4
CUT GENERATION ALGORITHM

4.1
Active cuts at the optimal solution

Depending on the percentage error threshold (and on the minimum and maximum scenarios' probabilities should the approximation cuts be determined as discussed at the end of Chapter 3) the number of necessary cuts may grow to be very large, leading to computational difficulties and slower performance of solution algorithms.

However, the observation that only a small fraction of these cuts will be active at the optimal solution of problem (3.12) – (3.17) – only $|S|$ cuts represented in the set of constraints (3.15) will be actually binding – naturally points towards the design of an algorithm that dynamically generates the cuts to construct the piecewise linear approximation to the exponential function.

Next, we follow the notation and terminology of Geoffrion (1972) [27]: the value of the objective function at the optimal solution of an optimization problem $(\cdot)$ is denoted by $v(\cdot)$ and its set of feasible solutions by $F(\cdot)$. Additionally, $\cdot^*$ denotes the value of variable $\cdot$ at the optimal solution.

4.2
Solution properties

The original problem (2.1) – (2.6) and its re-formulated linear counterpart (3.12) – (3.17) have exactly the same set of feasible solutions (or, more precisely, any feasible solution to one may be mapped into the feasible solution space of the other), which may be expressed by $F(P) = F(P_2)$. In addition, if we denote the true second-stage cost function by $Q(x, \xi(x))$ and its piecewise linear approximation by $\hat{Q}(x, \xi(x))$ then, by construction, the following relation holds for all feasible x:

$$\hat{Q}\big(x,\xi(x)\big) \leq Q\big(x,\xi(x)\big), \tag{4.1}$$

Consequently, the value of the optimal solution of problem (2.1) – (2.6) will always be greater or equal to the optimal value of problem (3.12) – (3.17), i.e.:

$$v(P_2) \leq v(P) \tag{4.2}$$

4.3
Approximation of the second-stage cost function

Based on the previous remarks, the following algorithm (ALG1) may be used in order to obtain a solution to the problem for which the percentage error of the approximation of the second-stage cost function is less or equal to ε:

Initialize the set of cuts $K = \emptyset$, the lower bound $LB = -inf$, upper bound $UB = +inf$ and define the maximum percentage error ε

While $|(UB - LB)/UB| > \varepsilon$

    Solve problem P_2 defined by (3.12) – (3.17) with the currently defined set of cuts K

    Set $LB = v(P_2) - \sum_{e \in E} r_e x_e^* \left(= \sum_{s \in S} g_s \cdot \hat{p}_s^*\right)$

    Set $UB = \sum_{s \in S} g_s \cdot \exp(w_s^*)$

    For each scenario $s \in S$

        Add the cut defined by $\alpha_k = \exp(w_s^*) \cdot (1 - w_s^*)$ and $\beta_k = \exp(w_s^*)$ to the cut set K

    End For

End While

The algorithm works by gradually constructing a better approximation of the second stage cost function through the addition of cuts around the optimal values of variables w_s^* found at each iteration. Following the discussion in Chapter 3, the addition of a cut centered on a specific value w_s^* provides an approximation that may also be useful (i.e., for which the percentage approximation error is smaller than ε) for other possible values of the same variable which may be part of the optimal solution found in subsequent iterations.

4.3.1
Convergence analysis

The following proposition determines the maximum number of iterations of the algorithm needed in order to obtain a solution for which the percentage error of the approximated second-stage cost function relative to the true function is no larger than ε.

Proposition 3. Let η^{+}, λ_s^e and μ_s^e be defined as in Chapter 3, then algorithm ALG1 converges to a solution of problem $\boldsymbol{P}$ for which the percentage gap of the approximated second-stage cost function relative to its exact counterpart is less or equal to ε in a number of iterations not larger than:

$$\sum_{s \in S} \left(\left\lceil \frac{(\ln(\prod_{e \in E} \mu_e^s) - \ln(\prod_{e \in E} \lambda_e^s))}{\eta^{+}} \right\rceil \right) \tag{4.3}$$

Proof. As per the result of Proposition 1, if the convergence criterium of the algorithm has not been met at a given iteration i, it means that there exists at least one $s \in S$ for which $\frac{(\exp(w_s^*) - \hat{p}_s^*)}{\exp(w_s^*)} > \varepsilon$. Since it can be verified that $|\eta^{+}| < |\eta^{-}|$, this implies the fact that there exists at least one variable w_s $(s \in S)$ which satisfies the relation:

$$\left| w_s^{*(i)} - w_s^{*(j)} \right| > \eta^{+}, \forall j < i \tag{4.4}$$

where $w_s^{*(i)}$ denotes the value of variable w_s at the optimal solution of problem $\boldsymbol{P_2}$ solved at iteration i ($w_s^{*(j)}$ are thus the points around which the piecewise linear approximation to the exponential function has been built in previous iterations).

Let $s \in S$ be a scenario for which relation (4.4) holds and let δ be the total length of the region(s) within the feasible interval of variable w_s for which the current approximation violates the maximum percentage error threshold. The addition of a cut around the value $w_s^{*(i)}$ reduces δ by at least η^+ (and, potentially, by $\eta^+ - \eta^-$). As discussed in Chapter 3, and repeated here for convenience, the condition of each edge is known for each scenario, thus allowing us to determine the feasible interval for each variable w_s as:

$$\left\{\ln(\prod_{e\in E}\lambda_e^s)\,,\ln(\prod_{e\in E}\mu_e^s)\right\} \tag{4.5}$$

At different iterations, each variable w_s ($s \in S$) may satisfy condition (4.4) at most $\lceil(\ln(\prod_{e\in E}\mu_e^s) - \ln(\prod_{e\in E}\lambda_e^s))/\eta^+\rceil$ times – since, after that, the approximation of the exponential function over all its feasible region will be so that the maximum percentage error is less or equal to ε. The result on the maximum number of iterations of the algorithm follows naturally.

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4.4
An algorithm considering the gap to the global optimal solution

The approximation of the second-stage cost function at the solution obtained by the algorithm presented in the previous Section is ensured to be within ε percentage points of the true function. However, the gap between the solution returned by the algorithm and the global optimal solution to the problem may be different since it depends on the first-stage cost function as well.

A slight modification to the algorithm may be introduced in order to account for the percentage gap between the solution of the problem solved using the approximation to the second-stage cost function and the global optimum, as shown below (ALG2):

1	Initialize the set of cuts $K = \emptyset$, the lower bound $LB = -inf$, upper bound $UB = +inf$ and define the maximum percentage error ε
2	While $\lvert(UB - LB)/UB\rvert > \varepsilon$

Solve problem P_2 defined by (3.12) – (3.17) with the currently defined set of cuts K

Set $LB = v(P_2)$

Set $UB_{aux} = \sum_{e \in E} r_e x_e^* + \sum_{s \in S} g_s \cdot \exp(w_s^*)$

If $UB_{aux} < UB$, set $UB = UB_{aux}$

For each scenario $s \in S$

    Add the cut defined by $\alpha_k = \exp(w_s^*) \cdot (1 - w_s^*)$ and $\beta_k = \exp(w_s^*)$ to the cut set K

End For

End While

The algorithm above works by (*i*) obtaining a series of feasible solutions for the original problem and (*ii*) progressively perfecting the approximation of the second stage cost function at each iteration, as in ALG1.

On the one hand, the series of feasible solutions provide a monotonically decreasing sequence of upper bounds. On the other hand, the series of values of the objective function at the optimal solution of the approximated problem solved at each iteration constitutes a monotonically increasing sequence of lower bounds, since $\hat{Q}_{i+1}(x, \xi(x)) \geq \hat{Q}_i(x, \xi(x))$ for all feasible x (where $\hat{Q}_i(x, \xi(x))$ denotes the piecewise linear approximation of the second stage cost function at iteration i).

In this case, a simple upper bound on the number of iterations until the convergence of the algorithm is given by $|S| \cdot 2^{|E|}$, which would correspond to a complete enumeration of the linear constraints that provide an exact representation of the exponential function at all possible values of each variable w_s.