we used to obtain the other results, if our algorithm have better performance than the global solver in better hardware, the algorithm should outperform them in the same computer as well.

Instances sizes are summarized in Table 4.1. Each generated instance is enumerated for the continuous and mixed-integer instances. For each group the number of decomposable subproblems, continuous variables, constraints, and integers variables are shown. It should be noted that, although small in size, these instances are difficulty to solve partially because of their density. In fact, when decomposed, each subproblem have both full quadratic (among continuous variables) and full linear density.

Table 4.2 and Table 4.3 show the relative gap and time results for the continuous instances, respectively. Table 4.4 and Table 4.5 show the relative gap and time results for the mixed-integer instances, respectively. "#N/A"

Table 4.2: Continuous instances - relative gap

instance	RNMDT	p-LR	LINDOGLOBAL	SCIP	COUENNE	ANTIGONE	BARON
1	0.0%	0.0%	153.6%	68.3%	0.0%	0.0%	0.0%
2	1.7%	0.0%	269.2%	201.7%	0.0%	3.9%	0.0%
3	8.5%	0.0%	267.5%	255.2%	14.9%	8.1%	4.4%
4	14.4%	0.0%	295.3%	275.7%	30.1%	9.4%	4.9%
5	17.2%	0.0%	298.1%	287.7%	40.0%	10.2%	6.2%
6	23.3%	0.0%	324.0%	300.3%	459.0%	13.5%	10.6%
7	27.2%	0.0%	306.8%	308.5%	57.1%	13.1%	11.4%
8	27.5%	0.0%	330.3%	322.0%	53.6%	13.5%	11.2%
9	28.5%	0.0%	325.9%	304.5%	170.0%	16.5%	15.7%
10	41.8%	0.0%	350.5%	362.0%	76.6%	18.8%	21.1%
11	47.5%	0.0%	395.1%	367.6%	83.3%	24.2%	24.6%
12	48.9%	0.0%	369.5%	378.9%	115.5%	21.2%	24.0%
13	52.1%	0.0%	498.3%	358.5%	85.1%	24.0%	27.8%
14	54.0%	0.0%	486.7%	373.8%	89.0%	24.7%	29.3%
15	52.8%	0.0%	504.6%	360.7%	89.0%	23.3%	30.2%
16	53.3%	0.0%	507.4%	374.7%	94.9%	24.9%	31.5%
17	55.9%	0.0%	500.9%	368.1%	97.3%	25.4%	33.9%
18	56.7%	0.0%	#N/A	358.4%	110.2%	24.8%	34.3%

denotes that the method was not able to obtain the lower or an upper bound, or both, for that instance. Table 4.6 shows how many instances each algorithm was able to converge before the time limit for the continuous and mixed-integer instances.

The proposed decomposition was compared to the commercial solver using the same methodology as in Chapter 3, i.e., relative and time performance profile based on Dolan [49] methodology. Figure 4.1 and Figure **??** shows the time and relative gap, respectively, for the continuous instances. Figure 4.3 and Figure 4.4 shows the time and relative gap, respectively, for the mixed integer instances p-LR was able to find and prove global optimality for all mixed integer instances with one exception that achieved a relative gap of less than 1%. All commercial solvers were dominated by p-LR in the numerical experiments.

The p-LR was able to converge in 35 of the 36 instances, not proving global optimality in only one mixed-integer instance where a relative gap of 0.3% was obtained. All others methods converged only for 4 or less instances. Since p-LR exploits these instances decomposable structure, it scales better with the number of decomposable subproblems, where the other methods scales worse since they are trying to solve the problem directly. It is interesting to notice that RNMDT with Algorithm 2 was able to compete wtih the benchmark solvers, but it scales badly for the mixed-integer instances when compared with the state-of-art commercial solvers Baron and Antigone. It is remarkable, that RNMDT obtained better solutions than the best open-source global solver available, i.e., Couenne.

Table 4.3: Continuous instances - time(s)

instance	RNMDT	p-LR	LINDOGLOBAL	SCIP	COUENNE	ANTIGONE	BARON
1	1.27E+02	2.29E+01	7.20E+03	7.20E+03	6.30E+01	1.77E+02	1.60E+01
2	7.20E+03	4.45E+01	7.20E+03	7.20E+03	2.10E+03	7.20E+03	2.20E+03
3	7.20E+03	1.27E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
4	7.20E+03	1.27E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
5	7.20E+03	7.66E+01	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
6	7.20E+03	1.93E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
7	7.20E+03	1.37E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
8	7.20E+03	2.13E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
9	7.20E+03	2.18E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
10	7.20E+03	3.70E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
11	7.20E+03	7.35E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
12	7.20E+03	4.10E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
13	7.20E+03	1.50E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
14	7.20E+03	2.37E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
15	7.20E+03	2.11E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
16	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
17	7.20E+03	4.17E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
18	7.20E+03	6.16E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03

Table 4.4: Mixed-integer instances - relative gap

instance	RNMDT	p-LR	LINDOGLOBAL	SCIP	COUENNE	ANTIGONE	BARON
1	23.2%	0.0%	290.5%	93.8%	0.0%	0.0%	0.0%
2	25.0%	0.0%	326.7%	182.5%	19.4%	5.8%	0.0%
3	29.9%	0.0%	327.2%	226.5%	239.4%	9.2%	5.7%
4	38.9%	0.0%	337.9%	271.2%	63.2%	11.4%	9.5%
5	45.2%	0.0%	331.8%	283.8%	74.1%	14.9%	13.5%
6	48.2%	0.0%	328.0%	281.1%	135.5%	14.8%	14.0%
7	48.5%	0.0%	334.8%	319.8%	75.5%	15.4%	16.2%
8	62.5%	0.0%	346.5%	316.7%	86.2%	18.1%	18.1%
9	65.5%	0.3%	421.9%	317.6%	85.1%	19.6%	20.1%
10	67.4%	0.0%	503.7%	365.8%	88.8%	19.3%	21.0%
11	85.5%	0.0%	436.8%	359.5%	82.4%	23.7%	27.8%
12	#N/A	0.0%	519.8%	331.1%	91.0%	23.6%	25.6%
13	#N/A	0.0%	709.7%	358.7%	85.2%	24.3%	29.0%
14	#N/A	0.0%	#N/A	358.3%	91.7%	24.8%	31.7%
15	#N/A	0.0%	6552.8%	355.7%	80.3%	24.5%	29.8%
16	124.6%	0.0%	#N/A	357.5%	87.1%	24.3%	32.7%
17	#N/A	0.0%	#N/A	361.6%	82.0%	25.9%	33.8%
18	#N/A	0.0%	#N/A	359.2%	81.8%	26.9%	32.6%

Table 4.5: Mixed-integer instances - time(s)

instance	RNMDT	p-LR	LINDOGLOBAL	SCIP	COUENNE	ANTIGONE	BARON
1	7.20E+03	3.76E+01	7.20E+03	7.20E+03	2.43E+02	5.42E+02	6.20E+01
2	7.20E+03	5.51E+01	7.20E+03	7.20E+03	7.20E+03	7.20E+03	2.98E+03
3	7.20E+03	5.94E+01	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
4	7.20E+03	4.92E+01	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
5	7.20E+03	6.70E+01	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
6	7.20E+03	6.36E+01	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
7	7.20E+03	1.20E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
8	7.20E+03	1.00E+02	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
9	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
10	7.20E+03	1.25E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
11	7.20E+03	1.17E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
12	7.20E+03	2.15E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
13	7.20E+03	2.08E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
14	7.20E+03	2.29E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
15	7.20E+03	4.85E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
16	7.20E+03	3.84E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
17	7.20E+03	3.28E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03
18	7.20E+03	4.33E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03	7.20E+03

Table 4.6: Solved instances

	RNMDT	p-LR	LINDOGLOBAL	SCIP	COUENNE	ANTIGONE	BARON	total
Continuous	1	18	0	0	2	1	2	18
Mixed-integer	0	17	0	0	1	1	2	18
total	**1**	**35**	**0**	**0**	**3**	**2**	**4**	**36**

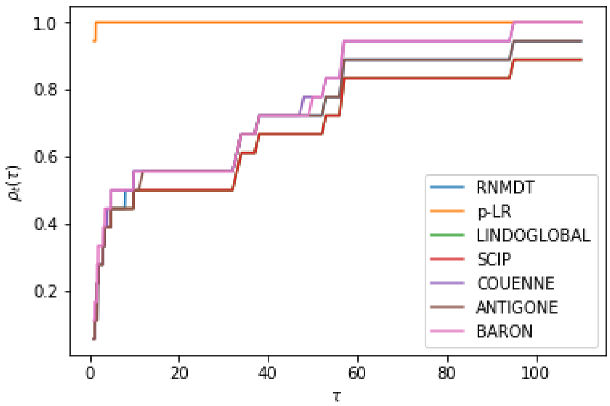


Figure 4.1: time performace profile for continuous instance

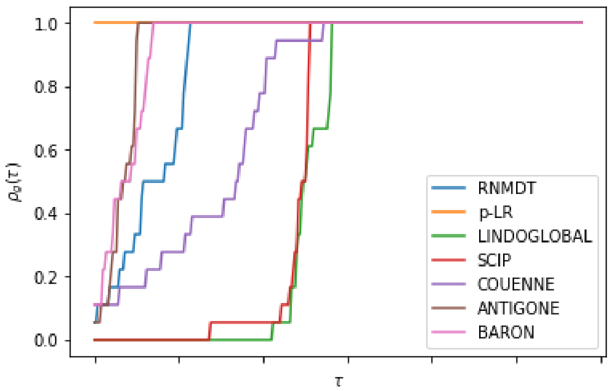


Figure 4.2: gap performace profile for continuous instance

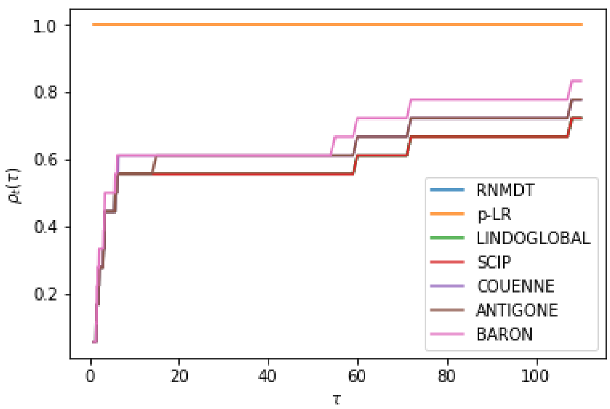


Figure 4.3: time performace profile for mixed-integer instance

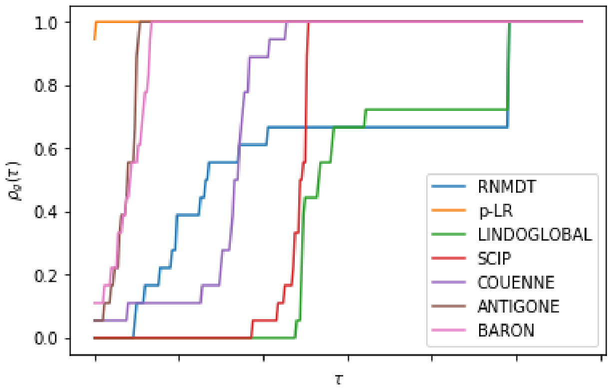


Figure 4.4: gap performace profile for mixed-integer instance

4.5 Discussion

In this chapter, we identified issues when using Lagrangian relaxations to decompose nonconvex quadratic problems, and, we addressed two of them, i.e., to obtain a finite dual bound and avoiding solve a nonconvex problem. This is achieved by only relaxing the complicated constraints, maintaining the quadratic and box constraints in the subproblem and by replacing the dual subproblem using a relaxation developed in Chapter 3. Then, we used these changes in the classic Lagrangian dual problem to define a class of dual problems and dual functions, and also proved relations between members of that class, including pointwise and uniform convergence to the classic Lagrangian dual problem.

Furthermore, we proposed an algorithm to solve a MIQCQP with the novel dual class that allows similar decomposition strategies then the classic Lagrangian decomposition. Then, we compared this algorithm with the algorithm proposed in Chapter 3 and with several commercial solvers and observed that it converges faster and it is particularity useful if the problem is decomposable in many subproblems.

5
Conclusions and further research

The main objective was to investigate and develop methods to solve nonconvex quadratically constrained quadratic programming, possibly with integer variables, ((MI)QCQP) problems. Thus, this study 1) review the existing methods and categorize them; 2) propose improvements in a relaxation method — NMDT — and in an algortihm based on it; 3) propose a set of dual functions that can be used to decompose a nonconvex (MI)QCQP as would be possible to do with a Lagrangian relaxation with the advantage of avoiding to solve nonconvex (MI)QCQP subproblems to global optimality.

Moreover, the proposed methods were compared with the state-of-art algorithms and solvers using both literature instances and random generated instances. Numerical experiments showed that the proposed methods are at least competitive and that taking advantage of special structure such as block angular structure can greatly improve the efficiency of the solution methods.

The short term future research plans are to publish the results found in Chapter 4 and to apply the p-LD method to solve a stochastic version of the refinery operational planning problem (ROPP). Longer term future plans include to improve the selection and the choice of how many binary variables will be used to discretized each continuous variables in RNMDT and how to close the dual gap presented in p-LD method.

Bibliography

[1] BURER, S.; LETCHFORD, A. N.. **Non-convex mixed-integer nonlinear programming: A survey**. Surveys in Operations Research and Management Science, 17(2):97–106, 2012.

[2] BAGAJEWICZ, M.. **A review of recent design procedures for water networks in refineries and process plants**. Computers & Chemical Engineering, 24(9-10):2093–2113, 2000.

[3] JEZOWSKI, J.. **Review of water network design methods with literature annotations**. Industrial & Engineering Chemistry Research, 49(10):4475–4516, 2010.

[4] MISENER, R.; FLOUDAS, C. A.. **Advances for the pooling problem: Modeling, global optimization, and computational studies**. Applied and Computational Mathematics, 8(1):3–22, 2009.

[5] VISWESWARAN, V.. **MINLP: Applications in blending and pooling problems.**, 2009.

[6] ANDRADE, T.; RIBAS, G. ; OLIVEIRA, F.. **A strategy based on convex relaxation for solving the oil refinery operations planning problem**. Industrial & Engineering Chemistry Research, 55(1):144–155, 2016.

[7] PARDALOS, P. M.; VAVASIS, S. A.. **Quadratic programming with one negative eigenvalue is NP-hard**. Journal of Global Optimization, 1(1):15–22, 1991.

[8] JEROSLOW, R.. **There cannot be any algorithm for integer programming with quadratic constraints**. Operations Research, 21(1):221–224, 1973.

[9] GAREY, M. R.; JOHNSON, D. S.. **Computers and intractability: A guide to the theory of NP-Completness**, volumen 29. wh freeman New York, 2002.

[10] LINDEROTH, J.. **A simplicial branch-and-bound algorithm for solving quadratically constrained quadratic programs**. Mathematical Programming, 103(2):251–282, 2005.

[11] THAKUR, L. S.. **Domain contraction in nonlinear programming: minimizing a quadratic concave objective over a polyhedron**. Mathematics of Operations Research, 16(2):390–407, 1991.

[12] GLEIXNER, A. M.; BERTHOLD, T.; MÜLLER, B. ; WELTGE, S.. **Three enhancements for optimization-based bound tightening**. Journal of Global Optimization, 67(4):731–757.

[13] TAWARMALANI, M.; SAHINIDIS, N. V.. **Global optimization of mixed-integer nonlinear programs: A theoretical and computational study**. Mathematical Programming, 99(3):563–591, 2004.

[14] MCCORMICK, G. P.. **Computability of global solutions to factorable nonconvex programs: Part I convex underestimating problems**. Mathematical programming, 10(1):147–175, 1976.

[15] AL-KHAYYAL, F. A.; FALK, J. E.. **Jointly constrained biconvex programming**. Mathematics of Operations Research, 8(2):273–286, 1983.

[16] BAO, X.; SAHINIDIS, N. V. ; TAWARMALANI, M.. **Multiterm polyhedral relaxations for nonconvex, quadratically constrained quadratic programs**. Optimization Methods & Software, 24(4-5):485–504, 2009.

[17] SHERALI, H. D.; ADAMS, W. P.. **A reformulation-linearization technique for solving discrete and continuous nonconvex problems**, volumen 31. Springer Science & Business Media, 2013.

[18] ANDROULAKIS, I. P.; MARANAS, C. D. ; FLOUDAS, C. A.. **αBB: A global optimization method for general constrained nonconvex problems**. Journal of Global Optimization, 7(4):337–363, 1995.

[19] FAMPA, M.; LEE, J. ; MELO, W.. **On global optimization with indefinite quadratics**. EURO Journal on Computational Optimization, 5.

[20] NOWAK, I.. **Dual bounds and optimality cuts for all-quadratic programs with convex constraints**. Journal of Global Optimization, 18(4):337–356, 2000.

[21] TUY, H.. **On solving nonconvex optimization problems by reducing the duality gap**. Journal of Global Optimization, 32(3):349–365, 2005.

[22] VAN VOORHIS, T.. **A global optimization algorithm using lagrangian underestimates and the interval newton method**. Journal of Global Optimization, 24(3):349–370, 2002.

[23] ROCKAFELLAR, R. T.. **Lagrange multipliers and optimality**. SIAM review, 35(2):183–238, 1993.

[24] DANDURAND, B.; BOLAND, N.; CHRISTIANSEN, J.; EBERHARD, A. ; OLIVEIRA, F.. **A parallelizable augmented Lagrangian method applied to large-scale non-convex-constrained optimization problems**. arXiv preprint arXiv:1702.00526, 2017.

[25] LOVÁSZ, L.. **On the shannon capacity of a graph**. IEEE Transactions on Information theory, 25(1):1–7, 1979.

[26] GOEMANS, M. X.; WILLIAMSON, D. P.. **Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming**. Journal of the ACM (JACM), 42(6):1115–1145, 1995.

[27] LUO, Z.-Q.; MA, W.-K.; SO, A. M.-C.; YE, Y. ; ZHANG, S.. **Semidefinite relaxation of quadratic optimization problems**. IEEE Signal Processing Magazine, 27(3):20–34, 2010.

[28] SHERALI, H. D.; FRATICELLI, B. M.. **Enhancing RLT relaxations via a new class of semidefinite cuts**. Journal of Global Optimization, 22(1-4):233–261, 2002.

[29] ANSTREICHER, K. M.. **Semidefinite programming versus the reformulation-linearization technique for nonconvex quadratically constrained quadratic programming**. Journal of Global Optimization, 43(2-3):471–484, 2009.

[30] BOMZE, I. M.; DÜR, M.; DE KLERK, E.; ROOS, C.; QUIST, A. J. ; TERLAKY, T.. **On copositive programming and standard quadratic optimization problems**. Journal of Global Optimization, 18(4):301–320, 2000.

[31] BOMZE, I. M.. **Copositive optimization–Recent developments and applications**. European Journal of Operational Research, 216(3):509–520, 2012.

[32] BALAS, E.. **Disjunctive programming**. Annals of Discrete Mathematics, 5:3–51, 1979.

[33] KARUPPIAH, R.; GROSSMANN, I. E.. **Global optimization for the synthesis of integrated water systems in chemical processes**. Computers & Chemical Engineering, 30(4):650–673, 2006.

[34] WICAKSONO, D. S.; KARIMI, I. A.. **Piecewise MILP under-and overestimators for global optimization of bilinear programs**. AIChE Journal, 54(4):991–1008, 2008.

[35] GOUNARIS, C. E.; MISENER, R. ; FLOUDAS, C. A.. **Computational comparison of piecewise- linear relaxations for pooling problems**. Industrial & Engineering Chemistry Research, 48(12):5742–5766, 2009.

[36] CASTRO, P. M.. **Tightening piecewise McCormick relaxations for bilinear problems**. Computers & Chemical Engineering, 72:300–311, 2015.

[37] CASTRO, P. M.. **Normalized multiparametric disaggregation: an efficient relaxation for mixed-integer bilinear problems**. Journal of Global Optimization, 64(4):765–784, 2016.

[38] GUPTE, A.; AHMED, S.; CHEON, M. S. ; DEY, S.. **Solving mixed integer bilinear problems using milp formulations**. SIAM Journal on Optimization, 23(2):721–744, 2013.

[39] CPLEX, I.. **Ilog cplex 12.6 optimization studio**, 2014.

[40] OPTIMIZATION, G.. **Inc.,?gurobi optimizer reference manual,? 2014**. URL: http://www. gurobi. com, 2014.

[41] FICO, T.. **Xpress optimization suite**. Xpress-Optimizer, Reference manual, Fair Isaac Corporation, 2009.

[42] LI, H.-L.; CHANG, C.-T.. **An approximate approach of global optimization for polynomial programming problems**. European Journal of Operational Research, 107(3):625–632, 1998.

[43] TELES, J. P.; CASTRO, P. M. ; MATOS, H. A.. **Multi-parametric disaggregation technique for global optimization of polynomial programming problems**. Journal of Global Optimization, 55(2):227–251, 2013.

[44] KOLODZIEJ, S.; CASTRO, P. M. ; GROSSMANN, I. E.. **Global optimization of bilinear programs with a multiparametric disaggregation technique**. Journal of Global Optimization, 57(4):1039–1063, 2013.

[45] TELES, J. P.; CASTRO, P. M. ; MATOS, H. A.. **Global optimization of water networks design using multiparametric disaggregation**. Computers & Chemical Engineering, 40:132–147, 2012.

[46] SAHINIDIS, N. V.. **BARON: A general purpose global optimization software package**. Journal of Global Optimization, 8(2):201–205, 1996.

[47] BELOTTI, P.. **COUENNE: a user's manual**. Technical report, Technical report, Lehigh University, 2009.

[48] LOUGEE-HEIMER, R.. **The common optimization interface for operations research: Promoting open-source software in the operations research community**. IBM Journal of Research and Development, 47(1):57–66, 2003.

[49] DOLAN, E. D.; MORÉ, J. J.. **Benchmarking optimization software with performance profiles**. Mathematical Programming, 91(2):201–213, 2002.

[50] DANTZIG, G. B.; WOLFE, P.. **Decomposition principle for linear programs**. Operations Research, 8(1):101–111, 1960.

[51] BENDERS, J. F.. **Partitioning procedures for solving mixed-variables programming problems**. Numerische Mathematik, 4(1):238–252, 1962.

[52] FISHER, M. L.. **The Lagrangian relaxation method for solving integer programming problems**. Management Science, 27(1):1–18, 1981.

[53] SHEPARDSON, F.; MARSTEN, R. E.. **A lagrangean relaxation algorithm for the two duty period scheduling problem**. Management Science, 26(3):274–281, 1980.

[54] GUIGNARD, M.; KIM, S.. **Lagrangean decomposition: A model yielding stronger lagrangean bounds**. Mathematical Programming, 39(2):215–228, 1987.

[55] GEOFFRION, A. M.. **Generalized benders decomposition**. Journal of Optimization Theory and Applications, 10(4):237–260, 1972.

[56] LI, X.; TOMASGARD, A. ; BARTON, P. I.. **Nonconvex generalized benders decomposition for stochastic separable mixed-integer nonlinear programs**. Journal of Optimization Theory and Applications, 151(3):425, 2011.

[57] BOYD, S.; VANDENBERGHE, L.. **Convex Optimization**. Cambridge Uuniversity Press, 2004.

[58] ROBINSON, S. M.. **First order conditions for general nonlinear optimization**. SIAM Journal on Applied Mathematics, 30(4):597–607, 1976.

[59] RUSZCZYŃSKI, A. P.. **Nonlinear optimization**, volumen 13. Princeton university press, 2006.

[60] PIETRZYKOWSKI, T.. **An exact potential method for constrained maxima**. SIAM Journal on numerical analysis, 6(2):299–304, 1969.

[61] ZANGWILL, W. I.. **Non-linear programming via penalty functions**. Management science, 13(5):344–358, 1967.

[62] COURANT, R.. **Variational methods for the solution of problems of equilibrium and vibrations**. Bulletin of the American Mathematical Society, 49(1):1–23, 1943.

[63] ROCKAFELLAR, R. T.; WETS, R. J.-B.. **Variational analysis**, volumen 317. Springer Science & Business Media, 2009.

[64] POWELL, M.. **A method for nonlinear constraints in minimization problems in optimization, r**. Academic Press, NY, p. 283–298.

[65] HESTENES, M. R.. **Multiplier and gradient methods**. Journal of Optimization Theory and Applications, 4(5):303–320, 1969.

[66] TAPPENDEN, R.; RICHTÁRIK, P. ; BÜKE, B.. **Separable approximations and decomposition methods for the augmented Lagrangian**. Optimization Methods and Software, 30(3):643–668, 2015.

[67] BAO, X.; SAHINIDIS, N. V. ; TAWARMALANI, M.. **Semidefinite relaxations for quadratically constrained quadratic programming: A review and comparisons**. Mathematical programming, 129(1):129–157, 2011.

[68] DENTCHEVA, D.; RÖMISCH, W.. **Duality gaps in nonconvex stochastic optimization**. Mathematical Programming, 101(3):515–535, 2004.

[69] BROOKE, A. K.; D MEERAUS, A.. **Gams release 2.25; a user's guide**. Technical report, GAMS Development Corporation, Washington, DC (EUA), 1996.

[70] DRUD, A.. **CONOPT: A GRG code for large sparse dynamic nonlinear optimization problems**. Mathematical Programming, 31(2):153–191, 1985.

[71] HELD, M.; WOLFE, P. ; CROWDER, H. P.. **Validation of subgradient optimization**. Mathematical Programming, 6(1):62–88, 1974.

[72] CAMERINI, P. M.; FRATTA, L. ; MAFFIOLI, F.. **On improving relaxation methods by modified gradient techniques**. In: NONDIFFERENTIABLE OPTIMIZATION, p. 26–34. Springer, 1975.

[73] CHENEY, E. W.; GOLDSTEIN, A. A.. **Newton's method for convex programming and Tchebycheff approximation**. Numerische Mathematik, 1(1):253–268, 1959.

[74] KELLEY, JR, J. E.. **The cutting-plane method for solving convex programs**. Journal of the Society for Industrial and Applied Mathematics, 8(4):703–712, 1960.

[75] MARSTEN, R.; HOGAN, W. ; BLANKENSHIP, J. W.. **The boxstep method for large-scale optimization**. Operations Research, 23(3):389–405, 1975.

[76] BARAHONA, F.; ANBIL, R.. **The volume algorithm: producing primal solutions with a subgradient method**. Mathematical Programming, 87(3):385–399, 2000.

[77] LEMARÉCHAL, C.. **An algorithm for minimizing convex functions.** In: IFIP CONGRESS, p. 552–556, 1974.

[78] ZHAO, X.; LUH, P. B.. **New bundle methods for solving lagrangian relaxation dual problems**. Journal of Optimization Theory and Applications, 113(2):373–397, 2002.

[79] RUDIN, W.. **Real and Complex Analysis**. Tata McGraw-Hill Education, 1987.

[80] LIN, Y.; SCHRAGE, L.. **The global solver in the LINDO API**. Optimization Methods & Software, 24(4-5):657–668, 2009.

[81] ACHTERBERG, T.. **SCIP: solving constraint integer programs**. Mathematical Programming Computation, 1(1):1–41, 2009.

[82] MISENER, R.; FLOUDAS, C. A.. **ANTIGONE: algorithms for continuous/integer global optimization of nonlinear equations**. Journal of Global Optimization, 59(2-3):503–526, 2014.

[83] CZYZYK, J.; MESNIER, M. P. ; MORÉ, J. J.. **The neos server**. IEEE Journal on Computational Science and Engineering, 5(3):68——75, 1998.

[84] DOLAN, E. D.. **The neos server 4.0 administrative guide**. Technical Memorandum ANL/MCS-TM-250, Mathematics and Computer Science Division, Argonne National Laboratory, 2001.

[85] GROPP, W.; MORÉ, J. J.. **Optimization environments and the neos server**. In: Buhman, M. D.; Iserles, A., editors, APPROXIMATION THEORY AND OPTIMIZATION, p. 167–182. Cambridge University Press, 1997.

[86] BENJAMINI, Y.. **Opening the box of a boxplot**. The American Statistician, 42(4):257–262, 1988.

A
Complete results for QCQP instances

The following table contains the complete results for the instances taken from the literature. The column "model" refers to which relaxation model and algorithm was used. The second to seventh columns contains the information about the instances name, seed used to generate the instance, number of continuous variables, number of constraints, density of quadratic terms and execution time in seconds. The last two columns are the lower and upper bound respectively.

model	instance	s.	#v.c.c	#c.	d.(%)	t(s)	LB	UB
NMDT + base 10	10_10_1_100	1	10	10	100	4.56	-14.1793	-14.179
NMDT + base 10	10_10_1_50	1	10	10	50	0.3	-7.38	-7.38
NMDT + base 10	10_10_2_100	2	10	10	100	1.76	-6.5478	-6.5468
NMDT + base 10	10_10_2_50	2	10	10	50	0.32	-6.95	-6.95
NMDT + base 10	10_10_3_100	3	10	10	100	1	-11.79	-11.79
NMDT + base 10	10_10_3_50	3	10	10	50	0.31	-10.3	-10.3
NMDT + base 10	10_15_1_100	1	10	15	100	8.67	-11.6368	-11.6365
NMDT + base 10	10_15_1_50	1	10	15	50	0.76	-5.39	-5.39
NMDT + base 10	10_15_2_100	2	10	15	100	0.4	-28.07	-28.07
NMDT + base 10	10_15_2_50	2	10	15	50	0.35	-11.6028	-11.6028
NMDT + base 10	10_15_3_100	3	10	15	100	1.03	-7.98	-7.98
NMDT + base 10	10_15_3_50	3	10	15	50	0.77	-7.83	-7.83
NMDT + base 10	10_20_1_100	1	10	20	100	1.41	-8.19	-8.19
NMDT + base 10	10_20_1_50	1	10	20	50	0.83	-5.4	-5.4
NMDT + base 10	10_20_2_100	2	10	20	100	2.43	-15.9451	-15.9442
NMDT + base 10	10_20_2_50	2	10	20	50	0.78	-3.13	-3.13
NMDT + base 10	10_20_3_100	3	10	20	100	3.14	-5.899	-5.8987
NMDT + base 10	10_20_3_50	3	10	20	50	0.32	-7.49	-7.49
NMDT + base 10	20_20_1_100	1	20	20	100	1001.16	-35.3384	-28.1906
NMDT + base 10	20_20_1_25	1	20	20	25	0.47	-12.86	-12.86
NMDT + base 10	20_20_1_50	1	20	20	50	2.36	-26.598	-26.5977
NMDT + base 10	20_20_2_100	2	20	20	100	1002.26	-25.679	-18.8752
NMDT + base 10	20_20_2_25	2	20	20	25	0.43	-9.88	-9.88
NMDT + base 10	20_20_2_50	2	20	20	50	3.08	-17.03	-17.03
NMDT + base 10	20_20_3_100	3	20	20	100	1198.7	-33.4183	-26.1
NMDT + base 10	20_20_3_25	3	20	20	25	0.46	-23.43	-23.43
NMDT + base 10	20_20_3_50	3	20	20	50	1.66	-18.54	-18.54
NMDT + base 10	20_30_1_100	1	20	30	100	1001.95	-27.4146	-17.08
NMDT + base 10	20_30_1_25	1	20	30	25	1.33	-13.3571	-13.3571
NMDT + base 10	20_30_1_50	1	20	30	50	2.96	-20.09	-20.09
NMDT + base 10	20_30_2_100	2	20	30	100	1001.73	-26.915	-19.9515
NMDT + base 10	20_30_2_25	2	20	30	25	0.54	-13.68	-13.68
NMDT + base 10	20_30_2_50	2	20	30	50	1.88	-24.7097	-24.7097
NMDT + base 10	20_30_3_100	3	20	30	100	1001.79	-28.2278	-24.0611
NMDT + base 10	20_30_3_25	3	20	30	25	0.55	-9.5442	-9.5442
NMDT + base 10	20_30_3_50	3	20	30	50	1.93	-28.83	-28.83
NMDT + base 10	20_40_1_100	1	20	40	100	1002.13	-28.4199	-19.3111

NMDT + base 10	20_40_1_25	1	20	40	25	0.68	-10.59	-10.59
NMDT + base 10	20_40_1_50	1	20	40	50	3.04	-21.97	-21.97
NMDT + base 10	20_40_2_100	2	20	40	100	1002.02	-30.5198	-30.51
NMDT + base 10	20_40_2_25	2	20	40	25	1.74	-10.8254	-10.8254
NMDT + base 10	20_40_2_50	2	20	40	50	10.16	-24.2649	-24.2644
NMDT + base 10	20_40_3_100	3	20	40	100	1002.3	-29.5042	-26.272
NMDT + base 10	20_40_3_25	3	20	40	25	2.34	-13.7731	-13.7731
NMDT + base 10	20_40_3_50	3	20	40	50	2.01	-27.4064	-27.4064
NMDT + base 10	28_28_1_25	1	28	28	25	0.81	-30.23	-30.23
NMDT + base 10	28_28_2_25	2	28	28	25	0.8	-27.1518	-27.1518
NMDT + base 10	28_28_3_25	3	28	28	25	6.06	-28.532	-28.5319
NMDT + base 10	28_42_1_25	1	28	42	25	5.04	-37.5291	-37.5283
NMDT + base 10	28_42_2_25	2	28	42	25	2.84	-26.78	-26.78
NMDT + base 10	28_42_3_25	3	28	42	25	1.05	-29.84	-29.84
NMDT + base 10	28_56_1_25	1	28	56	25	3.36	-17.9769	-17.9769
NMDT + base 10	28_56_2_25	2	28	56	25	3.39	-25.01	-25.01
NMDT + base 10	28_56_3_25	3	28	56	25	7.19	-32.6527	-32.652
NMDT + base 10	30_30_1_100	1	30	30	100	1003.61	-92.5073	-44.1885
NMDT + base 10	30_30_1_50	1	30	30	50	12.92	-54.6614	-54.6605
NMDT + base 10	30_30_2_100	2	30	30	100	1003.41	-88.6063	-44.7318
NMDT + base 10	30_30_2_50	2	30	30	50	207.84	-48.5095	-48.5095
NMDT + base 10	30_30_3_100	3	30	30	100	1003.32	-84.1821	-38.23
NMDT + base 10	30_30_3_50	3	30	30	50	137.13	-44.45	-44.45
NMDT + base 10	30_45_1_100	1	30	45	100	1004.51	-110.637	-80
NMDT + base 10	30_45_1_50	1	30	45	50	198.41	-53.2051	-53.2051
NMDT + base 10	30_45_2_100	2	30	45	100	1004.59	-103.455	-77.05
NMDT + base 10	30_45_2_50	2	30	45	50	9.61	-56.0508	-56.0504
NMDT + base 10	30_45_3_100	3	30	45	100	1004.71	-87.5204	-47.5897
NMDT + base 10	30_45_3_50	3	30	45	50	70.99	-53.9703	-53.9701
NMDT + base 10	30_60_1_100	1	30	60	100	1007.36	-102.618	-72.0805
NMDT + base 10	30_60_1_50	1	30	60	50	1004.1	-47.1663	-35.6243
NMDT + base 10	30_60_2_100	2	30	60	100	1006.58	-95.3213	-43.9133
NMDT + base 10	30_60_2_50	2	30	60	50	1003.38	-40.9757	-33.8916
NMDT + base 10	30_60_3_100	3	30	60	100	1192.88	-111.778	-61.3861
NMDT + base 10	30_60_3_50	3	30	60	50	108.75	-47.71	-47.71
NMDT + base 10	40_40_1_100	1	40	40	100	1013.25	-188.011	-90.82
NMDT + base 10	40_40_1_25	1	40	40	25	31.16	-48.3719	-48.3719
NMDT + base 10	40_40_1_50	1	40	40	50	1007.93	-87.2517	-51.9
NMDT + base 10	40_40_2_100	2	40	40	100	1008.11	-180.393	-71.364
NMDT + base 10	40_40_2_25	2	40	40	25	16.88	-44.65	-44.65
NMDT + base 10	40_40_2_50	2	40	40	50	1003.92	-89.4937	-54.29
NMDT + base 10	40_40_3_100	3	40	40	100	1010.35	-180.058	-97.6901
NMDT + base 10	40_40_3_25	3	40	40	25	2.41	-62.16	-62.16
NMDT + base 10	40_40_3_50	3	40	40	50	1004.06	-94.1587	-53.6238
NMDT + base 10	40_60_1_100	1	40	60	100	1011.43	-173.874	-52.2255
NMDT + base 10	40_60_1_25	1	40	60	25	1003.21	-48.7138	-48.0569
NMDT + base 10	40_60_1_50	1	40	60	50	1005.68	-98.5651	-76.663
NMDT + base 10	40_60_2_100	2	40	60	100	1012.03	-189.595	-74.4395
NMDT + base 10	40_60_2_25	2	40	60	25	14.06	-41.76	-41.76
NMDT + base 10	40_60_2_50	2	40	60	50	1005.43	-79.7596	-54.87
NMDT + base 10	40_60_3_100	3	40	60	100	1011.73	-167.841	-50.2828
NMDT + base 10	40_60_3_25	3	40	60	25	1003.14	-38.8982	-38.8627
NMDT + base 10	40_60_3_50	3	40	60	50	1005.32	-92.6473	-56.4343
NMDT + base 10	40_80_1_100	1	40	80	100	1014.97	-199.818	-97.7667
NMDT + base 10	40_80_1_25	1	40	80	25	1003.79	-39.6674	-32.7279
NMDT + base 10	40_80_1_50	1	40	80	50	1006.86	-96.3103	-85.6393
NMDT + base 10	40_80_2_100	2	40	80	100	1014.77	-191.639	-102.591
NMDT + base 10	40_80_2_25	2	40	80	25	1003.99	-48.4642	-48.421
NMDT + base 10	40_80_2_50	2	40	80	50	1006.97	-93.9692	-59.9291

NMDT + base 10	40_80_3_100	3	40	80	100	1014.83	-190.519	-81.3659
NMDT + base 10	40_80_3_25	3	40	80	25	1003.84	-34.1173	-30.8041
NMDT + base 10	40_80_3_50	3	40	80	50	1006.89	-89.0017	-50.3836
NMDT + base 10	48_48_1_25	1	48	48	25	1003.67	-66.809	-52.1083
NMDT + base 10	48_48_2_25	2	48	48	25	1003.55	-56.5626	-50.62
NMDT + base 10	48_48_3_25	3	48	48	25	1003.59	-71.3836	-54.516
NMDT + base 10	48_72_1_25	1	48	72	25	1004.98	-58.6024	-49.15
NMDT + base 10	48_72_2_25	2	48	72	25	1005.05	-72.9573	-63.7748
NMDT + base 10	48_72_3_25	3	48	72	25	1004.9	-70.6443	-59.7569
NMDT + base 10	48_96_1_25	1	48	96	25	868.12	-70.6765	-70.6761
NMDT + base 10	48_96_2_25	2	48	96	25	1006.43	-55.5084	-38.798
NMDT + base 10	48_96_3_25	3	48	96	25	1006.28	-55.0782	-37.1034
NMDT + base 10	50_100_1_100	1	50	100	100	1032.06	-274.533	-94.9017
NMDT + base 10	50_100_1_50	1	50	100	50	1015.77	-152.025	-86.3921
NMDT + base 10	50_100_2_100	2	50	100	100	1032.08	-300.149	-98.3512
NMDT + base 10	50_100_2_50	2	50	100	50	1015.28	-145.297	-78.2427
NMDT + base 10	50_100_3_100	3	50	100	100	1033.85	-308.527	-133.875
NMDT + base 10	50_100_3_50	3	50	100	50	1015.67	-138.847	-67.486
NMDT + base 10	50_50_1_100	1	50	50	100	1015.59	-280.978	-100.967
NMDT + base 10	50_50_1_50	1	50	50	50	1007.58	-139.117	-57.9588
NMDT + base 10	50_50_2_100	2	50	50	100	1015.03	-262.43	-76.5882
NMDT + base 10	50_50_2_50	2	50	50	50	1007.73	-149.666	-78.6604
NMDT + base 10	50_50_3_100	3	50	50	100	1014.58	-303.548	-118.338
NMDT + base 10	50_50_3_50	3	50	50	50	1007.26	-148.652	-77.2531
NMDT + base 10	50_75_1_100	1	50	75	100	1022.61	-315.536	-140.491
NMDT + base 10	50_75_1_50	1	50	75	50	1010.9	-151.543	-92.8247
NMDT + base 10	50_75_2_100	2	50	75	100	1021.05	-276.031	-72.7486
NMDT + base 10	50_75_2_50	2	50	75	50	1010.88	-144.781	-66.858
NMDT + base 10	50_75_3_100	3	50	75	100	1022.15	-285.581	-107.373
NMDT + base 10	50_75_3_50	3	50	75	50	1011.95	-139.093	-65.5265
NMDT + base 10	8_12_1_25	1	8	12	25	0.82	-6.44	-6.44
NMDT + base 10	8_12_2_25	2	8	12	25	0.3	-2.14	-2.14
NMDT + base 10	8_12_3_25	3	8	12	25	0.35	-9.21	-9.21
NMDT + base 10	8_16_1_25	1	8	16	25	0.31	-2.49	-2.49
NMDT + base 10	8_16_2_25	2	8	16	25	0.29	-2.79	-2.79
NMDT + base 10	8_16_3_25	3	8	16	25	0.3	-2.44	-2.44
NMDT + base 10	8_8_1_25	1	8	8	25	0.33	-3.97	-3.97
NMDT + base 10	8_8_2_25	2	8	8	25	0.48	-0.56	-0.56
NMDT + base 10	8_8_3_25	3	8	8	25	0.36	-5.3	-5.3
NMDT + base 2	10_10_1_100	1	10	10	100	4.64	-14.1797	-14.179
NMDT + base 2	10_10_1_50	1	10	10	50	0.26	-7.38	-7.38
NMDT + base 2	10_10_2_100	2	10	10	100	2.92	-6.5475	-6.5468
NMDT + base 2	10_10_2_50	2	10	10	50	0.25	-6.95	-6.95
NMDT + base 2	10_10_3_100	3	10	10	100	0.81	-11.79	-11.79
NMDT + base 2	10_10_3_50	3	10	10	50	0.25	-10.3	-10.3
NMDT + base 2	10_15_1_100	1	10	15	100	7.51	-11.6372	-11.6365
NMDT + base 2	10_15_1_50	1	10	15	50	0.54	-5.39	-5.39
NMDT + base 2	10_15_2_100	2	10	15	100	0.3	-28.07	-28.07
NMDT + base 2	10_15_2_50	2	10	15	50	0.26	-11.6028	-11.6028
NMDT + base 2	10_15_3_100	3	10	15	100	0.71	-7.98	-7.98
NMDT + base 2	10_15_3_50	3	10	15	50	0.69	-7.83	-7.83
NMDT + base 2	10_20_1_100	1	10	20	100	1.17	-8.19	-8.19
NMDT + base 2	10_20_1_50	1	10	20	50	0.63	-5.4	-5.4
NMDT + base 2	10_20_2_100	2	10	20	100	3.55	-15.9449	-15.9442
NMDT + base 2	10_20_2_50	2	10	20	50	0.71	-3.13	-3.13
NMDT + base 2	10_20_3_100	3	10	20	100	2.64	-5.8993	-5.8987
NMDT + base 2	10_20_3_50	3	10	20	50	0.26	-7.49	-7.49
NMDT + base 2	20_20_1_100	1	20	20	100	960.23	-28.1913	-28.1906
NMDT + base 2	20_20_1_25	1	20	20	25	0.3	-12.86	-12.86

NMDT + base 2	20_20_1_50	1	20	20	50	2.91	-26.5985	-26.5977
NMDT + base 2	20_20_2_100	2	20	20	100	1000.67	-18.8821	-18.8752
NMDT + base 2	20_20_2_25	2	20	20	25	0.32	-9.88	-9.88
NMDT + base 2	20_20_2_50	2	20	20	50	0.94	-17.03	-17.03
NMDT + base 2	20_20_3_100	3	20	20	100	41.68	-26.1	-26.1
NMDT + base 2	20_20_3_25	3	20	20	25	0.32	-23.43	-23.43
NMDT + base 2	20_20_3_50	3	20	20	50	1.14	-18.54	-18.54
NMDT + base 2	20_30_1_100	1	20	30	100	350.53	-17.08	-17.08
NMDT + base 2	20_30_1_25	1	20	30	25	1.88	-13.3572	-13.3571
NMDT + base 2	20_30_1_50	1	20	30	50	1.26	-20.09	-20.09
NMDT + base 2	20_30_2_100	2	20	30	100	750.02	-19.9523	-19.9515
NMDT + base 2	20_30_2_25	2	20	30	25	0.38	-13.68	-13.68
NMDT + base 2	20_30_2_50	2	20	30	50	0.94	-24.7097	-24.7097
NMDT + base 2	20_30_3_100	3	20	30	100	414.42	-24.0614	-24.0611
NMDT + base 2	20_30_3_25	3	20	30	25	0.36	-9.5442	-9.5442
NMDT + base 2	20_30_3_50	3	20	30	50	1	-28.83	-28.83
NMDT + base 2	20_40_1_100	1	20	40	100	1001.01	-19.3224	-19.3111
NMDT + base 2	20_40_1_25	1	20	40	25	0.39	-10.59	-10.59
NMDT + base 2	20_40_1_50	1	20	40	50	1.87	-21.97	-21.97
NMDT + base 2	20_40_2_100	2	20	40	100	82.55	-30.51	-30.51
NMDT + base 2	20_40_2_25	2	20	40	25	1.32	-10.8254	-10.8254
NMDT + base 2	20_40_2_50	2	20	40	50	7.18	-24.2647	-24.2644
NMDT + base 2	20_40_3_100	3	20	40	100	939.57	-26.2728	-26.272
NMDT + base 2	20_40_3_25	3	20	40	25	0.89	-13.7731	-13.7731
NMDT + base 2	20_40_3_50	3	20	40	50	1.14	-27.4064	-27.4064
NMDT + base 2	28_28_1_25	1	28	28	25	0.47	-30.23	-30.23
NMDT + base 2	28_28_2_25	2	28	28	25	0.47	-27.1518	-27.1518
NMDT + base 2	28_28_3_25	3	28	28	25	4.77	-28.5326	-28.5319
NMDT + base 2	28_42_1_25	1	28	42	25	6.27	-37.529	-37.5283
NMDT + base 2	28_42_2_25	2	28	42	25	1.33	-26.78	-26.78
NMDT + base 2	28_42_3_25	3	28	42	25	0.58	-29.84	-29.84
NMDT + base 2	28_56_1_25	1	28	56	25	1.56	-17.9769	-17.9769
NMDT + base 2	28_56_2_25	2	28	56	25	1.5	-25.01	-25.01
NMDT + base 2	28_56_3_25	3	28	56	25	10.95	-32.6526	-32.652
NMDT + base 2	30_30_1_100	1	30	30	100	1002.57	-67.5587	-44.1885
NMDT + base 2	30_30_1_50	1	30	30	50	11.07	-54.6611	-54.6605
NMDT + base 2	30_30_2_100	2	30	30	100	1002.37	-67.335	-44.7318
NMDT + base 2	30_30_2_50	2	30	30	50	4.62	-48.5095	-48.5095
NMDT + base 2	30_30_3_100	3	30	30	100	1001.63	-59.6437	-37.886
NMDT + base 2	30_30_3_50	3	30	30	50	3.23	-44.45	-44.45
NMDT + base 2	30_45_1_100	1	30	45	100	1001.85	-80.7549	-80
NMDT + base 2	30_45_1_50	1	30	45	50	6.24	-53.2051	-53.2051
NMDT + base 2	30_45_2_100	2	30	45	100	1004.1	-79.0787	-77.1362
NMDT + base 2	30_45_2_50	2	30	45	50	8.98	-56.0513	-56.0504
NMDT + base 2	30_45_3_100	3	30	45	100	1003.35	-63.1731	-47.5897
NMDT + base 2	30_45_3_50	3	30	45	50	41.93	-53.9708	-53.9701
NMDT + base 2	30_60_1_100	1	30	60	100	1002.69	-74.7214	-72.0805
NMDT + base 2	30_60_1_50	1	30	60	50	743.58	-35.9506	-35.9506
NMDT + base 2	30_60_2_100	2	30	60	100	1002.93	-69.7283	-43.3869
NMDT + base 2	30_60_2_50	2	30	60	50	1001.33	-33.895	-33.8916
NMDT + base 2	30_60_3_100	3	30	60	100	1002.74	-80.5275	-61.424
NMDT + base 2	30_60_3_50	3	30	60	50	5.33	-47.71	-47.71
NMDT + base 2	40_40_1_100	1	40	40	100	1003.58	-151.371	-90.82
NMDT + base 2	40_40_1_25	1	40	40	25	4.24	-48.3719	-48.3719
NMDT + base 2	40_40_1_50	1	40	40	50	1001.5	-54.6839	-51.9
NMDT + base 2	40_40_2_100	2	40	40	100	1003.42	-152.545	-71.364
NMDT + base 2	40_40_2_25	2	40	40	25	3.6	-44.65	-44.65
NMDT + base 2	40_40_2_50	2	40	40	50	1001.46	-58.3125	-55.7089
NMDT + base 2	40_40_3_100	3	40	40	100	1003.25	-147.586	-97.6901

NMDT + base 2	40_40_3_25	3	40	40	25	0.86	-62.16	-62.16
NMDT + base 2	40_40_3_50	3	40	40	50	1001.53	-61.6025	-54.09
NMDT + base 2	40_60_1_100	1	40	60	100	1004.8	-139.903	-52.2255
NMDT + base 2	40_60_1_25	1	40	60	25	698.32	-48.0574	-48.0569
NMDT + base 2	40_60_1_50	1	40	60	50	1002.21	-76.8747	-76.8059
NMDT + base 2	40_60_2_100	2	40	60	100	1004.74	-155.669	-73.1024
NMDT + base 2	40_60_2_25	2	40	60	25	4.71	-41.76	-41.76
NMDT + base 2	40_60_2_50	2	40	60	50	1002.11	-56.6075	-55.07
NMDT + base 2	40_60_3_100	3	40	60	100	1005.22	-132.388	-50.2946
NMDT + base 2	40_60_3_25	3	40	60	25	205.02	-38.8634	-38.8627
NMDT + base 2	40_60_3_50	3	40	60	50	1002.1	-61.1726	-56.4343
NMDT + base 2	40_80_1_100	1	40	80	100	1005.74	-165.528	-97.7667
NMDT + base 2	40_80_1_25	1	40	80	25	415.79	-32.7279	-32.7279
NMDT + base 2	40_80_1_50	1	40	80	50	805.67	-85.6402	-85.6393
NMDT + base 2	40_80_2_100	2	40	80	100	1005.83	-157.418	-102.602
NMDT + base 2	40_80_2_25	2	40	80	25	439.02	-48.4218	-48.421
NMDT + base 2	40_80_2_50	2	40	80	50	1002.64	-61.665	-59.9291
NMDT + base 2	40_80_3_100	3	40	80	100	1005.82	-159.246	-82.6464
NMDT + base 2	40_80_3_25	3	40	80	25	61.88	-30.899	-30.899
NMDT + base 2	40_80_3_50	3	40	80	50	1002.84	-62.2514	-50.4984
NMDT + base 2	48_48_1_25	1	48	48	25	1001.8	-52.143	-52.1085
NMDT + base 2	48_48_2_25	2	48	48	25	23.01	-50.9941	-50.9941
NMDT + base 2	48_48_3_25	3	48	48	25	1001.51	-55.8781	-55.7666
NMDT + base 2	48_72_1_25	1	48	72	25	17.32	-49.15	-49.15
NMDT + base 2	48_72_2_25	2	48	72	25	198.51	-63.7757	-63.7748
NMDT + base 2	48_72_3_25	3	48	72	25	99.84	-59.7569	-59.7569
NMDT + base 2	48_96_1_25	1	48	96	25	177.22	-70.6769	-70.6761
NMDT + base 2	48_96_2_25	2	48	96	25	1002.66	-41.6305	-37.3316
NMDT + base 2	48_96_3_25	3	48	96	25	1002.56	-40.84	-38.3101
NMDT + base 2	50_100_1_100	1	50	100	100	1012.96	-243.255	-94.9017
NMDT + base 2	50_100_1_50	1	50	100	50	1006.63	-124.257	-86.3921
NMDT + base 2	50_100_2_100	2	50	100	100	1013.28	-265.367	-98.3512
NMDT + base 2	50_100_2_50	2	50	100	50	1006.68	-115.592	-78.2427
NMDT + base 2	50_100_3_100	3	50	100	100	1012.8	-276.776	-133.875
NMDT + base 2	50_100_3_50	3	50	100	50	1006.5	-107.268	-67.486
NMDT + base 2	50_50_1_100	1	50	50	100	1007.18	-249.336	-91.9634
NMDT + base 2	50_50_1_50	1	50	50	50	1004.21	-105.939	-57.9588
NMDT + base 2	50_50_2_100	2	50	50	100	1006.7	-230.176	-76.5882
NMDT + base 2	50_50_2_50	2	50	50	50	1004.56	-118.224	-78.6604
NMDT + base 2	50_50_3_100	3	50	50	100	1007.24	-268.952	-120.102
NMDT + base 2	50_50_3_50	3	50	50	50	1003.97	-118.168	-77.2531
NMDT + base 2	50_75_1_100	1	50	75	100	1009.31	-280.17	-140.823
NMDT + base 2	50_75_1_50	1	50	75	50	1004.68	-121.676	-92.8517
NMDT + base 2	50_75_2_100	2	50	75	100	1009.5	-242.711	-72.7486
NMDT + base 2	50_75_2_50	2	50	75	50	1004.95	-116.186	-66.858
NMDT + base 2	50_75_3_100	3	50	75	100	1009.85	-253.045	-107.373
NMDT + base 2	50_75_3_50	3	50	75	50	1004.91	-109.091	-65.5265
NMDT + base 2	8_12_1_25	1	8	12	25	0.25	-6.44	-6.44
NMDT + base 2	8_12_2_25	2	8	12	25	0.22	-2.14	-2.14
NMDT + base 2	8_12_3_25	3	8	12	25	0.24	-9.21	-9.21
NMDT + base 2	8_16_1_25	1	8	16	25	0.25	-2.49	-2.49
NMDT + base 2	8_16_2_25	2	8	16	25	0.22	-2.79	-2.79
NMDT + base 2	8_16_3_25	3	8	16	25	0.27	-2.44	-2.44
NMDT + base 2	8_8_1_25	1	8	8	25	0.25	-3.97	-3.97
NMDT + base 2	8_8_2_25	2	8	8	25	0.27	-0.56	-0.56
NMDT + base 2	8_8_3_25	3	8	8	25	0.33	-5.3	-5.3
RNMDT + Alg 1	10_10_1_100	1	10	10	100	5.31	-14.1798	-14.179
RNMDT + Alg 1	10_10_1_50	1	10	10	50	0.28	-7.38	-7.38
RNMDT + Alg 1	10_10_2_100	2	10	10	100	3.58	-6.5475	-6.5468

RNMDT + Alg 1	10_10_2_50	2	10	10	50	0.27	-6.95	-6.95
RNMDT + Alg 1	10_10_3_100	3	10	10	100	0.7	-11.79	-11.79
RNMDT + Alg 1	10_10_3_50	3	10	10	50	0.27	-10.3	-10.3
RNMDT + Alg 1	10_15_1_100	1	10	15	100	6.93	-11.6371	-11.6365
RNMDT + Alg 1	10_15_1_50	1	10	15	50	0.86	-5.39	-5.39
RNMDT + Alg 1	10_15_2_100	2	10	15	100	0.32	-28.07	-28.07
RNMDT + Alg 1	10_15_2_50	2	10	15	50	0.26	-11.6028	-11.6028
RNMDT + Alg 1	10_15_3_100	3	10	15	100	0.63	-7.98	-7.98
RNMDT + Alg 1	10_15_3_50	3	10	15	50	0.84	-7.83	-7.83
RNMDT + Alg 1	10_20_1_100	1	10	20	100	1.11	-8.19	-8.19
RNMDT + Alg 1	10_20_1_50	1	10	20	50	0.61	-5.4	-5.4
RNMDT + Alg 1	10_20_2_100	2	10	20	100	3.17	-15.9449	-15.9442
RNMDT + Alg 1	10_20_2_50	2	10	20	50	0.59	-3.13	-3.13
RNMDT + Alg 1	10_20_3_100	3	10	20	100	2.94	-5.8993	-5.8987
RNMDT + Alg 1	10_20_3_50	3	10	20	50	1.96	-7.49	-7.49
RNMDT + Alg 1	20_20_1_100	1	20	20	100	773.92	-28.1913	-28.1906
RNMDT + Alg 1	20_20_1_25	1	20	20	25	0.33	-12.86	-12.86
RNMDT + Alg 1	20_20_1_50	1	20	20	50	2.64	-26.5986	-26.5977
RNMDT + Alg 1	20_20_2_100	2	20	20	100	1000.73	-18.8821	-18.8752
RNMDT + Alg 1	20_20_2_25	2	20	20	25	0.32	-9.88	-9.88
RNMDT + Alg 1	20_20_2_50	2	20	20	50	0.9	-17.03	-17.03
RNMDT + Alg 1	20_20_3_100	3	20	20	100	38.63	-26.1	-26.1
RNMDT + Alg 1	20_20_3_25	3	20	20	25	0.32	-23.43	-23.43
RNMDT + Alg 1	20_20_3_50	3	20	20	50	0.86	-18.54	-18.54
RNMDT + Alg 1	20_30_1_100	1	20	30	100	173.07	-17.0801	-17.08
RNMDT + Alg 1	20_30_1_25	1	20	30	25	1.75	-13.3572	-13.3571
RNMDT + Alg 1	20_30_1_50	1	20	30	50	0.93	-20.09	-20.09
RNMDT + Alg 1	20_30_2_100	2	20	30	100	626.01	-19.9523	-19.9515
RNMDT + Alg 1	20_30_2_25	2	20	30	25	0.34	-13.68	-13.68
RNMDT + Alg 1	20_30_2_50	2	20	30	50	0.91	-24.7097	-24.7097
RNMDT + Alg 1	20_30_3_100	3	20	30	100	481.01	-24.0614	-24.0611
RNMDT + Alg 1	20_30_3_25	3	20	30	25	0.51	-9.5442	-9.5442
RNMDT + Alg 1	20_30_3_50	3	20	30	50	1.41	-28.83	-28.83
RNMDT + Alg 1	20_40_1_100	1	20	40	100	1000.99	-19.3167	-19.3111
RNMDT + Alg 1	20_40_1_25	1	20	40	25	0.35	-10.59	-10.59
RNMDT + Alg 1	20_40_1_50	1	20	40	50	2.03	-21.97	-21.97
RNMDT + Alg 1	20_40_2_100	2	20	40	100	72.11	-30.51	-30.51
RNMDT + Alg 1	20_40_2_25	2	20	40	25	1.27	-10.8254	-10.8254
RNMDT + Alg 1	20_40_2_50	2	20	40	50	6.18	-24.2647	-24.2644
RNMDT + Alg 1	20_40_3_100	3	20	40	100	713.46	-26.2728	-26.272
RNMDT + Alg 1	20_40_3_25	3	20	40	25	0.93	-13.7731	-13.7731
RNMDT + Alg 1	20_40_3_50	3	20	40	50	0.87	-27.4064	-27.4064
RNMDT + Alg 1	28_28_1_25	1	28	28	25	0.44	-30.23	-30.23
RNMDT + Alg 1	28_28_2_25	2	28	28	25	0.45	-27.1518	-27.1518
RNMDT + Alg 1	28_28_3_25	3	28	28	25	3.9	-28.5326	-28.5319
RNMDT + Alg 1	28_42_1_25	1	28	42	25	5.91	-37.529	-37.5283
RNMDT + Alg 1	28_42_2_25	2	28	42	25	1.18	-26.78	-26.78
RNMDT + Alg 1	28_42_3_25	3	28	42	25	0.51	-29.84	-29.84
RNMDT + Alg 1	28_56_1_25	1	28	56	25	1.68	-17.9769	-17.9769
RNMDT + Alg 1	28_56_2_25	2	28	56	25	1.29	-25.01	-25.01
RNMDT + Alg 1	28_56_3_25	3	28	56	25	9.32	-32.6526	-32.652
RNMDT + Alg 1	30_30_1_100	1	30	30	100	1003.04	-66.5364	-44.1885
RNMDT + Alg 1	30_30_1_50	1	30	30	50	13.51	-54.6611	-54.6605
RNMDT + Alg 1	30_30_2_100	2	30	30	100	1005.18	-65.3151	-44.7318
RNMDT + Alg 1	30_30_2_50	2	30	30	50	4.92	-48.5095	-48.5095
RNMDT + Alg 1	30_30_3_100	3	30	30	100	1000.27	-59.2925	-37.886
RNMDT + Alg 1	30_30_3_50	3	30	30	50	3.78	-44.45	-44.45
RNMDT + Alg 1	30_45_1_100	1	30	45	100	1001.12	-80.755	-80
RNMDT + Alg 1	30_45_1_50	1	30	45	50	4.01	-53.2051	-53.2051

RNMDT + Alg 1	30_45_2_100	2	30	45	100	1000.66	-79.0787	-77.1362
RNMDT + Alg 1	30_45_2_50	2	30	45	50	9.32	-56.0513	-56.0504
RNMDT + Alg 1	30_45_3_100	3	30	45	100	1000.45	-62.8187	-47.5897
RNMDT + Alg 1	30_45_3_50	3	30	45	50	42.34	-53.9708	-53.9701
RNMDT + Alg 1	30_60_1_100	1	30	60	100	1005.9	-74.6338	-72.0805
RNMDT + Alg 1	30_60_1_50	1	30	60	50	703.92	-35.9506	-35.9506
RNMDT + Alg 1	30_60_2_100	2	30	60	100	1006.39	-69.8079	-40.7145
RNMDT + Alg 1	30_60_2_50	2	30	60	50	1001.23	-33.8935	-33.8916
RNMDT + Alg 1	30_60_3_100	3	30	60	100	1001.55	-79.4183	-61.424
RNMDT + Alg 1	30_60_3_50	3	30	60	50	5.71	-47.71	-47.71
RNMDT + Alg 1	40_40_1_100	1	40	40	100	1002.23	-152.087	-90.82
RNMDT + Alg 1	40_40_1_25	1	40	40	25	4.49	-48.3719	-48.3719
RNMDT + Alg 1	40_40_1_50	1	40	40	50	1001.35	-54.6838	-51.9
RNMDT + Alg 1	40_40_2_100	2	40	40	100	1002.33	-152.52	-71.364
RNMDT + Alg 1	40_40_2_25	2	40	40	25	4.02	-44.65	-44.65
RNMDT + Alg 1	40_40_2_50	2	40	40	50	1001.05	-58.3125	-54.96
RNMDT + Alg 1	40_40_3_100	3	40	40	100	1002.02	-147.531	-97.6901
RNMDT + Alg 1	40_40_3_25	3	40	40	25	0.85	-62.16	-62.16
RNMDT + Alg 1	40_40_3_50	3	40	40	50	1001.17	-61.6025	-53.8089
RNMDT + Alg 1	40_60_1_100	1	40	60	100	1003.24	-140.075	-52.2255
RNMDT + Alg 1	40_60_1_25	1	40	60	25	642.94	-48.0575	-48.0569
RNMDT + Alg 1	40_60_1_50	1	40	60	50	1001.59	-76.8747	-76.8059
RNMDT + Alg 1	40_60_2_100	2	40	60	100	1003.29	-155.755	-73.1024
RNMDT + Alg 1	40_60_2_25	2	40	60	25	6.97	-41.76	-41.76
RNMDT + Alg 1	40_60_2_50	2	40	60	50	1001.34	-56.6075	-55.07
RNMDT + Alg 1	40_60_3_100	3	40	60	100	1003.82	-131.883	-50.2946
RNMDT + Alg 1	40_60_3_25	3	40	60	25	224.51	-38.8634	-38.8627
RNMDT + Alg 1	40_60_3_50	3	40	60	50	1001.41	-61.1726	-56.4343
RNMDT + Alg 1	40_80_1_100	1	40	80	100	1004.07	-165.071	-97.7667
RNMDT + Alg 1	40_80_1_25	1	40	80	25	452.48	-32.7279	-32.7279
RNMDT + Alg 1	40_80_1_50	1	40	80	50	660.91	-85.6402	-85.6393
RNMDT + Alg 1	40_80_2_100	2	40	80	100	1004.33	-157.541	-102.602
RNMDT + Alg 1	40_80_2_25	2	40	80	25	495.49	-48.4217	-48.421
RNMDT + Alg 1	40_80_2_50	2	40	80	50	1001.98	-61.665	-59.9291
RNMDT + Alg 1	40_80_3_100	3	40	80	100	1003.81	-159.106	-82.6464
RNMDT + Alg 1	40_80_3_25	3	40	80	25	80.59	-30.899	-30.899
RNMDT + Alg 1	40_80_3_50	3	40	80	50	1002.58	-62.2514	-50.8809
RNMDT + Alg 1	48_48_1_25	1	48	48	25	1001.39	-52.143	-52.1085
RNMDT + Alg 1	48_48_2_25	2	48	48	25	26.1	-50.9941	-50.9941
RNMDT + Alg 1	48_48_3_25	3	48	48	25	1001.53	-55.8781	-55.7666
RNMDT + Alg 1	48_72_1_25	1	48	72	25	20.43	-49.15	-49.15
RNMDT + Alg 1	48_72_2_25	2	48	72	25	203.92	-63.7757	-63.7748
RNMDT + Alg 1	48_72_3_25	3	48	72	25	117.03	-59.7569	-59.7569
RNMDT + Alg 1	48_96_1_25	1	48	96	25	230.4	-70.6769	-70.6761
RNMDT + Alg 1	48_96_2_25	2	48	96	25	1000.6	-41.6305	-38.0908
RNMDT + Alg 1	48_96_3_25	3	48	96	25	1002.97	-42.2775	-38.3101
RNMDT + Alg 1	50_100_1_100	1	50	100	100	1007.6	-242.947	-94.9017
RNMDT + Alg 1	50_100_1_50	1	50	100	50	1028.42	-121.444	-86.3921
RNMDT + Alg 1	50_100_2_100	2	50	100	100	1010.47	-265.039	-98.3512
RNMDT + Alg 1	50_100_2_50	2	50	100	50	1004.63	-112.548	-78.2427
RNMDT + Alg 1	50_100_3_100	3	50	100	100	1009.24	-277.731	-133.875
RNMDT + Alg 1	50_100_3_50	3	50	100	50	1004.17	-105.21	-67.486
RNMDT + Alg 1	50_50_1_100	1	50	50	100	1004.13	-249.381	-100.967
RNMDT + Alg 1	50_50_1_50	1	50	50	50	1002.66	-104.553	-57.9588
RNMDT + Alg 1	50_50_2_100	2	50	50	100	1004.49	-230.262	-76.5882
RNMDT + Alg 1	50_50_2_50	2	50	50	50	1014.13	-117.096	-78.6604
RNMDT + Alg 1	50_50_3_100	3	50	50	100	1008.72	-267.17	-120.102
RNMDT + Alg 1	50_50_3_50	3	50	50	50	1004.48	-114.668	-77.2531
RNMDT + Alg 1	50_75_1_100	1	50	75	100	1008.37	-279.682	-140.598

RNMDT + Alg 1	50_75_1_50	1	50	75	50	1003.73	-121.956	-92.8247
RNMDT + Alg 1	50_75_2_100	2	50	75	100	1010.07	-241.871	-72.7486
RNMDT + Alg 1	50_75_2_50	2	50	75	50	1003.9	-113.065	-66.858
RNMDT + Alg 1	50_75_3_100	3	50	75	100	1007.75	-252.722	-107.373
RNMDT + Alg 1	50_75_3_50	3	50	75	50	1006.72	-106.684	-65.5265
RNMDT + Alg 1	8_12_1_25	1	8	12	25	0.25	-6.44	-6.44
RNMDT + Alg 1	8_12_2_25	2	8	12	25	0.22	-2.14	-2.14
RNMDT + Alg 1	8_12_3_25	3	8	12	25	0.23	-9.21	-9.21
RNMDT + Alg 1	8_16_1_25	1	8	16	25	0.27	-2.49	-2.49
RNMDT + Alg 1	8_16_2_25	2	8	16	25	0.24	-2.79	-2.79
RNMDT + Alg 1	8_16_3_25	3	8	16	25	0.28	-2.44	-2.44
RNMDT + Alg 1	8_8_1_25	1	8	8	25	0.25	-3.97	-3.97
RNMDT + Alg 1	8_8_2_25	2	8	8	25	0.33	-0.56	-0.56
RNMDT + Alg 1	8_8_3_25	3	8	8	25	0.23	-5.3	-5.3
RNMDT + Alg 2	10_10_1_100	1	10	10	100	5.1	-14.1797	-14.179
RNMDT + Alg 2	10_10_1_50	1	10	10	50	0.25	-7.38	-7.38
RNMDT + Alg 2	10_10_2_100	2	10	10	100	3.2	-6.5475	-6.5468
RNMDT + Alg 2	10_10_2_50	2	10	10	50	0.25	-6.95	-6.95
RNMDT + Alg 2	10_10_3_100	3	10	10	100	0.59	-11.79	-11.79
RNMDT + Alg 2	10_10_3_50	3	10	10	50	0.27	-10.3	-10.3
RNMDT + Alg 2	10_15_1_100	1	10	15	100	5.88	-11.6368	-11.6365
RNMDT + Alg 2	10_15_1_50	1	10	15	50	0.83	-5.39	-5.39
RNMDT + Alg 2	10_15_2_100	2	10	15	100	0.29	-28.07	-28.07
RNMDT + Alg 2	10_15_2_50	2	10	15	50	0.26	-11.6028	-11.6028
RNMDT + Alg 2	10_15_3_100	3	10	15	100	0.68	-7.98	-7.98
RNMDT + Alg 2	10_15_3_50	3	10	15	50	0.69	-7.83	-7.83
RNMDT + Alg 2	10_20_1_100	1	10	20	100	1.06	-8.19	-8.19
RNMDT + Alg 2	10_20_1_50	1	10	20	50	0.59	-5.4	-5.4
RNMDT + Alg 2	10_20_2_100	2	10	20	100	2.92	-15.9449	-15.9442
RNMDT + Alg 2	10_20_2_50	2	10	20	50	0.56	-3.13	-3.13
RNMDT + Alg 2	10_20_3_100	3	10	20	100	2.57	-5.8994	-5.8987
RNMDT + Alg 2	10_20_3_50	3	10	20	50	0.28	-7.49	-7.49
RNMDT + Alg 2	20_20_1_100	1	20	20	100	332.13	-28.1912	-28.1906
RNMDT + Alg 2	20_20_1_25	1	20	20	25	0.31	-12.86	-12.86
RNMDT + Alg 2	20_20_1_50	1	20	20	50	2.16	-26.5985	-26.5977
RNMDT + Alg 2	20_20_2_100	2	20	20	100	1000.68	-18.8766	-18.8752
RNMDT + Alg 2	20_20_2_25	2	20	20	25	0.28	-9.88	-9.88
RNMDT + Alg 2	20_20_2_50	2	20	20	50	1.32	-17.03	-17.03
RNMDT + Alg 2	20_20_3_100	3	20	20	100	32.36	-26.1	-26.1
RNMDT + Alg 2	20_20_3_25	3	20	20	25	0.29	-23.43	-23.43
RNMDT + Alg 2	20_20_3_50	3	20	20	50	0.77	-18.54	-18.54
RNMDT + Alg 2	20_30_1_100	1	20	30	100	213.21	-17.0801	-17.08
RNMDT + Alg 2	20_30_1_25	1	20	30	25	1.92	-13.3572	-13.3571
RNMDT + Alg 2	20_30_1_50	1	20	30	50	1.4	-20.09	-20.09
RNMDT + Alg 2	20_30_2_100	2	20	30	100	298.06	-19.9523	-19.9515
RNMDT + Alg 2	20_30_2_25	2	20	30	25	0.34	-13.68	-13.68
RNMDT + Alg 2	20_30_2_50	2	20	30	50	0.84	-24.7097	-24.7097
RNMDT + Alg 2	20_30_3_100	3	20	30	100	154.64	-24.0614	-24.0611
RNMDT + Alg 2	20_30_3_25	3	20	30	25	0.35	-9.5442	-9.5442
RNMDT + Alg 2	20_30_3_50	3	20	30	50	0.88	-28.83	-28.83
RNMDT + Alg 2	20_40_1_100	1	20	40	100	1000.84	-19.3125	-19.3111
RNMDT + Alg 2	20_40_1_25	1	20	40	25	0.36	-10.59	-10.59
RNMDT + Alg 2	20_40_1_50	1	20	40	50	2.26	-21.97	-21.97
RNMDT + Alg 2	20_40_2_100	2	20	40	100	65.78	-30.51	-30.51
RNMDT + Alg 2	20_40_2_25	2	20	40	25	1.61	-10.8254	-10.8254
RNMDT + Alg 2	20_40_2_50	2	20	40	50	5.95	-24.2647	-24.2644
RNMDT + Alg 2	20_40_3_100	3	20	40	100	265.85	-26.2728	-26.272
RNMDT + Alg 2	20_40_3_25	3	20	40	25	1.2	-13.7731	-13.7731
RNMDT + Alg 2	20_40_3_50	3	20	40	50	0.87	-27.4064	-27.4064

RNMDT + Alg 2	28_28_1_25	1	28	28	25	0.42	-30.23	-30.23
RNMDT + Alg 2	28_28_2_25	2	28	28	25	0.43	-27.1518	-27.1518
RNMDT + Alg 2	28_28_3_25	3	28	28	25	3.41	-28.5326	-28.5319
RNMDT + Alg 2	28_42_1_25	1	28	42	25	4.77	-37.529	-37.5283
RNMDT + Alg 2	28_42_2_25	2	28	42	25	1.02	-26.78	-26.78
RNMDT + Alg 2	28_42_3_25	3	28	42	25	0.5	-29.84	-29.84
RNMDT + Alg 2	28_56_1_25	1	28	56	25	2.19	-17.9769	-17.9769
RNMDT + Alg 2	28_56_2_25	2	28	56	25	1.22	-25.01	-25.01
RNMDT + Alg 2	28_56_3_25	3	28	56	25	7.39	-32.6526	-32.652
RNMDT + Alg 2	30_30_1_100	1	30	30	100	1001.45	-62.8013	-44.1885
RNMDT + Alg 2	30_30_1_50	1	30	30	50	10.01	-54.6611	-54.6605
RNMDT + Alg 2	30_30_2_100	2	30	30	100	1001.52	-61.5984	-44.7318
RNMDT + Alg 2	30_30_2_50	2	30	30	50	7.74	-48.5095	-48.5095
RNMDT + Alg 2	30_30_3_100	3	30	30	100	1001.45	-55.3	-37.886
RNMDT + Alg 2	30_30_3_50	3	30	30	50	6.78	-44.45	-44.45
RNMDT + Alg 2	30_45_1_100	1	30	45	100	1001.71	-81.1574	-80
RNMDT + Alg 2	30_45_1_50	1	30	45	50	9.51	-53.2051	-53.2051
RNMDT + Alg 2	30_45_2_100	2	30	45	100	1001.75	-78.924	-77.1362
RNMDT + Alg 2	30_45_2_50	2	30	45	50	12.45	-56.0513	-56.0504
RNMDT + Alg 2	30_45_3_100	3	30	45	100	1001.89	-57.4972	-47.5897
RNMDT + Alg 2	30_45_3_50	3	30	45	50	28.14	-53.9708	-53.9701
RNMDT + Alg 2	30_60_1_100	1	30	60	100	1002.61	-76.7451	-72.0805
RNMDT + Alg 2	30_60_1_50	1	30	60	50	456.04	-35.9506	-35.9506
RNMDT + Alg 2	30_60_2_100	2	30	60	100	1002.46	-62.8088	-43.4413
RNMDT + Alg 2	30_60_2_50	2	30	60	50	761.98	-33.8924	-33.8916
RNMDT + Alg 2	30_60_3_100	3	30	60	100	1002.39	-73.7557	-61.424
RNMDT + Alg 2	30_60_3_50	3	30	60	50	9.74	-47.71	-47.71
RNMDT + Alg 2	40_40_1_100	1	40	40	100	1003.31	-139.651	-90.82
RNMDT + Alg 2	40_40_1_25	1	40	40	25	8.03	-48.3719	-48.3719
RNMDT + Alg 2	40_40_1_50	1	40	40	50	1001.76	-57.322	-51.9
RNMDT + Alg 2	40_40_2_100	2	40	40	100	1003.33	-141.109	-71.364
RNMDT + Alg 2	40_40_2_25	2	40	40	25	4.83	-44.65	-44.65
RNMDT + Alg 2	40_40_2_50	2	40	40	50	1001.49	-58.3125	-55.7089
RNMDT + Alg 2	40_40_3_100	3	40	40	100	1003.47	-136.067	-97.6901
RNMDT + Alg 2	40_40_3_25	3	40	40	25	0.79	-62.16	-62.16
RNMDT + Alg 2	40_40_3_50	3	40	40	50	1001.91	-61.6025	-54.09
RNMDT + Alg 2	40_60_1_100	1	40	60	100	1004.49	-127.295	-52.2255
RNMDT + Alg 2	40_60_1_25	1	40	60	25	291.29	-48.0576	-48.0569
RNMDT + Alg 2	40_60_1_50	1	40	60	50	1001.84	-76.8268	-76.8059
RNMDT + Alg 2	40_60_2_100	2	40	60	100	1004.51	-144.426	-73.6973
RNMDT + Alg 2	40_60_2_25	2	40	60	25	7.1	-41.76	-41.76
RNMDT + Alg 2	40_60_2_50	2	40	60	50	1001.96	-56.6075	-55.07
RNMDT + Alg 2	40_60_3_100	3	40	60	100	1004.48	-121.182	-50.2828
RNMDT + Alg 2	40_60_3_25	3	40	60	25	109.49	-38.8634	-38.8627
RNMDT + Alg 2	40_60_3_50	3	40	60	50	1002.2	-61.1726	-56.4343
RNMDT + Alg 2	40_80_1_100	1	40	80	100	1005.45	-153.768	-97.7667
RNMDT + Alg 2	40_80_1_25	1	40	80	25	520.8	-32.7279	-32.7279
RNMDT + Alg 2	40_80_1_50	1	40	80	50	258.72	-85.6402	-85.6393
RNMDT + Alg 2	40_80_2_100	2	40	80	100	1005.43	-134.923	-102.591
RNMDT + Alg 2	40_80_2_25	2	40	80	25	246.53	-48.4214	-48.421
RNMDT + Alg 2	40_80_2_50	2	40	80	50	1002.41	-61.665	-59.9291
RNMDT + Alg 2	40_80_3_100	3	40	80	100	1006.13	-149.088	-83.9075
RNMDT + Alg 2	40_80_3_25	3	40	80	25	111.33	-30.899	-30.899
RNMDT + Alg 2	40_80_3_50	3	40	80	50	1002.89	-62.2514	-50.2957
RNMDT + Alg 2	48_48_1_25	1	48	48	25	1001.31	-52.1259	-52.1083
RNMDT + Alg 2	48_48_2_25	2	48	48	25	71.27	-50.9941	-50.9941
RNMDT + Alg 2	48_48_3_25	3	48	48	25	574.96	-55.8037	-55.8037
RNMDT + Alg 2	48_72_1_25	1	48	72	25	25.1	-49.15	-49.15
RNMDT + Alg 2	48_72_2_25	2	48	72	25	242.81	-63.7757	-63.7748

RNMDT + Alg 2	48_72_3_25	3	48	72	25	147.88	-59.7569	-59.7569
RNMDT + Alg 2	48_96_1_25	1	48	96	25	140.23	-70.6771	-70.6761
RNMDT + Alg 2	48_96_2_25	2	48	96	25	1002.61	-42.94	-38.101
RNMDT + Alg 2	48_96_3_25	3	48	96	25	1002.43	-41.635	-38.3101
RNMDT + Alg 2	50_100_1_100	1	50	100	100	1013.06	-218.9	-94.9017
RNMDT + Alg 2	50_100_1_50	1	50	100	50	1006.04	-119.153	-86.3921
RNMDT + Alg 2	50_100_2_100	2	50	100	100	1012.23	-216.467	-98.3512
RNMDT + Alg 2	50_100_2_50	2	50	100	50	1005.78	-109.148	-78.2427
RNMDT + Alg 2	50_100_3_100	3	50	100	100	1011.19	-236.287	-133.875
RNMDT + Alg 2	50_100_3_50	3	50	100	50	1005.9	-103.49	-67.6457
RNMDT + Alg 2	50_50_1_100	1	50	50	100	1006.24	-209.703	-100.493
RNMDT + Alg 2	50_50_1_50	1	50	50	50	1003.03	-98.7119	-57.9588
RNMDT + Alg 2	50_50_2_100	2	50	50	100	1006.04	-211.492	-76.7843
RNMDT + Alg 2	50_50_2_50	2	50	50	50	1003.88	-114.782	-76.334
RNMDT + Alg 2	50_50_3_100	3	50	50	100	1006.24	-224.163	-120.102
RNMDT + Alg 2	50_50_3_50	3	50	50	50	1003.39	-111.522	-78.2193
RNMDT + Alg 2	50_75_1_100	1	50	75	100	1008.83	-255.05	-140.67
RNMDT + Alg 2	50_75_1_50	1	50	75	50	1004.72	-117.785	-92.8247
RNMDT + Alg 2	50_75_2_100	2	50	75	100	1008.8	-198.693	-74.6084
RNMDT + Alg 2	50_75_2_50	2	50	75	50	1004.37	-110.371	-67.653
RNMDT + Alg 2	50_75_3_100	3	50	75	100	1008.29	-209.269	-107.373
RNMDT + Alg 2	50_75_3_50	3	50	75	50	1004.27	-105.641	-65.6177
RNMDT + Alg 2	8_12_1_25	1	8	12	25	0.23	-6.44	-6.44
RNMDT + Alg 2	8_12_2_25	2	8	12	25	0.22	-2.14	-2.14
RNMDT + Alg 2	8_12_3_25	3	8	12	25	0.21	-9.21	-9.21
RNMDT + Alg 2	8_16_1_25	1	8	16	25	0.22	-2.49	-2.49
RNMDT + Alg 2	8_16_2_25	2	8	16	25	0.21	-2.79	-2.79
RNMDT + Alg 2	8_16_3_25	3	8	16	25	0.24	-2.44	-2.44
RNMDT + Alg 2	8_8_1_25	1	8	8	25	0.21	-3.97	-3.97
RNMDT + Alg 2	8_8_2_25	2	8	8	25	0.26	-0.56	-0.56
RNMDT + Alg 2	8_8_3_25	3	8	8	25	0.2	-5.3	-5.3

B
Box plots

B.1
Literature instances

The average relative gap could, in principle, be influenced by one or two instances that were outliers compared to the others of the same class of size and density. To verify if the same observed behavior of the average is the same for the entire class, Tukey box plots [86] were plotted in Figures B.1, Figure B.2 and Figure B.3 for the small, medium and larger instances respectively.

The same trend observed for the average analysis was confirmed in the box plot analysis. The proposed dynamic algorithm is particular relevant in the presence of high density and larger instances, i.e., in the presence of a large number of quadratic terms and continuous variables to be discretized as can be observed by the almost steps-like box plots for the medium 100% density and large with 50% and 100% densities.

Figure B.1: Small instances - Relative gaps - box plots

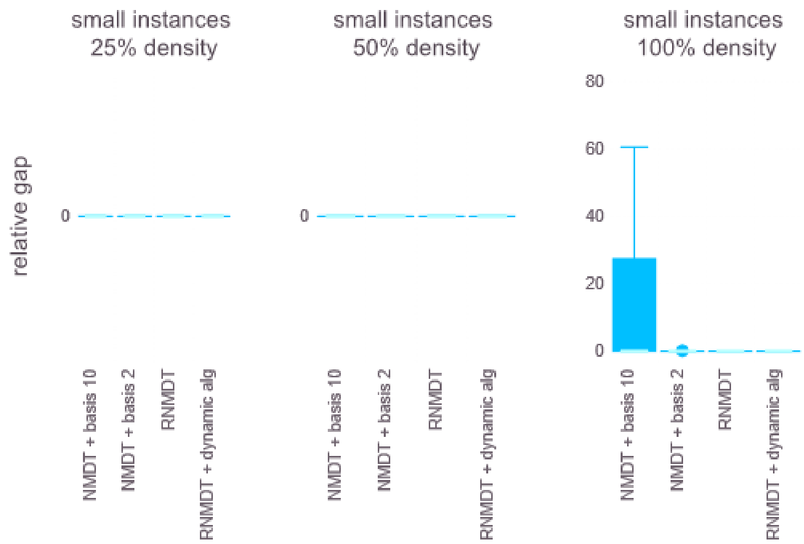


Figure B.2: Medium instances - Relative gaps - box plots

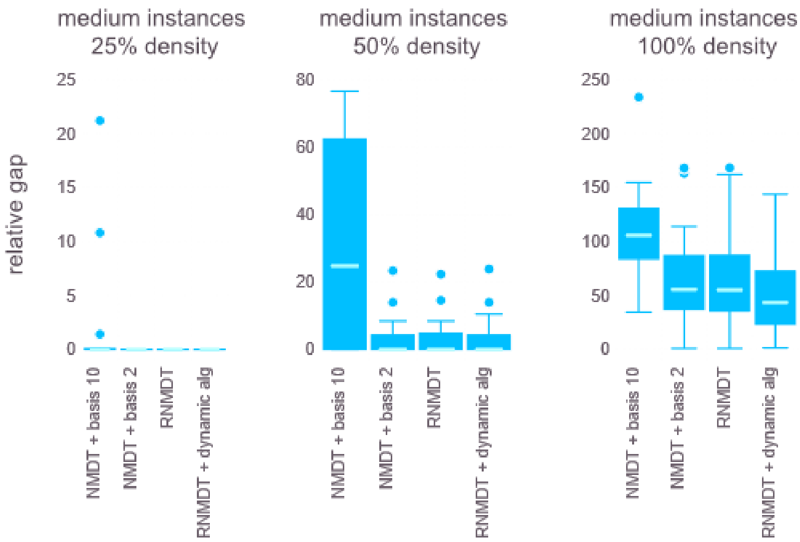
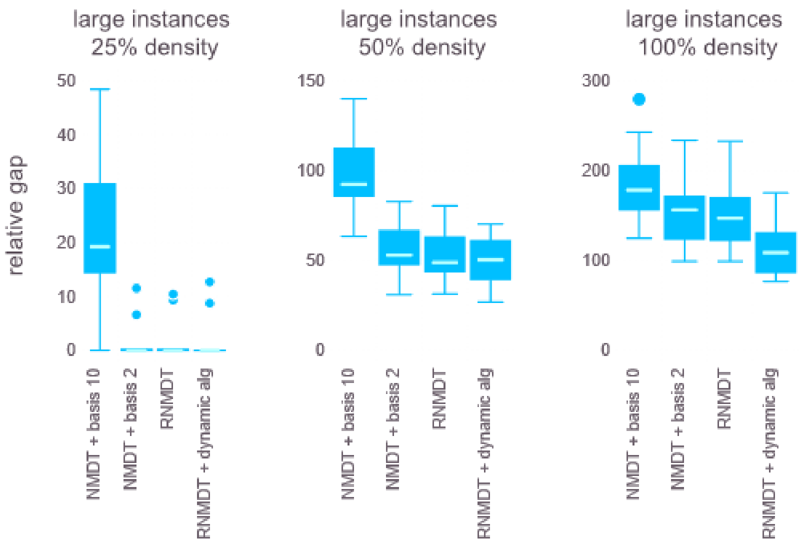


Figure B.3: Large instances - Relative gaps - box plots



B.2
Generated instances

Figure B.4 shows the box plot for the relative gap obtained in this instances comparing the 3 proposed improvements with the version found in [37]. The same behavior of the QCQP instances is seem for the MIQCP, with the two largest improvement in the relative gap being due to the change of the decimal expansion for the binary expansion and with an additional improvement in the relative gaps with Algorithm 2. RNMDT with Algorithm 1 was almost the same as NMDT with base 2 with the exception of one instance.

Figure B.4: Relative gap for mixed integer instances