3
REFORMULATION SCHEME

In this Chapter, a reformulation scheme which overcomes the difficulties associated with the existence of non-linear terms in the problem formulation will be presented. Section 3.1 describes the argument which allows the elimination of the product between first and second stage variables while Section 3.2 proposes a linearization technique that eliminates the products among binary variables; Section 3.3 is dedicated to a discussion of the approximation error and of how it can be managed when solving a problem.

3.1
Separability of second stage problems

The product between variables p_s and y_{es} in the objective function may be removed by observing that the feasible regions of the second-stage problems – sets of constraints (2.3) and (2.5) – are decoupled from first-stage variables. The second-stage problem of each scenario may then be solved independently of the others:

$$\forall s \in S, g_s = \quad Min \quad \sum_{e \in E} c_e y_{es} + \sum_{i \in N} d_i z_{is} \tag{3.1}$$

$$subject\ to: \quad W_s y_s + z_s = h_s \tag{3.2}$$

$$y_{es} \leq u_e \xi_{es} \qquad \forall e \in E \tag{3.3}$$

$$y, z \in \mathbb{R}^+ \tag{3.4}$$

As shown above, we denote by g_s the value of the optimal solution of problem (3.1) – (3.4) for a given scenario s, which then allows us to re-write problem (2.1) – (2.6) as follows:

$$(\boldsymbol{P_1}) \qquad Min \qquad \sum_{e \in E} r_e x_e + \sum_{s \in S} p_s g_s \tag{3.5}$$

$$subject\ to: \quad Ax \leq b \tag{3.6}$$

$$p_s = \prod_{e \in E} (\mathrm{p}_{es}^C + (\mathrm{p}_{es}^I - \mathrm{p}_{es}^C) \cdot x_e) \qquad \forall s \in S \tag{3.7}$$

$$x \in \{0,1\}^{|E|} \tag{3.8}$$

In the following we will assume that $g_s \geq 0, \forall s \in S$. However, this hypothesis comes without loss of generality, as shown in Appendix A.

3.2 Polynomials in binary variables

A remaining difficulty in solving problem (3.5) – (3.8) lies on the product of binary variables x_e in the definition of variables p_s – each equation defined in the set of constraints (3.7) is a polynomial of order $|E|$.

There has been a significant amount of research on the linearization of the product of binary variables. Following the initial article of Glover in 1975 [29], there have been related works focused on quadratic functions – Hansen and Meyer (2009) [34], Balas and Mazzola (1984) in [4] and [5], Gueye and Michelon (2005) [32] – but some authors have also considered the case of cubic and higher-degree polynomials – c.f., Adams and Forrester (2005) 0, Chang (2000) [17], Chang and Chang (2000) [18], Oral and Ketani (1990 and 1992) [44] and [45]. Essentially, the proposed techniques resort to the addition of auxiliary variables and constraints to linearize each non-linear term in the problem. Since the definition of each variable p_s implies an exponential number of nonlinear terms ($\sum_{k=2}^{|E|} \binom{|E|}{k}$ or, equivalently, $2^{|E|} - |E| - 1$), these methods result impractical for the class of problems under consideration. The special structure of the polynomials defined in the set of constraints (3.7) – specifically, the fact that they may be written as the product of linear terms in the form $a \cdot x + b$, where $a > 0$ and $a + b > 0$ – allows for the straightforward application of the linearization technique proposed in this thesis, described below.

3.2.1
Proposed linearization technique

By relying on the fact that $a = b \cdot c \rightarrow a = \exp(\ln b + \ln c)$, each equation in (3.7) may be re-written as:

$$\mathrm{p}_s = \exp\left(\sum_{e \in E} \ln(\mathrm{p}_{es}^C + (\mathrm{p}_{es}^I - \mathrm{p}_{es}^C) \cdot x_e)\right) \tag{3.9}$$

Since x is a vector of binary variables, the expression within the summation operator may also be re-written in such a way that variables x_e are not part of the logarithmic expression. This is accomplished by observing that the argument of each logarithm is p_{es}^C if x_e is equal to 0 and p_{es}^I otherwise, leading to:

$$\mathrm{p}_s = \exp\left(\sum_{e \in E} \{\ln(\mathrm{p}_{es}^C) + [\ln(\mathrm{p}_{es}^I) - \ln(\mathrm{p}_{es}^C)] \cdot x_e\}\right) \tag{3.10}$$

A continuous variable may be defined as the logarithm of the probability of each scenario, thus being an affine function of variables x_e (this auxiliary variable is introduced for ease of presentation but it is not strictly necessary):

$$w_s = \ln(\mathrm{p}_s) = \sum_{e \in E} \{\ln(\mathrm{p}_{es}^C) + [\ln(\mathrm{p}_{es}^I) - \ln(\mathrm{p}_{es}^C)] \cdot x_e\} \tag{3.11}$$

Having the value of the natural logarithm of the probability of a scenario given by expression (3.11), the actual value of its probability (i.e., the value of p_s) may be obtained by a piecewise linear approximation of the exponential function. Since the optimization sense of the problem is minimization and the exponential function is convex, this approximation may be represented by a set of linear constraints which can be incorporated into the problem.

Example. Let there be a network connecting cities A, B and C composed of two links (AB and BC), as shown in Figure 3-1 below. Suppose the current (i.e., pre-investment) survival probability of each link is given by and . If a reinforcement investment is made on each link, these probabilities increase to and , respectively. There are obviously four possible scenarios of network configuration and we will use the one where both links are operational to illustrate the proposed linearization technique.

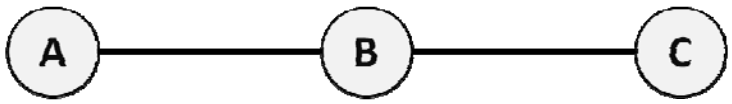

Figure 3-1 – Two-link network example

The probability of the scenario in which both links survive ( ) is given by the following expression:

(3.12)

The application of the logarithm to both sides of equation (3.12) results in:

(3.13)

The first term of the right-hand side of expression (3.13) is equivalent to (or, written in a slightly different form,). An analogous transformation may be applied to the second term of the right-hand side of expression (3.13), resulting in:

(3.14)

The values of the logarithm of the probability of occurrence of scenario (i.e., the possible values of variable defined above) are given in the Table below for all possible values of the investment decision variables and .

Table 3-1 – Probability of occurrence of scenario according to the investment decisions

x_{AB}	x_{BC}	$ln(P_{AB})$	$ln(P_{BC})$	$ln(P_{s_1})$
0	0	$\ln(50\%)$	$\ln(60\%)$	$\ln(30\%) = -1.204$
0	1	$\ln(50\%)$	$\ln(90\%)$	$\ln(45\%) = -0.799$
1	0	$\ln(70\%)$	$\ln(60\%)$	$\ln(42\%) = -0.868$
1	1	$\ln(70\%)$	$\ln(90\%)$	$\ln(63\%) = -0.462$

The scenario's actual probability of occurrence, represented by variable $\hat{p}_{s_1}$, may be obtained by adding to the problem the inequalities corresponding to the first order terms of the Taylor series expansion of the exponential function around the possible values of variable w_{s_1} – specified below and represented by the linear segments depicted in the following Figure:

$$\hat{p}_{s_1} \geq 30\% + 30\% \cdot (w_{s_1} - \ln(30\%)) \tag{3.15}$$

$$\hat{p}_{s_1} \geq 42\% + 42\% \cdot (w_{s_1} - \ln(42\%)) \tag{3.16}$$

$$\hat{p}_{s_1} \geq 45\% + 45\% \cdot (w_{s_1} - \ln(45\%)) \tag{3.17}$$

$$\hat{p}_{s_1} \geq 63\% + 63\% \cdot (w_{s_1} - \ln(63\%)) \tag{3.18}$$

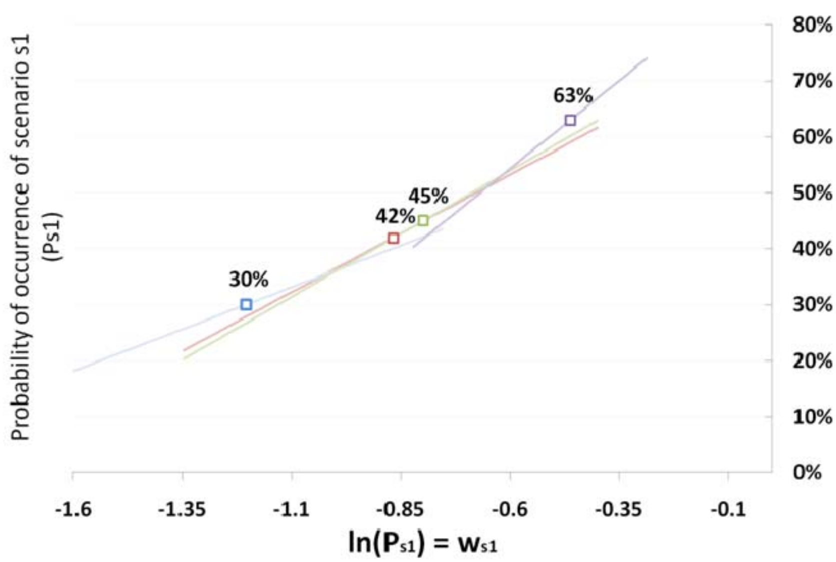

Figure 3-2 – Inequalities that provide a piecewise linear approximation to the exponential function

This would ensure that for every possible combination of the values of variables x_{AB} and x_{BC}, the value of $\hat{p}_{s_1}$ would be exactly equal to the probability of occurrence of scenario s_1.

3.2.2
Reformulation

Following the linearization technique proposed in the previous Section, we re-write problem (3.5) – (3.8), eliminating the non-linearities:

$$(\boldsymbol{P_2}) \quad Min \quad \sum_{e \in E} r_e x_e + \sum_{s \in S} g_s \hat{p}_s \tag{3.19}$$

$$subject\ to: \quad Ax \leq b \tag{3.20}$$

$$w_s = \sum_{e \in E} \{\ln(\mathrm{p}_{es}^C) + [\ln(\mathrm{p}_{es}^I) - \ln(\mathrm{p}_{es}^C)] \cdot x_e\} \qquad \forall s \in S \tag{3.21}$$

$$\hat{p}_s \geq \alpha_k + \beta_k \cdot w_s \qquad \forall s \in S, \forall k \in K \tag{3.22}$$

$$\hat{p} \in \mathbb{R}^+, w \in \mathbb{R} \tag{3.23}$$

$$x \in \{0,1\}^{|E|} \tag{3.24}$$

where:

K	set of linear constraints that approximate the exponential function
α_k, β_k	coefficients of the k-th segment used to approximate the exponential function
w_s	continuous variable equal to the natural logarithm of the probability of scenario s
$\hat{p}_s$	continuous variable equal to the approximation of the probability of scenario s

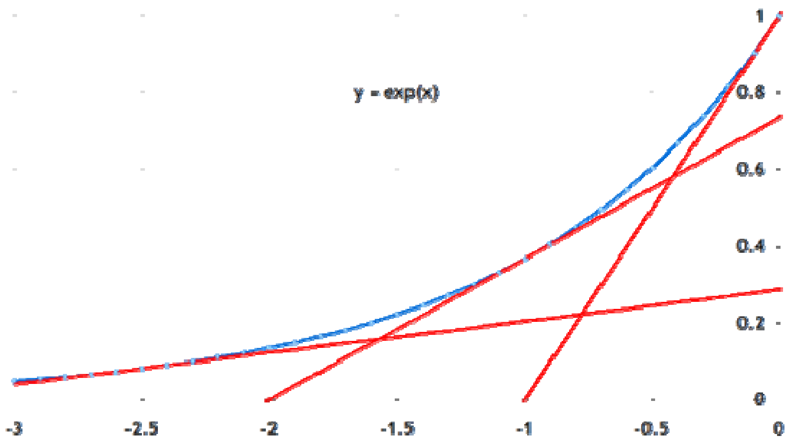

Figure 3-3 – Piecewise linear approximation of the exponential function

Given an approximation to the exponential function (i.e., given a set of cuts in the form $y \geq \exp(w_0) + \exp(w_0) \cdot (w - w_0)$ that provide a piecewise linear approximation to the exponential function) and assuming it is computationally feasible to enumerate and solve the second stage problems for all possible network configurations, one is able to solve problem (3.19) – (3.24) using commercially available solvers. The following sub-sections discuss the necessary number of additional constraints for an exact solution to the problem and the generation of constraints for a given error tolerance level ε.

3.2.3
Additional constraints

According to the set of constraints (3.21), each variable w_s is equal to the sum of the logarithm of the probability of the availability status of each edge e in scenario s. Each of these logarithms may take one out of two possible values depending on whether an investment is made on the corresponding edge and, consequently, variables w_s may potentially assume $2^{|E|}$ different values. In order to guarantee that the optimal solution to the reformulated problem corresponds to the global optimum of the original problem, there must be a cut to approximate the exponential function centered on each one of these values and, therefore, each equation defined in (3.21) requires the addition of $2^{|E|}$ constraints to the problem.

3.3
Approximation

As described above, the exact representation of the nonlinear terms of the problem requires an exponential number of additional constraints and this may cause the problem to grow prohibitively large even for medium-sized instances. In this section we discuss an approximation to the problem which allows it to be solved for larger instances whilst maintaining the approximation error bounded.

3.3.1
Generation of cuts for an error tolerance threshold ε

The solution of any two-stage stochastic program is essentially related to the determination of the optimal trade-off between deterministic first-stage costs and expected (probabilistic) second-stage costs. Therefore, the quality of the optimal solution of problem (3.19) – (3.24), which results from the application of the proposed linearization technique, relies on the quality of the piecewise linear approximation of the exponential function. Given a set K of linear constraints and a solution to the corresponding problem, the absolute error (i.e., the difference between the true value of the second-stage cost function and its approximation) is equal to:

$$\sum_{s \in S} g_s(\exp(w_s) - \hat{p}_s) \tag{3.25}$$

The percentage error is obtained by dividing the absolute error by the true value of the second-stage function at a solution:

$$\frac{\sum_{s \in S} g_s \cdot (\exp(w_s) - \hat{p}_s)}{\sum_{s \in S} g_s \cdot \exp(w_s)} \tag{3.26}$$

An approximation which guarantees the maximum percentage error to be below a given tolerance level ε may be constructed based on the following proposition:

Proposition 1. Let F be the set of elements $\{a_i/b_i\}_{i=1}^{N}$ and $\varepsilon^{MAX} = \max_i\{a_i/b_i\}$, then

$$\frac{\sum_{i=1}^{N} a_i}{\sum_{i=1}^{N} b_i} \leq \varepsilon^{MAX} \tag{3.27}$$

Proof.

$$\frac{\sum_{i=1}^{N} a_i}{\sum_{i=1}^{N} b_i} \leq \frac{\sum_{i=1}^{N} \epsilon^{MAX} b_i}{\sum_{i=1}^{N} b_i} = \epsilon^{MAX} \cdot \frac{\sum_{i=1}^{N} b_i}{\sum_{i=1}^{N} b_i} = \varepsilon^{MAX} \tag{3.28}$$

■

This result ensures that provided $(\exp(w_s) - \hat{p}_s)/\exp(w_s) \leq \varepsilon, \forall s \in S$ (i.e., the percentage error of the piecewise linear approximation is less or equal to ε for each scenario) for all scenarios and possible values of w_s, then the percentage error of the approximation to the second stage cost function is also not greater than ε.

For a given set of cuts K one can easily verify in $O(|K|)$ whether the condition is satisfied (since the largest error between two adjancent cuts occurs at the point where they intersect) which allows for various heuristic/iterative methods for generating a piecewise linear approximation that guarantees that a maximum percentage error threshold is not violated. In the next sub-section, the minimum number of cuts necessary for an ε-approximation of the second stage cost function along with a method for generating them will be shown.

3.3.2
Minimum number of cuts

The following proposition establishes the minimum number of cuts necessary for an approximation of the second stage cost function whose percentage error is not greater than ε.

Proposition 2. Let $q_e^C \xi_{es} + (1 - q_e^C) \cdot (1 - \xi_{es})$ and $q_e^I \xi_{es} + (1 - q_e^I) \cdot (1 - \xi_{es})$ be the two possible values for the probability of the availability status of edge e in scenario s and $W_s = \{\ln(p_s^i)\}_{i=1}^{2^{|E|}}$ be the set of all possible values that the logarithm of the probability of scenario s may assume (given by all combinations of the product of the edges' probabilities). Also, let η^+ and η^- be, respectively, the positive and negative roots of the equation $1 - \exp(\eta) + \eta \exp(\eta) = \varepsilon$. Then, the minimum number of additional constraints necessary for an approximation which guarantees the percentage error to be less or equal to ε is given by $\sum_{s \in S} \varphi_s$, where φ_s is the optimal value of the following optimization problem:

$$\varphi_s = \quad Min \quad \sum_{j=1}^{2^{|E|}} z_j \tag{3.29}$$

$$subject\ to: \quad y_j \geq M_1(1 - z_j) - M_1 z_j \qquad \forall j = 1, \dots, 2^{|E|} \tag{3.30}$$

$$\ln(p_s^i) \geq y_j - \eta^+ - M_2(1 - x_{ij}) \qquad \forall i, j = 1, \dots, 2^{|E|} \tag{3.31}$$

$$\ln(p_s^i) \leq y_j - \eta^- + M_3(1 - x_{ij}) \qquad \forall i, j = 1, \dots, 2^{|E|} \tag{3.32}$$

$$x_{ij} \leq z_j \qquad \forall i, j = 1, \dots, 2^{|E|} \tag{3.33}$$

$$\sum_{j=1}^{2^{|E|}} x_{ij} \geq 1 \qquad \forall i = 1, \dots, 2^{|E|} \tag{3.34}$$

$$z, x \in \{0, 1\} \tag{3.35}$$

$$y \in \mathbb{R} \tag{3.36}$$

Proof.

The percentage error of the approximation provided by a cut centered at point w_0 is given by the following expression:

$$\frac{\exp(w) - (\exp(w_0) + \exp(w_0) \cdot (w - w_0))}{\exp\,(w)} \tag{3.37}$$

where $\exp(w)$ is the true value of the exponential function (i.e., the true value of the probability of occurrence of a scenario) and $(\exp(w_0) + \exp(w_0) \cdot (w - w_0))$ is the approximation provided by a cut centered on w_0 as discussed in Section 3.2.1.

By rearranging the terms, this expression may be re-written as:

$$1 - \exp(w_0 - w) + (w_0 - w) \cdot \exp(w_0 - w) \qquad (3.38)$$

which is a strictly concave function and analogous to the equation defined in the Proposition by defining $\eta = w_0 - w$. Observe that the percentage error depends not on specific values of w or w_0 individually, but solely on the difference between the point in question w and the point at which the approximation is centered on, w_0. Consequently, the percentage error within the interval $\{w_0 - \eta^+, w_0 - \eta^-\}$ resulting from an approximation centered on any given point w_0 is less or equal to ε. This is illustrated in the Figures 3.4 and 3.5.

Regarding the optimization problem (3.29) – (3.36) corresponding to a given scenario s, binary variables z_j indicate the addition of a cut centered on the value of continuous variable y_j; variables x_{ij} indicate that a given point $\ln(p_s^i)$ is assigned to the cut centered on y_j, which – according to constraints (3.31) and (3.32) – can only occur if $\ln(p_s^i)$ is within the interval $\{y_j - \eta^+, y_j - \eta^-\}$ (i.e., if the approximation error at point $\ln(p_s^i)$ provided by the cut centered on y_j is less or equal to ε); M_1, M_2 and M_3 are sufficiently large positive numbers. Objective function (3.29) represents the number of cuts which are effectively needed to ensure the approximation error for all elements of the set W_s is no larger than ε. Contraint (3.30) ensures that a cut can only provide a useful approximation if the the corresponding variable z_j is properly set to 1; constraint (3.33)[1] determines that a given point $\ln(p_s^i)$ may only be assigned to a valid cut and constraint (3.34) requires each element of the set W_s to be assigned to at least one cut.

The solution of such problem determines not only the number of necessary cuts that provide an approximation for which the error is not larger than ε ($\sum_{j=1,\dots,2^{|E|}} z_j$) but also the exact points at which they should be centered on ($\{y_j | z_j = 1\}_{j=1}^{2^{|E|}}$).

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[1] Constraint (3.26) is actually redundant, given the set of constraints (3.23) to (3.25).

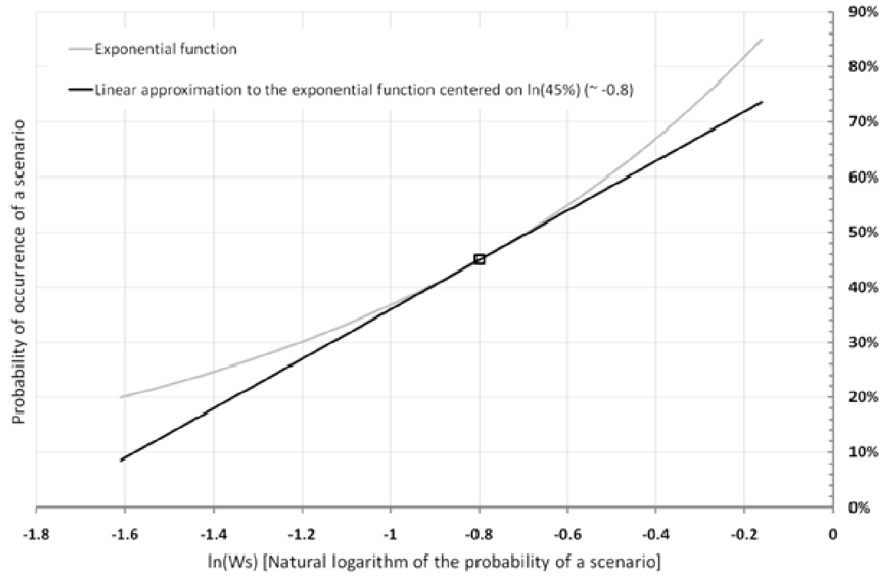

Figure 3-4 – Linear approximation to the exponential function provided by a cut centered on ln(45%)

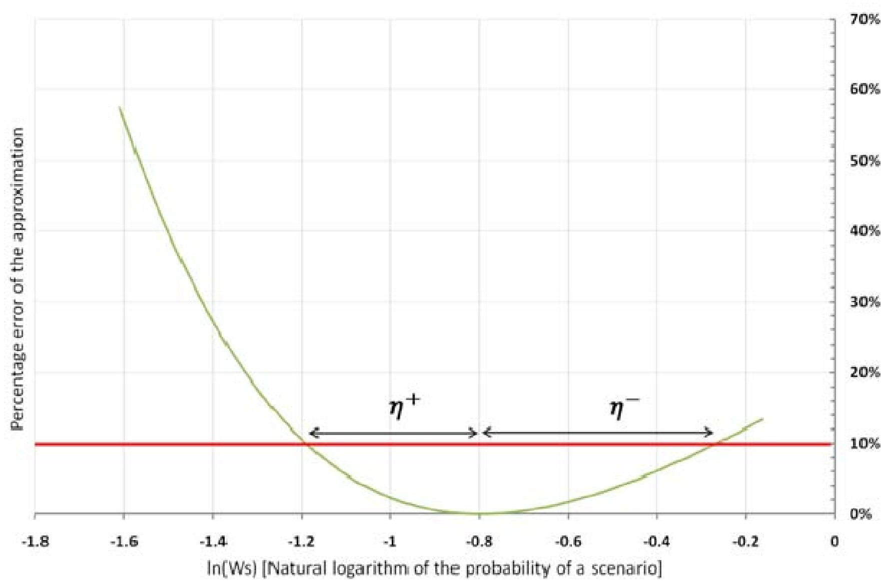

Figure 3-5 – Percentage error provided by a linear approximation centered on ln(45%) and illustration of $\boldsymbol{\eta}^+$ and $\boldsymbol{\eta}^-$ for $\boldsymbol{\varepsilon} = \mathbf{10}\%$

The size of each optimization problem defined in the previous Proposition, including the number of binary variables, grows exponentially with the number of edges of the graph corresponding to the transportation network of a given instance of the humanitarian logistics problem. This may cause the computational burden to be excessively large and ultimately render its solution to optimality very unlikely. Next, we discuss a relatively simpler approach to determining a set of cuts that provide an approximation that does not violate the bound on the maximum error and is much easier to compute since it does not require the full enumeration of the elements of the sets W_s.

3.3.3
An easier way to generate the cuts

Since the condition of each edge (i.e., whether each edge is active or failed) is known for each scenario, the feasible interval for each variable w_s is given by:

$$\left\{\ln\left(\prod_{e\in E}\lambda_e^s\right),\ln\left(\prod_{e\in E}\mu_e^s\right)\right\} \tag{3.39}$$

where $\lambda_e^s = \min\,\{q_e^C\xi_{es} + (1-q_e^C)\cdot(1-\xi_{es}),\ q_e^I\xi_{es} + (1-q_e^I)\cdot(1-\xi_{es})\}$ and $\mu_e^s = \max\,\{q_e^C\xi_{es} + (1-q_e^C)\cdot(1-\xi_{es}),\ q_e^I\xi_{es} + (1-q_e^I)\cdot(1-\xi_{es})\}$. The interval defined in expression (3.32) thus contains all the possible values of a given variable w_s and an approximation that ensures that the percentage error is not violated at any point within this range may be easily computed by adding the cuts corresponding to the first order Taylor's expansion of the exponential function around the points $\{\ln(\prod_{e\in E}\lambda_e^s) + \eta^+ + k\cdot(\eta^+ - \eta^-)\}_{k=0}^{\tau_s-1}$ (or, alternatively, $\{\ln(\prod_{e\in E}\mu_e^s) + \eta^- - k\cdot(\eta^+ - \eta^-)\}_{k=0}^{\tau_s-1}$), where τ_s is defined as follows:

$$\tau_s = \left\lceil\frac{(\ln(\prod_{e\in E}\mu_e^s) - \ln(\prod_{e\in E}\lambda_e^s))}{(\eta^+ - \eta^-)}\right\rceil \tag{3.40}$$

For each scenario, this procedure results in a number of inequalities which is, obviously, an upper bound to the optimal solution of the optimization problem defined in Proposition 2 and in a total number of cuts equal to:

$$\sum_{s\in S}\left(\left\lceil\frac{(\ln(\prod_{e\in E}\mu_e^s) - \ln(\prod_{e\in E}\lambda_e^s))}{(\eta^+ - \eta^-)}\right\rceil\right) \tag{3.41}$$